CHAPTER 9

SOLVING ARITHMETIC-ALGEBRAIC PROBLEMS

OVERVIEW

The chapter begins with an Introduction in which we shall refer to the work of other authors who broach different aspects of arithmetic-algebraic problem solving. Those aspects include the arithmetic or algebraic nature of the problems; the relation of that nature to the underlying structure of the word problem and to the processes of translating the problem into a mathematical sign system; and the entrenched nature of arithmetic-type reasoning that may eventually inhibit implementation of algebraic solution strategies or methods.

Afterwards we refer to classical methods, such as the Cartesian Method, and non-traditional methods for solving problems in order to discuss cognitive aspects such as that of the problem of transference, the competent use of the logic-semiotic outline, the strata of mathematical sign systems used as representations, and the use of primitive methods and their relationship with the use of memory.

The sense of the text of a word problem with the use of the *Method of Successive Analytic Inferences* (MSAI) is determined by the logical numeric structure presented in the problem situation. This thesis permeates all others to be discussed subsequently.

We present also results from empirical studies concerning the competencies that are necessary for the use of four teaching models based on four methods for solving arithmetic algebraic word problems: the *Method of Successive Analytical Inferences* (MSAI), the *Analytic Method of Successive Explorations* (AMSE), the *Spreadsheets Method* (SM), and the *Cartesian Method* (CM). We stress the need to be competent in increasingly abstract and general uses of the representations required to attain full competency in the algebraic method par excellence, which here is called the CM, contrasting it with the competencies required by the other three methods, which are rooted more in arithmetic. These methods are related to competency in usage of different *language strata* of the *algebra sign system* and to the appearance of cognitive tendencies within this context of solving word problems.

1. INTRODUCTION

The topic of arithmetic-algebraic problem solving has been extensively studied, both in the curricular realm (Bell, 1996) and with regard to the cognitive and change of focus demands represented by the activity for students in their transition from arithmetic to algebra (Bednarz and Janvier, 1996; Puig and Cerdán, 1990; Filloy, Rojano and Rubio, 2001). The researchers who have dealt with these problems have at the same time had to face the difficulty of specifying the differences between arithmetic and algebraic problems. The foregoing has led to discussions as to whether or not it is possible to make a dichotomic classification of this type because the elements that make up a word problem are apparently insufficient for its characterization. It seems that it is the relationship of those elements with the solution strategies put into play by the problem solvers that defines the arithmetic or algebraic nature of the entire activity.

One of the studies in algebra dealing with problem/strategy relations is that undertaken by N. Bednarz, L. Radford, B. Janvier, and A. Lapage (1989). Their findings show the influence wielded by by the structural factors of a problem in the solution strategies applied by pre-algebraic students. In their work, the latter researchers use an analytical framework for the problem types, in which they consider the "relational claculation" proposed by Vergnaud (1982) on the one hand, and the result of analyzing the problems that correspond to the arithmetics and algebra sections in textbooks, on the other. The results of their empirical work, developed within the foregoing framework, suggest the existence of differences between the "relational calculation" upon which an arithmetic mode of thought is based and the "relational calculation" upon which an algebraic mode of thought is based. That is to say, the mode of thought –be it arithmetic or algebraic– appears to be determined by the type of "relational calculation" that underlies the problem structure. And it is in this sense that the authors give themselves leave to speak of "algebraic" problems and of "arithmetic" problems (Bednarz and Janvier, 1996).

L. Puig and F. Cerdán refer to the nature of word problem solution, analyzing the processes of translating the text expressed in natural language into an expression through which the problem can be solved. Depending on whether the translation process leads to an expression that only involves givens or an expression that involves an unknown in the chain of operations (equation), the problem solution is said to be of an arithmetic or of an algebraic nature, respectively. The latter authors resort to two general methods in order to analyze said translation processes: the analysis and synthesis method and the Cartesian Method (Puig and Cerdán, 1990). Later in this same chapter, we expound upon what those methods consist of and how they are used to classify the forms of translation.

However in studying the transition toward algebraic thought within in the field of problem solving, the issue of deep-seated attachment to arithmetic modes of solution also arises as an unavoidable topic. One of those modes is that of proportional reasoning. Studies such as that carried out by L. Verschaffel et al. (2000) demonstrate that students 12 years of age and older tend to apply proportionality in an over-generalised manner, for instance to non-linear cases or to cases that require purely algebraic procedures. What is more, an extrapolation of those methods has been found to be present in students beyond the secondary level in their attempts to solve probability or differential calculus problems (van Dooren et al., 2003). In this chapter, we analyze the relations between arithmetic and algebraic problems solving methods, specifically taking students' tendency to remain anchored to an arithmetic mode of thought into consideration. The classification of problems is focused on the Family of Problems idea and the evolution toward mastery of the Cartesian (or algebraic) method is analyzed by way of progressive symbolization usage and of overcoming the difficulties that arise from a series of cognitive tendencies that act as obstructors of algebraic-type reasoning.

For consistency as regards the theoretical elements presented in previous chapters, we use the notion of a "more abstract" or "less abstract" mathematical sign system (MSS) and to that of intermediate strata of MSS in order to undertake our analysis of the evolution toward the Cartesian Method.

2. THE SOLUTION OF PROBLEM SITUATIONS IN ALGEBRA. COGNITIVE ASPECTS

To approach this issue we shall consider three classic methods for solving problems, and shall return to them later, when we describe an empirical study on a specific use of these methods in the solution of arithmetic/algebraic problems.

1) What we call the Method of Successive Analytic Inferences (MSAI), which is in fact the Classic Analytic Method for solving problems. In this method, the statements of the problems are conceived as descriptions of "real situations" or "possible states of the world," and consequently these texts are transformed by means of analytic sentences, i.e., using "facts" that are valid in "any possible world." These analytic sentences constitute logical inferences that act as descriptions of transformations of the "possible situations" until the solver comes to one that is recognized as the solution of the problem.

2) What we call the Analytic Method of Successive Explorations (AMSE). In this solving method the solver uses explorations with particular data to set in motion the analysis of the problem and thereby its solution.
3) What we call the Cartesian Method (CM), which is the usual approach to problem solving in current algebra texts. In this method, some of the unknown elements in the text are represented by means of expressions belonging to a more abstract MSS, and the text of the problem is then translated to a series of relations expressed in that MSS that lead to one or various texts, the decoding of which, via a regression in the translation to the original MSS, produces the solution of the problem.

We are interested in describing what kind of difficulties, obstacles, and facilities are produced by the use of each of these three methods for solving the word problems that appear in textbooks. But we are particularly interested in what kind of competences are generated by the use of the AMSE, in order that the user may come to be competent in the use of the CM, and which of the competences generated by the MSAI are necessary for competent use of the CM, given that the teaching objective is competence in the use of the CM.

Some experts and many beginners, when they use an MSS, spontaneously resort to the use of particular values and operate with them in order to explore and thus solve certain problems, since the use of particular data and their operation spontaneously provides meanings in a more concrete MSS to the relations that are immersed in a problem, and in many cases this produces more possibilities that the logical analysis may be set in motion. With the use of a more abstract MSS it is hard to capture the sense of the symbolic representations, as they are more abstract, and therefore it is hard to find strategies for solving the problem.

To solve more complex problems it is necessary to advance in the competence to make logical analyses of problem situations. But to be able to set in motion the analytic reasoning required for problem solving, it is necessary that certain obstructers should not be present, and that there should not be uncertainty about the tactics that need to be used to solve the problem, and in order to progress in all this it is necessary to advance in intermediate tactics immersed in the uses of the strata of the intermediate MSSs that are being used. We will explore these matters in what follows.

2.1. Competent use and cognitive tendencies

Competent use of the CM for solving problems implies an evolution in the use of symbolization, in which a competent user can eventually make sense of a symbolic representation of problems that is detached from the particular concrete examples given in the teaching process, thus creating Families of Problems, the members of which are problems identified by a particular scheme of solution. The sense of the CM will arrive when the user becomes aware that by using it he is going to be able to solve such Families of Problems. The sense of the CM for solving problems is not achieved by exemplifying it separated out, with example after example disarticulated, as is encouraged by the usual traditional teaching. The integral conception of the method requires the confidence of the user that the general application of its steps necessarily leads to the solution of such Families of Problems.

A competent use prevents the user from lapsing into certain cognitive tendencies that obstruct the possibility of making appropriate use of the CM to solve problems. Examples of this would be (1) the presence of calling mechanisms that lead to the setting in motion of incorrect solving processes, for example, if, in the solution of a problem, a type of mathematical text appears that the user does not know how to decode; (2) the presence of obstructions derived from semantics that affect syntax and viceversa, for example, when solving problems and endowing signs with meanings, which predisposes the user to a good use of syntax; and (3) the presence of inhibitory mechanisms, for example, when the values of certain data are changed in a problem from a Family of Problems that has already been solved (see latter in this chapter).

2.2. Mastery of intermediate tactics and cognitive tendencies

Mastery of intermediate tactics must contribute to the development of positive cognitive tendencies that present themselves in the processes for learning more abstract concepts, such as (1) the return to more concrete situations when an analysis situation presents itself, analysis being a necessary part for advancing in competence with the CM, or (2) the presence of a process of abbreviation of a concrete text in order to be able to produce new rules of syntax, for example, in problem solving when one is operating with the particular values assigned to the unknown in a problem in each exploration of the AMSE, and one then gradually operates on the abstract text with the rules of the more abstract MSS, no longer making reference to the concrete situation.

Symbolic representations of problems in the CM make the use of the working memory more efficient. When the student succeeds in creating relations between given values and unknowns the information is integrated, making more complex chunks of information. At the point when the student succeeds in creating these relations, the use of syntax avoids the need to burden the working memory with semantic descriptions bound up with the statement of the problems.

2.3. The problem of transference

With respect to the difference in solving problems that exists between an expert and a novice, some researchers establish that a competent user has preformed mental schemes that enable him or her to recognize a problem from the very first words, and when he recognizes it he realizes what kind of strategy has to be followed to solve it. Other researchers state that the formation of schemes enables users to classify problems on the basis of general principles, ignoring superficial aspects, in a process in which the outline of the problem that is obtained is brought into agreement with the mental scheme that is stored in the user's long-term memory.

From this it would seem that it is as a result of such schemes that a way of working forward is set in motion, in which what is produced is a synthesis of the problem rather than an analysis. However, although this may happen in many problem situations, it would not be of much help in the explanation of more complex processes that might enable us to say why some individuals can transfer the solution that appears in one kind of problem to another that has not been dealt with, i.e., the transfer of the use of a method from one MSS to another.

2.4. Competent use of the logical/semiotic outline

The most competent individuals in formal terms generally use the CM to solve certain kinds of problems that are presented to them. However, when they are solving some problems they first go through a brief phase of reflection, in which they themselves evaluate whether they are able to anticipate the steps of the solution, i.e., in which they make a logical/semiotic outline of the situation that includes, among other things, clarification or identification of what is "unknown" and discrimination of the central relations involved in the problem, for this purpose using an MSS stratum that often is not really the sign system required by the CM but a more concrete MSS stratum, for example the MSS of the MSAI or a stratum of the sort of MSSs that are used in the explorations of the AMSE.

To produce this outline one can set out from the given values and from there arrive at the value of the unknown, or else one can make a logical analysis that involves the establishment of relations in which one operates with the unknown, either in a particular form, as in the AMSE, or else with the unknown being represented directly by the MSS of the CM.

2.5. The pertinent use of certain intermediate strata

Formal competence in problem solving is not necessarily due to the formation of a large number of mental schemes referring to types of problems. In other words, although it may be possible to identify someone as being competent in problem solving because he or she uses strata of the MSS of the CM to solve them, apparently making automatic use of previously acquired mental schemes concerning the solving of different families of problems, if one wishes to make a better characterization of formal competence in problem solving one must consider a user's progress in the ability to make a logical/semiotic analysis of problem situations.

This means that a competent user of a more abstract MSS really is competent if he is also competent in other, more concrete MSS strata that enable him to have a greater possibility of setting in motion the logical analysis of a problem situation, tackling it by using MSS strata that are not necessarily the most abstract, but using the MSS stratum that enables him to understand the problem and thereby set in motion a logical analysis of it.

2.6. The logical/semiotic outline, the MSS strata used as representation

By the use of certain strata of the MSS required by the CM, users generate intermediate senses linked only to those levels: this enables them to simplify the solution of some Families of Problems. Once these senses are mastered, the use of this new sign system, solely with these levels, brings about the simplification of certain problems (see, for example, in Krutetskii (1976) the case of problems of the "chickens and rabbits" variety, the statement of which is in Section 2.1.2 of Chapter 6). Thus, by teaching a method such as the AMSE one is trying to make ad hoc use of intermediate strata, which can be identified among the more concrete strata required by the CM in order to simplify the analysis of the problem (although the more abstract strata also appear). The aim is to generate senses progressively for such representations which will be implemented by the use of the CM. Each Family of Problems determines the levels of representation —MSS strata— required for its solution.

2.7. The level of representation and the use of memory

To solve a problem such as the problem of the chickens and the rabbits, for example, with primitive methods, a high level of competence may be required. Consequently, the natural tendency is to use the method of trial and error, trying to find a way round the series of consecutive analytic inferences required by the arithmetic logical analysis of the situation. These inferences require representations that permit an analysis, which in turn demands a more advanced use of the sign systems involved. In other words, mathematics and natural language become interwoven and are set to work, and then competences are needed in order to produce logical/semiotic outlines that will make the solving strategy meaningful. What makes this analysis and logical/semiotic outline complicated is the fact that for some problems intense use of working memory is required, and this implies training that only expert problem solvers possess.

2.8. The use of primitive methods and the use of memory

When primitive methods are used, what is generated is not a unique representation of a certain style, but rather the representation changes with each Family of Problems. Moreover, with the use of a new MSS more advanced methods are used as a means for writing, arranging, and working, and the representation is produced using *canonic forms*. This constitutes part of the sense of the use of such an MSS. When a primitive method is used, representations must be invented for each Family of Problems, and this will call for a certain competent use of working memory in order to go on representing the solving actions proposed in the logical/semiotic outline, subsequently leaving new marks and indicators —or new chunks in the memory— by means of which the previous results can be grouped together and not left drifting. Other more advanced methods require the students to learn how to leave marks that progressively release units of memory, thus enabling the user to make use of these units in setting in motion the analysis and solution of the problem.

Intermediate representations arrange the information in chunks of more complex organization, even though it may not be possible to distinguish this from the signs produced by the user. Thus, during the interviews some students reached a representation of the problem in which they very probably made calculations —for example, by means of a calculator or a computer— and in the end they simply wrote down the numeric solution of the problem.

2.9. Personal codes

An important aspect to be considered is the use of personal notation (codes) to indicate the actions already carried out and the actions yet to be performed on the elements of the solving process. This suggests the existence of a stage prior to the operational stage. Obstructions also appear in this stage, imposed by these personal notations when the complexity of the situation increases, generating what are later considered to be *natural mistakes of syntax* in subsequent studies: the inappropriate use of equals signs or their absence, the forgetting of certain terms, etc.

2.10. Problem solving and syntax

Empirical evidence can be found to show that the process of analyzing a typical problem situation, expressed in natural language, leads to the appearance of phenomena of reading of the situation that inhibit the setting in motion of algorithms that a few moments before were carried out immediately and correctly. Thus, in the presence of an expression written in the usual algebraic language of a first-degree equation, the student is unable to decode it as such and is therefore unable to use the brilliant operational abilities that the same student had exhibited a few moments before with the same equation. Examples of problem situations can be quoted —in those parts in which translations are made from ordinary language to an MSS— that show the existence of a tension between the interpretation of the expression (decoding of the text) given by a reading that comes from a context belonging to the MSS, and the practices of mechanisation of operations (syntax), inhibiting the necessary reading given by the semantic interpretation that the concrete situation gives it in the word problem. Once again, a syntactic reading inhibits the reading of the concrete context in which the problem is situated, not allowing these expressions to be given an interpretation that will make it possible to go on with the correct solving strategy (which will lead to the solution), and which would include that part of the translation as one of its tactics.

2.11. Mechanization and practice

It is at this point in the discussion that some of the theoretical preoccupations of Thorndike (1923) and their implications in teaching acquire a new presence, because of the peremptory need for the automation of certain operations that come from the decoding of a concrete problem situation (problems of ages, mixtures, alloys, coins, work, etc.), since neither the sense of the algorithms required, nor the semantic interpretation in terms of the contexts in which the operations have been performed, nor the mechanisms of anticipation —especially inhibitory mechanisms— must obstruct the setting in motion of a solving strategy. moreover, when this strategy is transferred to the short-term memory, it is necessary that the length of time that it may remain there should not obstruct the possibility of considering all the intermediate tactics required for the solution proposed and should allow the concatenation of all the tactics —before all the steps necessary for the achievement of these partial goals have been carried out— to be performed in that part of the short-term memory, which it is difficult to keep activated for such a long time. it could be said that this ability to make considerable quantities of information remain for a long time, so as to be able to move out of that part of the memory and bring in important new information, is generally hard to find among middle school students, because it calls for substantial resources that typical teaching does not provide. in this situation, mechanisation as a result of intense practice permits optimum use of the expressions and operations customary in the mss, and thus it breaks away from the anticipatory mechanisms that inhibit the setting in motion of the necessary solving strategies.

3. SOLVING ARITHMETIC-ALGEBRAIC PROBLEMS

The sense of the text of a word problem as it is understood in Section 4 of this chapter —with the use of the Method of Successive Analytic Inferences (MSAI)— is determined by the logico-numeric structure presented in the problem situation. This thesis permeates all the others that will be discussed later.

In this chapter we present experimental results concerning the competences that are necessary for the use of four teaching models based on four methods for solving arithmetic-algebraic word problems: the Method of Successive Analytic Inferences (MSAI), the Analytic Method of Successive

Explorations (AMSE), the Spreadsheet Method (SM), and the Cartesian Method (CM). Definitions of these methods, with the exception of the SM, have already appeared in Section 2.

We will stress the need to be competent in increasingly abstract and general uses of the representations required to attain full competence in the algebraic method par excellence, which here is called the CM, contrasting it with the competences required by the other three methods, which are more rooted in arithmetic: the MSAI (see Section 4), the AMSE, and the SM (see Section 5). In Chapter 6 we analyzed the strengths and weaknesses of introducing the Cartesian Method with students who have just acquired competence in solving equations using as a teaching model the abstraction of operations via the concrete model that we gave in Chapters 4 and 6.

3.1. The "Solving Arithmetic-Algebraic Problems" project

It can be said that from the first stage, in which an exploratory study was carried out, the work was done with a set of results from which empirical results were obtained, and these were then converted into theses that in turn were put to the test in the experimental study in the final stage of the project. Thus the original theses not only evolved but were also gradually extended and modified when they were used as reference elements in the new study, serving as (1) an instrument of analysis of the exploratory questionnaires used to characterize the students who were selected for the clinical interviews, (2) an instrument of analysis of the performance of the students in the teaching sequence, and (3) an instrument of analysis of the performance of the students in the clinical interviews. Furthermore, in this use of the theses as one of the tools for interpretation in the experimental study other theses emerged, which were then put to the test in the final parts of the experimental study, as also were the clinical interviews in the case study —see Filloy and Rojano (2001), in which one of these cases is presented.

The theses that appear in this chapter may be considered as the empirical results that emerged from the final research, and as such they can be put to the test by means of other experimental studies, or else by the observations that emerge from teaching practice. This may make it possible to advance them, or to modify or even reject them.

The series of theses presented may have interrelations and similarities in some aspects, but our intention is to present them just as they were used as an instrument for interpretation in the research. It must be clearly understood that some of these theses were gradually refined, others evolved, having been enriched, made more precise and even modified during the study, and others simply come from the results that were obtained in the final phase of the research, that is, from the analysis of clinical interviews.

It must also be said that these theses have drawn both implicitly and explicitly on ideas from various fields of knowledge, such as history, epistemology, teaching, psychology, mathematics, etc. However, these suppositions were obtained both from the interaction with the students in the classroom when solving problems and from the interpretation of the results of their execution via the teaching model proposed. Let's begin by stating, in the following section, some theses concerning the behavior observable in middle school students when they solve arithmetic-algebraic word problems.

3.2. Some preliminary observations

3.2.1. A cognitive tendency: resistance to producing sense for an algebraic representation when one is in a numeric context

A cognitive tendency that is observed in a considerable part of the student population of this age consists in a resistance to producing sense for an algebraic representation when one is in a dynamic of numeric solving. For example, when one proposes systematic use of the AMSE and the SM after showing their virtues for solving certain problems by a procedure of trial and error, some students come to understand the operations that they have performed only after obtaining the numeric solution of the problem and establishing the equation that represents it.

This tendency to obtain an equation only from an equality, in order to follow the requirements of the teaching process, leads to a situation in which the representation of an unknown quantity or magnitude in the problem is only used as a label because, when it is used, the solution of the problem has already been obtained, and at best the students relate the value found by other means to the letter that appears in the equation. In these cases, the letters that are used in the equation obtained have the status of a name, and these letters are not associated with the supposed numeric values that were used to find the representation of the solution of the problem.

3.2.2. Concerning the natural tendency to use numeric values to explore problems

When using the language of algebra, some expert students and many beginners resort spontaneously to the use of numeric values (and arithmetic operations) to explore and thus solve certain algebraic word problems, since the use of numbers and arithmetic operations spontaneously gives meaning to the relations that are immersed in a problem, which in many cases increases the possibilities of being able to set in motion the logical analysis of the problem. With algebraic language it is more difficult to capture the sense of symbolic representations because they are more abstract, and therefore it is more difficult to find strategies to solve the problem.

3.2.3. The relationship between competence to make a logical analysis and mastery of intermediate tactics

In order to solve more complex problems it is necessary to advance in the competence to make a logical analysis of problem situations. However, to be able to set in motion the analytic reasonings required for solving problems, it is necessary that there should be no obstructers such as the following:

a) When they make a logical analysis of a problem, some students do not accept the operativity of the unknowns; in other words, when they try to make the analysis, they tend to use or give values to the unknowns and not to manipulate them as such, even in problems with situations that can be performed with concrete objects. When they come to make the analysis, the students cannot follow the train of thought that carries out concrete actions (which, separately, they accept without any difficulty), because when they think of something unknown, such as a number of children, they cannot follow the logical implications that derive from it.
b) There is great difficulty in being able to represent one unknown in terms of another, even among students who have overcome the difficulty expressed in the previous point.
c) There is uncertainty about the tactics that have to be used when solving the problem.

In turn, in order that these obstructers should not appear it is necessary to advance in the use of intermediate strategies immersed in the uses of:

a) algebraic expressions,
b) proportionality,
c) percentages,
d) multiplication within the schemes

$$\square \times A = B;\ A \times B = \square;\ A \times \square = B,$$

e) negative numbers.

Only a competent use of all this can prevent the user from lapsing into cognitive tendencies such as tendencies 7, 8, and 9, which obstruct the possibility of making use of the CM to solve word problems, as we indicated in Section 2.

3.2.4. One way of observing the complexity of a problem is through the difficulty that a user has in inventing a problem of the same family

One way of observing the complexity of a problem consists in analyzing the difficulties that are produced when one invents problems similar to a problem that has been solved previously, because varying the data allows one to observe whether the student perceives that the problems are the same from a logical viewpoint and that the difficulty depends only on finding relations between data and unknowns.

Similarly, the complexity of the relations of the problem can be observed when one invents problems similar to one that has been solved by setting out from the solution, that is, knowing the value of the unknown or unknowns – setting out from assigning a value to the unknown when one invents a similar problem is not a natural tendency in users. This process tests the establishment of the relations made previously and opens up the way to recognition of the family of problems because of the need to create the data of an analogous problem. The creation of problems similar to one solved previously tests the forms of mental representation or comprehension that were used when analyzing the original problem.

3.2.5. For a user to be competent in a more abstract MSS, he must also be competent in other, more concrete MSSs

For a student to become a competent user of the mathematical sign system of algebra (MSS_{al}), which, formally, is the most abstract sign system in our study, it is necessary that he or she should be competent in other, less abstract sign systems, such as the mathematical sign system of arithmetic (MSS_a), which is used in the MSAI, and the sign system in between these two, (MSS_i), which is used by the AMSE and the SM.

3.2.6. The sense of the CM is related both to the capability of going back to more concrete MSSs and also to the aptitude for recognizing the algebraic expressions used to solve the problem as expressions that involve unknowns

To give a full sense of use to the CM for solving algebraic word problems it is necessary that the (competent) user should have the capability of going back to sign systems with a greater semantic load; for example, to the intermediary mathematical sign system (MSS_i) associated with the AMSE and the SM, or to the most concrete mathematical sign system, that of arithmetic (MSS_a).

The sense of the CM in problem solving requires that users should recognize the algebraic expressions used in the solution of the problem as expressions that involve unknowns. It can be said that there is a competent use of expressions with unknowns when it makes sense to perform operations between the unknown and the data of the problem. In steps prior to competent use of the CM, the pragmatics of the more concrete sign systems leads to using the letters as variables, passing through a stage in which the letters are only used as names and representations of generalized numbers, and a subsequent stage in which they are used only for representing what is unknown in the problem. These last two stages, both clearly distinct, are predecessors of the use of letters as unknowns and using algebraic expressions as relations between magnitudes, in particular as functional relations.

3.3. Four teaching models

The use of the four teaching models associated with the MSAI, the AMSE, the SM, and the CM comes after observations made in the classroom over several years: when students with similar characteristics (students with knowledge of elementary algebra, from official secondary schools) begin subsequent levels of study they generally show a tendency to tackle word problems by means of the mathematical sign system of arithmetic (MSS_a).

The solution of problems by using the language of algebra causes great difficulties, even with problems equivalent to others that have been solved previously. The students cannot establish correct meanings for the algebraic relations of word problems. An attempt was made to find an answer to this situation by proposing teaching models associated with methods for solving word problems by a numeric approach: this not only helps to solve the problems but also, especially, aids the student to produce meanings.

This teaching proposal was formulated with the intention of facilitating the setting in motion of the analysis of word problems and making it possible to link the students' pre-algebraic tendencies for tackling problems with the learning of the model socially demanded, which is the CM, that is, the model in which problems are represented and solved by means of the language of algebra. For this purpose we took into account four mathematical sign systems (with their signs, way of operating on the unknown, strategies, actions, ideas, etc.): the MSS_a of arithmetic, linked with the MSAI; the MSS_{al} of algebra, related to the CM; the MSS_i corresponding to the intermediary sign system between these two, that is, the one associated with the AMSE; and, finally, the MSS that pertains to spreadsheets.

It seemed to us important to use the four teaching models, starting out from the theoretical analysis made in the first phase of the project and the empirical observations made up to that point. One aspect that should be pointed out has to do with the different uses that students make of the unknown when they are trying to represent and solve arithmetic-algebraic word problems, and also the difficulties that those uses generate.

The earlier empirical results indicated that students such as those in the study (teenagers 13 to 16 yeas of age, with prior knowledge of arithmetic and algebra) generally showed serious obstacles that hampered the setting in motion of the logical analysis of various families of arithmetic-algebraic word problems when using an algebraic expression with the use of a letter, usually *x*, to represent the unknown. Moreover, in general there was also a certain inability to make use of the mathematical sign system of algebra (MSS_{al}) and, therefore, to represent and solve word problems by means of the teaching model associated with the CM.

The students also had great difficulty in making the arithmetic logical analysis of many kinds of problem, and, consequently, in solving them with the MSAI. This classic arithmetic method requires great competence both in the use of the MSS_a and in making an analysis of the situations presented in the problems, especially in those whose text involves assertions that are not expressed in terms of unknowns, so that the analysis is then made with reasonings that involve unknown magnitudes or quantities.

See Section 4 for a more detailed presentation of the MSAI teaching model and the problems in its use, where it is shown that situations such as those mentioned make logical analysis of them more complex, especially if the MSS_a of arithmetic is used. This is even so in simple problems like those in Section 4: to solve them by applying the MSAI calls for a much greater capability of logical analysis of the situation than if one tackles it with numeric explorations as in the AMSE or the SM, and even greater if one approaches it with the CM.

Moreover, the teaching models associated with the AMSE and the SM contain elements that facilitate the setting in motion of the logical analysis of certain kinds of problem. As they use hypothetical numeric values for the unknown, arithmetic operations are performed between them and the data, and as these operations have a greater semantic load than algebraic relations, they make it more likely that the user may be able to produce meanings for these operations in accordance with the conditions of the problem. This last point is a key factor in problems in which it is complex to set the logical analysis in motion using the MSS_a of arithmetic.

One of the aims that we propose in Section 5 is to observe whether, with the use of the teaching model based on the AMSE and the SM, the user can: (1) start to relinquish the use of the mathematical sign system of arithmetic with which the MSAI is associated; (2) begin to break away from the arithmetic use that is made of the unknown by solving problems and operating with it by reasoning and making inferences with a representation of one's own; and (3) moving on to a use of the unknown in which one operates with a representation of one's own, through the solution of families of problems, for an understanding of which, and therefore for the setting in motion of their analysis, one requires the use of hypothetical numeric values for the unknown in natural form, knowing beforehand that in problems of this kind it is more complex to set their analysis in motion by the use of the MSS_a of arithmetic associated with the MSAI than by the use of the other two methods.

To sum up, the use of the four teaching models based on the MSAI, the AMSE, the CM, and the SM for solving arithmetic-algebraic word problems takes account of the fact that, in order to solve arithmetic-algebraic word problems with the CM and, in general, to achieve competent use of the mathematical sign system of algebra (MSS_{al}), it is necessary to consider the competences, adjustments, and limitations that other, more concrete mathematical sign systems, in this case the MSS_a of arithmetic and those associated with the AMSE and the SM, may impose on the more abstract sign system that it is socially desired to teach, that of algebra, which one usually wishes to use as a method for solving problems in algebraic form —a method identified in this work as the Cartesian Method.

3.4. The Cartesian Method

It is worth mentioning that any of the indicative procedures that are usually proposed in teaching or in textbooks for solving word problems by translating them to the MSS_{al} of algebra take into account, in some way, what we have called the Cartesian Method.

The reason for calling the method Cartesian is that part of Descartes's *Regulæ ad directionem ingenii (Rules for the direction of the mind)*[1] can be interpreted as an examination of the nature of the work of translating an arithmetic-algebraic word problem to the MSS_{al} of algebra and its solution in that MSS. This is how it was understood by Polya, who, in the chapter "The Cartesian Pattern" in his book *Mathematical Discovery*, rewrote the pertinent Cartesian rules in such a way that they could be seen as problem solving principles that use the MSS_{al} of algebra. Polya's paraphrase of Descartes's rules is as follows:

(1) First, having well understood the problem, reduce it to the determination of certain unknown quantities (Rules XIII–XVI).[2]
[…]
(2) Survey the problem in the most natural way, taking it as solved and visualizing in suitable order all the relations that must hold between the unknowns and the data according to the condition (Rule XVII).[3]
[…]
(3) Detach a part of the condition according to which you can express the same quantity in two different ways and so obtain an equation between the unknowns. Eventually you should split the condition into as many parts, and so obtain a system of as many equations, as there are unknowns (Rule XIX).[4]
[…]
(4) Reduce the system of equations to one equation (Rule XXI).[5] (Polya, 1966, pp. 27–28)

In Puig and Cerdán (1990) the process of solving arithmetic-algebraic word problems modeled by the analysis and synthesis method is compared with the process modeled by the Cartesian method. In the process modeled by the analysis and synthesis method one works from the unknown in the problem and concludes when one does not come to further unknown quantities (auxiliary unknowns) but rather known quantities (data of the problem), that is, when the unknown has been reduced to data. The product of the analysis is then a set of relations between the quantities of the problem linked in such a way that they can be represented in the form of a tree that leads from the unknown to the data of the problem. The synthesis then consists in making one's way through this diagram in the opposite direction, from the data to the unknown, performing the corresponding arithmetic operations or, if one wishes, writing the arithmetic expression to solve the problem. Therefore, when the analysis and synthesis method is used for solving problems of this kind and leads to their solution, it does so in the MSS_a of arithmetic.

To illustrate this we now present the statement of a problem, the representation of its analysis in a diagram, and the arithmetic expression that results from the synthesis.

The suit cloth problem

Four pieces of cloth, each 50 meters long, are going to be used to make 20 suits, each of which needs 3 meters of cloth. The rest of the cloth will be used to make coats. If each coat needs 4 meters, how many coats can be made?

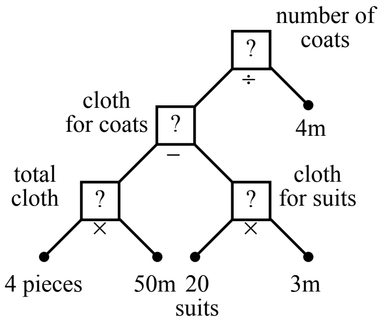

Figure 9.1

Problems like this can also be solved with the use of the MSS of algebra, by translating the statement into an equation and then solving it. Thus, in the problem just stated, it can be considered, when following the third of the rules rewritten by Polya, that the quantity that can be expressed in two different ways is the "total cloth," so that one writes the equation $4x + 20 \cdot 3 = 4 \cdot 50$, the solution of which is precisely the arithmetic expression given by the synthesis above. What makes the (indicated) solution of this equation coincide with this arithmetic expression is the fact that the sequence of operations indicated in the equation can be inverted, with the inverse operations affecting only known quantities. In other words, one can go through the set of quantities and relations expressed in this equation by proceeding from the unknown to data, as one does in analysis and synthesis. Observe that this equation is in fact one of those that we have called "arithmetic" equations.

It is not hard to realize that the equations that we have called "algebraic," that is, those in which the unknown appears on both sides of the equation, cannot be inverted in the same way, because it is necessary to operate on the unknown in order to solve them. So that one cannot go through the set of quantities and relations expressed in such an equation by proceeding from the unknown to the data as one does in analysis and synthesis.

Let us take as an example an equation such as $\frac{x}{217} + 171 = \frac{x}{198}$. If we try to trace the path of the analysis from the unknown, using the relations between quantities that are expressed in this equation, this does not reduce the unknown to data, and instead one returns to the unknown when one uses the relation that corresponds to its second appearance. We will show this by using a word problem in one of whose solutions this equation appears. Thus we will be able to name the quantities and relations expressed in the equation in accordance with their meanings in the context of the story that the statement of the problem tells.

The problem comes from Kalmykova (1975) and has already been used in Puig and Cerdán (1988) and Puig (1996). We will call it “the hay problem.”

The hay problem

A collective farm assumed that some hay stockpiled for cattle would last for 198 days, but the hay lasted for 217 days since it was of the highest quality and they used 171 kg less per day than they thought they would. How much hay had been prepared on the farm? (Kalmykova, 1975, p. 90)

If this problem can be translated into the equation $\frac{x}{217}+171=\frac{x}{198}$, it is because, from the story that the statement tells, we have extracted the known quantities “days planned,” D_p (198), “actual days,” D_a (217) and “daily reduction in consumption of hay,” C_r (171), the unknown quantities “planned daily consumption,” C_p, “actual daily consumption,” C_a, and “hay stockpiled,” T, and the relations between these quantities $D_p \times C_p = T$, $D_a \times C_a = T$ and $C_a + C_r = C_p$.

However, the use of these quantities and relations in the analysis of the unknown leads to one of the following two diagrams, which cannot end with data because once again the unknown appears, so that the analysis cannot conclude in such a way that the solution of the problem is an arithmetic expression obtained by synthesis.

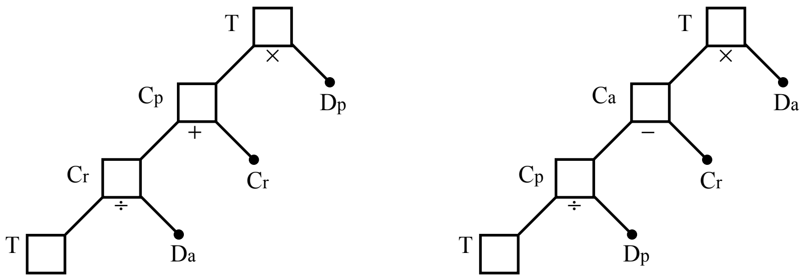

Figure 9.2

The diagrams in Figure 9.2 correspond to the two possible attempts to isolate each of the occurrences of x by inverting the operations indicated in the equation, which lead to $\left(\frac{x}{217}+171\right)198=x$ and $x=\left(\frac{x}{198}+171\right)217$, but not to x equals an arithmetic expression. In fact, with such equations it is not sufficient to invert the operations indicated in order to isolate the unknown, but rather it also necessary to operate on the unknown.

If, instead of solving the hay problem with the CM (and therefore the MSS_{al} of algebra), we had tried to solve it with the analysis and synthesis method (and the MSS_a of arithmetic) and the relations $D_p \times C_p = T$, $D_a \times C_a = T$ and $C_a + C_r = C_p$ had been established in the analysis, then it would not have been possible to solve the problem because this analysis does not allow us to reduce the unknown to data. The diagrams in Figure 9.2 show this clearly.

The fundamental difference between the analysis of the statement in the analysis and synthesis method and in the CM lies in the fact that the logic-semiotic outline that one makes when one uses the CM anticipates the use of the MSS_a of algebra. This entails not only the use of letters to designate the quantities that are determined in the analysis but also new meanings for arithmetic operations and relations, particularly the equals sign, which belong to that MSS_{al}. Consequently, when making this analysis the known and unknown quantities are considered in the same way —Descartes himself indicated that the whole art of the method lay in this.[6] In contrast, in the analysis and synthesis method the analysis is developed by situating oneself in the unknown of the problem and considering on what data one would have to operate in order to obtain it, and in the logic-semiotic outline one does not contemplate the possibility of operating other than on known quantities.

The diagrams shown so far, which reflect a solution modeled by the analysis and synthesis method, are not suitable for giving an account of the analytic reading in the CM. There is a different kind of diagram, which *is* suitable, however —one in the form of a graph that we have adapted from Fridman (1990) and that Cerdán (in preparation) studies and uses. These graphs represent the analytic reading of the statement of an arithmetic-algebraic word problem characteristic of the CM because their vertices represent quantities and their edges represent relations between quantities, so that the graph shows the network of relations between quantities that has been determined in this analytic reading. Moreover, the vertices corresponding to the data of the problem are represented by black circles (which we will call "dark vertices"), and the vertices corresponding to the unknown quantities (the unknowns of the problem or auxiliary unknowns) are represented by unfilled squares (which we will call "light vertices"). As the four basic arithmetic operations are binary, the corresponding relations are ternary, so that in the most common arithmetic-algebraic problems the edges have three vertices.[7]

Thus, the analytic reading of the hay problem (taken as a textual space) which produces the known quantities D_p (198), D_a (217), and C_r (171), the unknown quantities C_p, C_a , and T, and the relations between these quantities

$D_p \times C_p = T$, $D_a \times C_a = T$ and $C_a + C_r = C_p$ (a new text) is represented by the graph in Figure 9.3.

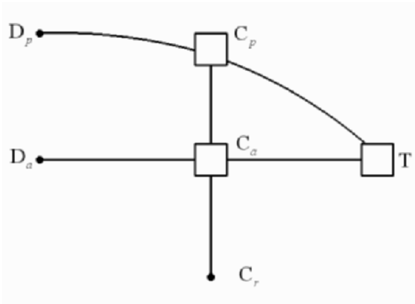

Figure 9.3

The analytic reading of the suit cloth problem in which the quantities and relations that we represented previously by a diagram are determined can also be represented by the graph in Figure 9.4. In it the known quantities are "number of pieces of cloth," N_p (4), "number of suits," N_s (20), "cloth per coat," C_c (4 m), "cloth per piece." C_p (50 m) and "cloth per suit," C_s (3 m); the unknown quantities are "number of coats," N_c, "cloth for the coats," T_c, "cloth for the pieces or total cloth," T_p, and "cloth for the suits," T_s, and the relations are $N_i \times G_i = T_i$, $i \in \{c, p, s\}$, $T_p = T_c + T_s$.

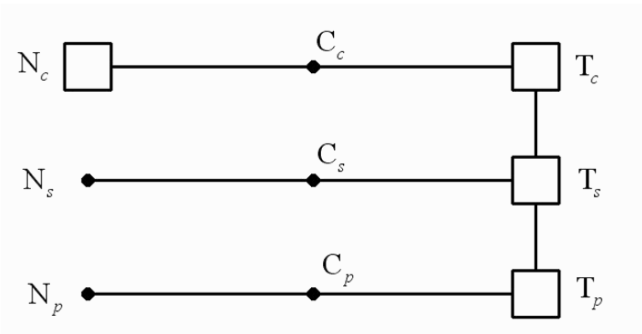

Figure 9.4

In these graphs it is also clear why one network of relations makes it possible to obtain the solution of the problem by using the MSS_a of arithmetic and the other does not. In fact, in order to avoid having to operate on unknown quantities it is necessary that on one ternary edge two of the vertices should be dark (should be known quantities), for then the light vertex can be converted into a dark vertex (the unknown quantity can be calculated from known quantities) by performing the corresponding arithmetic operation. The unknown of the problem can be obtained from the data as long as there is a way of progressively converting light vertices into dark vertices until one arrives at the unknown. In the graph corresponding to the analytic reading of the suit cloth problem this path exists and it coincides with the path described by the analysis and synthesis diagram in Figure 9.1. In the diagram of the reading of the hay problem the path cannot exist because there is no edge that has two dark vertices. Consequently, it is consistent with the terminology introduced earlier to describe as "arithmetic" those graphs that share with the graph in Figure 9.4 the property that we have explained, and as "algebraic" those that do not have it (such as the one in Figure 9.3), just as is done by Cerdán (in preparation).

These graphs represent the analytic reading of the statements of the problems when they are solved by means of the CM, but this analytic reading is only the first step in the method. To obtain the equation $4x + 20 \cdot 3 = 4 \cdot 50$ or the equation $\frac{x}{217} + 171 = \frac{x}{198}$ it is necessary to complete three further steps.

The second step consists in choosing a quantity (or several quantities) which one designates with a letter (or several different letters).

The third step consists in writing algebraic expressions to designate the other quantities, using the letter (or letters) introduced in the second step and the relations found in the analytic reading made in the first step.

The fourth step consists in writing an equation (or as many independent equations as the number of letters introduced in the second step) based on the observation that two (non-equivalent) algebraic expressions written in the third step designate the same quantity.

In Figures 9.5, 9.6, and 9.7 we show how these steps can also be represented in graphs.

Second step

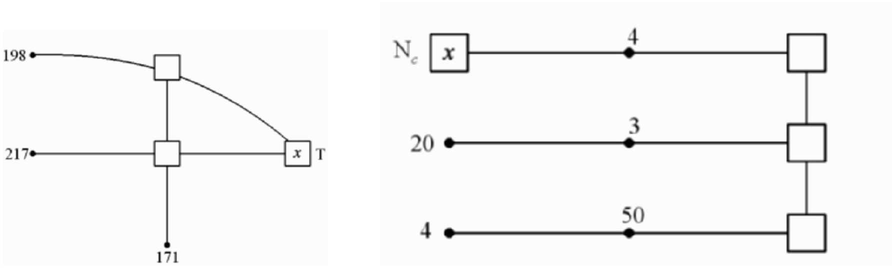

Figure 9.5

Third step

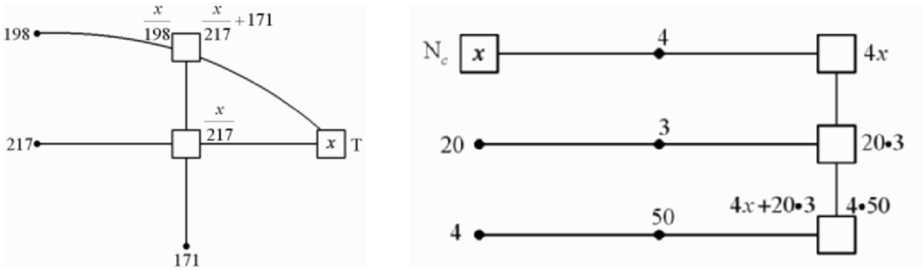

Figure 9.6

Fourth step

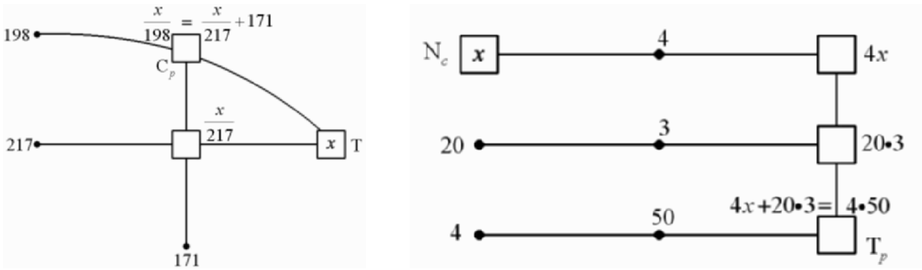

Figure 9.7

Observe that the equation obtained is arithmetic if the graph is arithmetic, and algebraic if it is algebraic. We have also seen that the corresponding analysis and synthesis diagrams either produce an arithmetic solution or they do not. However, this does not mean that we can describe the corresponding problems as "arithmetic" (the suit cloth problem) and "algebraic" (the hay problem). In fact, in the case of the hay problem what is not arithmetic is a solving process carried out by the analysis and synthesis method (represented by the analysis and synthesis diagram) and an analytic reading that constitutes the first step of the CM (represented by the graph); but it is possible that there may be another solving process or another analytic reading that determines another network of relations between quantities that *is* arithmetic. Such is indeed the case: if the analytic reading determines not only the quantities determined previously but also the unknown quantities "additional days," D_{m}, "consumption on the additional days," C_{Dm} and "total saving," S_{t}, and the new relations $D_{\mathrm{p}} \times C_{\mathrm{r}} = S_{\mathrm{t}}$, $D_{\mathrm{m}} \times C_{\mathrm{a}} = C_{\mathrm{Dm}}$, D_{p} $D_{\mathrm{m}} = D_{\mathrm{a}}$ and $C_{\mathrm{Dm}} = S_{\mathrm{t}}$, the corresponding graph is arithmetic, as can be seen in Figure 9.8, and the unknown is determined by means of the arithmetic expression $\left(\frac{198\cdot 171}{217-198}+171\right)\cdot 198$.

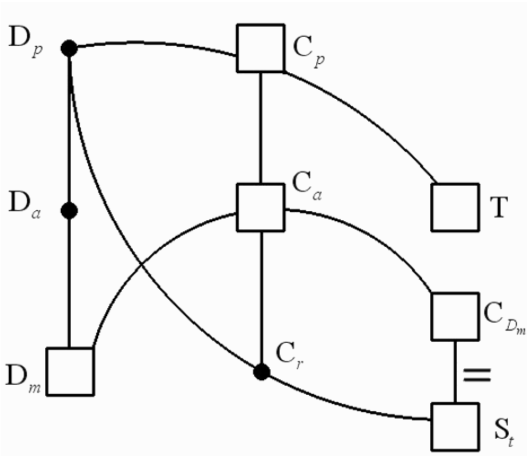

Figure 9.8

What can be described as arithmetic or algebraic, therefore, is the solving process (represented by the analysis and synthesis diagram), the analytic reading (represented by the graph), or the equation that translates the statement, but not the problem.

The splitting of the CM into steps, as presented, describes the competent behaviour of the ideal subject. Only in this sense is it possible that each step begins with the completion of the previous one. In fact, there are obvious connections between the termination of one step and the commencement of the next. For example, the writing of algebraic expressions (step two) is complete precisely when two expressions have been written that designate the same quantity, which in turn makes step four possible. What this splitting into steps clearly shows is that the CM is the algebraic method par excellence, because each of the steps makes sense only with the use of the MSS_{al} of algebra.

3.5. Spreadsheets used to solve word problems

As we saw at the beginning, some of the theses presented indicate the virtues of using numeric values to explain the solving of word problems. When a method such as the AMSE is put into practice in a context such as that of the computer spreadsheet, the search for the value of the unknown is done only with the use of numeric values. In other words, unlike what happens with the AMSE, in the SM one does not make an explicit formulation of the equation.

When one observes students using the SM to represent and solve word problems, the spreadsheet medium influences their preliminary solving strategies, but it can also be said that earlier experience in solving problems has an impact on the strategies used in the SM.

When this method is applied, once the problem has been expounded in the MSS of the spreadsheet the students have at their disposal a means with which to explore the possible solving strategies.

We will now indicate some of the observations that we have gathered concerning what happens when students use the SM.

1) Most students do not think spontaneously in terms of an algebraic experience when they first work in an environment such as the one provided by the SM.
2) The SM stimulates students to stop focusing on a specific example and move on to considering a general relation.
3) The SM also stimulates students to accept working with an unknown. The use of one cell in the spreadsheet to represent the unknown is established, and by using the mouse they can then express the various relations stated in the problem in terms of the cell used in the first place.
4) After using the SM there is a greater awareness of the relations between the unknowns, and between the unknowns and the data of the problem.
5) Before a sequence of sessions in the use of the SM one can observe an evolution toward a more general algebraic method consisting in proceeding from the unknown to the given.
6) In the SM one can see an integration of various solving strategies, such as the refinement of the whole and part strategy and trial and error.

4. THE METHOD OF SUCCESSIVE ANALYTIC INFERENCES

4.1. An example of the use of the MSAI

We have already described the MSAI in Section 2. We indicated there that the use of the MSAI for solving arithmetic-algebraic problems presents itself as a product of logical inferences which act as descriptions of the transformations of the possible situations of the problem until one comes to one that is recognized as the solution of the problem. We will illustrate this kind of inference with the problem that we call "the typist problem":

The typist problem

A typist has to type 1200 pages in a certain number of days. If she types 40 more pages a day, she will finish the work 8 days sooner. How many pages a day does she type and how many days is she expected to take to finish the work?

Solution

If □ is the number of pages that the typist normally types, the number of pages not typed in the 8 days will be 8 times □, which will have to be made up by typing 40 extra pages a day. The days that the typist will work will be 8 less than the number of days that she would take to do the work normally, which we can calculate by dividing 1200 by □. From all this we obtain the following equalities:

$$8\square = 40\left(\frac{1200}{\square} - 8\right)$$

$$\square = 5\left(\frac{1200}{\square} - 8\right) = \frac{6000}{\square} - 40$$

Multiplying both sides by the quantity □ we obtain the equality

$$\square^2 = 6000 - 40\square$$

which is the same as

$$\square^2 + 40\square = 6000\,.$$

In order to calculate the quantity we use the seventh proposition in the first book of Jordanus de Nemore's *De Numeris Datis*, written in the early 13th century:

If one divides a number into two parts, one of which has been given, and the product of the one that has not been given by itself and by the one that has been given is a given number, then the divided number will have been given.[8]

In Nemore's book each proposition has three parts: the first is the statement, in which he affirms that if some numbers (or ratios) have been given, and certain relations between them have also been given, then other numbers (or ratios) have also been given; the second part is the proof, in which he makes transformations of the numbers (or ratios) and the relations which either show that the numbers are in fact given or else convert them into the numbers and relations of the hypothesis of some previous proposition; and the third part is the calculation of an example with concrete numbers, which therefore has the value of an algorithm.

If we use the algorithm that Nemore presents in the third part of proposition I-7, we will solve the problem in the following manner:

One of the parts is 40, and the other part squared and by 40 is 6000. Double 6000 and double it again, giving 24000. Add to this the square of 40, which is 1600, making 25600, the square root of which is 160. From this take 40 and halve the result, giving 60. This is the unknown part (in our case, □). So that the number divided is 40 + 60 = 100.[9]

Verification

If the typist does 60 pages a day, she would do 1200 in 1200 ÷ 60 = 20 days. If she does 40 more pages a day, she will do 100, so that she will take 1200 ÷ 100 = 12 days, which is 8 less than 20.

4.2. Difficulties in the use of the MSAI

4.2.1. The tendency not to admit the possibility of making inferences about something that is unknown

There are middle school students who do not admit the possibility of making inferences about something that is unknown. A case of a similar nature is that of other students who simply avoid operating on the unknown. In other words, some students show resistance to bringing into play operations on the representation of something unknown.

4.2.2. Lack of knowledge of concepts as an obstructer

When a user solves a problem by means of the MSAI, this enables him to recognize that there is a logical structure in the problem. As a result of this recognition he may become aware that concepts that he has not mastered are involved in the logical relations of the problem, and this may become an obstructer for recognizing and generating problems of the same family as the one that he has just solved. On the other hand, if the student is competent in the concepts that are involved in the implications obtained from the logical outline of the problem, then he or she is capable of representing the unknown and making use of that representation, even if the unknown parts vary. This happens, for example, if one passes from using a directly proportional relation to using several directly proportional relations in the same problem.

4.2.3. Families of problems determine their level of representation.

Each family of problems determines the levels of representation that their solution requires. For example, when a student is really competent in a more abstract MSS and is presented with a problem of mixtures (see the mixtures problem that we present in Section 4.3.2), the solution may lead naturally to the use of the MSS_a of arithmetic. However, this kind of problem is usually solved algebraically.

4.2.4. The use of trial and error to avoid the difficulty of the inferences of the MSAI

To be able to solve certain families of problems with the MSAI an expert level of competence is required, and therefore there is a natural tendency to use trial and error —for example, to get away from the series of successive analytic inferences that logical analysis of the situation requires.

With the use of trial and error it is actually possible to simplify the difficulty of the inferences of the MSAI. This is due to the fact that there are problems in which, in order to tackle them with the MSAI, the series of successive inferences required in order to make the analysis of the situation calls for representations that involve competence in more advanced uses of the arithmetic sign system —the more complicated problems require a greater mastery of the codes that relate syntax and semantics, both in natural language and in the MSS_a of arithmetic, and also in their pragmatics, that is, in the uses that permit crossing between the two sign systems.

4.2.5. The need for intensive use of memory as an obstructer

A factor that complicates the establishment in the MSAI of the logical outline in the MSS_a of arithmetic is the fact that some problems require an intensive use of working memory, and this implies a training that only expert solvers have had. Moreover, when one uses the MSAI to solve word problems one must invent the representations problem by problem, and this calls for a certain capability of using working memory in order to represent the actions proposed in the logical outline of the solution and leave new markers and indicators —or new groupings in the memory— for preliminary results, and so not leave them isolated or forgotten.

4.2.6. The singularity of the representation of each problem in the MSAI as opposed to representation using canonic forms in the CM

When one uses the MSAI, one does not generate one sole representation of a certain style, but rather the representation changes for each problem, or at any rate for families of problems; on the other hand, when one uses the MSS_{al} of algebra or the CM, one always uses the representation provided by certain expressions which belong to that MSS_{al}, and those representations are reduced to canonic forms in order to solve them.

4.3. Advances with the MSAI

4.3.1. Modification of the natural tendency to tackle arithmetic-algebraic problems by means of arithmetic, and its relation to the representation of the unknown.

The natural tendency to tackle arithmetic-algebraic problems by means of arithmetic weakens when one tries to solve certain families of problems that are difficult to solve with the MSAI. When variations in the value of what is unknown are brought into play, it is possible to propose families that will require the student to use representations of a different kind, in which unknown quantities have to be represented so that inferences can then be made with them (see the problem in Section 4.3.2). In the end, with the Cartesian Method it will be necessary to operate on the representation of what is unknown in the problem.

The needs of representation generate new senses, which bring the possibility of making more abstract uses of the MSSs used to make the representation of the problem on the basis of the outline of the solution. The essential difference between the traditional introduction to solving problems with algebra and these preliminary approaches, such as the MSAI, lies in the fact that, in the solution of the problems (1) the unknown is represented, but one does not operate on it; (2) inferences are made that refer to the representation of the unknown; (3) if one operates on something, one always operates on data; (4) if one speak of unknowns, one does so in terms of the results of the operations that are performed on the data.

4.3.2. On the processes of abstraction and generalization

As more complex families of problems are solved, the sign systems used gradually become more abstract. These processes of generalization and abstraction operate on the families of problems, either by finding common elements —which we will call "generalization" —or by making negations in part of the members of the family —in which case we will speak of a process of abstraction.

Thus, for example, a mechanism to explain why mixture problems are more difficult than problems of other families can be found in observation of the need to break away from the use of only inferring from the representation of something unknown in order to be able to use the representation in which the unknown parts also vary.

As an example of what we have just said, we will use the MSAI to solve a mixture problem.

The mixture problem

A man wants to change the mixture of water and antifreeze in the radiator of his car, which contains 20% antifreeze. He has discovered that the best mixture is one that contains 50% antifreeze, so he has to remove a certain quantity of the mixture in the radiator and then add antifreeze until it represents 50% of the mixture. The radiator has a capacity of 30 liters. What quantity of mixture must he replace?

Solution

The mixture in the car initially contains 6 L of antifreeze and 24 L of water. It is necessary to remove 9 L of water so that only 15 L remains. In order to do this we note that any quantity of mixture is always 80% water and the rest is antifreeze. We have to remove a quantity of mixture such that 80% of it is 9 L, that is, $9 \div 0.8 = 11.25$ L.

Note that the MSAI allows one to envisage a family of problems in which the radiator is of any capacity and the initial mixture can be of any proportion. Here, in solving the problem we vary both the water and the antifreeze, both of which are initially unknown quantities.

4.3.3. With time, the MSAI requires representations similar to those of the AMSE and the SM

When this happens, representations are established that show a distancing from the use of the MSS_a of arithmetic that is used in the MSAI. In other words, in the AMSE and the SM one brings into use a representation of what is not known with a view to operating on the unknown that represents it, whereas in the MSAI what is not known is only represented and inferences are made that speak of that representation, but one never operates on the representation of what is not known. This is one of the greatest differences between the use of what is not known in the MSAI and the use that one seeks to provide in the intermediate $MSSs_i$ that are used in the AMSE and the SM.

4.3.4. The use of numeric trial and error in the arithmetic MSS stratum can enable the user to correct a faulty analysis made with the MSAI

4.3.5. The succinctness of the use of the MSAI

So that the reader may recognize the power of a solution obtained with the MSAI, we are going to solve the following problem, also solved with the AMSE in Filloy, Rojano, and Rubio (2001), which we will call the "teacher problem."

The teacher problem

A teacher at Kinder has 120 chocolates and 192 toffees. She is going to distribute them fairly among the students. If each student receives 3 more toffees than chocolates, how many students are there?

Solution

We begin by giving one chocolate and one toffee to each student. As there are fewer chocolates, they run out before the toffees, so that now only toffees remain to be given out: the 192 – 120 = 72 that remain.

Now we give them out, knowing that each student receives 3 of them; therefore there are 72 ÷ 3 = 24 students.

Verification

Each student receives 192 ÷ 24 = 8 toffees and 120 ÷ 24 = 5 chocolates, i.e., three fewer.

The reader can compare this solution with the one given in Filloy, Rojano, and Rubio (2001) using the AMSE, and will be able to see the succinctness of the solution just given, and also appreciate that from this solution one can generate a family of problems similar to that of the teacher. One has only to vary the number of toffees, chocolates, etc. One can also then see the kind of restrictions that have to be made so that these quantities produce a real problem, that is, a problem that has a solution.

5. TOWARD THE CM VIA THE MSAI, THE AMSE AND THE SM

In this section we are going to present the two other methods that are more deeply rooted in arithmetic: the Analytic Method of Successive Explorations (AMSE) and the Spreadsheet Method (SM). In Filloy, Rojano, and Rubio (2001) we gave examples of how they are used with students. Here we will give a series of reasons that make these two methods, together with the MSAI presented in the last section, suitable precursors before trying to get middle school students to become competent in the method traditionally used, which we have here called the Cartesian Method (CM).

It could be said that, hitherto, making students competent in the use of the CM has been the only aim indicated in traditional algebra courses in the chapters that talk about solving word problems. In Filloy and Rojano (2001) we showed the difficulties of introducing the CM when one has just taught how to solve equations. The results presented in that work are gratifying, although it presents only one case to analyze the difficulties and the successes.

We will now present results that endorse the appropriateness of using the MSAI, the AMSE, and the SM as vehicles for achieving the competences that the CM requires.

5.1. The AMSE and the SM as a bridge to unite syntactic and semantic development

The teaching models based on the AMSE and the SM serve as a bridge to unite syntactic and semantic development through the production of meanings for arithmetic-algebraic operations in the transition from the use of the notion of variable to that of unknown.

5.2. The MSAI, the AMSE, and the SM serve as precursors for creating the meanings of algebraic relationships

The meanings of arithmetic operations, their properties, and their results, as used in the MSAI, the AMSE, and the SM, serve as precursors for creating the meanings of the algebraic relations established in the use of expressions with unknowns and data, and even the meanings of the more complex expressions involving unknowns of a word problem that are presented in the use of the CM. The meanings of the arithmetic operations within strata of the MSS_a of arithmetic serve as precursors of more abstract representations —algebraic, for example. However, correctly signifying arithmetic operations, their properties and their results with the MSAI, the AMSE, and the SM in order to create meaning for the algebraic relations of the CM also implies the need to make competent use of them.

5.3. The AMSE and the SM encourage different algebraic interpretations of the word problem

The algebraic interpretations encouraged by the AMSE and the SM do not generally represent the relations between data and unknown in the order in which they appear in the statement of the problem, something that does usually happen in the teaching sequences with which the CM is illustrated.

Indeed, this freedom in interpretation is based on the student's natural tendency to manipulate only one unknown in problems that may involve the manipulation of two or more, in contrast to the classic teaching strategies, which are generally versions of the CM, tending to use two unknowns in the solution of such problems.

5.4. Dimensional analysis of equations serves as an element of control

Making a dimensional analysis of the equations obtained from numeric relations that are established between quantities involved in a word problem helps one to understand the notion of relation between quantities and magnitudes that emerge from the word problem, and therefore serves as an element of control of the representation of the problem and as a means for producing sense for the notion of equivalence between algebraic expressions.

In general, however, it serves to create senses that lead to the notion of equivalence between algebraic expressions that involve the use of unknowns as the common element in two algebraic relations, this being expressed in an equation.

5.5. For some problems the MSAI is more efficient than the AMSE or the SM

To embark on a plan to find a solution via the CM or the AMSE is not always the best path to the solution. An easier solving strategy may be one that is set in motion on the basis of a direct logic-arithmetic analysis, as in the MSAI (see, for example, the teacher problem and the mixture problem, presented in Sections 4.3.5 and 4.3.2). In the problems in which this happens, the intermediate character of the MSAI is not seen.

5.6. The relationship between representations in the CM and the efficient use of working memory

Symbolic representations of problems in the CM make the use of the working memory more efficient. When the student succeeds in making relations between data and unknowns he combines the information, making more complex packets of information. At the point when the student succeeds in making these relations, the use of syntax obviates the need to burden the working memory with semantic descriptions bound up with the posing of the problems.

5.7. The competent use of the CM and its relation to the various uses of algebraic expressions

To simplify the more complex problems of arithmetic and medium level algebra one requires a competent use of the MSS_{al} of algebra and therefore of the CM. Part of the order of complexity of families of arithmetic-algebraic word problems comes from the difficulties presented by their logic-arithmetic analysis.

To understand progress in the competent use of the CM one must:

a) Explore the tensions that exist between the uses of the concepts of name, representation of a generalized number, representation of what is not known, unknown, variable, and relation. To do so, one must understand what happens with the difficulty of a problem and of the solution of the equation that represents it when the data of the problem are varied.
b) Analyze the relation between the complexity of a family of problems and the development of algebraic syntax and semantics: the operation of negatives, use of rational numbers, simplification of algebraic expressions, solution of equations, etc. This has to do with the clarification of what we might call the various uses of the algebraic expressions indicated in (a).

5.8. The logical outline, the analysis of the problem, and other competences of students

In the first stage of the AMSE and the SM (and probably of any method), which consists in the reading and understanding of the text of the problem, it is necessary to make a logical outline of the problem situation. This outline involves, among other things, a logical mental representation of the problem that contains the basic information of the problem situation and that identifies the relations that are central for the possibility of setting any solving strategy in motion. However, having an understanding or overall logic-mental representation of the problem is not enough to enable one to set the AMSE and the CM in motion; one must also, as part of the logical analysis of the problem, have developed competences to:

1) Make a breakdown of the principal question of a problem that is given generically. In the AMSE and the SM, this becomes stage 1 of the teaching model based on this method: explanation of each of the unknowns of the problem.
2) Split up the problem in such a way that if there is an implicit unknown it is made explicit, and even becomes the principal unknown (ability to change the unknown).
3) Create new unknowns, based on the problem situation, with which one can set solving strategies in motion.
4) Represent relations between the various unknowns.

5) Identify representations of relations to find an element common to one or more of these relations.
6) Represent this identification via an equation.
7) Use algebraic procedures for the solving of equations as a tactic for the search for the unknown in the problem situation.

All of these competences are an important and necessary part of the sense of the Cartesian method.

5.9. The AMSE and the SM use special markers in their representations to release units of memory that allow the progressive setting in motion of the analysis

The AMSE and the SM require the student to learn to leave markers that release units of memory, enabling him or her to use them in the progressive setting in motion of the analysis and subsequently the solution of the problem. These markers can enable him or her to recover the senses of the relations that are established between the data and the unknown quantities.

Some students do not create enough markers, so that in their system of representation only a few of the equations that they propose are correct. These intermediate representations group the information into packets that have a complex organization, although this cannot be distinguished in the notation produced by these students.

5.10. The solution of some problems depends on whether the logical outline establishes a suitable representation

Proposing a more abstract representation is not sufficient to solve some problems. There are problems whose solution depends more on whether the representation established by the logical outline is suitable than on whether it is more abstract.

If one is using arithmetic methods, it is also not sufficient to be capable of retaining everything that one produces in the working memory. In certain families of problems some solvers try to get closer and closer to the result, yet by this path it is very difficult for them to find an equality as they get progressively closer.

In this case the problem is not a matter of not having records of the calculations; the difficulty lies in the fact that one is not taking the logical outline of the problem as a basis for seeking an equation as a representation of what is happening in the problem. Consequently, in order to progress one will require either a more abstract representation or, at least, a more articulated representation of the problem solving process, that is, a representation in which what one seeks is to be in possession of a process in which one carries out a series of stages that enable one gradually to clarify the relations between the data, so that some of these mutual relations can then be identified: in other words, a representation in which the aim is not to find the numeric solution of the problem but to establish the linear equation that models it.

5.11. Some abbreviations that use natural language are related to the production of mistaken representation in the MSSs

Some abbreviations that make use of natural language, referring to arithmetic relations established in the process of solving a problem or in the logical outline of it, possibly combined with the limitations of working memory, encourage non-competent users to produce mistaken representations, both when they use the MSS_a of arithmetic and when they use the MSS_{al} of algebra.

5.12. In some contexts one finds a cognitive tendency to make transfers (mistaken or otherwise) from one problem to another as a result of immediate recognition.

When one presents a problem after another problem with a statement that speaks of similar things but is not of the same family, there is a natural tendency to bring into play automatic processes that are based on a mistaken recognition of known forms or schemes in the statement, with the result that the user produces generalizations that lead him or her to represent the problem in the same way as the preceding one.

This tendency is related to the reading of problems not as the kind of texts that arithmetic-algebraic word problems are, but as narrative texts. The use of this kind of reading to replace the reading that constitutes the logical analysis of the problem situation may lead to errors in the representation of the problem.

5.13. The articulation of mistaken generalizations

When the student gets stuck in readings based on the use of certain parts of the arithmetic-algebraic language that do not enable him or her to solve the problem situation, he tends to get round the difficulty by the device of extending a rule to other contexts where its application does not make sense.

The context may play the part of an obstructer or encourager of an incorrect coding of a representation of a concept in which the user is trying to acquire formal competence through the teaching process. The cognitive tendency of getting stuck in readings made within an MSS prevents the setting in motion of a solving process by means of a different MSS stratum.

SUMMARY

In this chapter we provide the results of an empirical study concerning the competences that are necessary for the use of four methods for solving arithmetic-algebraic word problems. The methods in question are the Method of Successive Analytic Inferences, the Analytic Method of Successive Explorations, the Spreadsheet Method, and the Cartesian Method. Emphasis is placed on the need to be competent in increasingly general and more abstract uses of representations required for mastery of the algebraic method par excellence, the Cartesian Method, and the competences of the Cartesian Method are contrasted with those required for mastery of the other three methods, which are more deeply rooted in arithmetic.

The next chapter concludes the book by describing ways to further study educational algebra.

ENDNOTES

[1] The canonic edition of the works of Descartes is the one by Charles Adam and Paul Tannery, *Œuvres de Descartes*, volume X of which includes the original Latin of the rules. The original posthumous edition is Descartes (1701), and the first French translation is contained in the eleventh volume of the edition by Victor Cousin, *Œuvres de Descartes*, which was published in 1826.

[2] Although Polya says that this sentence paraphrases four of Descartes's rules, rule XIII really contains all that it paraphrases: "Quand nous comprenons parfaitement une question, il faut la dégager de toute conception superflue, la réduire au plus simple, la subdiviser le plus possible au moyen de l'énumération." (Descartes, 1826, p. 284). Previously (rule VII) Descartes has already stated the importance of "énumération", which he defines as "la recherche attentive et exacte de tout ce qui a rapport à la question proposée. […] cette recherche doit être telle que nous puissions conclure avec certitude que nous n'avons rien mis à tort" (Descartes, 1826, p. 235). Rule XIV speaks of the understanding of "l'étendue réelle des corps" and says that the preceding rule also applies to it. Rules XV and XVI are advice for the mind to pay attention to the essential and for the memory not to weary itself with what may be necessary but does not require the attention of the mind. Rule XV recommends drawing figures to keep the mind attentive: "Souvent il est bon de tracer ces figures, et de les montrer aux sens externes, pour tenir plus facilement notre esprit attentif." (Descartes, 1826, p. 313). Rule XVI recommends not using complete figures, but mere jottings in order to unburden the memory, when the attention of the mind is not needed: "Quant à ce qui n'exige pas l'attention de l'esprit, quoique nécessaire pour la conclusion, il vaut mieux le désigner par de courtes notes que par des figures entières. Par ce moyen la mémoire ne pourra nous faire défaut, et cependant la pensée ne sera pas distraite, pour le retenir, des autres opérations auxquelles elle est occupée" (Descartes, 1826, p. 313).

[3] "Il faut parcourir directement la difficulté proposée, en faisant abstraction de ce que quelques uns de ses termes sont connus et les autres inconnus, et en suivant, par la marche véritable, la mutuelle dépendance des unes et des autres" (Descartes, 1826, p. 319).

[4] "C'est par cette méthode qu'il faut chercher autant de grandeurs exprimées de deux manières différentes que nous supposons connus de termes inconnus, pour parcourir directement la difficulté; car, par ce moyen, nous aurons autant de comparaisons entre deux choses égales" (Descartes, 1826, p. 328).

[5] "S'il y a plusieurs équations de cette espèce, il faudra les réduire toutes à une seule, savoir à celle dont les termes occuperont le plus petit nombre de degrés, dans la série des grandeurs en proportion continue, selon laquelle ces termes eux-mêmes doivent être disposés" (Descartes, 1826, p. 329).

[6] "[…] tout l'art en ce lieu doit consister à pouvoir, en supposant connu ce qui ne l'est pas, nous munir d'un moyen facile et direct de recherche même dans les difficultés les plus embarrassées. […] Si […] nous les mettions, quoique inconnues, au nombre des choses connues, pour en déduire, graduellement et par la vraie route, le connu même comme s'il étoit inconnu, nous remplirons tout ce que cette règle exige" (Descartes, 1826, pp. 320–321).

[7] In fact, Fridman (1990) considers only trinomial graphs, that is, graphs with all the edges having three vertices. However, in order to give an account of all the arithmetic-algebraic problems that are set in primary and secondary school it is necessary to consider other kinds of edges: with four vertices (e.g., for relations of proportionality), with two vertices (e.g., for the relation of equality between two quantities), and others (e.g., to give an account of the relations corresponding to the operations of raising to powers and extracting roots –see Nassar, 2001).

[8] This book by Jordanus de Nemore has been published by Barnabas Hughes in Latin, with an English translation and a transcription into the language of modern algebra (Hughes, 1981).

Here we give a different translation of Nemore's propositions. The reasons for this are set out in Puig (1994). The Latin text of the statement of proposition I-7 is as follows (Hughes, 1981, p. 59):
Si dividatur numerus in duo, quorum alterum tantum datum, ex non dato autem in se et in datum provenerit numerus datus, erit et numerus qui divisus fuerat datus.
[9] Our translation is not literal, and in it we have replaced the numbers of Nemore's example with those of the typist problem. The Latin text is given below (Hughes, 1981, p. 59):
Huius operatio est verbi gratia. Sit vi unum dividentium, et ex reliquo in se et in vi fiant xl quorum duplum id est lxxx duplicentur, et erunt clx, quibus addatur quadratum vi hoc est xxxvi, et fient cxcvi, cuius radix est xiiii, de quo sublatis vi et reliquo mediato fient iiii, qui est reliquum. Eritque totus divisus x, coniunctis iiii et vi.