Lecture 16

Existence and Uniqueness of Solutions of Systems (Contd.)

In this lecture we shall continue extending the results for the initial value problem (15.4) some of which are analogous to those proved in earlier lectures for the problem (7.1).

Theorem 16.1 (Continuation of Solutions). Assume that $g(x, u)$ is continuous in E and $u(x)$ is a solution of the problem (15.4) in an interval J. Then $u(x)$ can be extended as a solution of (15.4) to the boundary of E.

Corollary 16.2. Assume that $g(x, u)$ is continuous in

$$E_1 \ = \ \{(x, u) \in E: \quad x_0 \leq x < x_0 + a, \quad a < \infty, \quad \|u\| < \infty\}.$$

If $u(x)$ is any solution of (15.4), then the largest interval of existence of $u(x)$ is either $[x_0, x_0 + a]$ or $[x_0, x_0 + \alpha)$, $\alpha < a$ and $\|u(x)\| \to \infty$ as $x \to x_0 + \alpha$.

Theorem 16.3 (Perron's Uniqueness Theorem). Let $f(x, y)$, $f(x, 0) \equiv 0$, be a nonnegative continuous function defined in the rectangle $x_0 \leq x \leq x_0 + a$, $0 \leq y \leq 2b$. For every $x_1 \in (x_0, x_0 + a)$, let $y(x) \equiv 0$ be the only differentiable function satisfying the initial value problem

$$y' \ = \ f(x, y), \quad y(x_0) \ = \ 0 \tag{16.1}$$

in the interval $[x_0, x_1)$. Further, let $g(x, u)$ be continuous in $\Omega_+ : x_0 \leq x \leq x_0 + a$, $\|u - u^0\| \leq b$ and

$$\|g(x, u) - g(x, v)\| \ \leq \ f(x, \|u - v\|) \tag{16.2}$$

for all (x, u), $(x, v) \in \Omega_+$. Then the problem (15.4) has at most one solution in $[x_0, x_0 + a]$.

Proof. Suppose $u(x)$ and $v(x)$ are any two solutions of (15.4) in $[x_0, x_0 + a]$. Let $y(x) = \|u(x) - v(x)\|$, then clearly $y(x_0) = 0$, and from Problem 11.5 it follows that

$$D^+ y(x) \ \leq \ \|u'(x) - v'(x)\| \ = \ \|g(x, u(x)) - g(x, v(x))\|. \tag{16.3}$$

R.P. Agarwal and D. O'Regan, *An Introduction to Ordinary Differential Equations*,
doi: 10.1007/978-0-387-71276-5_16,

Using inequality (16.2) in (16.3), we obtain $D^+y(x) \le f(x, y(x))$. Therefore, from Theorem 11.6 it follows that $y(x) \le r(x),\ x \in [x_0, x_1)$ for any $x_1 \in (x_0, x_0 + a)$, where $r(x)$ is the maximal solution of (16.1). However, from the hypothesis $r(x) \equiv 0$, and hence $y(x) \equiv 0$ in $[x_0, x_1)$. This proves the theorem. ∎

In Theorem 16.3 the function $f(x, y) = h(x)y$, where $h(x) \ge 0$ is continuous in $[x_0, x_0 + a]$ is admissible, i.e., it includes the Lipschitz uniqueness criterion.

For our next result we need the following lemma.

Lemma 16.4. Let $f(x, y)$ be a nonnegative continuous function for $x_0 < x \le x_0 + a,\ 0 \le y \le 2b$ with the property that the only solution $y(x)$ of the DE $y' = f(x, y)$ in any interval (x_0, x_1) where $x_1 \in (x_0, x_0 + a)$ for which $y'_+(x_0)$ exists, and

$$y(x_0) \ = \ y'_+(x_0) \ = \ 0 \tag{16.4}$$

is $y(x) \equiv 0$. Further, let $f_1(x, y)$ be a nonnegative continuous function for $x_0 \le x \le x_0 + a,\ 0 \le y \le 2b,\ f_1(x, 0) \equiv 0$ and

$$f_1(x, y) \ \le \ f(x, y), \quad x \ne x_0. \tag{16.5}$$

Then for every $x_1 \in (x_0, x_0 + a),\ y_1(x) \equiv 0$ is the only differentiable function in $[x_0, x_1)$, which satisfies

$$y_1' \ = \ f_1(x, y_1), \quad y_1(x_0) \ = \ 0. \tag{16.6}$$

Proof. Let $r(x)$ be the maximal solution of (16.6) in $[x_0, x_1)$. Since $f_1(x, 0) \equiv 0,\ y_1(x) \equiv 0$ is a solution of the problem (16.6). Thus, $r(x) \ge 0$ in $[x_0, x_1)$. Hence, it suffices to show that $r(x) = 0$ in $[x_0, x_1)$. Suppose, on the contrary, that there exists a $x_2,\ x_0 < x_2 < x_1$ such that $r(x_2) > 0$. Then because of the inequality (16.5), we have

$$r'(x) \ \le \ f(x, r(x)), \quad x_0 < x \le x_2.$$

If $\rho(x)$ is the minimal solution of

$$y' \ = \ f(x, y), \quad y(x_2) \ = \ r(x_2),$$

then an application of Problem 11.6 implies that

$$\rho(x) \ \le \ r(x) \tag{16.7}$$

as long as $\rho(x)$ exists to the left of x_2. The solution $\rho(x)$ can be continued to $x = x_0$. If $\rho(x_3) = 0$, for some $x_3,\ x_0 < x_3 < x_2$, we can affect the continuation by defining $\rho(x) = 0$ for $x_0 < x < x_3$. Otherwise, (16.7) ensures the possibility of continuation. Since $r(x_0) = 0$, $\lim_{x\to x_0^+} \rho(x) = 0$, and we define $\rho(x_0) = 0$. Furthermore, since $f_1(x, y)$ is continuous at $(x_0, 0)$ and $f_1(x_0, 0) = 0$, $r'_+(x_0)$ exists and is equal to zero. This, because of (16.7), implies that $\rho'_+(x_0)$ exists and $\rho'_+(x_0) = 0$. Thus, $\rho'(x) = f(x, \rho(x))$, $\rho(x_0) = 0$, $\rho'_+(x_0) = 0$, and hence from the hypothesis on $f(x, y)$ it follows that $\rho(x) \equiv 0$. This contradicts the assumption that $\rho(x_2) = r(x_2) > 0$. Therefore, $r(x) \equiv 0$. ■

Theorem 16.5 (Kamke's Uniqueness Theorem). Let $f(x, y)$ be as in Lemma 16.4, and $g(x, u)$ as in Theorem 16.3, except that the condition (16.2) holds for all (x, u), $(x, v) \in \Omega_+$, $x \neq x_0$. Then the problem (15.4) has at most one solution in $[x_0, x_0 + a]$.

Proof. Define the function

$$f_g(x, y) \ = \ \sup_{\|u-v\|=y} \|g(x, u) - g(x, v)\| \tag{16.8}$$

for $x_0 \le x \le x_0 + a$, $0 \le y \le 2b$. Since $g(x, u)$ is continuous in Ω_+, the function $f_g(x, y)$ is continuous for $x_0 \le x \le x_0 + a$, $0 \le y \le 2b$. From (16.8) it is clear that the condition (16.2) holds for the function $f_g(x, y)$. Moreover, $f_g(x, y) \le f(x, y)$ for $x_0 < x \le x_0 + a$, $0 \le y \le 2b$. Lemma 16.4 is now applicable with $f_1(x, y) = f_g(x, y)$ and therefore $f_g(x, y)$ satisfies the assumptions of Theorem 16.3. This completes the proof. ■

Kamke's uniqueness theorem is evidently more general than that of Perron and it includes as special cases many known criteria, e.g., the following:

1. Osgood's criterion in the interval $[x_0, x_0 + a]$: $f(x, y) = w(y)$, where the function $w(y)$ is as in Lemma 10.3.

2. Nagumo's criterion in the interval $[x_0, x_0 + a]$: $f(x, y) = ky/(x - x_0)$, $k \le 1$.

3. Krasnoselski–Krein criterion in the interval $[x_0, x_0 + a]$:

$$f(x, y) \ = \ \min\left\{\frac{ky}{x - x_0},\ Cy^\alpha\right\}, \quad C > 0, \quad 0 < \alpha < 1, \quad k(1 - \alpha) < 1.$$

Theorem 16.6 (Continuous Dependence on Initial Conditions). Let the following conditions hold:

(i) $g(x, u)$ is continuous and bounded by M in a domain E containing the points (x_0, u^0) and (x_1, u^1).

(ii) $g(x, u)$ satisfies a uniform Lipschitz condition (15.5) in E.

(iii) $h(x, u)$ is continuous and bounded by M_1 in E.

(iv) $u(x)$ and $v(x)$ are the solutions of the initial value problems (15.4) and

$$v' \ = \ g(x, v) + h(x, v), \quad v(x_1) \ = \ u^1,$$

respectively, which exist in an interval J containing x_0 and x_1.

Then for all $x \in J$, the following inequality holds:

$$\begin{aligned}\|u(x) - v(x)\| \;\;\leq\;\; & \left(\|u^0 - u^1\| + (M + M_1)|x_1 - x_0| + \frac{1}{L}M_1\right) \\ & \times\; \exp\left(L|x - x_0|\right) - \frac{1}{L}M_1.\end{aligned}$$

Theorem 16.7 (Differentiation with Respect to Initial Conditions). Let the following conditions be satisfied:

(i) $g(x,u)$ is continuous and bounded by M in a domain E containing the point (x_0, u^0).

(ii) The matrix $\partial g(x,u)/\partial u$ exists and is continuous and bounded by L in E.

(iii) The solution $u(x, x_0, u^0)$ of the initial value problem (15.4) exists in an interval J containing x_0.

Then the following hold:

1. The solution $u(x, x_0, u^0)$ is differentiable with respect to u^0, and for each j $(1 \leq j \leq n)$, $v^j(x) = \partial u(x, x_0, u^0)/\partial u^0_j$ is the solution of the initial value problem

$$v' \;=\; \frac{\partial g}{\partial u}(x, u(x, x_0, u^0))v \tag{16.9}$$

$$v(x_0) \;=\; e^j \;=\; (0, \ldots, 0, 1, 0, \ldots, 0). \tag{16.10}$$

2. The solution $u(x, x_0, u^0)$ is differentiable with respect to x_0 and $v(x) = \partial u(x, x_0, u^0)/\partial x_0$ is the solution of the differential system (16.9), satisfying the initial condition

$$v(x_0) \;=\; -\,g(x_0, u^0). \tag{16.11}$$

Finally, in this lecture we shall consider the differential system

$$u' \;=\; g(x, u, \lambda), \tag{16.12}$$

where $\lambda = (\lambda_1, \ldots, \lambda_m) \in \mathbb{R}^m$ is a parameter.

If in (16.12) we treat $\lambda_1, \ldots, \lambda_m$ as new variables, then

$$\frac{d\lambda_i}{dx} \;=\; 0, \quad 1 \leq i \leq m. \tag{16.13}$$

Thus, the new system consisting of (16.12) and (16.13) is exactly of the form (15.1), but instead of n, now it is $(n+m)$-dimensional. Hence, for the initial value problem

$$u' \;=\; g(x, u, \lambda), \quad u(x_0) \;=\; u^0, \tag{16.14}$$

the result analogous to Theorem 12.4 can be stated as follows.

Theorem 16.8. Let the following conditions be satisfied:

(i) $g(x, u, \lambda)$ is continuous and bounded by M in a domain $E \subset \mathbb{R}^{n+m+1}$ containing the point (x_0, u^0, λ^0).

(ii) The matrix $\partial g(x, u, \lambda)/\partial u$ exists and is continuous and bounded by L in E.

(iii) The $n \times m$ matrix $\partial g(x, u, \lambda)/\partial \lambda$ exists and is continuous and bounded by L_1 in E.

Then the following hold:

1. There exist positive numbers h and ϵ such that for any λ satisfying $\|\lambda - \lambda^0\| \leq \epsilon$, there exists a unique solution $u(x, \lambda)$ of the problem (16.14) in the interval $|x - x_0| \leq h$.
2. For all λ^i such that $\|\lambda^i - \lambda^0\| \leq \epsilon,\ i = 1, 2$, and x in $|x - x_0| \leq h$ the following inequality holds:

$$\|u(x, \lambda^1) - u(x, \lambda^2)\| \ \leq \ \frac{L_1}{L}\|\lambda^1 - \lambda^2\|(\exp(L|x - x_0|) - 1).$$

3. The solution $u(x, \lambda)$ is differentiable with respect to λ and for each j $(1 \leq j \leq m)$, $v^j(x, \lambda) = \partial u(x, \lambda)/\partial \lambda_j$ is the solution of the initial value problem

$$v'(x, \lambda) \ = \ \frac{\partial g}{\partial u}(x, u(x, \lambda), \lambda)v(x, \lambda) + \frac{\partial g}{\partial \lambda_j}(x, u(x, \lambda), \lambda) \qquad (16.15)$$

$$v(x_0, \lambda) \ = \ 0. \qquad (16.16)$$

Problems

16.1. Solve the following problems by using Picard's method of successive approximations:

(i) $u' = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} u, \quad u(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$

(ii) $u' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} u + \begin{bmatrix} x \\ x \end{bmatrix}, \quad u(0) = \begin{bmatrix} 2 \\ -2 \end{bmatrix}.$

16.2. Show that the problem (1.6), (1.8) is equivalent to the integral equation

$$y(x) = \sum_{i=0}^{n-1} \frac{(x-x_0)^i}{i!} y_i + \frac{1}{(n-1)!}\int_{x_0}^{x} (x-t)^{n-1} f(t, y(t), y'(t), \ldots, y^{(n-1)}(t))dt.$$

16.3. Let the following conditions hold:

(i) $f(x, \phi_0, \ldots, \phi_{n-1})$ is continuous in

$$\Omega_1 : \ |x - x_0| \le a, \quad \sum_{i=0}^{n-1} |\phi_i - y_i| \le b$$

and hence there exists a $M > 0$ such that $\sup_{\Omega_1} |f(x, \phi_0, \ldots, \phi_{n-1})| \le M$.
(ii) $f(x, \phi_0, \ldots, \phi_{n-1})$ satisfies a uniform Lipschitz condition in Ω_1, i.e., for all $(x, \phi_0, \ldots, \phi_{n-1})$, $(x, \psi_0, \ldots, \psi_{n-1}) \in \Omega_1$ there exists a constant L such that

$$|f(x, \phi_0, \ldots, \phi_{n-1}) - f(x, \psi_0, \ldots, \psi_{n-1})| \ \le \ L \sum_{i=0}^{n-1} |\phi_i - \psi_i|.$$

Show that the problem (1.6), (1.8) has a unique solution in the interval $J_h : |x - x_0| \le h = \min\{a, b/M_1\}$, where $M_1 = M + b + \sum_{i=0}^{n-1} |y_i|$.

16.4. Let $y(x)$ and $z(x)$ be two solutions of the DE

$$y^{(n)} + p_1(x) y^{(n-1)} + \cdots + p_n(x) y \ = \ r(x) \tag{16.17}$$

in the interval J containing the point x_0. Show that for all x in J

$$u(x_0) \exp(-2K|x - x_0|) \ \le \ u(x) \ \le \ u(x_0) \exp(2K|x - x_0|),$$

where

$$K \ = \ 1 + \sum_{i=1}^{n} \sup_{x \in J} |p_i(x)| \quad \text{and} \quad u(x) \ = \ \sum_{i=0}^{n-1} (y^{(i)}(x) - z^{(i)}(x))^2.$$

***16.5.** Consider the initial value problem

$$y'' + \alpha(y, y') y' + \beta(y) \ = \ f(x), \quad y(0) \ = \ y_0, \quad y'(0) \ = \ y_1$$

where $\alpha(y, y')$, $\beta(y)$ are continuous together with their first-order partial derivatives, and $f(x)$ is continuous and bounded on $\mathbb{R}$, $\alpha \ge 0$, $y\beta(y) \ge 0$. Show that this problem has a unique solution and it can be extended to $[0, \infty)$.

16.6. Using an example of the form

$$\begin{aligned} u_1' &= u_2 \\ u_2' &= -u_1 \end{aligned}$$

observe that a generalization of Theorem 11.1 to systems of first-order DEs with inequalities interpreted component-wise is in general not true.

Answers or Hints

16.1. (i) $(\sin x, \cos x)^T$. (ii) $(e^x + 2e^{-x} - x - 1, e^x - 2e^{-x} - x - 1)^T$.

16.2. Use Taylor's formula.

16.3. Write (1.6), (1.8) in system form and then apply Theorem 15.2.

16.4. Use the inequality $2|a||b| \leq a^2 + b^2$ to get $-2Ku(x) \leq u'(x) \leq 2Ku(x)$.

16.6. Let $J = [0, \pi),\ u(x) = (\sin x,\ \cos x)^T$ and $v(x) = (-\epsilon,\ 1 - \epsilon)^T,\ 0 < \epsilon < 1$.