CHAPTER 10

WIDENING PERSPECTIVES

OVERVIEW

In this chapter we put forward points of interest from the perspective of future research, in seven sections: (1) The history of algebraic ideas. (2) The dialectics of syntax and semantics in the study of the generation of errors and in new points of interest concerning the representation of the unknown, based on something that is also unknown. This leads us to analyze the usual treatments for teaching systems of equations and to propose new ways of teaching them based on the use of trial and error, canceling, comparison, and substitution. (3) The study of the new problems posed by information and communication technology (ICT) by analyzing the new roles in the classroom or in the communication between learner and teacher. (4) An analysis of the MSS of Jordanus de Nemore's *De Numeris Datis*, which makes it possible (a) to propose teaching models for second-degree equations; (b) to study the reason for the greater difficulty in the use of factorization compared with the learning of algebraic identities; and (c) to study the difficulties in using an algorithm as a subroutine in another more complex algorithm. (5) Early algebra, a rich new field for research with a huge past of research results. (6) Investigation of theoretical questions that involve the use of research into the instruction imparted by teachers. And (7), in the interests of (6), the need to develop the theoretical aspects of the communication model for the classroom using ICT.

1. HISTORICAL ANALYSIS OF ALGEBRAIC IDEAS

In Chapter 3 we showed the part played by the analysis of the history of algebra in our research, and consequently what kind of historical analysis is of interest to us.

Without repeating all that was said there, our use of history has two fundamental features. On the one hand, it is concerned with an analysis of algebraic ideas. As a result, there is very little interest for us in, for example, questions of dating or of priority in developing the concepts of algebra. What we are basically interested in is identifying the algebraic ideas that are brought into play in a specific text and the evolution of those ideas, which can be seen by comparing texts; in this context we can consider historical texts as cognitions and analyze them as we analyze the performance of pupils, whose productions also constitute mathematical texts (provided that one uses a notion of text such as the one we presented in Chapter 5).

The second feature creates a close bond between historical research and research into mathematics education, which allows us to state that our historical research belongs to research into mathematics education, and is characterized by a two-way movement between historical texts and school systems:

1) The problematics of the teaching and learning of algebra is what determines which texts must be sought out in history and what questions should be addressed to them.
2) The examination of historical texts leads to (a) considering new items that have to form part of the model of competence, (b) having new ways of understanding the performance of pupils and, therefore, of developing the model of cognition, and, lastly, (c) developing teaching models. With all these things teaching experiments are organized and the performance of students is analyzed.
3) Attention is redirected to the historical texts in order to question them once again, now using the results obtained with students, that is, the results derived from the performance of students when all that has been extracted from the analysis of algebraic ideas is incorporated into the teaching model and the analysis of the teaching and learning processes.
4) And so on, repeatedly.

1.1. Current and future research

In this section we indicate some of the directions in which we are turning to history as a result of issues present in the current problematics of research in educational algebra, and another one is mentioned especially in Section 3.

The use of spreadsheets to teach how to solve problems that students were traditionally taught to solve by using some version of the Cartesian method, whether with the aim of serving as an intermediary for the teaching of the Cartesian method or with a view to replacing it with a new method, poses the question of the ways of naming unknown quantities.

In fact, in a spreadsheet one can refer to unknown quantities by assigning one or more cells for one or more unknown quantities, like the way in which one assigns one or more letters to one or more unknown quantities in the second step of the Cartesian Method, and it is advisable to place the cell beneath a cell in which the name of the quantity is written in the vernacular, either complete or else abbreviated in some way. The relationships between the quantities can then be represented as operations expressed in spreadsheet language. In that language the cells are designated by a matrix code consisting of a letter and a number, e.g., B3, which indicates the column and row of the cell, and the cell name then becomes the name of the corresponding quantity in the spreadsheet language. The cell name can be written explicitly using the keyboard or generated by clicking on it with the mouse.

When the spreadsheet is used to solve word problems its language has considerably more complexity than what we have just described, but this brief outline shows that the way in which the unknown is named in this language is different from that of the language of modern school algebra. This has consequences for the use of the spreadsheet in teaching how to solve problems and in the performance of pupils when they are taught in this way. It is therefore worth turning now to historical texts in which the language of modern school algebra had not yet been developed, in order to see how unknown quantities are named in them and what effects the way of naming them has on the way of representing the relationships between quantities that are translated from the statement of the problem and the relationships generated in the course of the calculations.

Therefore, both in the case of the use of the spreadsheet and in that of the teaching of problem solving by using the Cartesian Method, at some point it is necessary to teach the use of more than one letter (or cell) to represent unknown quantities, together with the corresponding way of handling expressions in the corresponding languages. As we pointed out in Chapter 3 and in Puig and Rojano (2004), most of the languages of algebra prior to Viète were incapable of this, and yet problems were solved that we would now naturally represent by using more than one letter. It would be interesting, therefore, to turn to the historical texts once again in order to examine the ways in which this was done.

Finally, the use of graphic calculators, with the possibility of collecting data with calculator-based ranger (CBR) or calculator-based laboratory (CBL) sensors, to teach the idea of family of functions as a means for organizing phenomena through the modeling of real situations poses the problems of the establishment of canonical forms in which the parameters express properties of each family of functions and, therefore, of the situations that are modeled. Bound up with this is the development of a calculation, that is, a set of algebraic transformations, which makes it possible to reduce the expressions obtained by modeling the real situation to one of the canonical forms. The history of the idea of canonical form (and that of calculation or algebraic transformations that is bound up with it) thus acquires a new perspective which is worth studying, in addition to the aspect studied in Chapter 2 and in Puig and Rojano (2004).

2. COGNITIVE TENDENCIES AND THE INTERACTION BETWEEN SEMANTICS AND ALGEBRAIC SYNTAX IN THE PRODUCTION OF SYNTACTIC ERRORS

The literature on algebraic errors in the learning of algebra concentrates mainly on their syntactic component (Matz, 1982; Kirshner, 1987; Drouhard, 1992). There are not many books like Booth (1984) and Bell (1996) that place this problematic component in a more general context such as problem solving. In this section we analyze the interaction between semantics and algebraic syntax as a source of syntactic errors when the interaction takes place in a teaching process that uses concrete models. We defend the view that an analysis of this kind provides a perspective that enables us to give explanations that are different from the usual explanations for the presence of certain typical errors of algebraic syntax.

Undertaking a semantic introduction of new algebraic concepts, objects and operations involves selecting a concrete situation (i.e., a situation which is familiar to the learner in some context) in which such objects and operations can be modeled. With this focus it is possible to make use of previous knowledge in order to achieve the acquisition of new knowledge. This is one of the guiding principles of modeling, the strengths and weaknesses of which appear as soon as a specific model is brought into operation (see Chapter 5, for example).

2.1. Different tendencies

In Chapter 4 we introduced the syntactic/semantic opposition with respect to cognition.

The antagonism of these two tendencies (Vt's and Ma's) was evident from mere observation of their respective interviews. However, from a comparative analysis of them, there are a couple of points concerning aspects common to both cases that deserve to be emphasized. One can see, on the one hand, that despite the antagonism just mentioned there is a common tendency to abbreviate the process (with the pupils going their own way to perform the abbreviation in each case); and, on the other hand, a certain number of obstacles and errors are generated during these abbreviation processes that are also common, and that can be considered as typical of the subsequent syntactic handling of symbolic algebra. In one case (Vt), the tendency to abbreviation consists in trying to ease the operations performed in the model, but remaining in it. To do this it is necessary to pay attention to the actions (translation, comparison, etc.) that are performed repeatedly. This reflection leads, in turn, to a process of abbreviation of those actions. It is precisely through this abbreviation that some parts of the concrete model are lost: on the one hand, the "base-line" (the linear dimension that corresponds to the unknown), and, on the other, the area condition of the constant term and the operational handling of it. This leads to a tendency to perform the addition of x with the terms of degree zero, resulting in an aberrant operation between terms of different degrees.

2.2. Syntactic errors

The generation of the same kind of syntactic errors in the two cases that we have just mentioned is not accidental and can be explained by an analysis of the general level, as we did in Chapter 6. When one teaches with models there is a danger that what constitutes the main virtue of any concrete model (i.e., the fact that it seeks the support of previous knowledge) may become the main obstruction for the acquisition of new knowledge. In the cases of the pupils interviewed, who were allowed to develop the use of the geometric model on their own, what happened was that the component of the model which tended to abbreviate, and therefore to conceal, the operation with the unknown persisted in both cases. In cases like Vt's, pupils who possess a strong semantic tendency allow this to happen because the automation of the actions in the model weakens the presence of the unknown throughout the whole procedure. In cases like Ma's, this tendency is due to the effects of the creation of personal codes, created to record intermediate states of the equation proposed originally. The corrections in each case are of a local nature and in accordance with the tendency of each pupil. Thus, when there is an inclination to remain in the model, the correction of the syntactic aberration mentioned earlier is performed in the model itself, because only semantic models can make such an aberration evident. In the case of the syntactic tendency, however, the correction is normally performed together with other events in the syntax, specifically through an essential modification of the notions of equation and unknown.

2.3. New studies needed

By way of conclusion, the interaction between semantics and algebraic syntax, which is presented throughout the processes of abstraction of the operations performed with algebraic objects (which have been endowed with meaning and sense in the context of a concrete model) when learning the language of algebra, is modulated by the tendencies of the individual and by features of the specific model which is being used. Nevertheless, there are some aspects of this interaction that remain constant when there is a change in the tendency of the individual or the type of model. These essential aspects of the relationship between semantics and algebraic syntax in turn reflect essential aspects of another interaction that appears between the two basic components of the model, namely, the reduction to the concrete and the relinquishing of the semantics of the concrete. The transfer of the problem of semantics versus algebraic syntax to the level of actions with the model enables us to close the breach that exists between these two domains of algebra. The analysis of the interaction between semantics and syntax on this new level points to the need to intervene with the teaching model at key moments in the early days of the use of algebraic language.

This dialectics of semantics and syntax and the theoretical description of the relationships between the deep and superficial forms of the syntax of the MSS of algebra, i.e., the generative and transformational aspects of the description of grammar (cf. Kirschner, 1987; Drouhard, 1992), may be linked to the explanation of why mistakes are made when, in order to follow a rule, it is necessary to use one or more rules that previously were used competently.

Future research will clarify this matter. Moreover, in this context it is possible to continue with the study of the solution of systems of equations when solving problems, as we indicated at the end of Chapter 8.

3. JORDANUS NEMORARIUS'S *DE NUMERIS DATIS* AS AN MSS. THE CONSTRUCTION OF A TEACHING MODEL FOR THE SECOND-DEGREE EQUATION AND THE INTRODUCTION OF CERTAIN ALGEBRAIC IDENTITIES

In this book we have proposed a certain kind of reading of the classic texts of the history of mathematics (see Chapter 3 and Section 1 in this chapter).

From that viewpoint we have studied the MSS of Jordanus de Nemore's book *De Numeris Datis* (cf. Puig, 1994, where there is also a detailed description of the propositions of Book I of that work and three propositions from Book IV which are equivalent to al-Khwârizmî's three compound canonical forms), seeking in it the characteristics of the mathematical sign system (MSS) —or mathematical sign systems— in which it is written, the way in which that language configures the objects that one can speak of with it, the problems that it poses and seeks to solve, and the operativity that the MSS has over the objects expressed in it. As we have already indicated in Chapter 6, we conceive the construction of symbolic algebra as the final identification, within a single language stratum, of earlier language strata that are irreducible from one stratum to another until the more abstract language has been developed. From this perspective, the interest in turning to a 13th-century text such as Nemorarius's work, which is prior to Viète's establishment of the language of symbolic algebra, lies in the possibility of taking it as a monument and describing one of the MSSs that are seen retrospectively, from the viewpoint of symbolic algebra, as being less abstract. This interest attaches to research into the didactics of mathematics as soon as one conceives that what students do when they learn symbolic algebra and are taught in educational systems can also[1] be described in terms of the use of MSSs —some of them idiosyncratic— which have to culminate in the competent use of the more abstract MSS of symbolic algebra —or, at least, that is the aim of educational systems.

We will now present an analysis of proposition I-7,[2] which shows how the results of the analysis of *De Numeris Datis* set out in Puig (1994) could be used to construct a teaching model (which includes the use of new technologies) to teach how to solve the second-degree equation, and to study the difference in the difficulty of the use of algebraic identities, on the one hand, and the factorization of algebraic expressions, on the other. To give some idea of the construction of such a teaching model we will also reproduce that proposition from *De Numeris Datis*, interpreted in a figure.

The statement of proposition I-7 is the following:

If we divide a number into two parts, one of which has been given, and the product of the other by itself and by the one that has been given is a given number, the divided number will also have been given.

This proposition is important in the general organization of Nemore's book, as many other propositions reduce to it. On the other hand, if it is translated into the MSS of modern algebra the statement reduces to the equation $x^2 + ux = v$, which is the first of al-Khwârizmî's canonical forms.

This translation is obtained if the analysis of the statement determines the following quantities:

> the number that is divided,
> the part given,
> the part not given,
> the other quantity given (the part not given by itself and by the other given part),

and one then decides to represent "the part not given" with the letter x, and the data with the letters u, "the part given," and v, "the other quantity given," and one constructs the expressions $x + u$, for "the number that is divided," and $x(x + u)$, for the other given quantity, so that one can construct the equation $x(x + u) = v$ (equivalent to al-Khwârizmî's first canonical form).

However, the argument that Jordanus de Nemore develops to prove this proposition goes as follows (the division into parts is ours):

> [1] Let the number be divided into *a* and *b*, *b* given
> [2] Let *a* by itself and by *b*, that is, by all of *ab*, be *d*, given
> [3] Let us also add *c* to *ab*, and let *c* be equal to *a*
> [4] Thus, all of *abc* is divided into *ab* and *c*
> [5] So that *ab* by *c* is *d*, given
> [6] and the difference between *ab* and *c* is *b*, given
> [7] *abc* and *c* will be given, and equally *a* and *ab*.

To understand this argument one must know that Nemore has already proved in proposition I-5 that "if a number is divided into two parts, the product and difference of which have been given, the two parts will also have been given", so that what Nemore does is to reinterpret the quantities of the statement of I-7 so that the situation is that of I-5. Table 10.1 shows the correspondence between the letters that Nemore uses in the argument, the quantities in the statement that they represent, the meaning that he gives them so as to be able to use I-5, and the corresponding algebraic expressions in the MSS of modern algebra, and Figure 10.1 gives a representation of them all as lengths and areas.

Table 10.1

The letters in Nemore's MSS	Meaning in the statement	Meaning in the interpretation	Expression in the MSS of modern school algebra
a	the part not given	the smaller part (*s*)	X
b	the part given	the difference (*d*)	U
ab	the number that is divided	the larger part (*l*)	$x + u$
c	a number equal to *a*	the smaller part (*s*)	X
d	the part not given by itself and by the part given: the other given quantity	the product (*p*)	$x(x + u) = v$
abc		the total (*t*)	$x^2 + ux = v$

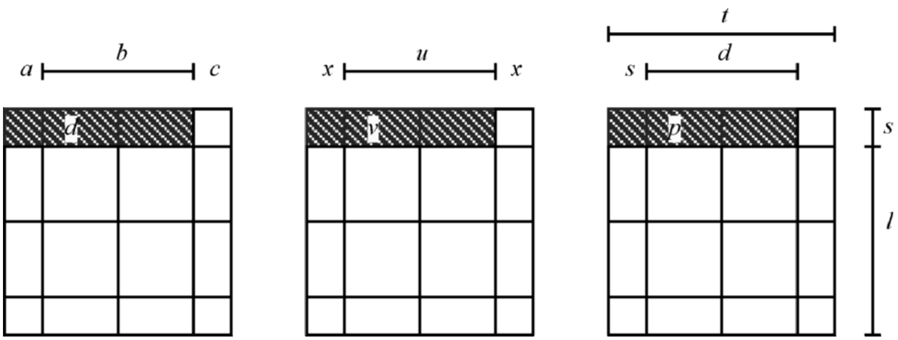

Figure 10.1

The divisions of the square in Figure 10.1 are selected in such a way that they easily represent the smaller part and the larger part into which a number has been divided (or, alternatively, the sum of two numbers), the difference of the parts, their product, the square of each of them, etc. Then the algebraic identities and the Babylonian procedures of cutting and pasting to solve quadratic problems are represented in this figure by shading parts of it and using the equalities between the subdivisions of the square, so that it can be used in the study of the teaching of these aspects of school algebra with a concrete model. By way of example, in Figure 10.2 we show the algebraic identity "sum by difference equals difference of squares."

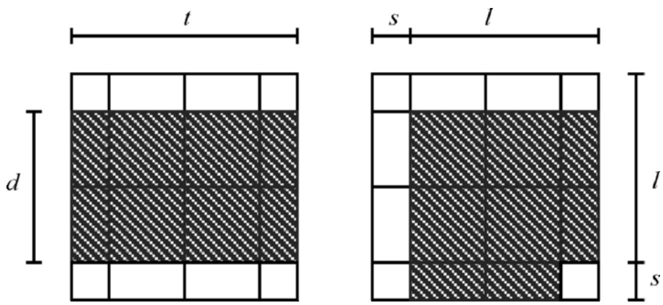

Figure 10.2

4. NEW USES OF TECHNOLOGY IN THE CLASSROOM AND THE COMMUNICATION MODEL

The powerful resources that are available nowadays, from calculators to computers, offer new ways of teaching mathematics. The numeric strategies and visual focuses that they provide represent a challenge to symbolic algebra as a means for obtaining the desired competences in solving problems. These numeric and visual resources enable us to design teaching activities by which students work with questions of a mathematical nature without having previously had any formal introduction to the mathematics involved in them.

Much stress has recently been laid on the use of computerised environments, but little light has been cast (from a theoretical viewpoint) on how this use has led to a new organization of classrooms. The result is undoubtedly beneficial, in view of the obsoleteness of teaching practices. And so it seems relevant that we should concentrate our discussion on the organization of the classroom environment.

A reflection on the relationship between theory and practice in the context of innovative approaches in the teaching and learning of algebra leads one to take into account both real classroom practice and curriculum contents such as those that result from the study of new proposals.

It is worth concentrating attention on the changes that have taken place in school classroom practice with the introduction of innovative techniques, whether or not they are based on the use of information and communication technology. It would be interesting to study the changes in the role of the teacher, the role of the students, and the role of the environments in which the teaching activities take place.

The continuation of studies such as the ones that we presented in Chapter 9 (about problem solving) and those that are connected with the processes of generalisation in algebra is bound up with the use of ICT. Much research has already been done on how the environments of technology can affect the learning and teaching of mathematics.

Specifically for the case of algebra, it is known that environments such as LOGO and spreadsheets allow the design of student activities for learning to express and manipulate the general. For example, it is known that young students are capable of perceiving the regularity of a pattern in sequences of numbers or figures, but that they find great difficulty in expressing the regularity with an algebraic formula, and it is even more difficult for them to manipulate such a formula (if they know it) in order to analyze the sequence, make predictions, generate new terms, etc. By using a spreadsheet the students can try to reproduce a given sequence, first numerically and then by introducing a general formula to generate the sequence and checking whether the formula is correct. This not only gives them the possibility of expressing the general in a language similar to that of algebra but also allows them to explore and discover properties of the sequence dynamically. That is to say, in this environment the transition to the expression and manipulation of the general can take place in an exploratory and experimental manner. In a similar way the regularities in a figure can be expressed in a LOGO program and the students can analyze them by using the program; and when they are the ones who write the program, they take control of processes of generalization that can be of great complexity.

In another area, symbolic manipulators (CAS) and even the spreadsheet have been used to help pupils to solve algebraic word problems, probably one of the most difficult items to teach in the curriculum. An argument widely accepted for the use of CAS is that the pupil concentrates on the phases of the analysis of the text of the problem and its translation into algebraic code without having to fight with the syntactic manipulation of the equation (or system of equations) to find the solution. Finding the solution is the responsibility of the automated processes of the symbolic manipulator. On the other hand, the spreadsheet has been applied to help students to organize the information of the text of a problem in cells and to develop formulas that express functional relationships between unknown and data and between unknowns (when there are two or more). The solution of the problem can then be found by means of the numerical variation of one of the unknowns. Thus the spreadsheet method also emphasizes the process of analysis of the text of the problem, but here the development of the formulas (in spreadsheet syntax) is done with an arbitrary initial numeric assignation to the value of one of the unknowns, which serves as an independent variable in the search for the solution. That is, unlike CASs, in the spreadsheet method the stage of posing and solving the problem is not done with the language of algebra, but it is done with a strong numeric support, which is favored by the predominance of arithmetic thinking in adolescent pupils.

The examples just given illustrate only a few of the uses of computational environments or technology that have been studied, proposed, and implemented for the teaching of algebra in some educational systems. However, the question of ICT in the teaching of algebra is far from being an exhausted topic in the field of research. The dynamic quality and immediate visual feedback of computer methods, so highly praised since they began to be used in education, do in fact play a fundamental part in the cases to which we have referred. But there is also widespread recognition of the fact that equally fundamental is the possibility of taking control of that dynamism by working with executable expressions, written and manipulated with a code (that of the software) which is very similar to algebraic code, but that is not, or does not behave, completely like symbolic algebra. This naturally raises the question of whether, in the long term, the use of these tools will tend to replace the manipulative aspects of algebra in teaching, or whether the intention is that the students' experiences with technological tools may eventually link up with the syntax of the algebra of pencil and paper.

These seemingly simple questions really give rise to fundamental considerations: on the one hand, concerning the importance of whether or not, in teaching, one conserves certain levels of competence in the use of the basic language of mathematics and science, and, on the other, concerning matters of situated learning, of the possibility of transferring knowledge and competence from one medium to another. These queries, subject to the specificity of the theoretical formulations that have been given in the preceding chapters, give rise to items in a research agenda, focusing on matters of interaction between languages, or rather of transition between mathematical sign systems of different levels of abstraction, inside and outside computer environments. This last aspect would include analysis, with this semiotic perspective, of the studies currently in progress, carried out with environments of simulation and graphing, where one is working with phenomena of variation and where the absence of the analytical representations of the functional relationships is deliberate.

5. EARLY ALGEBRA

For the presentation of this field we consider the following question to be of capital importance: What forms does syntactic competence adopt in early algebra? Has it been expelled from early algebra or has it only been hidden? In Section 5.3 we will give a reply that points to the need, when researching in this field, to take both the syntactic competences and the cognitive tendencies generated into account.

5.1 Generalization

To set bounds to this point of view, we will initially confine ourselves to one of the subsidiary areas most studied currently, generalization, given its importance in the implementation of curriculum developments for the youngest learners, for whom the aim is to start to lay the foundations of what will ultimately be developed in algebra in later years. See Ginsburg, Inoue and Seo (1999); Usiskin (1999); Cuevas and Yeatts (2001); Friel, Rachlin and Doyle (2001); Greenes, Cavanagh, Dacey, Findell and Small (2001); Malara and Navarra (2003).

5.2. Implementation of the teaching model

We can analyze the kind of tasks connected with generalization that have recently preoccupied the research community with seven examples taken from research papers (presented in strictly alphabetical order): (1) Iwasaki and Yamaguchi (1997), (2) Lee (1996), (3) Mason (1996), (4) Radford (2000b), (5) Stacey (1989), (6) Zazkis and Liljeddahl (2001) and (7) Zazkis and Liljeddahl (2002).

In general, in these articles we find:

1) Exchange of messages with mathematical texts (T_n), in an attempt to get the learners to acquire the desired competences. Usually these articles show a good performance on the part of the learners presented in parts of the series of texts T_n and the probable appearance of personal codes in the performances.
2) Some learners are successful in producing sense for the series of texts T_n and they thereby become competent users of the language game proposed by the series T_n (the Process of Generalization).
3) In all cases, what has been said in (1) and (2) implies the use of a stratum of the MSS in which syntactic competences in arithmetic and the rudiments of algebra are necessary.
4) Point (3) forces the appearance in the users of cognitive tendencies that have been well known for some time: those described in (a) Vergnaud (1981), (b) Carpenter, Moser and Romberg (eds.) (1982), (c) Fuson (1998), (d) Kieran (1981), (e) Booth (1984), (f) Figueras, Filloy and Valdemoros (1985, 1986), (g) Filloy and Lema (1996), (h) Filloy and Rojano (1989) (reverse of multiplication syndrome, polysemy of x, very many cognitive tendencies indicated in this book), (i) Filloy and Rubio (1991, 1993a, 1993b), (j) Filloy, Rojano and Rubio (2001), etc.

5.3. Grammatical change

To sum up what has been said: the reorganization of the new conceptual field developed on the basis of achieving the necessary competences in the use of the language set involves a change in the logical syntax of the new stratum of the MSS.

5.4. Research needed

We therefore have to put forward new local theoretical models (LTMs) to be used with these K–8 students to study the processes of abstraction and visualization characteristic of the use of ICT: electronic blackboards, symbolic calculators, specific software (spreadsheets, Cabri, SimCalc, etc.), programming languages (LOGO, the language of electronic calculators, Visual Basic, etc.), optical readers, videotaped interviews, and so on.

The cognition components of these new LTMs will have to take into account all the cognitive tendencies detected in the past, which were presented summarily in point (4) of 5.2.

6. RESULTS OF RECENT RESEARCH INTO PROBLEMS OF LEARNING ALGEBRA USED AS THE CORE FOR IN-SERVICE COURSES IN THE TEACHING OF MATHEMATICS

The need to apply a theoretical focus to the problem of teaching mathematics began to become evident in the middle of the last century. From the outset, this new awareness drew the attention of groups of mathematicians, educators, psychologists, and epistemologists, stimulating the introduction of new curricula at all levels of the educational system. This attitude resulted in the appearance of many areas for investigation that had not been studied previously and which posed unexpected problems.

The changes in the mathematics curriculum made it essential for teachers to have some knowledge that fit in with the new ideas about the teaching of mathematics. The need for researchers in this field also emerged.

6.1. Some topics for discussion

The development of a theoretical focus in research has led to the posing of questions such as the following:

1) What is the role of a theory in the didactics of mathematics?
2) How can a teacher obtain some benefit from learning theories of the didactics of mathematics or research methods used in the didactics of mathematics?

3) To what extent are research results relevant for conditions in the real world? To what extent are they useful for teachers in their day-to-day activities in the classroom?

This first group of questions arises as a result of the development of theoretical perspectives that, because of their complexity, are not easily accessible for teachers. Moreover, there seems to be a view that "didactic objectivity" can exist independently of the real conditions in which the processes of teaching and learning take place in current school systems. What, then, is the point of studying "didactics"? Of what use is the discourse in which its concepts are expressed? Could one sensibly imagine a course of study on "didactics" that did not aim to change the real practices that characterise the teaching and learning of mathematics?

It is certainly significant in this respect that five of our studies on algebra which are presented in this book were developed with the assistance of teachers and pupils in the educational systems in the countries where the authors live.

We ought to point out that in each of those studies the attitudes of the teachers involved underwent a transformation as the experimental work advanced. It is appropriate, therefore, to contrast the use of theories in the design of experiments for research and even the interpretation of the results with the intermediate products of the experiment as it progresses. In the experiments in the studies just mentioned, the teachers were involved directly as a fundamental part of the experiment and they produced a great abundance of ideas about the immediate results.

It is worth asking oneself how profound and penetrating an idea a teacher needs to have of the fundamentals of a didactic theory or research methodology to be influenced by the results of the experiment in the direction of changing his basic attitudes toward his work and getting the corresponding improvement from it. The examples in our studies suggest that a complete understanding of the theory used is not essential for the experiment to produce a wealth of useful data for the teachers who take part in the experiment, or for those data to be provided to other teachers. If it were essential we would be in a no-win situation, as it would be necessary, before a teacher could use the results of the study, for him to become an expert in the theoretical fields on which it was based: psychology, linguistics, artificial intelligence, epistemology, the history of ideas, etc., and also on the processing and interpretation of data, including advanced techniques of data analysis or methods for processing observations, etc.

6.2 Research needed

The following questions should be the focus of future research:

1) Why and how can many of the immediate results of research in educational mathematics be expressed in the direct discourse that teachers use in their ordinary speech?
2) Why are those results perceived as new information, previously unknown to the teachers who take part in the experiments and in the discussions held at the end of each experiment with other teachers not involved in them?
3) Why is this information perceived by the teachers as decisive for a new way of seeing their problems and changing what they previously thought was normal (in their activities in day-to-day teaching)?
4) Why is it that, as a result of a practical experiment, previous practices of teaching are seen by the teachers as inconsistent with the immediate results produced by the study that is being carried out?

In order to do research to answer questions such as these we will have to separate the theoretical framework that supports semiotic research in educational algebra (with teachers involved in it) from the theoretical framework on the basis of which the teachers play an active part in designing the research. To achieve this we will have to develop our ideas about the communication model. In the following section we put forward some ideas on which to work in the future. It is also necessary for what is proposed in Sections 3 and 5 of this chapter.

7. OBSERVATION IN THE CLASSROOM. A SEMIOTIC PERSPECTIVE

The only occasion on which the communication model has been used in this book is in the final example in Chapter 8. It is worth reproducing the conclusion that we reached in point 4.4.3, entitled “Different levels of abstraction: the case of names”: “The communication model enables us to establish the difference in the readings given by the interviewer and the student […] This difference in the meanings attributed to algebraic expressions is present in any process of teaching algebra where the teacher is a competent user and the students are learners, generating difficulties such as those reported here (tendencies 2 and 5).”

The words just quoted could be the most general thesis that guides the design of any communication model. In this book, as the observations reported concentrated on the results of the case studies, concluding with clinical interviews, the communication model has not been used much, and consequently it has not been introduced much; we have merely mentioned a few general considerations concerning it in Chapter 8. But if one is trying to analyze situations that occur in the classroom, or any situation in which there is a teacher (human or an interactive electronic teaching model) and learners, it is necessary to develop its theoretical aspects further.

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Chapter 10 brings the book to a close, proposing a research agenda that includes questions and topics that arise from the discussions and analyses presented in the preceding chapters, from the perspective of local theoretical models, mathematical sign systems, and historical analysis. To start with, we propose going back to history to study the evolution of certain algebraic ideas, analyzing historical texts as cognitions in the same way that we analyze students' productions, which in turn constitute mathematical texts in the sense used in Chapter 5. We also express an interest in making a deeper study of operating with the unknown, on a second level of representation, that is, when it is expressed in terms of another unknown (a typical situation in the solution of systems of equations with two or more unknowns) and in exploring new methods for solving systems of two equations with two unknowns, going beyond the classical methods of substitution and comparison. Also in the field of teaching, it is of interest to explore teaching models for solving the quadratic equation, based on analysis of Jordanus de Nemore's work *De Numeris Datis*. We also propose the need for a greater theoretical development with respect to models of communication in the classroom for the case of the teaching of algebra. In particular, it is of interest to investigate how the use of ICT affects patterns of communication in the classroom (the communication component in local theoretical models). This last point leads naturally to the theme of using the results of research in teacher training. Finally, the research trend which has acquired importance in recent years concerning the early introduction to algebra is also put forward, with questions such as: What forms does syntactic competence adopt in early algebra? Is this competence avoided in early algebra or is it present implicitly? We invite the reader to engage with these questions from the theoretical perspective expounded throughout this book, in which the identification of “the algebraic” in these early approaches is closely connected with explicit handling of the mathematical sign system of symbolic algebra.

ENDNOTES

[1] The MSSs on which the more abstract MSS is erected, present in history and in the history of each individual, cannot be the same; nor, therefore, can the paths toward the more abstract MSS. In the case of algebra, one need only take account of the fact that modern school arithmetic is not written in the vernacular but in an MSS imbued with signs and even rules of syntax that come from the MSS of symbolic algebra and which have come down from it to arithmetic.

[2] This proposition has already appeared in Chapter 9. There we used the third part of the proposition, the problem with which the algorithm obtained for the solution is exemplified, whereas here we will analyze the second part, which develops the proof of the proposition.