CHAPTER 2

CURRICULUM DESIGN AND DEVELOPMENT FOR STUDENTS, TEACHERS, AND RESEARCHERS

OVERVIEW

We first suggest that it is necessary to make the conception of the nature of mathematics explicit, as it underlies curriculum organization and curriculum development, and show some of the risks that appear when this is not done.

Section 2 explains what we understand by theoretical model through four basic characteristics, distinguishing it from other uses of the same term, and then introducing the methodological concept of the local theoretical model (LTM) and its four interrelated components. We discuss the contrast between the local and the general, and of the methodological nature of local modeling, setting out from the need to design ad hoc observation settings to study specific phenomena. We also explain the recursive character of the application of local models and, in Chapter 3 explain the ephemeral quality of certain theoretical theses in this application.

The major part of the chapter describes our manner of understanding the phenomenological analysis of mathematical concepts (or mathematical structures) that Freudenthal proposed in his book *Didactical Phenomenology of Mathematical Structures* (Freudenthal, 1983). For this purpose we outline the essential characteristics of a conception of the nature of mathematics that is compatible with our way of understanding Freudenthal's phenomenology and that also includes the idea of the generation of concepts from proofs, which is characteristic of the work of Lakatos. We also discuss Freudenthal's distinction between mental objects and concepts, and the consequences for curriculum development, which derive from the opposition that Freudenthal proposed between the constitution of mental objects and the acquisition of concepts. In this discussion, we use our semiotic viewpoint as a basis for interpreting the distinction established by Freudenthal, using as an example some considerations for a LTM for studying the uses of natural numbers. In the context of these considerations, we present the distinction between three types of sign —icons, indices and symbols— which Peirce himself used to describe algebraic expressions as iconic, while the letters in them are indices, and signs such as those of operation or equality are symbols.

1. INTRODUCTION

Any reflection about the elements with which one tries to structure plans and syllabuses for the teaching of mathematics entails, consciously or spontaneously, a conception of the fundamentals of mathematics. To try to free oneself from this discussion, which is far removed from the requirements of the usual practices of drawing up a curriculum, one tends to set out a list with various ways of analyzing the practices that take place in the teaching of mathematics in school systems –in other words, mathematics education. Thus one speaks of mathematics as (see Filloy and Sutherland, 1996):

- A corpus of knowledge to be learned.
- A set of techniques for solving problems.
- The study of certain structures: arithmetic-algebraic, geometric, etc.
- A language with a given sign system that intertwines with natural language.
- A formal science with a highly formalized language.
- A scientific activity, that of mathematicians, that has existed for centuries and that, at present, has developed specific practices very remote from those that can be found in educational systems.
- An activity in which phenomena belonging to the natural and social sciences are modeled.
- A collection of procedures for performing practical calculations to measure, classify, predict, count, etc.
- A part of natural language in which judgments are expressed about the progress of society, the economy, the climate, voting forecasts, etc.
- A collection of ways of talking about random or repeated phenomena with a view to predicting certain future events.
- An essential element of the culture of all historical ages.
- A symbolic system in which one can formulate expressions that give an account of general patterns so that one can make generalized calculations.
- A symbolic system in which generalizations and abstractions are expressed, and that permits representations with operational capability.
- A symbolic system in which one can express phenomena of iteration and recursion for the expression of algorithms.
- A system of mental abilities, such as spatial imagination, the ability to reason hypothetically and deductively, etc.
- Certain structures of the intellect, an internalization of the properties of actions that are performed with real objects.
- A list (even longer than the foregoing) of teaching activities such as is provided in mathematics textbooks.

1.1. Partial and eclectic points of view

The above list is clearly not exhaustive; but it can easily be multiplied if one simply thinks of the many different ways of interpreting some of the terms that we have used, within the various theoretical frameworks of psychology, for instance.

Of course, from some viewpoints the range of mathematical competences that one tries to teach to young students in the basic levels of our present educational systems is all these things and many others. Therefore, if some viewpoints are favored at the expense of others, this leads to the design of curricula that leaves much to be desired because of their partiality, limiting the possibilities of using the curriculum to achieve rich, novel teaching that contemplates a transformation of the vitiated practices. Such practices occur in the current educational systems and they are the direct cause of the poor progress of students and of the rejection of mathematics by the general population.

As a result of proceeding partially, placing some aspects above others, a false dilemma appears, in which the relational aspects of mathematical thinking work to the disadvantage of its instrumental use and viceversa. Similarly, the adoption of a particular bias makes the dilemma between understanding and mere mechanization more acute, in relation not only to mathematical operations but to mathematical thinking in general. As an example one can think of the risks entailed by an unduly narrow design of the curriculum for the teaching of mathematics, thinking of it simply as knowledge about given (ideal) objects the properties and relations of which must be gradually discovered, or the opposing risks introduced by other radical tendencies, which maintain the attitude that all mathematical knowledge is gradually constructed from the first interactions between individuals and reality. In both cases there is an exclusion of all the social aspects that intervene in the processes by which students become competent in the use of mathematical language and results, both for thinking and for producing practical knowledge that can be communicated to any other competent individual.

But perhaps the most common mistake is an extreme eclecticism, by trying to give the same weight to all of the aspects indicated in the preceding list. This generally leads to the production of curriculum designs in which the confusion reaches the most elementary strands in the curriculum. Of course, plunging spontaneously into the design of a curriculum can have even worse results, in which the path followed by the curriculum design leads to a tangle of contradictions and lines of force that interweave, mingle, or clash without rhyme or reason.

All this is no more than a preamble to the need to clarify the conception of the nature of the mathematics that is brought into play in the curriculum.

2. LOCAL THEORETICAL MODELS

2.1. Four characteristics of LTMs

The term *model* has a wide range of meanings: it can refer to many things, from a physical scale model to a set of abstract ideas. Here we examine the use that we make of this term in mathematics education.

We use the term *theoretical models*, or simply models, without claiming that everything given the name of model may be a model in this sense. In fact, in this usage models differ considerably from what is given the same name in other applications. Our aim in this book is to analyze how the various examples have certain common characteristics, which is why we call them models. To begin, we point out four characteristics.

The first characteristic is the fact that a theoretical model consists of a set of *assumptions* about some concept or system.

First, it is necessary to distinguish theoretical models from diagrams, illustrations, or physical models, which, although sometimes useful to represent the model, must not be identified with the model itself. Second, it is true that at times, albeit not always, what is called a model is also termed a theory.

This interchangeability of names is possible because, in such cases, the terms "model" and "theory" refer to the same set of assumptions, although the same things are not suggested about this set when we call it a model as when we call it a theory. Some of the differences, and also the reasons why not all models are called theories, must be analyzed. The second characteristic has precisely to do with this.

The second characteristic is the fact that a theoretical model describes a type of object or system by attributing to it what might be called an internal structure, a composition or mechanism that, when taken as a reference, will explain various properties of that object or system.

A theoretical model, therefore, analyzes a phenomenon that exhibits certain known regularities by reducing it to more basic components, and not simply by expressing those regularities in quantitative terms or by relating the known properties to those of different objects or systems. Accordingly, the term "theory" in this sense is broader than "model," because not all theories are formulated with the aim of providing structural analyses, which are typical of models.

The third characteristic is the fact that a theoretical model is considered an approximation that is useful for certain purposes.

The value of a particular model can be judged from two different but related viewpoints: how well it serves the purposes for which it is employed and the completeness and accuracy of the representation that it provides.

The fact that a theoretical model may be proposed as a way of representing the structure of an object or system for certain purposes explains why various models are often used alternately. This represents another difference between the use of the terms "model" and "theory." To propose something as a model of something is equivalent to suggesting it as a representation that provides at least some approximation to the real situation; further, it means admitting the possibility of alternative representations that may be useful for different purposes. To propose something as a theory, however, is equivalent to suggesting that something is governed by certain specified principles, and not just that it is useful for certain purposes to represent it as being governed by those principles or that those principles approximate to the principles that actually apply. Consequently, someone who proposes something as a theory is obliged to maintain that any alternative theories must be discarded or modified, or that they will be valid only in special cases.

Finally, the fourth characteristic is the fact that a theoretical model is often formulated and developed and perhaps even named on the basis of an analogy between the object or system that it describes and some other object or different system.

This implies a comparison in which one observes properties and principles that are similar in certain aspects, which fits in with the previous observation that theoretical models have the aim of providing a useful representation of a system. To provide such a representation, it is often helpful to establish an analogy between the system in question and some known system that is governed by rules or principles that are understood, and one supposes that some of those rules, or others like them, also govern the system that one is trying to describe with the model. Reasoning of this kind, based as it is on argument by analogy, is never considered sufficient to establish the principles in question, but only to suggest that they may be considered as first approximations, subject to proof and subsequent modification. In each case, however, the model itself can be distinguished from any analogy on the basis of which it was developed.

Theoretical models can fulfil the same functions as theories: they can be used for purposes of explanation, prediction, calculation, systematization, derivation of principles, and so on. The difference between the use of a model and the use of a theory does not lie in the kind of function for which it can be used, but in the way in which it fulfils that function. Theoretical models provide explanations; but these explanations are based on assumptions that may be simplified, and this condition must be borne in mind when one compares them with theories. It is often said of explanation and systematization by means of a theory that they are more profound and penetrating, which reflects the belief that the principles that constitute a theory are more accurate than those of a model and take more known magnitudes into account. So why not always use the theory, which is more complete? In what follows we briefly discuss why we prefer a local approach and not a general one, but first we mention some semiotic terms that we use repeatedly throughout the book.

2.2. Semantics and pragmatics

It is not our intention here to develop with any precision the kind of theoretical model that is presented throughout the book. We content ourselves with calling on the reader's intuitive concepts concerning terms such as semantics, syntax, semantic load, a more concrete or more abstract level of language, and of the reading level of a text. Even though one consequence of the interpretations obtained in the corresponding empirical studies — described later in the book— is precisely the fact that many of the mistakes that are usually made when using new expressions come from the anticipatory mechanisms of the individual who is decoding a situation that needs to be modeled in that mathematical sign system (MSS), where the semantic load — the custom of certain uses— produced by the individual's prior experience plays a decisive part in possible conceptual errors or mistakes in the syntactic use of the new signs. Nevertheless, we are confident that the approach that we offer for some of the problems proposed is valid in itself, even if it is read from the viewpoint of other theoretical frameworks, and that the "facts" that we describe have an intrinsic interest, even if considered in terms of other interpretations.

We pay more attention, therefore, to the pragmatic viewpoint, which consists in pointing out the meaning given by use, instead of placing greater emphasis on meaning in the abstract. As we have indicated, this approach diverts observation in mathematics education away from the competence of users of a MSS and toward performance, and it also has fundamental implications for the way in which MSSs are studied. Essentially, it is claimed that grammar (the abstract formal system) and pragmatics (the principles of the use of MSSs) are complementary domains in the observation of teaching processes with the various teaching models (innovative and traditional) that are used to achieve the aim of guiding students so that they become competent users of an MSS.

Consequently, this viewpoint not only includes the central role of formal grammar, but also recognizes that it should be incorporated in a broader framework that combines formal and functional explanations. In other words, that in order to interpret the complete meaning of some mathematical messages during normal teaching/learning processes, alongside the strictly formal meaning of the mathematical text in question we also have to admit some other meanings of certain other (logical) messages that are not explicitly communicated either by the sender or by the receiver. We refer to the so-called presuppositions (of which there are various kinds) or the immediate consequences or implications —all this requires the incorporation of some "natural logic" that takes the relation between these meanings into account.

Also, following the same direction of this idea, we are forced to distinguish the difference between competence to decode a message and competence to communicate it (many studies in mathematics education concentrate on this result). It is necessary that our theoretical approach should take these two different kinds of activity into account: the production of mathematical messages and their decoding.

Empirical observations of how a MSS is used during the exchange of messages within teaching/learning processes and the corresponding situation when those MSSs are used by an individual who is thinking out the solution of a problem situation show that the cognitive processes involved interweave the formal level of competence with the pragmatic level. There is a pragmatic component, which comes from the teaching environment in which the learning process takes place. This component is bound up with institutionalized social contracts, so that it is necessary to take into account not only the traditional, customary ways in which the messages of an MSS are emitted in the educational system, but also —and this seems more important— the presence of the entire historical evolution of such sign systems. Notation is the first aspect that appears, but it is not the only one of all the particular ways in which nowadays, after a historical evolution, we tend to use MSSs and their applications to problems in present-day science, technology, and social information processes.

Together with these pragmatic tendencies, there is a component that is due to an individual's cognitive mechanisms that appear in each stage of intellectual development, which gives preference to different mechanisms for proceeding, various ways of coding and decoding the mathematical messages pertinent for the stage in question, various strategies for solving problems, and so on. For example, think of all the evidence that has been accumulated about the tendencies of students to maintain the arithmetic interpretations of many algebraic situations despite their progression to advanced stages of algebra.

2.3. The components of LTMs

The stability of these phenomena of mathematics education and the well established replicability of the experimental designs that have been used to study them are such that we cannot fail to include these observations among the components that are important for any theoretical model for observation in mathematics education. Thus we have a need to propose theoretical components that deal with different types of (1) teaching models, together with (2) models for the cognitive processes, both related to (3) models of formal competence that simulate the competent performance of an ideal user of an MSS, and (4) models of communication, to describe the rules of communicative competence, formation and decoding of texts, and contextual and circumstantial disambiguation.

2.4. Local versus general, the reason for the local in our theoretical models

From the point of view maintained by some authors devoted to problem solving, close to cognitive psychology, one could infer that, to decode a problem situation, experts proceed according to a synthetic process, that is, from the data to the unknown. In several of these studies, in general, when competent users are presented with a problem situation, they recognize "types of problems," because they have formed schemes of them. Thus one could say that, when an expert is presented with a problem situation, in time he would make an integration of the information, in which he would recognize what the central relations of the situation are, comparing them with others that are already in his long-term memory, where there are also specific strategies to be followed. With all of these he is finally able to go on to represent the problem by means of mathematical texts and then decode them for the solution of the problem.

However, from our empirical observations about the decoding of mathematical problem situations it follows that any solution, however fast and fleeting it may be, necessarily passes through an initial logical analysis or logico-semiotic outline of the problem situation, conscious or unconscious, which makes it possible to sketch out the solution. That is, one shows the path that has to be followed to solve the problem in accordance with some mathematical text produced with the use of a certain stratum of an MSS, in which one can establish the direction that the solving process is going to take, and with which one can give analytic or synthetic reasoning processes. Thus an expert or a novice confronted with a problem tends to do work that may proceed from the unknown to the data or viceversa, but in that work the competence to decode the problem situation is determined more by the competence to produce the logico-semiotic outline of the problem situation — which includes strategies of analysis and synthesis— than by the mere recognition of some previously learnt scheme.

Thus, when a competent user performs a logico-semiotic outline of a problem situation, he or she may bring into play cognitive mechanisms that enable him (a) to anticipate the central relations in the problem and also (b) to decide in which stratum of an MSS to outline all the steps of the solution, or decide between one MSS and another more specific MSS, subsequently going on to a process of analysis and synthesis with which he finally obtains the decoding of the problem situation.

To the foregoing we could add many other examples of how, with a global approach, using the results of some general theory of certain branches of knowledge, the analyses of the phenomena that belong to mathematics education, performed thus, reduce the field of investigation very substantially, preventing a clear understanding of the specific phenomenon that one is trying to observe. For example, consider what we would achieve if we wished to use only a general linguistic theory to construct a useful semiotics for mathematics education.

Therefore, instead of arguing in favor of giving preferential consideration to certain components —"grammar," "logic," "mathematics," "teaching models," "models of cognition," "pragmatics," "communication"— we have to concentrate on local theoretical models, appropriate only for specific phenomena but capable of taking into account all four of the components indicated earlier. The idea is to propose ad hoc experimental designs that cast light on the interrelations and oppositions that take place during the evolution of all the relevant processes related to each of the four components.

2.5. The component of formal competence

Earlier, we gave reasons for the need to have models for cognitive processes; this is reinforced later, when we analyze teaching models (Chapter 5). When we introduce our framework of (semiotic) interpretation, MSSs, the need to have models of communication is also underpinned.

As we observe both thinking processes (cognitive component) and the exchange of messages (communicative component) between individuals with various degrees of competence in the use of the MSSs employed to create the mathematical texts (teaching model) relevant for the teaching/learning process, the need for these three models would seem obvious: the model of cognitive processes, the model of communication, and the teaching model.

The need for the model of formal competence comes from the requirement for a description of the situations observed by means of a more abstract MSS, to make it possible to decode all the texts produced in an exchange of messages in which the actors have various degrees of competence in the use of the MSSs in question. Later we see that we interpret the teaching/learning process in this way, hence the advisability that the observer should possess competence in a more abstract MSS that encompasses all the MSSs used in the process observed. In the most extreme case, we might suppose that the model of formal competence is the one with which the epistemic individual would decode the situations observed, that is, the decoding of someone who has all the competences created during the whole historical process of the construction of mathematical knowledge. Fortunately, it is sufficient for the observer to have a model of formal competence described in a more abstract MSS than the one used by all the individuals observed: the learners, the teachers, and the observer himself when he is involved in the exchange of messages (for example, in the clinical interview).

Let us emphasize the importance of the component of formal competence with a paragraph concerning what is stated about Freudenthal's didactical phenomenology presented later in this chapter. The order in which the various kinds of phenomenological analysis must be developed begins with pure phenomenology (the component of formal competence), for which what is of prime importance is knowledge of mathematics and its applications; it is completed with a historical phenomenology; then there is a didactical phenomenology (for which what has to be known is the process of teaching and learning); and in all cases it concludes with a genetic phenomenology. No phenomenological analysis can be effective when teaching is subsequently organized on the basis of it if it is not supported by a sound analysis of pure phenomenology (in other words, the component of formal competence).

3. A GENERAL FRAMEWORK FOR CURRICULUM DEVELOPMENT FOR THE STUDY OF AN LTM

It is advisable to begin the design of a curriculum of a teaching model with a general framework that is broad but based on certain clearly established attitudes, with the intention that various approaches may be obtained from them. Thus, the emphasis placed on them will come from one or another of these central theses, with the aim that the tensions between one viewpoint and another will consequently be diluted by the need to provide a response in each case to the demands of the theses selected, converted in this way into lines of force that promote certain decisions and not others, in making those decisions meaningful.

It is in this spirit that we put forward the following reflections to regulate the criteria for the design of the teaching models that it is decided to use. In the following two sections, we introduce ideas derived from the works of Freudenthal that are pertinent for curriculum development, and also the relation of those ideas to the generation of concepts through proving that is found in the work of Lakatos.

3.1. Concepts

School mathematics is articulated in a series of interrelated conceptual networks, with the characteristic that, with time, students succeed in becoming competent in the use of increasingly abstract general networks —competences that call on many previously mastered competences.

3.2. The relation with reality. Teaching mathematization

The first elementary mathematical concepts are a response to the interaction that children have with the real world. The first notions about quantity, magnitudes, classification, distribution, division, etc. are developed directly from the children's experiences in the real world, but they are also a response to the work of getting hold of the socially established codes for the symbolic manipulation of all these processes, including those inherent in the individual, such as understanding, analysis, and thought. That is why the first mathematical texts have the manipulation of objects and reflection on their interaction as their physical forms of expression. Therefore, a curriculum design that does not set out from the need to move from the concrete to the abstract and that does not then complete the inverse action will tend to result in the students producing MSSs that do not have the sense that one wished to give them socially.

In modern versions, this to and fro between the concrete and the abstract, between the real world and its representations in a mathematical sign system (quantitative modeling, a particular case of mathematization), has played a decisive part not only in science but also in education. Through quantitative modeling it is feasible to "interpret the world with numbers" (Boohan, 1994), using algebraic relations to calculate the numeric value of dependent variables and thus be able to make short- and long-term predictions about the behaviour of phenomena.

One gateway to the learning of algebra is modeling. In this kind of approach the emphasis is on the role of the sign system of algebra as a means to express relations between variables that correspond to phenomena or situations in the physical world, and the corresponding didactic paths contemplate the complete cycle: (1) translation of "concrete" situations or situations expressed in natural language (word problems) to algebraic code; (2) analysis of relations between variables, based on manipulation of the algebraic expressions produced (syntactic level); and (3) interpretation of the "concrete" situation in the light of the results of the work with algebraic syntax. The argument in favor of the virtues of this approach to algebra is that in step (1) meaning is given to algebraic expressions, and in steps (2) and (3) the syntactic manipulation of those expressions becomes meaningful.

With the characteristics just described, the teaching of algebra as a means of modeling tends to promote in students the production of signs in a socially accepted MSS, that of symbolic algebra. In more recent proposals, in the framework of teaching by modeling, other MSSs are also brought into play, such as those of making graphs, numeric tables of variation, spreadsheets, and mathematical narrative (Nemirovsky, 1996). The last of these has succeeded in facilitating processes that can present great difficulty in modeling, such as the translation of relations in a "concrete" situation to algebra.

3.3. Practical knowledge

On the basis of the knowledge obtained from experiences in the real world and the representation of that relation with a sign system that intertwines with natural language, mathematical concepts are used to perform measurements, calculations, and representations. Such concepts are immensely useful and no member of modern society who wishes to pursue a normal intellectual development can disregard them. Nowadays, to be able to analyze the events that take place in the daily lives of individuals and society, one requires certain competences in the use of the MSSs that are taught in mathematics classes. One important component of the curriculum must aim at making it possible for students to use their mathematical knowledge in their daily lives to solve the problems that are presupposed by modern educational systems and that refer to those with which society presents them every day (for example, in the reading of newspapers).

3.4. The analytic and instrumental function for other areas of knowledge

An important feature of elementary mathematics consists in the fact that many other areas of knowledge have gradually, but increasingly intensively, been making use of its symbolic systems to represent the various explanatory models that are found in those areas. Thus, school mathematics is required to describe and understand phenomena from a great diversity of sources. Mastery of the more abstract and general parts of the basic curriculum provides students with a symbolic system in which analytic capability is reinforced by language strata in which not only is it possible to model the phenomena that one is trying to understand and master, but also, precisely there, in the symbolic, one has the operational capability of advancing in the prediction of what will happen when the modeled phenomena take place in time, or when some variable evolves in a particular way. That is why the final parts of algebra, geometry, probability, and statistics, which are traditionally taught in the last two years of the secondary school (13-15 years of age), are of such importance for the future of individuals and for society, which demands competence in such matters if one is to master understanding of natural phenomena and progress in one's societal roles.

4. PHENOMENOLOGICAL ANALYSIS AS A COMPONENT OF DIDACTICAL ANALYSIS. HANS FREUDENTHAL'S APPROACH TO CURRICULUM DEVELOPMENT

4.1. Phenomenological analysis

The didactical analysis of mathematics, i.e., the analysis of the contents of mathematics that is performed for the sake of the organization of the teaching of mathematics in educational systems, has various components, which organize the various teaching models presented in this book. One of the components takes its name from Hans Freudenthal's book *Didactical Phenomenology of Mathematical Structures* (Freudenthal, 1983) and is the subject of this section. We here set out the characteristic features and some of the consequences of what we understand by phenomenological analysis of mathematics as a component of its didactical analysis. The exposition repeatedly refers to Freudenthal's work, taking some liberties with the terminology that he uses and introducing other terminology that is not his.

The phenomenological analysis of a concept or a mathematical structure consists of describing the phenomena for which it is the means of organization and the relation that the concept or structure has to those phenomena.

The description of the phenomena for which it is a means of organization must consider the totality of the phenomena for which this is so at the time, that is, it must take mathematics in its present state of development and in its present use; but it is also advisable to indicate the phenomena for the organization of which it was created and the phenomena to which it has extended subsequently.

The phenomenological analysis developed by Freudenthal is fashioned to serve teaching. However, Freudenthal distinguishes various types of phenol-menology, all important from the viewpoint of teaching, but only one of them is described as didactical. These types are: phenomenology, didactical phenol-menology, genetic phenomenology, and historical phenomenology.

The first thing that characterizes each of these phenomenological analyses is the phenomena that they take into consideration with respect to the concept that is analyzed. In the first case they are the phenomena that are organized in mathematics taken in its state at the present moment and assuming its present use. In the didactical case they are the phenomena present in the world of the students and the phenomena that are proposed in the teaching sequences. In the genetic case, the phenomena are considered with respect to the learners' cognitive development. In the historical case, special attention is paid to the phenomena for the organization of which the concept in question was created, and how it has extended to other phenomena.

The description of the relations between the phenomena and the concept takes into consideration, in the first case, the relations that are established, and in the other three how those relations were brought about, acquired or formed, in the educational system, with respect to cognitive development or in history, respectively.

Moreover, in the case of pure phenomenology the concepts or mathematical structures are treated as cognitive *products*, whereas in the case of didactical phenomenology they are treated as cognitive *processes*, i.e., situated in the educational system as teaching material and being learned by students. Freudenthal says that when writing a didactical phenomenology one may think that it should be based on a genetic phenomenology, but this idea is mistaken. The order in which the various types of phenomenological analysis must be used begins with pure phenomenology (for which it is sufficient to know mathematics and its applications); this is completed with a historical phenomenology, followed by a didactical phenomenology (for which it is necessary to know the teaching and learning process), and in all cases genetic phenomenology comes last. No phenomenological analysis can be effective when teaching is subsequently organized on the basis of it if it is not supported by a sound analysis of pure phenomenology.

Freudenthal's phenomenological analysis aims to serve as a basis for the organization of the teaching of mathematics and does not set out to elaborate an explanation of the nature of mathematics. It might be possible to use it without adopting any epistemological or ontological commitment about mathematics, that is, accepting mathematics as a means for the organization of phenomena, without maintaining that things really are so. However, the ideas that students form about the nature of mathematics and the ideas that teachers have exert a very considerable influence on how both students and teachers conceive the mathematical activity that has to be performed in class, and the knowledge that students produce and that teachers try to teach. This is also why we think it necessary to outline a conception of the nature of mathematics that is compatible with the interpretation that we make of Freudenthal's phenomenological analysis.

We set out, therefore, from the statement that mathematical concepts are means of organization for phenomena of the world. However, this characterization does not tell us much if we do not specify to what we are referring when we speak of the world, and if we do not establish which phenomena are organized by mathematical concepts. Nevertheless, one of the tasks of phenomenology is precisely to investigate which phenomena are organized by mathematical concepts, by analyzing those concepts, so that one cannot seek to know in advance which they are. Nor can one seek to characterize in advance the *kind* of phenomena organized by mathematics, because to do so one would need to have linked the phenomenology of mathematics to a general phenomenology in which one establishes a typology of phenomena — a task that, in our view, could be approached by means of Peirce's phenolmenology. Consequently, we can have an idea of the kind of phenomena involved only on the basis of the concrete analyses that we perform.

On the other hand, it is possible to interpret that from the foregoing statement it follows that mathematics lies in a separate world from the world whose phenomena it organizes, which is the world around us, the real world. This, however, is not the most appropriate interpretation.

In fact, if we place ourselves at the origin, or at the lowest level, we could say that the phenomena that are going to be organized by mathematical concepts are phenomena of this real, physical, everyday world. Our experiences with this physical world have to do with the objects of the world, their properties, the actions that we perform on them, and the properties that those actions have. Hence the phenomena that mathematics is to organize are the objects of the world, their properties, the actions that we perform on them or the properties of those actions, when objects, properties, actions, or properties of actions are seen as what is organized by those means of organisation and are considered in their relation to them.

This first interpretation establishes the idea that mathematical concepts do not actually reside in an ideal world whose reflection we study, nor do they have an existence prior to mathematical activity, nor does that activity consist, therefore, in the discovery of the geography of the world in which those objects are. Yet they are also not installed in a world foreign to our experience, inasmuch as they are created as a means of organization of phenomena of the world. The previous interpretation is not felicitous in this respect, because it does not take into account the fact that Freudenthal does not remain at the lowest level, describing mathematical activity simply as an interplay between phenomena of the world and means of organization in mathematics, in which phenomena seek to be organized and means for this are created in mathematics. On the contrary, Freudenthal accompanies the process of creation of mathematical objects as means of organization with a process by which the means of organization become objects that are situated in a field of phenomena. Consequently, mathematical objects are incorporated into the world of our experience, which they enter as phenomena in a new relation of phenomena/means of organization in which new mathematical concepts are created, and this process is repeated again and again.

Mathematics is therefore in the same world as the phenomena that it organizes: there are not two worlds but one, which grows with each product of mathematical activity. The phenomena that mathematical concepts organize are the phenomena of the world that contains the products of human cognition and, particularly, the products of mathematical activity itself; the phenomena that are organized by mathematical concepts are the objects of that world, their properties, the actions that we perform on them, and the properties of those actions, inasmuch as they are contained in the first term of a phenomena/means of organisation pair.

The staggered progression of phenomena/means of organization pairs entails two processes: the process of creation of mathematical concepts as means of organization, which is indicated by each pair, and the process by which a means of organization is objectified in such a way that it can become part of a new pair, this time in the position of phenomena. The staggered progression draws a picture of the production of more abstract mathematical objects on an ever higher level, and it shows that mathematical activity generates its own content.

4.2. Constitution of mental objects versus acquisition of concepts

We speak of mathematical concepts, of their creation in a relation of phenomena/means of organization, of the objectification of the means of organization and their entry into a phenomena/means of organization relation on a higher level; we speak of transformations of concepts as a consequence of the mathematical activities of proving theorems, solving problems, organizing in a deductive system and the process of defining. All this is accompanied by the affirmation that mathematical concepts do not have an existence independent from the mathematical activity that creates them. But we also bring into the arena a new idea developed by Freudenthal that will oblige us to rethink the relations that concepts establish in these ladders of concepts/means of organization: this is the idea of a mental object as opposed to a concept.

This idea is important primarily because it is on the basis of it that Freudenthal adopts a didactic attitude: the aim of educational activity in the school system must basically be the constitution of mental objects, and only secondarily the acquisition of concepts —which is in second place in terms of both time and order of importance. This attitude is also particularly important for the period of compulsory education, because one must consider what part of mathematics must be offered in it to the population as a whole. But it is also important for the phenomenological analysis of mathematical concepts, all the more so if the analysis is a didactical phenomenology and one has in mind the idea that the analysis is prior to the organization of teaching and is performed with that purpose. This is the aspect that we deal with here.

In a first approach, the contrast between mental object and concept that Freudenthal proposes can be seen as the consequence of considering the people who conceive or use mathematics in contrast to mathematics as a discipline or set of historically, socially, and culturally established knowledge. In the foregoing sections, when speaking of mathematical concepts we have considered them basically within the discipline, and we have hardly introduced the intervention of real people; what has appeared is, at best, a semblance of them, the ideal subject who performs actions with powers superior to those that we possess. We can set out, therefore, from an initial image: the contrast of mental object and concept is a contrast between what is in people's heads (mental objects) and what is in mathematics as a discipline (concepts).

As this is the sense in which Freudenthal uses these terms and in which we are going to use them here, it is worth pointing out before we go on that the term "mental object" does not appear in normal usage. The customary practice is to speak of the concept that someone has —of number or triangle or anything else, whether it belongs to mathematics or not— or to use the term "conception" instead of "concept" and speak of the conception that someone has of circumference, for example; but in this case one generally wishes to emphasize that what is in the person's mind is part of a concept or a way of seeing that concept.

4.3. Considerations for an LTM for studying the uses of natural numbers

Peirce also speaks of a certain progression in the types of signs we treated in Chapter 1:

A regular progression of one, two, three may be remarked in the three orders of signs, Icon, Index, Symbol. The Icon has no dynamical connection with the object it represents; it simply happens that its qualities resemble those of that object, and excite analogous sensations in the mind for which it is a likeness. But it really stands unconnected with them. The index is physically connected with its object; they form an organic pair, but the interpreting mind has nothing to do with this connection, except remarking it, after it is established. The symbol is connected with its object by virtue of the idea of symbol-using mind, without which no such connection would exist. (Peirce, *CP*, 2.299, pp. 168-169.)

4.3.1. The first arithmetic signs

It seems that the first written signs were arithmetic signs. Let us look at some of the characteristics of signs that we have just expounded at work in those primitive signs.

It has actually been determined that the first written signs were arithmetic signs as a result of a step-by-step reconstruction of the development of two systems of writing that had their beginning in about 3500 BC and that were created by Sumerians in the south of Mesopotamia and by Elamites in Susa (located in what is now Iran).[1]

These first signs were marked with a stylus on the outside of hollow balls of soft clay, and they always corresponded both in form and number to pebbles of various shapes contained inside the balls. These marks were thus icons that represented the hidden pebbles, and one had only to break the ball if one wished to confirm that they really did stand for the objects that they represented. The marks on the balls are icons because they resemble in form and number the objects they represent, so that they signify even if the balls are empty. These signs have what we might call a primitive way of working, because the code that the person who closes up the balls and makes marks on them has to share with the person into whose hands they come is not very well established socially, or, at any rate, is subject to doubt.

Interesting as these first written marks are in so patently possessing two of the natural characteristics of signs, they become even more interesting when we discover that they have the antecedents and consequents that will now be explained.

The marked balls that have been found in the excavations are from the second stage of this temporal series. Before that stage, the remains correspond to hollow balls containing pebbles but without any external mark. After the second stage the pebbles disappear and only the marks remain, and the hollow balls, which no longer have to contain anything, become flat tablets.

First of all, therefore, there are objects hidden in a hollow ball, then the first written signs, with the objects that they represent present but hidden, and finally only the written signs without the objects that they represent.

But these objects, in turn, are signs —although not signs belonging to a system of writing— because, in each of the three historical stages, the balls or tablets are records or trading transactions; they are accounts. The objects represented by the written marks are also arithmetic signs, because, by their shape and quantity, they represent a certain number of objects. What the archaeologists have reconstructed tells us that these pebbles were used to record an account in the course of a commercial transaction, and once the matter was settled they were placed inside a hollow ball to record the agreement between the trading parties concerning the quantity involved in the transaction. These first arithmetic signs stood for other arithmetic signs that had a different medium of expression and they eventually replaced them in the records, but only in the records, because if the traders probably continued using pebbles to do their accounts, they had no operational capability.

These mathematical signs on clay tablets led to the development of Sumerian cuneiform writing. We know that later, in the palaeo-Babylonian era (2000 to 1600 BC), genuine mathematical texts were written on tablets similar to these primitive specimens (and not only in Sumerian but also in Akkadian, a Semitic language), but that is another story, which we will not go into here.[2]

4.3.2. The signs used in the Roman number system

Although the arithmetic signs that are at the origin of cuneiform writing fell into disuse thousands of years ago, the Etruscan herdsmen, far from commercial transactions and the schools of scribes in the fertile crescent, by making notches on a stick, one for each head counted, created a number system that we still use, albeit only marginally: the one known as the Roman number system.

The signs that we have inherited from them for the representation of numbers actually seem to have developed as a result of their physical inscription on a linear record. Thus the primitive repetition of notches, |||||||…, became structured by means of special marks every five notches, with a view to making it easier to count the expression: a slanting mark in the fifth position, a cross-shaped mark in the tenth position, etc., giving rise to marks such as ||||/||||X||||/||||X||| to record a herd of twenty-three animals. The primary marks and structural marks eventually became the alphabetical letters I, V, and X, becoming integrated into the system of writing and identified with the letters that they most resembled.

As they were positions in a series, V and X did not signify the cardinal numbers "five" and "ten," but the fifth and tenth positions in the series. In fact, the first written forms for "five" and "ten" were not V and X but IIIIV and IIIIVIIIIX, which do indeed represent cardinal numbers, and in which both I and V represent a unit. It was only later that considerations of economy led to the use of V to represent IIIIV and thus five units. The signs V and X initially functioned as reference points in the series in yet another sense: IV came to signify "four," not as the result of a rule of subtraction between the cardinal numbers designated by I and V, but because from the presence of the sign V one could understand that the mark immediately before V in the series was being designated. Similarly, VI did not come to signify "six" as the result of a rule of addition, but because it designated the mark immediately after V. It was only when the signs V and X acquired a cardinal meaning —standing for IIIIV and IIIIVIIIIX— that the earlier rules, which had to do with positions in a series, i.e., with …IV… or …VI…, were reinterpreted as rules of addition and subtraction between cardinals. In this historical account, the transformations that took place in the expression as a result of the processes of abbreviation gave new senses both to the elementary signs and to the rules for the formation of compound signs, senses that correspond to the meanings now taught in schools.

These marks are indices of the action of counting. Puig (1997) points out that the phenomena that organize mathematical concepts are objects, properties, actions, and properties of actions. This is one of the clearest examples of a mathematical concept that organizes a phenomenon that does not belong to the domain of objects or properties of objects, but to the domain of actions and properties of actions (which does not do away with the fact that in the corresponding triadic relation the action of counting is the object of the sign for a mind, that is, for an interpretant). As a result of transformations of the expression, the indices become symbols.

4.3.3. Algebraic expressions

It is common to refer to algebraic expressions as "symbolic language" —for example, when one speaks of putting a problem into equations, one usually describes this as a "transition from natural language to symbolic language." However, if we use Peirce's terminology, algebraic expressions are not symbols but icons, strange as this may seem at first sight. Let us see how Peirce himself explains it:

> […] thus, an algebraic formula is an icon, rendered such by the rules of commutation, association, and distribution of the symbols. It may seem at first glance that it is an arbitrary

classification to call an algebraic expression an icon; that it might as well, or better, be regarded as a compound conventional sign [symbol]. But it is not so. Because a great distinguishing property of the icon is that by direct observation of it other truths concerning its object can be discovered than those which suffice to determine its construction. […] This capacity of revealing unexpected truth is precisely that wherein the utility of algebraic formulae consists, so that the iconic character is the prevailing one. (Peirce, *CP*, 2.279, p. 158.)

Algebraic expressions are icons, and this is precisely what makes them powerful, because as signs they have the properties that their objects have. However, the letters in algebraic expressions, taken in isolation, are not icons but indices, each letter being an index of a quantity. They are also not symbols. If the algebraic expression is the result of the translation of the verbal statement of an arithmetic-algebraic problem, each specific letter represents a specific quantity as a result of the convention established by the person who produced the translation, but each letter refers to a quantity even if there is no interpretant, because any interpretant who is not aware of the convention established will assign the letters to the right quantities, since the algebraic expression as a whole will require that the corresponding quantity be assigned to each letter. So are there no symbols in algebraic expressions? Yes, there are. The signs +, =, etc. are symbols in Peirce's sense.

Algebraic expressions are thus an example of the imbrication of three kinds of signs in mathematical writing: the letters are *indices*; the signs +, =, etc. are *symbols*; and the expression taken as a whole is an *icon*.

4.3.4. Uses of numbers in different contexts

The students in whom teachers attempted to instil the concept of number in the years of what was known as "modern mathematics" —in a school version of Cantor's construction of cardinals— would have left school without being able to count if they had not created a mental object of number apart from what the official syllabuses wished them to be taught. We will use this complex, multiple concept as an example to show the difference between mental object and concept, describing it in semiotic terms instead of as Freudenthal does.

If we consider the ordinary activity of people and not just the mathematical activities of mathematicians or the scholastic activities of students in mathematics classes, the use of number, or rather numbers, appears in very diverse contexts. A list of them might include the contexts of sequence, counting, cardinal, ordinal, measurement, label, written numeral, magic, and calculation. A description of the characteristics of each context is not our purpose here: the list is worth mentioning solely in order to show that it is possible to distinguish a considerable quantity of contexts. Following Wittgenstein for a moment, we understand meaning as being constituted by the use that one makes of a term, that use not being an arbitrary use, the product of what someone takes it into his or her head to do with the term in question, but a practice subject to rules.

The uses of numbers in each of these contexts follow rules. For example, when one says “My telephone number is three, eight, six, four, five, eight, six,” the number refers to an object and does not describe any property of it or its relation to other objects but serves to identify it. This is the context of label, and in it, when the expression is oral, the digits that make up the number are generally expressed separately, as in the example given. In an ordinal context, the number refers to an object that is in an ordered set of objects, and it describes what place it occupies—“he came third” or “he’s the one that makes three.” In a cardinal context, the number refers to a set of objects (without order, or whose order is not taken into consideration), and it describes the numerousness of the set— “there are three.” And so on.

The totality of the uses of numbers in all contexts constitutes the *semantic field* of “number,” the encyclopedic meaning of “number.” The identification of the context in which number is being used enables someone who is reading a text or receiving a message to abide by the *semantic restriction* that the context establishes and thus interpret it appropriately. However, the person who reads a text or has to interpret a message does not operate in the whole encyclopedia —i.e., the totality of the uses produced in a culture or an episteme— but in his personal semantic field, which he has gradually built up by producing sense —senses that becomes meanings if the interpretation is felicitous— in situations or contexts that demanded of him new uses for “number” or numbers.

In this semiotic description, what Freudenthal calls “mental object ‘number’” corresponds to this “personal semantic field.” Freudenthal’s didactic attitude in favor of the constitution of mental objects means that the aim of educational systems, expressed in the terms that we are using, should be that the student’s personal semantic field should be sufficiently rich —should embrace the encyclopedia sufficiently— to enable him to interpret appropriately all the situations in which it proves necessary to use “number” or numbers.

The contexts of the ordinary use of numbers are the various places in which we can experience the phenomena that have been organized by means of the concept of number, both the phenomena for which it was originally created and those to which it has now been extended. The idea of mental object that we have just introduced must also be seen, therefore, as a means of organisation of phenomena: with the mental object “number” people are able to count, among other things. Mental objects are constituted in chains of phenomena/means of organization, in the same way as with concepts, with the consequent increase in level —in fact, the contexts of the ordinary use of numbers that we have mentioned are situated on the lowest levels, and to realise the phenomenological richness of number in secondary school one must take other contexts into consideration, including contexts that have already been mathematized.

4.4. Relation between mental object and concept

This is an initial explanation of what a mental object is and how it is constituted, but what Freudenthal calls mental object could simply have been called the concept that a person has of number. To justify the introduction of a term that distinguishes it, it is necessary to explain for what other thing the term "concept" has been reserved, and how it differs from what we have just called "mental object." We have already said that the first distinction is that mental objects are in people's minds and concepts are in mathematics. But this would hardly be sufficient reason to oppose mental object to concept if we thought that the mental object is the reflection of the concept in people's minds. The relation between mental object and concept, however, is not a mirror-like relation. Once again we will explain it in semiotic terms.

We have identified the mental object "number" with the personal semantic field, which comes from all the uses of numbers in all the contexts in which they are used, from a semantic field consisting of all the culturally established meanings. The mathematical concepts of natural number —and we use the plural in order to emphasize the fact that we consider the concepts developed by Peano, Cantor and Benacerraf, for example, as different— in the form in which they exist in current mathematics are the product of a long history, with processes of creation and modification of concepts. In terms of the semiotic description that we are using now, any mathematical concept of number that one wishes to examine once it has been created appears as the result of the process of defining that has incorporated it into a system organized deductively as a *narrowing of the semantic field*. Thus, for example, the concept of natural number developed by Peano —especially in its more modern versions— can be seen as the breaking down of the meaning that pertains to the context of sequence and its presentation in the form of a series of axioms that give an exhaustive account of its components. The concept of natural number that is derived from Cantor's construction, on the other hand, is ascribed, in the very name that Cantor gave in his original intention, to the cardinal context.

In this explanation, concepts appear to be directly related to a part of the mental object, given that, in the process of defining, part of the meaning that the mental object embraces is selected. We will immediately point out that this is not the only difference, and that we do not wish to give the impression that the relation between mental object and concept is a relation between a part of the content of the mental object and the totality of its content. But we wish, rather, to indicate that what this explanation establishes provides a foundation for the attitude taken by Freudenthal that we have mentioned: the acquisition of the concept is a secondary school objective and can be left until after mental objects have been soundly constituted, and in any case it does come afterwards.

The relation between mental object and concept is more complex than is shown by the explanation that we have just given using the example of number, because the explanation was limited to comparing the deployment of the semantic field of number and Peano's definition, as if there were not centuries of history that have produced both contexts of use —which we are now going to find with traces of their organization by concepts of number— and Peano's definition. Taking into account the processes of creation and modification of concepts that are present in that history, the relation between the mental object that can be constituted from the contexts mentioned and the content of the concept of number created by Peano's definition cannot be reduced to a relation between part and whole.

Constituting a mental object implies being able to give an account with it of all the uses in all the contexts or being able to organize all the corresponding phenomena, in which case the mental object is well constituted. The aim of educational systems that Freudenthal indicates is this constitution of good mental objects. Acquiring the concept implies examining how it was established in mathematics organized locally or globally in a deductive system. The particular relation that each mathematical concept has to the corresponding mental object determines how the constitution of the mental object relates to the acquisition of the concept. The constituents of the good mental object are determined by means of the phenomenological analysis of the corresponding concept.

4.5. From phenomena to mental objects and concepts through teaching

The relation between mental objects and concepts is varied. Both are constituted as means of organization of phenomena, mental objects precede concepts, and concepts do not replace mental objects but contribute to the formation of new mental objects that contain them or with which they are compatible.

The distance between the mental object, or rather the first mental object, and the concept can be an abyss: this is the case with the mental object "curve" and Jordan's concept of curve, for example. In general, in topology mental objects do not lead very far, and it is necessary to form concepts, by means of a formation of concepts that involves more than a local organization. These concepts enter a field of phenomena that are organized on a higher level by mental objects such as spaces and varieties of arbitrary dimension, which in turn are converted into concepts by means of new processes of organization and the creation of more abstract sign systems to describe them. As this example shows, by introducing the idea of mental object the process of a progressive rise through the chain of phenomena/means of organization pairs links up with a process of transformation of mental objects into concepts.

The analyses of didactical phenomenology must be based on analyses of pure phenomenology, bearing in mind that, in many more cases than one might imagine, the distance between the mental object and the concept is so great that it is not possible to build bridges between them by didactic means in secondary school.

For the constitution of mental objects through teaching while bearing concepts in mind, the distance between them and the various forms that this distance adopts are therefore of importance. It is worth mentioning a few cases, such as those that are set out in the following paragraphs.

Sometimes there are components that are essential for the formation of the concept but are not pertinent for the constitution of the mental object. This is the case with the cardinal number: the comparison of sets without structure is essential for the concept, but it plays almost no part in the constitution of the mental object because, in the real situations in which a person experiences the phenomenon that is organized with the mental object "number" in its cardinal sense, the sets of objects are rarely without structure, and, moreover, the structure is a means for making the comparison, rather than something that must be removed in order to make it.

Sometimes, what a didactical phenomenology shows is that the phenol-mena organized by the concept are so varied that in fact different mental objects are constituted, depending on the field of phenomena that is selected for exploration in teaching, or several mental objects if several kinds of phenomena are explored. For the acquisition of the concept it is necessary, therefore, to integrate these different mental objects into a single mental object. This is the case with the concept of area, for example.

Indeed, lengths, areas, and volumes are the magnitudes that are measured in elementary geometry. It is therefore necessary that these concepts should be acquired as part of the learning of measurement and measuring. The comparison between qualities of objects is the beginning of the activity of measuring. This becomes measurement through the intermediary of the establishment of a unit and consideration of objects that are treated as objects of which one can predicate that quality —for example, one can predicate that they have length if it makes sense to say of them that they are "long."

However, as concepts, length, area, and volume are problematic because of the variety of approaches for the constitution of the mental object "area" (or "volume"). Indeed, plane figures can be compared with respect to area directly, if one is part of the other, or indirectly, after transformation by cutting and pasting, congruences, and other applications that preserve area; or else by measuring both of them. The measuring can be done by covering the figure with units of area, or by means of interior and exterior approximations; for this one uses the additivity of the area beneath the composition of plane figures that are mutually disjoint except for their boundaries (of dimension one), or convergence of the areas by approximation. It is not clear that these approaches lead to the same result, and in fact the proof that the result of measuring by following all these procedures is the same is not simple. The constitution of the mental object "volume" also has the additional complication of considering phenomena corresponding to capacity, which are usually measured with different units.

Sometimes it is difficult even to distinguish the mental object from the concept, at least if one wishes to have a unitary mental object: only by means of access to the concept is it possible to unify a heterogeneous set of mental objects. This is the case with the concept of function.

Finally, there are mental objects whose field of phenomena appears only in a mathematical or mathematized context. An example of this in secondary school is provided by the concepts of analytical geometry.

Indeed, in history, global location by using coordinates leads to the algebrization of geometry. Whereas the system of polar coordinates used to describe the sky and the Earth's surface has served to systematize location, the system of Cartesian coordinates is particularly efficacious for describing geometric figures and mechanical movements and, later, functions in general. A figure can be translated algebraically into a relation between coordinates, a movement in a function that depends on time, and a geometric application in a system of functions of a certain number of variables.

The phenomena that are proper to analytical geometry are thus phenomena produced by the expression of geometric properties in the complex sign system in which algebraic expressions and Cartesian representation refer to one another. They are, therefore, phenomena that can be explored only in contexts previously mathematized by the use of those sign systems.

4.6. Concepts generated by proving

We have seen that mathematical concepts are created in the phenomena/-means of organization process, but this does not mean that once created they remain immutable. On the contrary, mathematical concepts alter in history as a result of their use and the new MSSs in which they are described. This does not imply, however, that alterations in a concept indicate that the original concept was mistaken and that we have to see the history of mathematical concepts as an advance toward truth, for we have rejected the view that mathematical objects have an existence prior to the process that creates them.

A different idea of the evolution of concepts in history was developed by Lakatos in his book *Proofs and Refutations* (Lakatos, 1976). What is of interest for us here is the fact that in this book Lakatos examines how concepts evolve under the pressure of the proof of theorems in which they are involved.

Lakatos tells that, after the establishment of the conjecture that for any polyhedron the relation $C + V = A + 2$ is true, and after its proof by Euler, examples of solids emerged that did not fit in with the proof that had been performed or, what was more important, with the theorem that had been proved. In terms of a conception of the nature of mathematical objects according to which there is a pre-existing ideal object that we call polyhedron and what mathematical activity does is to discover its properties, the matter is quite clear: these solids are not true polyhedrons, or else the proof is wrong. The reconstruction of history that Lakatos makes is not this.

Lakatos separates the two types of counter-examples that I have just mentioned and calls them local and global counter-examples, respectively. A local counter-example is one that has characteristics that cause the proof not to be applicable to it, but that verifies the relation. These counter-examples do not refute the conjecture: what they do is to indicate that in the proof a property was used that was assumed to be valid for all polyhedrons, but it is not so. What is refuted, therefore, is a lemma that has been used implicitly, and therefore the proof. The presence of these counter-examples introduces a difference in the concepts that was not present before.

The effects of the appearance of global counter-examples have more importance for what we are examining. A counter-example is global when it refutes the conjecture. As first global counter-examples of the theorem proposed by Euler, Lakatos presents the solid that consists of a cube with a cube-shaped hollow inside it, and a solid formed by two tetrahedrons joined by one edge or one corner; later he presents the even more interesting case of a star-shaped solid, which does or does not verify the relation depending whether or not one considers that its faces are star-shaped polygons. The presence of these solids as counter-examples produces a tension between the concept, the theorem, and its proof. This tension can be resolved in various ways, which all affect the concept of polyhedron. The most elementary are:

1) Monster-barring.

The counter-examples presented are considered to be not genuine examples of the concept of polyhedron but monsters, i.e., beings whose existence is possible but not desired. The possibility of their existence is determined by the definition of polyhedron that is being used, whether explicitly or implicitly, so that, in order to preserve the theorem, a new definition of the concept of polyhedron that explicitly excludes them is produced.

2) Exception-barring.

The counter-examples presented are considered to be examples of the concept whose existence had not been foreseen when the conjecture was stated. The conjecture is modified with the intention of withdrawing to safe ground. To do so, a difference that separates these examples is introduced in the concept.

3) Monster-adjustment.

The objects are looked at in a different way so that they cease to be counter-examples; this is the case with the two ways of looking at star-shaped polyhedrons: as being composed of star-shaped polygons or not.

Although these are only the most elementary ways of confronting the tension created, even with them we can see that the concept of polyhedron is affected in all cases. Whether the counter-examples are accepted or excluded as examples of the concept, the semantic field is expanded. In one case, because the content of the expression increases, or, to put it differently, because the field of phenomena for which the concept had been created — which is what constitutes its semantic field— did not contain the phenomena corresponding to the objects and properties that are now present, and it is extended to include them. In the other case, because the concept enters into an interplay of relations to these new objects from which it explicitly disassociates itself in the new definition, which also form a constitutive part of its content.

The full story is more complex, and it also features progressively richer and more abstract mathematical sign systems to which the concepts initially expressed in other, less rich or less abstract mathematical sign systems are translated, and it leads Lakatos to state that the concepts generated by the proof do not improve the original concepts, they are not specifications or generalisations of them, but they convert them into something totally different, they create new concepts. This is precisely what we wish to emphasize: the result of the process that Lakatos presents, a process of tension between concepts, theorems, and proofs, is not the delimitation of the true concept of polyhedron that supposedly corresponds to the pre-existing ideal object, but the creation of new concepts.

4.7. Problem solving, defining, and other processes that also generate concepts

From Lakatos we have just extracted the idea that mathematical concepts do not remain immutable once created. We have also outlined how concepts change, impelled by the tension produced in them by their application in proofs and refutations. However, mathematical activity does not consist only in proving theorems. One of the fundamental driving forces in the development of mathematics is problem solving, and this includes the proving of theorems, but also other activities.

Problem solving includes the proving of theorems in two senses. In the first sense, problem solving includes the proving of theorems considered globally, because, if we follow the terminology of Polya (1957) and, instead of distinguishing between problems and theorems as was first done by Greek mathematicians, we call them all problems and distinguish between "problems to find" and "problems to prove," then the proving of theorems is simply one kind of problem solving: the solving of problems to prove.

In the second and more important sense, problem solving includes the proving of theorems in the solving of each problem in particular; indeed, what characterizes problem solving in mathematics, even with problems to find, is the fact that the obtaining of the result must be accompanied by an argument that substantiates the fact that the result obtained verifies the conditions of the problem, i.e., any problem is a problem to prove or, if it is a problem to find, it contains a problem to prove —the problem to prove that the result found verifies the conditions of the statement.

This obliges us to extend the terrain in which concepts are submitted to a tension that modifies them beyond the proving of theorems to the solving of problems. But it becomes even more necessary to do so if we take into consideration other parts of problem solving that do not involve the proving of theorems —specifically, the proposal of new problems or the study of families of problems.

Problem solving also does not exhaust the field of mathematical activities or the field of mathematical activities that generate concepts. Other activities that are responsible for the creation of many great mathematical concepts in the form in which we know them now have to do with the *organization* of sets of results of varying extent —obtained in the activity of solving problems and proving theorems— in a *deductive system*. This systematic organization has adopted different forms in the course of history, and it may be more local or more global, more or less axiomatic or formalized, but in any case it has constituted an essential component of mathematics since mathematicians moved from accumulating results and techniques for obtaining them to writing "elements." Indeed, although we do not detail that set of activities here, one essential characteristic of it is that it has transformed the sense in which definitions are used in mathematics. "In mathematics a definition does not serve just to explain to people what is meant by a certain word," as Freudenthal says, but rather, when we consider the mathematical activities by means of which deductive systems are organized, "definitions are *links in deductive chains.*"

The process of defining is, therefore, a means of deductive organization of the properties of a mathematical object, which brings into the foreground the properties that are deemed to make it possible to constitute a mathematical system, local or global, in which that mathematical object is incorporated. However, emphasizing certain properties such as those that define a concept is not an innocent operation, a neutral operation with respect to the concept, because, on the one hand, it makes the concept appear as originally created to organize the corresponding phenomena, and, on the other, it makes the content of the concept be, from then on, what is derived from that definition in the deductive system in which it has been incorporated. Therefore, this process of defining also creates new concepts, just as proving theorems do.

SUMMARY

In this chapter we have presented the phenomenological analysis (based on the work of Hans Freudenthal) as an approach to curricular development for teachers, students, and researchers. The content is basic for the remainder of the chapters since it deals essentially with establishing the difference between acquiring concepts and building mental objects in mathematics, as well as how one goes from phenomena to mental objects and to concepts through teaching. The ideas are illustrated through the case of uses of natural numbers. We also refer to the work of Lakatos "Proofs and Refutations" in order to make evident that *tests*, *definitions*, and *problem solving* are concept generators.

We have also dealt with the concepts of *mathematical sign systems* and *local theoretical model,* thus adding further to their introduction in Chapter 1. Dealing with these concepts has enabled us to refer to the phenomenological analysis as a framework for developing teaching models, as components of a local theoretical model.

In the next chapter, we deal with the methodological aspect of LTMs, and we present both an historical and phenomenological analysis of school algebra.

ENDNOTES

[1] This reconstruction is recounted in full detail in Ifrah (1994), vol. I, pp. 233–263. See also Schmandt-Besserat (1992) and Glassner (2000).

[2] Although it would be worth doing so. In the texts of problems that appear on tablets written in Akkadian, the words "long" and "wide" are in Sumerian and are used to designate unknown quantities, even though the problem is not geometric. One can imagine that the strangeness of the signs of another language in a text written in Akkadian favored the use of those signs as what Høyrup (2002a) calls "a functionally abstract representation." Indeed, although "long" and "wide" continue to retain the original geometric meaning, the sense that they have is no more than that of two quantities that can form part of a calculation – that is, these words are precursors of the objects of algebra.