CHAPTER 1

# INTRODUCTION

## OVERVIEW

This book is based on an experience of ours in which the need to interpret unanticipated phenomena observed in empirical studies on the transition toward algebraic thought conducted in the 1980s, triggered a long-term research program that in turn led to a theoretical formulation that emphasizes local analyses.

To illustrate that experience, we briefly examine a few of the phenomena observed in the transition from arithmetic to algebra, which represent an essential part of pre-algebra. The observations dealing with cognition are presented in Section 4.2.1, the reverse of multiplication syndrome in 4.2.2, different uses of the notion of equality. Polysemy of $x$; and in 4.2.3, difficulties in translations. We begin by indicating the role of historical analysis in Section 4.1, and complete the section with an example of a dialogue that took place during a clinical interview, in which additional phenomena appeared in translating algebraic language to natural language (Section 4.3).

The book presents the theoretical elements developed and shows how the theory of local models, through their different components, has enabled a deeper study of phenomena in the field of acquiring algebraic language, considering aspects that are relevant to learning, teaching, and research.

Use of the term "educational algebra" in the title of the book, instead of the more usual term "school algebra" is appropiate given the broad-based nature of the educational aspects we deal with. As will become patent in the rest of the book, besides working with children and teachers in schools we have used other sources as well to design and develop empirical studies: semiotics, epistemological analysis (primarily history of mathematical ideas), phenomenological analysis (mainly Freudenthal's approach to curriculum development), formal mathematics, cognitive theories, etc. The term "educational algebra" is sufficiently broad to encompass the aspects that are educational, albeit not necessarily school-related.

We also introduce two central terms, "mathematical sign systems" and "local theoretical models", which are used throughout the book. They are discussed more extensively in Chapter 2 and in other chapters, where they are used in the description of concrete examples arising from the empirical studies.

We conclude with a review of other literature on the subject of mathematical language and language and mathematics to place our work within a context and to demonstrate its contribution therein.

The desire to achieve a profound understanding of both the origin and nature of the difficulties confronting those who seek to gain access to algebraic thinking has set in motion great ideas and inquiries about them over the past three decades. The vast amount of literature produced from all this research activity makes the task of surveying and updating the state of the art in this field increasingly difficult. It is not so difficult, however, to identify a series of studies that concentrate on studying symbolic algebra as a language, together with the details of its acquisition. Because of the abstract nature of algebraic language and the highly syntactic competences required for its use, many of these studies use approaches that include semiotic concepts and linguistic analyses. This book is devoted to setting out a path of theoretical development for educational algebra, in which this very perspective is adopted and in which an historical element becomes a contributing factor.

Despite the deliberately theoretical character of this work, its direction differs from that of general models. This work incorporates elements that make it possible to develop local frameworks of analysis and methodological design for the study of specific phenomena. In these frameworks, it is possible to include evidence connected with such phenomena, the interpretation of which escapes general treatments. Such is the case when individuals who are beginning the study of algebra, produce personal sign systems that are located on a level prior to the mathematical sign system that is to be learned (that of symbolic algebra in this case).

After the worldwide acknowledgment in the late 1970s that the educational system had largely failed to teach algebra in secondary schools, one of the great ideas put forward was Hans Freudenthal's proposal. Freudenthal stressed the need to analyze the language of algebra by comparing it with other languages, such as natural language and the language of arithmetic, both of which were considered means of support (Freudenthal, 1983, ch. 16). His dissertation was followed by many other studies dealing with mathematics education seen through a linguistic hue.

Most of the research carried out recently on the didactics of mathematics lacks paradigmatic theoretical models, even if one uses the term paradigm (somewhat in the sense of Kuhn, 1962) not as a synonym of theory, but in a more general sense, i.e, as the set of basic assumptions that one can make about the nature and limits of the actual subject to be studied, the method for studying it, and the decision as to what will be accepted as evidence. Nor has a consensus been reached about which of the basic assumptions should determine the form to be taken by the theoretical frameworks for interpreting specific phenomena and for proposing new experimental designs that will carry theory further forward to embrace other evidence or new unrelated evidence. In short, it is still necessary to speak of the boundaries of many research projects.

As a start, other disciplines have already begun research on the very subjects that pervade most of the work on which mathematics educators have reported. Some of these subjects include linguistics, logic, psycholinguistics, semiotics, general cognitive psychology, the psychology of mathematics, the epistemology of mathematics, the history of mathematics, the psychology of education, the theory of the development of mathematics curricula, and the didactics of mathematics.

Many research studies have recently incorporated the results of these disciplines and have redefined results within their own theoretical frameworks. Here we interpret various recent theoretical assumptions to reorganize the research undertaken on the processes of teaching/learning algebra during the past few years. To accomplish this, it is necessary to work with a good deal of new terminology to be able to describe recent research.

To this end, in Chapter 2 we introduce the methodological concept of local theoretical models (LTMs). Although LTMs are dealt with in greater detail in Chapter 2, we can state here that the subject is considered in terms of four interrelated components: (1) teaching models, (2) models for the cognitive processes, (3) models of formal competence and (4) models of communication. Here we shall refer only to their local character.

## 1. ON THE LOCAL CHARACTER OF THE THEORETICAL FORMULATION AND ITS COMPONENTS

One of the chief reasons for resorting to local theoretical elaborations was the need to interpret phenomena that arose during the study. These phenomena could not have been anticipated from the design of the observation and did not fit into schemes of analysis based on general theories derived from mathematics education itself or from neighboring disciplines such as psychology, pedagogy, sociology, history, epistemology, or linguistics. Studies on the transition from arithmetical thought to algebraic thought carried out in the 1980s came up against this situation, giving rise to a long-term research program that envisaged the development of theoretical elements that would make it possible to refine the analysis of such phenomena. An initial hypothesis is that although we set out from a general notion —that of the mathematical sign system— it is the local character of the theoretical elaboration that makes it possible to delve deeper and thus generate new knowledge about the subject. Hence LTMs (Filloy, 1990) represent the central idea in this work. Rather than partializing the problems of mathematics education research, LTMs open up paths of communication between the various components that usually contribute to them. In fact, each local model contemplates the study of cognitive aspects, formal mathematical competence, teaching, and communication. This comprehensive approach offers possibilities of making a substantive contribution to a highly focused study, based on a multiplicity of disciplines and drawing on the work of specialists and communities connected with those disciplinary fields. The contents of this book are the result of progress in the research agenda that we set ourselves when, in our studies on algebraic thought among adolescents in the 1980s, we were first faced with the limitations of general analytical schemes in trying to interpret the phenomenon of the polysemy of $x$ or that of the reverse of multiplication syndrome, for instance. Later in this chapter we provide a detailed description of those phenomena, as well as others that arose during our research. For our descriptions, we shall be using the notion of mathematical sign system (MSS), a brief introduction to which is provided in the following section.

## 2. MATHEMATICAL SIGN SYSTEMS

### *2.1. Sign*

This section discusses the phenomena that take place in mathematics education, using the jargon of semiotics. We do so not to embellish our observations with cryptic language, but because we consider these phenomena as processes of signification and communication, and semiotics deals with processes of precisely this type.

The fact that semiotics studies these processes rather than signs is especially clear in the semiotics developed by Charles Sanders Peirce. In Peirce's semiotics, this emphasis on processes is present even in the very idea of sign. Peirce gave countless definitions of "sign" throughout his extensive writings, in which he repeatedly outlined the concept. In all of his definitions, three characteristics are worthy of special emphasis. The first is the fact that the sign is not characterized by a dyadic relation such as that of Saussure's signifier/signified pair; the relationship to which any sign belongs is triadic. And one of the elements, which Peirce calls the "interpretant," is the *cognition* produced in a mind. The second is the fact that the sign is not a static entity but is instead open within a series. Since all cognition is in turn a sign, that sign therefore stands within a triadic relationship to another interpretant (which is another cognition), and so on and so forth. The third is the fact that the sign is not arbitrary or rather that the triadic relation to which it belongs is not arbitrary.

In a manuscript written in 1873, Peirce gives his briefest and most compact definition of a sign:

A sign is an object which stands for another to some mind (Peirce, *W* 3, p. 66).[1]

The relation is established between the sign (S), its object (O), and a mind for which the sign is related to its object in such a way that, for certain purposes, it can be treated as if it were that other.[2] Let us see how Peirce defines the interpretant (I):

A sign […] addresses somebody, that is, creates in the mind of that person an equivalent sign, or perhaps a more developed sign. That sign which it creates I call the *interpretant* of the first sign. The sign stands for something, its *object*. (Peirce, *CP*, 2.228, p. 135.)

The triadic relation (S, O, I) is, therefore, a relation in which both S and I are signs, so that I is a new sign, S', which enters into another triadic relation, i.e., it creates in a mind another sign as interpretant, I', of object O, a new cognition I', such that object O links the two triadic relations (S, O, I) and (S', O, I'). This leads to the open nature of the sign in a process of semiosis that has no end. Peirce expressed it thus in another definition, subsequent to that quoted above:

*Sign* [Lat. *signum*, a mark, a token]: Ger. *Zeichen*; Fr. *signe*; It. *segno*. (I) Anything which determines something else (its *interpretant*) to refer to an object to which itself refers (its *object*) in the same way, the interpretant becoming in turn a sign, and so on *ad infinitum*. (Peirce, *CP*, 2.303, p. 169.)

Also present in this definition is the third aforementioned characteristic: the fact that the relation is not arbitrary. The sign forces the interpretant to refer to the same object as the one to which it refers. In a more extensive definition, quoted later, Peirce is even more exigent and adds that the sign forces the interpretant to refer to the same object and, furthermore, *in the same way* as it refers. Moreover, there must also be an interpretant, $I_1$, of interpretant I, which has as object $O_1$, the *relation* between the sign and its object.

A Sign, or *Representamen*, is a First which stands in such a genuine triadic relation to a Second, called its *Object*, as to be capable of determining a Third, called its *Interpretant*, to assume the same triadic relation to its Object in which it stands itself to the same Object. The triadic relation is *genuine*, that is its three members are bound together by it in a way that does not consist in any complexus of dyadic relations. […] The Third must indeed stand in such a relation, and thus must be capable of determining a Third of its own; but besides that, it must have a second triadic relation in which the Representamen, or rather the relation thereof to its Object, shall be its own (the Third's) Object, and must be capable of determining a Third to this relation. All this must equally be true of the Third's Third and so on endlessly […] (Peirce, *CP*, 2.274, p. 156).

### *2.2. Mathematical sign systems*

The examples presented throughout the book have enabled us to make use of Peirce's concept of the sign and its typology, and to explore the sense through which it casts light on what we wish to examine. The examples also show something else: the signs that are used in mathematics are not all of a linguistic nature, which makes it advisable not to use the terminology or concept of the sign that belong to linguistics (derived, to a greater or lesser extent, from the work of Saussure), and therefore not to speak of the signifier/signified pair. In the preceding text we have not done so, using instead the term "expression," from the expression/content pair —terminology that has been introduced in semiotics (the science of signs in general, and not just of linguistic signs). This is also very convenient because in mathematics one is accustomed to speaking of "algebraic expressions" or "arithmetic expressions" to refer to the corresponding written forms.

However, in putting the emphasis on individual signs, what we have seen so far may conceal the crucial fact that there are no isolated signs in any text (whether mathematical or not).

It is very common for a description of the language in which mathematical texts are written to distinguish between two subsets of signs: one consisting of signs conceived as strictly mathematical and another consisting of signs in some vernacular language. From the viewpoint of signification processes, however, this distinction ceases to be crucial although it can still be made. What seems to be crucial is the sign system taken as a whole, and what must be described as mathematical is the system and not the signs, because the system is responsible for the meaning of the texts. One must therefore understand the term "mathematical sign systems" as mathematical systems of signs and not systems of mathematical signs,[3] for what is of a mathematical nature is the system and not just the individual signs. Consequently, what is of interest for the development of mathematics education is to study the characteristics of these (mathematical) sign systems which are due not just to the fact that they are sign systems but also precisely to the fact that they are mathematical systems.

Filloy (1990) and Kieran and Filloy (1989) introduced the need to use a sufficiently broad notion of mathematical sign systems. It had to serve as a tool to analyze the texts produced by students when they are taught mathematics in school systems —and those texts are conceived as the result of processes of production of sense— as well as to analyze historical mathematical texts, taken as monuments, petrifactions of human action, or processes of cognition belonging to an episteme. In taking these mathematical texts as the object of study, rather than supposedly ideal texts conceived as manifestations of "mathematical language" or texts that are measured by them, the notions of mathematical sign systems and of text must both open up in various directions.

Thus one must speak of mathematical sign systems, with their corresponding code, when there is a socially conventionalized possibility of generating sign functions (by the use of a sign functor, see Chapter 7), even when the functional correlations have been established in the use of didactic artefacts in a teaching situation with the intention that they should be impermanent. But one must also consider the sign systems or strata of sign systems that learners produce in order to give sense to what is presented to them in the teaching model,[4] although they may be governed by a system of correspondences that has not been socially established but is idiosyncratic.

## 3. DIFFERENT ANSWERS TO SAME QUESTIONS?

We point out, however, that not only semiotics but also information processing theory and the didactics of mathematics (Brousseau, 1997) have done important work on the notion of code. This notion is emerging as a key concept to interpret what comes from using the idea of representation in the models that explain the cognitive problems presented by alternative teaching approaches or technology learning environments. Or, to provide another example, consider the emphasis that psycholinguistics and artificial intelligence place on a process-based model of human capabilities and relate it to the way in which the model explains how and why users of mathematical language naturally and commonly make mistakes in its syntactic procedures. To these developments, one must add the attention that a pragmatic viewpoint has given to meaning in use rather than formal meaning.

By accumulation, these approaches —and others of a similar nature— have led to a change of direction in recent work, which is shifting away from the competence of mathematical language users and moving toward performance. This change of viewpoint has basic and essential implications for the manner in which mathematical language is seen. Essentially, the claim is that grammar —the formal abstract system— and pragmatics —the principles of language usage— are complementary domains in our studies. In addition, that both are domains related to the various teaching models, be they innovative or traditional, that are used to achieve the objective of guiding students in order for them to become competent users of the language of mathematics. Since one of our aims is to observe what happens in mathematics classrooms, however, we must also confront the complexities of teaching and learning phenomena within that particular setting.

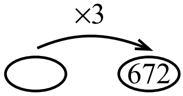
Indeed, one of the simplest phenomena demonstrated by classroom observation, for instance, reading level permanence among children who have just finished primary education (approximately 12 years old), is what arises when they are confronted with questions like those in Figure 1.1, which shows the evolution of the equation $Ax = B$ in school teaching.

Evolution of the equation $Ax = B$

1) $3 \times \square = 12$

2) $3 \times \square = 672$

3)




4) $3 \times x = 672$

5) $3x = 672$

Between the ages of 10 and 12 it is easy to direct some students so that all the questions are read like [2]: What is the number that, when multiplied by 3, gives 672?

Figure 1.1

When one analyzes the responses of children within this age group, many issues arise. Apart from the fact that these questions are deemed as different because some can be answered and others cannot, we also find that it is fairly easy to get a certain student profile bogged down in their use of the preferred arithmetic method, trial and error. It is even quite easy to induce them to continue using that method for a considerable time despite the fact that the numbers become progressively larger, which eventually means that they no longer have sufficient arithmetic skills to be able to answer the question without making mistakes. We call this phenomenon "the reverse of multiplication syndrome."

To be able to observe phenomena of this kind experimentally, we therefore need an experimental framework that will enable us to interpret the facts and propose new observations that will unravel the relations existing between the different components in play.

## 4. THE TRANSITION FROM ARITHMETIC TO ALGEBRA. PRE-ALGEBRA. SOME OBSERVATIONS ABOUT COGNITION

Several studies have indicated conceptual and/or symbolic changes that mark the difference between arithmetic and algebraic thought. Examples of the foregoing are those related to the various interpretations of letters (Booth, 1984), those dealing with the notion of equality (Kieran, 1980, 1981), and those produced with respect to the symbolic or graphic conventions for coding operations and transformations in solving of equations (Matz, 1982). From such indications, one can imagine paths of evolution from arithmetic to algebraic thought that correspond to the representative notions and forms for the objects and operations involved in the mechanisms of change. Thus, the changes deemed essential for a person to attain algebraic knowledge can be visualized along each of these paths as points where there is a cut between one kind of thought and another.

One of the foregoing points that is of particular interest to the topic of equation solving is suggested by analyzing the strategies and methods used to solve equation systems in texts of pre-symbolic algebra from the 13th to 15th centuries. An important factor in this analysis for developing solution strategies and methods is that of operating unknowns. This arises as a result of the limitations imposed by the frameworks that belong to the pre-symbolic representation of equations and their characteristic elements. Thus, for example, the solution that led to equations that we now write as $x^2 + c = 2bx$ and $x^2 = 2bx + c$ are completely different in each case. Yet this would not happen if the rules for transposing terms from one side of an equation to the other were known because, for instance, then it would be possible to reduce the case of one of the equations on a syntactic level to that of the other, which would correspond to a more developed level of operation on unknowns.

### *4.1. The role of historical analysis*

The propositions contained in Leonardo of Pisa's *Liber Quadratorum* (Book of Squares)[5] can quickly be proved using the mathematical sign system of secondary school algebra. Indeed, they are propositions that can be proved in less space than that taken up by one of the pages of this text, and their mathematical content does not go beyond what is presently learned in secondary school. Nevertheless, it is easy to perceive the intensity of thought required to follow the reasoning depicted in Leonardo's book. And it is not as if he were rhetorically playing with trivial matters. On the contrary, his work is very possibly the pinnacle of mathematical thought in Middle Age Europe. Frozen in time on those pages one finds reasoning that drew —and still draws— great admiration because of its freshness and intensity. It is a thought that comes to us from the 13th century and that shows us how a mathematical sign system predetermines the ways in which we analyze problems, advocates our solution strategies, and draws the lines of strength that guide the sense of all our inferences. This strikes us as odd within the context of developing the thought processes of children, but it is even more amazing in the thought process of a first-class mathematician, perhaps the greatest mathematician of that era. At the same time it provides us with the opportunity to discover unknown terrain on which we can observe and describe the same cognitive tendencies as we find in today's children as they attempt to become competent users of the mathematical sign systems that they are taught in secondary school. It further enables us to draw plausible hypotheses, and then observe those hypotheses in the behavior of present-day students as one tries to have them make competent use of the sign systems currently used to articulate the messages through which today's mathematics education is communicated.

As one confronts mathematical texts such as those of Leonardo, one's attention is first drawn to the fact that no one speaks that language now. Were they translated into the mathematical sign system of current symbolic algebra they might appear to be advanced problems typical of a modern textbook. They, however, differ in that their solution strategies do not conform to customs. In addition, today one would not perform many of the operations and intermediate steps that seem to be necessarily present in those texts. The language of the *abbacus* books[6] is today a dead language. Their translation to modern algebraic language fill us with amazement for the novel actions that led to the same results as ours, but that follow unheard of paths. Their very presence in problem after problem and in book after book are indicative of skills unrelated to those we have developed in building and using our algebraic language —skills, one might add, that we have never felt the need to build, develop, or use when confronting problems with our arithmetic abilities and knowledge.

Clearly, as we build new conceptual apparatuses that have imposed themselves upon us without the possibility of erecting them within a proper structure, connecting them to others that have previously been firmly rooted, those new skills tend to overshadow older skills. Moreover, given the fragile means at one's disposal to use at that point any new resources and solution techniques, even problems that had been mastered for quite some time are now difficult to model in the new language within which the infant conceptual apparatus that is in the process of being constructed is expressed.

Nonetheless, well anchored intellectual structures tend to perpetuate themselves and compel us to reconsider situations that, when modeled in the new language, could be solved with simple, routine operations.

Indeed we do realize that we began speaking of reading and interpreting ancient texts, and have now taken a leap onto the plane of psychological processes. We now feel confident in saying that it is precisely this leap, forward and backward, that enables us to produce hypotheses founded on the development of general knowledge and to convert them into hypotheses about the didactics of mathematics, which then seeks to reconsider that process on the level of individuals –children, in this case. The MSS of arithmetic has to make way for that of algebra, and this has become an increasingly pertinent matter even for situations that have always before been modelled in arithmetic.

Building the new MSS, whose point of departure must necessarily be elementary arithmetic operations, will involve the need to operate on new objects. These objects will signify not only numbers but also numerical representations, whether as individual items (e.g., unknowns), sets of numbers (e.g., coefficients of equations), an expression of relations between sets of numbers (e.g., proportional variation), or as functions, etc. The algebraic MSS will have to be structured on new objects whose operations will not be completely determined until the outlines of the new world of objects become more precise. What is more, the objects will not be totally outlined and well defined until the new operations have been completely structured in terms of both their semantic and syntactic aspects.

Such profound changes in arithmetic habits and notions do not take place spontaneously in individuals simply because they are confronted with the need for change. The intervention of teaching, at that point of transition from arithmetic to algebraic knowledge, can be crucial for most students who are learning algebra for the first time.

Although it is necessary to modify some arithmetic notions in order to acquire the new —algebraic— knowledge, it is also necessary to preserve the previous knowledge —arithmetic, in this case. Even in the single example of equations previously presented, there is a need for arithmetic equations to subsequently be recognized as such in order to preserve the entire operativity acquired beforehand for their solution —an operativity situated on an intermediate level of knowledge between arithmetic and algebraic knowledge, that is to say the level of pre-algebraic knowledge.

### *4.2. Mathematics lessons at the beginning of secondary education*

We present three types of situations that generally arise when students have just completed elementary education and are beginning secondary education:

1) The reverse of multiplication syndrome.
2) Different uses of the notion of equality. The phenomenon of contextual ambiguity.
3) Difficulty translating from natural language to algebra and viceversa.

*4.2.1. The reverse of multiplication syndrome*

As mentioned in Section 3, some students get stuck using the arithmetic method of their preference, which is trial and error when solving the equation $Ax = B$, and even go on using this method when it has become inefficient because the $B$ numbers are too large for them to perform arithmetic trial and error without making mistakes.

During the first year of secondary school (in the Mexican Educational System), most students end up preferring the method of dividing $B$ by $A$ in order to solve the equation $Ax = B$, which is the objective of the mathematics syllabus at that stage. However, the same trial and error strategy reappears in the work of students who had already achieved operativity to solve all first-degree equations, when the context in which the equation $Ax = B$ appears comes from an analytical process while the student is solving a word problem.

Even more surprisingly, at times when the expression $Ax = B$ is written by the very person who is being observed, the signs are not recognized as the expression of an equation that a few moments before the student knew how to deal with operationally to find the solution. The context in which the equation appears, even in its written form, makes the student "forget" the operativity achieved previously and revert to preferring the arithmetic method of trial and error or, in some cases, become unable to bring any method of solution into play. A more detailed description of what happens in the latter case shows that the interpretation of the sign $x$ is crucial in interpretation of the expression $Ax = B$: interpreting the $x$ as *an unknown* makes the student not know what to do, because "it is something that is not known," in the student's own words. In addition, it is important to recall that we are at a point in teaching when we are trying to have students begin to use the knowledge they have learned about solving first-degree equations in order to solve application problems that appear in mathematics lessons as well as in physics, chemistry, and other subjects.

### *4.2.2. Different uses of the notion of equality. Polysemy of* x

Several ways of interpreting equality can be distinguished among the uses made by 12- to 13-year-old children, as follows:

*A) As an arithmetic equality* ($E_{Ar}$)

In this case, the student tries to combine the terms on the right side of the equation or read them as a single number before giving any type of answer or performing any operation. Those who make this interpretation carry out one of the following procedures:

i) Completion: “This [the independent term on the left side] needs this much to be equal to this [the complete right side]”.
ii) Direct isolation: this procedure predominates in students who perform well, but it also appears in the other cases. In items such as $x + \frac{141}{16} = 17 + \frac{141}{16}$, $x + \sqrt{17} = 42 + \sqrt{17}$ and $x + \frac{x}{4} = 6 + \frac{x}{4}$, students who make this interpretation *($E_{Ar}$)* and who try to isolate $x$ face serious difficulties. For some students, the fact that they do not know the value of $\sqrt{17}$ prevents them from tackling the $x + \sqrt{17} = 42 + \sqrt{17}$.

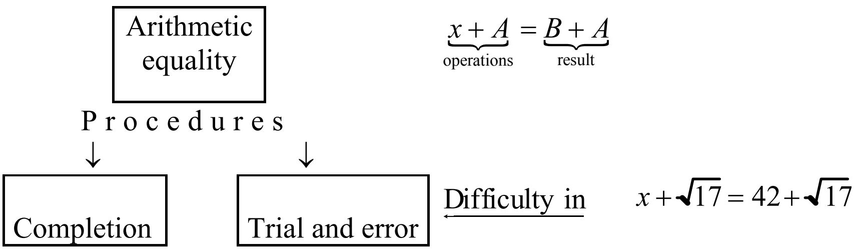



Figure 1.2

*B) Equality of the left side (as a whole) with the right side (also as a whole)*($C_0$)

This interpretation also allows for two procedures:

i) Completion, which in many cases is more visual than arithmetic.

ii) Isolation. In some children of mid-level performance, the $C_0$ interpretation precedes the appearance of the $E_{Ar}$ interpretation. In other cases, it appears on its own.

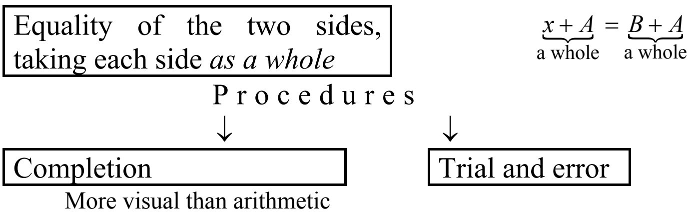



Figure 1.3

*C) Equality term by term (C₁)*

With this interpretation it is possible to solve these equations very quickly, except the items $x+\frac{x}{4}=6+\frac{x}{4}$ and $x+5=x+x$, on which we will comment later. This interpretation predominates in students with a high achievement level, although in some cases it is preceded by the $C_0$ interpretation or (in children of mid-level performance) by rearrangement of the terms with respect to the = sign.

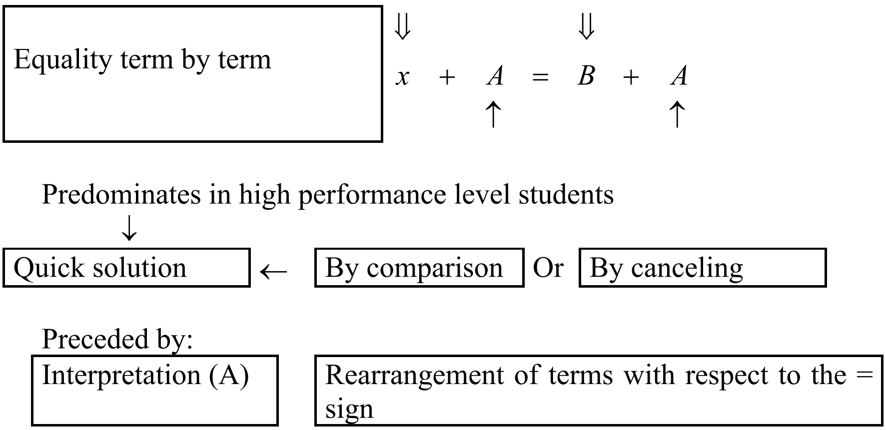



Figure 1.4

In the items $x+\frac{x}{4}=6+\frac{x}{4}$ and $x + 5 = x + x$ there is a tendency to give a $C_1$ interpretation, but also to assign different values to different occurrences of $x$. The typical response is:

This $x$ $(\overset{\downarrow}{x} + \frac{x}{4} = 6 + \frac{x}{4})$ equals 6 and these $(x + \overset{\downarrow}{\frac{x}{4}} = 6 + \overset{\downarrow}{\frac{x}{4}})$ can be any number.

In the item $x + 5 = x + x$, the same kind of response appears:

This $\overset{\downarrow}{x}$ $(x + 5 = x + \overset{\downarrow}{x})$ equals 5 and these $(\overset{\downarrow}{x} + 5 = \overset{\downarrow}{x} + x)$ can have any value.

In some cases, the students were asked to invent a problem that could be solved with this equation. Others were presented with a problem of the same kind to make them see that, within the context of one and the same problem, all occurrences of an unknown represent that same unknown. However, this clarification was not always successful.

We have denominated this phenomenon of unknown multivalence as "polysemy of *x*," because it involves a reading of the same sign in various contexts in which *x* is an unknown and in which *x* is a generalized number. Moreover, what "unites" these two interpretations or readings is the numerical equality of both sides of the equation.

### *4.2.3. Difficulties in translations*

After secondary school students have received pre-algebra MSS instruction and been introduced to elementary algebra so as to solve linear equations and decode arithmetic-algebraic texts, yet before receiving systematic teaching on usage of open expressions, equivalence of expressions, and how to solve equation systems, the task of reading or writing algebraic language is very difficult for them. This is so much the case that one can almost see the tension mount in them as they struggle between using the arithmetic MSS to read and express themselves and their need to give mathematical signs new meanings within the context of the algebraic MSS. This is yet another indicator of the fact that the boundary between arithmetic and algebra cannot be avoided given that it would lead to false conceptions about the processes for acquiring the algebraic MSS, and, consequently, about the role of teaching within those processes. Furthermore, this highlights just how important it is for reading and writing algebraic MSS to be considered a decisive educational goal for middle school students.

The forms of notation used in algebra just happen to be basically the same as those in arithmetic, i.e., numbers, operation symbols, the equal sign, and letters. However, their meanings and the way they are operated essentially differ in the two fields. Consider the following example:

The two expressions $A = b \times h$ and $y = ax$ are syntactically equivalent, yet the way of reading them —their interpretation— determines the actions that subsequently have to be performed.

One (conventional) way of reading $A = b \times h$ is "the area (of a rectangle) is equal to the base times the height," which refers readers to the field of arithmetic-geometric MSS. Whereas $y = ax$, read in context, could mean "$y$ is a linear function of $x$, with parameter $a$." Or else in the realm of analytical geometry, it could be read as "the geometric place that corresponds to the straight line that passes through the origin with slope $a$."

In the latter example, clearly texts expressed by means of the same MSS have no lack of ambiguity because those very texts can be read as texts of different related MSSs.

As another example: in the expression $(3 + 5 - 2) - (7 - 3 - 2) = 4$, the equal sign functions as an indicator of the result of actions performed with signs that belong to the arithmetic MSS.

$$\underbrace{(3+5-2)-(7-3-2)}_{\substack{\text{OPERATIONS} \\ \text{(actions to be performed)}}} \quad = \quad \underbrace{4}_{\substack{\downarrow \\ \text{RESULT}}}$$

Figure 1.5

However, the equal sign that appears in $4x + 2 = 5x - 3$ denotes a relation between expressions (between texts). This relation is algebraic, but it is numeric for a specific value of $x$, which leads to a numeric identity when substituted in the equation: $x = 5$ gives $4(5) + 2 = 5(5) - 3$; $22 = 22$.

$$\begin{gathered} 4\text{x} + 2 = 5\text{x} - 3 \\ 4(5) + 2 = 5(5) - 3 \\ 22 = 22 \\ \downarrow \\ \text{SYMMETRY} \end{gathered}$$

Figure 1.6

In this case, = denotes symmetry and the numeric value found for $x$ does not appear on the right hand side of the equalization as a result of the actions carried out; rather it is in a relation of identity with itself.

In an expression syntactically equivalent to the latter, such as $3x + 5 = 2x + x + 3 + 2$, the equal sign also denotes algebraic equivalence, but in this case it is tautological. In other words, the equality is valid for any value of *x*.

The examples given show that mathematical signs do not have one single interpretation, and therefore their correct reading requires a reconceptualization of the mathematical objects that these signs represent, as one steps from one context to another —from arithmetic to algebra or to geometry.

### *4.3. Algebraic and natural language translations*

In order to research translation from algebraic to natural language and viceversa, researchers can ask questions in which students are taught to use algebraic language to write sentences previously written in natural language —originally Spanish, although here we are presenting an English translation of the transcriptions. For example, one can ask a student to write phrases such as "*a* increased by 2" using signs, which we illustrate with part of a dialogue between the interviewer (I) and a student (S):

S: I don't understand that.
I: They give you a sentence and they ask you to write it using letters and signs representing operations.
S: In other words, symbolizing something …
I: Yes, but what do you mean by symbolizing?
S: …
I: Give me a sentence in Spanish in which you use "increasing."
S: You've increased speed.
I: And what does that mean?
S: That the person is now, well, going faster?
I: And another sentence in Spanish which also includes "two"?
S: Well … I went on increasing my speed for two days.
[…]
S: In the last two days he's increased in weight.
[…]
S: His weight increased by two kilos.

In this case, the student makes use of meanings taken from colloquial language in order to answer the teacher's questions. It is obvious that he needs to give meaning to the phrase presented before proceeding to symbolise it, and in all replies the student is inconsistent in terms of the varying interpretations of the phrase, which will lead to its incorrect symbolisation. This inconsistency derives from the fact that in the original phrase "*a*" and "two" are measurements or quantities of the same thing, and here the student assigns them to different things (speed and days, weight and time), except in the sentence “His weight increased by two kilos,” in which he seems to speak of an initial weight (*a*) and the *two* kilos by which it has increased.

In another case, a student responds to the same question with the drawing that appears in Figure 1.7.

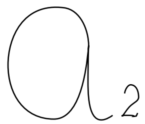


Figure 1.7

Here the word “increase” is clearly not identified with the mathematical action of adding, but corresponds to a real action of enlarging or expanding, an action that affects the letter (sign) “a”.

By putting questions such as these to students, one can observe the interaction of mathematical language with natural language. In this regard very interesting analyses have been undertaken that illustrate once again that, at the ages in reference, the meanings of the words in natural language predominate and that these meanings inhibit translation of phrases that consist of those words to the MSS of algebra.

Here is another example. When students were asked to read open expressions such as

$$\frac{a+b}{2},\ ab,\ 3ab,\ a^2,$$

in addition to producing textual readings like

“*a* plus *b* over two”

some students tended to associate geometric meanings with these expressions, and therefore they produced non-algebraic readings. We will now illustrate this with part of a dialogue between the interviewer and a student:

I: Read the next expression aloud. [Pointing to the first one in the list shown above].
S: Broader side over two.
I: And now, if you stop thinking of it as a formula, what would you read there, in what situations have you come across it?
S: Well, it’s for finding a result.
I: Such as?
S: Well …

Here, after a long pause, the interviewer intervenes and asks the question again, this time referring to the expression $a + b$.

S: As numbers … for example, 50 + 20.
I: So that if this [indicating $a + b$] is there on the blackboard, it means 50 + 20.
S: No, I mean, it could also mean something else.
I: Such as what?
S: Another unknown.
I: What unknown?
S: For example, $a$ equals, no … well, if $a$ equals 20, what does $b$ equal?
I: And there …
S: The unknown is $b$.

What we see in this dialogue is that the student tends to interpret open expressions as geometric formulae (for example, "broader side over two") or else tends to close them, either seeking a result by assigning specific numeric values to the letters, or converting one of the letters into a given and the other into an unknown, which is a characteristic of the closed expressions of algebra, i.e., equations.

In both cases, we see that the letters and operation symbols still suggest to the student meanings associated with those signs in primary school. In other words, expressions that include letters or that are formulae or that are simple equations, even when the equals sign —necessary in these two cases— is absent. It is the student who completes the expression in order to be able to read it within contexts that are familiar to him or her.

This example points to the kind of semantic antecedents of the MSSs used by 12- to 14-year old students, which are the foundations upon which their algebraic language skills must be developed. It is there that open expressions, for instance, will denote new mathematical objects at a higher level of generality, involving more general concepts such as that of generalized number ($a$ and $b$ in $a + b$) and that of suspended operation (the addition in $a + b$).

## 5. ALGEBRA AS A LANGUAGE: APPROACHES FROM LINGUISTIC, SEMIOTIC, AND HISTORICAL PERSPECTIVES

In this section, we review several works on mathematical language and language and mathematics that are especially relevant to the issues raised and analyzed in this book.

From a broader perspective than one that solely encompasses algebra, David Pimm has carried out an analysis of the language spoken and written in the maths classroom, expressed in his book *Speaking Mathematically*, which was published in the late 1980s (Pimm, 1987). In this work, Pimm tackled the task of examining school mathematical discourse by applying analytical techniques from theoretical linguistics. The language of the students, the language of the teacher, and the discussions in the mathematics classroom expressed through the authentic output of those actors constitute the main corpus of analysis in Pimm's research.

As far as theoretical analytical instruments are concerned, Pimm turns to the linguistic concept of register in order to approach the concept of metaphor in mathematics, to which he gives special importance, since he sets out from the recognition that the part played by this concept in the learning of mathematics is as fundamental as the part it plays in the learning of natural language (Pimm, 1987). When treating the theme of the formalism of written language, Pimm necessarily touches on the subject of symbolic algebra, inasmuch as it is an essential reference when speaking of a system of symbols in mathematics, and of their syntax and grammar. His analysis also tackles the theme of the role of natural language in teaching and learning in mathematics, with special emphasis on how the meanings assigned to words in colloquial language are spontaneously transferred by children to mathematics.

The fact that Pimm concentrates his analysis of language on the maths classroom, in its various expressions (speaking, writing, reading) and through the output of various actors (pupils, teacher), is a manifestation of his clear interest in matters of communication, which places him among researchers with a conception of mathematics as a social activity.

Raymond Duval, on the other hand, in his book *Sémiosis et Pensée Humaine. Registres sémiotiques et apprentissages intellectuels* (Duval, 1995), tackles the subject of learning in mathematics from a semiotic perspective, based on the relationship between semiosis (apprehension or production of a semiotic representation) and noesis (cognitive acts such as conceptual apprehension, understanding of inferences or discrimination of differences). Duval emphasizes the role of this relationship in the cognitive functioning of thinking and in its implications for the learning of mathematics and the native language. The variety of semiotic systems of representation in mathematics (graphs, formulae, tables, geometric figures, etc.) and the conversions between them are the material analyzed in Duval's works, which indicates that one of the greatest problems in semiosis has to do with the phenomena of non-congruence, which arise precisely in processes of conversion between representations. One of the central theses in this work is that coordination of registers of representation by learners is a necessary condition for conceptual apprehension in mathematics.

Among the studies on learning in mathematics with a semiotic perspective, Duval's is characterized by its theoretical analysis of the relationship between semiotic representations and mental representations, in cognitive development and in the exercise of cognitive activities. It is also characteristic of this author to give prominence to the cognitive activities of reasoning and the comprehension of texts, which prompts him to expound specific aspects of argumentation and proof in mathematics and to tackle the subject of sense. This last aspect is considered basically in relation to the orientation and conscious control of the fundamental cognitive activities by learners.

Such a wide-ranging treatise as Duval's could not fail to apply its theoretical analysis to the learning of the mother tongue and its connection with learning in mathematics. Within this framework, the author also deals with the differences and relationships between natural language and formal languages, taking geometry and logic as cases illustrating translations between the native language and formal language. The case of algebra is not used in this sense, but there is no doubt that it could be very relevant for the analysis of specific situations of "putting into equations," that is, the translation of the text of a problem (written in natural language) into algebraic language. There is a clear allusion by the author to symbolic algebra in the chapter devoted to conversion between registers, in connection with conversions between algebraic expressions and Cartesian graphs, but without devoting an *ex professo* treatment to algebraic language, with respect to the major themes that he develops, such as congruence and conversions between semiotic representations; the comprehension of texts and sense; and the relationships between noesis and semiosis, natural language and formal language, and mental representations and semiotic representations.

More recently, in his book *Mathematics Education and Language* (Brown, 2001), Tony Brown has presented a theoretical study in which elements of hermeneutics, linguistics, poststructuralism, and social phenomenology are combined to analyze the instrumental character of language in the development of mathematical understanding. Brown uses examples taken from research on mathematics education to examine how language influences the activity developed in the normative framework of a given situation. One of the implications of this analysis is that learning can be seen as a reconciliation between the conventional ways and potential ways (for learners and teachers) of describing such a situation.

Accordingly, Brown pays special attention to the role of pupil and teacher narratives. Specifically, in Chapter 8, "Narratives of learning mathematics," he analyzes a theoretical perspective concerning the ways in which pupils progress in learning mathematics, for the particular case of progressing from arithmetic thinking to algebraic thinking. He goes back to data collected by other authors in various studies on this transition and proposes that they be revised, including their discourses and suppositions. For this purpose he makes use of Ricœur's analysis of time and narrative in order to form an analytical approach to the treatment of notions such as transition, development and progression in the learning of mathematics. From this new perspective, the results of previous studies on the transition toward algebraic thinking attain another dimension, that of the view of the individuals who experience the transition and who use their own resources of expression to narrate their appreciation of the boundary between arithmetic and algebra. Moreover, according to Brown, in light of what Ricœur calls semantic innovation, adding a new narrative is interpreted as an extension of the familiar comprehensions of an individual with respect to the actions that he performs to incorporate figures of speech that will enable him to grasp a mental experience that has not yet fitted into previous versions of his linguistic usage. In fact, the cases of transition from arithmetic to algebra that Brown reanalyzes contribute elements that recreate this part of Ricœur's theory concerning semantic innovation.

The studies by Pimm, Duval, and Brown to which we have referred respond to the need to develop theory in order to analyze phenomena of the learning and teaching of mathematics closely connected with language in a broad sense. In these approaches, the analysis of mathematics as a language —in its various expressions, oral and written; with its different semiotic representations, through formulae, graphs, tables, etc.; used by different actors, pupils and teachers; through conventional expressions or potential expressions (narratives) — is as important as the analysis of its intricate relationship with natural language. The theoretical advances reported in these three works draw on theories from other disciplines, such as linguistics, semiotics, critical sociology, and hermeneutics. Similarly, this book forms part of attempts to theorize about mathematics, language and education, with a specialised focus on the language of symbolic algebra, assuming, as we indicated earlier, a theoretical view in which two main elements participate, semiotic and historical, and adopting a perspective based on pragmatics, favoring the study of meaning in use rather than formal meaning. In this way the focus of attention is shifted toward the activity of individuals with the language of algebra. Essentially, grammar, as the formal system, and pragmatics, as the set of principles of using language, are conceived as complementary domains, especially when they are related with models of teaching algebra.

Other works that emphasize algebra's character of written language have been devoted to the task of analyzing algebraic syntax and semantics, taking elements from support theories, such as linguistics and semiotics. The work done by David Kirshner makes use of generative and transformational grammar (Kirshner, 1987) to generate simple algebraic expressions and perform transformations with them, all based on descriptions of the superficial forms and deep forms of those expressions. In transformational grammar, the transformations of the expressions take place in the corresponding deep forms, which reveal the structure of the forms produced with respect to the operations that constitute them and their hierarchy.

Jean-Philippe Drouhard, on the other hand, develops a notion of signification, with which he associates four aspects: reference, which corresponds to the function of algebraic evaluation; sense, which is given by the set of transformations applicable to the expression; interpretation, which corresponds to the various readings given to the expression in the different contexts in which it may appear (such as number theory, analytical geometry, etc.); and connotation, which corresponds to psychological signification (depending on each individual) (Drouhard, 1992). The analysis of the meaning, or significations, of algebraic writing is then approached theoretically by the application of this subdivision into these four aspects.

Finally, in this brief survey of the studies most directly related to ours, we must mention the work of Luis Radford, who shares with us a semiotic perspective and an interest in historical analysis, which he proposes from an anthropological viewpoint (Radford, 2000a, 2003, 2004). Radford takes from Vygotsky the idea that human cognitions are tied to usage of signs, so that it is no longer central to consider what signs represent but rather what they enable one to do; furthermore, these signs belong to sign systems that are part of a culture and therefore transcend individual cognitions (Radford, 2000b). From this viewpoint he analyzes both the emergence of algebraic thought in pupils who are starting to study algebra and the emergence of algebraic symbolism in history.

Developments such as those just described have proved to be valuable materials in the applications of the theoretical formulation that is discussed here, the connection of which with teaching forms part of its essential characteristics as it envisages the need to develop local models (to interpret specific phenomena), which comprise components of formal competence, teaching, cognitive processes and communication.

## SUMMARY

Throughout the book we emphasize our adoption of the pragmatic perspective of meaning in use rather than formal meaning, which has led many studies, and this one in particular, to concentrate attention on the user's performance with the mathematical sign system (MSS). In the case that concerns us, the theme of the algebraic sign system and its relationship with the sign systems of arithmetic and the native language and with personal output is approached on the basis of the notion of MSSs and strata of MSSs. Before introducing these notions in greater depth, here we have presented some basic aspects of Charles S. Peirce's semiotics that are pertinent for an understanding of the sense in which we use the notion of sign, in particular, the triadic conception of the sign, with the introduction of the interpretant as the third fundamental element, the idea of unlimited semiosis.

In the next chapter, we shall develop the notion of the local theoretical model in the context of curricular design and development for students, teachers, and researchers. We shall also stress the crucial role of Freudenthal's didactical phenomenology, both in this context of curricular design and development and in the context of experimental design, which we present in Chapter 3. In Chapter 4 we describe a study conducted following such a design.

All chapters of this book shall have the same structure as this Introduction. They all begin with an Overview and end with a Summary (which includes mention of the topic to be discussed in the chapter that follows).

## ENDNOTES

[1] We have given the references to Peirce's works not by indicating the year of publication but by using the abbreviation *W* followed by the volume number in the case of the ongoing publication by the University of Indiana entitled *Writings of Charles S. Peirce: A Chronological Edition*, or by the abbreviation *CP* followed by the paragraph number in the case of the now classic collection of his works entitled *Collected Papers of Charles Sanders Peirce*.
[2] This is how Peirce explains the meaning of "represent" or "stand in relation to" in Peirce, *CP*, 2.273, p. 155.
[3] The ambiguity of the English expression "mathematical sign systems" would not exist if English used brackets as one of its structuring mechanisms, as is the case in the sign system of algebra. Then one would only need to write "mathematical (sign systems)" rather than "(mathematical sign) systems." Freudenthal (1983) analyzed this difference between the sign system of written English (and most vernacular languages) and mathematical sign systems in the chapter entitled "The Algebraic Language," and he illustrates this ambiguity of English with the example of the expression "pretty little girls schools," "which according to the places of the —lacking— brackets can have 17 different meanings" (Freudenthal, 1983, p. 471). Fortunately, our expression "mathematical sign systems" cannot be interpreted in so many ways, and we are using it in only one sense, which is the one specified by the brackets in "mathematical (sign systems)."
[4] We will come back to the idea of sense as opposed to meaning and the idea of giving sense throughout the book, particularly in Chapters 7 and 8.
[5] We have consulted the French translation (Ver Eecke, 1952) and the English translation (Sigler, 1987), both done from the original Latin, and the Spanish version done from the French version cited (Ver Eecke, 1973). A detailed analysis of this book from the viewpoint that interests us here can be found in Filloy (1993a).
[6] We remark that Fibonacci's Liber Abaci (L. E. Sigler, 2002) is one of the most famous of this collection of books. We come back to them in Chapter 3, Section 3.3.