Lecture 15

Existence and Uniqueness of Solutions of Systems

So far we have concentrated on the existence and uniqueness of solutions of scalar initial value problems. It is natural to extend these results to a system of first-order DEs and higher-order DEs. We consider a system of first-order DEs of the form

$$\begin{aligned} u_1' &= g_1(x, u_1, \ldots, u_n) \\ u_2' &= g_2(x, u_1, \ldots, u_n) \\ \cdots & \\ u_n' &= g_n(x, u_1, \ldots, u_n). \end{aligned} \tag{15.1}$$

Such systems arise frequently in many branches of applied sciences, especially in the analysis of vibrating mechanical systems with several degrees of freedom. Furthermore, these systems have mathematical importance in themselves, e.g., each nth-order DE (1.6) is equivalent to a system of n first-order equations. Indeed, if we take $y^{(i)} = u_{i+1},\ 0 \le i \le n-1$, then the equation (1.6) can be written as

$$\begin{aligned} u_i' &= u_{i+1}, \quad 1 \le i \le n-1 \\ u_n' &= f(x, u_1, \ldots, u_n), \end{aligned} \tag{15.2}$$

which is of the type (15.1).

Throughout, we shall assume that the functions $g_1, \ldots, g_n$ are continuous in some domain E of $(n+1)$-dimensional space $\mathbb{R}^{n+1}$. By a solution of (15.1) in an interval J we mean a set of n functions $u_1(x), \ldots, u_n(x)$ such that (i) $u_1'(x), \ldots, u_n'(x)$ exist for all $x \in J$, (ii) for all $x \in J$ the points $(x, u_1(x), \ldots, u_n(x)) \in E$, and (iii) $u_i'(x) = g_i(x, u_1(x), \ldots, u_n(x))$ for all $x \in J$. In addition to the differential system (15.1) there may also be given initial conditions of the form

$$u_1(x_0) = u_1^0, \quad u_2(x_0) = u_2^0, \ldots, u_n(x_0) = u_n^0, \tag{15.3}$$

where x_0 is a specified value of x in J and $u_1^0, \ldots, u_n^0$ are prescribed numbers such that $(x_0, u_1^0, \ldots, u_n^0) \in E$. The differential system (15.1) together with the initial conditions (15.3) forms an initial value problem.

To study the existence and uniqueness of the solutions of (15.1), (15.3), there are two possible approaches, either directly imposing sufficient conditions on the functions $g_1, \ldots, g_n$ and proving the results, or alternatively

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using vector notations to write (15.1), (15.3) in a compact form and then proving the results. We shall prefer to use the second approach since then the proofs are very similar to the scalar case.

By setting

$$u(x) \ = \ (u_1(x), \ldots, u_n(x)) \quad \text{and} \quad g(x,u) \ = \ (g_1(x,u), \ldots, g_n(x,u))$$

and agreeing that differentiation and integration are to be performed component-wise, i.e., $u'(x) = (u_1'(x), \ldots, u_n'(x))$ and

$$\int_\alpha^\beta u(x)dx \ = \ \left(\int_\alpha^\beta u_1(x)dx, \ldots, \int_\alpha^\beta u_n(x)dx \right),$$

the problem (15.1), (15.3) can be written as

$$u' \ = \ g(x,u), \quad u(x_0) \ = \ u^0, \tag{15.4}$$

which is exactly the same as (7.1) except now u and u' are the functions defined in J; and taking the values in $\mathbb{R}^n$, $g(x,u)$ is a function from $E \subseteq \mathbb{R}^{n+1}$ to $\mathbb{R}^n$ and $u^0 = (u_1^0, \ldots, u_n^0)$.

The function $g(x,u)$ is said to be continuous in E if each of its components is continuous in E. The function $g(x,u)$ is defined to be uniformly Lipschitz continuous in E if there exists a nonnegative constant L (Lipschitz constant) such that

$$\|g(x,u) - g(x,v)\| \ \le \ L\|u - v\| \tag{15.5}$$

for all (x,u), (x,v) in the domain E. For example, let $g(x,u) = (a_{11}u_1 + a_{12}u_2,\ a_{21}u_1 + a_{22}u_2)$ and $E = \mathbb{R}^3$, then

$$\begin{aligned}
&\|g(x,u) - g(x,v)\| \\
&\quad = \ \|(a_{11}(u_1 - v_1) + a_{12}(u_2 - v_2),\ a_{21}(u_1 - v_1) + a_{22}(u_2 - v_2))\| \\
&\quad = \ |a_{11}(u_1 - v_1) + a_{12}(u_2 - v_2)| + |a_{21}(u_1 - v_1) + a_{22}(u_2 - v_2)| \\
&\quad \le \ |a_{11}||u_1 - v_1| + |a_{12}||u_2 - v_2| + |a_{21}|u_1 - v_1| + |a_{22}||u_2 - v_2| \\
&\quad = \ [|a_{11}| + |a_{21}|]|u_1 - v_1| + [|a_{12}| + |a_{22}|]|u_2 - v_2| \\
&\quad \le \ \max\{|a_{11}| + |a_{21}|,\ |a_{12}| + |a_{22}|\}[|u_1 - v_1| + |u_2 - v_2|] \\
&\quad = \ \max\{|a_{11}| + |a_{21}|,\ |a_{12}| + |a_{22}|\}\|u - v\|.
\end{aligned}$$

Hence, the Lipschitz constant is

$$L \ = \ \max\{|a_{11}| + |a_{21}|,\ |a_{12}| + |a_{22}|\}.$$

The following result provides sufficient conditions for the function $g(x, u)$ to satisfy the Lipschitz condition (15.5).

Theorem 15.1. Let the domain E be convex and for all (x,u) in E the partial derivatives $\partial g/\partial u_k,\ k=1,\ldots,n$ exist and $\|\partial g/\partial u\| \le L$. Then the function $g(x,u)$ satisfies the Lipschitz condition (15.5) in E with Lipschitz constant L.

Proof. Let (x,u) and (x,v) be fixed points in E. Then since E is convex, for all $0\le t\le 1$ the points $(x, v+t(u-v))$ are in E. Thus, the vector-valued function $G(t)=g(x,v+t(u-v)),\ 0\le t\le 1$ is well defined, also

$$\begin{aligned} G'(t) &= (u_1-v_1)\frac{\partial g}{\partial u_1}(x,v+t(u-v))+\cdots \\ &\quad +(u_n-v_n)\frac{\partial g}{\partial u_n}(x,v+t(u-v)) \end{aligned}$$

and hence

$$\begin{aligned} \|G'(t)\| &\le \sum_{i=1}^{n}\left|\frac{\partial g_i}{\partial u_1}(x,v+t(u-v))\right| |u_1-v_1|+\cdots \\ &\quad +\sum_{i=1}^{n}\left|\frac{\partial g_i}{\partial u_n}(x,v+t(u-v))\right| |u_n-v_n| \\ &\le L[|u_1-v_1|+\cdots+|u_n-v_n|] \ = \ L\|u-v\|. \end{aligned}$$

Now from the relation

$$g(x,u)-g(x,v) \ = \ G(1)-G(0) \ = \ \int_0^1 G'(t)dt$$

we find that

$$\|g(x,u)-g(x,v)\| \ \le \ \int_0^1 \|G'(t)\|dt \ \le \ L\|u-v\|. \qquad \blacksquare$$

As an example once again we consider

$$g(x,u) \ = \ (a_{11}u_1+a_{12}u_2,\ a_{21}u_1+a_{22}u_2).$$

Since

$$\frac{\partial g}{\partial u_1} \ = \ (a_{11},\ a_{21}), \qquad \frac{\partial g}{\partial u_2} \ = \ (a_{12},\ a_{22}),$$

$$\left\|\frac{\partial g}{\partial u}\right\| \ = \ \max\{|a_{11}|+|a_{21}|,\ |a_{12}|+|a_{22}|\} \ = \ L,$$

as it should be.

Next arguing as in Theorem 7.1, we see that if $g(x,u)$ is continuous in the domain E, then any solution of (15.4) is also a solution of the integral equation

$$u(x) \ = \ u^0+\int_{x_0}^{x} g(t,u(t))dt \tag{15.6}$$

and conversely.

To find a solution of the integral equation (15.6) the Picard method of successive approximations is equally useful. Let $u^0(x)$ be any continuous function which we assume to be an initial approximation of the solution, then we define approximations successively by

$$u^{m+1}(x) \;=\; u^0 + \int_{x_0}^{x} g(t, u^m(t))dt, \quad m = 0, 1, \ldots \tag{15.7}$$

and, as before, if the sequence of functions $\{u^m(x)\}$ converges uniformly to a continuous function $u(x)$ in some interval J containing x_0 and for all $x \in J$, the points $(x, u(x)) \in E$, then this function $u(x)$ will be a solution of the integral equation (15.6).

Example 15.1. For the initial value problem

$$\begin{array}{rcl} u_1' &=& x + u_2 \\ u_2' &=& x + u_1 \\ u_1(0) &=& 1, \quad u_2(0) \;=\; -1 \end{array} \tag{15.8}$$

we take $u^0(x) = (1, -1)$, to obtain

$$\begin{aligned}
u^1(x) &= (1,-1) + \int_0^x (t-1, t+1)dt = \left(1 - x + \frac{x^2}{2}, -1 + x + \frac{x^2}{2}\right) \\
u^2(x) &= (1,-1) + \int_0^x \left(t - 1 + t + \frac{t^2}{2},\ t + 1 - t + \frac{t^2}{2}\right) dt \\
&= \left(1 - x + \frac{2x^2}{2} + \frac{x^3}{3!},\ -1 + x + \frac{x^3}{3!}\right) \\
u^3(x) &= \left(1 - x + \frac{2x^2}{2} + \frac{x^4}{4!},\ -1 + x + \frac{2x^3}{3!} + \frac{x^4}{4!}\right) \\
u^4(x) &= \left(1 - x + \frac{2x^2}{2} + \frac{2x^4}{4!} + \frac{x^5}{5!},\ -1 + x + \frac{2x^3}{3!} + \frac{x^5}{5!}\right) \\
&= \Bigg(-(1+x) + \left(2 + \frac{2x^2}{2!} + \frac{2x^4}{4!} + \frac{x^5}{5!}\right), \\
&\qquad -(1+x) + \left(2x + \frac{2x^3}{3!} + \frac{x^5}{5!}\right)\Bigg) \\
&\cdots.
\end{aligned}$$

Hence, the sequence $\{u^m(x)\}$ exists for all real x and converges to $u(x) = (-(1+x) + e^x + e^{-x},\ -(1+x) + e^x - e^{-x})$, which is the solution of the initial value problem (15.8).

Now we shall state several results for the initial value problem (15.4) which are analogous to those proved in earlier lectures for the problem (7.1).

Theorem 15.2 (Local Existence Theorem). Let the following conditions hold:

(i) $g(x,u)$ is continuous in $\Omega : |x-x_0| \le a,\ \|u-u^0\| \le b$ and hence there exists a $M>0$ such that $\|g(x,u)\| \le M$ for all $(x,u) \in \Omega$.

(ii) $g(x,u)$ satisfies a uniform Lipschitz condition (15.5) in Ω.

(iii) $u^0(x)$ is continuous in $|x-x_0| \le a$ and $\|u^0(x)-u^0\| \le b$.

Then the sequence $\{u^m(x)\}$ generated by the Picard iterative scheme (15.7) converges to the unique solution $u(x)$ of the problem (15.4). This solution is valid in the interval $J_h : |x-x_0| \le h = \min\{a, b/M\}$. Further, for all $x \in J_h$, the following error estimate holds

$$\|u(x)-u^m(x)\| \ \le\ Ne^{Lh}\min\left\{1, \frac{(Lh)^m}{m!}\right\}, \qquad m=0,1,\ldots$$

where $\|u^1(x)-u^0(x)\| \le N$.

Theorem 15.3 (Global Existence Theorem). Let the following conditions hold:

(i) $g(x,u)$ is continuous in $\Delta : |x-x_0| \le a,\ \|u\| < \infty$.

(ii) $g(x,u)$ satisfies a uniform Lipschitz condition (15.5) in Δ.

(iii) $u^0(x)$ is continuous in $|x-x_0| \le a$.

Then the sequence $\{u^m(x)\}$ generated by the Picard iterative scheme (15.7) exists in the entire interval $|x-x_0| \le a$, and converges to the unique solution $u(x)$ of the problem (15.4).

Corollary 15.4. Let $g(x,u)$ be continuous in $\mathbb{R}^{n+1}$ and satisfy a uniform Lipschitz condition (15.5) in each $\Delta_a : |x| \le a,\ \|u\| < \infty$ with the Lipschitz constant L_a. Then the problem (15.4) has a unique solution which exists for all x.

Theorem 15.5 (Peano's Existence Theorem). Let $g(x,u)$ be continuous and bounded in Δ. Then the problem (15.4) has at least one solution in $|x-x_0| \le a$.

Definition 15.1. Let $g(x,u)$ be continuous in a domain E. A function $u(x)$ defined in J is said to be an ϵ-approximate solution of the differential system $u' = g(x,u)$ if (i) $u(x)$ is continuous for all x in J, (ii) for all $x \in J$ the points $(x, u(x)) \in E$, (iii) $u(x)$ has a piecewise continuous derivative in J which may fail to be defined only for a finite number of points, say, $x_1, x_2, \ldots, x_k$, and (iv) $\|u'(x) - g(x,u(x))\| \le \epsilon$ for all $x \in J,\ x \ne x_i,\ i = 1,2,\ldots,k$.

Theorem 15.6. Let $g(x,u)$ be continuous in Ω, and hence there exists a $M>0$ such that $\|g(x,u)\| \le M$ for all $(x,u) \in \Omega$. Then for any $\epsilon > 0$, there exists an ϵ-approximate solution $u(x)$ of the differential system $u' = g(x, u)$ in the interval J_h such that $u(x_0) = u^0$.

Theorem 15.7 (Cauchy–Peano's Existence Theorem). Let the conditions of Theorem 15.7 be satisfied. Then the problem (15.4) has at least one solution in J_h.