CHAPTER 5

TEACHING MODELS

OVERVIEW

From the standpoint of symbolic algebra as a language, we characterize teaching models as successions of *mathematical texts* that are exchanged between pupil and teacher. Said characterization involves notions such as that of *text* and of *textual space*, the differentiation of which corresponds to the difference between *meaning* and *sense*, given that once one understands that a *text* is the result of reading a *textual space*, teaching and learning in mathematics class may be interpreted as a repeated reading process – transformation of *textual spaces* into *texts*, which are in turn taken as *textual spaces* to be read, and so on and so forth. This theoretical treatment of teaching models is completed by use of the notions of *mathematical sign system* and of *language strata*, to be applied to the case of concrete modeling introduced in the previous chapter, as well as to the analysis of syntactic models in algebra and of the semantic – syntactic relationship in algebra, the discussion of which was also begun in the preceding chapter.

1. INTRODUCTION

The structuralist movement of the 1960s advocated teaching a mathematics in which school algebra was conceived as the explanation of the structural properties of numbers and of arithmetic-algebraic operations. In the texts and materials produced in that period there were many different presentations, for example, of the laws of commutation and association, which referred first to numbers (or a specific number system) and second to letters. This is an example of how the transition from arithmetic to algebra was reduced to a mere paraphrase of the laws that were valid for numbers, but applied on this occasion to algebraic expressions.

This conception of algebra as simply an extension of arithmetic knowledge denies the conceptual and qualitative changes in the way of operating and of solving problems that the appropriation of algebraic language presupposes, and in the teaching of mathematics at middle school levels it gives rise to what one might call “a forgotten boundary” (Chevallard, 1983): the boundary between arithmetic and an algebraic way of thinking, which is eliminated from the aspect of the structural properties common to both —for example, commutativity and associativity are properties that are equally valid for numbers and letters— since that viewpoint hides the characteristics that differentiate them.

Despite the years that have elapsed and the fact that the structuralist movement is no longer in vogue, it is still necessary to talk about this in curriculum development because our teaching plans and syllabuses for mathematics are still influenced by it. With the reforms that have been carried out since then, perhaps the approach proposed has changed, and consequently there has been a development of syllabuses with a greater inclusion of the need to use the solving of problem situations. Nevertheless, there is still a need to insist on more profound changes, which have not yet taken place. In this book we discuss various problems that will have to be taken into account in the future, in the design of those parts of syllabuses that have to do with solving first-degree equations and arithmetic-algebraic word problems.

The importance of algebra as a language of generalizations and as a method is precisely what distinguishes it from arithmetic, and what for centuries has set it in a privileged place in education. However, algebra has ceased to play that role in our current syllabuses. One cannot yet see a proper recovery of the significance of school algebra as a symbolic language whose potential lies in its use as a means for expressing situations and for solving problems posed in various areas of knowledge.

In the last chapter we talked about a clinical study of 12- to 14-year-old children that showed the difficulties that secondary school students face when they have to read or write algebraic language. At the time of the observation, the children had already received instruction in pre-algebra and had been introduced to elementary algebra through solving linear equations and the corresponding word problems, but they had not yet received systematic instruction on the use of open expressions, the equivalence of expressions, or solving systems of equations. At this level it was still possible to see a tension in the students between the way of reading and expressing themselves using the language of arithmetic and the need to produce new meanings for mathematical texts in the context of algebra. The latter aspect is yet another indicator that the arithmetic-algebraic boundary cannot be avoided, because that would lead to false conceptions about the processes of acquiring the language of algebra and, consequently, about the role of teaching in such processes. On the other hand, the importance of considering the reading and writing of symbolic algebra as an educational goal for learners at middle school level is reaffirmed.

1.1. Problem-solving ability and competence in the use of the Mathematical Sign Systems (MSSs) of algebra

Problem solving is an objective that has remained in the school curriculum despite all the educational reforms that have taken place, and at present it has particular importance in the curriculum of mathematics at middle school levels. Moreover, research on mathematics education has always considered that it was a matter worth studying in depth, and the many studies that have been performed in this field have constantly pointed out the role played by symbolization in problem solving.

It might be said that the first tasks of mathematical symbolization that the learner performs at a higher level of generality than that of arithmetic come when he tries to solve a problem with the tool of algebra, and that then there is the beginning of a process of combined evolution of symbolization and problem solving that involves using algebra as a language in which to model and solve problems derived from various branches of knowledge (physics, biology, geometry, financial affairs, etc.), subsequently culminating in the use of algebra as a basic language for expressing statements and procedures performed in other branches of mathematics (analytic geometry, calculus, mathematical analysis, etc.). In Chapter 9 we explore the possibility of attaining the competences required for the use of the Cartesian method for solving problems when syntactic competences have recently been acquired using a concrete teaching model. In this chapter we also deal with concrete teaching models in general terms.

However, in addition to recognizing in algebra this fundamental role as a means of scientific expression, it is also necessary to recognize its importance in school education, that is, in the realm of teaching. Yet it is precisely in this area that the assimilation of the language of algebra by students presents difficulties that come from the interaction between this language which is in the process of being constructed and two languages that have already been mastered, namely, the language of arithmetic and natural language. In the translation between mathematical sign systems and natural language these difficulties were shown through the predominance of the meanings given to signs and words in the two languages in which the students were competent, natural language and the language of arithmetic prior to the sign system of algebra. The students would have to overcome these difficulties, therefore, in order to attain to the reading and writing of algebra and thus become competent users of the language of algebra. On the one hand, this would help them to achieve one of the goals of educational systems, which is precisely the mastery of the language of mathematics, and on the other it would assist them to satisfy one of the most ancient social requirements of human beings, the capability of solving problems in a general, systematic way.

Indeed, if we admit the above-mentioned suppositions about the fundamental role that algebra plays in the school curriculum, we must also admit the need to recover the conception of school algebra as a language which is essentially different from that of arithmetic, and as a language whose symbolic level made it the first language in the history of mathematics that was capable of explaining itself, and one that has subsequently served as a basis for the symbolic development of mathematics as a whole, to the point of achieving the algorithmic and expressive autonomy that characterize it now.

1.2. The rest of the chapter

In what follows we shall make a series of observations about mathematical texts in which we shall make use of the notions about mathematical sign systems that we introduced in the Introduction (Chapter 1) and Chapter 2. This will enable us to characterize teaching models as successions of mathematical texts (all this is treated in greater depth and with greater generality in Chapter 8), which are exchanged between the learner and the teacher. Having done this, we take a look at concrete teaching models and their strengths and weaknesses.

2. MATHEMATICAL TEXTS AND TEACHING MODELS

2.1. A teaching model is a sequence of mathematical texts

Because we do not conceive mathematical texts as manifestations of mathematical language, and also because, in order to be able to give an account of those that are present in the processes of teaching and learning, we cannot identify them with written texts, it is pertinent to use a notion of text that conceives it as "the result of a reading/transformational labor made over the textual space" (Talens and Company, 1984, p. 32). Indeed, this idea was introduced in order to provide a notion of text that could be used in the analysis of any practice of production of sense (for example, the work of a learner with a teaching model, although this example may be rather far removed from the concerns of Talens and Company in their article), and for this purpose it is useful to introduce a distinction between "textual space"

(TS) and "text" (T), which corresponds to a distinction between "meaning" and "sense." A text, therefore, is the result of a reading/transformational labor made with a textual space, the aim of which is not to extract or unravel a meaning inherent in the textual space, but to produce sense. The textual space has an empirical existence; it is a system that imposes a semantic restriction on the person who reads it; the text is a new articulation of that space, individual and unrepeatable, made by a person as a result of an act of reading.

Moreover, the distinction between TS and T is a distinction between positions in a process, because any T resulting from a reading of a TS is immediately in the position of a TS for a new reading —and so on ad infinitum.

Both the work of mathematicians and that of students in mathematics classes can be described from the aspect of this repeated process of reading/transformation of textual spaces into texts. In particular, from this viewpoint a teaching model is a sequence of texts that are taken as a TS to be read/transformed into other TSs as the learners create sense in their readings.

2.2. Mathematical texts are produced by means of stratified mathematical sign systems and with heterogeneous matters of expression

In saying this we wish, first, to go against the idea of the existence of a text written in a totally formalized language that, although never actualized, is on the horizon as the text alluded to by the text that is really produced, by operations that are conceived as "abuses of language."[1] But we also want to contrast it with Rotman's idea that there is a rigorous text always present as the text belonging to a Code[2] that establishes the rules of the rigorous mathematical text, but that is enveloped in an informal text organized by the metaCode, although Rotman states that, contrary to the previous case, the text of the metaCode is unavoidable because it is the only way of guaranteeing the *persuasion* that, according to him, is an intrinsic need pertaining to any mathematical text.

Moreover, for Rotman, the fact that one cannot do without the metaCode "opens up mathematics to the sort of critical activity familiar in the humanities." However, according to Rotman "it by no means follows from this that mathematics' ways of making sense, communicating, signifying and allowing interpretations to be multiplied can be assimilated to those of conventionally written texts in the humanities," because in mathematical texts there are signs that are not those of natural language. So, having avoided the danger of reduction of the mathematical text to the ideal text, it seems that for Rotman it is a question of avoiding the symmetrical danger of reduction to text written in the vernacular, since he asks himself "[w]hat, in short, is unsayable (in fact, unthinkable, unwritable) except via mathematical symbols." We, however, do not find it so special to analyze a text in which not only natural language appears, since semiotics has set about the analysis of films, music or dance, for example, the expression of which is heterogeneous in that it combines matters of various origins; and we find that it is more suitable to study what type of combination of heterogeneous matter of expression is characteristic of mathematical texts than to undertake a search for something that can be expressed only by means of an expressive matter that is specific to mathematics.

However, abandoning the idea of a formalized or rigorous text as the background that in one way or another governs the analysis of mathematical texts does not make us deny the role effectively played in practice by the illusion of the formalized text, because this illusion has formed part of the idea that mathematicians have had of the rules of their practice. The way of combining matters of expression from different languages and the way of forming relations between the strata of mathematical sign systems is determined by this non-discursive component of the practice of mathematics, among other things, as are the texts produced in a given historical period among all those that might have been produced.

2.3. The heterogeneity of the matter of expression is revealed in the presence in the texts of segments of natural language, algebraic language, geometric figures and other diagrams, etc.

Although these segments come from languages with which it is possible to produce texts according to systems of rules that belong to each of them, they are not governed separately in mathematical texts by the rules of each of those languages. What really happens is that the rules of some languages contaminate those of others, so that mathematical sign systems are governed by new rules, created from those of the various languages that they incorporate.

We shall show this contamination between languages with an example in which the rules of natural language have been modified by copying them from the rules of arithmetic language. An expression such as "siete menos cuatro" [seven minus four], for example, is constructed by importing the form of the arithmetic expression 7 – 4 into Spanish. The way of expressing the task of subtracting one number from another in Spanish is what we have just used in this sentence: "sustraer, quitar o restar cuatro de siete" [subtract, remove or take four from seven], a phrase in which the operation appears first —and not between the numbers— and the numbers appear in reverse order.

The strangeness in Spanish of expressions such as this, which we may not notice now, is evident when we examine school texts from the 19th century and see that, in any of them, these expressions are introduced as something whose meaning has to be explained by resorting to the expression in the vernacular "restar tanto de tanto" [take so much from so much]. Thus, in the school text most frequently used in Spain in the 19th century, Vallejo writes: "la espresión 5 – 3 = 2, quiere decir que después de quitar 3 unidades del 5 quedan 2, y se lee cinco menos tres igual ó es igual á dos" [the expression 5 – 3 = 2 means that, after taking 3 units from 5, 2 remain, and it is read as five minus three equals or is equal to two] (Vallejo, 1841, p. 26). Freudenthal (1983) points to this phenomenon in other languages, such as German and Dutch. Thus, in German, until the early 20th century subtractions were formulated with the expression "vier von sieben" [four from seven], until textbooks began to introduce "sieben minus vier" [seven minus four] for élite schools and "sieben weniger vier" [seven less four], for ordinary schools — expressions that were foreign to German in both cases.

2.4. Inscribed in mathematical texts there are deictics that refer to elements of segments of different natures

Thus, for example, in a text in which the expression "point *A*, point *B*, segment *AB*" is accompanied by the corresponding geometric figure, whether drawn physically or imagined, the letters *A* and *B* link together words, figures, and expressions formed exclusively by these letters, and manipulation of the letters or the figures in the expression itself makes up for the lack of manipulation of natural language.

2.5. Through these deictics, indications of translations between elements that refer to each other are inscribed in the text, which are marks, borne by the text itself, of the semantic field that the reader has to use to produce sense

Unless one admits a drift toward aberrant readings, these indications are necessary because any reading of a mathematical text constitutes a learning process, in a non-trivial sense, for the empirical reader. Thus, in a school text in which Pythagoras' Theorem is stated, the many references between the expression "the hypotenuse *c*," the letter *c* written next to one of the sides of a triangle drawn on the page, and the algebraic expression $a^2 + b^2 = c^2$ enable one to understand that the text stipulates that the drawn figure which *looks like* a right-angled triangle effectively represents that geometric object, and that $a^2 + b^2 = c^2$ states Pythagoras' Theorem.

2.6. The objects with which mathematics deals are created in a movement of phenomena/means of organization by the mathematical sign systems that describe them

Since this movement of promotion from phenomena to means of organization does not always develop on the same level, that is, what is taken as phenomena asking to be organized by new means is not in an immutable world whose collection of phenomena is the subject of study of mathematics, mathematics generates its own content (see Section 4.1 in Chapter 2). An important aspect of this movement can be called "abstraction." The stratification of the mathematical sign systems with which mathematical texts are produced has to do with these processes of abstraction.

2.7. The fact that mathematical sign systems are the product of a process of progressive abstraction, whether in the history of mathematics or in the personal history of an empirical subject, has the effect that the ones that are really used are made up of strata that come from different points in the process, interrelated by the correspondences that it has established

In the Introduction (Chapter 1) and in Chapter 4 we have dealt with various phenomena that show this use of different strata of an MSS.

2.8. The reading/transformation of a text/textual space can therefore be performed using different strata of the mathematical sign system, making use of concepts, actions, or properties of concepts or actions that are described in one of those strata

The texts produced by readings that use different strata or a different combination of strata can be translated into one another and recognized as "equivalent" on condition that the pertinent correspondences between the elements used are also described in the mathematical sign system. When this is not the case, only the creation of a new MSS will make it possible. The process of creating new MSSs for this purpose is actually a process of abstraction, and the new MSS is more abstract than the preceding ones.

To express this with more precision, if it happens that two textual spaces ET and ET' cannot be read/transformed by means of a stratified mathematical sign system L by using the same concepts, actions, or properties of concepts or actions as are described in one of the strata, whereas in another mathematical sign system M it can be done, then M is "more abstract" than L with respect to ET and ET'. This is what happens in the book *De Numeris Datis*, for example, with two propositions that Jordanus Nemorarius transforms by means of different procedures, but that could be transformed in the same way using the sign system of modern elementary algebra. The MSS of this 13th-century text is less abstract than the MSS of modern elementary algebra, and in the history of mathematics the creation of the latter MSS was a process of abstraction that resulted, among other things, in the fact that texts such as those that could not be seen as equivalent for Jordanus Nemorarius are now equivalent (see Chapter 3 and Puig, 1994).

The creation of more abstract MSSs that takes place in the history of mathematics in this way has its correspondence in school systems. Indeed, during a teaching and learning process a student is sometimes incapable of transforming a textual space ET' by means of a stratified mathematical sign system L, using the same concepts, actions, or properties of concepts or actions as those with which he transformed a textual space ET; the breaking down of this impossibility is precisely what is sought by the teaching model and what constitutes true learning, and it occurs when the student modifies the language stratum in which the means of transformation are described, creating a new mathematical sign system M, in which the textual spaces ET and ET' are identified as being transformable by the same means (see Chapter 4). The creation of this M is a "process of abstraction" that also entails the creation of "more abstract" concepts or actions (the ones described in the modified language stratum).

2.9. In these modifications of language strata that lead to identifying concepts or actions, an important part is played by the autonomization of the trans-formations of the expression with respect to the content

The importance of this autonomization resides in the fact that these transformations can then be made in accordance with the rules without having to verify the result of the transformations of the expression with respect to the content in each of the steps, but only occasionally or once the complete set of transformations has been concluded.

Umberto Eco points out that in algebraic expressions, as in all the signs that he calls “diagrams” and which for us, following Peirce, are icons, “there are one-to-one correspondences between expression and content,” so that “the operations that I perform on the expression modify the content. If these operations are performed following certain rules, the result provides me with new information about the content” (Eco, 1984, p. 16). Geometric figures are also diagrams in this sense, whether they are drawn to represent geometric objects —as in Euclid’s *Elements*— or to represent algebraic quantities. In the didactic device that we describe at the end of this chapter, geometric figures are used precisely for this purpose. Al-Khwârizmî had already done so in his *Concise Book of the Calculation of Al-jabr and Al-muqâbala*,[3] in which he used geometric figures to prove the correctness of the algorithmic rules that he gave to solve the six canonic forms of equations that we now call first- and second-degree equations, in what he called “proofs by means of figures” —and not “geometric proofs,” since he did not make use of the propositions of Euclid’s *Elements*.

However, we would not say, with Eco, that what the result of the transformations of the expression provides can always be described as “new information about the content.” Sometimes, producing sense for the result of a transformation in the expression involves expanding the semantic field of the objects or actions involved, as is shown by the simple example of the identification of a^0 with 1, by virtue of the fact that certain rules produce $a^n/a^n = a^{n-n} = a^0$ and others produce $a^n/a^n = 1$, so that the expression a^0, literally meaning “*a* multiplied by itself zero times,” which does not mean anything, is given sense by expanding the semantic field of “multiply” and “times.” The autonomization of the expression thus brings with it a power to generate content.

Since the inscription of the first written arithmetic signs, which, as we indicated in Chapter 2, lacked operational capability, during the course of history mathematicians have gradually developed sign systems the expression of which has had increasingly greater power to generate content. Hence, as we see it, examining mathematics as a sign system and showing the crucial role played by the autonomization of the expression does not have to lead to Russell’s famous conclusion that “the propositions of logic and mathematics are purely linguistic, and they are concerned with syntax” (Russell, 1973, p. 306).

2.10. The development of new competences in mathematics can be seen as the result of working with an MSS that one has already mastered to some extent

This happens both in the history of mathematics and in the history of individuals. In the school system, this work consists in an exchange of messages between teacher and student that is produced by means of the reading/transformation of the sequence of texts that we call a teaching model. As a result of this reading/transformation, new concepts are produced through the production of new senses and the establishment of new meanings for the MSS (or MSSs) in which what is taught is described and produced, which even entail the creation of new MSSs.

In his *Remarks on the Foundations of Mathematics* (Part III, 31), Wittgenstein wrote that "the proof changes the grammar of our language, changes our concepts. It makes new connections, and it creates the concept of these connexions. (It does not establish that they are there; they do not exist until it makes them.) [der Beweis ändert die Grammatik unserer Sprache, ändert unsere Begriffe. Er macht neue Zusammenhänge, und er schafft den Begriff dieser Zusammenhänge. (Er stellt nicht fest, daß sie da sind, sondern sie sind nicht da, ehe er sie nicht macht.)]" This observation by Wittgenstein about the effect of proof in the grammar of our language and in our concepts can be paraphrased by transferring it to what we have just expounded and simply replacing "proof" by "work with an MSS" and "our language" by "an MSS that we have mastered." Thereby one is being at the same time more general and less precise. One is more general because proving is obviously a kind of work with an MSS and it is not only this kind of work that changes mathematical concepts (see Section 4.7 in Chapter 2). One is less precise because we do not specify what kind of work with an MSS changes concepts and MSSs and we are not claiming that it is any kind.

However, Wittgenstein's remark is about the work of mathematicians and not about the work of students in the school system. As our viewpoint and our area of interest is the school system, we will have to use a version of Wittgenstein's remark adapted to the fact that we are only dealing with the processes of teaching and learning mathematics in school systems where mathematical concepts are not created for the first time but have to be recreated— or "reinvented," to use Freudenthal's expression —by students using the guide of the teaching process. In this sense, the aim of the teaching model, of the sequence of texts that are read and transformed, must be that the new senses produced by the students should be felicitous, that is, that they should be in agreement with the socially established meanings, and that the new, "more abstract" MSSs created should become non-idiosyncratic MSSs.

2.11. A teaching model is a sequence of problem situations. This is the sense of teaching through problem solving

As the teaching model is a sequence of texts, produced both by the teacher and by the student, and those texts are the result of the work of both in teaching situations that are in fact problem situations (which are taken as textual spaces), it is pertinent also to add what we have learned from our studies and inquiries about problem solving. In particular, we have evidence that, when a problem is solved, one inevitably makes an initial logical conscious or unconscious analysis (which in Chapter 9 we will call a "logico-semiotic outline"), however quick and fleeting it may be, which seeks to rough out the solution, that is, to indicate the path that must be followed in the solution of the problem in accordance with some mathematical text produced with the use of a certain MSS.

A competent user who makes such a logic-semiotic outline uses cognitive mechanisms that enable him to anticipate the key relations of the problem and, from various MSSs or strata of MSSs, decide which one, more abstract or more concrete, he is going to use to outline the steps of the solution. Only then does he develop a process of analysis and synthesis that enables him to decode the problem situation.

Along these lines, the age-old idea in reforming declarations of basing teaching on problem solving can begin to make sense for us. A teaching model is also a sequence of problem situations, a sequence of mathematical texts T_n, the production and decoding of which by the learner finally enables him to interpret all the texts T_n in a more abstract MSS. This "changes the grammar of our language," because the new, more abstract MSS is of such a nature that its code makes it possible to decode the texts T_n as messages with a socially established mathematical code, precisely the code proposed by the educational aims that fixed the model of competence that the teaching model pursues.

Sense is produced in the new MSS by the use of new signs in each step of the analysis and solution in the way in which they have to be used —as Wittgenstein says: "I go through the proof and say: 'Yes, this is how it *has* to be; I must fix the use of my language in this way' [Ich gehe den Beweis durch and sage: 'Ja, so muß es sein; ich muß den Gebrauch meiner Sprache so festlegen'.]" (Wittgenstein, 1956, III, 30). This is possible when the MSS as a whole is bound by the concatenation of the actions set in motion during the problem-solving processes in all the problem situations that were previously seen as different and irreducible, but that now, thanks to the new MSS, are solved by means of processes that are established as being the same, i.e., that are transferred from one problem to another, at the same time converting what was a diversity of problems into a family of problems.

Teaching organizes the transition from an MSS that has to some extent been mastered by the learner, through its use in problem situations and the chain of readings/transformations ET/T = ET'/T' = ET"/T" …, to a new MSS from which the previous one is seen as being more concrete, and with which what was previously described as separate and unconnected is now described as being the same, and as a result is produced as new concepts and new signs.

3. CONCRETE MODELING

In discussions about the kind of teaching resources that should be used in the curricular development of any teaching model, two conflicting positions usually appear. In the case of the solution of equations, one of the positions proposes modeling the new operations and new objects in (more) concrete contexts (with "concrete" understood as contexts that are familiar for the learner), with the aim of endowing them with meanings and constructing the first elements of manipulative operations, taking this context as a starting point. A contrasting position proposes starting from the syntactic level and teaching the rules of syntax so that they can later be applied in the solving of equations and problems. This is the traditional treatment in the teaching of the solving of equations, based on the syntactic models of Viète —transposition of terms from one side of the equation to the other— and Euler —addition and multiplication of the additive and multiplicative inverses, respectively, in the two sides of the equation.

If one adopts the first of the two positions just indicated in order to develop teaching strategies at the beginnings of the acquisition of the competences of a MSS, it is necessary to possess knowledge about the processes that intervene between the actions performed on a more concrete level —i.e., the actions in the model— and the corresponding elements of syntax that may be obtained from them. These processes, which we will here call processes "of abstraction of operations," and that are processes of recovery, on a syntactic level, of the elements common to the actions performed in the repeated use of a model or a concrete teaching situation, present regular characteristics in the course of their development by individuals; but they also move along paths that may differ greatly from one individual to another, owing to the presence in individuals of tendencies with respect to their use and learning of mathematics (we have looked at this area in more detail at the end of Chapter 4).

Moreover, although there is a set of regular characteristics or characteristics that are repeated from individual to another in these processes of abstraction of operations, some of them may vary with variations in the concrete situation from which one sets out in order to obtain or construct the corresponding syntactic elements —or in the model from which one sets out.

3.1. Algebraic semantics versus syntax

In the study expounded in other chapters, in which a teaching model concerning the solution of first-degree equations was developed, the interrelations between two overall strategies for the design of learning sequences that occupy long periods of time in the middle school algebra curriculum were basic. These strategies were:

a) Modeling of more abstract situations in more concrete languages in order to develop syntactic abilities.
b) Producing codes to develop problem-solving abilities and using syntactic abilities to develop solving strategies.

Broadly speaking, in (a) the objective was to give meanings to new expressions and operations, modeling them on more concrete situations and operations. In (b) the objective was to give senses to the new expressions and operations so that problem-solving codes would be generated, setting out from the supposition of the presence of certain abilities of syntactic use of the new signs and their utilization as a more abstract language. In the Introduction (Chapter 1) we show the problems that learners present when they have just finished primary education.

In what follows we will see that the development of syntax and semantics produces a dialectic relation in which an advance in one of these two aspects is necessary for an advance in the other, although sometimes the development of one may inhibit development of the other.

3.2. Components of concrete modeling

If one thinks about the introduction of certain mathematical notions by means of models (as is done in Chapter 4 for the solution of algebraic equations), it is advisable to bear in mind some of the main components of modeling, especially two components that are fundamental. The first component is translation, by means of which sense and meaning are given in a more concrete context to the new objects and operations that are introduced, which, from a more abstract viewpoint, are the same as those that appear in more abstract situations. In other words, through translation these objects and operations are related to elements of a more abstract situation and, on the basis of what is known about the solution of such situations on the more concrete level, operations are introduced which, although carried out on the concrete level, are also intended to be performed on the corresponding objects on the more abstract level. Consequently, there is a need for a two-way translation between one context and the other, so that it may be possible to identify each operation on the more abstract level with the corresponding operation in the concrete model.

The second component is the separation of the new objects and operations from the more concrete meanings with which they were introduced. In other words, in the modeling one also seeks to relinquish the semantics of the concrete model, because what one wishes to achieve ultimately is not the solution of a situation that one already knows can be solved, but the discovery of ways of solving more abstract situations by means of more abstract operations. This second component is a driving principle that directs the modeling function toward the construction of a syntax external to the model.

3.3. Concrete modeling versus mechanization and practice

In his book *The Psychology of Algebra*, published in the early 1920s, Thorndike proposed the integration of everything that seemed pertinent at the time so that the teaching of algebra could advance. That aspiration can still be seen now as a program yet to be fulfilled for any other theoretical and experimental approach —leaving aside, perhaps, certain emphases and pre-occupations belonging only to the theoretical perspective, in accordance with the psychological knowledge of the time. Among matters that are still of great relevance today we find the central motivation:

> Algebraic computation as actually found is then emphatically an intellectual ability. It is not so indicative of intellect as problem solving, partly because it involves less abstraction, selection, and original thinking, partly because it involves only numbers, not numbers and words. It is, however, far above the reproach of being a mechanical routine which can be learned and operated without thought. (Thorndike, 1923, p. 451.)

During the 84 years that have elapsed since then the emphasis placed by researchers has varied greatly, leading, in the middle of the last century, to the granting of total pre-eminence, not to what is called "problem solving" in the remark just quoted, but to the structural components of the matter studied. The result of this was that, in French middle school syllabuses, it was possible to find a so-called algebra in which what had been the traditional teaching situations until then, based on considering algebra as a continuation of arithmetic, did not appear anywhere. As a reaction to this, there was a swing toward the use of teaching models based on situations similar to those proposed by Thorndike, but more concrete, mechanizing the handling of algebraic expressions, with an expeditious use of the rules of syntax.

3.4. Syntactic models

The idea of a concrete teaching model can be extended to the strategies proposed in the 1920s, which we here call "syntactic models," in contrast to concrete models, which we call "semantic," because in them emphasis is placed on working with a considerable semantic load in all the signs and operations involved. In the syntactic model, conversely, the emphasis is placed on the general rule used to construct the habits that set the operations in motion.

With respect to these models, empirical evidence indicates that, apart from the generation of private semantics of the individual that give meaning to the terms proposed by the general rule and to the operations involved, phenomena of reading of the situations proposed appear, guided by the senses given to the rules that must be set in motion in order to carry out the syntactic task. For example, when someone first comes across equations of the type $Ax - B = C$ (A, B, $C > 0$), he may always attribute the positive sign to B and the negative sign to A, guided by the sense that he has obtained from previous practices performed with the solution of equations of the type $Ax + B = C$. In other words, a syntactic context guides a mistaken but natural reading, due to the individual's anticipatory mechanisms —a cognitive tendency that we presented in the last chapter.

In this respect, the emphasis placed not only on mechanisation but also on the concern for practice, and the consequences that this has on the practice times that learning experiments propose, acquire a new sense in view of the need to correct spontaneous readings, here generated not by semantics but by syntax.

3.5. Modeling and teaching algebra

The results described in this book allow us to state that the correction of mistakes of algebraic syntax and the operational mishaps that appear amid complex processes of solving problems or equations generated during the learning of algebra cannot be left to the spontaneity with which students make use of the first elements with which they are provided in order to penetrate into the realm of algebra. The paths traced by those spontaneous developments are not directed toward what the teaching of algebra seeks to achieve: that is precisely why this correction is a task of teaching. So that, if one thinks of introducing certain notions of algebra by means of models (including the syntactic model), it is advisable to bear in mind the main components of modeling, as described above.

The studies described in this book show that mastery of the first of the components of modeling (translation) may weaken or inhibit the development of the second: such is the case with learners such as Vt, mentioned in Chapter 4, who achieve a good command of the concrete model, but as a result also develop a tendency to remain and progress within that context, and this anchoring to the model goes against the other component, that of the abstraction of operations toward a syntactic level, which involves breaking away from the semantics of the concrete model.

This indication about the interaction between the two basic components of modeling does not depend on the tendency of the individual, for even in cases of a syntactic tendency, such as that of Mt, mentioned in Chapter 4, during the processes of abbreviation of actions and production of intermediate notations (between the concrete situation and the level of algebraic syntax) obstructions to the processes of abstraction of the operations effected in the concrete model are generated as a result of not possessing, in that period of transition, suitable ways of representing the results or states to which the operations lead. Once again, this is a deficiency in the second of the components of the action of modeling.

The obstructions indicated earlier constitute a kind of essential insufficiency, in the sense that, if modeling is left to spontaneous development by the learner, one of its components is strengthened, and this tends to hide precisely what one is essentially trying to teach, which is *new concepts and operations* (a more detailed description can be found in Chapter 4).

This kind of dialectic between the processes that correspond to the two components of modeling must be taken into account by teaching, which should try to develop the two kinds of process harmoniously, so that neither obstructs the other. Indeed, from the analysis of the cases presented in Chapter 4 it is clear that this is a task of teaching, given that this second aspect of modeling, that of breaking away from the earlier notions and operations on which the introduction of the new knowledge is based, is a process that consists in the negation of parts of the semantics of the model, and these partial negations take place during the transfer of the use of the model from one problem situation to another —in the case of the geometric model it is a transfer of its application from one variety of equation to another. However, when this generalization in the use of the model is at the expense of spontaneous development by the learner, the partial negations may take place in essential parts of it —in the geometric model, the presence of the unknown and operation on it are negated. Consequently, intervention with teaching becomes necessary in the development of these processes of relinquishing and negation of the model, in order to channel them toward the construction of the new notions.

The transfer of the problematic of algebraic semantics versus syntax to a level of actions of modeling makes it possible to narrow the distance between teaching and this problematic, since analysis of the interaction on this other level reveals didactic phenomena that show the need for the intervention of teaching at key points in the processes set in motion at the beginnings of the acquisition of the language of algebra.

SUMMARY

In this chapter we use the notions of textual space and stratified mathematical sign systems (from “less abstract” to “more abstract”) to describe teaching models in terms of sequences of mathematical texts (produced by the teacher or pupils) and in terms of sequences of problem situations. These theoretical notions generalize the examples of teaching models used to teach the syntax required for solving first-degree equations with the unknown appearing on both sides of the equality, presented in Chapter 4. In this chapter we speak of concrete models (the balance scales and a geometric model) and of “abstract” or syntactic models (the model of “doing the same on both sides” and the model of transposing terms). In the study “Operating on the Unknown” these models were used to observe the processes of transferring actions performed in simple cases to cases of equations with more complex characteristics, and also the processes of abstraction of actions performed in all the cases of equations presented to the pupils. In the next chapter we analyze the first steps toward the use of algebraic syntax in problem solving.

ENDNOTES

[1] The illusion of a text written in a formalized mathematical language, which is never present but to which the text that is really written refers, could not be better expressed than it is in the Introduction to Book I of Nicolas Bourbaki's *Éléments de Mathématique* (Bourbaki, 1966): "Nous abandonnerons donc très tôt la Mathématique formalisée […] Les facilités qu'apportent les premiers "abus de langage" ainsi introduits nous permettront d'écrire le reste de ce Traité […] comme le sont en pratique tous les textes mathématiques, c'est-à-dire en partie en langage courant et en partie au moyen de formules constituant des formalisations partielles, particulières et incomplètes, et dont celles du calcul algébrique fournissent l'exemple le plus connu. Souvent même on se servira du langage courant d'une manière bien plus libre encore, par des abus de langage volontaires, par l'omission pur et simple des passages qu'on présume pouvoir être restitués aisément par un lecteur tant soit peu exercé, par des indications intraduisibles en langage formalisé […] Ainsi, rédigé suivant la méthode axiomatique, et conservant toujours présente, comme une sorte d'horizon, la possibilité d'une formalisation totale, notre Traité vise à une rigueur parfaite […]" (pp. 6-7). The expression "abuse of language," which describes the fundamental operation that makes it possible to abandon the writing of the formalized text and refer to it, appears repeatedly throughout the treatise.

[2] Rotman presented a first version of his semiotic model of mathematical activity in Rotman (1988). A more recent version, modified and more extensive, is in Chapter 3 of Rotman (1993), which begins by announcing that "What I propose here is a semiotic model of mathematical activity fabricated around the idea of a thought experiment. The model identifies mathematical reasoning in its entirety —proofs, justifications, validation, demonstrations, verifications – with the carrying out of chains of imagined actions that detail the step-by-step realization of a certain kind of symbolically instituted, mentally experienced narrative" (Rotman, 1993, p. 66). His distinction between Code and metaCode seeks to account for the fact that "contemporary mathematicians divide their activity […] into two modes: the formal and the informal" (p. 69). Code is, therefore, "the unified system of all such rules, conventions, protocols, and associated linguistic devices which sanction what is to be understood as a correct or acceptable use of signs by the mathematical community," metaCode is a "heterogeneous and divergent collection of semiotic and discursive means" which give an account of "the mass of signifying and communicational activities that in practice accompany the first mode of presenting mathematics" (p. 69). In his model there is also a third element, which Rotman calls the "subCode" or "virtual Code," and three characters: the Subject, who uses the signs of the Code; the Person, who uses those of the metaCode; and the Agent, who uses those of the virtual Code. Rotman (1988) is also included in Rotman (2000) as its first chapter.

[3] As was usual in the 9th century in the Arab world, this book by Muhammad ibn Mûsa al-Khwârizmî did not have a title. Two manuscripts of it have been conserved, one of which was edited and translated into English by Frederic Rosen, with the title *The Algebra of Mohammed ben Musa* (Rosen, 1831). According to Høyrup (1991), both this manuscript and Rosen's translation are less close to the original text than the Latin translation produced by Gerardo de Cremona in the 12th century in the school of translators in Toledo. There is a recent edition of this manuscript (Hughes, 1986). Gerardo de Cremona heads his translation with the words "Liber Maumeti filli Moysi alchoarismi de algebra et almuchabala incipit" ["here begins the book of algebra and almuchabala by Mahomet the son of Moses alchoarismi"], leaving the Arabic words *al-jabr* and *al-muqâbala* untranslated, as we have just done. It is precisely because *al-jabr* remained untranslated that this part of mathematics, which in a sense al-Khwârizmî founded, was eventually called algebra. See an analysis of one of al-Khwârizmî's proofs by means of figures in Puig (1998).