CHAPTER 3

EXPERIMENTAL DESIGN

OVERVIEW

We begin analyzing the diagram of an experimental setting design for the observation of algebra learning and teaching phenomena. In this chapter we discuss a manner of studying the evolution and development of algebraic ideas through historical and epistemological analysis (based on the analysis of ancient pre-symbolic algebra texts), which in turn serves as a point of departure for experimental design in mathematical education for the particular case of the transition toward algebraic thought. The phenomenological analysis, as presented in general terms in Chapter 2, is applied to the case of algebraic language and to that of school algebra (didactic phenomenology). Here once again the notions of *mathematical sign system* and of *language strata* become relevant, especially when the historical analysis touches upon the genesis of modern algebra thus re-broaching the elements that correspond to said notions presented in Chapter 2. The chapter consists of the following sections: 1, Introduction; 2, Experimental observation; 3, On the role of historical analysis; and 4, The phenomenological analysis of school algebra.

1. INTRODUCTION

In this chapter we present two diagrams that give a general description of the design of a study in accordance with the guidelines of our research program (diagram A), and the general form of the development of the study (diagram B). In the rest of the chapter, we specify some of the terms used in those diagrams and set out in more detail how the historical analysis of algebraic ideas and phenomenological analysis intervene in it.

2. EXPERIMENTAL OBSERVATION

2.1. The design and development of the experiment

Both the design and the development of the experiment are presented in the form of a flow diagram (see Figures 3.1. and 3.2). We merely wish to emphasize that we have introduced our theoretical elements —local theoretical models (LTMs) and mathematical sign systems (MSSs)— as the theoretical counterpart with which the experimental observations are designed and interpreted. For this is a theory produced to provide support for observation, and that is how it should be interpreted. These ways of designing and developing experimentation are exemplified throughout the book, and they are in use in several research works (see Chapters 4, 6, 7, 8, and 9).

2.2. Recursiveness in the use of LTMs and the ephemeral quality of certain theses

Note that in diagram A there is a recurrence: the diagram begins with a box that represents the area under investigation, and at the end of the entire process there is a return to the beginning. In the case of diagram B the starting point is a local theoretical model, designed in the stages of diagram A, and after the performance of an experimental study, in which the theses of this first LTM are confronted with what occurs in the empirical development of the experiment, one finally comes to a phase of analysis and interpretation. On the basis of the results of this phase, the initial problem area is framed within the perspective of a new LTM, the design of which returns to the first stages of diagram A, so as to be able once again to start the process described in diagram B.

In this recursiveness, it may well happen that the theoretical theses framed in the first LTM prove to be insufficient to study and interpret the empirical observations made in the stage of empirical development (see, for example, Chapter 9), or else some of the theses as elaborated might have to be discarded or differentiated into others that provide a better fit for the interpretation of what has been observed. In this respect one could speak of the ephemeral quality of certain theses that do not stand up to verification with the empirical facts observed.

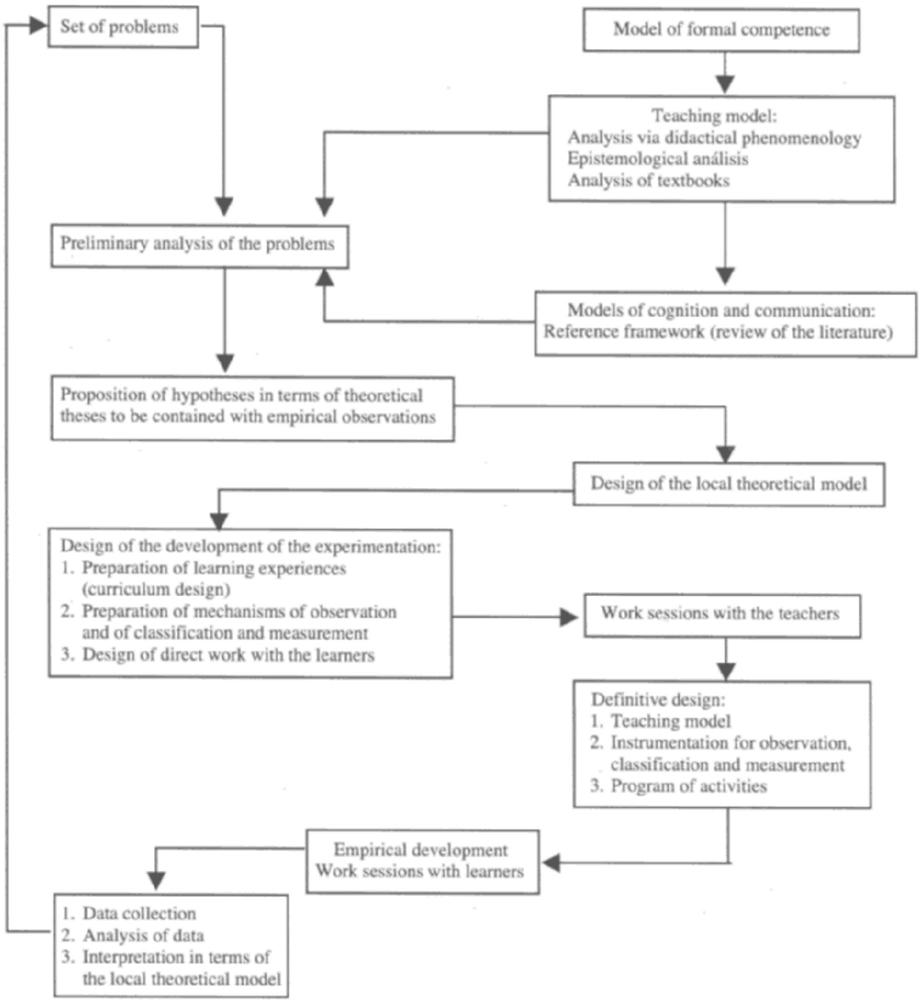

Figure 3.1. Diagram A of the design of the study

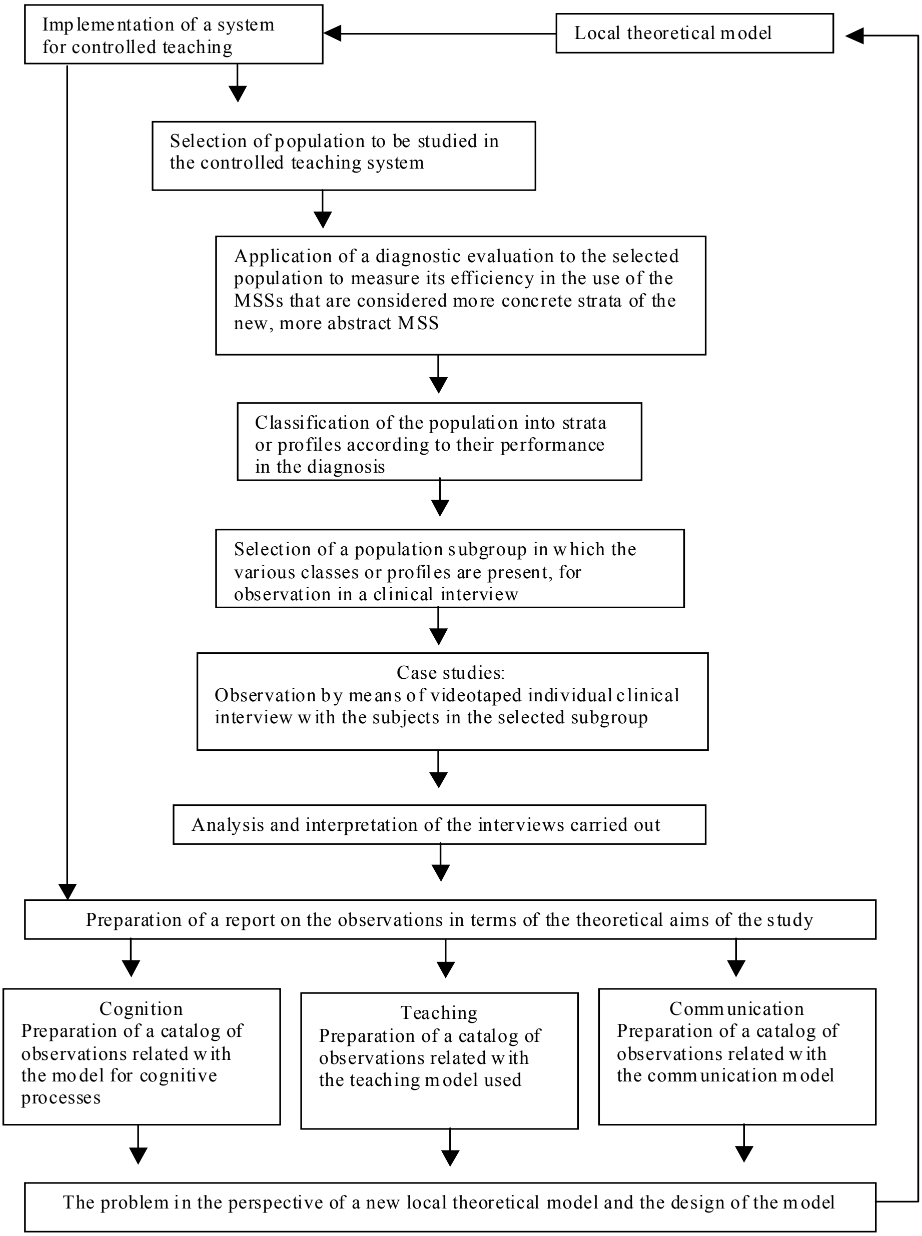

Figure 3.2. Diagram B of the development of the study

2.3. On the didactic cut

We mention first that it is advisable to choose the moment of the experimental observation at some point in the mathematics curriculum at which what has been learned (of the material taught up to that point) does not make it possible for the next topic that is to be taught to be discovered spontaneously without the intervention of the teaching that is to come. The ideal situation is to find a conceptual area in which, when the competences of the population with regard to the uses of those concepts are diagnosed, one sees that those competences lead to performances very far from what is expected (the aims of the education). For example, in the case of Thales' Theorem in Chapter 7, through what had been learned previously *the entire population* had developed tendencies that caused *all the learners* to have mistaken responses concerning ratio and proportion when faced with the most elementary questions that can be asked in this field, questions that are the basis of the whole future development of trigonometry.

Another example is the solving of equations and the transition from arithmetic to algebra, which is discussed in Chapter 4. On the other hand, in the ongoing studies that are mentioned at the end of Chapter 8, there are already indications that suggest that what is being studied in them would also constitute a didactic cut.[1]

2.4. On controlled teaching

Second, it is advisable that the population being studied should comprise several cohorts of the same age, belonging to the same grade level, at the same school, and that they should receive instruction in mathematics within a system of controlled teaching. This means that the population being studied receives instruction in mathematics with materials that allow them to do individual work in class, at their own pace, that there is monitoring of advances made by individuals and groups of students, that there is the possibility of intervening with supplementary teaching material where it is required.

2.5. On diagnosis

The groups that receive the controlled teaching constitute the totality of the population being studied. During the period in which the controlled teaching is developed, mechanisms of measurement and classification are prepared and refined so as to make it possible eventually to construct a diagnostic test by which classes or profiles of individuals can be obtained. The diagnosis provides a detailed description of the performances of the students and has the further aim of delimiting the profiles so that one can see which students it would be interesting to observe in greater depth. For this purpose a case study is designed in which the clinical interview plays the main part with a view to setting up the observation environment.

In general, it is advisable to classify the population in relation to three axes. The first has to do with the syntactic competences of the individuals in the use of the more concrete MSSs. The second has to do with the competences concerning the use of the semantics of those MSSs, when applying it to the solution of problem situations. The third axis of competence seeks to group together the competences that have to do with the intuitive and spontaneous uses of the strata of the more concrete MSSs that will be used in the decoding of the new teaching situations which the teaching model that is being used will require.

We will see an illustration of this principle in Chapter 4, which contains a description of a study in which the population is classified by means of a written test on pre-algebra consisting of three subsections: arithmetic equations with literal notation (e.g., $5x + 3 = 90$), arithmetic equations without literal notation (e.g., $\square - 95 = 23$), and problems corresponding to arithmetic equations.

The classification of the population in relation to these three axes makes it possible subsequently to select pupils for the clinical interview who have different profiles with regard to one or more of the axes, and who therefore correspond to aspects of the MSSs brought into play in the teaching.

2.6. On the clinical interview

To be able to observe the phenomena studied with greater precision one needs an experimental situation that makes it possible to monitor certain disturbing factors that are always present in the classroom, and one needs observation mechanisms that allow a more exhaustive and precise analysis. However, this must be done in such a way that what is observed has to do with the problems presented by the individual being observed and also that the components that the teaching brings into play are present. That is the nature of clinical interviews with teachers.

The clinical interviews have a structured format, but the interviewer moves freely between the various steps that have been designed previously, allowing the line of thought of the interviewee to define each of the subparts of the interview. The first part of the interview is usually devoted to confirming that the interviewee has the profile given by the diagnosis.

Except in cases where the interviewee has no difficulty in solving the problem that is set, the interviewer intervenes to put further questions that, through a process of discovery, help the interviewee to learn the problem that he was initially unable to solve. It is a question of discovering the difficulties presented by the learning of beginnings of algebra, given the ways in which one seeks to teach it nowadays. In these clinical interviews the focuses of observation are the ways of teaching and the particular ways of learning (with their typical obstructions and difficulties) that are seen in the students.

2.7. On the preliminary analysis of the problems

If we look at diagram A of the design of the experiment, in step 2, “Preliminary analysis of the problems,” many general disciplines combine to make it possible to perform the analysis: psychology, historical analysis, epistemology, mathematics, sociology, education in mathematics, etc. Many research studies nowadays favor one or more of these focuses, or else, in the case of the design of the experiment, there tends to be a tension between the studies that favor a quantitative approach (via the use of statistics) and those that favor a more qualitative approach (via the use of clinical observation).

However, in favoring some general focus, such as the analysis of the history of mathematical ideas, it is possible that all the other items that are described in diagrams A and B may be left out. One might think, therefore, that such a study is a valid contribution only in the field of the general discipline with which it is concerned; nevertheless, experience shows that studies of this kind are ultimately of little interest in the general discipline, where there is a preference for certain working habits and focuses and for using all the antecedents so far established in that discipline. Moreover, they also generally prove to be of little importance in mathematics sducation, the main results of which are intended to be useful for students and teachers in the present educational systems. In the next section we show a way of using the analysis of mathematical ideas, which, in our view, makes it fruitful for research in mathematics education.

3. ON THE ROLE OF HISTORICAL ANALYSIS

It is clear that any analysis that seeks to clarify educational problems — analysis being the prime driving force in our research— must be performed in the specific framework of our educational systems; but also, as a unitary counterpart, such analysis cannot help seeking to transform the conditions in which the teaching of mathematics is taking place in our countries. This clearly conditions the problems and therefore the methodology of the study; but also, in an aspect usually overlooked, it imprints on the results the need to be used, to be put to the test in the very place where they supposedly seek to cast light, where their modifications will have to be taken into account in order to advance, to go deeper into the facts being investigated, to be able to construct new hypotheses that take the work done into account.

This makes it necessary for the problems, in at least some of their aspects, to be closely linked with the actual process of teaching. However, this does not mean abandoning somewhat theoretical problems and their appropriate logical methods; rather, the studies take place within larger programs in which direct contact with students and teachers is present.

In this section we show that the historic-critical analysis of the development of mathematical ideas makes it possible, for example, to construct learning sequences that reflect the achievements of theoretical research, and that it becomes fully meaningful when, in turn, in theoretical research the history of ideas is enriched by the new hypotheses formulated by putting teaching sequences to the test in educational systems. Then we will rightly be able to maintain that we are speaking of studies in the field of mathematics education and not in that of the history or epistemology of mathematics.

3.1. Epistemological analysis

At one time history was relegated to being a pastime of mathematicians, although with the production of dazzling works, such as Van der Waerden (1954), or general views seen through new eyes, such as Boyer (1968). Now, however, it has regained its proper stature and has even made its way into the textbooks (see Edwards, 1979).

Even earlier, however, Boyer (1959) had offered us more profound attempts to capture other more intense moments: those of the evolution of ideas. Many titles could be added here to illustrate this great return of history as an instrument with which to view the present. We will only indicate, by way of example, that our ideas about the nature of the rudimentary processes of constructing mathematical models have changed completely as a result of historical studies concerning the Babylonians (Neugebauer, 1969); that our conceptions about the origin of the theory of proportions, deduction, and axiomatization have begun to acquire subtle tonalities that we did not perceive before, thanks to Szabó (1977); and that Jens Høyrup, pursuing this evolution of algebraic ideas from Babylon to medieval Italian algebra in numerous studies, has made us see it in a different way (see, for example, Høyrup, 1985, 1986, 1987, 1991, 1999a, 2002a, 2002b).

This re-encounter between history and epistemology through the history of ideas has also begun to benefit the didactics of mathematics.

3.2. The reading of texts

The new approach consists of analyzing problems of teaching and learning mathematics with the historic-critical method, and then of putting the theoretical findings to the test in the educational systems so that, after this experimentation, one will once again have a new view of the problematics of the history of ideas that corresponds to the teaching results.

A first example, taken from Filloy (1980), will make this idea clearer. Analyses of Diophantus's *Arithmetic*, Bombelli's *Algebra* and the contrast between it and Viète's *The Analytic Art*[2] lead to interesting hypotheses about the development of the first notions of algebra in secondary school (with pupils aged 12 to 15), as one can gather from the works of Jacob Klein (1968), for example. From these results one can infer that the most significant change in symbolization, in that stage of the beginning of algebra, is the step from the mathematical concept of unknown to the mathematical concept of variable. A transition that involves not only the feat of solving complicated arithmetic problems, already achieved in Diophantus's *Arithmetic* in one sense more efficiently than by Viète, but also reflection on the operations that are always performed to solve such problems. This reflection on operations suggested to Viète the need to speak not only of unknowns but also of the fact that the coefficients of the equations that result from making the *zetetic* analysis of the problems are also variables; that is, such coefficients have to operate on each other, not just representing a number, unknown or not but ultimately only one number, but rather representing all the numbers that could come from equations resulting from the analysis of arithmetic problems.

These facts would seem to complete the picture, especially when the analysis is continued along these lines, as is done by Klein (1968) or Jones (1978). This change of perspective (in Viète) immediately generated others, owing to the problems posed by operating on measurements, as can be seen clearly in the work of Stevin. A change is generated, as we were saying, in the very concept of number, that is, new (ideal) objects become numbers: numbers that can be operated on in the same way; for example, decimals become numbers provided that they obtain the category of mathematical objects, the main argument in Stevin's works, *Arithmetic* and *Disme* (Stevin, 1634).

But when one constructs teaching sequences that try to follow this connecting theme, as in Alarcón and others (1981–1982), and one observes the behaviour of the population (in the statistical sense) in the distributions that indicate the evolution of mathematical abilities, one finds that there are other elements which have not been taken into account. It then becomes apparent that first of all one would have to seek out the history of operational aspects, of the syntax of arithmetic-algebraic language, in its development in the East, and also, secondly, study the evolution of another history, apparently unconnected but one that in practice is revealed to be totally related to that of operational aspects: the history of the analysis of variation and change; either, in the first instance, purely arithmetic methods (such as those of proportional variation), or, on a deeper level, those entailed by pictographic representation of the first and second variations of movement, of the change in the intensity of light, or of the propagation of heat.

At this point it would seem to be very important to go back to history and analyze the works of the Middle Ages in regard to this. Our debt to historians (see Grant 1969, 1971; Clagett 1959, 1968; Van Egmond, 1980; Hughes, 1981; and Høyrup, 1999b, for example) is inestimable in this context, for their compilations, translations, and commentaries provide us with living material which is waiting for us to go to them with new eyes: those of the problematics of the teaching of algebra, at the very point where algebra was to make it possible to introduce analytical ideas in geometry, and, immediately afterwards, the methods of infinite calculus. Similarly, in order to understand the jump between arithmetic and algebra (and the appearance of arithmetic-algebraic language) it is necessary to cast light on the period immediately before the publication of Bombelli's and Viète's books.

In Viète's *The Analytic Art* we find the construction of an algebraic language in which, in addition to being able to model the problem situations solved by the languages used by Bombelli and Diophantus, we can also find a language in which one can describe the syntheses and algebraic properties of the operations introduced in the older texts. What is new in Viète's language lies in the fact that, whereas in those earlier texts operations were used only by performing them or employing them problem by problem, in Viète there is the possibility of describing the syntheses (algebraic theorems) and the syntactic properties of operations, because they can all be described with that language stratum and they can also be added to the store of knowledge on which someone who has mastered that new language stratum can draw.

In the following sections we describe language strata prior to the introduction of the language of Viète's *The Analytic Art*. As examples we use certain differences between the *abbacus* books and Jordanus de Nemore's *De Numeris Datis*.

3.3. The abbacus *books*

As can be seen in the work of compilation by Van Egmond (1980), the *abbacus* books represent the most feasible path for the assimilation of the mathematics of the East by Western European civilization; and in this adaptation of Indo-Arabic mathematics to the problems characteristic of a society with a vigorously rising economy (the society of Italy in the 15th and 16th centuries) a new kind of mathematics was born.

This mathematics was present and ready to be applied in the so-called *abbacus* books, the content of which essentially comprised the presentation of the Indo-Arabic positional system of numeration, the four elementary arithmetic operations, and the solution of commercial problems. These problems involved the four elementary operations, and also the use of the simple and compound rule of three, simple and compound interest, and the solution of some simple algebraic equations. Some books also included multiplication tables, tables of monetary equivalents, and tables of weights and measures.

The first *abbacus* book of which we know was written in Latin in the Near East (Greece) and was introduced in Western Europe, in its first vernacular version, between the 12th and 13th centuries.

The meaning of the word *abbacus* in the name of these books was that of "the art of calculating, counting and arithmetic." The term was first used in this sense by Leonardo of Pisa, better known as Fibonacci, who in the 13th century wrote a compendium on the mathematical practice known up to that point. This happened naturally, because his father was a merchant from Pisa who visited and stayed in Arab countries in the East and in the Maghreb, particularly in the town now known as Bejaia (in what is now Algeria), and so Fibonacci was taught by Arabic teachers and learned the Arabic systems, both of commerce and of mathematics, with the result that his book contained knowledge of practical commercial mathematics, in accordance with the Indo-Arabic system, and with a particular influence exercised by his own experience in merchant life and by his instruction in a great variety of Arabic texts on algebra, geometry, and commercial mathematics.

The production of *abbacus* books increased greatly in Italy in the 15th century; it is estimated that there were then 400, with about 400 different problems solved in each one; so, with regard to problems, even if we eliminate the repetitions between books, the production was of the order of tens of thousands.

The first *abbacus* schools appeared in the West almost at the same time as the arrival of the first *abbacus* books. It is known that the first school was founded in 1284 in the commune of Verona, and that these schools were attended mainly by the sons of merchants and, in general, by men of affairs, in order to practice commercial mathematics and continue their basic education in grammar. The *abbacus* schools tended to proliferate in the 14th century; it is known that in Florence alone, in about 1343, there were six schools in which over a thousand students were taught. And, although this growth did not remain stable in subsequent centuries, there are references for about three or four schools in each important city (Florence, Milan, Pisa, Venice, Lucca), which functioned continuously from the 14th century and throughout the Renaissance.

The most plausible historical explanation (see Van Egmond, 1980) for the appearance and multiplication of *abbacus* books, schools, and teachers in the West is of a social and economic nature. With the so-called Commercial Revolution in Italy in the 13th century there was a substantial social change: monetary power began to count more than feudal power, with the result that there was a greater desire for control of trading and financial activities, together with the skills required for their performance, than for possession even of land. Consequently, the rise of this new social class that came to power imposed the need to create the means to make this new kind of inheritance effective: the skills required in order to be able to participate in commercial power. These skills naturally included the contents of the *abbacus* books, originally produced to serve as reference books for the accountants and merchants of the time, and the need to make them accessible to the merchants' sons led to the creation of *abbacus* schools and teachers, financed, initially at least, by the parents themselves.

The children who attended the *abbacus* schools were 10 or 11 years old, and they were trained in the basic principles of arithmetic and practical mathematics (writing Indo-Arabic digits, the four operations with integers and fractions, solving commercial problems, and handling monetary equivalents and weights and measures), and also in grammar. It might be considered that the *abbacus* schools functioned as a kind of basic secondary education, acting as a bridge between basic education (the classical Roman school) and university (the first universities having been founded in Europe in the 12th and 13th centuries).

Although the *abbacus* schools might be viewed as an integral part of school education at that time, in reality they constituted a genuine innovation in educational circles because, unlike the universities, which provided instruction for the elites and functioned primarily as places for discussion and reflection about knowledge, the *abbacus* schools acted as transmitters of knowledge applicable to daily life. In the 15th century commercial activity was not just transacted between merchants and men of affairs, but rather those activities began to form part of the everyday life of what had become an urban population.

Thus the *abbacus* schools and contents served to satisfy a social demand in the new Europe civilization, with such success that they became a tradition that endured for centuries as a companion to the new pattern of culture, the mentality created by the commercial revolution.

To appreciate the full extent of the social and educational role of the *abbacus* books one would only need to review some of the mathematical contents of current textbooks for basic education in any country in the world to realise that, essentially, they are the same as what could be extracted from a typical *abbacus* book (see Van Egmond, 1980). This gave them the character of assimilators of Eastern mathematics to the needs of the new Western culture (which now stretches back over more than five centuries) through school education.

3.4. An abbacus *problem*

In the section of recreational problems in the *Trattato di Fioretti* (Mazzinghi, 1967), we can find problems such as the following:

Fa’ di 19, 3 parti nella proportionalità chontinua che, multiplichato la prima chontro all’altre 2 e lla sechonda parte multiplichato all’altre 2 e lla terza parte multiplichate all’altre 2, e quelle 3 somme agunte insieme faccino 228. Adimandasi qualj sono le dette parti. [From 19 make 3 continually proportional parts such that, if the first is multiplied by the other 2 and the second part is multiplied by the other 2 and the third part is multiplied by the other 2 and those 3 are added, together they make 228. The question is what the aforementioned parts are.]

We can state this problem by translating it into the MSS of current algebra as follows:

Find three numbers x, y, z such that

$$\left.\begin{array}{l} x+y+z=19 \\ \dfrac{x}{y}=\dfrac{y}{z} \\ x(y+z)+y(x+z)+z(x+y)=228 \end{array}\right\}$$

In Puig and Rojano (2004) there is a transcription of the original version in old Italian of the solution of this problem, accompanied by a translation into the MSS of algebra. For our present purposes, the solution presented in the treatise consists in applying a series of rules, in particular a rule of doubles[3] and the Babylonian method of completing squares. In both cases, but more obviously in the rule of doubles, each time that the rule is used it is reworded for the specific numbers with which it is necessary to operate. We will see that this is one of the characteristics that make the MSSs of the *abbacus* books more concrete than Viète's MSSs, but also more concrete than that of Jordanus de Nemore's book, *De Numeris Datis*.

3.5. De Nemore and his work

The bibliographic information about Jordanus de Nemore is very diffuse, but the authenticity of his work has been established. He lived during the period that ranges from the middle of the 12th century to the middle of the 13th century, and on the basis of annotations in the margins of his writings it is believed that he taught at the University of Toulouse. Research on his life and work has led him to be considered, since the last century, one of the most prestigious natural philosophers of the 13th century. It is also known that he devoted himself to physics-mathematics, laying the foundations for the whole area of medieval statics. Among his mathematical works, those devoted to arithmetic (and algebra) continued to be reproduced until the 16th century.

If we consider only the treatises of a strictly mathematical character, we can identify six works attributed to Jordanus: *Demonstratio de algorismo*, which is a practical explanation of the Arabic number system with regard to integers and their use; *Demonstratio de minutiis*, which deals with fractions; *De elementis arithmetice artis*, which became the classic source of theoretical arithmetic in the Middle Ages; *Liber philotegni de triangulis*, which stands out in medieval Latin geometry particularly because it gives geometric proofs of theorems; *Demonstratio de plana sphera*, which consists of five multi-partite propositions that clarify various aspects of stereographic projection; and, lastly, *De numeris datis*, considered the first book of advanced algebra written in Western Europe, after Diophantus's *Arithmetic* (which was written in about 250 BC but did not reappear in the Christian West until the 15th century, whereas in the Islamic East there is an Arabic translation of the 9th century; see Sesiano, 1982, and Rashed, 1984).

3.6. De Numeris Datis

Our description of this work is based on the version edited, translated, and interpreted by Barnabas Hughes and published by the University of Berkeley (Hughes, 1981). The book includes a critical edition in Latin of the complete *De Numeris Datis*, together with an English translation of the entire text and a translation into modern symbology of the statement and canonical form of each proposition (it does not include the symbolic translation of the solving procedure). In Puig (1994) there is a detailed description of the MSSs of this work, together with a translation of parts of book one, more literal than the translation by Hughes, precisely with the aim of bringing out the characteristics of the MSSs. Here we will limit ourselves to outlining what is of interest for our present purpose.

Unlike the *abbacus* books, employed as elementary algebra texts in secondary education for use in commercial life, *De Numeris Datis* was a text aimed at university students of the time, with the intention of setting them non-routine "algebraic" problems and teaching how to solve them. Indeed, *De Numeris Datis* presents a treatment of quadratic, simultaneous and proportional equations which presupposes handling contents equivalent to those of al-Khwârizmî's *Concise book of the calculation of al-jabr and al-muqâbala* (Rosen, 1831) and Fibonacci's *Liber abbaci* (Boncompagni, ed., 1857; Sigler, ed., 2002). Both texts begin with some definitions and the development of the equations $x^2 = bx$, $x^2 = c$ and $bx = c$, very rapidly arriving at the equations $x^2 + bx = c$, $x^2 + c = bx$ and $bx + c = x^2$.

The part played by *De Numeris Datis* in the history of mathematics is comparable to that of Euclid's *Data* (Taisbak, 2003), in Hughes's opinion, in the sense that the former constitutes the first book of advanced algebra, in the same way that the latter is the first book of advanced geometry and implies a good knowledge of fundamental geometry (contained in the *Elements*), confronting the ambitious student with the proof and solution of non-standard problems by the method of analysis.

The propositions of *De Numeris Datis* are useful for analysis, therefore, just as a box of tools is, but the very structure of the book is also an exercise in analysis. In fact, unlike what happens in the problem of the *Trattato di Fioretti*, which we have just mentioned and that can be taken as representative of the *abbacus* type of problem, in the propositions of *De Numeris Datis* it is a question of finding numbers for which some numerical relationships are known, but these relationships are given by constants, that is, the book says, for example, that the sum of three numbers has been given instead of saying that the sum of three numbers is equal to a certain specific number, 228 in the case mentioned, as it appears in an *abbacus* book. In fact, the statements of the propositions in *De Numeris Datis* are not problems but theorems, as they always have the form "if such numbers or ratios and relationships between them have been given, then such numbers or ratios have been given," and they are proved like theorems, and are accompanied by a particular problem with specific numbers that is solved with the rule derived from the steps of the proof of the theorem or from the steps of the theorems to which this theorem is reduced.

This second point is fundamental for the character of *De Numeris Datis* that interests us here: the sequence in which the propositions in *De Numeris Datis* are solved explicitly shows the reduction of each proposition to one that has been proved previously, and, therefore, the solution of the corresponding problems to the solutions of others solved previously. This kind of sequence is not entirely absent in *abbacus* texts, that is, in *abbacus* problems we also see repeated application of rules or algorithms when the solving procedure has led to a well-identified situation in which the application is feasible: this is the case with the rule of doubles or the Babylonian method in the problem mentioned. However, this aim of reducing to situations or forms already encountered and solved previously does not appear explicitly in the *abbacus* texts, whereas in *De Numeris Datis* it forms part of the method of solution. This might be due to the fact that expressions that we would write as $x+y+z=a$ and $x+y+z=b$ with $a \neq b$ are not fully identified as equivalents for the purposes of the solving procedures and strategies, which in the *abbacus* books depend strongly on the specific properties of the specific number a (or b) and its relationships with the other numbers that appear in the other equations of the system in question.

It is in this sense that *De Numeris Datis* might be located in a more evolved stage, as it makes it possible to group problems that can be solved in the same way into large families by identifying more general forms. By this we do not wish to suggest that the actual strategies and skills required for the solution of problems in *De Numeris Datis* are on a higher level of abstraction or a more evolved level in terms of symbolization than those developed by the *abbacus* texts. These ideas about establishing a clear difference between levels of symbolization and solving strategies and skills are developed in Filloy and Rojano (1983). The point of view developed there considers the construction of symbolic algebra as the final identification within a single language of earlier strata of that language in which the absence of abstract symbolism causes the posing of the problems and the procedures for solving them to be carried out in the vernacular (Latin, Italian). This imprints peculiarities on the operations performed, peculiarities that vary from one stratum to another and that cause those operations to be irreducible from one stratum to another unless one has developed what we call a more abstract MSS.

4. THE PHENOMENOLOGICAL ANALYSIS OF SCHOOL ALGEBRA

Modern algebra organizes phenomena that have to do with the structural properties of arbitrary sets of objects in which there are defined operations. Those properties and those objects come from the objectification of means of organisation of other phenomena of a lower level and they are the product of a long history with successive rises in level.

4.1. Characteristics of algebra in al-Khwârizmî

One way of viewing this history consists in placing oneself in the 9th century at the time when al-Khwârizmî wrote the *Concise Book of the Calculation of Al-jabr and Al-muqâbala* and taking that event as the birth of algebra as a clearly defined discipline within mathematics. What al-Khwârizmî did, and what separates his work from all the others that have been seen as algebra after him, was that he began by establishing "all the types or species of numbers that are required for calculations."

The context in which he seems to have examined those species is that of the exchange of money in trading or inheritances, and from it he takes the names that he uses for the species of numbers. The world of commercial problems and inheritances is linear or quadratic: in the course of the calculations there are numbers that are multiplied by themselves, in which case they are "roots" of other numbers, and the numbers that result from multiplying a number by itself are *mâl*, literally "possession" or "treasure"; other numbers are not multiplied by themselves and are not the result of multiplying a number by itself, and therefore they are neither roots nor treasures, they are "simple numbers" or *dirhams* (the monetary unit). Treasures, roots, and simple numbers are thus the species of numbers that al-Khwârizmî considers.

In his *Arithmetic*, Diophantus had already distinguished different species (*eidei*) of numbers, with a different conceptualization (ways in which a number may have been given), using the names *monas*, *arithmos*, *dynamis*, *cubos*, *dynamodynamis*, *dynamocubos*, etc., and thus a longer series than al-Khwârizmî's.

Calculating with al-Khwârizmî's or Diophantus's species of numbers follows similar rules: what is obtained is always an expression equivalent to our polynomials or rational expressions, as the numbers of the same species are added together, or are taken that many times, or that many parts are taken, and the result is a number of that species a certain number of times or a certain number of parts of a number of that species; and if numbers of different species are added, the sum cannot be performed and is simply indicated. Thus, "four ninths of treasure and nine dirhams minus four roots, equal to one root" (Rosen, 1831, p. 41 of the text in Arabic) is an algebraic equation in al-Khwârizmî's book, since al-Khwârizmî's MSS uses vernacular language (Arabic in his case) exclusively; and

$\Delta^{\Upsilon}\overline{\beta}\,\overset{\circ}{M}\,\overline{\sigma}$ ἴσα $\overset{\circ}{M}\,\overline{\sigma\eta}$ (Tannery, 1893, vol. I, p. 64, l. 8)

is an equation in Diophantus's MSS, which is read as "dynamis 2 monas 200 equals monas 208," since Diophantus uses abbreviations for the names of the species of numbers, which in this case consist of the first two letters of the word, and the Greek system of numeration uses the letters of the alphabet marked with a horizontal stroke, in a system that is not positional but additive, with codes for the nine units, the nine tens and the nine hundreds. There is almost no conceptual difference between the algebraic expressions and the equations of the two authors, as what is represented in them is the names of the species, the specific numbers that indicate how many of each species there are, the operations indicated between the quantities of each species, and the relationship of equality between quantities.

Al-Khwârizmî's book might thus be seen as more elementary or situated one step behind Diophantus, as the set of species of numbers is smaller and the expression uses only the signs of the vernacular. However, what is new in al-Khwârizmî's book is that it suggests having a complete set of possibilities of combinations of the different kinds of numbers. It is clear that initially the possibilities are infinite, and that therefore it is necessary to reduce them to canonical forms in order to be able to consider obtaining a complete set. But al-Khwârizmî's aim then is also to find an algorithmic rule that makes it possible to solve each of the canonical forms, and to establish a set of operations of calculation with the expressions that makes it possible to reduce any equation consisting of those species of numbers to one of the canonical forms. All the possible equations would then be soluble in his calculation. Moreover, al-Khwârizmî also establishes a method for translating any (quadratic) problem into an equation expressed in terms of those species, so that all quadratic problems would then be soluble in his calculation.

Al-Khwârizmî obtains the set of canonical forms by combining all the possible forms of the three species, taken two at a time and taken three at a time. He thus obtains the three forms which he calls "simple," making the species equal two at a time:

treasure equal to roots
treasure equal to numbers
roots equal to numbers

and the three forms that he calls "compound," adding two of them without taking order into account and making them equal to the third:

treasure and roots equal to numbers
treasure and numbers equal to roots
roots and numbers equal to treasure.

As al-Khwârizmî is able to present an algorithm to solve each of these canonical forms simply by collecting and justifying methods that are established and that have been in use since the time of the Babylonians, all that remains is to establish a procedure for translating the statements of the problems into their algebraic expressions and a calculation that makes it possible to transform any equation into one of the canonical forms.

The species of numbers refer to concrete numbers with which calculations are performed, so that in order to be able to translate the statements of the problems into those algebraic expressions it is necessary to be able to refer also to unknown quantities as if they were concrete numbers and calculate with them, that is, it is necessary to name the unknown and treat it like a known number. What al-Khwârizmî does to achieve this is to use the word *shay'*, literally "thing," to name an unknown quantity. He then uses it to perform the calculations which the analysis of the quantities and relationships present in the problem indicates to him as being necessary, and in the course of the calculations he sees what species of number that thing is: a root if it is multiplied by itself, or a treasure if it is the result of a quantity that has been multiplied by itself; so that he can translate the statement of the problem into two expressions that represent the same quantity and make them equal so as to have an equation. In Chapter 11 we will see that these are in fact the steps of the Cartesian method.

"Thing," incidentally, is a common noun for representing any unknown quantity, not the proper name of a specific unknown quantity, unlike what is established by the Cartesian method; in fact, al-Khwârizmî does not say "the thing" but "thing," that is, "a thing," when he refers to the unknown quantity which he calls "thing." In the course of the construction of the equation that translates the problem, however, “thing” is bound to one of the unknown quantities, functioning as the proper name of that quantity.

The operations in the calculation are algebraic transformations of the equations that seek to obtain one of the canonical forms. However, the canonical forms have three features that characterize them (and that cause the complete set of canonical forms to have 6 items), and the operations are directed at achieving each of those three features.

The first is that there are no negative terms, or, to use al-Khwârizmî’s terminology, there is nothing “that is lacking” on either of the two sides of the equation.

In fact, in al-Khwârizmî’s or Diophantus’s algebraic expressions there are quantities that are being subtracted from other quantities. There are not positive and negative quantities, but quantities that are being added to others (additive quantities) and quantities that are being subtracted from others, and the latter cannot be conceived on their own but only as being subtracted from others. Thus, al-Khwârizmî may even go so far as to speak of “minus thing” when he is explaining the sign rules, but he is always referring to a situation in which that thing is being subtracted from something:

> When you say ten minus thing by ten and thing, you say ten by ten, a hundred, and minus thing by ten, ten “subtractive” things, and thing by ten, ten “additive” things, and minus thing by thing, “subtractive” treasure; therefore, the product is a hundred dirhams minus one treasure. (Rosen, 1831, p. 17 of the text in Arabic)

However, as the subtractive quantities are conceived as something that has been subtracted from something, an expression in which there is a subtractive quantity represents a quantity with a defect, a quantity in which something is lacking. Diophantus’s sign system expresses this way of conceiving the subtractive in an especially explicit way, as in his work all the additive quantities are written together, juxtaposed in a sequence one after another, and all the subtractive quantities are written afterwards, also juxtaposed, preceded by the word *leipsis* (what is lacking). Thus, the algebraic expression

$$x^3 - 3x^2 + 3x - 1$$

is written as

$$K^{\Upsilon}\overline{\alpha}\,\varsigma\overline{\gamma}\ \pitchfork\ \Delta^{\Upsilon}\overline{\gamma}\,\overset{\circ}{M}\overline{\alpha}$$ (Tannery, 1893, vol. I, p. 424, l. 10),

an abbreviation of “cubos 1 arithmos 3 what is lacking dynamis 3 monas 1,” in which the expressions corresponding to x^3 and $3x$ are juxtaposed on one side, and x^2 and 1 on the other, separated by the abbreviation for “what is lacking.”

It is precisely this idea that there is something lacking in the quantity that is directly responsible for the form adopted by the operation that eventually gave its name to algebra. In fact, the objective of the operation that al-Khwârizmî calls *al-jabr* is that nothing should be "lacking" on either side of the equation. That is why the operation is called *al-jabr*, literally "restoration," because it restores what is lacking. In terms of the language of modern algebra, *al-jabr* eliminates the negative terms in an equation by adding them to the other side, but *al-jabr* is not equivalent to the transposition of terms because the modern transposition of terms can also transfer a positive term to the other side by making it negative, which goes against the intention of the *al-jabr* operation (but is consistent with the fact that the canonical form that one now seeks to attain with algebraic transformations is $ax^2 + bx + c = 0$, with a, b, and c being real numbers).

The second characteristic feature of al-Khwârizmî's canonical forms is that each species of number appears only once. The algebraic transformation that this pursues is *al-muqâbala*, literally "opposition." As al-Khwârizmî always performs this operation after *al-jabr*, at this point there is nothing lacking; there are no negative terms in the equation. The operation consists in compensating for the number of times that a given species of number appears on each side of the equation, leaving the difference on the appropriate side.

Lastly, the third characteristic is that there is only one treasure, or, in modern terms, that the coefficient of the treasure is 1. This is achieved by means of two operations that al-Khwârizmî calls "reduction" (*radd*) and "completion" (*ikmâl* or *takmîl*). "Reduction" is used when the coefficient of the treasure is greater than one, and it consists in dividing the complete equation by the coefficient; and "completion" is used when the coefficient of the treasure is less than one (it is "part of a treasure," in al-Khwârizmî's words), and it consists in multiplying the complete equation by the inverse of the coefficient.

The first two operations, *al-jabr* and *al-muqâbala*, appear in the title of al-Khwârizmî's book as the characteristic operations of calculation, and they are also mentioned, although not by name, in the introduction to Diophantus's *Arithmetic* (Tannery, 1893, vol. I, p. 14, ll. 16–20).

What makes all these calculations meaningful, therefore, is the idea of the establishment of a complete set of canonical forms, which then organizes algebraic expressions through transformations, and it organizes problems into families of problems that are solved in the same way.

4.2. Steps toward modern algebra

Al-Khwârizmî's complete set of canonical forms was complete only with the condition of restricting the species of numbers to the three that he considered.

The continuation, including the cube as the fourth species, was developed by ᶜUmar al-Khayyâm, who established that the complete set of canonical forms had 25 items, but that he could not find an algorithm for solving the 25. What al-Khayyâm did as a result of his inability to give a strictly algebraic solution for the matter was to show how the solution of the canonical forms could be constructed in the cases that resisted him by means of intersecting conical sections.[4] As a response to the same inability, Sharaf al-Dîn al-Tûsî added to this the establishment of procedures for the approximate calculation of roots.[5] For the historical phenomenology that we are outlining, these non-algebraic responses to the lack of ability to find algorithms for all the canonical forms do not interest us. Nor are we interested in the fact that eventually algorithms were found not only for al-Khayyâm's 25 canonical forms but also for fourth-degree equations. What interests us is the response given to the inability to find algorithms for the canonical forms of equations of a degree higher than the fourth from Lagrange onwards.

In fact, in his memoir written in 1771, "Réflexions sur la résolution algébrique des équations" (Lagrange, 1899, vol. III, pp. 205-424), Lagrange explicitly proposed an aim which was not that of continuing to search for algorithms, but of examining why it had been possible to find them.

> I propose to examine the various methods that have been found so far for the algebraic solution of equations, and reduce them to general principles and show a priori why these methods succeed for the third and fourth degree and are lacking for higher degrees. (Lagrange, 1899, vol. III, p. 206)

Here, therefore, Lagrange explicitly takes the methods themselves as the object of study, so that the problematics of algebra is shifted to a higher level, beyond the organization of problems into families by the establishment of canonical forms in a more abstract MSS than that of the problems themselves. Now it is the characteristics of the canonical forms themselves that have to be organized in order to account for the success or failure of algebraic methods of solution. What Lagrange does is to make a critique of the methods, a critique in the sense of establishing limits. To do this, he studies the relationships in the methods between a given equation that one is trying to solve and the reduced equation, a second-degree auxiliary equation that can therefore be solved algebraically, to which one can proceed from the given equation by a rational relationship; and, on the basis of this study, in a crucial movement he reverses the relationship by finding a way of expressing the reduced equation in terms of the roots of the given equation (what Lagrange calls the resolvent). From this point he is able to establish the reason for the success of the methods, and also the fact that the same reason cannot exist for degrees higher than the fourth (which does not exclude the possibility of an algebraic solution, but does rule out the possibility of it belonging to the same structure).

The shift made by Lagrange, from the search for methods of solution to the explanation of why they are successful or not, led Abel in 1824 to jump to a new level, in his *Mémoire sur les équations algébriques, où l'on démontre l'impossibilité de la résolution de l'équation générale du cinquième degré* (Abel, 1881, I, pp. 28-33), in which he shows, as the title says, that the inability to find an algebraic method of solution for equations of a degree higher than the fourth really is an impossibility, thereby giving the previously insoluble problem a formulation in which it is soluble, changing the problem of finding a method into the problem of proving whether such a method exists.

Galois's works provided the final and definitive jump in level, by linking the solubility of an equation to the properties of the equation's group and tackling the problem by studying the properties of those groups, so that what is studied is not what equations are soluble but what groups are soluble. He shows this clearly in a memoir written in 1831, *Sur les conditions de résolubilité des équations par radicaux*, where he says:

> Problem. "In what case is an equation soluble by simple radicals?"
> First of all I will observe that in order to solve an equation one must lower its group progressively until it contains only one permutation.
> […] let us seek the condition that must be satisfied by an equation's group so that it can thus be lowered by the adjunction of radical quantities. (Galois, 1846, p. 426)

With this step, from Galois onward algebra becomes modern algebra. As Vuillemin says,

> […] Galois's theory has shifted the interest of algebra: whereas, essentially, it set out to solve equations, in future it will tend rather to seek the nature of the magnitudes that must be added to the base field in order to determine the factorising field in which it becomes possible to express and ascertain roots rationally. (Vuillemin, 1962, p. 247)

4.3. The phenomenological analysis of the language of algebra

After Galois we enter a different history, that of modern algebra, which is absent from current school algebra, yet the historical phenomenology that we have expounded in the two previous sections does not exhaust the phenol-menology of school algebra. It is at least necessary to consider what phenomena are organized by the language of algebra, and in what way it organizes them. Once again, this can be done as historical phenomenology or as didactical phenomenology. The historical view is developed in Section 1.3, "Algebraic Language: A History of Symbolisation," in Puig and Rojano (2004); the didactical phenomenology can be found in Freudenthal's *Didactical Phenomenology of Mathematical Structures*, Chapter 16, "The Algebraic Language." In this section we refer to what is expounded in the two texts.

4.3.1. The representation of unknown quantities and species of numbers

In Puig and Rojano (2004) there is an analysis of how the central core of the evolution of the language of algebra has to do with the way in which unknown quantities, on the one hand, and species of numbers, on the other, are represented in algebraic expressions and therefore in equations.

In most of the sign systems of medieval algebra there is only one name to represent the unknown, "thing," which is in fact a common noun although used as a proper name. Consequently, those MSSs cannot represent different unknown quantities with different proper names. Instead, once an unknown quantity has been named as "a thing," the others have to be named with compound names constructed more or less algorithmically from the relationships between it and each new unknown quantity (for example, "ten minus thing" is the name that one could give to an unknown quantity of which it is known that when it is added to "thing" the result is ten). However, the network of relationships between the quantities in the problem might be so complex that it is extremely intricate, or even impossible, to name all the quantities with compound names: for these problems, the fact that only the term "thing" is available makes the sign system not very efficient.

Medieval algebraists resorted to various devices to get around this. Sometimes they used the term "thing" again, but with a qualifier. This is the case with Abû Kâmil, who in one problem in his book of algebra (cf. Levey, ed. 1966, pp. 142-144) uses the names "large thing" and "small thing" ("res magna" and "res parva" in the Latin version edited by Sesiano, 1993, p. 388). Sometimes they used names of coins for the other unknown quantities. This is also the case with Abû Kâmil, who uses *dînâr* and *fals* (cf. Levey, ed. 1966, p. 133, n. 140, although on this occasion Abû Kâmil is expounding a different solution for a problem that has already been solved using "thing" on its own), or with Leonardo of Pisa, who uses *denaro*, as well as *res* (cf. Boncompagni, 1857, pp. 435-436 and p. 455). In the part devoted to inheritances in al-Khwârizmî's book, at one point he does not even use the term "thing" but calls the inheritance *mâl*, treasure, using it in its vernacular sense, and he calls what corresponds to each of the heirs "share" or "part share," and he constructs the indeterminate linear equation "five shares and two parts of eleven of share equal to the treasure." According to Anbouba (1978), in the same part of al-Khwârizmî's book there is also a problem in which he constructs a linear system of two equations using "thing" and "part of thing" to name two different unknown quantities.[6]

Moreover, what appears in the algebraic expressions is the names of the species of numbers (simple number or *dirham*, root, treasure, cube, etc.; or, in the translation into Latin, *numerus*, *radix*, *census*,[7] *cubus*, etc.), but the quantity by which this species is qualified is not named. From the identification of "thing" with "root" it is assumed that the treasure is the thing multiplied by itself, but there is no way of expressing another quantity represented with another proper name that has been multiplied by itself. The algebraic expressions of these sign systems do not say "five treasures of thing" but just "five treasures," unlike the sign system of modern algebra, which uses $5x^2$ to say "five times the square of x," and, therefore, is structurally prepared for designating another unknown quantity with another proper name, y, and saying "five times the square of y", $5y^2$.

The sign system of Indian algebra does have proper names for different unknown quantities (it uses names of colors for this purpose), and it forms algebraic expressions by juxtaposing the name of the unknown quantity and the name of the species (cf. Colebrooke, ed., 1817), but this system did not have any impact on medieval Arabic algebra, or therefore, on algebra in the Christian West. It was not until Viète that a sign system was developed in which there were proper names for different quantities, together with the names of the species. But Viète's sign system also used letters as proper names, and not just for unknown quantities but also for known quantities. This freed the algebraic expressions from ambiguities and made them capable of providing a direct representation of the quantities analyzed in the statements of the problems.

However, in Puig and Rojano (2004) it is shown that Viète's sign system lacks full operational capacity on the syntactic level because the species of numbers are represented by words or abbreviations of them, although these words are constructed algorithmically from certain basic words. It is also shown that this syntactic operativity is attained when one combines the representation of quantities by letters, introduced by Viète, with the representation of species by means of numbers that indicate the position of the species in the series of species (in continual proportion).[8] The algorithmic rules for the construction of the names of the species can then be replaced by those numbers and converted into part of the calculation.

4.3.2. Aspects of the didactical phenomenology of the language of algebra

In the "Variables in the Vernacular" section of his phenomenological analysis of the language of algebra, Hans Freudenthal recounts that

> When my daughter was at the age when children play the game "what does this mean?" and I asked her what is "thing" she answered: Thing is if you mean something and you do not know what is its name. (Freudenthal, 1983, p. 474)

The didactical phenomenology of the language of algebra that Freudenthal expounds is based precisely on the examination of the phenomena that are organized by the language of algebra, seen with regard to how those phenomena are organized in the vernacular and in the language of arithmetic, which are the languages that provide the starting point or context from which pupils have to acquire the language of algebra.

We will not repeat Freudenthal's observations here, but simply indicate some of the aspects that he analyzes.

1) The rules of transformation in languages

We have already seen that the need for the development of rules of transformation in the language of algebra comes from the aim of being able to solve all problems without needing to have a specific algorithm for each one, and that this is done by the establishment of canonical forms and calculation on the syntactic level. In teaching, only awareness of the overall aim can give sense to the use of such syntactic transformations. Freudenthal examines the fact that rules of transformation also exist in the vernacular, but that the correctness of the transformations performed in the vernacular cannot generally be decided without resorting to the contextual meaning, whereas in the language of algebra the part played by the context in this sense is generally nil.

2) The algorithmic construction of proper names

We have seen that this is an outstanding aspect of the language of algebra. Freudenthal points out that algorithmic features are not unusual in vernacular languages. But these algorithmic procedures of sign construction are not very systematic and are not generalized (plurals, conjugations and declensions, etc.). The first experience that children have of an algorithmic construction of proper names is the learning of numbers in their mother tongue: an area of contact between the vernacular and the language of arithmetic.

3) Structuring devices

The rules of transformation and the algorithmic construction of proper names are based on the structure of the language. The language of algebra has a wide range of structuring devices, many of them shared with the language of arithmetic, especially parentheses, priority between operations, and the arrangement of signs in relation to the text line (exponents, subscripts and superscripts, the fraction bar and the positions above and below the text line that it determines, roots, etc.). Once again, Freudenthal analyzes the existence of such structuring devices in the vernacular, and the fact that there they are based on content, whereas this is not the case (or not so much) in the language of algebra.

4) Variables in the vernacular versus algebra variables

We have already analyzed the use of "thing" in the language of algebra, and the differences between it and the variables of modern school algebra. Freudenthal points out that the use of letters must also be examined in geometry, where Euclid's *Elements* already used letters to refer to points, lines, and figures, and he indicates the origin of the expression "point A," in which A is the proper name of the point, in an abbreviation of an earlier expression, "the point at A," which simply describes a drawn figure to which letters have been added in order to be able to refer to it in the oral discourse which was customary in teaching.

Freudenthal also examines the fact that in order to use a variable as a proper name it is necessary to bind the variable. "Variables," says Freudenthal (1983, pp. 474-475), "can be bound independently of any context, by linguistic logic devices, or in dependence of a context." The logic devices are the universal and existential quantifiers, the definite article (including "the thing" as opposed to "thing"), the set former, the function or species former and the interrogative, whereas the devices that depend on context are the demonstratives.

5) Formal substitution and algebraic transformations

Formal substitution is the culminating point in the constitution of the MSS in the teaching of school algebra. For this to take place it is necessary that the algebraic expressions should have completely relinquished the character of representing actions that their antecedents in the MSS of arithmetic possess, and should have completely acquired the static character of a relationship. One of the key elements in this transition from language as action to the language of algebra is the exceedingly well-known change of meaning from the arithmetic equals sign to the algebraic equals sign.

However, in the context of algebraic transformations, which are performed between expressions with a static character that represent relationships, the meaning of the arithmetic equals sign reappears. The algebraic transformation par excellence consists in "reducing" an expression to a simpler form or a canonical form, so that $(x + a)(x - a) =$ is an indication that an action must be performed and that the result of the transformation is expected on the other side of the equals sign; it is not just the construction of an equivalence. Yet the reversibility of algebraic transformations may give that appearance: for example, the action that is the opposite of "reducing" is "factorizing" (and one would have to decide which is simpler, the classical canonical forms resulting from reducing, or the expressions that explicitly show the roots that result from factoring).

At the origin of formal substitution there is the possibility that the letter that names a quantity may be replaced by a compound expression that names the same quantity. This makes it necessary for the user of the MSS to accept the fact that, as the letter and the compound expression represent the same thing, not only can they be made equal but also the calculations or relationships represented in an expression in which the letter appears can also be carried out with the equivalent expression and the new expression will represent the same thing. On the other hand, the user will have to face syntax problems[9] that derive from the structuring devices, such as the priority between operations, which sometimes makes it necessary to introduce other structuring devices such as parentheses where they were not present; or the problems posed by having to replace a letter with an expression in which that letter may also appear. This is the case with the difficulty that pupils find in replacing n with $n + 1$, for example, when using the method of complete induction.

However, the substitution becomes definitively formal when the expressions are no longer the result of the translation of the statement of a problem but are algebraic expressions which are studied as such.

SUMMARY

This chapter goes over part of a diagram of the design of the experimental setting for the observation of phenomena of learning and teaching algebra.

In the next chapter we shall apply the methodological diagram to the study "Operating on the Unknown," with a view to studying the processes of transition from arithmetic thinking to algebraic thinking at the point when pupils first encounter the need to operate on what is represented. In order to locate this point (called a "didactic cut" in the study), we use historical and epistemological analysis of mathematical sign systems found in old texts on algebra from the pre-symbolic period (before the appearance of François Viète's *The Analytic Art*). This analysis and the phenomenological analysis of algebraic language illustrate the power of the methodology proposed by local models, in the part corresponding to the choice of the moment of observation.

ENDNOTES

[1] See Filloy, Rojano and Solares (2004)
[2] The canonic edition of the Greek text of Diophantus's *Arithmetic* is the one by Paul Tannery (Tannery, 1893); we have also consulted the French translation by Paul Ver Eecke (Ver Eecke, 1959). The canonic edition of Bombelli's *Algebra* is the one by Ettore Bortolotti (Bortolotti, 1966). The original Latin text of Viète's book, *In Artem Analyticen Isagoge*, is included in the complete works of Viète compiled and published by Franciscus van Schooten (Van Schooten, 1646); there is an English translation included as an appendix in Klein (1968), and another one in Witmer, ed. (1983).
[3] In the text of the *Trattato di Fioretti* the rule is not stated generally but with reference to the concrete case to which it is applied, as follows: "E a multiplichare la seconda parte nella somma di tutte e 3 due volte è chome a multiplichare la seconda parte nel doppio della somma di tutte a 3, overo quanto a multiplichare lo doppio della seconda parte nella somma di tutte et 3." [And multiplying the second part by the sum of the 3 two times is like multiplying the second by double the sum of all 3, or like multiplying double the second part by the sum of all 3] (Mazzinghi, 1967, p. 16). Stated in a general form, the rule would say: "multiplying one quantity by another one twice is equal to multiplying the first quantity by double the second one, or also multiplying double the first quantity by the second."
[4] There is a recent edition of the Arabic text of al-Khayyâm's *Treatise on Algebra*, accompanied by a translation into French, in Rashed and Vahebzadeh (1999). One can also consult the English translation by Kasir (1931).
[5] There is an edition by Roshdi Rashed of the Arabic text of Sharaf al-Dîn al-Tûsî's *Treatise on Equations*, accompanied by a translation into French, in al-Tûsî (1986).
[6] Diophantus also has a single name for unknown quantities (*arithmos*). In problem 28 in Book II of his *Arithmetic* (Tannery, 1893, vol. I, pp. 124–127), he resorts to the device of saying that a second unknown quantity is one unit (*monas* 1), performing the calculations using this supposition, and then in the result changing the units to *arithmos* and calculating again.
[7] "Census" was the term chosen by Gerardo de Cremona for *mâl*, treasure, in his translation of al-Khwârizmî's book of algebra, and it was the one that caught on in the Christian Mediaeval West (cf. the edition by Hughes, 1986).
[8] This is already present in Chuquet's *Triparty*, written in French in 1484. However, this book by Chuquet remained unpublished and was therefore scarcely known until the end of the 19th century, when Aristide Marre published it (Marre, 1880). Bombelli used the same kind of representation in his *Algebra*, from which it became more widely known among algebraists.
[9] See our ongoing work reported in Chapter 8 and in Filloy, Rojano and Solares (2004).