Lecture 10

Uniqueness Theorems

In our previous lectures we have proved that the continuity of the function $f(x, y)$ in the closed rectangle $\overline{S}$ is sufficient for the existence of at least one solution of the initial value problem (7.1) in the interval J_h, and to achieve the uniqueness (i.e., existence of at most one solution) some additional condition on $f(x, y)$ is required. In fact, continuous functions $f(x, y)$ have been constructed (see Lavrentev [30], Hartman [20]) so that from any given point (x_0, y_0) the equation $y' = f(x, y)$ has at least two solutions in every neighborhood of (x_0, y_0). In Theorem 8.1 this additional condition was assumed to be the Lipschitz continuity. In the following, we shall provide several such conditions which are sufficient for the uniqueness of the solutions of (7.1).

Theorem 10.1 (Lipschitz Uniqueness Theorem). Let $f(x, y)$ be continuous and satisfy a uniform Lipschitz condition (7.3) in $\overline{S}$. Then (7.1) has at most one solution in $|x - x_0| \le a$.

Proof. In Theorem 8.1 the uniqueness of the solutions of (7.1) is proved in the interval J_h; however, it is clear that J_h can be replaced by the interval $|x - x_0| \le a$. ∎

Theorem 10.2 (Peano's Uniqueness Theorem). Let $f(x, y)$ be continuous in $\overline{S}_+ :\ x_0 \le x \le x_0 + a,\ |y - y_0| \le b$ and nonincreasing in y for each fixed x in $x_0 \le x \le x_0 + a$. Then (7.1) has at most one solution in $x_0 \le x \le x_0 + a$.

Proof. Suppose $y_1(x)$ and $y_2(x)$ are two solutions of (7.1) in $x_0 \le x \le x_0 + a$ which differ somewhere in $x_0 \le x \le x_0 + a$. We assume that $y_2(x) > y_1(x)$ in $x_1 < x < x_1 + \epsilon \le x_0 + a$, while $y_1(x) = y_2(x)$ in $x_0 \le x \le x_1$, i.e., x_1 is the greatest lower bound of the set A consisting of those x for which $y_2(x) > y_1(x)$. This greatest lower bound exists because the set A is bounded below by x_0 at least. Thus, for all $x \in (x_1, x_1 + \epsilon)$ we have $f(x, y_1(x)) \ge f(x, y_2(x))$; i.e., $y_1'(x) \ge y_2'(x)$. Hence, the function $z(x) = y_2(x) - y_1(x)$ is nonincreasing, since if $z(x_1) = 0$ we should have $z(x) \le 0$ in $(x_1, x_1 + \epsilon)$. This contradiction proves that $y_1(x) = y_2(x)$ in $x_0 \le x \le x_0 + a$. ∎

Example 10.1. The function $|y|^{1/2}\mathrm{sgn}\ y$, where $\mathrm{sgn}\ y = 1$ if $y \ge 0$, and -1 if $y < 0$ is continuous, nondecreasing, and the initial value problem $y' = |y|^{1/2}\mathrm{sgn}\ y,\ y(0) = 0$ has two solutions $y(x) \equiv 0,\ y(x) = x^2/4$ in the

R.P. Agarwal and D. O'Regan, *An Introduction to Ordinary Differential Equations*,
doi: 10.1007/978-0-387-71276-5_10,

interval $[0, \infty)$. Thus, in Theorem 10.2 "nonincreasing" cannot be replaced by "nondecreasing."

For our next result, we need the following lemma.

Lemma 10.3. Let $w(z)$ be continuous and increasing function in the interval $[0, \infty)$, and $w(0) = 0,\ w(z) > 0$ for $z > 0$, with also

$$\lim_{\epsilon \to 0^+} \int_\epsilon \frac{dz}{w(z)} = \infty. \tag{10.1}$$

Let $u(x)$ be a nonnegative continuous function in $[0, a]$. Then the inequality

$$u(x) \le \int_0^x w(u(t))dt, \quad 0 < x \le a \tag{10.2}$$

implies that $u(x) \equiv 0$ in $[0, a]$.

Proof. Define $v(x) = \max_{0 \le t \le x} u(t)$ and assume that $v(x) > 0$ for $0 < x \le a$. Then $u(x) \le v(x)$ and for each x there is an $x_1 \le x$ such that $u(x_1) = v(x)$. From this, we have

$$v(x) = u(x_1) \le \int_0^{x_1} w(u(t))dt \le \int_0^x w(v(t))dt;$$

i.e., the nondecreasing function $v(x)$ satisfies the same inequality as $u(x)$ does. Let us set

$$\overline{v}(x) = \int_0^x w(v(t))dt,$$

then $\overline{v}(0) = 0,\ v(x) \le \overline{v}(x),\ \overline{v}'(x) = w(v(x)) \le w(\overline{v}(x))$. Hence, for $0 < \delta < a$, we have

$$\int_\delta^a \frac{\overline{v}'(x)}{w(\overline{v}(x))}dx \le a - \delta < a.$$

However, from (10.1) it follows that

$$\int_\delta^a \frac{\overline{v}'(x)}{w(\overline{v}(x))}dx = \int_\epsilon^\alpha \frac{dz}{w(z)}, \quad \overline{v}(\delta) = \epsilon, \quad \overline{v}(a) = \alpha$$

becomes infinite when $\epsilon \to 0$ $(\delta \to 0)$. This contradiction shows that $v(x)$ cannot be positive, so $v(x) \equiv 0$, and hence $u(x) = 0$ in $[0, a]$. ∎

Theorem 10.4 (Osgood's Uniqueness Theorem). Let $f(x, y)$ be continuous in $\overline{S}$ and for all $(x, y_1),\ (x, y_2) \in \overline{S}$ it satisfies

$$|f(x, y_1) - f(x, y_2)| \le w(|y_1 - y_2|), \tag{10.3}$$

where $w(z)$ is the same as in Lemma 10.3. Then (7.1) has at most one solution in $|x - x_0| \le a$.

Proof. Suppose $y_1(x)$ and $y_2(x)$ are two solutions of (7.1) in $|x-x_0| \leq a$. Then from (10.3) it follows that

$$|y_1(x) - y_2(x)| \ \leq \ \left| \int_{x_0}^{x} w(|y_1(t) - y_2(t)|) dt \right|.$$

For any x in $[x_0, x_0+a]$, we set $u(x) = |y_1(x_0+x) - y_2(x_0+x)|$. Then the nonnegative continuous function $u(x)$ satisfies the inequality (10.2), and therefore, Lemma 10.3 implies that $u(x) = 0$ in $[0, a]$, i.e., $y_1(x) = y_2(x)$ in $[x_0, x_0+a]$. If x is in $[x_0-a, x_0]$, then the proof remains the same except that we need to define the function $u(x) = |y_1(x_0-x) - y_2(x_0-x)|$ in $[0, a]$. ∎

For our next result, we shall prove the following lemma.

Lemma 10.5. Let $u(x)$ be nonnegative continuous function in $|x-x_0| \leq a$, and $u(x_0) = 0$, and let $u(x)$ be differentiable at $x = x_0$ with $u'(x_0) = 0$. Then the inequality

$$u(x) \ \leq \ \left| \int_{x_0}^{x} \frac{u(t)}{t - x_0} dt \right| \tag{10.4}$$

implies that $u(x) = 0$ in $|x - x_0| \leq a$.

Proof. It suffices to prove the lemma only for $x_0 \leq x \leq x_0 + a$. We define

$$v(x) \ = \ \int_{x_0}^{x} \frac{u(t)}{t - x_0} dt.$$

This integral exists since

$$\lim_{x \to x_0} \frac{u(x)}{x - x_0} \ = \ u'(x_0) \ = \ 0.$$

Further, we have

$$v'(x) \ = \ \frac{u(x)}{x - x_0} \ \leq \ \frac{v(x)}{x - x_0}$$

and hence $d/dx[v(x)/(x - x_0)] \leq 0$, which implies that $v(x)/(x - x_0)$ is nonincreasing. Since $v(x_0) = 0$, this gives $v(x) \leq 0$, which is a contradiction to $v(x) \geq 0$. So, $v(x) \equiv 0$, and hence $u(x) = 0$ in $[x_0, x_0 + a]$. ∎

Theorem 10.6 (Nagumo's Uniqueness Theorem). Let $f(x, y)$ be continuous in $\overline{S}$ and for all $(x, y_1),\ (x, y_2) \in \overline{S}$ it satisfies

$$|f(x, y_1) - f(x, y_2)| \ \leq \ k|x - x_0|^{-1}|y_1 - y_2|, \quad x \neq x_0, \quad k \leq 1. \tag{10.5}$$

Then (7.1) has at most one solution in $|x - x_0| \leq a$.

Proof. Suppose $y_1(x)$ and $y_2(x)$ are two solutions of (7.1) in $|x-x_0| \leq a$. Then from (10.5) it follows that

$$|y_1(x) - y_2(x)| \ \leq \ \left| \int_{x_0}^{x} |t - x_0|^{-1} |y_1(t) - y_2(t)| dt \right|.$$

We set $u(x) = |y_1(x) - y_2(x)|$; then the nonnegative function $u(x)$ satisfies the inequality (10.4). Further, since $u(x)$ is continuous in $|x - x_0| \le a$, and $u(x_0) = 0$, from the mean value theorem we have

$$\begin{aligned} u'(x_0) &= \lim_{h\to 0} \frac{u(x_0+h) - u(x_0)}{h} \\ &= \lim_{h\to 0} \frac{|y_1(x_0) + hy_1'(x_0+\theta_1 h) - y_2(x_0) - hy_2'(x_0+\theta_2 h)|}{h}, \\ & \qquad\qquad\qquad\qquad\qquad\qquad 0 < \theta_1,\ \theta_2 < 1 \\ &= (\mathrm{sgn}\ h) \lim_{h\to 0} |y_1'(x_0+\theta_1 h) - y_2'(x_0+\theta_2 h)| \ = \ 0. \end{aligned}$$

Thus, the conditions of Lemma 10.5 are satisfied and $u(x) \equiv 0$, i.e., $y_1(x) = y_2(x)$ in $|x - x_0| \le a$. ■

Example 10.2. It is easy to verify that the function

$$f(x,y) = \begin{cases} 0 & 0 \le x \le 1, \quad y \le 0 \\ \dfrac{(1+\epsilon)y}{x} & 0 \le x \le 1, \quad 0 < y < x^{1+\epsilon}, \quad \epsilon > 0 \\ (1+\epsilon)x^{\epsilon} & 0 \le x \le 1, \quad x^{1+\epsilon} \le y \end{cases}$$

is continuous and satisfies the condition (10.5) (except $k = 1 + \epsilon > 1$) in $\overline{S} : [0,1] \times \mathbb{R}$. For this function the initial value problem (7.1) with $(x_0, y_0) = (0, 0)$ has an infinite number of solutions $y(x) = cx^{1+\epsilon}$, where c is an arbitrary constant such that $0 < c < 1$. Thus, in condition (10.5) the constant $k \le 1$ is the best possible, i.e., it cannot be replaced by $k > 1$.

Theorem 10.7 (Krasnoselski–Krein Uniqueness Theorem). Let $f(x, y)$ be continuous in $\overline{S}$ and for all $(x, y_1),\ (x, y_2) \in \overline{S}$ it satisfies

$$|f(x,y_1) - f(x,y_2)| \le k|x - x_0|^{-1}|y_1 - y_2|, \quad x \ne x_0, \quad k > 0 \tag{10.6}$$

$$|f(x,y_1) - f(x,y_2)| \le C|y_1 - y_2|^{\alpha}, \quad C > 0, \quad 0 < \alpha < 1, \quad k(1-\alpha) < 1. \tag{10.7}$$

Then (7.1) has at most one solution in $|x - x_0| \le a$.

Proof. Suppose $y_1(x)$ and $y_2(x)$ are two solutions of (7.1) in $|x - x_0| \le a$. We shall show that $y_1(x) = y_2(x)$ only in the interval $[x_0, x_0 + a]$. For this, from (10.7) we have

$$u(x) = |y_1(x) - y_2(x)| \le \int_{x_0}^{x} Cu^{\alpha}(t)dt$$

and hence Problem 7.5 gives that

$$u(x) \le [C(1-\alpha)(x - x_0)]^{(1-\alpha)^{-1}} \le [C(x - x_0)]^{(1-\alpha)^{-1}}.$$

Thus, the function $v(x) = u(x)/(x - x_0)^k$ satisfies the inequality

$$0 \ \leq \ v(x) \ \leq \ C^{(1-\alpha)^{-1}}(x - x_0)^{(1-\alpha)^{-1}-k}. \tag{10.8}$$

Since $k(1-\alpha) < 1$, it is immediate that $\lim_{x \to x_0} v(x) = 0$. Hence, if we define $v(x_0) = 0$, then the function $v(x)$ is continuous in $[x_0, x_0 + a]$. We wish to show that $v(x) = 0$ in $[x_0, x_0+a]$. If $v(x) > 0$ at any point in $[x_0, x_0+a]$, then there exists a point $x_1 > x_0$ such that $0 < m = v(x_1) = \max_{x_0 \leq x \leq x_0+a} v(x)$. However, from (10.6) we obtain

$$\begin{aligned} m \ = \ v(x_1) \ &\leq \ (x_1 - x_0)^{-k} \int_{x_0}^{x_1} k(t - x_0)^{-1} u(t)dt \\ &\leq \ (x_1 - x_0)^{-k} \int_{x_0}^{x_1} k(t - x_0)^{k-1} v(t)dt \\ &< \ m(x_1 - x_0)^{-k} \int_{x_0}^{x_1} k(t - x_0)^{k-1} dt \\ &= \ m(x_1 - x_0)^{-k}(x_1 - x_0)^k \ = \ m, \end{aligned}$$

which is the desired contradiction. So, $v(x) \equiv 0$, and hence $u(x) = 0$ in $[x_0, x_0 + a]$. ∎

Theorem 10.8 (Van Kampen Uniqueness Theorem). Let $f(x, y)$ be continuous in $\overline{S}$ and for all $(x, y) \in \overline{S}$ it satisfies

$$|f(x, y)| \ \leq \ A|x - x_0|^p, \quad p > -1, \quad A > 0. \tag{10.9}$$

Further, let for all $(x, y_1),\ (x, y_2) \in \overline{S}$ it satisfies

$$|f(x, y_1) - f(x, y_2)| \ \leq \ \frac{C}{|x - x_0|^r}|y_1 - y_2|^q, \quad q \geq 1, \quad C > 0 \tag{10.10}$$

with $q(1 + p) - r = p,\ \rho = C(2A)^{q-1}/(p + 1)^q < 1$. Then (7.1) has at most one solution in $|x - x_0| \leq a$.

Proof. Suppose $y_1(x)$ and $y_2(x)$ are two solutions of (7.1) in $|x - x_0| \leq a$. We shall show that $y_1(x) = y_2(x)$ only in the interval $[x_0 - a, x_0]$. For this, from (10.9) we have

$$\begin{aligned} u(x) \ = \ |y_1(x) - y_2(x)| \ &\leq \ \int_x^{x_0} |f(t, y_1(t)) - f(t, y_2(t))|dt \\ &\leq \ 2A \int_x^{x_0} (x_0 - t)^p dt \ = \ \frac{2A}{p + 1}(x_0 - x)^{p+1}. \end{aligned}$$

Using this estimate and (10.10), we obtain

$$\begin{aligned} u(x) \ &\leq \ C \int_x^{x_0} \frac{1}{(x_0 - t)^r} u^q(t)dt \\ &\leq \ C \left(\frac{2A}{p + 1}\right)^q \int_x^{x_0} (x_0 - t)^{q(p+1)-r} dt \ = \ \rho \left(\frac{2A}{p + 1}\right)(x_0 - x)^{p+1}. \end{aligned}$$

Now using this new estimate and (10.10), we get

$$u(x) \ \le \ \rho^{1+q}\left(\frac{2A}{p+1}\right)(x_0-x)^{p+1}.$$

Continuing in this way, we find

$$u(x) \ \le \ \rho^{1+q+q^2+\cdots+q^m}\left(\frac{2A}{p+1}\right)(x_0-x)^{p+1}, \quad m=1,2,\ldots.$$

Since $q \ge 1$ and $\rho < 1$, it follows that $u(x) = 0$ in $|x_0 - a, x_0]$. ∎

Problems

10.1. Consider the initial value problem

$$\begin{aligned} y' \ &= \ f(x,y) \ = \ \begin{cases} \dfrac{4x^3y}{x^4+y^2}, & (x,y)\ne(0,0) \\ 0, \quad (x,y)=(0,0) \end{cases} \\ y(0) \ &= \ 0. \end{aligned} \tag{10.11}$$

Show that the function $f(x,y)$ is continuous but does not satisfy the Lipschitz condition in any region containing the origin (see Problem 7.4). Further, show that (10.11) has an infinite number of solutions.

10.2. Given the equation $y' = xg(x,y)$, suppose that g and $\partial g/\partial y$ are defined and continuous for all (x,y). Show the following:

(i) $y(x) \equiv 0$ is a solution.

(ii) If $y = y(x),\ x \in (\alpha,\beta)$ is a solution and if $y(x_0) > 0,\ x_0 \in (\alpha,\beta)$, then $y(x) > 0$ for all $x \in (\alpha,\beta)$.

(iii) If $y = y(x),\ x \in (\alpha,\beta)$ is a solution and if $y(x_0) < 0,\ x_0 \in (\alpha,\beta)$, then $y(x) < 0$ for all $x \in (\alpha,\beta)$.

10.3. Let $f(x,y)$ be continuous and satisfy the generalized Lipschitz condition

$$|f(x,y_1) - f(x,y_2)| \ \le \ L(x)|y_1 - y_2|$$

for all $(x,y_1),\ (x,y_2)$ in $\overline{S}$, where the function $L(x)$ is such that the integral $\int_{x_0-a}^{x_0+a} L(t)dt$ exists. Show that (7.1) has at most one solution in $|x - x_0| \le a$.

10.4. Give some examples to show that the Lipschitz condition in Theorem 10.1 is just a sufficient condition for proving the uniqueness of the solutions of (7.1) but not the necessary condition.

10.5. Let $f(x,y)$ be continuous in $\overline{S}_+$ and for all $(x,y_1),\ (x,y_2)$ in $\overline{S}_+$ with $y_2 \ge y_1$ satisfy one sided Lipschitz condition

$$f(x,y_2) - f(x,y_1) \ \le\ L(y_2 - y_1).$$

Show that (7.1) has at most one solution in $x_0 \le x \le x_0 + a$.

10.6. Let $f(x,y)$ be continuous in $\overline{S}_- : x_0 - a \le x \le x_0,\ |y - y_0| \le b$ and nondecreasing in y for each fixed x in $x_0 - a \le x \le x_0$. Show that (7.1) has at most one solution in $x_0 - a \le x \le x_0$.

10.7. Show that the functions $w(z) = Lz^\alpha$ $(\alpha \ge 1)$, and

$$w(z) \ =\ \begin{cases} -z\ln z, & 0 \le z \le e^{-1} \\ e^{-1}, & z > e^{-1} \end{cases}$$

satisfy the conditions of Lemma 10.3.

10.8. Consider the function $f(x,y)$ in the strip $T:\ \ -\infty < x \le 1$, $-\infty < y < \infty$ defined by

$$f(x,y) \ =\ \begin{cases} 0 & -\infty < x \le 0, \quad -\infty < y < \infty \\ 2x & 0 < x \le 1, \quad -\infty < y < 0 \\ 2x - \dfrac{4y}{x} & 0 < x \le 1, \quad 0 \le y \le x^2 \\ -2x & 0 < x \le 1, \quad x^2 < y < \infty. \end{cases}$$

Show that the problem $y' = f(x,y),\ y(0) = 0$ has a unique solution in the interval $-\infty < x \le 1$. Further, show that the Picard iterates with $y_0(x) \equiv 0$ for this problem do not converge.

10.9. Consider the function $f(x,y)$ in the strip $T:\ 0 \le x \le 1,\ -\infty < y < \infty$ defined by

$$f(x,y) = \begin{cases} 0 & 0 \le x \le 1, \quad x^{1/(1-\alpha)} < y < \infty, \quad 0 < \alpha < 1 \\ kx^{\alpha/(1-\alpha)} - k\dfrac{y}{x} & 0 \le x \le 1, \quad 0 \le y \le x^{1/(1-\alpha)}, \quad k > 0 \\ kx^{\alpha/(1-\alpha)} & 0 \le x \le 1, \quad -\infty < y < 0, \quad k(1-\alpha) < 1. \end{cases}$$

Show that the problem $y' = f(x,y),\ y(0) = 0$ has a unique solution in $[0,1]$.

***10.10 (Rogers' Uniqueness Theorem).** Let $f(x,y)$ be continuous in the strip $T: 0 \le x \le 1,\ -\infty < y < \infty$ and satisfy the condition

$$f(x,y) \ =\ o\left(e^{-1/x}x^{-2}\right)$$

uniformly for $0 \le y \le \delta,\ \delta > 0$ arbitrary. Further, let for all $(x,y_1),\ (x,y_2)$ $\in T$ it satisfy

$$|f(x,y_1) - f(x,y_2)| \ \le\ \frac{1}{x^2}|y_1 - y_2|.$$

Show that the problem $y' = f(x, y),\ y(0) = 0$ has at most one solution in $[0, 1]$.

10.11. Consider the function $f(x, y)$ in the strip $T: 0 \le x \le 1,\ -\infty < y < \infty$ defined by

$$f(x,y) = \begin{cases} \left(1 + \dfrac{1}{x}\right) e^{-1/x} & 0 \le x \le 1,\quad xe^{-1/x} \le y < \infty \\ \dfrac{y}{x^2} + e^{-1/x} & 0 \le x \le 1,\quad 0 \le y \le xe^{-1/x} \\ e^{-1/x} & 0 \le x \le 1,\quad -\infty < y \le 0. \end{cases}$$

Show that the problem $y' = f(x, y),\ y(0) = 0$ has a unique solution in $[0, 1]$.

10.12. Consider the function $f(x, y)$ in the strip $T: 0 \le x \le 1,\ -\infty < y < \infty$ defined by

$$f(x,y) = \begin{cases} 0 & 0 \le x \le 1,\quad -\infty < y \le 0 \\ \dfrac{y}{x^2} & 0 \le x \le 1,\quad 0 \le y \le e^{-1/x} \\ \dfrac{e^{-1/x}}{x^2} & 0 \le x \le 1,\quad e^{-1/x} \le y < \infty. \end{cases}$$

Show that the problem $y' = f(x, y),\ y(0) = 0$ has an infinite number of solutions in $[0, 1]$.

Answers or Hints

10.1. $y = c^2 - \sqrt{x^4 + c^4}$, where c is arbitrary.

10.2. (i) Verify directly. (ii) Use Theorem 10.1. (iii) Use Theorem 10.1.

10.3. Since $\int_{x_0-a}^{x_0+a} L(t)dt$ exists, Corollary 7.4 is applicable.

10.4. Consider the Problem 8.3(i), or $y' = y \ln(1/y),\ y(0) = \alpha \ge 0$.

10.5. Suppose two solutions $y_1(x)$ and $y_2(x)$ are such that $y_2(x) > y_1(x)$, $x_1 < x < x_1 + \epsilon \le x_0 + a$ and $y_1(x) = y_2(x),\ x_0 \le x \le x_1$. Now apply Corollary 7.4.

10.6. The proof is similar to that of Theorem 10.2.

10.7. Verify directly.

10.8. The given function is continuous and bounded by 2 in T. Also it satisfies the conditions of Theorem 10.2 (also, see Problem 10.6). The only solution is $y(x) = \begin{cases} 0, & -\infty < x \le 0 \\ x^2/3, & 0 < x \le 1. \end{cases}$ The successive approximations are $y_{2m-1}(x) = x^2,\ y_{2m}(x) = -x^2,\ m = 1, 2, \ldots.$

10.9. Show that conditions of Theorem 10.7 are satisfied. The only solution is $y(x) = k(1-\alpha)x^{1/(1-\alpha)}/[k(1-\alpha)+1]$.

10.11. Show that conditions of Problem 10.10 are satisfied. The only solution is $y(x) = xe^{-1/x}$.

10.12. Show that the condition $f(x, y) = o\left(e^{-1/x}x^{-2}\right)$ of Problem 10.10 is not satisfied. For each $0 \leq c \leq 1,\ y(x) = ce^{-1/x}$ is a solution.