Eugenio Filloy
Luis Puig
Teresa Rojano

Educational Algebra

A Theoretical and Empirical Approach

Mathematics Education Library

Springer

Educational Algebra

Mathematics Education Library

VOLUME 43

The titles published in this series are listed at the end of this volume.

Eugenio Filloy
Luis Puig
Teresa Rojano
(Authors)

Educational Algebra

A Theoretical and Empirical Approach

Springer

Eugenio Filloy
Centre for Research and
Advanced Studies
Mexico City
Mexico
SMMEef@aol.com

Luis Puig
University of Valencia
Valencia
Spain
Luis.Puig@uv.es

Teresa Rojano
Centre for Research and
Advanced Studies
Mexico City
Mexico
trojano@cinvestav.mx

Series Editor:
Alan Bishop
Monash University
Melboure 3800
Australia
Alan.Bishop@Education.monash.edu.au

Library of Congress Control Number: 2007930472

ISBN -13: 978-0-387-71253-6 e-ISBN-13: 978-0-387-71254-3

Printed on acid-free paper.

9 8 7 6 5 4 3 2 1

springer.com

CONTENTS

CHAPTER 1

INTRODUCTION

OVERVIEW

This book is based on an experience of ours in which the need to interpret unanticipated phenomena observed in empirical studies on the transition toward algebraic thought conducted in the 1980s, triggered a long-term research program that in turn led to a theoretical formulation that emphasizes local analyses.

To illustrate that experience, we briefly examine a few of the phenomena observed in the transition from arithmetic to algebra, which represent an essential part of pre-algebra. The observations dealing with cognition are presented in Section 4.2.1, the reverse of multiplication syndrome in 4.2.2, different uses of the notion of equality. Polysemy of *x*; and in 4.2.3, difficulties in translations. We begin by indicating the role of historical analysis in Section 4.1, and complete the section with an example of a dialogue that took place during a clinical interview, in which additional phenomena appeared in translating algebraic language to natural language (Section 4.3).

The book presents the theoretical elements developed and shows how the theory of local models, through their different components, has enabled a deeper study of phenomena in the field of acquiring algebraic language, considering aspects that are relevant to learning, teaching, and research.

Use of the term "educational algebra" in the title of the book, instead of the more usual term "school algebra" is appropiate given the broad-based nature of the educational aspects we deal with. As will become patent in the rest of the book, besides working with children and teachers in schools we have used other sources as well to design and develop empirical studies: semiotics, epistemological analysis (primarily history of mathematical ideas), phenomenological analysis (mainly Freudenthal's approach to curriculum development), formal mathematics, cognitive theories, etc. The term "educational algebra" is sufficiently broad to encompass the aspects that are educational, albeit not necessarily school-related.

We also introduce two central terms, "mathematical sign systems" and "local theoretical models", which are used throughout the book. They are discussed more extensively in Chapter 2 and in other chapters, where they are used in the description of concrete examples arising from the empirical studies.

We conclude with a review of other literature on the subject of mathematical language and language and mathematics to place our work within a context and to demonstrate its contribution therein.

The desire to achieve a profound understanding of both the origin and nature of the difficulties confronting those who seek to gain access to algebraic thinking has set in motion great ideas and inquiries about them over the past three decades. The vast amount of literature produced from all this research activity makes the task of surveying and updating the state of the art in this field increasingly difficult. It is not so difficult, however, to identify a series of studies that concentrate on studying symbolic algebra as a language, together with the details of its acquisition. Because of the abstract nature of algebraic language and the highly syntactic competences required for its use, many of these studies use approaches that include semiotic concepts and linguistic analyses. This book is devoted to setting out a path of theoretical development for educational algebra, in which this very perspective is adopted and in which an historical element becomes a contributing factor.

Despite the deliberately theoretical character of this work, its direction differs from that of general models. This work incorporates elements that make it possible to develop local frameworks of analysis and methodological design for the study of specific phenomena. In these frameworks, it is possible to include evidence connected with such phenomena, the interpretation of which escapes general treatments. Such is the case when individuals who are beginning the study of algebra, produce personal sign systems that are located on a level prior to the mathematical sign system that is to be learned (that of symbolic algebra in this case).

After the worldwide acknowledgment in the late 1970s that the educational system had largely failed to teach algebra in secondary schools, one of the great ideas put forward was Hans Freudenthal's proposal. Freudenthal stressed the need to analyze the language of algebra by comparing it with other languages, such as natural language and the language of arithmetic, both of which were considered means of support (Freudenthal, 1983, ch. 16). His dissertation was followed by many other studies dealing with mathematics education seen through a linguistic hue.

Most of the research carried out recently on the didactics of mathematics lacks paradigmatic theoretical models, even if one uses the term paradigm (somewhat in the sense of Kuhn, 1962) not as a synonym of theory, but in a more general sense, i.e, as the set of basic assumptions that one can make about the nature and limits of the actual subject to be studied, the method for studying it, and the decision as to what will be accepted as evidence. Nor has a consensus been reached about which of the basic assumptions should determine the form to be taken by the theoretical frameworks for interpreting specific phenomena and for proposing new experimental designs that will carry theory further forward to embrace other evidence or new unrelated evidence. In short, it is still necessary to speak of the boundaries of many research projects.

As a start, other disciplines have already begun research on the very subjects that pervade most of the work on which mathematics educators have reported. Some of these subjects include linguistics, logic, psycholinguistics, semiotics, general cognitive psychology, the psychology of mathematics, the epistemology of mathematics, the history of mathematics, the psychology of education, the theory of the development of mathematics curricula, and the didactics of mathematics.

Many research studies have recently incorporated the results of these disciplines and have redefined results within their own theoretical frameworks. Here we interpret various recent theoretical assumptions to reorganize the research undertaken on the processes of teaching/learning algebra during the past few years. To accomplish this, it is necessary to work with a good deal of new terminology to be able to describe recent research.

To this end, in Chapter 2 we introduce the methodological concept of local theoretical models (LTMs). Although LTMs are dealt with in greater detail in Chapter 2, we can state here that the subject is considered in terms of four interrelated components: (1) teaching models, (2) models for the cognitive processes, (3) models of formal competence and (4) models of communication. Here we shall refer only to their local character.

1. ON THE LOCAL CHARACTER OF THE THEORETICAL FORMULATION AND ITS COMPONENTS

One of the chief reasons for resorting to local theoretical elaborations was the need to interpret phenomena that arose during the study. These phenomena could not have been anticipated from the design of the observation and did not fit into schemes of analysis based on general theories derived from mathematics education itself or from neighboring disciplines such as psychology, pedagogy, sociology, history, epistemology, or linguistics. Studies on the transition from arithmetical thought to algebraic thought carried out in the 1980s came up against this situation, giving rise to a long-term research program that envisaged the development of theoretical elements that would make it possible to refine the analysis of such phenomena. An initial hypothesis is that although we set out from a general notion —that of the mathematical sign system— it is the local character of the theoretical elaboration that makes it possible to delve deeper and thus generate new knowledge about the subject. Hence LTMs (Filloy, 1990) represent the central idea in this work. Rather than partializing the problems of mathematics education research, LTMs open up paths of communication between the various components that usually contribute to them. In fact, each local model contemplates the study of cognitive aspects, formal mathematical competence, teaching, and communication. This comprehensive approach offers possibilities of making a substantive contribution to a highly focused study, based on a multiplicity of disciplines and drawing on the work of specialists and communities connected with those disciplinary fields. The contents of this book are the result of progress in the research agenda that we set ourselves when, in our studies on algebraic thought among adolescents in the 1980s, we were first faced with the limitations of general analytical schemes in trying to interpret the phenomenon of the polysemy of x or that of the reverse of multiplication syndrome, for instance. Later in this chapter we provide a detailed description of those phenomena, as well as others that arose during our research. For our descriptions, we shall be using the notion of mathematical sign system (MSS), a brief introduction to which is provided in the following section.

2. MATHEMATICAL SIGN SYSTEMS

2.1. Sign

This section discusses the phenomena that take place in mathematics education, using the jargon of semiotics. We do so not to embellish our observations with cryptic language, but because we consider these phenomena as processes of signification and communication, and semiotics deals with processes of precisely this type.

The fact that semiotics studies these processes rather than signs is especially clear in the semiotics developed by Charles Sanders Peirce. In Peirce's semiotics, this emphasis on processes is present even in the very idea of sign. Peirce gave countless definitions of "sign" throughout his extensive writings, in which he repeatedly outlined the concept. In all of his definitions, three characteristics are worthy of special emphasis. The first is the fact that the sign is not characterized by a dyadic relation such as that of Saussure's signifier/signified pair; the relationship to which any sign belongs is triadic. And one of the elements, which Peirce calls the "interpretant," is the *cognition* produced in a mind. The second is the fact that the sign is not a static entity but is instead open within a series. Since all cognition is in turn a sign, that sign therefore stands within a triadic relationship to another interpretant (which is another cognition), and so on and so forth. The third is the fact that the sign is not arbitrary or rather that the triadic relation to which it belongs is not arbitrary.

In a manuscript written in 1873, Peirce gives his briefest and most compact definition of a sign:

A sign is an object which stands for another to some mind (Peirce, *W* 3, p. 66).[1]

The relation is established between the sign (S), its object (O), and a mind for which the sign is related to its object in such a way that, for certain purposes, it can be treated as if it were that other.[2] Let us see how Peirce defines the interpretant (I):

A sign […] addresses somebody, that is, creates in the mind of that person an equivalent sign, or perhaps a more developed sign. That sign which it creates I call the *interpretant* of the first sign. The sign stands for something, its *object*. (Peirce, *CP*, 2.228, p. 135.)

The triadic relation (S, O, I) is, therefore, a relation in which both S and I are signs, so that I is a new sign, S', which enters into another triadic relation, i.e., it creates in a mind another sign as interpretant, I', of object O, a new cognition I', such that object O links the two triadic relations (S, O, I) and (S', O, I'). This leads to the open nature of the sign in a process of semiosis that has no end. Peirce expressed it thus in another definition, subsequent to that quoted above:

Sign [Lat. *signum*, a mark, a token]: Ger. *Zeichen*; Fr. *signe*; It. *segno*. (I) Anything which determines something else (its *interpretant*) to refer to an object to which itself refers (its *object*) in the same way, the interpretant becoming in turn a sign, and so on *ad infinitum*. (Peirce, *CP*, 2.303, p. 169.)

Also present in this definition is the third aforementioned characteristic: the fact that the relation is not arbitrary. The sign forces the interpretant to refer to the same object as the one to which it refers. In a more extensive definition, quoted later, Peirce is even more exigent and adds that the sign forces the interpretant to refer to the same object and, furthermore, *in the same way* as it refers. Moreover, there must also be an interpretant, I_1, of interpretant I, which has as object O_1, the *relation* between the sign and its object.

A Sign, or *Representamen*, is a First which stands in such a genuine triadic relation to a Second, called its *Object*, as to be capable of determining a Third, called its *Interpretant*, to assume the same triadic relation to its Object in which it stands itself to the same Object. The triadic relation is *genuine*, that is its three members are bound together by it in a way that does not consist in any complexus of dyadic relations. […] The Third must indeed stand in such a relation, and thus must be capable of determining a Third of its own; but besides that, it must have a second triadic relation in which the Representamen, or rather the relation thereof to its Object, shall be its own (the Third's) Object, and must be capable of determining a Third to this relation. All this must equally be true of the Third's Third and so on endlessly […] (Peirce, *CP*, 2.274, p. 156).

2.2. Mathematical sign systems

The examples presented throughout the book have enabled us to make use of Peirce's concept of the sign and its typology, and to explore the sense through which it casts light on what we wish to examine. The examples also show something else: the signs that are used in mathematics are not all of a linguistic nature, which makes it advisable not to use the terminology or concept of the sign that belong to linguistics (derived, to a greater or lesser extent, from the work of Saussure), and therefore not to speak of the signifier/signified pair. In the preceding text we have not done so, using instead the term "expression," from the expression/content pair —terminology that has been introduced in semiotics (the science of signs in general, and not just of linguistic signs). This is also very convenient because in mathematics one is accustomed to speaking of "algebraic expressions" or "arithmetic expressions" to refer to the corresponding written forms.

However, in putting the emphasis on individual signs, what we have seen so far may conceal the crucial fact that there are no isolated signs in any text (whether mathematical or not).

It is very common for a description of the language in which mathematical texts are written to distinguish between two subsets of signs: one consisting of signs conceived as strictly mathematical and another consisting of signs in some vernacular language. From the viewpoint of signification processes, however, this distinction ceases to be crucial although it can still be made. What seems to be crucial is the sign system taken as a whole, and what must be described as mathematical is the system and not the signs, because the system is responsible for the meaning of the texts. One must therefore understand the term "mathematical sign systems" as mathematical systems of signs and not systems of mathematical signs,[3] for what is of a mathematical nature is the system and not just the individual signs. Consequently, what is of interest for the development of mathematics education is to study the characteristics of these (mathematical) sign systems which are due not just to the fact that they are sign systems but also precisely to the fact that they are mathematical systems.

Filloy (1990) and Kieran and Filloy (1989) introduced the need to use a sufficiently broad notion of mathematical sign systems. It had to serve as a tool to analyze the texts produced by students when they are taught mathematics in school systems —and those texts are conceived as the result of processes of production of sense— as well as to analyze historical mathematical texts, taken as monuments, petrifactions of human action, or processes of cognition belonging to an episteme. In taking these mathematical texts as the object of study, rather than supposedly ideal texts conceived as manifestations of "mathematical language" or texts that are measured by them, the notions of mathematical sign systems and of text must both open up in various directions.

Thus one must speak of mathematical sign systems, with their corresponding code, when there is a socially conventionalized possibility of generating sign functions (by the use of a sign functor, see Chapter 7), even when the functional correlations have been established in the use of didactic artefacts in a teaching situation with the intention that they should be impermanent. But one must also consider the sign systems or strata of sign systems that learners produce in order to give sense to what is presented to them in the teaching model,[4] although they may be governed by a system of correspondences that has not been socially established but is idiosyncratic.

3. DIFFERENT ANSWERS TO SAME QUESTIONS?

We point out, however, that not only semiotics but also information processing theory and the didactics of mathematics (Brousseau, 1997) have done important work on the notion of code. This notion is emerging as a key concept to interpret what comes from using the idea of representation in the models that explain the cognitive problems presented by alternative teaching approaches or technology learning environments. Or, to provide another example, consider the emphasis that psycholinguistics and artificial intelligence place on a process-based model of human capabilities and relate it to the way in which the model explains how and why users of mathematical language naturally and commonly make mistakes in its syntactic procedures. To these developments, one must add the attention that a pragmatic viewpoint has given to meaning in use rather than formal meaning.

By accumulation, these approaches —and others of a similar nature— have led to a change of direction in recent work, which is shifting away from the competence of mathematical language users and moving toward performance. This change of viewpoint has basic and essential implications for the manner in which mathematical language is seen. Essentially, the claim is that grammar — the formal abstract system— and pragmatics —the principles of language usage— are complementary domains in our studies. In addition, that both are domains related to the various teaching models, be they innovative or traditional, that are used to achieve the objective of guiding students in order for them to become competent users of the language of mathematics. Since one of our aims is to observe what happens in mathematics classrooms, however, we must also confront the complexities of teaching and learning phenomena within that particular setting.

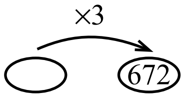
Indeed, one of the simplest phenomena demonstrated by classroom observation, for instance, reading level permanence among children who have just finished primary education (approximately 12 years old), is what arises when they are confronted with questions like those in Figure 1.1, which shows the evolution of the equation $Ax = B$ in school teaching.

Evolution of the equation $Ax = B$

1) $3 \times \square = 12$

2) $3 \times \square = 672$

3)



4) $3 \times x = 672$

5) $3x = 672$

Between the ages of 10 and 12 it is easy to direct some students so that all the questions are read like [2]: What is the number that, when multiplied by 3, gives 672?

Figure 1.1

When one analyzes the responses of children within this age group, many issues arise. Apart from the fact that these questions are deemed as different because some can be answered and others cannot, we also find that it is fairly easy to get a certain student profile bogged down in their use of the preferred arithmetic method, trial and error. It is even quite easy to induce them to continue using that method for a considerable time despite the fact that the numbers become progressively larger, which eventually means that they no longer have sufficient arithmetic skills to be able to answer the question without making mistakes. We call this phenomenon "the reverse of multiplication syndrome."

To be able to observe phenomena of this kind experimentally, we therefore need an experimental framework that will enable us to interpret the facts and propose new observations that will unravel the relations existing between the different components in play.

4. THE TRANSITION FROM ARITHMETIC TO ALGEBRA. PRE-ALGEBRA. SOME OBSERVATIONS ABOUT COGNITION

Several studies have indicated conceptual and/or symbolic changes that mark the difference between arithmetic and algebraic thought. Examples of the foregoing are those related to the various interpretations of letters (Booth, 1984), those dealing with the notion of equality (Kieran, 1980, 1981), and those produced with respect to the symbolic or graphic conventions for coding operations and transformations in solving of equations (Matz, 1982). From such indications, one can imagine paths of evolution from arithmetic to algebraic thought that correspond to the representative notions and forms for the objects and operations involved in the mechanisms of change. Thus, the changes deemed essential for a person to attain algebraic knowledge can be visualized along each of these paths as points where there is a cut between one kind of thought and another.

One of the foregoing points that is of particular interest to the topic of equation solving is suggested by analyzing the strategies and methods used to solve equation systems in texts of pre-symbolic algebra from the 13th to 15th centuries. An important factor in this analysis for developing solution strategies and methods is that of operating unknowns. This arises as a result of the limitations imposed by the frameworks that belong to the pre-symbolic representation of equations and their characteristic elements. Thus, for example, the solution that led to equations that we now write as $x^2 + c = 2bx$ and $x^2 = 2bx + c$ are completely different in each case. Yet this would not happen if the rules for transposing terms from one side of an equation to the other were known because, for instance, then it would be possible to reduce the case of one of the equations on a syntactic level to that of the other, which would correspond to a more developed level of operation on unknowns.

4.1. The role of historical analysis

The propositions contained in Leonardo of Pisa's *Liber Quadratorum* (Book of Squares)[5] can quickly be proved using the mathematical sign system of secondary school algebra. Indeed, they are propositions that can be proved in less space than that taken up by one of the pages of this text, and their mathematical content does not go beyond what is presently learned in secondary school. Nevertheless, it is easy to perceive the intensity of thought required to follow the reasoning depicted in Leonardo's book. And it is not as if he were rhetorically playing with trivial matters. On the contrary, his work is very possibly the pinnacle of mathematical thought in Middle Age Europe. Frozen in time on those pages one finds reasoning that drew —and still draws— great admiration because of its freshness and intensity. It is a thought that comes to us from the 13th century and that shows us how a mathematical sign system predetermines the ways in which we analyze problems, advocates our solution strategies, and draws the lines of strength that guide the sense of all our inferences. This strikes us as odd within the context of developing the thought processes of children, but it is even more amazing in the thought process of a first-class mathematician, perhaps the greatest mathematician of that era. At the same time it provides us with the opportunity to discover unknown terrain on which we can observe and describe the same cognitive tendencies as we find in today's children as they attempt to become competent users of the mathematical sign systems that they are taught in secondary school. It further enables us to draw plausible hypotheses, and then observe those hypotheses in the behavior of present-day students as one tries to have them make competent use of the sign systems currently used to articulate the messages through which today's mathematics education is communicated.

As one confronts mathematical texts such as those of Leonardo, one's attention is first drawn to the fact that no one speaks that language now. Were they translated into the mathematical sign system of current symbolic algebra they might appear to be advanced problems typical of a modern textbook. They, however, differ in that their solution strategies do not conform to customs. In addition, today one would not perform many of the operations and intermediate steps that seem to be necessarily present in those texts. The language of the *abbacus* books[6] is today a dead language. Their translation to modern algebraic language fill us with amazement for the novel actions that led to the same results as ours, but that follow unheard of paths. Their very presence in problem after problem and in book after book are indicative of skills unrelated to those we have developed in building and using our algebraic language —skills, one might add, that we have never felt the need to build, develop, or use when confronting problems with our arithmetic abilities and knowledge.

Clearly, as we build new conceptual apparatuses that have imposed themselves upon us without the possibility of erecting them within a proper structure, connecting them to others that have previously been firmly rooted, those new skills tend to overshadow older skills. Moreover, given the fragile means at one's disposal to use at that point any new resources and solution techniques, even problems that had been mastered for quite some time are now difficult to model in the new language within which the infant conceptual apparatus that is in the process of being constructed is expressed.

Nonetheless, well anchored intellectual structures tend to perpetuate themselves and compel us to reconsider situations that, when modeled in the new language, could be solved with simple, routine operations.

Indeed we do realize that we began speaking of reading and interpreting ancient texts, and have now taken a leap onto the plane of psychological processes. We now feel confident in saying that it is precisely this leap, forward and backward, that enables us to produce hypotheses founded on the development of general knowledge and to convert them into hypotheses about the didactics of mathematics, which then seeks to reconsider that process on the level of individuals –children, in this case. The MSS of arithmetic has to make way for that of algebra, and this has become an increasingly pertinent matter even for situations that have always before been modelled in arithmetic.

Building the new MSS, whose point of departure must necessarily be elementary arithmetic operations, will involve the need to operate on new objects. These objects will signify not only numbers but also numerical representations, whether as individual items (e.g., unknowns), sets of numbers (e.g., coefficients of equations), an expression of relations between sets of numbers (e.g., proportional variation), or as functions, etc. The algebraic MSS will have to be structured on new objects whose operations will not be completely determined until the outlines of the new world of objects become more precise. What is more, the objects will not be totally outlined and well defined until the new operations have been completely structured in terms of both their semantic and syntactic aspects.

Such profound changes in arithmetic habits and notions do not take place spontaneously in individuals simply because they are confronted with the need for change. The intervention of teaching, at that point of transition from arithmetic to algebraic knowledge, can be crucial for most students who are learning algebra for the first time.

Although it is necessary to modify some arithmetic notions in order to acquire the new —algebraic— knowledge, it is also necessary to preserve the previous knowledge —arithmetic, in this case. Even in the single example of equations previously presented, there is a need for arithmetic equations to subsequently be recognized as such in order to preserve the entire operativity acquired beforehand for their solution —an operativity situated on an intermediate level of knowledge between arithmetic and algebraic knowledge, that is to say the level of pre-algebraic knowledge.

4.2. Mathematics lessons at the beginning of secondary education

We present three types of situations that generally arise when students have just completed elementary education and are beginning secondary education:

1) The reverse of multiplication syndrome.
2) Different uses of the notion of equality. The phenomenon of contextual ambiguity.
3) Difficulty translating from natural language to algebra and viceversa.

4.2.1. The reverse of multiplication syndrome

As mentioned in Section 3, some students get stuck using the arithmetic method of their preference, which is trial and error when solving the equation $Ax = B$, and even go on using this method when it has become inefficient because the B numbers are too large for them to perform arithmetic trial and error without making mistakes.

During the first year of secondary school (in the Mexican Educational System), most students end up preferring the method of dividing B by A in order to solve the equation $Ax = B$, which is the objective of the mathematics syllabus at that stage. However, the same trial and error strategy reappears in the work of students who had already achieved operativity to solve all first-degree equations, when the context in which the equation $Ax = B$ appears comes from an analytical process while the student is solving a word problem.

Even more surprisingly, at times when the expression $Ax = B$ is written by the very person who is being observed, the signs are not recognized as the expression of an equation that a few moments before the student knew how to deal with operationally to find the solution. The context in which the equation appears, even in its written form, makes the student "forget" the operativity achieved previously and revert to preferring the arithmetic method of trial and error or, in some cases, become unable to bring any method of solution into play. A more detailed description of what happens in the latter case shows that the interpretation of the sign x is crucial in interpretation of the expression $Ax = B$: interpreting the x as *an unknown* makes the student not know what to do, because "it is something that is not known," in the student's own words. In addition, it is important to recall that we are at a point in teaching when we are trying to have students begin to use the knowledge they have learned about solving first-degree equations in order to solve application problems that appear in mathematics lessons as well as in physics, chemistry, and other subjects.

4.2.2. Different uses of the notion of equality. Polysemy of x

Several ways of interpreting equality can be distinguished among the uses made by 12- to 13-year-old children, as follows:

A) As an arithmetic equality (E_{Ar})

In this case, the student tries to combine the terms on the right side of the equation or read them as a single number before giving any type of answer or performing any operation. Those who make this interpretation carry out one of the following procedures:

i) Completion: “This [the independent term on the left side] needs this much to be equal to this [the complete right side]”.
ii) Direct isolation: this procedure predominates in students who perform well, but it also appears in the other cases. In items such as $x+\frac{141}{16}=17+\frac{141}{16}$, $x+\sqrt{17}=42+\sqrt{17}$ and $x+\frac{x}{4}=6+\frac{x}{4}$, students who make this interpretation *(E_{Ar})* and who try to isolate x face serious difficulties. For some students, the fact that they do not know the value of $\sqrt{17}$ prevents them from tackling the $x+\sqrt{17}=42+\sqrt{17}$.

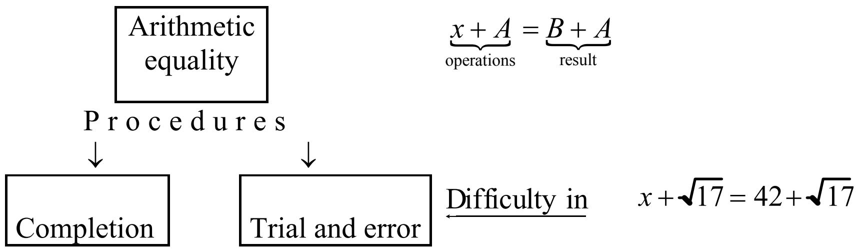


Figure 1.2

B) Equality of the left side (as a whole) with the right side (also as a whole)(C_0)

This interpretation also allows for two procedures:

i) Completion, which in many cases is more visual than arithmetic.

ii) Isolation. In some children of mid-level performance, the C_0 interpretation precedes the appearance of the E_{Ar} interpretation. In other cases, it appears on its own.

Equality of the two sides, taking each side *as a whole*

$$\underbrace{x + A}_{\text{a whole}} = \underbrace{B + A}_{\text{a whole}}$$

Procedures

↓ Completion (More visual than arithmetic)

↓ Trial and error

Figure 1.3

C) Equality term by term (C₁)

With this interpretation it is possible to solve these equations very quickly, except the items $x+\frac{x}{4}=6+\frac{x}{4}$ and $x+5=x+x$, on which we will comment later. This interpretation predominates in students with a high achievement level, although in some cases it is preceded by the C_0 interpretation or (in children of mid-level performance) by rearrangement of the terms with respect to the = sign.

Equality term by term

$$\overset{\Downarrow}{x} + \underset{\uparrow}{A} = \overset{\Downarrow}{B} + \underset{\uparrow}{A}$$

Predominates in high performance level students

↓

Quick solution ← By comparison Or By canceling

Preceded by:

Interpretation (A)

Rearrangement of terms with respect to the = sign

Figure 1.4

In the items $x+\frac{x}{4}=6+\frac{x}{4}$ and $x + 5 = x + x$ there is a tendency to give a C_1 interpretation, but also to assign different values to different occurrences of x. The typical response is:

This x ($\overset{\downarrow}{x} + \frac{x}{4} = 6 + \frac{x}{4}$) equals 6 and these ($x + \overset{\downarrow}{\frac{x}{4}} = 6 + \overset{\downarrow}{\frac{x}{4}}$) can be any number.

In the item $x + 5 = x + x$, the same kind of response appears:

This $\overset{\downarrow}{x}$ ($x + 5 = x + \overset{\downarrow}{x}$) equals 5 and these ($\overset{\downarrow}{x} + 5 = \overset{\downarrow}{x} + x$) can have any value.

In some cases, the students were asked to invent a problem that could be solved with this equation. Others were presented with a problem of the same kind to make them see that, within the context of one and the same problem, all occurrences of an unknown represent that same unknown. However, this clarification was not always successful.

We have denominated this phenomenon of unknown multivalence as "polysemy of *x*," because it involves a reading of the same sign in various contexts in which *x* is an unknown and in which *x* is a generalized number. Moreover, what "unites" these two interpretations or readings is the numerical equality of both sides of the equation.

4.2.3. Difficulties in translations

After secondary school students have received pre-algebra MSS instruction and been introduced to elementary algebra so as to solve linear equations and decode arithmetic-algebraic texts, yet before receiving systematic teaching on usage of open expressions, equivalence of expressions, and how to solve equation systems, the task of reading or writing algebraic language is very difficult for them. This is so much the case that one can almost see the tension mount in them as they struggle between using the arithmetic MSS to read and express themselves and their need to give mathematical signs new meanings within the context of the algebraic MSS. This is yet another indicator of the fact that the boundary between arithmetic and algebra cannot be avoided given that it would lead to false conceptions about the processes for acquiring the algebraic MSS, and, consequently, about the role of teaching within those processes. Furthermore, this highlights just how important it is for reading and writing algebraic MSS to be considered a decisive educational goal for middle school students.

The forms of notation used in algebra just happen to be basically the same as those in arithmetic, i.e., numbers, operation symbols, the equal sign, and letters. However, their meanings and the way they are operated essentially differ in the two fields. Consider the following example:

The two expressions $A = b \times h$ and $y = ax$ are syntactically equivalent, yet the way of reading them —their interpretation— determines the actions that subsequently have to be performed.

One (conventional) way of reading $A = b \times h$ is "the area (of a rectangle) is equal to the base times the height," which refers readers to the field of arithmetic-geometric MSS. Whereas $y = ax$, read in context, could mean "y is a linear function of x, with parameter a." Or else in the realm of analytical geometry, it could be read as "the geometric place that corresponds to the straight line that passes through the origin with slope a."

In the latter example, clearly texts expressed by means of the same MSS have no lack of ambiguity because those very texts can be read as texts of different related MSSs.

As another example: in the expression $(3 + 5 - 2) - (7 - 3 - 2) = 4$, the equal sign functions as an indicator of the result of actions performed with signs that belong to the arithmetic MSS.

$$\underbrace{(3+5-2)-(7-3-2)}_{\substack{\text{OPERATIONS} \\ \text{(actions to be performed)}}} = \underbrace{4}_{\substack{\downarrow \\ \text{RESULT}}}$$

Figure 1.5

However, the equal sign that appears in $4x + 2 = 5x - 3$ denotes a relation between expressions (between texts). This relation is algebraic, but it is numeric for a specific value of x, which leads to a numeric identity when substituted in the equation: $x = 5$ gives $4(5) + 2 = 5(5) - 3$; $22 = 22$.

$$4x + 2 = 5x - 3$$
$$4(5) + 2 = 5(5) - 3$$
$$22 = 22$$
$$\downarrow$$
SYMMETRY

Figure 1.6

In this case, = denotes symmetry and the numeric value found for x does not appear on the right hand side of the equalization as a result of the actions carried out; rather it is in a relation of identity with itself.

In an expression syntactically equivalent to the latter, such as $3x + 5 = 2x + x + 3 + 2$, the equal sign also denotes algebraic equivalence, but in this case it is tautological. In other words, the equality is valid for any value of x.

The examples given show that mathematical signs do not have one single interpretation, and therefore their correct reading requires a reconceptualization of the mathematical objects that these signs represent, as one steps from one context to another —from arithmetic to algebra or to geometry.

4.3. Algebraic and natural language translations

In order to research translation from algebraic to natural language and viceversa, researchers can ask questions in which students are taught to use algebraic language to write sentences previously written in natural language —originally Spanish, although here we are presenting an English translation of the transcriptions. For example, one can ask a student to write phrases such as "*a* increased by 2" using signs, which we illustrate with part of a dialogue between the interviewer (I) and a student (S):

S: I don't understand that.
I: They give you a sentence and they ask you to write it using letters and signs representing operations.
S: In other words, symbolizing something …
I: Yes, but what do you mean by symbolizing?
S: …
I: Give me a sentence in Spanish in which you use "increasing."
S: You've increased speed.
I: And what does that mean?
S: That the person is now, well, going faster?
I: And another sentence in Spanish which also includes "two"?
S: Well … I went on increasing my speed for two days.
[…]
S: In the last two days he's increased in weight.
[…]
S: His weight increased by two kilos.

In this case, the student makes use of meanings taken from colloquial language in order to answer the teacher's questions. It is obvious that he needs to give meaning to the phrase presented before proceeding to symbolise it, and in all replies the student is inconsistent in terms of the varying interpretations of the phrase, which will lead to its incorrect symbolisation. This inconsistency derives from the fact that in the original phrase "*a*" and "two" are measurements or quantities of the same thing, and here the student assigns them to different things (speed and days, weight and time), except in the sentence "His weight increased by two kilos," in which he seems to speak of an initial weight (*a*) and the *two* kilos by which it has increased.

In another case, a student responds to the same question with the drawing that appears in Figure 1.7.

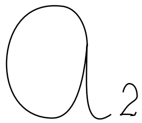


Figure 1.7

Here the word "increase" is clearly not identified with the mathematical action of adding, but corresponds to a real action of enlarging or expanding, an action that affects the letter (sign) "a".

By putting questions such as these to students, one can observe the interaction of mathematical language with natural language. In this regard very interesting analyses have been undertaken that illustrate once again that, at the ages in reference, the meanings of the words in natural language predominate and that these meanings inhibit translation of phrases that consist of those words to the MSS of algebra.

Here is another example. When students were asked to read open expressions such as

$$\frac{a+b}{2},\ ab,\ 3ab,\ a^2,$$

in addition to producing textual readings like

"*a* plus *b* over two"

some students tended to associate geometric meanings with these expressions, and therefore they produced non-algebraic readings. We will now illustrate this with part of a dialogue between the interviewer and a student:

I: Read the next expression aloud. [Pointing to the first one in the list shown above].
S: Broader side over two.
I: And now, if you stop thinking of it as a formula, what would you read there, in what situations have you come across it?
S: Well, it's for finding a result.
I: Such as?
S: Well …

Here, after a long pause, the interviewer intervenes and asks the question again, this time referring to the expression $a + b$.

S: As numbers … for example, 50 + 20.
I: So that if this [indicating $a + b$] is there on the blackboard, it means 50 + 20.
S: No, I mean, it could also mean something else.
I: Such as what?
S: Another unknown.
I: What unknown?
S: For example, a equals, no … well, if a equals 20, what does b equal?
I: And there …
S: The unknown is b.

What we see in this dialogue is that the student tends to interpret open expressions as geometric formulae (for example, "broader side over two") or else tends to close them, either seeking a result by assigning specific numeric values to the letters, or converting one of the letters into a given and the other into an unknown, which is a characteristic of the closed expressions of algebra, i.e., equations.

In both cases, we see that the letters and operation symbols still suggest to the student meanings associated with those signs in primary school. In other words, expressions that include letters or that are formulae or that are simple equations, even when the equals sign —necessary in these two cases— is absent. It is the student who completes the expression in order to be able to read it within contexts that are familiar to him or her.

This example points to the kind of semantic antecedents of the MSSs used by 12- to 14-year old students, which are the foundations upon which their algebraic language skills must be developed. It is there that open expressions, for instance, will denote new mathematical objects at a higher level of generality, involving more general concepts such as that of generalized number (a and b in $a + b$) and that of suspended operation (the addition in $a + b$).

5. ALGEBRA AS A LANGUAGE: APPROACHES FROM LINGUISTIC, SEMIOTIC, AND HISTORICAL PERSPECTIVES

In this section, we review several works on mathematical language and language and mathematics that are especially relevant to the issues raised and analyzed in this book.

From a broader perspective than one that solely encompasses algebra, David Pimm has carried out an analysis of the language spoken and written in the maths classroom, expressed in his book *Speaking Mathematically*, which was published in the late 1980s (Pimm, 1987). In this work, Pimm tackled the task of examining school mathematical discourse by applying analytical techniques from theoretical linguistics. The language of the students, the language of the teacher, and the discussions in the mathematics classroom expressed through the authentic output of those actors constitute the main corpus of analysis in Pimm's research.

As far as theoretical analytical instruments are concerned, Pimm turns to the linguistic concept of register in order to approach the concept of metaphor in mathematics, to which he gives special importance, since he sets out from the recognition that the part played by this concept in the learning of mathematics is as fundamental as the part it plays in the learning of natural language (Pimm, 1987). When treating the theme of the formalism of written language, Pimm necessarily touches on the subject of symbolic algebra, inasmuch as it is an essential reference when speaking of a system of symbols in mathematics, and of their syntax and grammar. His analysis also tackles the theme of the role of natural language in teaching and learning in mathematics, with special emphasis on how the meanings assigned to words in colloquial language are spontaneously transferred by children to mathematics.

The fact that Pimm concentrates his analysis of language on the maths classroom, in its various expressions (speaking, writing, reading) and through the output of various actors (pupils, teacher), is a manifestation of his clear interest in matters of communication, which places him among researchers with a conception of mathematics as a social activity.

Raymond Duval, on the other hand, in his book *Sémiosis et Pensée Humaine. Registres sémiotiques et apprentissages intellectuels* (Duval, 1995), tackles the subject of learning in mathematics from a semiotic perspective, based on the relationship between semiosis (apprehension or production of a semiotic representation) and noesis (cognitive acts such as conceptual apprehension, understanding of inferences or discrimination of differences). Duval emphasizes the role of this relationship in the cognitive functioning of thinking and in its implications for the learning of mathematics and the native language. The variety of semiotic systems of representation in mathematics (graphs, formulae, tables, geometric figures, etc.) and the conversions between them are the material analyzed in Duval's works, which indicates that one of the greatest problems in semiosis has to do with the phenomena of non-congruence, which arise precisely in processes of conversion between representations. One of the central theses in this work is that coordination of registers of representation by learners is a necessary condition for conceptual apprehension in mathematics.

Among the studies on learning in mathematics with a semiotic perspective, Duval's is characterized by its theoretical analysis of the relationship between semiotic representations and mental representations, in cognitive development and in the exercise of cognitive activities. It is also characteristic of this author to give prominence to the cognitive activities of reasoning and the comprehension of texts, which prompts him to expound specific aspects of argumentation and proof in mathematics and to tackle the subject of sense. This last aspect is considered basically in relation to the orientation and conscious control of the fundamental cognitive activities by learners.

Such a wide-ranging treatise as Duval's could not fail to apply its theoretical analysis to the learning of the mother tongue and its connection with learning in mathematics. Within this framework, the author also deals with the differences and relationships between natural language and formal languages, taking geometry and logic as cases illustrating translations between the native language and formal language. The case of algebra is not used in this sense, but there is no doubt that it could be very relevant for the analysis of specific situations of "putting into equations," that is, the translation of the text of a problem (written in natural language) into algebraic language. There is a clear allusion by the author to symbolic algebra in the chapter devoted to conversion between registers, in connection with conversions between algebraic expressions and Cartesian graphs, but without devoting an *ex professo* treatment to algebraic language, with respect to the major themes that he develops, such as congruence and conversions between semiotic representations; the comprehension of texts and sense; and the relationships between noesis and semiosis, natural language and formal language, and mental representations and semiotic representations.

More recently, in his book *Mathematics Education and Language* (Brown, 2001), Tony Brown has presented a theoretical study in which elements of hermeneutics, linguistics, poststructuralism, and social phenomenology are combined to analyze the instrumental character of language in the development of mathematical understanding. Brown uses examples taken from research on mathematics education to examine how language influences the activity developed in the normative framework of a given situation. One of the implications of this analysis is that learning can be seen as a reconciliation between the conventional ways and potential ways (for learners and teachers) of describing such a situation.

Accordingly, Brown pays special attention to the role of pupil and teacher narratives. Specifically, in Chapter 8, "Narratives of learning mathematics," he analyzes a theoretical perspective concerning the ways in which pupils progress in learning mathematics, for the particular case of progressing from arithmetic thinking to algebraic thinking. He goes back to data collected by other authors in various studies on this transition and proposes that they be revised, including their discourses and suppositions. For this purpose he makes use of Ricœur's analysis of time and narrative in order to form an analytical approach to the treatment of notions such as transition, development and progression in the learning of mathematics. From this new perspective, the results of previous studies on the transition toward algebraic thinking attain another dimension, that of the view of the individuals who experience the transition and who use their own resources of expression to narrate their appreciation of the boundary between arithmetic and algebra. Moreover, according to Brown, in light of what Ricœur calls semantic innovation, adding a new narrative is interpreted as an extension of the familiar comprehensions of an individual with respect to the actions that he performs to incorporate figures of speech that will enable him to grasp a mental experience that has not yet fitted into previous versions of his linguistic usage. In fact, the cases of transition from arithmetic to algebra that Brown reanalyzes contribute elements that recreate this part of Ricœur's theory concerning semantic innovation.

The studies by Pimm, Duval, and Brown to which we have referred respond to the need to develop theory in order to analyze phenomena of the learning and teaching of mathematics closely connected with language in a broad sense. In these approaches, the analysis of mathematics as a language —in its various expressions, oral and written; with its different semiotic representations, through formulae, graphs, tables, etc.; used by different actors, pupils and teachers; through conventional expressions or potential expressions (narratives) — is as important as the analysis of its intricate relationship with natural language. The theoretical advances reported in these three works draw on theories from other disciplines, such as linguistics, semiotics, critical sociology, and hermeneutics. Similarly, this book forms part of attempts to theorize about mathematics, language and education, with a specialised focus on the language of symbolic algebra, assuming, as we indicated earlier, a theoretical view in which two main elements participate, semiotic and historical, and adopting a perspective based on pragmatics, favoring the study of meaning in use rather than formal meaning. In this way the focus of attention is shifted toward the activity of individuals with the language of algebra. Essentially, grammar, as the formal system, and pragmatics, as the set of principles of using language, are conceived as complementary domains, especially when they are related with models of teaching algebra.

Other works that emphasize algebra's character of written language have been devoted to the task of analyzing algebraic syntax and semantics, taking elements from support theories, such as linguistics and semiotics. The work done by David Kirshner makes use of generative and transformational grammar (Kirshner, 1987) to generate simple algebraic expressions and perform transformations with them, all based on descriptions of the superficial forms and deep forms of those expressions. In transformational grammar, the transformations of the expressions take place in the corresponding deep forms, which reveal the structure of the forms produced with respect to the operations that constitute them and their hierarchy.

Jean-Philippe Drouhard, on the other hand, develops a notion of signification, with which he associates four aspects: reference, which corresponds to the function of algebraic evaluation; sense, which is given by the set of transformations applicable to the expression; interpretation, which corresponds to the various readings given to the expression in the different contexts in which it may appear (such as number theory, analytical geometry, etc.); and connotation, which corresponds to psychological signification (depending on each individual) (Drouhard, 1992). The analysis of the meaning, or significations, of algebraic writing is then approached theoretically by the application of this subdivision into these four aspects.

Finally, in this brief survey of the studies most directly related to ours, we must mention the work of Luis Radford, who shares with us a semiotic perspective and an interest in historical analysis, which he proposes from an anthropological viewpoint (Radford, 2000a, 2003, 2004). Radford takes from Vygotsky the idea that human cognitions are tied to usage of signs, so that it is no longer central to consider what signs represent but rather what they enable one to do; furthermore, these signs belong to sign systems that are part of a culture and therefore transcend individual cognitions (Radford, 2000b). From this viewpoint he analyzes both the emergence of algebraic thought in pupils who are starting to study algebra and the emergence of algebraic symbolism in history.

Developments such as those just described have proved to be valuable materials in the applications of the theoretical formulation that is discussed here, the connection of which with teaching forms part of its essential characteristics as it envisages the need to develop local models (to interpret specific phenomena), which comprise components of formal competence, teaching, cognitive processes and communication.

SUMMARY

Throughout the book we emphasize our adoption of the pragmatic perspective of meaning in use rather than formal meaning, which has led many studies, and this one in particular, to concentrate attention on the user's performance with the mathematical sign system (MSS). In the case that concerns us, the theme of the algebraic sign system and its relationship with the sign systems of arithmetic and the native language and with personal output is approached on the basis of the notion of MSSs and strata of MSSs. Before introducing these notions in greater depth, here we have presented some basic aspects of Charles S. Peirce's semiotics that are pertinent for an understanding of the sense in which we use the notion of sign, in particular, the triadic conception of the sign, with the introduction of the interpretant as the third fundamental element, the idea of unlimited semiosis.

In the next chapter, we shall develop the notion of the local theoretical model in the context of curricular design and development for students, teachers, and researchers. We shall also stress the crucial role of Freudenthal's didactical phenomenology, both in this context of curricular design and development and in the context of experimental design, which we present in Chapter 3. In Chapter 4 we describe a study conducted following such a design.

All chapters of this book shall have the same structure as this Introduction. They all begin with an Overview and end with a Summary (which includes mention of the topic to be discussed in the chapter that follows).

ENDNOTES

[1] We have given the references to Peirce's works not by indicating the year of publication but by using the abbreviation *W* followed by the volume number in the case of the ongoing publication by the University of Indiana entitled *Writings of Charles S. Peirce: A Chronological Edition*, or by the abbreviation *CP* followed by the paragraph number in the case of the now classic collection of his works entitled *Collected Papers of Charles Sanders Peirce*.
[2] This is how Peirce explains the meaning of "represent" or "stand in relation to" in Peirce, *CP*, 2.273, p. 155.
[3] The ambiguity of the English expression "mathematical sign systems" would not exist if English used brackets as one of its structuring mechanisms, as is the case in the sign system of algebra. Then one would only need to write "mathematical (sign systems)" rather than "(mathematical sign) systems." Freudenthal (1983) analyzed this difference between the sign system of written English (and most vernacular languages) and mathematical sign systems in the chapter entitled "The Algebraic Language," and he illustrates this ambiguity of English with the example of the expression "pretty little girls schools," "which according to the places of the —lacking— brackets can have 17 different meanings" (Freudenthal, 1983, p. 471). Fortunately, our expression "mathematical sign systems" cannot be interpreted in so many ways, and we are using it in only one sense, which is the one specified by the brackets in "mathematical (sign systems)."
[4] We will come back to the idea of sense as opposed to meaning and the idea of giving sense throughout the book, particularly in Chapters 7 and 8.
[5] We have consulted the French translation (Ver Eecke, 1952) and the English translation (Sigler, 1987), both done from the original Latin, and the Spanish version done from the French version cited (Ver Eecke, 1973). A detailed analysis of this book from the viewpoint that interests us here can be found in Filloy (1993a).
[6] We remark that Fibonacci's Liber Abaci (L. E. Sigler, 2002) is one of the most famous of this collection of books. We come back to them in Chapter 3, Section 3.3.

CHAPTER 2

CURRICULUM DESIGN AND DEVELOPMENT FOR STUDENTS, TEACHERS, AND RESEARCHERS

OVERVIEW

We first suggest that it is necessary to make the conception of the nature of mathematics explicit, as it underlies curriculum organization and curriculum development, and show some of the risks that appear when this is not done.

Section 2 explains what we understand by theoretical model through four basic characteristics, distinguishing it from other uses of the same term, and then introducing the methodological concept of the local theoretical model (LTM) and its four interrelated components. We discuss the contrast between the local and the general, and of the methodological nature of local modeling, setting out from the need to design ad hoc observation settings to study specific phenomena. We also explain the recursive character of the application of local models and, in Chapter 3 explain the ephemeral quality of certain theoretical theses in this application.

The major part of the chapter describes our manner of understanding the phenomenological analysis of mathematical concepts (or mathematical structures) that Freudenthal proposed in his book *Didactical Phenomenology of Mathematical Structures* (Freudenthal, 1983). For this purpose we outline the essential characteristics of a conception of the nature of mathematics that is compatible with our way of understanding Freudenthal's phenomenology and that also includes the idea of the generation of concepts from proofs, which is characteristic of the work of Lakatos. We also discuss Freudenthal's distinction between mental objects and concepts, and the consequences for curriculum development, which derive from the opposition that Freudenthal proposed between the constitution of mental objects and the acquisition of concepts. In this discussion, we use our semiotic viewpoint as a basis for interpreting the distinction established by Freudenthal, using as an example some considerations for a LTM for studying the uses of natural numbers. In the context of these considerations, we present the distinction between three types of sign —icons, indices and symbols— which Peirce himself used to describe algebraic expressions as iconic, while the letters in them are indices, and signs such as those of operation or equality are symbols.

1. INTRODUCTION

Any reflection about the elements with which one tries to structure plans and syllabuses for the teaching of mathematics entails, consciously or spontaneously, a conception of the fundamentals of mathematics. To try to free oneself from this discussion, which is far removed from the requirements of the usual practices of drawing up a curriculum, one tends to set out a list with various ways of analyzing the practices that take place in the teaching of mathematics in school systems –in other words, mathematics education. Thus one speaks of mathematics as (see Filloy and Sutherland, 1996):

- A corpus of knowledge to be learned.
- A set of techniques for solving problems.
- The study of certain structures: arithmetic-algebraic, geometric, etc.
- A language with a given sign system that intertwines with natural language.
- A formal science with a highly formalized language.
- A scientific activity, that of mathematicians, that has existed for centuries and that, at present, has developed specific practices very remote from those that can be found in educational systems.
- An activity in which phenomena belonging to the natural and social sciences are modeled.
- A collection of procedures for performing practical calculations to measure, classify, predict, count, etc.
- A part of natural language in which judgments are expressed about the progress of society, the economy, the climate, voting forecasts, etc.
- A collection of ways of talking about random or repeated phenomena with a view to predicting certain future events.
- An essential element of the culture of all historical ages.
- A symbolic system in which one can formulate expressions that give an account of general patterns so that one can make generalized calculations.
- A symbolic system in which generalizations and abstractions are expressed, and that permits representations with operational capability.
- A symbolic system in which one can express phenomena of iteration and recursion for the expression of algorithms.
- A system of mental abilities, such as spatial imagination, the ability to reason hypothetically and deductively, etc.
- Certain structures of the intellect, an internalization of the properties of actions that are performed with real objects.
- A list (even longer than the foregoing) of teaching activities such as is provided in mathematics textbooks.

1.1. Partial and eclectic points of view

The above list is clearly not exhaustive; but it can easily be multiplied if one simply thinks of the many different ways of interpreting some of the terms that we have used, within the various theoretical frameworks of psychology, for instance.

Of course, from some viewpoints the range of mathematical competences that one tries to teach to young students in the basic levels of our present educational systems is all these things and many others. Therefore, if some viewpoints are favored at the expense of others, this leads to the design of curricula that leaves much to be desired because of their partiality, limiting the possibilities of using the curriculum to achieve rich, novel teaching that contemplates a transformation of the vitiated practices. Such practices occur in the current educational systems and they are the direct cause of the poor progress of students and of the rejection of mathematics by the general population.

As a result of proceeding partially, placing some aspects above others, a false dilemma appears, in which the relational aspects of mathematical thinking work to the disadvantage of its instrumental use and viceversa. Similarly, the adoption of a particular bias makes the dilemma between understanding and mere mechanization more acute, in relation not only to mathematical operations but to mathematical thinking in general. As an example one can think of the risks entailed by an unduly narrow design of the curriculum for the teaching of mathematics, thinking of it simply as knowledge about given (ideal) objects the properties and relations of which must be gradually discovered, or the opposing risks introduced by other radical tendencies, which maintain the attitude that all mathematical knowledge is gradually constructed from the first interactions between individuals and reality. In both cases there is an exclusion of all the social aspects that intervene in the processes by which students become competent in the use of mathematical language and results, both for thinking and for producing practical knowledge that can be communicated to any other competent individual.

But perhaps the most common mistake is an extreme eclecticism, by trying to give the same weight to all of the aspects indicated in the preceding list. This generally leads to the production of curriculum designs in which the confusion reaches the most elementary strands in the curriculum. Of course, plunging spontaneously into the design of a curriculum can have even worse results, in which the path followed by the curriculum design leads to a tangle of contradictions and lines of force that interweave, mingle, or clash without rhyme or reason.

All this is no more than a preamble to the need to clarify the conception of the nature of the mathematics that is brought into play in the curriculum.

2. LOCAL THEORETICAL MODELS

2.1. Four characteristics of LTMs

The term *model* has a wide range of meanings: it can refer to many things, from a physical scale model to a set of abstract ideas. Here we examine the use that we make of this term in mathematics education.

We use the term *theoretical models*, or simply models, without claiming that everything given the name of model may be a model in this sense. In fact, in this usage models differ considerably from what is given the same name in other applications. Our aim in this book is to analyze how the various examples have certain common characteristics, which is why we call them models. To begin, we point out four characteristics.

The first characteristic is the fact that a theoretical model consists of a set of *assumptions* about some concept or system.

First, it is necessary to distinguish theoretical models from diagrams, illustrations, or physical models, which, although sometimes useful to represent the model, must not be identified with the model itself. Second, it is true that at times, albeit not always, what is called a model is also termed a theory.

This interchangeability of names is possible because, in such cases, the terms "model" and "theory" refer to the same set of assumptions, although the same things are not suggested about this set when we call it a model as when we call it a theory. Some of the differences, and also the reasons why not all models are called theories, must be analyzed. The second characteristic has precisely to do with this.

The second characteristic is the fact that a theoretical model describes a type of object or system by attributing to it what might be called an internal structure, a composition or mechanism that, when taken as a reference, will explain various properties of that object or system.

A theoretical model, therefore, analyzes a phenomenon that exhibits certain known regularities by reducing it to more basic components, and not simply by expressing those regularities in quantitative terms or by relating the known properties to those of different objects or systems. Accordingly, the term "theory" in this sense is broader than "model," because not all theories are formulated with the aim of providing structural analyses, which are typical of models.

The third characteristic is the fact that a theoretical model is considered an approximation that is useful for certain purposes.

The value of a particular model can be judged from two different but related viewpoints: how well it serves the purposes for which it is employed and the completeness and accuracy of the representation that it provides.

The fact that a theoretical model may be proposed as a way of representing the structure of an object or system for certain purposes explains why various models are often used alternately. This represents another difference between the use of the terms “model” and “theory.” To propose something as a model of something is equivalent to suggesting it as a representation that provides at least some approximation to the real situation; further, it means admitting the possibility of alternative representations that may be useful for different purposes. To propose something as a theory, however, is equivalent to suggesting that something is governed by certain specified principles, and not just that it is useful for certain purposes to represent it as being governed by those principles or that those principles approximate to the principles that actually apply. Consequently, someone who proposes something as a theory is obliged to maintain that any alternative theories must be discarded or modified, or that they will be valid only in special cases.

Finally, the fourth characteristic is the fact that a theoretical model is often formulated and developed and perhaps even named on the basis of an analogy between the object or system that it describes and some other object or different system.

This implies a comparison in which one observes properties and principles that are similar in certain aspects, which fits in with the previous observation that theoretical models have the aim of providing a useful representation of a system. To provide such a representation, it is often helpful to establish an analogy between the system in question and some known system that is governed by rules or principles that are understood, and one supposes that some of those rules, or others like them, also govern the system that one is trying to describe with the model. Reasoning of this kind, based as it is on argument by analogy, is never considered sufficient to establish the principles in question, but only to suggest that they may be considered as first approximations, subject to proof and subsequent modification. In each case, however, the model itself can be distinguished from any analogy on the basis of which it was developed.

Theoretical models can fulfil the same functions as theories: they can be used for purposes of explanation, prediction, calculation, systematization, derivation of principles, and so on. The difference between the use of a model and the use of a theory does not lie in the kind of function for which it can be used, but in the way in which it fulfils that function. Theoretical models provide explanations; but these explanations are based on assumptions that may be simplified, and this condition must be borne in mind when one compares them with theories. It is often said of explanation and systematization by means of a theory that they are more profound and penetrating, which reflects the belief that the principles that constitute a theory are more accurate than those of a model and take more known magnitudes into account. So why not always use the theory, which is more complete? In what follows we briefly discuss why we prefer a local approach and not a general one, but first we mention some semiotic terms that we use repeatedly throughout the book.

2.2. Semantics and pragmatics

It is not our intention here to develop with any precision the kind of theoretical model that is presented throughout the book. We content ourselves with calling on the reader's intuitive concepts concerning terms such as semantics, syntax, semantic load, a more concrete or more abstract level of language, and of the reading level of a text. Even though one consequence of the interpretations obtained in the corresponding empirical studies — described later in the book— is precisely the fact that many of the mistakes that are usually made when using new expressions come from the anticipatory mechanisms of the individual who is decoding a situation that needs to be modeled in that mathematical sign system (MSS), where the semantic load — the custom of certain uses— produced by the individual's prior experience plays a decisive part in possible conceptual errors or mistakes in the syntactic use of the new signs. Nevertheless, we are confident that the approach that we offer for some of the problems proposed is valid in itself, even if it is read from the viewpoint of other theoretical frameworks, and that the "facts" that we describe have an intrinsic interest, even if considered in terms of other interpretations.

We pay more attention, therefore, to the pragmatic viewpoint, which consists in pointing out the meaning given by use, instead of placing greater emphasis on meaning in the abstract. As we have indicated, this approach diverts observation in mathematics education away from the competence of users of a MSS and toward performance, and it also has fundamental implications for the way in which MSSs are studied. Essentially, it is claimed that grammar (the abstract formal system) and pragmatics (the principles of the use of MSSs) are complementary domains in the observation of teaching processes with the various teaching models (innovative and traditional) that are used to achieve the aim of guiding students so that they become competent users of an MSS.

Consequently, this viewpoint not only includes the central role of formal grammar, but also recognizes that it should be incorporated in a broader framework that combines formal and functional explanations. In other words, that in order to interpret the complete meaning of some mathematical messages during normal teaching/learning processes, alongside the strictly formal meaning of the mathematical text in question we also have to admit some other meanings of certain other (logical) messages that are not explicitly communicated either by the sender or by the receiver. We refer to the so-called presuppositions (of which there are various kinds) or the immediate consequences or implications —all this requires the incorporation of some "natural logic" that takes the relation between these meanings into account.

Also, following the same direction of this idea, we are forced to distinguish the difference between competence to decode a message and competence to communicate it (many studies in mathematics education concentrate on this result). It is necessary that our theoretical approach should take these two different kinds of activity into account: the production of mathematical messages and their decoding.

Empirical observations of how a MSS is used during the exchange of messages within teaching/learning processes and the corresponding situation when those MSSs are used by an individual who is thinking out the solution of a problem situation show that the cognitive processes involved interweave the formal level of competence with the pragmatic level. There is a pragmatic component, which comes from the teaching environment in which the learning process takes place. This component is bound up with institutionalized social contracts, so that it is necessary to take into account not only the traditional, customary ways in which the messages of an MSS are emitted in the educational system, but also —and this seems more important— the presence of the entire historical evolution of such sign systems. Notation is the first aspect that appears, but it is not the only one of all the particular ways in which nowadays, after a historical evolution, we tend to use MSSs and their applications to problems in present-day science, technology, and social information processes.

Together with these pragmatic tendencies, there is a component that is due to an individual's cognitive mechanisms that appear in each stage of intellectual development, which gives preference to different mechanisms for proceeding, various ways of coding and decoding the mathematical messages pertinent for the stage in question, various strategies for solving problems, and so on. For example, think of all the evidence that has been accumulated about the tendencies of students to maintain the arithmetic interpretations of many algebraic situations despite their progression to advanced stages of algebra.

2.3. The components of LTMs

The stability of these phenomena of mathematics education and the well established replicability of the experimental designs that have been used to study them are such that we cannot fail to include these observations among the components that are important for any theoretical model for observation in mathematics education. Thus we have a need to propose theoretical components that deal with different types of (1) teaching models, together with (2) models for the cognitive processes, both related to (3) models of formal competence that simulate the competent performance of an ideal user of an MSS, and (4) models of communication, to describe the rules of communicative competence, formation and decoding of texts, and contextual and circumstantial disambiguation.

2.4. Local versus general, the reason for the local in our theoretical models

From the point of view maintained by some authors devoted to problem solving, close to cognitive psychology, one could infer that, to decode a problem situation, experts proceed according to a synthetic process, that is, from the data to the unknown. In several of these studies, in general, when competent users are presented with a problem situation, they recognize "types of problems," because they have formed schemes of them. Thus one could say that, when an expert is presented with a problem situation, in time he would make an integration of the information, in which he would recognize what the central relations of the situation are, comparing them with others that are already in his long-term memory, where there are also specific strategies to be followed. With all of these he is finally able to go on to represent the problem by means of mathematical texts and then decode them for the solution of the problem.

However, from our empirical observations about the decoding of mathematical problem situations it follows that any solution, however fast and fleeting it may be, necessarily passes through an initial logical analysis or logico-semiotic outline of the problem situation, conscious or unconscious, which makes it possible to sketch out the solution. That is, one shows the path that has to be followed to solve the problem in accordance with some mathematical text produced with the use of a certain stratum of an MSS, in which one can establish the direction that the solving process is going to take, and with which one can give analytic or synthetic reasoning processes. Thus an expert or a novice confronted with a problem tends to do work that may proceed from the unknown to the data or viceversa, but in that work the competence to decode the problem situation is determined more by the competence to produce the logico-semiotic outline of the problem situation — which includes strategies of analysis and synthesis— than by the mere recognition of some previously learnt scheme.

Thus, when a competent user performs a logico-semiotic outline of a problem situation, he or she may bring into play cognitive mechanisms that enable him (a) to anticipate the central relations in the problem and also (b) to decide in which stratum of an MSS to outline all the steps of the solution, or decide between one MSS and another more specific MSS, subsequently going on to a process of analysis and synthesis with which he finally obtains the decoding of the problem situation.

To the foregoing we could add many other examples of how, with a global approach, using the results of some general theory of certain branches of knowledge, the analyses of the phenomena that belong to mathematics education, performed thus, reduce the field of investigation very substantially, preventing a clear understanding of the specific phenomenon that one is trying to observe. For example, consider what we would achieve if we wished to use only a general linguistic theory to construct a useful semiotics for mathematics education.

Therefore, instead of arguing in favor of giving preferential consideration to certain components —"grammar," "logic," "mathematics," "teaching models," "models of cognition," "pragmatics," "communication"— we have to concentrate on local theoretical models, appropriate only for specific phenomena but capable of taking into account all four of the components indicated earlier. The idea is to propose ad hoc experimental designs that cast light on the interrelations and oppositions that take place during the evolution of all the relevant processes related to each of the four components.

2.5. The component of formal competence

Earlier, we gave reasons for the need to have models for cognitive processes; this is reinforced later, when we analyze teaching models (Chapter 5). When we introduce our framework of (semiotic) interpretation, MSSs, the need to have models of communication is also underpinned.

As we observe both thinking processes (cognitive component) and the exchange of messages (communicative component) between individuals with various degrees of competence in the use of the MSSs employed to create the mathematical texts (teaching model) relevant for the teaching/learning process, the need for these three models would seem obvious: the model of cognitive processes, the model of communication, and the teaching model.

The need for the model of formal competence comes from the requirement for a description of the situations observed by means of a more abstract MSS, to make it possible to decode all the texts produced in an exchange of messages in which the actors have various degrees of competence in the use of the MSSs in question. Later we see that we interpret the teaching/learning process in this way, hence the advisability that the observer should possess competence in a more abstract MSS that encompasses all the MSSs used in the process observed. In the most extreme case, we might suppose that the model of formal competence is the one with which the epistemic individual would decode the situations observed, that is, the decoding of someone who has all the competences created during the whole historical process of the construction of mathematical knowledge. Fortunately, it is sufficient for the observer to have a model of formal competence described in a more abstract MSS than the one used by all the individuals observed: the learners, the teachers, and the observer himself when he is involved in the exchange of messages (for example, in the clinical interview).

Let us emphasize the importance of the component of formal competence with a paragraph concerning what is stated about Freudenthal's didactical phenomenology presented later in this chapter. The order in which the various kinds of phenomenological analysis must be developed begins with pure phenomenology (the component of formal competence), for which what is of prime importance is knowledge of mathematics and its applications; it is completed with a historical phenomenology; then there is a didactical phenomenology (for which what has to be known is the process of teaching and learning); and in all cases it concludes with a genetic phenomenology. No phenomenological analysis can be effective when teaching is subsequently organized on the basis of it if it is not supported by a sound analysis of pure phenomenology (in other words, the component of formal competence).

3. A GENERAL FRAMEWORK FOR CURRICULUM DEVELOPMENT FOR THE STUDY OF AN LTM

It is advisable to begin the design of a curriculum of a teaching model with a general framework that is broad but based on certain clearly established attitudes, with the intention that various approaches may be obtained from them. Thus, the emphasis placed on them will come from one or another of these central theses, with the aim that the tensions between one viewpoint and another will consequently be diluted by the need to provide a response in each case to the demands of the theses selected, converted in this way into lines of force that promote certain decisions and not others, in making those decisions meaningful.

It is in this spirit that we put forward the following reflections to regulate the criteria for the design of the teaching models that it is decided to use. In the following two sections, we introduce ideas derived from the works of Freudenthal that are pertinent for curriculum development, and also the relation of those ideas to the generation of concepts through proving that is found in the work of Lakatos.

3.1. Concepts

School mathematics is articulated in a series of interrelated conceptual networks, with the characteristic that, with time, students succeed in becoming competent in the use of increasingly abstract general networks —competences that call on many previously mastered competences.

3.2. The relation with reality. Teaching mathematization

The first elementary mathematical concepts are a response to the interaction that children have with the real world. The first notions about quantity, magnitudes, classification, distribution, division, etc. are developed directly from the children's experiences in the real world, but they are also a response to the work of getting hold of the socially established codes for the symbolic manipulation of all these processes, including those inherent in the individual, such as understanding, analysis, and thought. That is why the first mathematical texts have the manipulation of objects and reflection on their interaction as their physical forms of expression. Therefore, a curriculum design that does not set out from the need to move from the concrete to the abstract and that does not then complete the inverse action will tend to result in the students producing MSSs that do not have the sense that one wished to give them socially.

In modern versions, this to and fro between the concrete and the abstract, between the real world and its representations in a mathematical sign system (quantitative modeling, a particular case of mathematization), has played a decisive part not only in science but also in education. Through quantitative modeling it is feasible to "interpret the world with numbers" (Boohan, 1994), using algebraic relations to calculate the numeric value of dependent variables and thus be able to make short- and long-term predictions about the behaviour of phenomena.

One gateway to the learning of algebra is modeling. In this kind of approach the emphasis is on the role of the sign system of algebra as a means to express relations between variables that correspond to phenomena or situations in the physical world, and the corresponding didactic paths contemplate the complete cycle: (1) translation of "concrete" situations or situations expressed in natural language (word problems) to algebraic code; (2) analysis of relations between variables, based on manipulation of the algebraic expressions produced (syntactic level); and (3) interpretation of the "concrete" situation in the light of the results of the work with algebraic syntax. The argument in favor of the virtues of this approach to algebra is that in step (1) meaning is given to algebraic expressions, and in steps (2) and (3) the syntactic manipulation of those expressions becomes meaningful.

With the characteristics just described, the teaching of algebra as a means of modeling tends to promote in students the production of signs in a socially accepted MSS, that of symbolic algebra. In more recent proposals, in the framework of teaching by modeling, other MSSs are also brought into play, such as those of making graphs, numeric tables of variation, spreadsheets, and mathematical narrative (Nemirovsky, 1996). The last of these has succeeded in facilitating processes that can present great difficulty in modeling, such as the translation of relations in a "concrete" situation to algebra.

3.3. Practical knowledge

On the basis of the knowledge obtained from experiences in the real world and the representation of that relation with a sign system that intertwines with natural language, mathematical concepts are used to perform measurements, calculations, and representations. Such concepts are immensely useful and no member of modern society who wishes to pursue a normal intellectual development can disregard them. Nowadays, to be able to analyze the events that take place in the daily lives of individuals and society, one requires certain competences in the use of the MSSs that are taught in mathematics classes. One important component of the curriculum must aim at making it possible for students to use their mathematical knowledge in their daily lives to solve the problems that are presupposed by modern educational systems and that refer to those with which society presents them every day (for example, in the reading of newspapers).

3.4. The analytic and instrumental function for other areas of knowledge

An important feature of elementary mathematics consists in the fact that many other areas of knowledge have gradually, but increasingly intensively, been making use of its symbolic systems to represent the various explanatory models that are found in those areas. Thus, school mathematics is required to describe and understand phenomena from a great diversity of sources. Mastery of the more abstract and general parts of the basic curriculum provides students with a symbolic system in which analytic capability is reinforced by language strata in which not only is it possible to model the phenomena that one is trying to understand and master, but also, precisely there, in the symbolic, one has the operational capability of advancing in the prediction of what will happen when the modeled phenomena take place in time, or when some variable evolves in a particular way. That is why the final parts of algebra, geometry, probability, and statistics, which are traditionally taught in the last two years of the secondary school (13-15 years of age), are of such importance for the future of individuals and for society, which demands competence in such matters if one is to master understanding of natural phenomena and progress in one's societal roles.

4. PHENOMENOLOGICAL ANALYSIS AS A COMPONENT OF DIDACTICAL ANALYSIS. HANS FREUDENTHAL'S APPROACH TO CURRICULUM DEVELOPMENT

4.1. Phenomenological analysis

The didactical analysis of mathematics, i.e., the analysis of the contents of mathematics that is performed for the sake of the organization of the teaching of mathematics in educational systems, has various components, which organize the various teaching models presented in this book. One of the components takes its name from Hans Freudenthal's book *Didactical Phenomenology of Mathematical Structures* (Freudenthal, 1983) and is the subject of this section. We here set out the characteristic features and some of the consequences of what we understand by phenomenological analysis of mathematics as a component of its didactical analysis. The exposition repeatedly refers to Freudenthal's work, taking some liberties with the terminology that he uses and introducing other terminology that is not his.

The phenomenological analysis of a concept or a mathematical structure consists of describing the phenomena for which it is the means of organization and the relation that the concept or structure has to those phenomena.

The description of the phenomena for which it is a means of organization must consider the totality of the phenomena for which this is so at the time, that is, it must take mathematics in its present state of development and in its present use; but it is also advisable to indicate the phenomena for the organization of which it was created and the phenomena to which it has extended subsequently.

The phenomenological analysis developed by Freudenthal is fashioned to serve teaching. However, Freudenthal distinguishes various types of phenolmenology, all important from the viewpoint of teaching, but only one of them is described as didactical. These types are: phenomenology, didactical phenolmenology, genetic phenomenology, and historical phenomenology.

The first thing that characterizes each of these phenomenological analyses is the phenomena that they take into consideration with respect to the concept that is analyzed. In the first case they are the phenomena that are organized in mathematics taken in its state at the present moment and assuming its present use. In the didactical case they are the phenomena present in the world of the students and the phenomena that are proposed in the teaching sequences. In the genetic case, the phenomena are considered with respect to the learners' cognitive development. In the historical case, special attention is paid to the phenomena for the organization of which the concept in question was created, and how it has extended to other phenomena.

The description of the relations between the phenomena and the concept takes into consideration, in the first case, the relations that are established, and in the other three how those relations were brought about, acquired or formed, in the educational system, with respect to cognitive development or in history, respectively.

Moreover, in the case of pure phenomenology the concepts or mathematical structures are treated as cognitive *products*, whereas in the case of didactical phenomenology they are treated as cognitive *processes*, i.e., situated in the educational system as teaching material and being learned by students. Freudenthal says that when writing a didactical phenomenology one may think that it should be based on a genetic phenomenology, but this idea is mistaken. The order in which the various types of phenomenological analysis must be used begins with pure phenomenology (for which it is sufficient to know mathematics and its applications); this is completed with a historical phenomenology, followed by a didactical phenomenology (for which it is necessary to know the teaching and learning process), and in all cases genetic phenomenology comes last. No phenomenological analysis can be effective when teaching is subsequently organized on the basis of it if it is not supported by a sound analysis of pure phenomenology.

Freudenthal's phenomenological analysis aims to serve as a basis for the organization of the teaching of mathematics and does not set out to elaborate an explanation of the nature of mathematics. It might be possible to use it without adopting any epistemological or ontological commitment about mathematics, that is, accepting mathematics as a means for the organization of phenomena, without maintaining that things really are so. However, the ideas that students form about the nature of mathematics and the ideas that teachers have exert a very considerable influence on how both students and teachers conceive the mathematical activity that has to be performed in class, and the knowledge that students produce and that teachers try to teach. This is also why we think it necessary to outline a conception of the nature of mathematics that is compatible with the interpretation that we make of Freudenthal's phenomenological analysis.

We set out, therefore, from the statement that mathematical concepts are means of organization for phenomena of the world. However, this characterization does not tell us much if we do not specify to what we are referring when we speak of the world, and if we do not establish which phenomena are organized by mathematical concepts. Nevertheless, one of the tasks of phenomenology is precisely to investigate which phenomena are organized by mathematical concepts, by analyzing those concepts, so that one cannot seek to know in advance which they are. Nor can one seek to characterize in advance the *kind* of phenomena organized by mathematics, because to do so one would need to have linked the phenomenology of mathematics to a general phenomenology in which one establishes a typology of phenomena — a task that, in our view, could be approached by means of Peirce's phenolmenology. Consequently, we can have an idea of the kind of phenomena involved only on the basis of the concrete analyses that we perform.

On the other hand, it is possible to interpret that from the foregoing statement it follows that mathematics lies in a separate world from the world whose phenomena it organizes, which is the world around us, the real world. This, however, is not the most appropriate interpretation.

In fact, if we place ourselves at the origin, or at the lowest level, we could say that the phenomena that are going to be organized by mathematical concepts are phenomena of this real, physical, everyday world. Our experiences with this physical world have to do with the objects of the world, their properties, the actions that we perform on them, and the properties that those actions have. Hence the phenomena that mathematics is to organize are the objects of the world, their properties, the actions that we perform on them or the properties of those actions, when objects, properties, actions, or properties of actions are seen as what is organized by those means of organisation and are considered in their relation to them.

This first interpretation establishes the idea that mathematical concepts do not actually reside in an ideal world whose reflection we study, nor do they have an existence prior to mathematical activity, nor does that activity consist, therefore, in the discovery of the geography of the world in which those objects are. Yet they are also not installed in a world foreign to our experience, inasmuch as they are created as a means of organization of phenomena of the world. The previous interpretation is not felicitous in this respect, because it does not take into account the fact that Freudenthal does not remain at the lowest level, describing mathematical activity simply as an interplay between phenomena of the world and means of organization in mathematics, in which phenomena seek to be organized and means for this are created in mathematics. On the contrary, Freudenthal accompanies the process of creation of mathematical objects as means of organization with a process by which the means of organization become objects that are situated in a field of phenomena. Consequently, mathematical objects are incorporated into the world of our experience, which they enter as phenomena in a new relation of phenomena/means of organization in which new mathematical concepts are created, and this process is repeated again and again.

Mathematics is therefore in the same world as the phenomena that it organizes: there are not two worlds but one, which grows with each product of mathematical activity. The phenomena that mathematical concepts organize are the phenomena of the world that contains the products of human cognition and, particularly, the products of mathematical activity itself; the phenomena that are organized by mathematical concepts are the objects of that world, their properties, the actions that we perform on them, and the properties of those actions, inasmuch as they are contained in the first term of a phenomena/means of organisation pair.

The staggered progression of phenomena/means of organization pairs entails two processes: the process of creation of mathematical concepts as means of organization, which is indicated by each pair, and the process by which a means of organization is objectified in such a way that it can become part of a new pair, this time in the position of phenomena. The staggered progression draws a picture of the production of more abstract mathematical objects on an ever higher level, and it shows that mathematical activity generates its own content.

4.2. Constitution of mental objects versus acquisition of concepts

We speak of mathematical concepts, of their creation in a relation of phenomena/means of organization, of the objectification of the means of organization and their entry into a phenomena/means of organization relation on a higher level; we speak of transformations of concepts as a consequence of the mathematical activities of proving theorems, solving problems, organizing in a deductive system and the process of defining. All this is accompanied by the affirmation that mathematical concepts do not have an existence independent from the mathematical activity that creates them. But we also bring into the arena a new idea developed by Freudenthal that will oblige us to rethink the relations that concepts establish in these ladders of concepts/means of organization: this is the idea of a mental object as opposed to a concept.

This idea is important primarily because it is on the basis of it that Freudenthal adopts a didactic attitude: the aim of educational activity in the school system must basically be the constitution of mental objects, and only secondarily the acquisition of concepts —which is in second place in terms of both time and order of importance. This attitude is also particularly important for the period of compulsory education, because one must consider what part of mathematics must be offered in it to the population as a whole. But it is also important for the phenomenological analysis of mathematical concepts, all the more so if the analysis is a didactical phenomenology and one has in mind the idea that the analysis is prior to the organization of teaching and is performed with that purpose. This is the aspect that we deal with here.

In a first approach, the contrast between mental object and concept that Freudenthal proposes can be seen as the consequence of considering the people who conceive or use mathematics in contrast to mathematics as a discipline or set of historically, socially, and culturally established knowledge. In the foregoing sections, when speaking of mathematical concepts we have considered them basically within the discipline, and we have hardly introduced the intervention of real people; what has appeared is, at best, a semblance of them, the ideal subject who performs actions with powers superior to those that we possess. We can set out, therefore, from an initial image: the contrast of mental object and concept is a contrast between what is in people's heads (mental objects) and what is in mathematics as a discipline (concepts).

As this is the sense in which Freudenthal uses these terms and in which we are going to use them here, it is worth pointing out before we go on that the term "mental object" does not appear in normal usage. The customary practice is to speak of the concept that someone has —of number or triangle or anything else, whether it belongs to mathematics or not— or to use the term "conception" instead of "concept" and speak of the conception that someone has of circumference, for example; but in this case one generally wishes to emphasize that what is in the person's mind is part of a concept or a way of seeing that concept.

4.3. Considerations for an LTM for studying the uses of natural numbers

Peirce also speaks of a certain progression in the types of signs we treated in Chapter 1:

A regular progression of one, two, three may be remarked in the three orders of signs, Icon, Index, Symbol. The Icon has no dynamical connection with the object it represents; it simply happens that its qualities resemble those of that object, and excite analogous sensations in the mind for which it is a likeness. But it really stands unconnected with them. The index is physically connected with its object; they form an organic pair, but the interpreting mind has nothing to do with this connection, except remarking it, after it is established. The symbol is connected with its object by virtue of the idea of symbol-using mind, without which no such connection would exist. (Peirce, *CP*, 2.299, pp. 168-169.)

4.3.1. The first arithmetic signs

It seems that the first written signs were arithmetic signs. Let us look at some of the characteristics of signs that we have just expounded at work in those primitive signs.

It has actually been determined that the first written signs were arithmetic signs as a result of a step-by-step reconstruction of the development of two systems of writing that had their beginning in about 3500 BC and that were created by Sumerians in the south of Mesopotamia and by Elamites in Susa (located in what is now Iran).[1]

These first signs were marked with a stylus on the outside of hollow balls of soft clay, and they always corresponded both in form and number to pebbles of various shapes contained inside the balls. These marks were thus icons that represented the hidden pebbles, and one had only to break the ball if one wished to confirm that they really did stand for the objects that they represented. The marks on the balls are icons because they resemble in form and number the objects they represent, so that they signify even if the balls are empty. These signs have what we might call a primitive way of working, because the code that the person who closes up the balls and makes marks on them has to share with the person into whose hands they come is not very well established socially, or, at any rate, is subject to doubt.

Interesting as these first written marks are in so patently possessing two of the natural characteristics of signs, they become even more interesting when we discover that they have the antecedents and consequents that will now be explained.

The marked balls that have been found in the excavations are from the second stage of this temporal series. Before that stage, the remains correspond to hollow balls containing pebbles but without any external mark. After the second stage the pebbles disappear and only the marks remain, and the hollow balls, which no longer have to contain anything, become flat tablets.

First of all, therefore, there are objects hidden in a hollow ball, then the first written signs, with the objects that they represent present but hidden, and finally only the written signs without the objects that they represent.

But these objects, in turn, are signs —although not signs belonging to a system of writing— because, in each of the three historical stages, the balls or tablets are records or trading transactions; they are accounts. The objects represented by the written marks are also arithmetic signs, because, by their shape and quantity, they represent a certain number of objects. What the archaeologists have reconstructed tells us that these pebbles were used to record an account in the course of a commercial transaction, and once the matter was settled they were placed inside a hollow ball to record the agreement between the trading parties concerning the quantity involved in the transaction. These first arithmetic signs stood for other arithmetic signs that had a different medium of expression and they eventually replaced them in the records, but only in the records, because if the traders probably continued using pebbles to do their accounts, they had no operational capability.

These mathematical signs on clay tablets led to the development of Sumerian cuneiform writing. We know that later, in the palaeo-Babylonian era (2000 to 1600 BC), genuine mathematical texts were written on tablets similar to these primitive specimens (and not only in Sumerian but also in Akkadian, a Semitic language), but that is another story, which we will not go into here.[2]

4.3.2. The signs used in the Roman number system

Although the arithmetic signs that are at the origin of cuneiform writing fell into disuse thousands of years ago, the Etruscan herdsmen, far from commercial transactions and the schools of scribes in the fertile crescent, by making notches on a stick, one for each head counted, created a number system that we still use, albeit only marginally: the one known as the Roman number system.

The signs that we have inherited from them for the representation of numbers actually seem to have developed as a result of their physical inscription on a linear record. Thus the primitive repetition of notches, |||||||…, became structured by means of special marks every five notches, with a view to making it easier to count the expression: a slanting mark in the fifth position, a cross-shaped mark in the tenth position, etc., giving rise to marks such as ||||/||||X||||/||||X||| to record a herd of twenty-three animals. The primary marks and structural marks eventually became the alphabetical letters I, V, and X, becoming integrated into the system of writing and identified with the letters that they most resembled.

As they were positions in a series, V and X did not signify the cardinal numbers “five” and “ten,” but the fifth and tenth positions in the series. In fact, the first written forms for “five” and “ten” were not V and X but IIIIV and IIIIVIIIIX, which do indeed represent cardinal numbers, and in which both I and V represent a unit. It was only later that considerations of economy led to the use of V to represent IIIIV and thus five units. The signs V and X initially functioned as reference points in the series in yet another sense: IV came to signify “four,” not as the result of a rule of subtraction between the cardinal numbers designated by I and V, but because from the presence of the sign V one could understand that the mark immediately before V in the series was being designated. Similarly, VI did not come to signify “six” as the result of a rule of addition, but because it designated the mark immediately after V. It was only when the signs V and X acquired a cardinal meaning —standing for IIIIV and IIIIVIIIIX— that the earlier rules, which had to do with positions in a series, i.e., with …IV… or …VI…, were reinterpreted as rules of addition and subtraction between cardinals. In this historical account, the transformations that took place in the expression as a result of the processes of abbreviation gave new senses both to the elementary signs and to the rules for the formation of compound signs, senses that correspond to the meanings now taught in schools.

These marks are indices of the action of counting. Puig (1997) points out that the phenomena that organize mathematical concepts are objects, properties, actions, and properties of actions. This is one of the clearest examples of a mathematical concept that organizes a phenomenon that does not belong to the domain of objects or properties of objects, but to the domain of actions and properties of actions (which does not do away with the fact that in the corresponding triadic relation the action of counting is the object of the sign for a mind, that is, for an interpretant). As a result of transformations of the expression, the indices become symbols.

4.3.3. Algebraic expressions

It is common to refer to algebraic expressions as “symbolic language” —for example, when one speaks of putting a problem into equations, one usually describes this as a “transition from natural language to symbolic language.” However, if we use Peirce’s terminology, algebraic expressions are not symbols but icons, strange as this may seem at first sight. Let us see how Peirce himself explains it:

> […] thus, an algebraic formula is an icon, rendered such by the rules of commutation, association, and distribution of the symbols. It may seem at first glance that it is an arbitrary

classification to call an algebraic expression an icon; that it might as well, or better, be regarded as a compound conventional sign [symbol]. But it is not so. Because a great distinguishing property of the icon is that by direct observation of it other truths concerning its object can be discovered than those which suffice to determine its construction. [...] This capacity of revealing unexpected truth is precisely that wherein the utility of algebraic formulae consists, so that the iconic character is the prevailing one. (Peirce, *CP*, 2.279, p. 158.)

Algebraic expressions are icons, and this is precisely what makes them powerful, because as signs they have the properties that their objects have. However, the letters in algebraic expressions, taken in isolation, are not icons but indices, each letter being an index of a quantity. They are also not symbols. If the algebraic expression is the result of the translation of the verbal statement of an arithmetic-algebraic problem, each specific letter represents a specific quantity as a result of the convention established by the person who produced the translation, but each letter refers to a quantity even if there is no interpretant, because any interpretant who is not aware of the convention established will assign the letters to the right quantities, since the algebraic expression as a whole will require that the corresponding quantity be assigned to each letter. So are there no symbols in algebraic expressions? Yes, there are. The signs +, =, etc. are symbols in Peirce's sense.

Algebraic expressions are thus an example of the imbrication of three kinds of signs in mathematical writing: the letters are *indices*; the signs +, =, etc. are *symbols*; and the expression taken as a whole is an *icon*.

4.3.4. Uses of numbers in different contexts

The students in whom teachers attempted to instil the concept of number in the years of what was known as "modern mathematics" —in a school version of Cantor's construction of cardinals— would have left school without being able to count if they had not created a mental object of number apart from what the official syllabuses wished them to be taught. We will use this complex, multiple concept as an example to show the difference between mental object and concept, describing it in semiotic terms instead of as Freudenthal does.

If we consider the ordinary activity of people and not just the mathematical activities of mathematicians or the scholastic activities of students in mathematics classes, the use of number, or rather numbers, appears in very diverse contexts. A list of them might include the contexts of sequence, counting, cardinal, ordinal, measurement, label, written numeral, magic, and calculation. A description of the characteristics of each context is not our purpose here: the list is worth mentioning solely in order to show that it is possible to distinguish a considerable quantity of contexts. Following Wittgenstein for a moment, we understand meaning as being constituted by the use that one makes of a term, that use not being an arbitrary use, the product of what someone takes it into his or her head to do with the term in question, but a practice subject to rules.

The uses of numbers in each of these contexts follow rules. For example, when one says “My telephone number is three, eight, six, four, five, eight, six,” the number refers to an object and does not describe any property of it or its relation to other objects but serves to identify it. This is the context of label, and in it, when the expression is oral, the digits that make up the number are generally expressed separately, as in the example given. In an ordinal context, the number refers to an object that is in an ordered set of objects, and it describes what place it occupies—“he came third” or “he’s the one that makes three.” In a cardinal context, the number refers to a set of objects (without order, or whose order is not taken into consideration), and it describes the numerousness of the set— “there are three.” And so on.

The totality of the uses of numbers in all contexts constitutes the *semantic field* of “number,” the encyclopedic meaning of “number.” The identification of the context in which number is being used enables someone who is reading a text or receiving a message to abide by the *semantic restriction* that the context establishes and thus interpret it appropriately. However, the person who reads a text or has to interpret a message does not operate in the whole encyclopedia —i.e., the totality of the uses produced in a culture or an episteme— but in his personal semantic field, which he has gradually built up by producing sense —senses that becomes meanings if the interpretation is felicitous— in situations or contexts that demanded of him new uses for “number” or numbers.

In this semiotic description, what Freudenthal calls “mental object ‘number’” corresponds to this “personal semantic field.” Freudenthal’s didactic attitude in favor of the constitution of mental objects means that the aim of educational systems, expressed in the terms that we are using, should be that the student’s personal semantic field should be sufficiently rich —should embrace the encyclopedia sufficiently— to enable him to interpret appropriately all the situations in which it proves necessary to use “number” or numbers.

The contexts of the ordinary use of numbers are the various places in which we can experience the phenomena that have been organized by means of the concept of number, both the phenomena for which it was originally created and those to which it has now been extended. The idea of mental object that we have just introduced must also be seen, therefore, as a means of organisation of phenomena: with the mental object “number” people are able to count, among other things. Mental objects are constituted in chains of phenomena/means of organization, in the same way as with concepts, with the consequent increase in level —in fact, the contexts of the ordinary use of numbers that we have mentioned are situated on the lowest levels, and to realise the phenomenological richness of number in secondary school one must take other contexts into consideration, including contexts that have already been mathematized.

4.4. Relation between mental object and concept

This is an initial explanation of what a mental object is and how it is constituted, but what Freudenthal calls mental object could simply have been called the concept that a person has of number. To justify the introduction of a term that distinguishes it, it is necessary to explain for what other thing the term "concept" has been reserved, and how it differs from what we have just called "mental object." We have already said that the first distinction is that mental objects are in people's minds and concepts are in mathematics. But this would hardly be sufficient reason to oppose mental object to concept if we thought that the mental object is the reflection of the concept in people's minds. The relation between mental object and concept, however, is not a mirror-like relation. Once again we will explain it in semiotic terms.

We have identified the mental object "number" with the personal semantic field, which comes from all the uses of numbers in all the contexts in which they are used, from a semantic field consisting of all the culturally established meanings. The mathematical concepts of natural number —and we use the plural in order to emphasize the fact that we consider the concepts developed by Peano, Cantor and Benacerraf, for example, as different— in the form in which they exist in current mathematics are the product of a long history, with processes of creation and modification of concepts. In terms of the semiotic description that we are using now, any mathematical concept of number that one wishes to examine once it has been created appears as the result of the process of defining that has incorporated it into a system organized deductively as a *narrowing of the semantic field*. Thus, for example, the concept of natural number developed by Peano —especially in its more modern versions— can be seen as the breaking down of the meaning that pertains to the context of sequence and its presentation in the form of a series of axioms that give an exhaustive account of its components. The concept of natural number that is derived from Cantor's construction, on the other hand, is ascribed, in the very name that Cantor gave in his original intention, to the cardinal context.

In this explanation, concepts appear to be directly related to a part of the mental object, given that, in the process of defining, part of the meaning that the mental object embraces is selected. We will immediately point out that this is not the only difference, and that we do not wish to give the impression that the relation between mental object and concept is a relation between a part of the content of the mental object and the totality of its content. But we wish, rather, to indicate that what this explanation establishes provides a foundation for the attitude taken by Freudenthal that we have mentioned: the acquisition of the concept is a secondary school objective and can be left until after mental objects have been soundly constituted, and in any case it does come afterwards.

The relation between mental object and concept is more complex than is shown by the explanation that we have just given using the example of number, because the explanation was limited to comparing the deployment of the semantic field of number and Peano's definition, as if there were not centuries of history that have produced both contexts of use —which we are now going to find with traces of their organization by concepts of number— and Peano's definition. Taking into account the processes of creation and modification of concepts that are present in that history, the relation between the mental object that can be constituted from the contexts mentioned and the content of the concept of number created by Peano's definition cannot be reduced to a relation between part and whole.

Constituting a mental object implies being able to give an account with it of all the uses in all the contexts or being able to organize all the corresponding phenomena, in which case the mental object is well constituted. The aim of educational systems that Freudenthal indicates is this constitution of good mental objects. Acquiring the concept implies examining how it was established in mathematics organized locally or globally in a deductive system. The particular relation that each mathematical concept has to the corresponding mental object determines how the constitution of the mental object relates to the acquisition of the concept. The constituents of the good mental object are determined by means of the phenomenological analysis of the corresponding concept.

4.5. From phenomena to mental objects and concepts through teaching

The relation between mental objects and concepts is varied. Both are constituted as means of organization of phenomena, mental objects precede concepts, and concepts do not replace mental objects but contribute to the formation of new mental objects that contain them or with which they are compatible.

The distance between the mental object, or rather the first mental object, and the concept can be an abyss: this is the case with the mental object "curve" and Jordan's concept of curve, for example. In general, in topology mental objects do not lead very far, and it is necessary to form concepts, by means of a formation of concepts that involves more than a local organization. These concepts enter a field of phenomena that are organized on a higher level by mental objects such as spaces and varieties of arbitrary dimension, which in turn are converted into concepts by means of new processes of organization and the creation of more abstract sign systems to describe them. As this example shows, by introducing the idea of mental object the process of a progressive rise through the chain of phenomena/means of organization pairs links up with a process of transformation of mental objects into concepts.

The analyses of didactical phenomenology must be based on analyses of pure phenomenology, bearing in mind that, in many more cases than one might imagine, the distance between the mental object and the concept is so great that it is not possible to build bridges between them by didactic means in secondary school.

For the constitution of mental objects through teaching while bearing concepts in mind, the distance between them and the various forms that this distance adopts are therefore of importance. It is worth mentioning a few cases, such as those that are set out in the following paragraphs.

Sometimes there are components that are essential for the formation of the concept but are not pertinent for the constitution of the mental object. This is the case with the cardinal number: the comparison of sets without structure is essential for the concept, but it plays almost no part in the constitution of the mental object because, in the real situations in which a person experiences the phenomenon that is organized with the mental object "number" in its cardinal sense, the sets of objects are rarely without structure, and, moreover, the structure is a means for making the comparison, rather than something that must be removed in order to make it.

Sometimes, what a didactical phenomenology shows is that the phenol-mena organized by the concept are so varied that in fact different mental objects are constituted, depending on the field of phenomena that is selected for exploration in teaching, or several mental objects if several kinds of phenomena are explored. For the acquisition of the concept it is necessary, therefore, to integrate these different mental objects into a single mental object. This is the case with the concept of area, for example.

Indeed, lengths, areas, and volumes are the magnitudes that are measured in elementary geometry. It is therefore necessary that these concepts should be acquired as part of the learning of measurement and measuring. The comparison between qualities of objects is the beginning of the activity of measuring. This becomes measurement through the intermediary of the establishment of a unit and consideration of objects that are treated as objects of which one can predicate that quality —for example, one can predicate that they have length if it makes sense to say of them that they are "long."

However, as concepts, length, area, and volume are problematic because of the variety of approaches for the constitution of the mental object "area" (or "volume"). Indeed, plane figures can be compared with respect to area directly, if one is part of the other, or indirectly, after transformation by cutting and pasting, congruences, and other applications that preserve area; or else by measuring both of them. The measuring can be done by covering the figure with units of area, or by means of interior and exterior approximations; for this one uses the additivity of the area beneath the composition of plane figures that are mutually disjoint except for their boundaries (of dimension one), or convergence of the areas by approximation. It is not clear that these approaches lead to the same result, and in fact the proof that the result of measuring by following all these procedures is the same is not simple. The constitution of the mental object "volume" also has the additional complication of considering phenomena corresponding to capacity, which are usually measured with different units.

Sometimes it is difficult even to distinguish the mental object from the concept, at least if one wishes to have a unitary mental object: only by means of access to the concept is it possible to unify a heterogeneous set of mental objects. This is the case with the concept of function.

Finally, there are mental objects whose field of phenomena appears only in a mathematical or mathematized context. An example of this in secondary school is provided by the concepts of analytical geometry.

Indeed, in history, global location by using coordinates leads to the algebrization of geometry. Whereas the system of polar coordinates used to describe the sky and the Earth's surface has served to systematize location, the system of Cartesian coordinates is particularly efficacious for describing geometric figures and mechanical movements and, later, functions in general. A figure can be translated algebraically into a relation between coordinates, a movement in a function that depends on time, and a geometric application in a system of functions of a certain number of variables.

The phenomena that are proper to analytical geometry are thus phenomena produced by the expression of geometric properties in the complex sign system in which algebraic expressions and Cartesian representation refer to one another. They are, therefore, phenomena that can be explored only in contexts previously mathematized by the use of those sign systems.

4.6. Concepts generated by proving

We have seen that mathematical concepts are created in the phenomena/-means of organization process, but this does not mean that once created they remain immutable. On the contrary, mathematical concepts alter in history as a result of their use and the new MSSs in which they are described. This does not imply, however, that alterations in a concept indicate that the original concept was mistaken and that we have to see the history of mathematical concepts as an advance toward truth, for we have rejected the view that mathematical objects have an existence prior to the process that creates them.

A different idea of the evolution of concepts in history was developed by Lakatos in his book *Proofs and Refutations* (Lakatos, 1976). What is of interest for us here is the fact that in this book Lakatos examines how concepts evolve under the pressure of the proof of theorems in which they are involved.

Lakatos tells that, after the establishment of the conjecture that for any polyhedron the relation $C + V = A + 2$ is true, and after its proof by Euler, examples of solids emerged that did not fit in with the proof that had been performed or, what was more important, with the theorem that had been proved. In terms of a conception of the nature of mathematical objects according to which there is a pre-existing ideal object that we call polyhedron and what mathematical activity does is to discover its properties, the matter is quite clear: these solids are not true polyhedrons, or else the proof is wrong. The reconstruction of history that Lakatos makes is not this.

Lakatos separates the two types of counter-examples that I have just mentioned and calls them local and global counter-examples, respectively. A local counter-example is one that has characteristics that cause the proof not to be applicable to it, but that verifies the relation. These counter-examples do not refute the conjecture: what they do is to indicate that in the proof a property was used that was assumed to be valid for all polyhedrons, but it is not so. What is refuted, therefore, is a lemma that has been used implicitly, and therefore the proof. The presence of these counter-examples introduces a difference in the concepts that was not present before.

The effects of the appearance of global counter-examples have more importance for what we are examining. A counter-example is global when it refutes the conjecture. As first global counter-examples of the theorem proposed by Euler, Lakatos presents the solid that consists of a cube with a cube-shaped hollow inside it, and a solid formed by two tetrahedrons joined by one edge or one corner; later he presents the even more interesting case of a star-shaped solid, which does or does not verify the relation depending whether or not one considers that its faces are star-shaped polygons. The presence of these solids as counter-examples produces a tension between the concept, the theorem, and its proof. This tension can be resolved in various ways, which all affect the concept of polyhedron. The most elementary are:

1) Monster-barring.

The counter-examples presented are considered to be not genuine examples of the concept of polyhedron but monsters, i.e., beings whose existence is possible but not desired. The possibility of their existence is determined by the definition of polyhedron that is being used, whether explicitly or implicitly, so that, in order to preserve the theorem, a new definition of the concept of polyhedron that explicitly excludes them is produced.

2) Exception-barring.

The counter-examples presented are considered to be examples of the concept whose existence had not been foreseen when the conjecture was stated. The conjecture is modified with the intention of withdrawing to safe ground. To do so, a difference that separates these examples is introduced in the concept.

3) Monster-adjustment.

The objects are looked at in a different way so that they cease to be counter-examples; this is the case with the two ways of looking at star-shaped polyhedrons: as being composed of star-shaped polygons or not.

Although these are only the most elementary ways of confronting the tension created, even with them we can see that the concept of polyhedron is affected in all cases. Whether the counter-examples are accepted or excluded as examples of the concept, the semantic field is expanded. In one case, because the content of the expression increases, or, to put it differently, because the field of phenomena for which the concept had been created — which is what constitutes its semantic field— did not contain the phenomena corresponding to the objects and properties that are now present, and it is extended to include them. In the other case, because the concept enters into an interplay of relations to these new objects from which it explicitly disassociates itself in the new definition, which also form a constitutive part of its content.

The full story is more complex, and it also features progressively richer and more abstract mathematical sign systems to which the concepts initially expressed in other, less rich or less abstract mathematical sign systems are translated, and it leads Lakatos to state that the concepts generated by the proof do not improve the original concepts, they are not specifications or generalisations of them, but they convert them into something totally different, they create new concepts. This is precisely what we wish to emphasize: the result of the process that Lakatos presents, a process of tension between concepts, theorems, and proofs, is not the delimitation of the true concept of polyhedron that supposedly corresponds to the pre-existing ideal object, but the creation of new concepts.

4.7. Problem solving, defining, and other processes that also generate concepts

From Lakatos we have just extracted the idea that mathematical concepts do not remain immutable once created. We have also outlined how concepts change, impelled by the tension produced in them by their application in proofs and refutations. However, mathematical activity does not consist only in proving theorems. One of the fundamental driving forces in the development of mathematics is problem solving, and this includes the proving of theorems, but also other activities.

Problem solving includes the proving of theorems in two senses. In the first sense, problem solving includes the proving of theorems considered globally, because, if we follow the terminology of Polya (1957) and, instead of distinguishing between problems and theorems as was first done by Greek mathematicians, we call them all problems and distinguish between "problems to find" and "problems to prove," then the proving of theorems is simply one kind of problem solving: the solving of problems to prove.

In the second and more important sense, problem solving includes the proving of theorems in the solving of each problem in particular; indeed, what characterizes problem solving in mathematics, even with problems to find, is the fact that the obtaining of the result must be accompanied by an argument that substantiates the fact that the result obtained verifies the conditions of the problem, i.e., any problem is a problem to prove or, if it is a problem to find, it contains a problem to prove —the problem to prove that the result found verifies the conditions of the statement.

This obliges us to extend the terrain in which concepts are submitted to a tension that modifies them beyond the proving of theorems to the solving of problems. But it becomes even more necessary to do so if we take into consideration other parts of problem solving that do not involve the proving of theorems —specifically, the proposal of new problems or the study of families of problems.

Problem solving also does not exhaust the field of mathematical activities or the field of mathematical activities that generate concepts. Other activities that are responsible for the creation of many great mathematical concepts in the form in which we know them now have to do with the *organization* of sets of results of varying extent —obtained in the activity of solving problems and proving theorems— in a *deductive system*. This systematic organization has adopted different forms in the course of history, and it may be more local or more global, more or less axiomatic or formalized, but in any case it has constituted an essential component of mathematics since mathematicians moved from accumulating results and techniques for obtaining them to writing "elements." Indeed, although we do not detail that set of activities here, one essential characteristic of it is that it has transformed the sense in which definitions are used in mathematics. "In mathematics a definition does not serve just to explain to people what is meant by a certain word," as Freudenthal says, but rather, when we consider the mathematical activities by means of which deductive systems are organized, "definitions are *links in deductive chains.*"

The process of defining is, therefore, a means of deductive organization of the properties of a mathematical object, which brings into the foreground the properties that are deemed to make it possible to constitute a mathematical system, local or global, in which that mathematical object is incorporated. However, emphasizing certain properties such as those that define a concept is not an innocent operation, a neutral operation with respect to the concept, because, on the one hand, it makes the concept appear as originally created to organize the corresponding phenomena, and, on the other, it makes the content of the concept be, from then on, what is derived from that definition in the deductive system in which it has been incorporated. Therefore, this process of defining also creates new concepts, just as proving theorems do.

SUMMARY

In this chapter we have presented the phenomenological analysis (based on the work of Hans Freudenthal) as an approach to curricular development for teachers, students, and researchers. The content is basic for the remainder of the chapters since it deals essentially with establishing the difference between acquiring concepts and building mental objects in mathematics, as well as how one goes from phenomena to mental objects and to concepts through teaching. The ideas are illustrated through the case of uses of natural numbers. We also refer to the work of Lakatos "Proofs and Refutations" in order to make evident that *tests*, *definitions*, and *problem solving* are concept generators.

We have also dealt with the concepts of *mathematical sign systems* and *local theoretical model,* thus adding further to their introduction in Chapter 1. Dealing with these concepts has enabled us to refer to the phenomenological analysis as a framework for developing teaching models, as components of a local theoretical model.

In the next chapter, we deal with the methodological aspect of LTMs, and we present both an historical and phenomenological analysis of school algebra.

ENDNOTES

[1] This reconstruction is recounted in full detail in Ifrah (1994), vol. I, pp. 233–263. See also Schmandt-Besserat (1992) and Glassner (2000).

[2] Although it would be worth doing so. In the texts of problems that appear on tablets written in Akkadian, the words “long” and “wide” are in Sumerian and are used to designate unknown quantities, even though the problem is not geometric. One can imagine that the strangeness of the signs of another language in a text written in Akkadian favored the use of those signs as what Høyrup (2002a) calls “a functionally abstract representation.” Indeed, although “long” and “wide” continue to retain the original geometric meaning, the sense that they have is no more than that of two quantities that can form part of a calculation – that is, these words are precursors of the objects of algebra.

CHAPTER 3

EXPERIMENTAL DESIGN

OVERVIEW

We begin analyzing the diagram of an experimental setting design for the observation of algebra learning and teaching phenomena. In this chapter we discuss a manner of studying the evolution and development of algebraic ideas through historical and epistemological analysis (based on the analysis of ancient pre-symbolic algebra texts), which in turn serves as a point of departure for experimental design in mathematical education for the particular case of the transition toward algebraic thought. The phenomenological analysis, as presented in general terms in Chapter 2, is applied to the case of algebraic language and to that of school algebra (didactic phenomenology). Here once again the notions of *mathematical sign system* and of *language strata* become relevant, especially when the historical analysis touches upon the genesis of modern algebra thus re-broaching the elements that correspond to said notions presented in Chapter 2. The chapter consists of the following sections: 1, Introduction; 2, Experimental observation; 3, On the role of historical analysis; and 4, The phenomenological analysis of school algebra.

1. INTRODUCTION

In this chapter we present two diagrams that give a general description of the design of a study in accordance with the guidelines of our research program (diagram A), and the general form of the development of the study (diagram B). In the rest of the chapter, we specify some of the terms used in those diagrams and set out in more detail how the historical analysis of algebraic ideas and phenomenological analysis intervene in it.

2. EXPERIMENTAL OBSERVATION

2.1. The design and development of the experiment

Both the design and the development of the experiment are presented in the form of a flow diagram (see Figures 3.1. and 3.2). We merely wish to emphasize that we have introduced our theoretical elements —local theoretical models (LTMs) and mathematical sign systems (MSSs)— as the theoretical counterpart with which the experimental observations are designed and interpreted. For this is a theory produced to provide support for observation, and that is how it should be interpreted. These ways of designing and developing experimentation are exemplified throughout the book, and they are in use in several research works (see Chapters 4, 6, 7, 8, and 9).

2.2. Recursiveness in the use of LTMs and the ephemeral quality of certain theses

Note that in diagram A there is a recurrence: the diagram begins with a box that represents the area under investigation, and at the end of the entire process there is a return to the beginning. In the case of diagram B the starting point is a local theoretical model, designed in the stages of diagram A, and after the performance of an experimental study, in which the theses of this first LTM are confronted with what occurs in the empirical development of the experiment, one finally comes to a phase of analysis and interpretation. On the basis of the results of this phase, the initial problem area is framed within the perspective of a new LTM, the design of which returns to the first stages of diagram A, so as to be able once again to start the process described in diagram B.

In this recursiveness, it may well happen that the theoretical theses framed in the first LTM prove to be insufficient to study and interpret the empirical observations made in the stage of empirical development (see, for example, Chapter 9), or else some of the theses as elaborated might have to be discarded or differentiated into others that provide a better fit for the interpretation of what has been observed. In this respect one could speak of the ephemeral quality of certain theses that do not stand up to verification with the empirical facts observed.

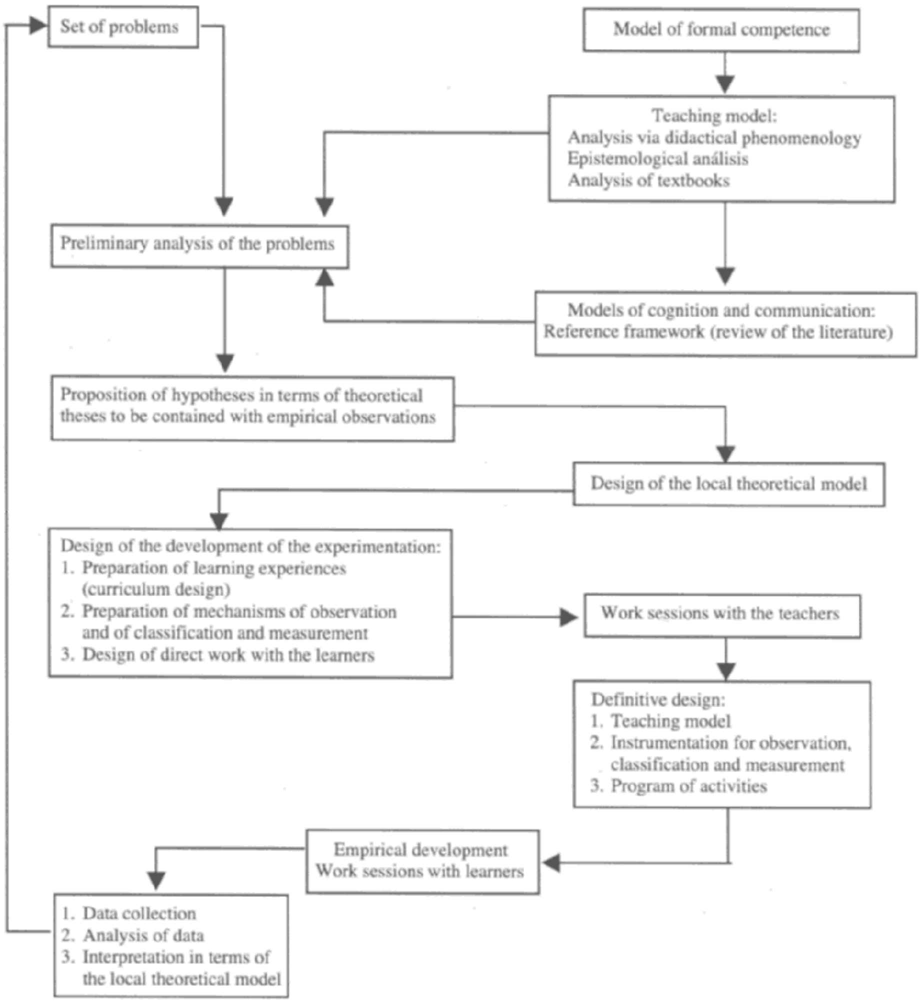


Figure 3.1. Diagram A of the design of the study

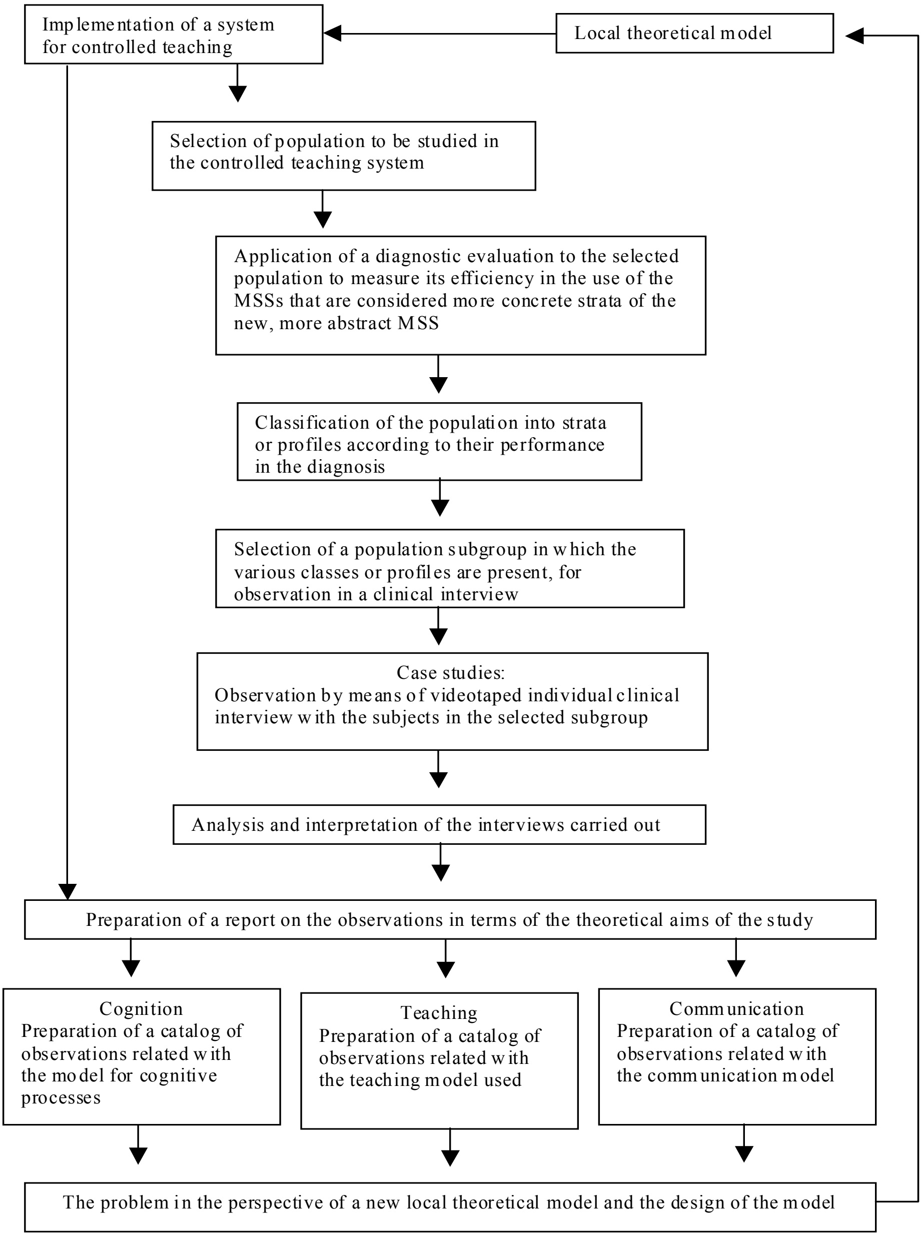


Figure 3.2. Diagram B of the development of the study

2.3. On the didactic cut

We mention first that it is advisable to choose the moment of the experimental observation at some point in the mathematics curriculum at which what has been learned (of the material taught up to that point) does not make it possible for the next topic that is to be taught to be discovered spontaneously without the intervention of the teaching that is to come. The ideal situation is to find a conceptual area in which, when the competences of the population with regard to the uses of those concepts are diagnosed, one sees that those competences lead to performances very far from what is expected (the aims of the education). For example, in the case of Thales' Theorem in Chapter 7, through what had been learned previously *the entire population* had developed tendencies that caused *all the learners* to have mistaken responses concerning ratio and proportion when faced with the most elementary questions that can be asked in this field, questions that are the basis of the whole future development of trigonometry.

Another example is the solving of equations and the transition from arithmetic to algebra, which is discussed in Chapter 4. On the other hand, in the ongoing studies that are mentioned at the end of Chapter 8, there are already indications that suggest that what is being studied in them would also constitute a didactic cut.[1]

2.4. On controlled teaching

Second, it is advisable that the population being studied should comprise several cohorts of the same age, belonging to the same grade level, at the same school, and that they should receive instruction in mathematics within a system of controlled teaching. This means that the population being studied receives instruction in mathematics with materials that allow them to do individual work in class, at their own pace, that there is monitoring of advances made by individuals and groups of students, that there is the possibility of intervening with supplementary teaching material where it is required.

2.5. On diagnosis

The groups that receive the controlled teaching constitute the totality of the population being studied. During the period in which the controlled teaching is developed, mechanisms of measurement and classification are prepared and refined so as to make it possible eventually to construct a diagnostic test by which classes or profiles of individuals can be obtained. The diagnosis provides a detailed description of the performances of the students and has the further aim of delimiting the profiles so that one can see which students it would be interesting to observe in greater depth. For this purpose a case study is designed in which the clinical interview plays the main part with a view to setting up the observation environment.

In general, it is advisable to classify the population in relation to three axes. The first has to do with the syntactic competences of the individuals in the use of the more concrete MSSs. The second has to do with the competences concerning the use of the semantics of those MSSs, when applying it to the solution of problem situations. The third axis of competence seeks to group together the competences that have to do with the intuitive and spontaneous uses of the strata of the more concrete MSSs that will be used in the decoding of the new teaching situations which the teaching model that is being used will require.

We will see an illustration of this principle in Chapter 4, which contains a description of a study in which the population is classified by means of a written test on pre-algebra consisting of three subsections: arithmetic equations with literal notation (e.g., $5x + 3 = 90$), arithmetic equations without literal notation (e.g., $\square - 95 = 23$), and problems corresponding to arithmetic equations.

The classification of the population in relation to these three axes makes it possible subsequently to select pupils for the clinical interview who have different profiles with regard to one or more of the axes, and who therefore correspond to aspects of the MSSs brought into play in the teaching.

2.6. On the clinical interview

To be able to observe the phenomena studied with greater precision one needs an experimental situation that makes it possible to monitor certain disturbing factors that are always present in the classroom, and one needs observation mechanisms that allow a more exhaustive and precise analysis. However, this must be done in such a way that what is observed has to do with the problems presented by the individual being observed and also that the components that the teaching brings into play are present. That is the nature of clinical interviews with teachers.

The clinical interviews have a structured format, but the interviewer moves freely between the various steps that have been designed previously, allowing the line of thought of the interviewee to define each of the subparts of the interview. The first part of the interview is usually devoted to confirming that the interviewee has the profile given by the diagnosis.

Except in cases where the interviewee has no difficulty in solving the problem that is set, the interviewer intervenes to put further questions that, through a process of discovery, help the interviewee to learn the problem that he was initially unable to solve. It is a question of discovering the difficulties presented by the learning of beginnings of algebra, given the ways in which one seeks to teach it nowadays. In these clinical interviews the focuses of observation are the ways of teaching and the particular ways of learning (with their typical obstructions and difficulties) that are seen in the students.

2.7. On the preliminary analysis of the problems

If we look at diagram A of the design of the experiment, in step 2, "Preliminary analysis of the problems," many general disciplines combine to make it possible to perform the analysis: psychology, historical analysis, epistemology, mathematics, sociology, education in mathematics, etc. Many research studies nowadays favor one or more of these focuses, or else, in the case of the design of the experiment, there tends to be a tension between the studies that favor a quantitative approach (via the use of statistics) and those that favor a more qualitative approach (via the use of clinical observation).

However, in favoring some general focus, such as the analysis of the history of mathematical ideas, it is possible that all the other items that are described in diagrams A and B may be left out. One might think, therefore, that such a study is a valid contribution only in the field of the general discipline with which it is concerned; nevertheless, experience shows that studies of this kind are ultimately of little interest in the general discipline, where there is a preference for certain working habits and focuses and for using all the antecedents so far established in that discipline. Moreover, they also generally prove to be of little importance in mathematics sducation, the main results of which are intended to be useful for students and teachers in the present educational systems. In the next section we show a way of using the analysis of mathematical ideas, which, in our view, makes it fruitful for research in mathematics education.

3. ON THE ROLE OF HISTORICAL ANALYSIS

It is clear that any analysis that seeks to clarify educational problems — analysis being the prime driving force in our research— must be performed in the specific framework of our educational systems; but also, as a unitary counterpart, such analysis cannot help seeking to transform the conditions in which the teaching of mathematics is taking place in our countries. This clearly conditions the problems and therefore the methodology of the study; but also, in an aspect usually overlooked, it imprints on the results the need to be used, to be put to the test in the very place where they supposedly seek to cast light, where their modifications will have to be taken into account in order to advance, to go deeper into the facts being investigated, to be able to construct new hypotheses that take the work done into account.

This makes it necessary for the problems, in at least some of their aspects, to be closely linked with the actual process of teaching. However, this does not mean abandoning somewhat theoretical problems and their appropriate logical methods; rather, the studies take place within larger programs in which direct contact with students and teachers is present.

In this section we show that the historic-critical analysis of the development of mathematical ideas makes it possible, for example, to construct learning sequences that reflect the achievements of theoretical research, and that it becomes fully meaningful when, in turn, in theoretical research the history of ideas is enriched by the new hypotheses formulated by putting teaching sequences to the test in educational systems. Then we will rightly be able to maintain that we are speaking of studies in the field of mathematics education and not in that of the history or epistemology of mathematics.

3.1. Epistemological analysis

At one time history was relegated to being a pastime of mathematicians, although with the production of dazzling works, such as Van der Waerden (1954), or general views seen through new eyes, such as Boyer (1968). Now, however, it has regained its proper stature and has even made its way into the textbooks (see Edwards, 1979).

Even earlier, however, Boyer (1959) had offered us more profound attempts to capture other more intense moments: those of the evolution of ideas. Many titles could be added here to illustrate this great return of history as an instrument with which to view the present. We will only indicate, by way of example, that our ideas about the nature of the rudimentary processes of constructing mathematical models have changed completely as a result of historical studies concerning the Babylonians (Neugebauer, 1969); that our conceptions about the origin of the theory of proportions, deduction, and axiomatization have begun to acquire subtle tonalities that we did not perceive before, thanks to Szabó (1977); and that Jens Høyrup, pursuing this evolution of algebraic ideas from Babylon to medieval Italian algebra in numerous studies, has made us see it in a different way (see, for example, Høyrup, 1985, 1986, 1987, 1991, 1999a, 2002a, 2002b).

This re-encounter between history and epistemology through the history of ideas has also begun to benefit the didactics of mathematics.

3.2. The reading of texts

The new approach consists of analyzing problems of teaching and learning mathematics with the historic-critical method, and then of putting the theoretical findings to the test in the educational systems so that, after this experimentation, one will once again have a new view of the problematics of the history of ideas that corresponds to the teaching results.

A first example, taken from Filloy (1980), will make this idea clearer. Analyses of Diophantus's *Arithmetic*, Bombelli's *Algebra* and the contrast between it and Viète's *The Analytic Art*[2] lead to interesting hypotheses about the development of the first notions of algebra in secondary school (with pupils aged 12 to 15), as one can gather from the works of Jacob Klein (1968), for example. From these results one can infer that the most significant change in symbolization, in that stage of the beginning of algebra, is the step from the mathematical concept of unknown to the mathematical concept of variable. A transition that involves not only the feat of solving complicated arithmetic problems, already achieved in Diophantus's *Arithmetic* in one sense more efficiently than by Viète, but also reflection on the operations that are always performed to solve such problems. This reflection on operations suggested to Viète the need to speak not only of unknowns but also of the fact that the coefficients of the equations that result from making the *zetetic* analysis of the problems are also variables; that is, such coefficients have to operate on each other, not just representing a number, unknown or not but ultimately only one number, but rather representing all the numbers that could come from equations resulting from the analysis of arithmetic problems.

These facts would seem to complete the picture, especially when the analysis is continued along these lines, as is done by Klein (1968) or Jones (1978). This change of perspective (in Viète) immediately generated others, owing to the problems posed by operating on measurements, as can be seen clearly in the work of Stevin. A change is generated, as we were saying, in the very concept of number, that is, new (ideal) objects become numbers: numbers that can be operated on in the same way; for example, decimals become numbers provided that they obtain the category of mathematical objects, the main argument in Stevin's works, *Arithmetic* and *Disme* (Stevin, 1634).

But when one constructs teaching sequences that try to follow this connecting theme, as in Alarcón and others (1981–1982), and one observes the behaviour of the population (in the statistical sense) in the distributions that indicate the evolution of mathematical abilities, one finds that there are other elements which have not been taken into account. It then becomes apparent that first of all one would have to seek out the history of operational aspects, of the syntax of arithmetic-algebraic language, in its development in the East, and also, secondly, study the evolution of another history, apparently unconnected but one that in practice is revealed to be totally related to that of operational aspects: the history of the analysis of variation and change; either, in the first instance, purely arithmetic methods (such as those of proportional variation), or, on a deeper level, those entailed by pictographic representation of the first and second variations of movement, of the change in the intensity of light, or of the propagation of heat.

At this point it would seem to be very important to go back to history and analyze the works of the Middle Ages in regard to this. Our debt to historians (see Grant 1969, 1971; Clagett 1959, 1968; Van Egmond, 1980; Hughes, 1981; and Høyrup, 1999b, for example) is inestimable in this context, for their compilations, translations, and commentaries provide us with living material which is waiting for us to go to them with new eyes: those of the problematics of the teaching of algebra, at the very point where algebra was to make it possible to introduce analytical ideas in geometry, and, immediately afterwards, the methods of infinite calculus. Similarly, in order to understand the jump between arithmetic and algebra (and the appearance of arithmetic-algebraic language) it is necessary to cast light on the period immediately before the publication of Bombelli's and Viète's books.

In Viète's *The Analytic Art* we find the construction of an algebraic language in which, in addition to being able to model the problem situations solved by the languages used by Bombelli and Diophantus, we can also find a language in which one can describe the syntheses and algebraic properties of the operations introduced in the older texts. What is new in Viète's language lies in the fact that, whereas in those earlier texts operations were used only by performing them or employing them problem by problem, in Viète there is the possibility of describing the syntheses (algebraic theorems) and the syntactic properties of operations, because they can all be described with that language stratum and they can also be added to the store of knowledge on which someone who has mastered that new language stratum can draw.

In the following sections we describe language strata prior to the introduction of the language of Viète's *The Analytic Art*. As examples we use certain differences between the *abbacus* books and Jordanus de Nemore's *De Numeris Datis*.

3.3. The abbacus *books*

As can be seen in the work of compilation by Van Egmond (1980), the *abbacus* books represent the most feasible path for the assimilation of the mathematics of the East by Western European civilization; and in this adaptation of Indo-Arabic mathematics to the problems characteristic of a society with a vigorously rising economy (the society of Italy in the 15th and 16th centuries) a new kind of mathematics was born.

This mathematics was present and ready to be applied in the so-called *abbacus* books, the content of which essentially comprised the presentation of the Indo-Arabic positional system of numeration, the four elementary arithmetic operations, and the solution of commercial problems. These problems involved the four elementary operations, and also the use of the simple and compound rule of three, simple and compound interest, and the solution of some simple algebraic equations. Some books also included multiplication tables, tables of monetary equivalents, and tables of weights and measures.

The first *abbacus* book of which we know was written in Latin in the Near East (Greece) and was introduced in Western Europe, in its first vernacular version, between the 12th and 13th centuries.

The meaning of the word *abbacus* in the name of these books was that of "the art of calculating, counting and arithmetic." The term was first used in this sense by Leonardo of Pisa, better known as Fibonacci, who in the 13th century wrote a compendium on the mathematical practice known up to that point. This happened naturally, because his father was a merchant from Pisa who visited and stayed in Arab countries in the East and in the Maghreb, particularly in the town now known as Bejaia (in what is now Algeria), and so Fibonacci was taught by Arabic teachers and learned the Arabic systems, both of commerce and of mathematics, with the result that his book contained knowledge of practical commercial mathematics, in accordance with the Indo-Arabic system, and with a particular influence exercised by his own experience in merchant life and by his instruction in a great variety of Arabic texts on algebra, geometry, and commercial mathematics.

The production of *abbacus* books increased greatly in Italy in the 15th century; it is estimated that there were then 400, with about 400 different problems solved in each one; so, with regard to problems, even if we eliminate the repetitions between books, the production was of the order of tens of thousands.

The first *abbacus* schools appeared in the West almost at the same time as the arrival of the first *abbacus* books. It is known that the first school was founded in 1284 in the commune of Verona, and that these schools were attended mainly by the sons of merchants and, in general, by men of affairs, in order to practice commercial mathematics and continue their basic education in grammar. The *abbacus* schools tended to proliferate in the 14th century; it is known that in Florence alone, in about 1343, there were six schools in which over a thousand students were taught. And, although this growth did not remain stable in subsequent centuries, there are references for about three or four schools in each important city (Florence, Milan, Pisa, Venice, Lucca), which functioned continuously from the 14th century and throughout the Renaissance.

The most plausible historical explanation (see Van Egmond, 1980) for the appearance and multiplication of *abbacus* books, schools, and teachers in the West is of a social and economic nature. With the so-called Commercial Revolution in Italy in the 13th century there was a substantial social change: monetary power began to count more than feudal power, with the result that there was a greater desire for control of trading and financial activities, together with the skills required for their performance, than for possession even of land. Consequently, the rise of this new social class that came to power imposed the need to create the means to make this new kind of inheritance effective: the skills required in order to be able to participate in commercial power. These skills naturally included the contents of the *abbacus* books, originally produced to serve as reference books for the accountants and merchants of the time, and the need to make them accessible to the merchants' sons led to the creation of *abbacus* schools and teachers, financed, initially at least, by the parents themselves.

The children who attended the *abbacus* schools were 10 or 11 years old, and they were trained in the basic principles of arithmetic and practical mathematics (writing Indo-Arabic digits, the four operations with integers and fractions, solving commercial problems, and handling monetary equivalents and weights and measures), and also in grammar. It might be considered that the *abbacus* schools functioned as a kind of basic secondary education, acting as a bridge between basic education (the classical Roman school) and university (the first universities having been founded in Europe in the 12th and 13th centuries).

Although the *abbacus* schools might be viewed as an integral part of school education at that time, in reality they constituted a genuine innovation in educational circles because, unlike the universities, which provided instruction for the elites and functioned primarily as places for discussion and reflection about knowledge, the *abbacus* schools acted as transmitters of knowledge applicable to daily life. In the 15th century commercial activity was not just transacted between merchants and men of affairs, but rather those activities began to form part of the everyday life of what had become an urban population.

Thus the *abbacus* schools and contents served to satisfy a social demand in the new Europe civilization, with such success that they became a tradition that endured for centuries as a companion to the new pattern of culture, the mentality created by the commercial revolution.

To appreciate the full extent of the social and educational role of the *abbacus* books one would only need to review some of the mathematical contents of current textbooks for basic education in any country in the world to realise that, essentially, they are the same as what could be extracted from a typical *abbacus* book (see Van Egmond, 1980). This gave them the character of assimilators of Eastern mathematics to the needs of the new Western culture (which now stretches back over more than five centuries) through school education.

3.4. An abbacus *problem*

In the section of recreational problems in the *Trattato di Fioretti* (Mazzinghi, 1967), we can find problems such as the following:

Fa' di 19, 3 parti nella proportionalità chontinua che, multiplichato la prima chontro all'altre 2 e lla sechonda parte multiplichato all'altre 2 e lla terza parte multiplichate all'altre 2, e quelle 3 somme agunte insieme faccino 228. Adimandasi qualj sono le dette parti. [From 19 make 3 continually proportional parts such that, if the first is multiplied by the other 2 and the second part is multiplied by the other 2 and the third part is multiplied by the other 2 and those 3 are added, together they make 228. The question is what the aforementioned parts are.]

We can state this problem by translating it into the MSS of current algebra as follows:

Find three numbers x, y, z such that

$$\left.\begin{aligned} &x+y+z=19 \\ &\frac{x}{y}=\frac{y}{z} \\ &x(y+z)+y(x+z)+z(x+y)=228 \end{aligned}\right\}$$

In Puig and Rojano (2004) there is a transcription of the original version in old Italian of the solution of this problem, accompanied by a translation into the MSS of algebra. For our present purposes, the solution presented in the treatise consists in applying a series of rules, in particular a rule of doubles[3] and the Babylonian method of completing squares. In both cases, but more obviously in the rule of doubles, each time that the rule is used it is reworded for the specific numbers with which it is necessary to operate. We will see that this is one of the characteristics that make the MSSs of the *abbacus* books more concrete than Viète's MSSs, but also more concrete than that of Jordanus de Nemore's book, *De Numeris Datis*.

3.5. De Nemore and his work

The bibliographic information about Jordanus de Nemore is very diffuse, but the authenticity of his work has been established. He lived during the period that ranges from the middle of the 12th century to the middle of the 13th century, and on the basis of annotations in the margins of his writings it is believed that he taught at the University of Toulouse. Research on his life and work has led him to be considered, since the last century, one of the most prestigious natural philosophers of the 13th century. It is also known that he devoted himself to physics-mathematics, laying the foundations for the whole area of medieval statics. Among his mathematical works, those devoted to arithmetic (and algebra) continued to be reproduced until the 16th century.

If we consider only the treatises of a strictly mathematical character, we can identify six works attributed to Jordanus: *Demonstratio de algorismo*, which is a practical explanation of the Arabic number system with regard to integers and their use; *Demonstratio de minutiis*, which deals with fractions; *De elementis arithmetice artis*, which became the classic source of theoretical arithmetic in the Middle Ages; *Liber philotegni de triangulis*, which stands out in medieval Latin geometry particularly because it gives geometric proofs of theorems; *Demonstratio de plana sphera*, which consists of five multi-partite propositions that clarify various aspects of stereographic projection; and, lastly, *De numeris datis*, considered the first book of advanced algebra written in Western Europe, after Diophantus's *Arithmetic* (which was written in about 250 BC but did not reappear in the Christian West until the 15th century, whereas in the Islamic East there is an Arabic translation of the 9th century; see Sesiano, 1982, and Rashed, 1984).

3.6. De Numeris Datis

Our description of this work is based on the version edited, translated, and interpreted by Barnabas Hughes and published by the University of Berkeley (Hughes, 1981). The book includes a critical edition in Latin of the complete *De Numeris Datis*, together with an English translation of the entire text and a translation into modern symbology of the statement and canonical form of each proposition (it does not include the symbolic translation of the solving procedure). In Puig (1994) there is a detailed description of the MSSs of this work, together with a translation of parts of book one, more literal than the translation by Hughes, precisely with the aim of bringing out the characteristics of the MSSs. Here we will limit ourselves to outlining what is of interest for our present purpose.

Unlike the *abbacus* books, employed as elementary algebra texts in secondary education for use in commercial life, *De Numeris Datis* was a text aimed at university students of the time, with the intention of setting them non-routine "algebraic" problems and teaching how to solve them. Indeed, *De Numeris Datis* presents a treatment of quadratic, simultaneous and proportional equations which presupposes handling contents equivalent to those of al-Khwârizmî's *Concise book of the calculation of al-jabr and al-muqâbala* (Rosen, 1831) and Fibonacci's *Liber abbaci* (Boncompagni, ed., 1857; Sigler, ed., 2002). Both texts begin with some definitions and the development of the equations $x^2 = bx$, $x^2 = c$ and $bx = c$, very rapidly arriving at the equations $x^2 + bx = c$, $x^2 + c = bx$ and $bx + c = x^2$.

The part played by *De Numeris Datis* in the history of mathematics is comparable to that of Euclid's *Data* (Taisbak, 2003), in Hughes's opinion, in the sense that the former constitutes the first book of advanced algebra, in the same way that the latter is the first book of advanced geometry and implies a good knowledge of fundamental geometry (contained in the *Elements*), confronting the ambitious student with the proof and solution of non-standard problems by the method of analysis.

The propositions of *De Numeris Datis* are useful for analysis, therefore, just as a box of tools is, but the very structure of the book is also an exercise in analysis. In fact, unlike what happens in the problem of the *Trattato di Fioretti*, which we have just mentioned and that can be taken as representative of the *abbacus* type of problem, in the propositions of *De Numeris Datis* it is a question of finding numbers for which some numerical relationships are known, but these relationships are given by constants, that is, the book says, for example, that the sum of three numbers has been given instead of saying that the sum of three numbers is equal to a certain specific number, 228 in the case mentioned, as it appears in an *abbacus* book. In fact, the statements of the propositions in *De Numeris Datis* are not problems but theorems, as they always have the form "if such numbers or ratios and relationships between them have been given, then such numbers or ratios have been given," and they are proved like theorems, and are accompanied by a particular problem with specific numbers that is solved with the rule derived from the steps of the proof of the theorem or from the steps of the theorems to which this theorem is reduced.

This second point is fundamental for the character of *De Numeris Datis* that interests us here: the sequence in which the propositions in *De Numeris Datis* are solved explicitly shows the reduction of each proposition to one that has been proved previously, and, therefore, the solution of the corresponding problems to the solutions of others solved previously. This kind of sequence is not entirely absent in *abbacus* texts, that is, in *abbacus* problems we also see repeated application of rules or algorithms when the solving procedure has led to a well-identified situation in which the application is feasible: this is the case with the rule of doubles or the Babylonian method in the problem mentioned. However, this aim of reducing to situations or forms already encountered and solved previously does not appear explicitly in the *abbacus* texts, whereas in *De Numeris Datis* it forms part of the method of solution. This might be due to the fact that expressions that we would write as $x+y+z=a$ and $x+y+z=b$ with $a \neq b$ are not fully identified as equivalents for the purposes of the solving procedures and strategies, which in the *abbacus* books depend strongly on the specific properties of the specific number a (or b) and its relationships with the other numbers that appear in the other equations of the system in question.

It is in this sense that *De Numeris Datis* might be located in a more evolved stage, as it makes it possible to group problems that can be solved in the same way into large families by identifying more general forms. By this we do not wish to suggest that the actual strategies and skills required for the solution of problems in *De Numeris Datis* are on a higher level of abstraction or a more evolved level in terms of symbolization than those developed by the *abbacus* texts. These ideas about establishing a clear difference between levels of symbolization and solving strategies and skills are developed in Filloy and Rojano (1983). The point of view developed there considers the construction of symbolic algebra as the final identification within a single language of earlier strata of that language in which the absence of abstract symbolism causes the posing of the problems and the procedures for solving them to be carried out in the vernacular (Latin, Italian). This imprints peculiarities on the operations performed, peculiarities that vary from one stratum to another and that cause those operations to be irreducible from one stratum to another unless one has developed what we call a more abstract MSS.

4. THE PHENOMENOLOGICAL ANALYSIS OF SCHOOL ALGEBRA

Modern algebra organizes phenomena that have to do with the structural properties of arbitrary sets of objects in which there are defined operations. Those properties and those objects come from the objectification of means of organisation of other phenomena of a lower level and they are the product of a long history with successive rises in level.

4.1. Characteristics of algebra in al-Khwârizmî

One way of viewing this history consists in placing oneself in the 9th century at the time when al-Khwârizmî wrote the *Concise Book of the Calculation of Al-jabr and Al-muqâbala* and taking that event as the birth of algebra as a clearly defined discipline within mathematics. What al-Khwârizmî did, and what separates his work from all the others that have been seen as algebra after him, was that he began by establishing "all the types or species of numbers that are required for calculations."

The context in which he seems to have examined those species is that of the exchange of money in trading or inheritances, and from it he takes the names that he uses for the species of numbers. The world of commercial problems and inheritances is linear or quadratic: in the course of the calculations there are numbers that are multiplied by themselves, in which case they are "roots" of other numbers, and the numbers that result from multiplying a number by itself are *mâl*, literally "possession" or "treasure"; other numbers are not multiplied by themselves and are not the result of multiplying a number by itself, and therefore they are neither roots nor treasures, they are "simple numbers" or *dirhams* (the monetary unit). Treasures, roots, and simple numbers are thus the species of numbers that al-Khwârizmî considers.

In his *Arithmetic*, Diophantus had already distinguished different species (*eidei*) of numbers, with a different conceptualization (ways in which a number may have been given), using the names *monas*, *arithmos*, *dynamis*, *cubos*, *dynamodynamis*, *dynamocubos*, etc., and thus a longer series than al-Khwârizmî's.

Calculating with al-Khwârizmî's or Diophantus's species of numbers follows similar rules: what is obtained is always an expression equivalent to our polynomials or rational expressions, as the numbers of the same species are added together, or are taken that many times, or that many parts are taken, and the result is a number of that species a certain number of times or a certain number of parts of a number of that species; and if numbers of different species are added, the sum cannot be performed and is simply indicated. Thus, "four ninths of treasure and nine dirhams minus four roots, equal to one root" (Rosen, 1831, p. 41 of the text in Arabic) is an algebraic equation in al-Khwârizmî's book, since al-Khwârizmî's MSS uses vernacular language (Arabic in his case) exclusively; and

$\Delta^{\Upsilon}\overline{\beta}\,\overset{\circ}{M}\,\overline{\sigma}\ \acute{\iota}\sigma\alpha\ \overset{\circ}{M}\,\overline{\sigma\eta}$ (Tannery, 1893, vol. I, p. 64, l. 8)

is an equation in Diophantus's MSS, which is read as "dynamis 2 monas 200 equals monas 208," since Diophantus uses abbreviations for the names of the species of numbers, which in this case consist of the first two letters of the word, and the Greek system of numeration uses the letters of the alphabet marked with a horizontal stroke, in a system that is not positional but additive, with codes for the nine units, the nine tens and the nine hundreds. There is almost no conceptual difference between the algebraic expressions and the equations of the two authors, as what is represented in them is the names of the species, the specific numbers that indicate how many of each species there are, the operations indicated between the quantities of each species, and the relationship of equality between quantities.

Al-Khwârizmî's book might thus be seen as more elementary or situated one step behind Diophantus, as the set of species of numbers is smaller and the expression uses only the signs of the vernacular. However, what is new in al-Khwârizmî's book is that it suggests having a complete set of possibilities of combinations of the different kinds of numbers. It is clear that initially the possibilities are infinite, and that therefore it is necessary to reduce them to canonical forms in order to be able to consider obtaining a complete set. But al-Khwârizmî's aim then is also to find an algorithmic rule that makes it possible to solve each of the canonical forms, and to establish a set of operations of calculation with the expressions that makes it possible to reduce any equation consisting of those species of numbers to one of the canonical forms. All the possible equations would then be soluble in his calculation. Moreover, al-Khwârizmî also establishes a method for translating any (quadratic) problem into an equation expressed in terms of those species, so that all quadratic problems would then be soluble in his calculation.

Al-Khwârizmî obtains the set of canonical forms by combining all the possible forms of the three species, taken two at a time and taken three at a time. He thus obtains the three forms which he calls "simple," making the species equal two at a time:

treasure equal to roots
treasure equal to numbers
roots equal to numbers

and the three forms that he calls "compound," adding two of them without taking order into account and making them equal to the third:

treasure and roots equal to numbers
treasure and numbers equal to roots
roots and numbers equal to treasure.

As al-Khwârizmî is able to present an algorithm to solve each of these canonical forms simply by collecting and justifying methods that are established and that have been in use since the time of the Babylonians, all that remains is to establish a procedure for translating the statements of the problems into their algebraic expressions and a calculation that makes it possible to transform any equation into one of the canonical forms.

The species of numbers refer to concrete numbers with which calculations are performed, so that in order to be able to translate the statements of the problems into those algebraic expressions it is necessary to be able to refer also to unknown quantities as if they were concrete numbers and calculate with them, that is, it is necessary to name the unknown and treat it like a known number. What al-Khwârizmî does to achieve this is to use the word *shay'*, literally "thing," to name an unknown quantity. He then uses it to perform the calculations which the analysis of the quantities and relationships present in the problem indicates to him as being necessary, and in the course of the calculations he sees what species of number that thing is: a root if it is multiplied by itself, or a treasure if it is the result of a quantity that has been multiplied by itself; so that he can translate the statement of the problem into two expressions that represent the same quantity and make them equal so as to have an equation. In Chapter 11 we will see that these are in fact the steps of the Cartesian method.

"Thing," incidentally, is a common noun for representing any unknown quantity, not the proper name of a specific unknown quantity, unlike what is established by the Cartesian method; in fact, al-Khwârizmî does not say "the thing" but "thing," that is, "a thing," when he refers to the unknown quantity which he calls "thing." In the course of the construction of the equation that translates the problem, however, “thing” is bound to one of the unknown quantities, functioning as the proper name of that quantity.

The operations in the calculation are algebraic transformations of the equations that seek to obtain one of the canonical forms. However, the canonical forms have three features that characterize them (and that cause the complete set of canonical forms to have 6 items), and the operations are directed at achieving each of those three features.

The first is that there are no negative terms, or, to use al-Khwârizmî’s terminology, there is nothing “that is lacking” on either of the two sides of the equation.

In fact, in al-Khwârizmî’s or Diophantus’s algebraic expressions there are quantities that are being subtracted from other quantities. There are not positive and negative quantities, but quantities that are being added to others (additive quantities) and quantities that are being subtracted from others, and the latter cannot be conceived on their own but only as being subtracted from others. Thus, al-Khwârizmî may even go so far as to speak of “minus thing” when he is explaining the sign rules, but he is always referring to a situation in which that thing is being subtracted from something:

> When you say ten minus thing by ten and thing, you say ten by ten, a hundred, and minus thing by ten, ten “subtractive” things, and thing by ten, ten “additive” things, and minus thing by thing, “subtractive” treasure; therefore, the product is a hundred dirhams minus one treasure. (Rosen, 1831, p. 17 of the text in Arabic)

However, as the subtractive quantities are conceived as something that has been subtracted from something, an expression in which there is a subtractive quantity represents a quantity with a defect, a quantity in which something is lacking. Diophantus’s sign system expresses this way of conceiving the subtractive in an especially explicit way, as in his work all the additive quantities are written together, juxtaposed in a sequence one after another, and all the subtractive quantities are written afterwards, also juxtaposed, preceded by the word *leipsis* (what is lacking). Thus, the algebraic expression

$$x^3 - 3x^2 + 3x - 1$$

is written as

$$K^{\Upsilon}\overline{\alpha}\,\varsigma\overline{\gamma}\ \pitchfork\ \Delta^{\Upsilon}\overline{\gamma}\,\mathring{M}\overline{\alpha}$$ (Tannery, 1893, vol. I, p. 424, l. 10),

an abbreviation of “cubos 1 arithmos 3 what is lacking dynamis 3 monas 1,” in which the expressions corresponding to x^3 and $3x$ are juxtaposed on one side, and x^2 and 1 on the other, separated by the abbreviation for “what is lacking.”

It is precisely this idea that there is something lacking in the quantity that is directly responsible for the form adopted by the operation that eventually gave its name to algebra. In fact, the objective of the operation that al-Khwârizmî calls *al-jabr* is that nothing should be "lacking" on either side of the equation. That is why the operation is called *al-jabr*, literally "restoration," because it restores what is lacking. In terms of the language of modern algebra, *al-jabr* eliminates the negative terms in an equation by adding them to the other side, but *al-jabr* is not equivalent to the transposition of terms because the modern transposition of terms can also transfer a positive term to the other side by making it negative, which goes against the intention of the *al-jabr* operation (but is consistent with the fact that the canonical form that one now seeks to attain with algebraic transformations is $ax^2 + bx + c = 0$, with a, b, and c being real numbers).

The second characteristic feature of al-Khwârizmî's canonical forms is that each species of number appears only once. The algebraic transformation that this pursues is *al-muqâbala*, literally "opposition." As al-Khwârizmî always performs this operation after *al-jabr*, at this point there is nothing lacking; there are no negative terms in the equation. The operation consists in compensating for the number of times that a given species of number appears on each side of the equation, leaving the difference on the appropriate side.

Lastly, the third characteristic is that there is only one treasure, or, in modern terms, that the coefficient of the treasure is 1. This is achieved by means of two operations that al-Khwârizmî calls "reduction" (*radd*) and "completion" (*ikmâl* or *takmîl*). "Reduction" is used when the coefficient of the treasure is greater than one, and it consists in dividing the complete equation by the coefficient; and "completion" is used when the coefficient of the treasure is less than one (it is "part of a treasure," in al-Khwârizmî's words), and it consists in multiplying the complete equation by the inverse of the coefficient.

The first two operations, *al-jabr* and *al-muqâbala*, appear in the title of al-Khwârizmî's book as the characteristic operations of calculation, and they are also mentioned, although not by name, in the introduction to Diophantus's *Arithmetic* (Tannery, 1893, vol. I, p. 14, ll. 16–20).

What makes all these calculations meaningful, therefore, is the idea of the establishment of a complete set of canonical forms, which then organizes algebraic expressions through transformations, and it organizes problems into families of problems that are solved in the same way.

4.2. Steps toward modern algebra

Al-Khwârizmî's complete set of canonical forms was complete only with the condition of restricting the species of numbers to the three that he considered.

The continuation, including the cube as the fourth species, was developed by ᶜUmar al-Khayyâm, who established that the complete set of canonical forms had 25 items, but that he could not find an algorithm for solving the 25. What al-Khayyâm did as a result of his inability to give a strictly algebraic solution for the matter was to show how the solution of the canonical forms could be constructed in the cases that resisted him by means of intersecting conical sections.[4] As a response to the same inability, Sharaf al-Dîn al-Tûsî added to this the establishment of procedures for the approximate calculation of roots.[5] For the historical phenomenology that we are outlining, these non-algebraic responses to the lack of ability to find algorithms for all the canonical forms do not interest us. Nor are we interested in the fact that eventually algorithms were found not only for al-Khayyâm's 25 canonical forms but also for fourth-degree equations. What interests us is the response given to the inability to find algorithms for the canonical forms of equations of a degree higher than the fourth from Lagrange onwards.

In fact, in his memoir written in 1771, "Réflexions sur la résolution algébrique des équations" (Lagrange, 1899, vol. III, pp. 205-424), Lagrange explicitly proposed an aim which was not that of continuing to search for algorithms, but of examining why it had been possible to find them.

> I propose to examine the various methods that have been found so far for the algebraic solution of equations, and reduce them to general principles and show a priori why these methods succeed for the third and fourth degree and are lacking for higher degrees. (Lagrange, 1899, vol. III, p. 206)

Here, therefore, Lagrange explicitly takes the methods themselves as the object of study, so that the problematics of algebra is shifted to a higher level, beyond the organization of problems into families by the establishment of canonical forms in a more abstract MSS than that of the problems themselves. Now it is the characteristics of the canonical forms themselves that have to be organized in order to account for the success or failure of algebraic methods of solution. What Lagrange does is to make a critique of the methods, a critique in the sense of establishing limits. To do this, he studies the relationships in the methods between a given equation that one is trying to solve and the reduced equation, a second-degree auxiliary equation that can therefore be solved algebraically, to which one can proceed from the given equation by a rational relationship; and, on the basis of this study, in a crucial movement he reverses the relationship by finding a way of expressing the reduced equation in terms of the roots of the given equation (what Lagrange calls the resolvent). From this point he is able to establish the reason for the success of the methods, and also the fact that the same reason cannot exist for degrees higher than the fourth (which does not exclude the possibility of an algebraic solution, but does rule out the possibility of it belonging to the same structure).

The shift made by Lagrange, from the search for methods of solution to the explanation of why they are successful or not, led Abel in 1824 to jump to a new level, in his *Mémoire sur les équations algébriques, où l'on démontre l'impossibilité de la résolution de l'équation générale du cinquième degré* (Abel, 1881, I, pp. 28-33), in which he shows, as the title says, that the inability to find an algebraic method of solution for equations of a degree higher than the fourth really is an impossibility, thereby giving the previously insoluble problem a formulation in which it is soluble, changing the problem of finding a method into the problem of proving whether such a method exists.

Galois's works provided the final and definitive jump in level, by linking the solubility of an equation to the properties of the equation's group and tackling the problem by studying the properties of those groups, so that what is studied is not what equations are soluble but what groups are soluble. He shows this clearly in a memoir written in 1831, *Sur les conditions de résolubilité des équations par radicaux*, where he says:

> Problem. "In what case is an equation soluble by simple radicals?"
> First of all I will observe that in order to solve an equation one must lower its group progressively until it contains only one permutation.
> […] let us seek the condition that must be satisfied by an equation's group so that it can thus be lowered by the adjunction of radical quantities. (Galois, 1846, p. 426)

With this step, from Galois onward algebra becomes modern algebra. As Vuillemin says,

> […] Galois's theory has shifted the interest of algebra: whereas, essentially, it set out to solve equations, in future it will tend rather to seek the nature of the magnitudes that must be added to the base field in order to determine the factorising field in which it becomes possible to express and ascertain roots rationally. (Vuillemin, 1962, p. 247)

4.3. The phenomenological analysis of the language of algebra

After Galois we enter a different history, that of modern algebra, which is absent from current school algebra, yet the historical phenomenology that we have expounded in the two previous sections does not exhaust the phenol-menology of school algebra. It is at least necessary to consider what phenomena are organized by the language of algebra, and in what way it organizes them. Once again, this can be done as historical phenomenology or as didactical phenomenology. The historical view is developed in Section 1.3, "Algebraic Language: A History of Symbolisation," in Puig and Rojano (2004); the didactical phenomenology can be found in Freudenthal's *Didactical Phenomenology of Mathematical Structures*, Chapter 16, "The Algebraic Language." In this section we refer to what is expounded in the two texts.

4.3.1. The representation of unknown quantities and species of numbers

In Puig and Rojano (2004) there is an analysis of how the central core of the evolution of the language of algebra has to do with the way in which unknown quantities, on the one hand, and species of numbers, on the other, are represented in algebraic expressions and therefore in equations.

In most of the sign systems of medieval algebra there is only one name to represent the unknown, "thing," which is in fact a common noun although used as a proper name. Consequently, those MSSs cannot represent different unknown quantities with different proper names. Instead, once an unknown quantity has been named as "a thing," the others have to be named with compound names constructed more or less algorithmically from the relationships between it and each new unknown quantity (for example, "ten minus thing" is the name that one could give to an unknown quantity of which it is known that when it is added to "thing" the result is ten). However, the network of relationships between the quantities in the problem might be so complex that it is extremely intricate, or even impossible, to name all the quantities with compound names: for these problems, the fact that only the term "thing" is available makes the sign system not very efficient.

Medieval algebraists resorted to various devices to get around this. Sometimes they used the term "thing" again, but with a qualifier. This is the case with Abû Kâmil, who in one problem in his book of algebra (cf. Levey, ed. 1966, pp. 142-144) uses the names "large thing" and "small thing" ("res magna" and "res parva" in the Latin version edited by Sesiano, 1993, p. 388). Sometimes they used names of coins for the other unknown quantities. This is also the case with Abû Kâmil, who uses *dînâr* and *fals* (cf. Levey, ed. 1966, p. 133, n. 140, although on this occasion Abû Kâmil is expounding a different solution for a problem that has already been solved using "thing" on its own), or with Leonardo of Pisa, who uses *denaro*, as well as *res* (cf. Boncompagni, 1857, pp. 435-436 and p. 455). In the part devoted to inheritances in al-Khwârizmî's book, at one point he does not even use the term "thing" but calls the inheritance *mâl*, treasure, using it in its vernacular sense, and he calls what corresponds to each of the heirs "share" or "part share," and he constructs the indeterminate linear equation "five shares and two parts of eleven of share equal to the treasure." According to Anbouba (1978), in the same part of al-Khwârizmî's book there is also a problem in which he constructs a linear system of two equations using "thing" and "part of thing" to name two different unknown quantities.[6]

Moreover, what appears in the algebraic expressions is the names of the species of numbers (simple number or *dirham*, root, treasure, cube, etc.; or, in the translation into Latin, *numerus*, *radix*, *census*,[7] *cubus*, etc.), but the quantity by which this species is qualified is not named. From the identification of "thing" with "root" it is assumed that the treasure is the thing multiplied by itself, but there is no way of expressing another quantity represented with another proper name that has been multiplied by itself. The algebraic expressions of these sign systems do not say "five treasures of thing" but just "five treasures," unlike the sign system of modern algebra, which uses $5x^2$ to say "five times the square of x," and, therefore, is structurally prepared for designating another unknown quantity with another proper name, y, and saying "five times the square of y", $5y^2$.

The sign system of Indian algebra does have proper names for different unknown quantities (it uses names of colors for this purpose), and it forms algebraic expressions by juxtaposing the name of the unknown quantity and the name of the species (cf. Colebrooke, ed., 1817), but this system did not have any impact on medieval Arabic algebra, or therefore, on algebra in the Christian West. It was not until Viète that a sign system was developed in which there were proper names for different quantities, together with the names of the species. But Viète's sign system also used letters as proper names, and not just for unknown quantities but also for known quantities. This freed the algebraic expressions from ambiguities and made them capable of providing a direct representation of the quantities analyzed in the statements of the problems.

However, in Puig and Rojano (2004) it is shown that Viète's sign system lacks full operational capacity on the syntactic level because the species of numbers are represented by words or abbreviations of them, although these words are constructed algorithmically from certain basic words. It is also shown that this syntactic operativity is attained when one combines the representation of quantities by letters, introduced by Viète, with the representation of species by means of numbers that indicate the position of the species in the series of species (in continual proportion).[8] The algorithmic rules for the construction of the names of the species can then be replaced by those numbers and converted into part of the calculation.

4.3.2. Aspects of the didactical phenomenology of the language of algebra

In the “Variables in the Vernacular” section of his phenomenological analysis of the language of algebra, Hans Freudenthal recounts that

> When my daughter was at the age when children play the game “what does this mean?” and I asked her what is “thing” she answered: Thing is if you mean something and you do not know what is its name. (Freudenthal, 1983, p. 474)

The didactical phenomenology of the language of algebra that Freudenthal expounds is based precisely on the examination of the phenomena that are organized by the language of algebra, seen with regard to how those phenomena are organized in the vernacular and in the language of arithmetic, which are the languages that provide the starting point or context from which pupils have to acquire the language of algebra.

We will not repeat Freudenthal’s observations here, but simply indicate some of the aspects that he analyzes.

1) The rules of transformation in languages

We have already seen that the need for the development of rules of transformation in the language of algebra comes from the aim of being able to solve all problems without needing to have a specific algorithm for each one, and that this is done by the establishment of canonical forms and calculation on the syntactic level. In teaching, only awareness of the overall aim can give sense to the use of such syntactic transformations. Freudenthal examines the fact that rules of transformation also exist in the vernacular, but that the correctness of the transformations performed in the vernacular cannot generally be decided without resorting to the contextual meaning, whereas in the language of algebra the part played by the context in this sense is generally nil.

2) The algorithmic construction of proper names

We have seen that this is an outstanding aspect of the language of algebra. Freudenthal points out that algorithmic features are not unusual in vernacular languages. But these algorithmic procedures of sign construction are not very systematic and are not generalized (plurals, conjugations and declensions, etc.). The first experience that children have of an algorithmic construction of proper names is the learning of numbers in their mother tongue: an area of contact between the vernacular and the language of arithmetic.

3) Structuring devices

The rules of transformation and the algorithmic construction of proper names are based on the structure of the language. The language of algebra has a wide range of structuring devices, many of them shared with the language of arithmetic, especially parentheses, priority between operations, and the arrangement of signs in relation to the text line (exponents, subscripts and superscripts, the fraction bar and the positions above and below the text line that it determines, roots, etc.). Once again, Freudenthal analyzes the existence of such structuring devices in the vernacular, and the fact that there they are based on content, whereas this is not the case (or not so much) in the language of algebra.

4) Variables in the vernacular versus algebra variables

We have already analyzed the use of "thing" in the language of algebra, and the differences between it and the variables of modern school algebra. Freudenthal points out that the use of letters must also be examined in geometry, where Euclid's *Elements* already used letters to refer to points, lines, and figures, and he indicates the origin of the expression "point A," in which A is the proper name of the point, in an abbreviation of an earlier expression, "the point at A," which simply describes a drawn figure to which letters have been added in order to be able to refer to it in the oral discourse which was customary in teaching.

Freudenthal also examines the fact that in order to use a variable as a proper name it is necessary to bind the variable. "Variables," says Freudenthal (1983, pp. 474-475), "can be bound independently of any context, by linguistic logic devices, or in dependence of a context." The logic devices are the universal and existential quantifiers, the definite article (including "the thing" as opposed to "thing"), the set former, the function or species former and the interrogative, whereas the devices that depend on context are the demonstratives.

5) Formal substitution and algebraic transformations

Formal substitution is the culminating point in the constitution of the MSS in the teaching of school algebra. For this to take place it is necessary that the algebraic expressions should have completely relinquished the character of representing actions that their antecedents in the MSS of arithmetic possess, and should have completely acquired the static character of a relationship. One of the key elements in this transition from language as action to the language of algebra is the exceedingly well-known change of meaning from the arithmetic equals sign to the algebraic equals sign.

However, in the context of algebraic transformations, which are performed between expressions with a static character that represent relationships, the meaning of the arithmetic equals sign reappears. The algebraic transformation par excellence consists in "reducing" an expression to a simpler form or a canonical form, so that $(x + a)(x - a) =$ is an indication that an action must be performed and that the result of the transformation is expected on the other side of the equals sign; it is not just the construction of an equivalence. Yet the reversibility of algebraic transformations may give that appearance: for example, the action that is the opposite of "reducing" is "factorizing" (and one would have to decide which is simpler, the classical canonical forms resulting from reducing, or the expressions that explicitly show the roots that result from factoring).

At the origin of formal substitution there is the possibility that the letter that names a quantity may be replaced by a compound expression that names the same quantity. This makes it necessary for the user of the MSS to accept the fact that, as the letter and the compound expression represent the same thing, not only can they be made equal but also the calculations or relationships represented in an expression in which the letter appears can also be carried out with the equivalent expression and the new expression will represent the same thing. On the other hand, the user will have to face syntax problems[9] that derive from the structuring devices, such as the priority between operations, which sometimes makes it necessary to introduce other structuring devices such as parentheses where they were not present; or the problems posed by having to replace a letter with an expression in which that letter may also appear. This is the case with the difficulty that pupils find in replacing n with $n + 1$, for example, when using the method of complete induction.

However, the substitution becomes definitively formal when the expressions are no longer the result of the translation of the statement of a problem but are algebraic expressions which are studied as such.

SUMMARY

This chapter goes over part of a diagram of the design of the experimental setting for the observation of phenomena of learning and teaching algebra.

In the next chapter we shall apply the methodological diagram to the study "Operating on the Unknown," with a view to studying the processes of transition from arithmetic thinking to algebraic thinking at the point when pupils first encounter the need to operate on what is represented. In order to locate this point (called a "didactic cut" in the study), we use historical and epistemological analysis of mathematical sign systems found in old texts on algebra from the pre-symbolic period (before the appearance of François Viète's *The Analytic Art*). This analysis and the phenomenological analysis of algebraic language illustrate the power of the methodology proposed by local models, in the part corresponding to the choice of the moment of observation.

ENDNOTES

[1] See Filloy, Rojano and Solares (2004)
[2] The canonic edition of the Greek text of Diophantus's *Arithmetic* is the one by Paul Tannery (Tannery, 1893); we have also consulted the French translation by Paul Ver Eecke (Ver Eecke, 1959). The canonic edition of Bombelli's *Algebra* is the one by Ettore Bortolotti (Bortolotti, 1966). The original Latin text of Viète's book, *In Artem Analyticen Isagoge*, is included in the complete works of Viète compiled and published by Franciscus van Schooten (Van Schooten, 1646); there is an English translation included as an appendix in Klein (1968), and another one in Witmer, ed. (1983).
[3] In the text of the *Trattato di Fioretti* the rule is not stated generally but with reference to the concrete case to which it is applied, as follows: "E a multiplichare la seconda parte nella somma di tutte e 3 due volte è chome a multiplichare la seconda parte nel doppio della somma di tutte a 3, overo quanto a multiplichare lo doppio della seconda parte nella somma di tutte et 3." [And multiplying the second part by the sum of the 3 two times is like multiplying the second by double the sum of all 3, or like multiplying double the second part by the sum of all 3] (Mazzinghi, 1967, p. 16). Stated in a general form, the rule would say: "multiplying one quantity by another one twice is equal to multiplying the first quantity by double the second one, or also multiplying double the first quantity by the second."
[4] There is a recent edition of the Arabic text of al-Khayyâm's *Treatise on Algebra*, accompanied by a translation into French, in Rashed and Vahebzadeh (1999). One can also consult the English translation by Kasir (1931).
[5] There is an edition by Roshdi Rashed of the Arabic text of Sharaf al-Dîn al-Tûsî's *Treatise on Equations*, accompanied by a translation into French, in al-Tûsî (1986).
[6] Diophantus also has a single name for unknown quantities (*arithmos*). In problem 28 in Book II of his *Arithmetic* (Tannery, 1893, vol. I, pp. 124–127), he resorts to the device of saying that a second unknown quantity is one unit (*monas* 1), performing the calculations using this supposition, and then in the result changing the units to *arithmos* and calculating again.
[7] "Census" was the term chosen by Gerardo de Cremona for *mâl*, treasure, in his translation of al-Khwârizmî's book of algebra, and it was the one that caught on in the Christian Mediaeval West (cf. the edition by Hughes, 1986).
[8] This is already present in Chuquet's *Triparty*, written in French in 1484. However, this book by Chuquet remained unpublished and was therefore scarcely known until the end of the 19th century, when Aristide Marre published it (Marre, 1880). Bombelli used the same kind of representation in his *Algebra*, from which it became more widely known among algebraists.
[9] See our ongoing work reported in Chapter 8 and in Filloy, Rojano and Solares (2004).

CHAPTER 4

CONCRETE MODELS AND ABSTRACTION PROCESSES

TEACHING TO OPERATE ON THE UNKNOWN

OVERVIEW

This chapter deals with concrete modeling in teaching the elements of algebraic syntax and the processes (abstraction) that arise both in modeling itself and in the use of the syntactic skills learned in order to solve word problems. The subject is approached by means of a clinical study case "Operación de la incógnita" ("Operating on the Unknown"), the design and experimental performance of which are within the theory of *local theoretical models*, following the recursive diagrams that appear in Chapter 3. The study deals with the transition from arithmetic to algebraic thought, and in the study the idea of *didactic cut* is introduced for this context. We also begin the discussion of the dialectic relationship between semantics and algebraic syntax. Throughout the chapter we utilize the phenomenological analysis of school algebra presented in the previous chapter and the notions of *mathematical sign system* and *language strata*, dealt with in Chapter 1.

1. INTRODUCTION

1.1. Observation in class

As we mentioned in the Introduction (Chapter 1), one of the simplest phenomena that observation in class shows about the permanence in a reading level with children who have just finished primary education (about 12 years old) is one that appears when they are confronted with questions of the kind that illustrate what we called the reverse of multiplication syndrome.

These observations can easily be made in the classroom, where it is possible to infer that these events are linked with many others, examples of the intrinsic difficulties that the learning of algebra presents: the usual syntax mistakes when one is working operationally with algebraic expressions, translation mistakes when one is using algebra to solve problems written in ordinary language, mistaken interpretations of the meaning of algebraic expressions in the different contexts in which they appear, the difficulty of finding any meaning in them, the impossibility of using algebra to solve ordinary problems, etc.

1.2. Experimental observation

In order to observe these phenomena with greater precision, one needs an experimental situation in which one can monitor various obstructions that are always present in the classroom, and observation mechanisms that allow a more exhaustive, precise analysis. But the task must also be undertaken in such a way that what is observed has to do with the problems presented by the person under observation, and also that the components that teaching brings into play are present.

Over a period of four school years an experiment was carried out at the Centro Escolar Hermanos Revueltas in Mexico City, in which the teaching of mathematics in the six years of the secondary education program was monitored from the viewpoint of the teaching aims that it strove to achieve, and also a check was kept on the teaching strategies used throughout the middle school stage. Moreover, a laboratory for clinical observation was set up in which individual or group interviews could be conducted and videotaped. The clinical interviews had a structured format, but the interviewer moved freely within the previously designed steps, allowing the line of thought of the person interviewed to define each of the subparts of the interview. Except in cases in which the interviewee had no difficulty in solving the task set, the interviewer intervened to set further questions that would help the interviewee to learn (by discovery) the task that he or she was unable to solve initially. The aim was to discover the difficulties that the beginnings of algebra present for learning, given the usual ways in which it is taught at present. These were clinical interviews, in which the focus of observation was the usual ways of teaching and the individual ways of learning (with their typical obstructions and difficulties) that the students presented.

This infrastructure formed the basis for the development of the project "Evolution of Symbolization in the Middle School Population," and within it the study "Acquisition of the Language of Algebra," concentrating on the interrelationships between two overall strategies for the design of learning sequences that cover long periods of time in the middle school algebra curriculum, which are:

a) Modeling of *more abstract* situations in *more concrete* languages in order to develop syntactic skills.

b) Production of codes to develop problem-solving skills. Use of syntactic skills for the development of solving strategies.

Broadly speaking, in (a) the aim is to make new expressions and operations meaningful, modeling them on concrete situations. In (b) the aim is to produce senses for new expressions and operations (in such a way as to generate problem-solving codes), setting out from the assumption of the presence of certain skills in the syntactic use of the new symbols and their use as a *more abstract* language.

1.3. The theoretical framework

Apart from empirical observations such as those indicated in the first section of this introduction, the theoretical lines that guided this project were drawn essentially from three sources: first, an epistemology based on analysis of texts from the Middle Ages and the Renaissance (a description of which can be found in Chapter 3); second, a line drawn from semiotics, in an attempt to make it a guide for the analysis of algebra on the basis of its conception as a mathematical sign system; and third, cognitive psychology in its recent developments concerning language acquisition and its relation to the pragmatics of language.

We shall try to approach various aspects concerning the interrelation between the semantic and syntactic components of the problem, seen from the viewpoint of teaching strategies of types (a) and (b), briefly described above. This chapter concentrates, as its title indicates, on type (a) strategies and on the point in teaching when one wishes to teach how to operate on the unknowns that appear in first-degree equations.

Here we are not going to go into an analysis of what happens when a totally syntactic model is used as a teaching strategy. In Chapter 5, Section 3.5, we show that the phenomena that appear in that case are of the same nature as those described here for concrete models. The reader will not fail to perceive that aspects of type (b) strategies also appear here in the description of the mechanisms that are brought into play when the processes of abstraction are set in motion. However, the approach focuses totally on type (a) teaching strategies, their relations to the appearance of the usual syntax mistakes, their differences model by model, and the relation that they have to the students' prior attitudes, especially in terms of the extreme positions between the clearly syntactic tendencies and clearly semantic tendencies that are seen in the students. Emphasis is placed on the processes of abstraction in the situations presented, and in the operations involved.

A general description of what is presented indicates that there is a dialectic relationship between syntactic and semantic advances, and that an advance in either of these two components implies an advance in the other. This analysis is made from a viewpoint that corresponds to the usual strategies of teaching algebra. The starting point is the belief that the facts described here are not taken into account in current teaching systems, merely being left to later corrections that students may manage to make to various misconceptions and mistakes in the use of algebraic properties —properties that one is trying to teach for the first time.

1.4. Reading Guide

The text of this chapter is divided into the following four parts:

1) *The solution of equations and the transition from arithmetic to algebra*. This part sets out the theoretical and empirical antecedents that are relevant for the problem presented, especially for the determination of the point in the development of the algebra curriculum at which the experimental observation was situated.
2) *Concrete modeling at a transition point*. This part describes the point when the observation took place from the perspective of the teaching that preceded it, and it also describes the population from which the individuals were taken in order to carry out the case studies that make up the clinical part of this research. The population is classified in terms of its abilities and prior knowledge, and there is a discussion of why the study described here was carried out only with individuals from the so-called "upper stratum."
3) *Processes of abstraction of operations using a concrete model to learn how to operate on the unknown*. Here we begin to describe how the individuals observed performed, after a phase of instruction in operating on the unknown based on the modeling of equations in *concrete* contexts. A brief description of the empirical results obtained is given, in order to present referents that will enable us to describe the processes of interaction between the semantic and syntactic aspects that appear in the acquisition of the first elements of the language of algebra.
4) *Semantics vs. algebraic syntax*. This section is devoted to making a comparative analysis of the differences between the use of two models (the balance scales and a geometric model) by part of the population with a better performance. The differences can thus be taken into account for the proposal of teaching strategies based on these

observations. The two most important are: first, the fact that there are differences in the translation of elements of the equation to the model which obstruct progress in its use; and second, the fact that some equations offer more natural translations in one model than in another.

5) Contrast between two cognitive tendencies in the learning and use of mathematics, with respect to the application of the same model for operating on the unknown.

2. THE SOLUTION OF EQUATIONS AND THE TRANSITION FROM ARITHMETIC TO ALGEBRA

In Sections 3 and 4 of the Introduction (Chapter 1) we pointed out that various clearly established research results mark distinct differences between ways of thinking rooted in arithmetic and others that are characteristic of algebra. One of them is the inability to operate with the unknown as if it were known, which can be seen in most students when one starts to teach them algebra.

This kind of operational insufficiency in what is represented in the pre-symbolic stage of algebra suggests the presence of a point of cut-off or change between operating on the unknown and not operating on it, here on the level of individual thinking. In the clinical study "Operating on the Unknown," carried out with children 12 or 13 years of age,[1] operating on the unknown does indeed seem to be a necessary action for the solution by means of non-spontaneous methods[2] of certain first-degree equations with at least two occurrences of the unknown, for the solution of which it is not sufficient to reverse the operations on the coefficients. The following equations are examples of this kind:

$$38x + 72 = 56x$$
$$3x + 20 = x + 164$$

According to the study, the step from the operational solution of equations such as $x + 27 = 58$ or $4 \times (x + 11) = 52$ to the solution of equations such as $3x + 8 = 7x$ and $7x + 2 = 3x + 6$, for example, is not immediate, and in between comes the construction (or acquisition) of certain elements of syntax which is algebraic, strictly speaking. The construction of these syntactic elements is carried out on the basis of a reasonably well consolidated knowledge of arithmetic, and, in turn, this construction is possible only if one succeeds in breaking away from certain notions that belong to the domain of arithmetic; hence the presence of a cut.

Take, for example, the notion of an equation: $Ax \pm B = C$. In arithmetic terms, the left side of an equation corresponds to a sequence of operations that are carried out on numbers (whether known or not), and the right side corresponds to the result of having carried out those operations: this is what one might call an arithmetic notion of equality (or of an equation). Starting from such a notion, an equation of the type $Ax \pm B = C$ (where A, B, and C are particular given numbers) can be solved by simply inverting the operations in the sequence on the left, starting from the result C. We will call equations of this type "arithmetic" equations.

However, the arithmetic notion of equality does not apply to an equation such as $Ax \pm B = Cx \pm D$ (where A, B, C, and D are particular given numbers), and therefore its operational solution involves operations outside the scope of arithmetic, such as operating on the unknown. In order that such operations may acquire sense for the individual and so be brought into use in the process of solving an equation, equations such as those of the form described here (which we will call "non-arithmetic" equations) must in turn be provided with some meaning; this, however, implies a basic modification to the notion of equation or numeric equality.

With respect to the *meaning* of the *new* equations, it must be understood that the expressions in both parts of the equality are of the same nature (or structure), and that there is a series of actions that give sense to the equality between them (such as the actions corresponding to the substitution of the numeric value of x).

Profound changes or modifications in arithmetic habits and notions do not occur in the individual spontaneously, simply as the result of being confronted with the need for such changes to take place.[3] Intervention with teaching, at this point of transition from arithmetic to algebraic knowledge, may prove crucial for most individuals who are learning algebra for the first time (Filloy and Rojano, 1984).

On the other hand, although some arithmetic notions have to be modified for the sake of the acquisition of a new knowledge, that of algebra, the earlier knowledge (of arithmetic, in this case) must also be preserved, as even in the single example of equations that we have presented it is necessary that arithmetic equations should subsequently continue to be recognized as such, in order to preserve all the previously acquired manipulative skills for their solution. These skills are situated at a level of knowledge between arithmetic and algebraic knowledge: that of pre-algebraic knowledge.

3. CONCRETE MODELING AT A TRANSITION POINT

As we noted in the previous section, changes in the conception of the operations performed on objects such as numbers are essential in order to prepare the way for the conception of operations on objects *other* than numbers (such as unknowns) and for the conception of objects themselves (what they represent or may come to represent). Therefore, at this point the teaching of algebra requires the use of teaching resources by means of which one can bring into play the relations between elements that participate in the realisation of these changes.

We will now describe some of the results obtained in the clinical study “Operating on the Unknown,” carried out with children 12 to 14 years of age who, at the time of observation, had not received instruction in solving linear equations with one unknown occurring two or more times, i.e., the equations that we have described as *non-arithmetic*. The teaching models used in the study are those presented in Section 5 of this chapter.

3.1. The study “Operating on the Unknown”

This study was preceded by work carried out in two different fields: (1) analysis of parts of mathematical texts and works of transition that came before the first work of symbolic algebra, *The Analytic Art*, by François Viète, and (2) experimentation with pedagogical sequences the writing of which was based, in turn, on works of historic-critical analysis of the development of mathematical ideas (see Chapter 3). On the basis of these prior studies we conjectured the existence and location of the didactic cut mentioned earlier, in the child’s line of development from arithmetical to algebraic thinking. This didactic cut corresponds (allowing for the differences between the two areas) to important changes in the history of the emergence of symbolic algebra, concerning the conception and use of objects such as unknowns. Thus, in one of its parts the research on the acquisition of the language of algebra focuses on the study of the processes of change that are brought about in a small neighborhood of the cut. The study “Operating on the Unknown” was situated at this point, and its preparation comprised two stages: the design and application of a teaching treatment prior to the clinical observation, so that it was possible to halt the teaching at the point indicated previously, and the part corresponding to the design and application of a diagnostic test of pre-algebraic efficiency, in order to select the individuals to be observed on the basis of their performance in the test; the results were also used in the design and setting up of the clinical observation.

In terms of teaching, the cut is situated

here
↓

The children have already learned to solve arithmetic equations.	The children have not received any instruction in solving the first non-arithmetic equations.
Examples are: $Ax \pm B = C$ $Ax(Bx \pm C) = D$ $\frac{x}{A} = B$ $\frac{x}{A} = \frac{B}{C}$	Examples are: $Ax \pm B = Cx$ $Ax \pm B = Cx \pm D$
They can be solved simply by inverting the operations (applied to the data of the problem).	They cannot be solved simply by inverting the operations (applied to the data of the problem).
There is no need to operate on the unknown	It is necessary to operate on what is represented

↑
The study was carried out at the cut-off point.

As far as the clinical observation was concerned, the aims of the study "Operating on the Unknown" were:

1) To analyze the children's spontaneous responses when faced with solving non-arithmetic equations for the first time.
2) To analyze the children's performance in the solution of non-arithmetic equations immediately after they had been provided (in the same interview) with a phase of instruction in operating on the unknown.

Aims 1 and 2 were directed at more general objectives of the study:

a) To corroborate the location and perception by the child of the didactic cut between operating and not operating on the unknown.
b) To isolate phenomena concerning behavior of anchoring in arithmetic knowledge, which might correspond to obstructions for the acquisition of the language of algebra.

c) To recognize problems in learning the new concepts, deriving from the way in which they are taught and from the teaching strategies used to teach pre-algebraic material.

We were interested in dealing with the area of the processes that are set in motion when new concepts and operations are introduced by means of a *concrete* model, and so we will refer only to the aims indicated in 2, (b) and (c), which are more directly related to teaching.

The population studied comprised three cohorts of children ages 12 to 13 in the second year of secondary education, all at the same school, receiving instruction in mathematics within a system of controlled teaching.[4]

The written test of pre-algebra comprised three subsections: arithmetic equations with literal notation (e.g., $5x + 3 = 90$), arithmetic equations without literal notation (e.g., $\square - 95 = 23$), and problems corresponding to arithmetic equations.

Once the criteria had been established for the classification of the population with respect to each of the axes (subsections) considered in the pre-algebra test, distributions were obtained for the test as a whole, as shown in Figure 4.1.

The group observed consisted basically of children located on the main diagonal, but also included some cases that contravened the order of some of the axes, i.e., children in categories corresponding to the other vertices of the cube.

With respect to the first aim of the study, i.e., the one concerned with the children's spontaneous responses to their first non-arithmetic equations, in each cohort of children we considered the three categories that appeared on the main diagonal, calling them the lower, middle, and upper strata respectively. Twenty-seven children were interviewed in all, and the interviews were videotaped.

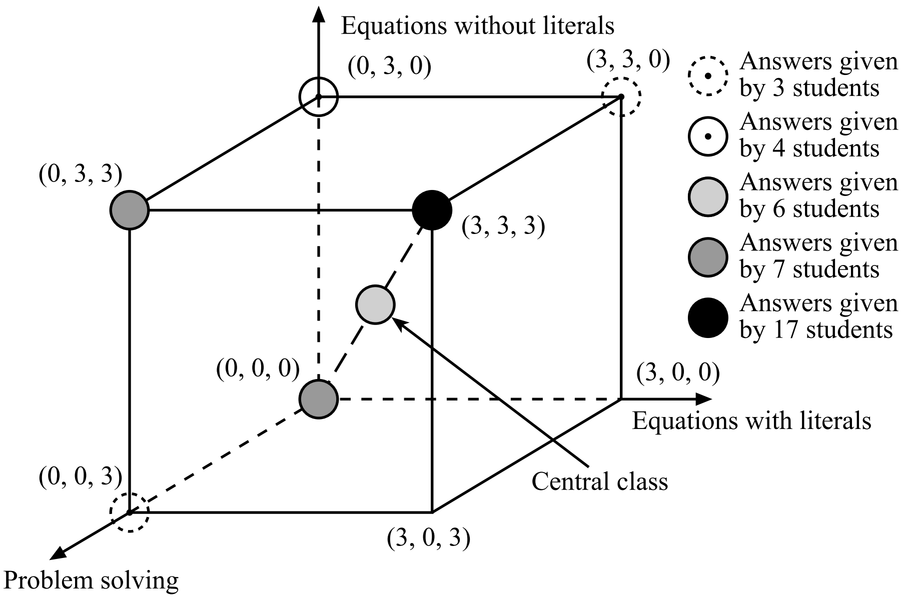


Figure 4.1

3.2. The clinical interview

Five sequences of items, series E, C, I, A, and P, make up the basic content of the clinical interview. It must be borne in mind, however, that, depending on how each interview developed, the order of the items and the order of some of the series were altered, and additional items were even created. This can be seen if one compares the items in series E, C, and I presented in this section with the ones presented in Chapter 6, which are the items actually used in the interview with Ma. Series A and P are not shown in detail here because their items coincide exactly with those in the interview with Ma and they are presented in Chapter 5.

Series E: Verification of the pre-test	*Arithmetic equations*
$x+5=8$	$13\times\square=39$
$x-4=8$	$3\times x=39$
$x+27=58$	$6\times\square=34434$
$x-15=143$	$(x+3)\times 6=48$
$x-1568=392$	$4\times(x+11)=52$

Series C: The equation as equivalence

$$x+5=5+2$$

$$x+\frac{141}{16}=7+\frac{141}{16}$$

$$x+\sqrt{17}=41+\sqrt{17}$$

$$x+\frac{x}{4}=6+\frac{x}{4}$$

Cancellation

$$x+5=2+5$$

$$x+2=2x+x$$

$$x+2=x+x$$

$$x+5=x+x$$

Series I: Operating on the unknown

$$x+2=2x$$

$$2x+4=4x$$

$$3x+8=7x$$

$$3x+8=6x$$

$$3+2x=5x$$

$$5x=2x+3$$

$$5x=3+2x$$

$$7x+2=3x+6$$

Non-arithmetic equations

$$7x+15=8x$$

$$38x+72=56x$$

$$37x+852=250x$$

$$2x+3=5x$$

$$13x+20=x+164$$

$$10x-18=4x$$

$$10x-8=4x+6$$

$$7x-20=5x+30$$

With respect to the first aim of the study, the cross-analysis of the interview series against the three strata of students produced interesting results that made it possible, on the one hand, to confirm the presence of the didactic cut (especially on the basis of the performance of the children in the upper stratum), and, on the other, to outline the characteristic approaches of each stratum to the situation represented by the cut, i.e., the spontaneous solution of non-arithmetic equations, series *C* and *I*.

In order to tackle the second aim of the study, concerning the children's performance after a phase of instruction in operating on the unknown based on the modeling of equations in concrete contexts, both the administration and the analysis of the second part of the clinical interview focused on the children in the upper stratum. This was essentially because it was necessary to be sure of a certain degree of mastery of arithmetic and pre-algebraic language so that genuine transition phenomena could be assimilated without running the risk that those phenomena might have a causal relationship to shortcomings in the basic knowledge on which the new language was to be constructed.

This part of the study began with a phase of instruction in operating on the unknown at the point at which the child stopped trying to solve the equations in series I using his or her own resources.

The following section provides a brief description of the results of this second part of the clinical study in order to obtain empirical referents for the description of the processes of interaction between the semantic and syntactic aspects in the acquisition of the first elements of algebraic language.

4. PROCESSES OF ABSTRACTION OF OPERATIONS, BASED ON THE USE OF A MODEL TO LEARN HOW TO OPERATE ON THE UNKNOWN

Although there are theoretical bases for feeling sure that an initial semantic approach to algebra is more helpful for subsequent good performance with algebra than a merely syntactic approach, this does not mean that the construction of algebraic syntax from this first approach is immediate; in between there are processes of abstraction of the operations performed with the elements of the concrete situation in which the new objects and operations are modeled. These processes, in turn, imply others, such as the process of generalization of actions in modeling and the process of discrimination of the various cases to be modeled, among others.

As was pointed out in an earlier section, for the purposes of this study we set out from the basis that one of the first algebraic operations, strictly speaking, is operating on the unknown to solve non-arithmetic linear equations, and we adopted the position of introducing this operation semantically by the use of concrete models.

Two models were used, the balance scales and a geometric model. A schematic description is given below.

5. TWO CONCRETE MODELS

In this section we give a schematic description of the concrete models used in the studies reported in other chapters of this book. These two concrete models were designed so that first-degree equations and the algebraic transformations that make it possible to solve them could be translated into the models. They consist of what we will call a “geometric” model and the balance scales model.

In the geometric model, the quantities represented by letters and algebraic expressions are represented as lengths and areas of rectangles (x and its coefficients are lengths, and their products and the independent terms are areas), the addition of algebraic expressions is represented as a juxtaposition of areas, and the equation, i.e., the equality of two algebraic expressions, by equality of areas. In this model one can also represent some algebraic transformations as actions of comparing, cutting, and pasting areas.

In the balance scales model, the equation is represented by the balance of weights in the two pans, so that what is placed in each pan has to represent the algebraic expression corresponding to each side of the equation. This is achieved by representing the x – the unknown – by an object of unknown weight, an expression such as Ax by A objects of the same unknown weight, and an independent term B by B objects of a given known weight. The algebraic transformations are represented by actions of adding and removing objects that do not alter the balance of the scales.

We will now present the use of these two models to solve equations of the type $Ax + B = Cx$, which are the simplest equations in which it is necessary to operate on the unknown.

5.1. The geometric model

The equation given is $Ax + B = Cx$, with A, B, and C being given positive integers and $C > A$ in this case.

The steps for solving the equation by using the model are:

1) Reproduction of the model (translation of the equation to the model).

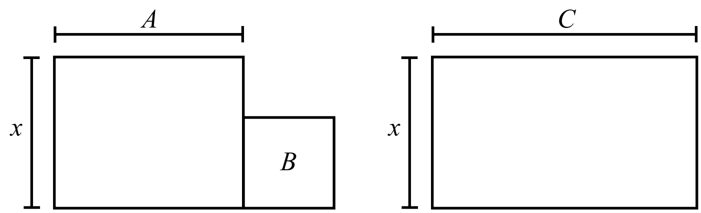


Figure 4.2

2) Comparison of areas:

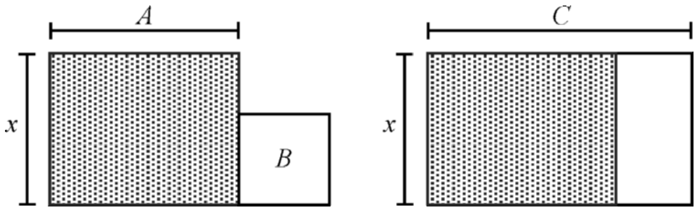


Figure 4.3

3) Production of the simplified equation: $(C - A)\,x = B$.
4) Solution of the simplified equation.
5) Verification of the answer.

5.2. The balance scales model

The equation given is $Ax + B = Cx$, with A, B, and C being positive integers and $C > A$ in this case.

The steps for solving the equation by using the model are:

1) Reproduction of the model (translation of the equation to the model).

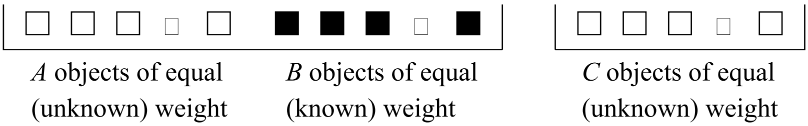


Figure 4.4

2) Repeated reduction of the objects of unknown weight while maintaining the balance, until all the objects of this type have been removed from one of the pans.

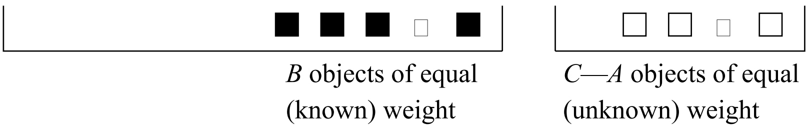


Figure 4.5

3) Production of the simplified equation: $(C - A)\,x = B$.
4) Solution of the simplified equation.
5) Verification of the answer.

In the case of the two models, the children of high pre-algebraic efficiency were provided only with the first elements of the model (the first step of translation), and were allowed to develop the following stages on their own, with as little help as possible from the interviewer. Once they had mastered the use of the model for an equation of the type $Ax + B = Cx$, they were given equations of increasing complexity ($Ax + B = Cx + D$; $Ax - B = Cx + D$, $Ax - B = Cx - D$, etc.), in order to observe the transfer of the use of the model to these types, and also the processes of abstraction of the operations performed repeatedly in the model.

5.3. Results

In the course of the interviews we saw processes of abstraction of the operations on new objects (in this case, unknowns), based on performing actions on them in the model and progressing to operating on them on the level of symbolic algebra. In these processes of abstraction we detected two kinds of phenomena, which we will now describe.

5.3.1. Momentary loss of earlier skills, accompanied by the presence of behavior anchored in arithmetic

The most frequent case was the apparent forgetting of manipulative skills for solving arithmetic equations when they appeared as intermediate steps in the process of solving non-arithmetic equations with the use of a model. This was a non-recognition of the simplified equation $(C - A)x = B$ as an equation that the student already knew how to solve syntactically. It obeys a phenomenon of getting stuck in the model that prevents the child from reading the simplified equation as an expression detached from the concrete meanings that the model gives it.

Example: Fragments of the interview[5] with Vt, age 13, upper stratum, who in series E proved very efficient at solving "arithmetic" equations, even with negative solutions.

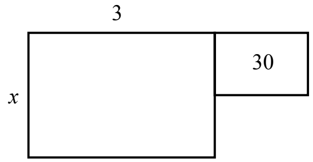
Equation set: $8x + 30 = 5x + 9$

By means of the geometric model, Vt arrived at the simplified equation $3x + 30 = 9$

Vt: What?
I: Do you think you could solve this now?
Vt: Nine minus … (points to the 9 on the right and then points to the 30 on the left) Nine minus a number bigger than nine, that will give me a negative number, divided by three …

Vt does not give the answer, and when the interviewer asks her to do so she starts using the geometric model (which is not appropriate in this case):

$3x + 30 = 9$



The interviewer asks Vt to solve the equation $3x + 30 = 9$ without the model.

I: Can you solve it?
Vt: No.

After a few minutes:

Vt: Well, if you like I can do it with … [She picks up the calculator and, recovering her previous manipulative skills, reverses the operations and obtains $x = -7$.]

5.3.2. Modification of the arithmetic notion of equation

This phenomenon appeared in various ways, which we now present.

1) As a confrontation with different kinds of equations, with a structure that did not necessarily agree with that of the examples used during the modeling

Example: Description of a series of items in the interview with Mt, age 13, upper stratum, who at this point had already abstracted the actions of the model on a syntactic level, and no longer used the model to solve equations.

Initially she did not recognize the equations $2x + 3 = 5x$, $3 + 2x = 5x$, $5x = 2x + 3$ and $5x = 3 + 2x$ as being the same.

I Mt 16 Mt operates with the unknown $2x$, as she has done in the previous items.	$2x + 3 = 5x$ $2x + 3 = 5x - 2x = 3x$
I Mt 17 The permutation of the terms on the left makes Mt uncertain about what she should do with x.	$3 + 2x = 5x$
I Mt 18 Mt is presented with the equation from item 16 so that she can compare them. She recognizes that they are the same equation except for the permutation of terms, but she also admits that she had not realized this.	$2x + 3 = 5x$
I Mt 19 Mt modifies the equation in order to solve it like the preceding one:	$5x = 2x + 3$ $5x = 2x + 3$ $2x + 3 = 5x$ $2x + 3 = 5x - 2x = 3x$
I Mt 20 Although Mt sees this equation as being the same as the one in the previous item, she does not assign to it the solution already found for that item; she applies the same method of solution to it:	$5x = 3 + 2x$ $5x = 3 + 2x \curvearrowright 3x \quad x = 1$

2) As a need not only to make the terms of the equation meaningful, but also to give sense to these new expressions and the operations required in order to use them

One way of giving sense to them is presented by the process of verification, giving new meaning to algebraic equations in which equality appears, such as those in which it is possible to perform a series of operations in order to obtain the value of an unknown, and then substitute it on the left side of the equation and perform the operations indicated, and also do the same on the right, and the results agree.

Example: Mt has solved 25 items, some with the geometric model and the last ones on a syntactic level. In the stage of checking the answer to $10x - 18 = 4x + 6$ she spontaneously gives a "more algebraic" interpretation of the equation:

I Mt 26 $\quad 10x - 18 = 4x + 6$

Mt: In other words, they're equivalent.
I: What do you mean by equivalent?
Mt: … If I take the value of x and do this operation (pointing to the left side of the equation), I get a result. That result has to be the same as this (pointing to the right side of the equation).

Mt goes on to solve the equation using the method of "operating on the unknown in the equation."

Note: At the beginning of series I, Mt interpreted a non-arithmetic equation as "two equivalent equations" (using this expression to refer to the two sides of the equation, because x appeared in both), and it was only in this item that she made her interpretation explicit in operative terms.

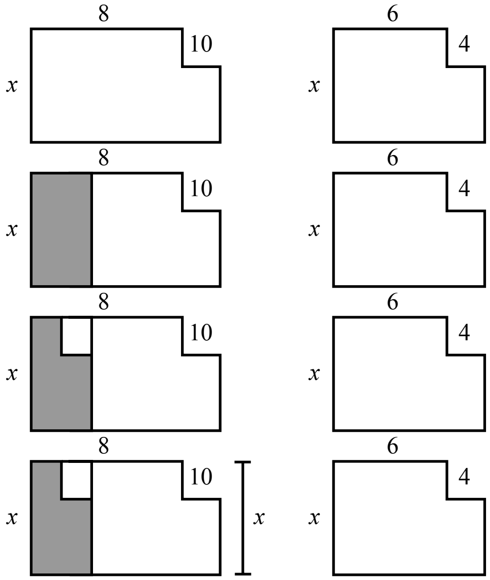
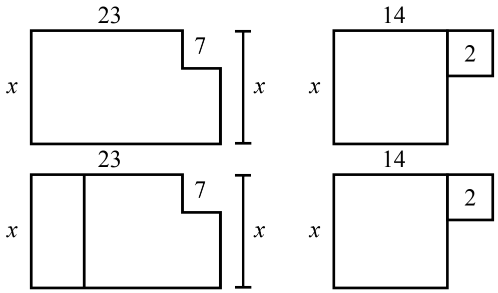
3) Use of personal notations (codes) to indicate the actions already performed and the actions still to be performed with elements of the equation during the solving process

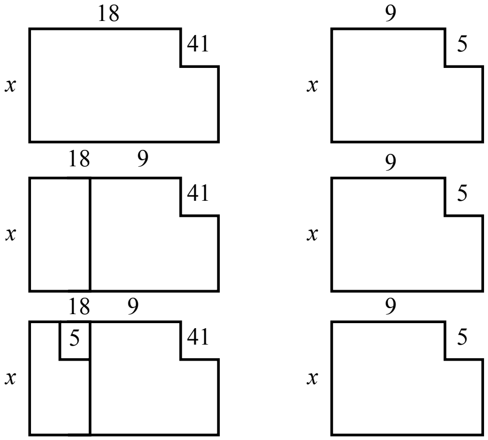
This suggests the existence of a stage prior to the operational algebraic stage. In this stage there are also obstructions, which these notations impose when the complexity of the equations increases, generating what are subsequently considered, in the later study of algebra, natural mistakes of syntax: inappropriate use of the equals sign, absence of the equals sign, forgetting certain terms, etc.

An illustrative example is that of Mt, who soon gave up using the concrete model to operate on the unknown and generated her own notations and codes to indicate the actions to be applied to the elements of the equation.

4) Clinging to the model (even in cases that are very complicated to represent), an attitude that underlies an apparent algebraic manipulative ability with the elements of the equation

Example: After 28 items of series I, Vt continued using the geometric model, without showing any sign of trying to do without it. The final items in the interview, which present an increasing degree of difficulty in their modelling, were the following:

I Vt 29 First attempt: Vt: Maybe, first I take away these parts (the parts corresponding to –10 and –4 in the drawing). Second attempt: Mistaken operation confusing length and area: Vt: … well, x … minus ten. With help from the interviewer: Vt: Two times x, minus ten, plus four. Vt: … must be equal to … I: Nothing, because you took away everything.	$8x - 10 = 6x - 4$  $2x - 10 + 4$ $2x - 10 + 4 = 0.$
I Vt 30 Spontaneous —unaided With help from the interviewer. Vt realizes that the area that remains now is different from zero. Vt: Nine times x, minus seven. I: Is equal to what? Vt: Two.	$23x - 7 = 14x + 2$  $9x - 7 = 2$

I Vt 31 Spontaneous —unaided Vt: But would it be … minus five? … Let's see, it would be nine times x, minus 41, plus five. I: That must be equal to … Vt: Zero. I: Zero, because you put it here.	$18x - 41 = 9x - 5$  $9x - 41 + 5 = 0$
I Vt 32 Spontaneous —unaided	$10x - 3 = 4x$

5) Relinquishing the model, transferring manipulation with coefficients to manipulation with terms that include unknowns

This leads them to make the usual mistake of adding monomials of degree one to monomials of degree zero (coefficients and constants); in other words, faulty operation on the unknown. The case of Mt is now presented, with item 11 from series I, in which she tries to do without the use of the geometric model and makes mistakes of syntax:

I Mt 11 $15x + 13 = 16x$

Mt operates on the unknown in two steps. The first is correct; the second is incorrect.

Mt: $16x$ minus $15x$ is 1 times, so 1 plus 13, 14.

The interviewer refers her to the geometric model:

$15x + 13 = 16x$ Š $15x$

Mt: … one x … [She expresses uncertainty about making "one x" equal to the constant term.]

With help from the interviewer:

Mt: So here we have to put one times x (pointing to the result of $16x - 15x$).
I: And that has to be equal to what?
Mt: Thirteen. (She writes $1x = 13$.)

6) The presence of obstructions inherent in each model, but also with the preservation of general behavior from which it is possible to learn about the difficulties that do not depend solely on the way in which the material is taught

7) Recognition, through the models, of the diversity of the types of first-degree equations. The solution of these equations ultimately requires the same operations, but they are not transferred from the simpler ones to the more complex ones (see the example illustrating result 5)

6. SEMANTICS VERSUS ALGEBRAIC SYNTAX

6.1. Comparison of the use of two different models to operate on the unknown

In a third phase of the clinical study "Operating on the Unknown," we made a comparative analysis of observations of the performance of upper-stratum children (with respect to pre-algebraic manipulation skills) in the solution of non-arithmetic equations, using the *balance scales model*, and observations concerning the processes of abstraction of the operations of the model toward a syntactic level, in the case of the *use of the geometric model*.

This comparison between models is interesting because it enables us to identify the phenomena that are stimulated during the processes of abstraction of the operations in modeling that do *not* depend on the specific model that is used; it is also interesting, however, to detect the variations from one model to another, so that they can be taken into account when proposing teaching strategies based on them.

Some of the more important results in terms of the aspects that varied from one model to another were the following:

1) There are specific ways (depending on each model) of translating the elements of the equation to the model, which represent an obstruction to progress in using it

In the geometric model we find phenomena such as those illustrated by the following items from the interview with Mt using series I:

I Mt 13 Mt: So x equals two (wrong answer).	$129x + 51 = 231x$ $129x + 51 = 231x$ Š $129x + 51 = 231x$ Š $129x = 102x$ $129x + 51 = 231x$ Š $129x = 102x$

I Mt 14 She applies the method of "operating on the unknown in the equation" and *abbacus*-style reading. Mt: What number multiplied by 213 gives us this (pointing to 852)?	$37x + 852 = 250x$ $37x + 852 = 250x$ Š $37x = 213x$

I Mt 15 Mt: One x must be equal to 5, in other words, x equals 5.	$x + 5 = 2x$ $x + 5 = 2x - x = 1x$ $x + 5 = 2x$ Š $x = 1$

In item I Mt 14, it is worth noting that Mt does not write down the simplified equation using algebraic notation, although she does work with the equality $102x = 51$, indicating it with an arrow and solving it (wrongly, because she uses a specific item wrongly). Mt creates her own signs to explain the simplified equation, and from the indications accompanying it one can see the actions that have already been performed and the actions that have yet to be carried out to solve the equation; this is a stage prior to the operational level in which the simplified equation is written using algebraic syntax.

This stage prior to the operational stage plays a very important part in the development of operations in mathematics; before having the operations established on an operational level, one guides oneself by means of personal signs created (by oneself) for the type of problem that one is attacking, and even later, although more reduced, more syntactic ways of performing the operations involved are created, this strategy of drawing (with personal signs) helps one greatly to see what one is doing. In fact, Mt is creating a way of indicating the process with a notation in which she writes down what she is doing, and what corresponds to what; really, what Mt is presenting is a simplified equation with many more signs, but one that enables her to see what has been done before and what has to be done afterwards.

The geometric approach, described above as an obstruction, consists in breaking down into linear dimensions the rectangular area that represents the constant middle term (B in $Ax + B = C$), which leads to the application of the method of "joining of linear dimensions" to solve the equation, i.e., to find b and h such that $b \times h = B$ and $b = C - A$ or $h = C - A$. This method is not applicable in the case in which B is not divisible by $C - A$, because, if $C - A$ does not divide into B, then it is not possible to find b and h with the desired characteristics, as can be seen in Figure 4.6:

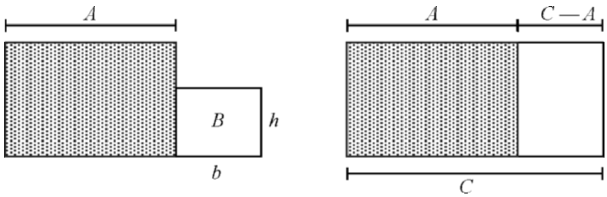


Figure 4.6

On the other hand, in the model of the balance scales, when an attempt is made to assign units of weight to the objects in the scales, confusion may be caused in the development of the initial "natural" strategy of the model by the repeated cancellation of identical weights, and this repeated cancellation also weakens the notion of the unknown in the context of the concrete situation. We will illustrate this with some items from Dr's interview with series I:

I Dr 22 $4x + 6 = 2x$

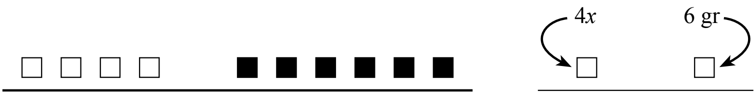
Dr: … the value of this x (pointing to the x on the left side) must be less … than the value of this x (pointing to the x on the right side).

Dr decides to use the balance scales model and translates the terms of the equation to the model correctly. But then he says:

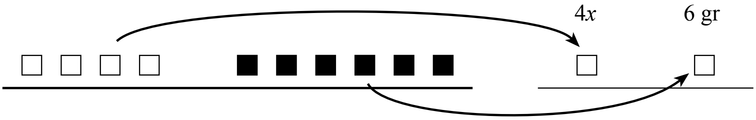
Dr: Let's say that this (pointing to one of the dots in the pan on the right) is equal to 6 grams.

[…]

Dr: … in other words, that this (pointing to the first dot in the pan on the right) is four x (writing "4x" and connecting it to the dot with an arrow), and this (indicating the second dot) is equal to 6 grams (writing "6 gr" and connecting it to the dot).



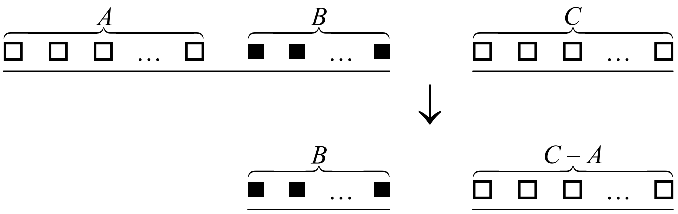
I: But aren't these the same? (Indicates the two dots on the right)
Dr: Uh-huh.
[…]
Dr: The weight of this (pointing to the right side) has to be equal to the weight of this (pointing to the left side), so I could simplify it like this: this (indicating one of the two dots on the right) … each of them could be 6 grams.
I: But why? Why 6 grams?
Dr: It's a way of saying it, as if this (indicating the four white dots) were one *x* and this (pointing to the six black dots) were the other *x*.
[…]
Dr: … But perhaps … Who knows?



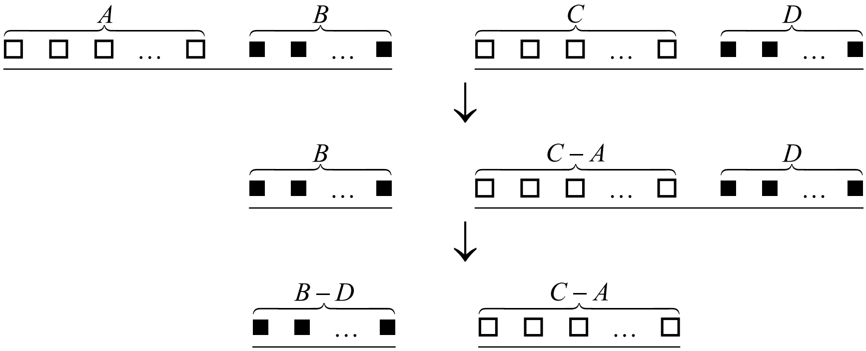
2. Some transfers from the use of a particular variety of model to a particular variety of equation are more natural *in one model than in another*

The transition from the variety $Ax + B = Cx$ to $Ax + B = Cx + D$ is more natural in the scales, given that the repeated cancellation is essentially the same in each case, and also, in this model, the simplified equation is presented in the model itself and can be solved without having to translate it to the notation of algebra.

From $Ax + B = Cx$

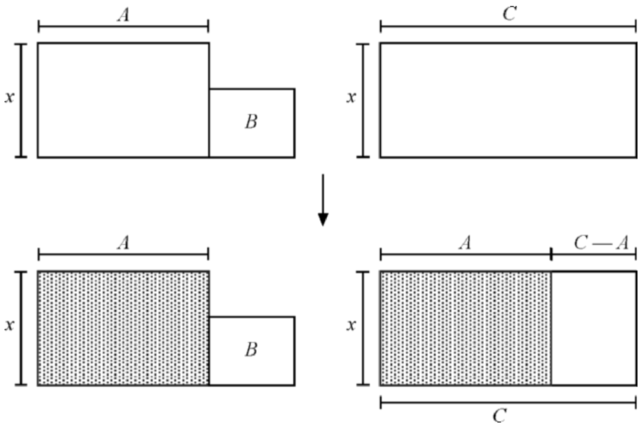


to $Ax + B = Cx + D$ (if $B > D$)

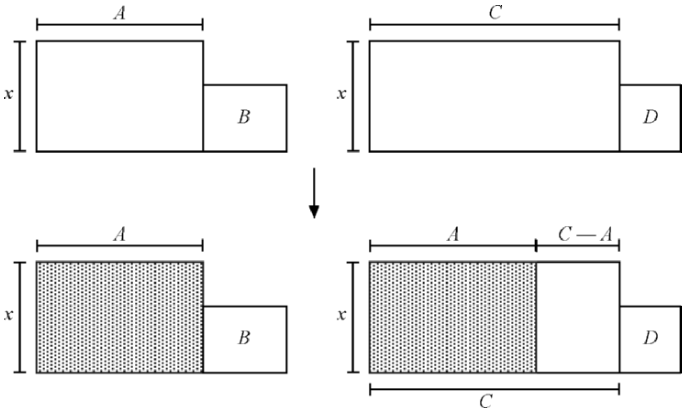


In the geometric model, however, it is necessary to realize that one has only to superimpose the areas corresponding to the first-degree terms, without performing any action on the areas corresponding to the constant terms, in order to produce the simplified equation, i.e., in this case the transfer from the use of a model to a new variety of equation is not trivial.

From $Ax + B = Cx$

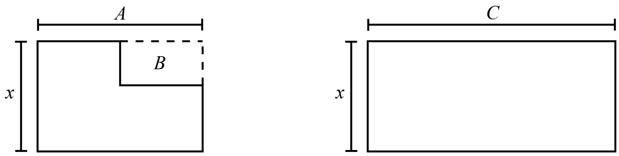


to $Ax + B = Cx + D$

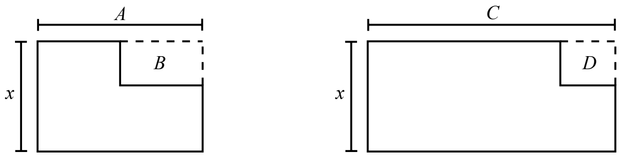


However, the transition to varieties such as $Ax - B = Cx$ or $Ax - B = Cx - D$ has to be interpreted, in the model, in terms of *negative* constant numbers, which do not have concrete representations that correspond to them in the scales (other than representations derived from mental actions, such as the re-establishment of equilibrium), whereas in the context of areas these terms can be interpreted as concrete actions of removing areas equivalent to the absolute values of the terms in question, without thereby violating the semantics of the model.

$Ax - B = Cx$



or $Ax - B = Cx - D$



Finally, in the case of a negative solution there is no way of carrying out actions in the concrete model to "simplify" the proposed equation in either of the two models. For example, the equation $4x + 6 = 2x$ can certainly be represented in the balance scales model and in the geometric model, but in both cases the actions of simplification *within the model* lead to situations that are lacking in sense on the concrete level, such as areas or weights that are equal to zero but do not appear thus in their concrete or graphic representations.

7. CONTRAST BETWEEN TWO COGNITIVE TENDENCIES IN THE LEARNING AND USE OF MATHEMATICS, WITH RESPECT TO THE APPLICATION OF THE SAME MODEL FOR OPERATING ON THE UNKNOWN

Results 1 to 7 correspond to the first phase of the study presented in Section 5 of this chapter, in which there is a description and analysis of certain general patterns that were observed in the performance of children of the upper stratum, before and after the phase of instruction in operating on the unknown (see also Filloy and Rojano, 1989). In carrying out the analysis, the natural characteristics of that stratum with respect to the others —the middle and lower strata and the cases that contravened that order— were taken into account, without considering individual differences in the children's attitudes or tendencies.

However, among those individual differences it is of particular interest to examine the contrast between algebraic syntax and semantics, which correspond to canonic tendencies that vary within a spectrum that ranges from a kind of operational tendency, at one extreme, to a kind of semantic tendency at the other. The individual tendencies in this spectrum show preferences for certain kinds of methods for solving or attacking problems, varying from the most operational and algorithmic to the most semantic and analytic. Irrespective of the origin of tendencies of this kind, they exist and are detected, generally, during the learning and use of mathematics —in the case of the children's performance, in the classroom or in the clinical interview— and not in results of tests with closed responses, multiple-choice questions, etc. In other words, these tendencies have been reported by teachers and researchers as a result of clinical or natural observations.

With regard to the interaction between syntax and semantics in the language of algebra, if the tendency of the person interviewed is taken into account as a factor, along the lines of the tendencies mentioned in the last paragraph, it makes it easier to carry out a more sharply defined analysis, in the sense that, once the interaction phenomena that are strongly linked to this factor are differentiated from the phenomena that are independent of it, one eliminates the risk of false generalizations with respect to an evolution from a concrete version of certain operations toward their syntactic version. Thus, in the interview corresponding to the post-instruction performance of two girls with extreme, opposite tendencies —Mt, upper stratum, 13 years old, with a marked operational tendency in mathematics, and Vt, also in the upper stratum and 13 years old, with a marked semantic tendency in her approach to pre-algebraic equations and problems— these tendencies were strongly confirmed, although in the two cases the same models had been used for the instruction in operating on the unknown (the models in question are presented at the end of the next chapter). Two results can be derived from this, which we will now present.

The first is that the spontaneous development of the use of the model to operate on the unknown is not uniform even in children of the same level of pre-algebraic efficiency, but rather this development is strongly bound up with tendencies in the learner that are of a general character and that vary within a spectrum that ranges from the syntactic or operational, at one extreme, to the semantic at the other. Indeed, we detected extreme cases with trajectories of development of the use of *the same model* that were quite different: in one case, this development took place with a continuance in the context of the model, even in varieties of equation the modeling of which was highly complex; in the other case, that of the operational tendency, there was a constant search for the elements of syntax in the actions in the model, which were repeated in the solving of each equation and in each variety of equation, and this search brought about a rapid relinquishing of the semantics of the model and a modeling of those actions in a more abstract language by the creation of personal forms of notation that did not belong to the model or to algebra but were on an intermediate level prior to manipulative algebra.

The second result is the fact that there are obstructers to the abstraction of the operations in the model toward a syntactic-algebraic level that do not depend either on the particular model used or on the tendencies of the learner such as those mentioned above, but that depend on the emphasis placed on the component of modeling that makes it possible to build on previous knowledge and on operations already mastered by the learner in order to introduce the new objects, concepts, and operations. This reduction of the new to the known brings with it the risk of concealing the difficulties of operating with the new objects and of bringing the new concepts into play. Indeed, in the processes of abbreviation and automation of the actions in the two models used here we observed a tendency to conceal the real operation on the unknown. In the geometric model, the abbreviation led to the disappearance of the areas that involve the unknown; in fact, the linear dimension that was lost was the one that represented the unknown, the operations were reduced to operations between the given values of the equation, and the unknown ceased to play any part.

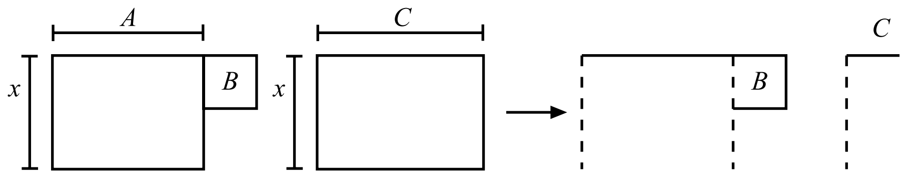


Figure 4.7

In the balance scales, as a result of the discretization of the coefficients of x and the constant terms, it is possible to perform operations of the same kind with both, these operations being between *numbers of objects* of known weight and *number of objects* of unknown weight.

Figure 4.8

This automation (in the two models) leads to the subsequent making of typical mistakes of algebraic syntax, such as actually adding (or subtracting) the coefficients of terms of different degrees. Even students with a pronounced operational tendency make the same kind of mistakes as a result of the use of their personal forms of notation, which are also generated in a process of automating actions.

$$Ax + B = Cx + D \;\rightarrow\; (A + B - D) + C.$$

Another error is the introduction of *artificial* brackets in some expressions, such as, for example, $B + Ax \rightarrow (B + A)x$, isolating the operations between numbers from their operation with the unknown.

SUMMARY

One of the central themes in research on school algebra is that of the approaches or models used for the teaching of specific themes or concepts. In the context of local models, this theme is located in the component of the teaching models. This chapter discusses this theme for the case of the teaching of syntactic procedures for solving linear equations by means of concrete models (the balance scales and a geometric model) in the framework of the study "Operating on the Unknown." Specifically, we study the strategies used by pupils when these models are applied. The focus of attention is the relationship between the pupils' strategies and the appearance of common mistakes of syntax and extreme tendencies, on the one hand toward preferring to operate on an abstract or syntactic level, and on the other toward preferring to keep the operations that are performed tied to referents on the concrete level of the teaching model. Throughout this analysis special emphasis is placed on the processes of abstraction that take place in learning with concrete models and in the operations that are involved.

In this chapter we have shown and analyzed results concerning the use of two specific teaching models. In the next chapter we deal with the issue of teaching models in general, that is, the teaching component of a local theoretical model. In Chapter 6 we come back to the use of the teaching models considered in this chapter to further analyze issues related with problem solving. In Chapter 9 the issue of problem solving is treated in a more general way.

ENDNOTES

[1] The observation took place at the Centro Escolar Hermanos Revueltas in Mexico City, with children receiving instruction in mathematics in a controlled teaching system (the 82/83, 83/84, 84/85 and 85/86 cohorts.) Early reports on this study are Filloy and Rojano (1984, 1985a, 1985b) and the doctoral dissertation of Rojano (1985).
[2] Trial and error and guessing the solution are examples of spontaneous solving methods.
[3] This was the case with the children in the study "Operating on the Unknown," who at the time of observation had not received instruction on moving the unknown in the solving of equations. When these children were faced for the first time with equations such as $Ax \pm B = Cx$, they tackled them with trial and error methods, without any indication of spontaneously operating on the x terms.
[4] The population studied received instruction in mathematics with materials that allowed them to work individually in class at their own pace. The progress of individuals and groups of students was monitored, and there was a possibility of intervention with supplementary teaching materials in cases where it was necessary.
[5] In the transcription of the interviews, I signifies the "interviewer" and the pupils are designated as Vt (in this case), Ma, etc.

CHAPTER 5

TEACHING MODELS

OVERVIEW

From the standpoint of symbolic algebra as a language, we characterize teaching models as successions of *mathematical texts* that are exchanged between pupil and teacher. Said characterization involves notions such as that of *text* and of *textual space*, the differentiation of which corresponds to the difference between *meaning* and *sense*, given that once one understands that a *text* is the result of reading a *textual space*, teaching and learning in mathematics class may be interpreted as a repeated reading process – transformation of *textual spaces* into *texts*, which are in turn taken as *textual spaces* to be read, and so on and so forth. This theoretical treatment of teaching models is completed by use of the notions of *mathematical sign system* and of *language strata*, to be applied to the case of concrete modeling introduced in the previous chapter, as well as to the analysis of syntactic models in algebra and of the semantic – syntactic relationship in algebra, the discussion of which was also begun in the preceding chapter.

1. INTRODUCTION

The structuralist movement of the 1960s advocated teaching a mathematics in which school algebra was conceived as the explanation of the structural properties of numbers and of arithmetic-algebraic operations. In the texts and materials produced in that period there were many different presentations, for example, of the laws of commutation and association, which referred first to numbers (or a specific number system) and second to letters. This is an example of how the transition from arithmetic to algebra was reduced to a mere paraphrase of the laws that were valid for numbers, but applied on this occasion to algebraic expressions.

This conception of algebra as simply an extension of arithmetic knowledge denies the conceptual and qualitative changes in the way of operating and of solving problems that the appropriation of algebraic language presupposes, and in the teaching of mathematics at middle school levels it gives rise to what one might call "a forgotten boundary" (Chevallard, 1983): the boundary between arithmetic and an algebraic way of thinking, which is eliminated from the aspect of the structural properties common to both —for example, commutativity and associativity are properties that are equally valid for numbers and letters— since that viewpoint hides the characteristics that differentiate them.

Despite the years that have elapsed and the fact that the structuralist movement is no longer in vogue, it is still necessary to talk about this in curriculum development because our teaching plans and syllabuses for mathematics are still influenced by it. With the reforms that have been carried out since then, perhaps the approach proposed has changed, and consequently there has been a development of syllabuses with a greater inclusion of the need to use the solving of problem situations. Nevertheless, there is still a need to insist on more profound changes, which have not yet taken place. In this book we discuss various problems that will have to be taken into account in the future, in the design of those parts of syllabuses that have to do with solving first-degree equations and arithmetic-algebraic word problems.

The importance of algebra as a language of generalizations and as a method is precisely what distinguishes it from arithmetic, and what for centuries has set it in a privileged place in education. However, algebra has ceased to play that role in our current syllabuses. One cannot yet see a proper recovery of the significance of school algebra as a symbolic language whose potential lies in its use as a means for expressing situations and for solving problems posed in various areas of knowledge.

In the last chapter we talked about a clinical study of 12- to 14-year-old children that showed the difficulties that secondary school students face when they have to read or write algebraic language. At the time of the observation, the children had already received instruction in pre-algebra and had been introduced to elementary algebra through solving linear equations and the corresponding word problems, but they had not yet received systematic instruction on the use of open expressions, the equivalence of expressions, or solving systems of equations. At this level it was still possible to see a tension in the students between the way of reading and expressing themselves using the language of arithmetic and the need to produce new meanings for mathematical texts in the context of algebra. The latter aspect is yet another indicator that the arithmetic-algebraic boundary cannot be avoided, because that would lead to false conceptions about the processes of acquiring the language of algebra and, consequently, about the role of teaching in such processes. On the other hand, the importance of considering the reading and writing of symbolic algebra as an educational goal for learners at middle school level is reaffirmed.

1.1. Problem-solving ability and competence in the use of the Mathematical Sign Systems (MSSs) of algebra

Problem solving is an objective that has remained in the school curriculum despite all the educational reforms that have taken place, and at present it has particular importance in the curriculum of mathematics at middle school levels. Moreover, research on mathematics education has always considered that it was a matter worth studying in depth, and the many studies that have been performed in this field have constantly pointed out the role played by symbolization in problem solving.

It might be said that the first tasks of mathematical symbolization that the learner performs at a higher level of generality than that of arithmetic come when he tries to solve a problem with the tool of algebra, and that then there is the beginning of a process of combined evolution of symbolization and problem solving that involves using algebra as a language in which to model and solve problems derived from various branches of knowledge (physics, biology, geometry, financial affairs, etc.), subsequently culminating in the use of algebra as a basic language for expressing statements and procedures performed in other branches of mathematics (analytic geometry, calculus, mathematical analysis, etc.). In Chapter 9 we explore the possibility of attaining the competences required for the use of the Cartesian method for solving problems when syntactic competences have recently been acquired using a concrete teaching model. In this chapter we also deal with concrete teaching models in general terms.

However, in addition to recognizing in algebra this fundamental role as a means of scientific expression, it is also necessary to recognize its importance in school education, that is, in the realm of teaching. Yet it is precisely in this area that the assimilation of the language of algebra by students presents difficulties that come from the interaction between this language which is in the process of being constructed and two languages that have already been mastered, namely, the language of arithmetic and natural language. In the translation between mathematical sign systems and natural language these difficulties were shown through the predominance of the meanings given to signs and words in the two languages in which the students were competent, natural language and the language of arithmetic prior to the sign system of algebra. The students would have to overcome these difficulties, therefore, in order to attain to the reading and writing of algebra and thus become competent users of the language of algebra. On the one hand, this would help them to achieve one of the goals of educational systems, which is precisely the mastery of the language of mathematics, and on the other it would assist them to satisfy one of the most ancient social requirements of human beings, the capability of solving problems in a general, systematic way.

Indeed, if we admit the above-mentioned suppositions about the fundamental role that algebra plays in the school curriculum, we must also admit the need to recover the conception of school algebra as a language which is essentially different from that of arithmetic, and as a language whose symbolic level made it the first language in the history of mathematics that was capable of explaining itself, and one that has subsequently served as a basis for the symbolic development of mathematics as a whole, to the point of achieving the algorithmic and expressive autonomy that characterize it now.

1.2. The rest of the chapter

In what follows we shall make a series of observations about mathematical texts in which we shall make use of the notions about mathematical sign systems that we introduced in the Introduction (Chapter 1) and Chapter 2. This will enable us to characterize teaching models as successions of mathematical texts (all this is treated in greater depth and with greater generality in Chapter 8), which are exchanged between the learner and the teacher. Having done this, we take a look at concrete teaching models and their strengths and weaknesses.

2. MATHEMATICAL TEXTS AND TEACHING MODELS

2.1. A teaching model is a sequence of mathematical texts

Because we do not conceive mathematical texts as manifestations of mathematical language, and also because, in order to be able to give an account of those that are present in the processes of teaching and learning, we cannot identify them with written texts, it is pertinent to use a notion of text that conceives it as “the result of a reading/transformational labor made over the textual space” (Talens and Company, 1984, p. 32). Indeed, this idea was introduced in order to provide a notion of text that could be used in the analysis of any practice of production of sense (for example, the work of a learner with a teaching model, although this example may be rather far removed from the concerns of Talens and Company in their article), and for this purpose it is useful to introduce a distinction between “textual space”

(TS) and "text" (T), which corresponds to a distinction between "meaning" and "sense." A text, therefore, is the result of a reading/transformational labor made with a textual space, the aim of which is not to extract or unravel a meaning inherent in the textual space, but to produce sense. The textual space has an empirical existence; it is a system that imposes a semantic restriction on the person who reads it; the text is a new articulation of that space, individual and unrepeatable, made by a person as a result of an act of reading.

Moreover, the distinction between TS and T is a distinction between positions in a process, because any T resulting from a reading of a TS is immediately in the position of a TS for a new reading —and so on ad infinitum.

Both the work of mathematicians and that of students in mathematics classes can be described from the aspect of this repeated process of reading/transformation of textual spaces into texts. In particular, from this viewpoint a teaching model is a sequence of texts that are taken as a TS to be read/transformed into other TSs as the learners create sense in their readings.

2.2. Mathematical texts are produced by means of stratified mathematical sign systems and with heterogeneous matters of expression

In saying this we wish, first, to go against the idea of the existence of a text written in a totally formalized language that, although never actualized, is on the horizon as the text alluded to by the text that is really produced, by operations that are conceived as "abuses of language."[1] But we also want to contrast it with Rotman's idea that there is a rigorous text always present as the text belonging to a Code[2] that establishes the rules of the rigorous mathematical text, but that is enveloped in an informal text organized by the metaCode, although Rotman states that, contrary to the previous case, the text of the metaCode is unavoidable because it is the only way of guaranteeing the *persuasion* that, according to him, is an intrinsic need pertaining to any mathematical text.

Moreover, for Rotman, the fact that one cannot do without the metaCode "opens up mathematics to the sort of critical activity familiar in the humanities." However, according to Rotman "it by no means follows from this that mathematics' ways of making sense, communicating, signifying and allowing interpretations to be multiplied can be assimilated to those of conventionally written texts in the humanities," because in mathematical texts there are signs that are not those of natural language. So, having avoided the danger of reduction of the mathematical text to the ideal text, it seems that for Rotman it is a question of avoiding the symmetrical danger of reduction to text written in the vernacular, since he asks himself "[w]hat, in short, is unsayable (in fact, unthinkable, unwritable) except via mathematical symbols." We, however, do not find it so special to analyze a text in which not only natural language appears, since semiotics has set about the analysis of films, music or dance, for example, the expression of which is heterogeneous in that it combines matters of various origins; and we find that it is more suitable to study what type of combination of heterogeneous matter of expression is characteristic of mathematical texts than to undertake a search for something that can be expressed only by means of an expressive matter that is specific to mathematics.

However, abandoning the idea of a formalized or rigorous text as the background that in one way or another governs the analysis of mathematical texts does not make us deny the role effectively played in practice by the illusion of the formalized text, because this illusion has formed part of the idea that mathematicians have had of the rules of their practice. The way of combining matters of expression from different languages and the way of forming relations between the strata of mathematical sign systems is determined by this non-discursive component of the practice of mathematics, among other things, as are the texts produced in a given historical period among all those that might have been produced.

2.3. The heterogeneity of the matter of expression is revealed in the presence in the texts of segments of natural language, algebraic language, geometric figures and other diagrams, etc.

Although these segments come from languages with which it is possible to produce texts according to systems of rules that belong to each of them, they are not governed separately in mathematical texts by the rules of each of those languages. What really happens is that the rules of some languages contaminate those of others, so that mathematical sign systems are governed by new rules, created from those of the various languages that they incorporate.

We shall show this contamination between languages with an example in which the rules of natural language have been modified by copying them from the rules of arithmetic language. An expression such as "siete menos cuatro" [seven minus four], for example, is constructed by importing the form of the arithmetic expression 7 – 4 into Spanish. The way of expressing the task of subtracting one number from another in Spanish is what we have just used in this sentence: "sustraer, quitar o restar cuatro de siete" [subtract, remove or take four from seven], a phrase in which the operation appears first —and not between the numbers— and the numbers appear in reverse order.

The strangeness in Spanish of expressions such as this, which we may not notice now, is evident when we examine school texts from the 19th century and see that, in any of them, these expressions are introduced as something whose meaning has to be explained by resorting to the expression in the vernacular "restar tanto de tanto" [take so much from so much]. Thus, in the school text most frequently used in Spain in the 19th century, Vallejo writes: "la espresión 5 – 3 = 2, quiere decir que después de quitar 3 unidades del 5 quedan 2, y se lee cinco menos tres igual ó es igual á dos" [the expression 5 – 3 = 2 means that, after taking 3 units from 5, 2 remain, and it is read as five minus three equals or is equal to two] (Vallejo, 1841, p. 26). Freudenthal (1983) points to this phenomenon in other languages, such as German and Dutch. Thus, in German, until the early 20th century subtractions were formulated with the expression "vier von sieben" [four from seven], until textbooks began to introduce "sieben minus vier" [seven minus four] for élite schools and "sieben weniger vier" [seven less four], for ordinary schools — expressions that were foreign to German in both cases.

2.4. Inscribed in mathematical texts there are deictics that refer to elements of segments of different natures

Thus, for example, in a text in which the expression "point *A*, point *B*, segment *AB*" is accompanied by the corresponding geometric figure, whether drawn physically or imagined, the letters *A* and *B* link together words, figures, and expressions formed exclusively by these letters, and manipulation of the letters or the figures in the expression itself makes up for the lack of manipulation of natural language.

2.5. Through these deictics, indications of translations between elements that refer to each other are inscribed in the text, which are marks, borne by the text itself, of the semantic field that the reader has to use to produce sense

Unless one admits a drift toward aberrant readings, these indications are necessary because any reading of a mathematical text constitutes a learning process, in a non-trivial sense, for the empirical reader. Thus, in a school text in which Pythagoras' Theorem is stated, the many references between the expression "the hypotenuse *c*," the letter *c* written next to one of the sides of a triangle drawn on the page, and the algebraic expression $a^2 + b^2 = c^2$ enable one to understand that the text stipulates that the drawn figure which *looks like* a right-angled triangle effectively represents that geometric object, and that $a^2 + b^2 = c^2$ states Pythagoras' Theorem.

2.6. The objects with which mathematics deals are created in a movement of phenomena/means of organization by the mathematical sign systems that describe them

Since this movement of promotion from phenomena to means of organization does not always develop on the same level, that is, what is taken as phenomena asking to be organized by new means is not in an immutable world whose collection of phenomena is the subject of study of mathematics, mathematics generates its own content (see Section 4.1 in Chapter 2). An important aspect of this movement can be called "abstraction." The stratification of the mathematical sign systems with which mathematical texts are produced has to do with these processes of abstraction.

2.7. The fact that mathematical sign systems are the product of a process of progressive abstraction, whether in the history of mathematics or in the personal history of an empirical subject, has the effect that the ones that are really used are made up of strata that come from different points in the process, interrelated by the correspondences that it has established

In the Introduction (Chapter 1) and in Chapter 4 we have dealt with various phenomena that show this use of different strata of an MSS.

2.8. The reading/transformation of a text/textual space can therefore be performed using different strata of the mathematical sign system, making use of concepts, actions, or properties of concepts or actions that are described in one of those strata

The texts produced by readings that use different strata or a different combination of strata can be translated into one another and recognized as "equivalent" on condition that the pertinent correspondences between the elements used are also described in the mathematical sign system. When this is not the case, only the creation of a new MSS will make it possible. The process of creating new MSSs for this purpose is actually a process of abstraction, and the new MSS is more abstract than the preceding ones.

To express this with more precision, if it happens that two textual spaces ET and ET' cannot be read/transformed by means of a stratified mathematical sign system L by using the same concepts, actions, or properties of concepts or actions as are described in one of the strata, whereas in another mathematical sign system M it can be done, then M is “more abstract” than L with respect to ET and ET'. This is what happens in the book *De Numeris Datis*, for example, with two propositions that Jordanus Nemorarius transforms by means of different procedures, but that could be transformed in the same way using the sign system of modern elementary algebra. The MSS of this 13th-century text is less abstract than the MSS of modern elementary algebra, and in the history of mathematics the creation of the latter MSS was a process of abstraction that resulted, among other things, in the fact that texts such as those that could not be seen as equivalent for Jordanus Nemorarius are now equivalent (see Chapter 3 and Puig, 1994).

The creation of more abstract MSSs that takes place in the history of mathematics in this way has its correspondence in school systems. Indeed, during a teaching and learning process a student is sometimes incapable of transforming a textual space ET' by means of a stratified mathematical sign system L, using the same concepts, actions, or properties of concepts or actions as those with which he transformed a textual space ET; the breaking down of this impossibility is precisely what is sought by the teaching model and what constitutes true learning, and it occurs when the student modifies the language stratum in which the means of transformation are described, creating a new mathematical sign system M, in which the textual spaces ET and ET' are identified as being transformable by the same means (see Chapter 4). The creation of this M is a “process of abstraction” that also entails the creation of “more abstract” concepts or actions (the ones described in the modified language stratum).

2.9. In these modifications of language strata that lead to identifying concepts or actions, an important part is played by the autonomization of the trans-formations of the expression with respect to the content

The importance of this autonomization resides in the fact that these transformations can then be made in accordance with the rules without having to verify the result of the transformations of the expression with respect to the content in each of the steps, but only occasionally or once the complete set of transformations has been concluded.

Umberto Eco points out that in algebraic expressions, as in all the signs that he calls "diagrams" and which for us, following Peirce, are icons, "there are one-to-one correspondences between expression and content," so that "the operations that I perform on the expression modify the content. If these operations are performed following certain rules, the result provides me with new information about the content" (Eco, 1984, p. 16). Geometric figures are also diagrams in this sense, whether they are drawn to represent geometric objects —as in Euclid's *Elements*— or to represent algebraic quantities. In the didactic device that we describe at the end of this chapter, geometric figures are used precisely for this purpose. Al-Khwârizmî had already done so in his *Concise Book of the Calculation of Al-jabr and Al-muqâbala*,[3] in which he used geometric figures to prove the correctness of the algorithmic rules that he gave to solve the six canonic forms of equations that we now call first- and second-degree equations, in what he called "proofs by means of figures" — and not "geometric proofs," since he did not make use of the propositions of Euclid's *Elements*.

However, we would not say, with Eco, that what the result of the transformations of the expression provides can always be described as "new information about the content." Sometimes, producing sense for the result of a transformation in the expression involves expanding the semantic field of the objects or actions involved, as is shown by the simple example of the identification of a^0 with 1, by virtue of the fact that certain rules produce $a^n/a^n = a^{n-n} = a^0$ and others produce $a^n/a^n = 1$, so that the expression a^0, literally meaning "*a* multiplied by itself zero times," which does not mean anything, is given sense by expanding the semantic field of "multiply" and "times." The autonomization of the expression thus brings with it a power to generate content.

Since the inscription of the first written arithmetic signs, which, as we indicated in Chapter 2, lacked operational capability, during the course of history mathematicians have gradually developed sign systems the expression of which has had increasingly greater power to generate content. Hence, as we see it, examining mathematics as a sign system and showing the crucial role played by the autonomization of the expression does not have to lead to Russell's famous conclusion that "the propositions of logic and mathematics are purely linguistic, and they are concerned with syntax" (Russell, 1973, p. 306).

2.10. The development of new competences in mathematics can be seen as the result of working with an MSS that one has already mastered to some extent

This happens both in the history of mathematics and in the history of individuals. In the school system, this work consists in an exchange of messages between teacher and student that is produced by means of the reading/transformation of the sequence of texts that we call a teaching model. As a result of this reading/transformation, new concepts are produced through the production of new senses and the establishment of new meanings for the MSS (or MSSs) in which what is taught is described and produced, which even entail the creation of new MSSs.

In his *Remarks on the Foundations of Mathematics* (Part III, 31), Wittgenstein wrote that "the proof changes the grammar of our language, changes our concepts. It makes new connections, and it creates the concept of these connexions. (It does not establish that they are there; they do not exist until it makes them.) [der Beweis ändert die Grammatik unserer Sprache, ändert unsere Begriffe. Er macht neue Zusammenhänge, und er schafft den Begriff dieser Zusammenhänge. (Er stellt nicht fest, daß sie da sind, sondern sie sind nicht da, ehe er sie nicht macht.)]" This observation by Wittgenstein about the effect of proof in the grammar of our language and in our concepts can be paraphrased by transferring it to what we have just expounded and simply replacing "proof" by "work with an MSS" and "our language" by "an MSS that we have mastered." Thereby one is being at the same time more general and less precise. One is more general because proving is obviously a kind of work with an MSS and it is not only this kind of work that changes mathematical concepts (see Section 4.7 in Chapter 2). One is less precise because we do not specify what kind of work with an MSS changes concepts and MSSs and we are not claiming that it is any kind.

However, Wittgenstein's remark is about the work of mathematicians and not about the work of students in the school system. As our viewpoint and our area of interest is the school system, we will have to use a version of Wittgenstein's remark adapted to the fact that we are only dealing with the processes of teaching and learning mathematics in school systems where mathematical concepts are not created for the first time but have to be recreated— or "reinvented," to use Freudenthal's expression —by students using the guide of the teaching process. In this sense, the aim of the teaching model, of the sequence of texts that are read and transformed, must be that the new senses produced by the students should be felicitous, that is, that they should be in agreement with the socially established meanings, and that the new, "more abstract" MSSs created should become non-idiosyncratic MSSs.

2.11. A teaching model is a sequence of problem situations. This is the sense of teaching through problem solving

As the teaching model is a sequence of texts, produced both by the teacher and by the student, and those texts are the result of the work of both in teaching situations that are in fact problem situations (which are taken as textual spaces), it is pertinent also to add what we have learned from our studies and inquiries about problem solving. In particular, we have evidence that, when a problem is solved, one inevitably makes an initial logical conscious or unconscious analysis (which in Chapter 9 we will call a "logico-semiotic outline"), however quick and fleeting it may be, which seeks to rough out the solution, that is, to indicate the path that must be followed in the solution of the problem in accordance with some mathematical text produced with the use of a certain MSS.

A competent user who makes such a logic-semiotic outline uses cognitive mechanisms that enable him to anticipate the key relations of the problem and, from various MSSs or strata of MSSs, decide which one, more abstract or more concrete, he is going to use to outline the steps of the solution. Only then does he develop a process of analysis and synthesis that enables him to decode the problem situation.

Along these lines, the age-old idea in reforming declarations of basing teaching on problem solving can begin to make sense for us. A teaching model is also a sequence of problem situations, a sequence of mathematical texts T_n, the production and decoding of which by the learner finally enables him to interpret all the texts T_n in a more abstract MSS. This "changes the grammar of our language," because the new, more abstract MSS is of such a nature that its code makes it possible to decode the texts T_n as messages with a socially established mathematical code, precisely the code proposed by the educational aims that fixed the model of competence that the teaching model pursues.

Sense is produced in the new MSS by the use of new signs in each step of the analysis and solution in the way in which they have to be used —as Wittgenstein says: "I go through the proof and say: 'Yes, this is how it *has* to be; I must fix the use of my language in this way' [Ich gehe den Beweis durch and sage: 'Ja, so muß es sein; ich muß den Gebrauch meiner Sprache so festlegen'.]" (Wittgenstein, 1956, III, 30). This is possible when the MSS as a whole is bound by the concatenation of the actions set in motion during the problem-solving processes in all the problem situations that were previously seen as different and irreducible, but that now, thanks to the new MSS, are solved by means of processes that are established as being the same, i.e., that are transferred from one problem to another, at the same time converting what was a diversity of problems into a family of problems.

Teaching organizes the transition from an MSS that has to some extent been mastered by the learner, through its use in problem situations and the chain of readings/transformations ET/T = ET'/T' = ET"/T" …, to a new MSS from which the previous one is seen as being more concrete, and with which what was previously described as separate and unconnected is now described as being the same, and as a result is produced as new concepts and new signs.

3. CONCRETE MODELING

In discussions about the kind of teaching resources that should be used in the curricular development of any teaching model, two conflicting positions usually appear. In the case of the solution of equations, one of the positions proposes modeling the new operations and new objects in (more) concrete contexts (with "concrete" understood as contexts that are familiar for the learner), with the aim of endowing them with meanings and constructing the first elements of manipulative operations, taking this context as a starting point. A contrasting position proposes starting from the syntactic level and teaching the rules of syntax so that they can later be applied in the solving of equations and problems. This is the traditional treatment in the teaching of the solving of equations, based on the syntactic models of Viète —transposition of terms from one side of the equation to the other— and Euler —addition and multiplication of the additive and multiplicative inverses, respectively, in the two sides of the equation.

If one adopts the first of the two positions just indicated in order to develop teaching strategies at the beginnings of the acquisition of the competences of a MSS, it is necessary to possess knowledge about the processes that intervene between the actions performed on a more concrete level —i.e., the actions in the model— and the corresponding elements of syntax that may be obtained from them. These processes, which we will here call processes "of abstraction of operations," and that are processes of recovery, on a syntactic level, of the elements common to the actions performed in the repeated use of a model or a concrete teaching situation, present regular characteristics in the course of their development by individuals; but they also move along paths that may differ greatly from one individual to another, owing to the presence in individuals of tendencies with respect to their use and learning of mathematics (we have looked at this area in more detail at the end of Chapter 4).

Moreover, although there is a set of regular characteristics or characteristics that are repeated from individual to another in these processes of abstraction of operations, some of them may vary with variations in the concrete situation from which one sets out in order to obtain or construct the corresponding syntactic elements —or in the model from which one sets out.

3.1. Algebraic semantics versus syntax

In the study expounded in other chapters, in which a teaching model concerning the solution of first-degree equations was developed, the interrelations between two overall strategies for the design of learning sequences that occupy long periods of time in the middle school algebra curriculum were basic. These strategies were:

a) Modeling of more abstract situations in more concrete languages in order to develop syntactic abilities.
b) Producing codes to develop problem-solving abilities and using syntactic abilities to develop solving strategies.

Broadly speaking, in (a) the objective was to give meanings to new expressions and operations, modeling them on more concrete situations and operations. In (b) the objective was to give senses to the new expressions and operations so that problem-solving codes would be generated, setting out from the supposition of the presence of certain abilities of syntactic use of the new signs and their utilization as a more abstract language. In the Introduction (Chapter 1) we show the problems that learners present when they have just finished primary education.

In what follows we will see that the development of syntax and semantics produces a dialectic relation in which an advance in one of these two aspects is necessary for an advance in the other, although sometimes the development of one may inhibit development of the other.

3.2. Components of concrete modeling

If one thinks about the introduction of certain mathematical notions by means of models (as is done in Chapter 4 for the solution of algebraic equations), it is advisable to bear in mind some of the main components of modeling, especially two components that are fundamental. The first component is translation, by means of which sense and meaning are given in a more concrete context to the new objects and operations that are introduced, which, from a more abstract viewpoint, are the same as those that appear in more abstract situations. In other words, through translation these objects and operations are related to elements of a more abstract situation and, on the basis of what is known about the solution of such situations on the more concrete level, operations are introduced which, although carried out on the concrete level, are also intended to be performed on the corresponding objects on the more abstract level. Consequently, there is a need for a two-way translation between one context and the other, so that it may be possible to identify each operation on the more abstract level with the corresponding operation in the concrete model.

The second component is the separation of the new objects and operations from the more concrete meanings with which they were introduced. In other words, in the modeling one also seeks to relinquish the semantics of the concrete model, because what one wishes to achieve ultimately is not the solution of a situation that one already knows can be solved, but the discovery of ways of solving more abstract situations by means of more abstract operations. This second component is a driving principle that directs the modeling function toward the construction of a syntax external to the model.

3.3. Concrete modeling versus mechanization and practice

In his book *The Psychology of Algebra*, published in the early 1920s, Thorndike proposed the integration of everything that seemed pertinent at the time so that the teaching of algebra could advance. That aspiration can still be seen now as a program yet to be fulfilled for any other theoretical and experimental approach —leaving aside, perhaps, certain emphases and pre-occupations belonging only to the theoretical perspective, in accordance with the psychological knowledge of the time. Among matters that are still of great relevance today we find the central motivation:

> Algebraic computation as actually found is then emphatically an intellectual ability. It is not so indicative of intellect as problem solving, partly because it involves less abstraction, selection, and original thinking, partly because it involves only numbers, not numbers and words. It is, however, far above the reproach of being a mechanical routine which can be learned and operated without thought. (Thorndike, 1923, p. 451.)

During the 84 years that have elapsed since then the emphasis placed by researchers has varied greatly, leading, in the middle of the last century, to the granting of total pre-eminence, not to what is called “problem solving” in the remark just quoted, but to the structural components of the matter studied. The result of this was that, in French middle school syllabuses, it was possible to find a so-called algebra in which what had been the traditional teaching situations until then, based on considering algebra as a continuation of arithmetic, did not appear anywhere. As a reaction to this, there was a swing toward the use of teaching models based on situations similar to those proposed by Thorndike, but more concrete, mechanizing the handling of algebraic expressions, with an expeditious use of the rules of syntax.

3.4. Syntactic models

The idea of a concrete teaching model can be extended to the strategies proposed in the 1920s, which we here call "syntactic models," in contrast to concrete models, which we call "semantic," because in them emphasis is placed on working with a considerable semantic load in all the signs and operations involved. In the syntactic model, conversely, the emphasis is placed on the general rule used to construct the habits that set the operations in motion.

With respect to these models, empirical evidence indicates that, apart from the generation of private semantics of the individual that give meaning to the terms proposed by the general rule and to the operations involved, phenomena of reading of the situations proposed appear, guided by the senses given to the rules that must be set in motion in order to carry out the syntactic task. For example, when someone first comes across equations of the type $Ax - B = C$ (A, B, $C > 0$), he may always attribute the positive sign to B and the negative sign to A, guided by the sense that he has obtained from previous practices performed with the solution of equations of the type $Ax + B = C$. In other words, a syntactic context guides a mistaken but natural reading, due to the individual's anticipatory mechanisms —a cognitive tendency that we presented in the last chapter.

In this respect, the emphasis placed not only on mechanisation but also on the concern for practice, and the consequences that this has on the practice times that learning experiments propose, acquire a new sense in view of the need to correct spontaneous readings, here generated not by semantics but by syntax.

3.5. Modeling and teaching algebra

The results described in this book allow us to state that the correction of mistakes of algebraic syntax and the operational mishaps that appear amid complex processes of solving problems or equations generated during the learning of algebra cannot be left to the spontaneity with which students make use of the first elements with which they are provided in order to penetrate into the realm of algebra. The paths traced by those spontaneous developments are not directed toward what the teaching of algebra seeks to achieve: that is precisely why this correction is a task of teaching. So that, if one thinks of introducing certain notions of algebra by means of models (including the syntactic model), it is advisable to bear in mind the main components of modeling, as described above.

The studies described in this book show that mastery of the first of the components of modeling (translation) may weaken or inhibit the development of the second: such is the case with learners such as Vt, mentioned in Chapter 4, who achieve a good command of the concrete model, but as a result also develop a tendency to remain and progress within that context, and this anchoring to the model goes against the other component, that of the abstraction of operations toward a syntactic level, which involves breaking away from the semantics of the concrete model.

This indication about the interaction between the two basic components of modeling does not depend on the tendency of the individual, for even in cases of a syntactic tendency, such as that of Mt, mentioned in Chapter 4, during the processes of abbreviation of actions and production of intermediate notations (between the concrete situation and the level of algebraic syntax) obstructions to the processes of abstraction of the operations effected in the concrete model are generated as a result of not possessing, in that period of transition, suitable ways of representing the results or states to which the operations lead. Once again, this is a deficiency in the second of the components of the action of modeling.

The obstructions indicated earlier constitute a kind of essential insufficiency, in the sense that, if modeling is left to spontaneous development by the learner, one of its components is strengthened, and this tends to hide precisely what one is essentially trying to teach, which is *new concepts and operations* (a more detailed description can be found in Chapter 4).

This kind of dialectic between the processes that correspond to the two components of modeling must be taken into account by teaching, which should try to develop the two kinds of process harmoniously, so that neither obstructs the other. Indeed, from the analysis of the cases presented in Chapter 4 it is clear that this is a task of teaching, given that this second aspect of modeling, that of breaking away from the earlier notions and operations on which the introduction of the new knowledge is based, is a process that consists in the negation of parts of the semantics of the model, and these partial negations take place during the transfer of the use of the model from one problem situation to another —in the case of the geometric model it is a transfer of its application from one variety of equation to another. However, when this generalization in the use of the model is at the expense of spontaneous development by the learner, the partial negations may take place in essential parts of it —in the geometric model, the presence of the unknown and operation on it are negated. Consequently, intervention with teaching becomes necessary in the development of these processes of relinquishing and negation of the model, in order to channel them toward the construction of the new notions.

The transfer of the problematic of algebraic semantics versus syntax to a level of actions of modeling makes it possible to narrow the distance between teaching and this problematic, since analysis of the interaction on this other level reveals didactic phenomena that show the need for the intervention of teaching at key points in the processes set in motion at the beginnings of the acquisition of the language of algebra.

SUMMARY

In this chapter we use the notions of textual space and stratified mathematical sign systems (from “less abstract” to “more abstract”) to describe teaching models in terms of sequences of mathematical texts (produced by the teacher or pupils) and in terms of sequences of problem situations. These theoretical notions generalize the examples of teaching models used to teach the syntax required for solving first-degree equations with the unknown appearing on both sides of the equality, presented in Chapter 4. In this chapter we speak of concrete models (the balance scales and a geometric model) and of “abstract” or syntactic models (the model of “doing the same on both sides” and the model of transposing terms). In the study “Operating on the Unknown” these models were used to observe the processes of transferring actions performed in simple cases to cases of equations with more complex characteristics, and also the processes of abstraction of actions performed in all the cases of equations presented to the pupils. In the next chapter we analyze the first steps toward the use of algebraic syntax in problem solving.

ENDNOTES

[1] The illusion of a text written in a formalized mathematical language, which is never present but to which the text that is really written refers, could not be better expressed than it is in the Introduction to Book I of Nicolas Bourbaki's *Éléments de Mathématique* (Bourbaki, 1966): "Nous abandonnerons donc très tôt la Mathématique formalisée […] Les facilités qu'apportent les premiers "abus de langage" ainsi introduits nous permettront d'écrire le reste de ce Traité […] comme le sont en pratique tous les textes mathématiques, c'est-à-dire en partie en langage courant et en partie au moyen de formules constituant des formalisations partielles, particulières et incomplètes, et dont celles du calcul algébrique fournissent l'exemple le plus connu. Souvent même on se servira du langage courant d'une manière bien plus libre encore, par des abus de langage volontaires, par l'omission pur et simple des passages qu'on présume pouvoir être restitués aisément par un lecteur tant soit peu exercé, par des indications intraduisibles en langage formalisé […] Ainsi, rédigé suivant la méthode axiomatique, et conservant toujours présente, comme une sorte d'horizon, la possibilité d'une formalisation totale, notre Traité vise à une rigueur parfaite […]" (pp. 6-7). The expression "abuse of language," which describes the fundamental operation that makes it possible to abandon the writing of the formalized text and refer to it, appears repeatedly throughout the treatise.

[2] Rotman presented a first version of his semiotic model of mathematical activity in Rotman (1988). A more recent version, modified and more extensive, is in Chapter 3 of Rotman (1993), which begins by announcing that "What I propose here is a semiotic model of mathematical activity fabricated around the idea of a thought experiment. The model identifies mathematical reasoning in its entirety —proofs, justifications, validation, demonstrations, verifications – with the carrying out of chains of imagined actions that detail the step-by-step realization of a certain kind of symbolically instituted, mentally experienced narrative" (Rotman, 1993, p. 66). His distinction between Code and metaCode seeks to account for the fact that "contemporary mathematicians divide their activity […] into two modes: the formal and the informal" (p. 69). Code is, therefore, "the unified system of all such rules, conventions, protocols, and associated linguistic devices which sanction what is to be understood as a correct or acceptable use of signs by the mathematical community," metaCode is a "heterogeneous and divergent collection of semiotic and discursive means" which give an account of "the mass of signifying and communicational activities that in practice accompany the first mode of presenting mathematics" (p. 69). In his model there is also a third element, which Rotman calls the "subCode" or "virtual Code," and three characters: the Subject, who uses the signs of the Code; the Person, who uses those of the metaCode; and the Agent, who uses those of the virtual Code. Rotman (1988) is also included in Rotman (2000) as its first chapter.

[3] As was usual in the 9th century in the Arab world, this book by Muhammad ibn Mûsa al-Khwârizmî did not have a title. Two manuscripts of it have been conserved, one of which was edited and translated into English by Frederic Rosen, with the title *The Algebra of Mohammed ben Musa* (Rosen, 1831). According to Høyrup (1991), both this manuscript and Rosen's translation are less close to the original text than the Latin translation produced by Gerardo de Cremona in the 12th century in the school of translators in Toledo. There is a recent edition of this manuscript (Hughes, 1986). Gerardo de Cremona heads his translation with the words "Liber Maumeti filli Moysi alchoarismi de algebra et almuchabala incipit" ["here begins the book of algebra and almuchabala by Mahomet the son of Moses alchoarismi"], leaving the Arabic words *al-jabr* and *al-muqâbala* untranslated, as we have just done. It is precisely because *al-jabr* remained untranslated that this part of mathematics, which in a sense al-Khwârizmî founded, was eventually called algebra. See an analysis of one of al-Khwârizmî's proofs by means of figures in Puig (1998).

CHAPTER 6

ALGEBRAIC SYNTAX AND SOLVING WORD PROBLEMS
FIRST STEPS

OVERVIEW

This chapter deals with advancing toward semantics, in the sense of applying or using a recently learned syntax (as of concrete models) toward solution of word problems. It is a continuation of the analysis of the results obtained in the study, introduced in Chapter 4, "Operación de la Incógnita" ("Operating on the Unknown"), in which through presentation of a case we revisit the discussion of the dialectic relationship between the syntactic and semantic components of symbolic algebra; we touch upon the issue of the need for important reconceptualizations in the dynamics of the relationship, such as the reconceptualization of equalizations, unknowns, and the equivalence of expressions; and we analyze the repercussions of this relationship in the field of teaching algebra.

1. INTRODUCTION

When one speaks of the semantics of elementary algebra, there is the possibility that one may be referring to the meanings that can be suggested by algebraic signs and expressions, with the involvement of processes of decoding of such signs and expressions, or that one may be talking of the semantics of the contexts in which the statements of problems that can be modelled and solved in algebra are immersed.

In behavioral terms, that is, in terms of behavior that is externalized, and even with respect to the thought processes that give rise to such behavior, this double interpretation of the term makes sense. Indeed, the reading and interpretation of the signs and expressions of algebra and their use in symbolizing and attacking problems are two kinds of process, but their starting points and the products that they generate are different.

However, if one accepts the thesis that is expounded in the last chapter (or that follows from what is expounded there) about the existence of a dialectic inherent in the interaction between the syntactic and semantic aspects of algebraic language, and about the decisive role that this interaction plays in the acquisition of that language, considering, for example, the tensions between the two basic components of modeling used in the teaching to which the interaction gives rise, then one also accepts as a consequence that the meanings that are made to correspond to the signs and expressions of algebra at the point when they are introduced (meanings that may be taken from the context of concrete models or even from syntactic models) will necessarily have to play some part in the subsequent use of them to model situations that come from richer semantic contexts, such as statements of problems. And, on the other hand, one will also accept that, in turn, the semantic experience obtained in the use of algebraic syntax for problem solving has an influence on the readings that may be given to signs, algebraic expressions, equations, etc. outside the contexts of those problems.

In view of these considerations, the term "semantics of elementary algebra" can be interpreted as referring to the semantic fields to which the learner has related the elements of syntax that he has acquired at a particular point. This includes both the meanings with which those elements were learned, which may also be in a syntactic context, and the meanings that they have acquired during their use, which may be a clearly operational use or their use in solving problems. In this way, the differences that a behavioral conception of the term poses would be evened out.

Although one may think of a kind of logical linking of assertions, as set out in the last paragraph, with respect to the interpretation(s) of the term algebraic semantics, the fact is that, within the framework of the present study, what has been presented (in the last chapter) is only what was observed empirically in the processes that were set in motion by a semantic introduction of elements of syntax —such as operations with the unknown in order to solve non-arithmetic equations. One would also have to establish the other links with empirical experience, that is, the transition to the semantic aspect of algebra that problem solving represents.

In the present chapter, this area is dealt with in terms of the transfer of the recently acquired manipulative skills to other contexts, such as those of statements of problems. Unlike the kind of observations made in series E and I in the clinical interview (see Chapter 4), in which the other aspect of the semantics of algebra —that of the introduction to syntax— was touched on with the aim of detecting the kind of difficulties that appear in the beginnings of the construction of syntax, for this second aspect the child was shown two series of statements of problems, series A with *abbacus*-style statements (statements of the "find a number …" variety) and series P with statements in other contexts, in which a process of progressive symbolization was needed in order to construct an equation that would solve the problem. In these cases in series P it was not a question of observing the child's spontaneous developments with respect to the way of attacking the problem and the solution procedures carried out, but rather the interviewer intervened with teaching in this phase of the solving activity. The aim of the observation in this series was to see how far it was possible, by teaching, to take the child along the path of the transfer from very limited, recently acquired algebraic manipulative skills to other contexts, different from those on which the instruction for introducing these skills was based. And the purpose of this was to provide those skills with senses —the senses given by becoming aware of the aim of mastering such skills— and to conclude a kind of sequence in the teaching, with which one would obtain an overall view of this transition toward the application of algebra, overall in the sense of its interconnections with the syntactic aspect and the semantic aspect of the learner's prior experience in the first incursions into the realm of algebra.

2. THE TRANSITION TO SEMANTICS (THE CASE OF MA)

Among the processes of transfer of a certain algebraic manipulation to contexts of problems it can be used to solve, there are those by means of which the instances of the solving procedure in which these manipulation skills can effectively be applied are identified. These processes of simple recognition are just part of the complex process of transference —which includes, among others, the processes of the analytic reading of the statement, the production of a strategy, and a system of representation. But although they are processes of simple recognition, they are not simple processes.

In the last chapter we made a point of reporting the observed fact that, in the middle of a complex process of reasoning, the presence of some distracting element is sufficient to make the student get stuck in a certain context in which this recognition of what is already known, what has already been mastered on operational levels, is not put into effect. The possibility that such phenomena of getting stuck may appear in the development of the procedure of problem solving cannot be excluded, and if they are, as has already been observed in other cases, it may disturb the whole procedure and even impede the possibility of finding a solution. The counterpart to obstructers of this kind is that level of transference of manipulation skills in which the elements of syntax that have already been mastered can be withdrawn from the semantics of the context in which the problem is posed (or solved).

In the clinical study carried out with 12- to 13-year-old children described in the previous chapter, this problem of using manipulation skills was taken into account, and two series of items in the interview (series A and P) were devoted to providing the possibility of observing phenomena of the transfer of elements of algebraic manipulation acquired by the child in the previous series (series I). Moreover, in order to be able to apply the P series of problems it was necessary that there should be a certain amount of algebraic manipulation on a syntactic level in the solving of linear equations with more than one occurrence of the unknown, precisely so that one could observe the transfer of these manipulation skills to other contexts. As can be seen in Chapter 4, this requirement could not be satisfied by all the children interviewed within the space of time that the interview lasted, not even by all the children in the upper stratum. From the reduced group of children who were confronted with the problems in series P we have chosen one of the most significant cases, that of Ma, in order to set out an analysis of the interview with her in this chapter.

The application of series A (*abbacus* problems) was intended to allow observation of the phenomena that arise in a process of minimal transfer of recently acquired manipulation skills, as it consisted of word problems corresponding to the same kind of equations as those that the child had been solving in series E and I. They were problems that could be translated directly into the corresponding equation. This series, A, was applied to most of the children in order to observe whether they were able to achieve this translation, even if they did not manage to solve the equation.

Series P consisted of a sequence of problems, the statements of which are not directly translatable to an equation but require a progressive symbolization of elements of the problem in order to construct the equation. In the interview, this progressive symbolization was achieved by the teaching phase. In fact, with this teaching two initial steps in the solving of any problem were obviated: the analysis of the statement, which leads to the production of a strategy for attacking the problem, and the representation of the elements of the problem that intervene in the solution, that is, in this case, a progressive symbolization. Once instruction had been provided in this translation phase, the child was allowed to perform the transformations of the equation in order to solve it, and the interviewer intervened with teaching only in those steps of algebraic manipulation that had not been taught previously.

The first problems in series P appeared in pairs: in the first problem of each pair the instruction described previously was provided, and in the second problem the child was allowed to tackle the problem and solve it completely. Finally, the last items in series P were problems that did not appear in pairs and that varied in contexts —moving to contexts such as geometr— and instruction was provided when necessary.

The objectives of confronting the child with the problems of series P were: (1) To observe another level of transference of the algebraic manipulation acquired in previous series (different from the level required in series A), taking account especially of the processes of identification or recognition of the instances of the solving procedure in which it was feasible to apply that algebraic manipulation. (2) To observe the processes of automation of actions by exploring the possibility that the corresponding problems might become routine operations. (3) To observe the processes of transfer of the translation and solving skills acquired in the first items of the series to the solution of the final items, which did not appear in pairs and were in varied contexts. (4) To detect phenomena of interaction between the syntax applied in the problems —whether it had already been mastered or not— and the semantics of their context.

Ma, the case presented here, belonged to the upper stratum of the population studied, observed in the period 1982/83, and was 13 at the time of the interview. The most important characteristics of the interview were: high manipulation skills in series E; very rapid abstraction of operations in the model toward the level of algebraic syntax in series I; consistently correct use of algebraic notation to record the steps in the solution of the equations of series I with and without use of the model; immediate transfer of the algebraic manipulation acquired in I to the solving of the problems in series A; immediate recognition and operational solution of equations like those of series I in other contexts (series P); and an advance in algebraic manipulation as a result of the need to develop new elements of algebraic syntax to solve the problems in series P.

We shall present the case of Ma in the following sections:

1) Map of the interview.
2) Performance prior to the phase of instruction in operating on the unknown.
3) Performance subsequent to the instruction phase.
4) Progress toward semantics.

2.1. Map of the interview

2.1.1. Data of the interviewee

Name:	Mariana (abbreviation, Ma)
Age:	13
School level:	Second year of secondary education
School year:	1982–1983
Level:	High efficiency in pre-algebra

Note: At the time of the interview, she had received no teaching in algebra as such.

2.1.2. Items in the interview

The items appear in the order in which they were presented to the student.

Series E

The aim of this series was to verify the results of the pre-test. Only arithmetic equations were included, i.e., equations with one occurrence of x. In what follows, the abbreviation E Ma n means the nth item of series E in the interview with Ma.

E Ma 1	$x + 5 = 8$	E Ma 6	$13x = 39$
E Ma 2	$x - 4 = 8$	E Ma 7	$6x = 37434$
E Ma 3	$x + 27 = 58$	E Ma 8	$(x + 3) \times 6 = 48$
E Ma 4	$x - 15 = 143$	E Ma 9	$4 \times (x + 11) = 52$
E Ma 5	$x - 1568 = 392$		

Series I

The aim of this series was to explore the student's difficulties when confronted with the first non-arithmetic equations (i.e., equations with occurrences of x on both sides) and having to operate on the unknown. Nine sub-series (I-a to I-i) and four intercalated items made up the complete series. In what follows, the abbreviation I Ma n means the nth item of series I in the interview with Ma —the items are not numbered with respect to the sub-series.

Sub-series I-a

Equations of the type $Ax + B = Cx$, with A, B, and C being given positive integers.

I Ma 1	$x + 2 = 2x$	I Ma 7	$4x + 12 = 6x$
I Ma 2	$x + 5 = 2x$	I Ma 8	$5x + 8 = 3x$
I Ma 3	$2x + 4 = 4x$	I Ma 9	$5x + 8 = 8x$
I Ma 4	$2x + 3 = 5x$	I Ma 10	$7x + 468 = 19x$
I Ma 5	$6x + 15 = 9x$	I Ma 11	$113x = 3328 + 321x$
I Ma 6	$4x + 12 = x$		

Sub-series I-f

A new variety of equation: two occurrences of x, one on each side, and two positive constant terms, one on each side. In other words, equations of the type $Ax + B = Cx + D$, with A, B, C, and D being positive integers.

I Ma 12	$7x + 2 = 3x + 6$	I Ma 13	$13x + 20 = x + 164$
I Ma 14	$8x + 12 = 4x + 52$	I Ma 15	$28x + 348 = 52x + 12$

Sub-series I-g

Equations of types $Ax + B = Cx – D$ and $Ax – B = Cx + D$, with A, B, C, and D being positive integers.

I Ma 16	$5x – 3 = 2x + 6$	I Ma 17	$15x + 1590 = 71x – 202$
I Ma 18	$11x + 687 = 45x – 27$		

Sub-series I-e

Operating with the unknown is suggested, i.e., the occurrences of x appear on the same side of the equation.

I Ma 19 $4x – 3x = 7$

Intercalation I-f

The characteristics of the equations are the same as in sub-series I-f.

I Ma 20 $8x – 7 = 4x + 13$

Intercalation I-g

The characteristics of the equations are the same as in sub-series I-g.

I Ma 21 $7x – 1114 = 3x + 1001$

Sub-series I-h

A variety of equation with two negative constant terms.

I Ma 22 $7x – 1114 = 3x – 1001$

Intercalation I-g

The characteristics of the equations are the same as in sub-series I-g.

I Ma 23 $7x + 1114 = 3x - 1001$ I Ma 24 $113x - 70 = 22x + 1022$

Series A

This series contains *abbacus*-style word problems with a view to observing the translation to equations or diagrams. It includes statements that correspond either to arithmetic equations or to non-arithmetic equations. In what follows, the abbreviation A Ma *n* means the *n*th item of series A in the interview with Ma.

A Ma 1 If you add five to a number and then take away thirteen and the result is forty-five, what is the number?
A Ma 2 If you add twelve to a number and then multiply it by nine and the result is a hundred and seventeen, what is the number?
A Ma 3 If you add two to a number the result is double the original number. What is the number?
A Ma 4 Seven times a number reduced by twelve is equal to three times that number. What is the number?
A Ma 5 If we add the triple of a number to forty-eight, the result is nineteen times that number. What is the number?
A Ma 6 Five times a number plus three is equal to twice the number plus twelve. What is the number?

Series C

The method of cancellation is sufficient to solve the equations in this series, and therefore it is not necessary to operate on the unknown. It includes equations with one occurrence of *x* and equations with two or more occurrences of *x*. In what follows, the abbreviation C Ma *n* means the *n*th item of series C in the interview with Ma.

C Ma 1 $x + 5 = 5 + 2$ C Ma 2 $x + \frac{141}{16} = 17 + \frac{141}{16}$

Series P

This series contains statements of problems in contexts different from those of the *abbacus*-type statements, with a view to observing the transfer of the recently acquired algebraic syntax. In what follows, the abbreviation P Ma *n* means the *n*th item of series P in the interview with Ma.

P Ma 1 There are chickens and rabbits in a yard. I count the heads and there are sixteen, I count the legs and there are fifty-two. How many chickens and how many rabbits are there in the yard?

Series S

This series includes an introduction to the syntactic manipulation required to continue with the process of solving the problem in item P Ma 1. In what follows, the abbreviation S Ma *n* means the *n*th item of series S in the interview with Ma.

S Ma 1	$2 \times (x + 16) =$	S Ma 5	$2 \times (x - 1) =$
S Ma 2	$3 \times (x + 16) =$	S Ma 6	$3 \times (x - 1) =$
S Ma 3	$2 \times (2x + 1) =$	S Ma 7	$4 \times (16 - x) =$
S Ma 4	$6 \times (2x + 5) =$		

Series P

The characteristics of the problems are those that we have just described.

P Ma 1 There are chickens and rabbits in a yard. I count the heads and there are sixteen, I count the legs and there are fifty-two. How many chickens and rabbits are there in the yard?

P Ma 2 There are chickens and rabbits in a yard. I count the heads and there are eight, I count the legs and there are twenty-six. How many chickens and how many rabbits are there in the yard?

P Ma 3 Mariana is thirteen, Eugenio is forty. When will Eugenio be twice as old as Mariana?

P Ma 4 Mariana is thirteen, Roberto is twenty-four. When will Roberto be twice as old as Mariana?

P Ma 5 José Luis asks his Uncle Juan: “How old are you?” As his uncle likes word puzzles, he answers: “Twice my age plus your sister’s age is equal to three times my age less your age.” If José Luis is fifteen and his sister is twenty-eight, how old is Uncle Juan?

P Ma 6 Three boys won nine hundred and sixty pesos. Enrique won twenty-four pesos less than Eduardo, and Esteban won ten times as much as Enrique. What did each of them win?

P Ma 7 How many meters is the perimeter of Figure 6.1 if both figures, 1 and 2, have the same perimeter? What is the perimeter of Figure 6.2?

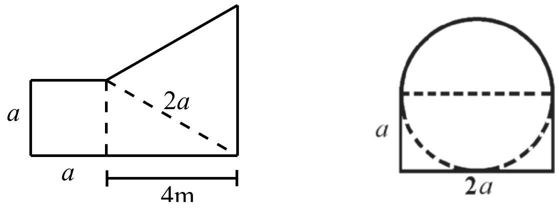


Figure 6.2

2.2. Performance prior to the instruction phase

In Ma's performance in the arithmetic equations of series E we observed that she used trial and error during the first items because of the simplicity of the equations that appeared in them, but when the numerical form of the equation became complex she applied the method of reversing operations without using the visual resource of the diagram, whether the equations consisted of one step or two. She generally used the calculator, and as it could handle negative numbers it was not possible to observe whether she found difficulty in operations of this kind. According to the results of the pre-test, she had had no difficulty in operating with negative numbers.

Except for the first items in the series, in which she was asked whether her answer was correct, Ma verified the result spontaneously and correctly by substituting in the equation that was given, without any indication of a polysemic operation with x.

Ma tackled the initial items in series I with the trial and error method, preceded by an *abbacus*-style reading of the equation. Like all the children in the upper stratum, Ma very soon stopped trying to solve the equations by the method of trial and error. Thus, she stopped trying to solve item I Ma 4 $2x + 3 = 5x$, which is easy to do by trial and error, in the expectation of being taught an operative solving method. At that point the phase of instruction in operating on the unknown took place, with the use of the geometric model.

2.3. Performance after the instruction phase

Ma's performance in the use of the geometric model to operate on the unknown was characterized by immediate abstraction of the operations carried out in the model toward the level of algebraic syntax, as can be seen in item I Ma 6, the first that she tackled after the instruction phase. One factor that influenced this very rapid abstraction was the parallel record that Ma kept, using signs of algebraic language, while she was performing the actions with the elements of the model. The procedure was based on a mental representation of the model, as Ma dispensed with the drawing from the outset. This, therefore, is a case in which the processes of abstraction of the operations of the model to algebra (in items of the same variety) were reduced to immediate translation of the actions in the model into algebraic language; in other words, the relinquishing of the semantics of the model was effected by means of this translation. In fact, the translation in the opposite direction (from the equation to the model) was not at all explicit, precisely because of the omission of the drawings. This way of using algebraic notation to record the development of the procedure almost step by step avoided the concealment of the real operation on the unknown, detected in cases where the processes of abstraction involved a prior automation of the actions in the model without making a record of those actions and of the automation. In some cases there was no parallel record of the actions in any non-model language; in others, a kind of intermediate language was created in which the elements of syntax that the student succeeded in recovering through the automation of the actions were reproduced. In both types of cases we observed concealment of the operation with the unknown and consequent mistakes of algebraic syntax.

The immediate abstraction of the actions in the model to algebraic language was not completely uniform in Ma's post-instruction performance: the changes to more complex varieties of equation or even changes to equations with a non-trivial numeric structure —such as those of the type $C < A$ in $Ax + B = Cx$— acted as disturbing elements in the manner of abstracting the operations. In these changes we saw a return to explicit use of the model followed by a repetition of the process of abstraction, via the translation into algebra of the steps carried out in the model. That is to say, when the syntax generated locally in one variety could not be transferred to new varieties, it was necessary to revert to the semantics of the model to reconstruct the actions in it and once again recover them on the syntactic level of algebra: this could be seen very clearly in the change from sub-series I-a to sub-series I-f, which began with item I Ma 12 $7x + 2 = 3x + 6$.

On the other hand, the overcoming of the obstacle presented by the new variety of equation presented in I Ma 12 was translated into an evolution in the solving method on a syntactic level. That is, Ma's habit of jotting down the steps of the procedure in algebraic language as she went along —even when that language was developed in the concrete model— enabled her, on the one hand, to realize that the method consisted in reducing the new type of equation to one that was already known; and, on the other hand, given that the steps of the reduction were written down using the signs of algebra, it was possible for her to begin an automation of the procedure of simplification of the equation on the level of algebraic syntax. The way in which Ma began the simplification of the original equation that she was given, operating on the constant terms before operating on the terms in x, led her to propose a first reduction of the equation as one of the intermediate steps in the process of simplification, in other words, producing an equation of the same variety as that of the items in sub-series I-a, with two occurrences of x and only one constant term.

I Ma 12 Ma reduces the equation to an equation of the first type	$7x + 2 = 3x + 6$ $7x = 3x + 4$

Ma did not tackle this equation immediately, despite having automated the operations for solving this kind of equation in sub-series I-a: this was a sort of regression in the line of evolution that we had been observing in her performance. Ma had to be referred to the geometric model, and there she solved this equation.

This pattern of solution was repeated in several subsequent items in the same sub-series, I-f, such as:

I Ma 13 Ma performs the first reduction syntactically and writes from right to left. Ma resorts to the geometric model and obtains the simplified equation.	$13x + 20 = x + 164$ $13x = 144 + x$ $12x = 144$

By the end of sub-series I-f she was performing all the operations on a syntactic level, without resorting to the model. Moreover, as Ma began the procedure in this case by operating first on the terms in x, the first reduction of the equation no longer appeared as a step in the procedure:

I Ma 15 The difficulty in placing the terms in the simplified equation disappears. She does the operations $(348 - 12) \div 24$ with a calculator.	$28x + 348 = 52x + 12$ $24x + 12 = 348$ $x = 14$

In this case, as in the others that we have analyzed, this level of evolution in the way of solving the new equations was impeded by the presence of disturbing elements. Indeed, in the last item of sub-series I-f we saw that Ma had progressed substantially as regards translating the actions to a syntactic-algebraic level, which she had been doing in a combined manner in the model and apart from it, in addition to having succeeded in overcoming the difficulty of placing the terms resulting from the reduction operations in this item. It was precisely this difficulty that reappeared in the change to the variety of equation in sub-series I-g, when Ma tried to transfer the syntactic rules used by her implicitly —she never actually stated them— in the previous varieties:

I Ma 16	$5x - 3 = 2x + 6$
Development by Ma	$5x - 3 =$
[Wrong]	$3x = - 3 + 6$
Correction made at the request of the interviewer	$3x - 3 = 6$
Ma says:	"*x* equals three"

The difficulty in placing the terms after the first transformations of the equation persisted in this sub-series (I-f) and in the following sub-series, although in some items in them it did not appear because Ma took the precaution of operating first on the terms in *x* and beginning to write down the simplified equation from right to left. This apparently helped her to relocate the terms according to their degree —one or zero— in the new equation.

After overcoming this difficulty, which was brought about by the presence of disturbing elements in the equation, such as, for example, the fact that the solution was negative or less than one, Ma achieved a high manipulative level in solving linear equations with one or more occurrences of *x*. Because of this degree of mastery that Ma had acquired in solving the new equations, she was subsequently given two further series of items in the interview: series A, consisting of *abbacus*-type statements, and series P, the context of which went beyond the context of the model with which she had been given instruction in operating on the unknown. In these two series, the statements corresponded to equations that were either arithmetic (like those of series E) or non-arithmetic (like those of series I) —see Section 2.1, "Map of the interview."

Theoretically, there could be several possible responses to these statements in the case of children in the upper stratum: translation of the statement to an equation (or a diagram, in those cases where it was meaningful) and not going on to solve it; translation to an equation and solving it by applying the recently acquired manipulative skills; solving the problem without translating it into an equation (or a diagram, where appropriate). In Ma's case, in the *abbacus*-style statements of series A we observed great fluency in translating the statement into the corresponding equation and solving it operationally in both cases (equations with one occurrence of *x* or with more than one), which denoted an immediate correct transfer of her recently acquired algebraic manipulation to word statements of the new equations. We now describe the development of some of the items in series A to illustrate what we have just indicated.

A Ma 1	If you add five to a number and then take away thirteen and the result is forty-five, what is the number?

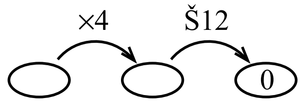
Equation Solution by reversing operations	$(x + 5) - 13 = 45$ $(45 + 13) - 5$

A Ma 6	Five times a number plus three is equal to twice the number plus twelve. What is the number?

Equation Solution by operating on the unknown in the equation	$5x + 3 = 2x + 12$ $3x + 3 = 12$ $x = 3$

In one item in series A we saw a momentary loss of earlier skills in Ma because of the presence of a disturbing element: a result equaling zero in the simplified equation.

A Ma 4	Seven times a number reduced by twelve is equal to three times that number. What is the number?

Equation Simplified equation Ma tries to solve the simplified equation by using trial and error, but when she is unsuccessful she soon resorts to using the visual representation of the diagram: Solution of the diagram with a calculator, by reversing the operations	$7x - 12 = 3x$ $4x - 12 = 0$  $x = 3$

Finally, in series P Ma was given problems for which the translation into an equation was not direct because two unknowns appeared in the statement —two quantities that had to be found— so that it was necessary to write one of the unknowns as a function of the other. Some types of problems appeared in pairs, where the second problem in each pair was obtained simply by changing the given values in the first problem. The pairs in series P were P

Ma 1 and P Ma 2, P Ma 3 and P Ma 4 (see Section 2.1, "Map of the interview"). The interviewer intervened in the first item of each pair in order to help with the formulation of the equation, whereas the second item of each pair was developed by Ma entirely on her own: once the equation had been formulated Ma was perfectly capable of applying her recently acquired manipulative skills to the solution of this equation, which had more than one occurrence of x.

This pattern of development, by pairs of problems, was repeated in part of series P, and it can be said that once the initial difficulty had been resolved, which in these cases had to do with the translation into an equation, and given that the earlier algebraic manipulation was transferred immediately to the solution of the equation, the problem posed, simply with the given values changed, immediately became a routine problem for Ma.

It must be pointed out that the phase of translating from the statement to the equation, in all cases, was a complex process of progressive symbolization, with the gradual construction of the equation corresponding to the statement, and that the fact that Ma managed to deal with these problems in a routine way so quickly was due to two things: first, the fact that she knew in advance that the idea was to produce the equation, and, second, the fact that, in the progressive symbolization, she recorded all the steps of the process with fairly tidy algebraic notation, which enabled her to re-read the process whenever she found it necessary, and to direct her development of the procedure toward obtaining the equation. Also, a third factor that counted in this very rapid transfer to other contexts was the operational confidence that Ma had already acquired in solving equations of this kind.

We ought to make it clear that this transfer of manipulation skills to other contexts took place in Ma on a level where there was no presupposition of the production of a strategy for attacking the problem by the person interviewed: it was a transfer on the level of recognizing equations that came from contexts of problems as equations that she already knew how to solve. In this specific case, these equations required operation on the unknown for their solution. This kind of transfer made it possible for the recently acquired algebraic manipulation to be considered as a potential tool for solving a larger family of problems, that is, a family including statements that led to the linear equations here described as non-arithmetic.

2.4. Progress toward semantics

Despite the confidence reflected, in Ma's case, in the ability to solve the new equations operationally when they appeared in other contexts, we must also consider the processes that come into play between understanding the statement and formulation of an equation, so as to be able to speak of a real transfer of that algebraic manipulation to the solution of problems. These processes include those of *representation* of the elements of the problem, which presuppose a reading and analysis of the statement that discriminates between what is given and what has to be found, so as to be able to recover the relevant information and set aside non-essential facts. This *representation* can also be preceded by the production of a strategy for attacking the problem, although sometimes it comes afterwards.

In Ma's case these processes were precluded, first because she was faced with these problems immediately after she had learned to solve the equations in series I, in other words, as a result of the sequence of the interview it may have been natural for her to apply what she had just learned; second, phases of instruction took place throughout the guided translation procedure, so that there was no opportunity to observe her own approaches to the problem or the difficulties that might have appeared in the solving procedure, to which her approaches might have given rise.

However, there are two notable features in Ma's performance in this guided solving of problems that were related to the interconnection of semantics and algebraic syntax. One of these features was Ma's recognition of the equation, once presented to her, as one of those that she had learned to solve, and the fact that she then went on to solve it. As can be seen in the analysis of series I, what usually happens is that, when certain situations that previously could be tackled operationally are presented in the middle of a complex process of reasoning, they are not read on a language level in which it is possible to recognize them and solve them as before: this is reported as "the momentary loss of earlier skills" resulting from a behavior of getting stuck. In Ma, this phenomenon did not appear in series P, and there are two plausible explanations for this: the certainty that she had from the outset that she was going to come to the formulation of an equation —as we mentioned earlier— and the fact that she had been able to give sense to the recently acquired manipulative skills at the point when she perceived that it corresponded to ways of attacking new problems. It is in this way, once again, that the consolidation of the first elements of algebraic syntax is supported by its connection to semantics outside algebra, in this case that of the statements of the problems.

A second notable feature in this part of the interview with Ma is the fact that, during the development of the solving of the problems, instruction phases were necessary in order to resolve certain operational steps that formed part of the transformations of the equation that had been formulated. These operational steps were elements of syntax that had not yet been taught to Ma within the context of algebra. Moreover, she had not yet been taught all the algebra that is included in the curriculum of secondary education. In these instruction phases the interviewer again made use of contexts outside algebra, such as in the case of the law of distribution when there are unknowns involved, the explanation and development of which was carried out in a geometric representation and the development was subsequently translated into algebraic language.

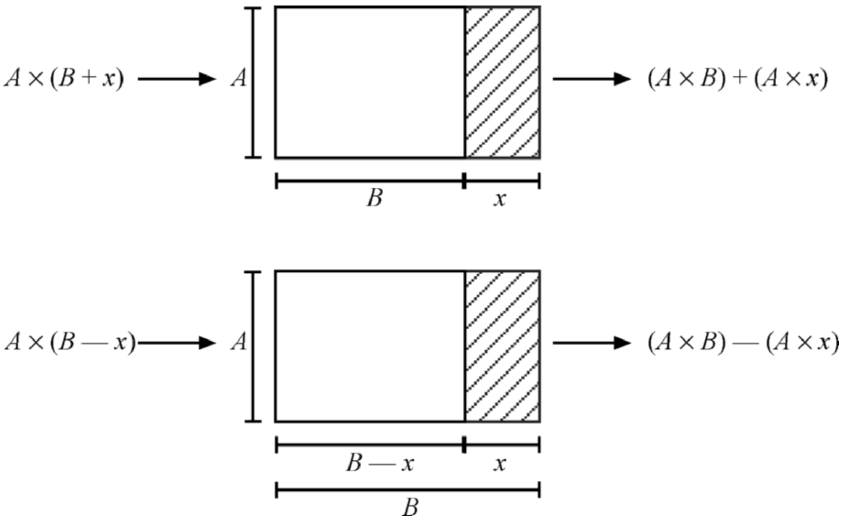


Figure 6.3

This law was known by Ma at the time of the interview. In fact, it was in what she had been taught previously, and there, too, it had been presented by means of a geometric representation of the expressions involved. However, she did not recognize it as a known situation in the middle of this procedure of transforming the equation, and she needed a phase of instruction. The explanation for this non-recognition is that, although expressions of the type $A \times (B + C)$ —which is how Ma had learned the development of these expressions, with A, B, and C being any numbers— and $A \times (B + x)$ are basically the same, there really is a difference in terms of the conceptualization and interpretation of the literal notations in the two types. This difference consists in the simple fact that in the first type A, B and C are generalized numbers, i.e., the letters represent a set of numbers (Booth, 1984), and in the second only A and B are generalized numbers, whereas x represents an unknown for Ma in the context of solving a problem.

In this case, therefore, it was not a matter of a momentary loss of an earlier skill, but the fact that the identification of one type of expression with the other requires a process of conceptualization in the learner that denies the condition of unknown to x and interprets it on the level of generalized numbers, which includes A, B, and C. The strategy used in the interview to lead Ma to this identification was to give the new type of expression, $A \times (B + x)$, a treatment analogous to that given to expressions of the type $A \times (B + C)$, i.e., to translate the expression into more concrete terms in a context of areas, so that the development performed in the concrete was recovered in algebraic language. In this way, by taking as a starting point a problem situation with a semantic context outside algebra (ages, numbers of rabbits, etc.), translating that situation to algebraic sentences (with the involvement of all the processes that were mentioned earlier) and attempting to apply skills of algebraic syntax to them, one reaches a point where it becomes necessary to progress on this syntactic level: the transformations of the equation require new elements of syntax, which in turn are based on substantial changes in the conception of the objects that those elements involve so that they can be developed. These changes sometimes require one to negate notions that, on the other hand, are necessary for translating the statement of a problem to algebra, such as the notion of the unknown, and thus to make way for other notions, such as that of generalized number, which are necessary for evolving in the development of certain elements of syntax, such as the manipulation of algebraic identities, for example.

Returning to the matter of the transfer of the algebraic manipulation acquired to broader contexts such as those of statements of problems, although Ma's case cannot be considered as an example of transfer understood in its full extension, one could speak of another level of partial transfer, which consisted in having the manipulative skills required in the solving of the problem well established, to a point where, on the one hand, there was an ability to recognize situations in which these skills could be applied (equations, in this case) and, on the other, the mere recognition of these situations acted as a motor for the setting in motion of the corresponding operational actions – even though an intervention with teaching was needed so that they could be carried out. This is a necessary, but not sufficient, level of transfer for the transition from semantics (that of the problem) to take place. In Ma's case, it seems that this first level of transfer was resolved, at least with respect to these first incursions into the terrain of algebraic operations, operations with the unknown.

In other respects, Ma's performance in the last items of series P, in which she had already operationalized the new elements of the syntax that she had been taught in the first problems in the series, makes it clear that progress in algebraic semantics (as far as its use in problem solving is concerned) also implies progress in syntax. The implication in the opposite sense, that is, that progress in syntax implies progress in the semantic aspect of algebra is also true, and this second implication seems to be a fairly widespread opinion, as it is always considered that a certain level of syntax is a factor that assists problem solving.

In the interview with Ma, this kind of regression toward a line of evolution in syntax shows the possibility of concluding the sequence that, in its most significant points, travels along a path that takes as its starting point the construction of minimal elements of algebraic syntax, endowing them with meanings in the context of a concrete model, then detaching them from those meanings in order to recover them on the more abstract level of the syntax of algebra, and finally endowing them with the *sense* that is provided by the possibility of applying them to the solution of problems.

In the realm of teaching, it might seem to be impossible to propose to conclude sequences such as the one just described in a relatively short time span: the interview with Ma shows not only that it is useful and necessary, but also that it is possible. In other words, although Ma presents certain characteristics that represent favorable conditions for making the attempt, the fact that in her case it was possible to cover in just one hour and forty minutes all the ground for the teaching of these first notions and operations of algebra opens up the possibility in the realm of teaching of trying to develop elements of algebraic syntax and their corresponding semantic versions in harmony — either as elements derived from contexts of the model(s) used to introduced them, or else as elements that are themselves used to model situations that emanate from statements of problems— and make them come together in the correct, congruent use of algebraic language, with respect to these two aspects, syntactic and semantic.

3. SOME POINTERS CONCERNING TEACHING

By way of conclusion, on the basis of the observations in Ma's case it can be said that the interactions between algebraic semantics and syntax can be channeled by means of appropriate teaching strategies toward the completion of sequences in the course of which the new algebraic notions and operations that are introduced can be endowed not only with meanings but also with senses. This kind of approach breaks away from the old idea that in the teaching of algebra it is possible to develop independently a syntax that will be used subsequently in the solution of problems. This traditional conception encourages the development of the "non-senses" of learning algebra in children, given that only a small proportion of the student population eventually acquires the language of algebra at the level of using it as a tool for solving problems. In other words, in order to provide senses for all this initial algebraic manipulation it is necessary somehow to conclude a kind of sequence, realizing why one is learning and automating all this syntax. In a traditional strategy, the concluding of the sequence is deferred in time “until the operational area has been developed sufficiently,” which excludes a large part of the population that studies algebra from ever reaching that “conclusion.”

On the other hand, the results of this study indicate that it is not solely a question of time, but that the transfer of algebraic manipulation to the context of the problems —which is usually not addressed specifically by teaching but rather is largely left to spontaneous developments— is not simple, and that processes are involved in which the semantic and syntactic aspects of the new language interact and are transformed into their conceptual elements. Indeed, as we have already pointed out in Ma’s case, the transfer processes are of various kinds and range from those that are usually taken into consideration in studies carried out under the heading of “problem solving,” such as the analytic reading of the statement, the production of a strategy of attack, the production of a representation of the elements that feature in the statement, etc., to the generation of new syntactic elements, on the basis of modifications in the learner’s conceptual apparatus, such as, for example, the possible negation of the condition of unknown in some of the signs that appear in the representation of the situation of the problem and in the syntactic operations that have to be carried out. On the other hand, bearing in mind the observations about the stage prior to the introduction of some elements of syntax, it is also necessary to complete the cycle if one wishes to achieve a certain degree of consolidation of the new operational knowledge and a certain degree of potential and actual mastery in its application to problem solving.

The foregoing indications are important for the area of teaching symbolic algebra, because, if they are taken into account when particular teaching strategies are proposed, this implies making basic modifications in the current conceptions of teaching in this area, or at least in these conceptions as they are reflected by the approaches that appear in syllabuses, in current textbooks and in some specialized reports on studies (whether experimental or not). They would be basic modifications because even an analysis of the cognitive aspect of certain central objectives in the teaching of algebra in secondary education, in the terms of the observations and results that we have expounded, would show that any significant advance in the learning of algebraic language, in any of the semantic or syntactic aspects, is based on substantial modifications of notions that are interlinked, such as the notions of the unknown, variable, generalized number, function, algebraic identity, proportional relation, etc. These notions in themselves play a leading role in achieving important objectives such as the solution of problems in which the unknowns do not appear explicitly in the statement and there is the involvement of other notions, such as proportional variation or function, and in the course of the solution it is necessary to negate that variability in order to be able to fix the relation and recover the unknown(s) of the problem. On the other hand, results set out in this study show that underlying the dynamics of such advances there is a dialectic interaction between the semantic and syntactic aspects of algebra, so that progress in one of these directions is supported by progress in the other, and the possibility of making use of this interaction in the development of teaching strategies is something that would also have a repercussion on a reconsideration of the current approaches to the teaching of elementary algebra.

SUMMARY

Perhaps one of the greatest challenges in the teaching of algebra, once pupils have acquired the rudimentary elements of algebraic syntax (for example, through concrete models), is that of managing to make the pupils use them in the solution of word problems. In the same way that, in the learning of syntax, one can see an intricate relationship between syntax and semantics (understood as a relationship between the operations performed on the purely syntactic level and the meanings that those operations have in the context of the concrete model), in the application of that syntax in the solution of problems one can also perceive a tension between the semantics that belongs to the context of the problem and the symbolic manipulation of the algebraic expressions that represent the situation of the problem. This chapter deals with this relationship and the elaboration of the thesis that states that there is a dialectic relationship between the semantic and syntactic aspects of algebraic language, in the sense that an advance in one of these aspects often takes place at the cost of a retreat or stagnation in the other. The final section of the chapter contains a reflection on the implications that the relationship between semantics and syntax has in the realm of teaching. Once again we have taken as a basis for the discussion observations made in the clinical study "Operating on the Unknown," presented in Chapter 4.

In the next chapter, we shall present the cognitive tendencies of pupils from the study "Operating on the Unknown," and from a study on the Thales theorem, in which another teaching model is in use and a geometrical sign system is involved.

CHAPTER 7

COGNITIVE TENDENCIES AND ABSTRACTION PROCESSES

OVERVIEW

This chapter is devoted to the subject of the cognitive processes that take place during the learning of algebraic language. We analyze its close relationship with cognitive tendencies that are observed in students during the learning of algebraic syntax and its use in solving problems. Eleven cognitive tendencies are described, well identified in empirical studies, some referring to the presence of readings by learners on different levels of language or sign systems, which may obstruct the possibility of solving an algebraic task.

We explore also the processes of abstraction in terms of the theoretical notions of meaning and sense, in relation to mathematical sign systems (MSSs), seen in strata. The material for this exploration is the basic protocol or plan of two interviews, one concerning the solution of linear equations and the other concerning proportional variation. The plan of the interviews is set out and analyzed in episodes, incorporating into each episode comments related to the transformations of texts (algebraic expressions or geometric figures or propositions) for the solution of the task presented. These texts are located in more abstract or less abstract strata of MSSs. The chapter ends with a list of cognitive tendencies identified in the various episodes.

1. INTRODUCTION

The cognitive processes that are set in action in order to carry out the forms of mathematical thinking and their communication with socially established codes gradually fine-tune the complex elements that are used in (1) perception, for example, in the case of the handling of geometric forms and their transformations; (2) the directing of attention and its relations to the processes of understanding; (3) the increasingly intensive use of memory; (4) the setting in motion of processes of analyzis and synthesis increasingly intertwined with the use of logic; (5) the heuristic conceptions used in the solving of problem situations, and (6) learning, closely bound up with the processes of generalization and abstraction and requiring novel uses of the MSSs of school mathematics.

We ourselves see these cognitive processes in teaching situations, particularly when, in a teaching situation, one is trying to move from a more concrete stratum of an MSS language to a more abstract one and various events take place, a brief list of which we present in this introduction. In Sections 2, 3, and 4 they are developed in more detail, and in Section 5 this list of events is exemplified with the two studies.

1.1. Cognitive tendencies toward a competent use of more abstract MSSs

1.1.1. The presence of a process of abbreviation of concrete texts in order to be able to produce new rules of syntax

1.1.2. The production of intermediate senses

1.1.3. The return to more concrete situations when an analysis situation presents itself

This feature is present in most of the actions of mathematical thinking and has been reported in many other studies.

1.1.4. The inability to set in motion operations that could be performed a few moments before

Behavior of this kind has even been seen in a student who was trying to solve the equation $Ax = B$, hard as it may be to believe in view of the simplicity of the equation. Situations also appear in which the operativity of fractions is inhibited by the presence of mistaken spontaneous readings of a geometric nature concerning the notions of ratio and proportion of magnitudes. In Section 4, where one teaching sequence is the proof of Thales' Theorem, we will see how and with what difficulties it is possible to give new senses to the uses of the various concepts that are embraced in the arithmetic of fractions.

1.1.5. Getting stuck in readings made in language levels that will not allow the problem situation to be solved

An example is the observation of the performance of 12- to 14-year-old students when they try to solve problem situations based on solving the equation $Ax = B$, which we called the "reverse of multiplication syndrome" in the Introduction (Chapter 1). In this chapter, examples can be found of a mistaken geometric reading concerning the order of magnitude between ratios of magnitudes.

1.1.6. The articulation of mistaken generalizations

The literature on mistakes made by students is full of this kind of behavior. The student tries to get away from the behavior mentioned in Section 1.1.5 by promoting a rule to other contexts where its application is meaningless: what is involved is an incorrect use of those concepts and operations.

1.1.7. The presence of calling mechanisms that cause the learner to get stuck in setting in motion mistaken solving processes

When we spoke of the "reverse of multiplication syndrome," we indicated the presence of this cognitive tendency. Another example appears when students try to find the side of a rectangle when the area and measurement of the base are known, using trial and error instead of the operation of division. Many of the phenomena that appear in 1.1.9 are also due to this behavior.

1.1.8. The presence of inhibiting mechanisms

In an extreme case, the examples in Section 1.1.7 are typical of this behavior, but also in the area of solving equations the presence of negative solutions gives rise to obstructions of syntactic rules that had been mastered previously. The insistence on not beginning to analyze a problem, the refusal to solve simple equations in which radicals appear, and the inability to use elements of syntax that have not been fully mastered in the intermediate steps of an analytic chain to solve a problem are further examples of this behavior.

1.1.9. The effect of obstructions derived from semantics on syntax and viceversa

When a person is solving problems and endowing algebraic signs with meanings, this brings about a predisposition to use syntax. Most of the phenomena mentioned in Section 1.1.4 can be interpreted in this way. A student may even write down a simple arithmetic equation in the middle of solving a problem and not recognize it as such, despite having spent years solving such equations with great skill. In the case of syntax, the tendency to get stuck on more concrete levels inhibits appropriate readings of more abstract texts.

1.1.10. The generation of syntactic errors due to the production of intermediate personal codes in order to produce senses for intermediate concrete actions

1.1.11. The need to produce senses for increasingly abstract networks of actions until they become operations

We will find these and some other cognitive tendencies again and again in Chapters 4 to 6, where empirical studies are presented. In Chapter 9, in which we analyze cognition and problem situations in mathematics, we will also see in each subsection that the presence of one or more of the cognitive tendencies that have just been listed can be observed. The same can be said of Sections 3 and 4 of the Introduction, (Chapter 1), devoted to pre-algebra and cognition.

2. SOLVING EQUATIONS AND THALES' THEOREM

When youngsters begin studying algebra and trigonometry they bring with them the notions and approaches that they used in arithmetic. However, algebra and trigonometry are not just a generalization of arithmetic. Learning this new material does not merely mean making explicit what was implicit in arithmetic. A change in the student's thinking is required, from concrete numeric situations to more general propositions about numbers, figures, and operations. The transition from what might be considered as an informal way of representing and solving problems to a new MSS proves difficult for many of those who begin to study algebra and geometry. These students go on using the methods that worked for them in arithmetic.

2.1. General description

We are going to explore the theoretical idea introduced in Section 2.10 of Chapter 5 and develop it further in the following chapter, to the effect that the acquisition of new competences in elementary mathematics can be considered as the product of the modification of concepts, actions, and procedures of MSSs for which competences have already been mastered to some degree.

When one observes how mathematics is learned, it is apparent that new rules are constantly being formed as learners find new paths that extend previously developed conceptual networks. A fundamental aspect of this viewpoint is the idea of sense, in contrast to that of meaning when one is talking of stratified MSSs. In this chapter we analyze in more detail what usually occurs in the clinical interviews that we used in Chapters 4 and 6.

In order to show that the cognitive tendencies that we introduced in Section 1 are present not only when one is studying algebra[1], in Sections 2 and 4 we advance in the use of algebra in order to introduce trigonometry and analytical geometry. We do so by describing an experimental study on the notion of the slope of a straight line, which we now introduce briefly and develop in the fourth section of this chapter, after a discussion about solving equations.

2.2. Thales' Theorem: Meaning and sense in an MSS

The aim of this study was to observe the natural obstructions to the use of an MSS in which the notions of geometric proportional variation could be presented. The analysis focused on observation of learners when they were presented with situations that simulated the demonstration of Thales' theorem. With respect to the two possible readings, in two different MSSs, it was concluded that neither of them could be reduced to the other. One reading was made with the Egyptian model for rational numbers and the other with the Greek model (later, in Section 4.3, we give a brief description of these two models of rational numbers). The notion of meaning was used throughout the study, in contrast to that of the sense of a text that uses a specific MSS.

In Chapter 8 we present the Fundamental Theorem of Geometric Proportionality as an example of the structuring of a formal competence model; in this chapter we illustrate the ad hoc teaching model by means of an example. The teaching model is based on an analysis of what happens in straight staircases with respect to the relation between how high one has climbed and the distance one has moved forward when one reaches any point on the incline of the staircase. The aim is to make it clear that this relation is constant and can be called the "slope" of the staircase.

In Section 4 we will see that when we used this teaching model with children who had just left primary school we came up against a great difficulty: breaking away from what geometric intuition indicated to the learners. In fact, when they were asked the question shown below, about the geometric problem that considers the incline of a straight staircase (which we will call question Q),

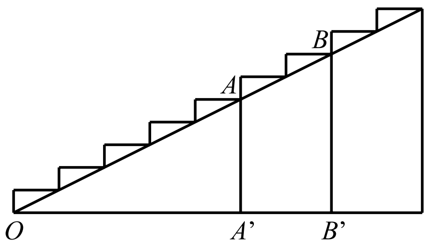


Figure 7.1

Compare the ratio between the height climbed and the distance traveled horizontally at A with the ratio between the height climbed and the distance traveled horizontally at B.

What is the relationship between the ratios $\frac{AA'}{OA'}$ and $\frac{BB'}{OB'}$?

Are they equal or is one greater than the other?

The "natural" response was that the second was greater because $\overline{OA'} < \overline{OB'}$ and $\overline{OA'} < \overline{OB'}$, that is, both the height climbed and the distance traveled horizontally in B is greater than the height climbed and the distance traveled horizontally in A.

In Section 4 we find other facts related to this and see how teaching can be used to prevent such cognitive tendencies that lead students to make mistakes that supposedly have already been overcome.

2.3. The rest of the chapter

We explore processes of abstraction and their relation to the theoretical notions of meaning and sense for MSSs: in Section 3, for the solving of equations, and in Section 4 for proportional variation and Thales' Theorem. Finally, in Section 5, we return to cognitive tendencies, recapitulating what has been said in this and earlier chapters.

3. SOLVING EQUATIONS. ANALYSIS OF A TYPICAL INTERVIEW BY EPISODES

In this section we present a description of a *typical* clinical interview in the form of episodes. As a teaching sequence it uses a strategy that sets out from a concrete model for teaching how to solve linear equations. The study in which these interviews were conducted is described in Chapters 4 and 6.

The teaching sequence is designed to provide the student with a series of problem situations that are stated in the language of symbolic algebra and are translated to a language (iconic and written) which is concrete (balance scales, piles of stones, exchange of plots of land, etc.). The aim is that by the end of the sequence the learners will solve linear equations syntactically. We will designate the MSS that it is desired to teach as MSS_a, using the subscript a to indicate that this MSS is more abstract than the MSS used to describe the problem situations that are given to the student, because in the more concrete MSS_c the signs have a greater direct relation to certain meanings that come from the situations presented: plots of land and geometric properties, balance scales, and properties of balancing, etc. The subscript c is used here to indicate that this MSS is more concrete.

Thus the first type of situation in MSS_a are texts of the following variety:

> $Ax + B = Cx$, where A, B, and C are positive integers and in this case $C > A$,

presented to the students with numeric coefficients.

On a more concrete level, i.e., in the MSS_c referred to, these texts take the form:

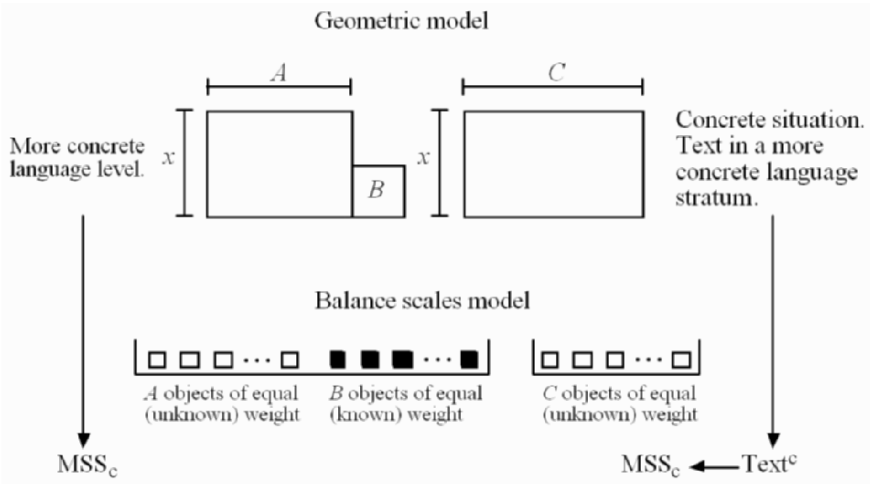


Figure 7.2

The texts of this type are called algebraic, in contrast to the ones that are called arithmetic, where it is not necessary to operate on the unknown in order to solve the equation (see Chapter 4).

3.1. First episode

Step A. Interpretation of Text^a as Text^c.

Step A: Transferring the equation to the model

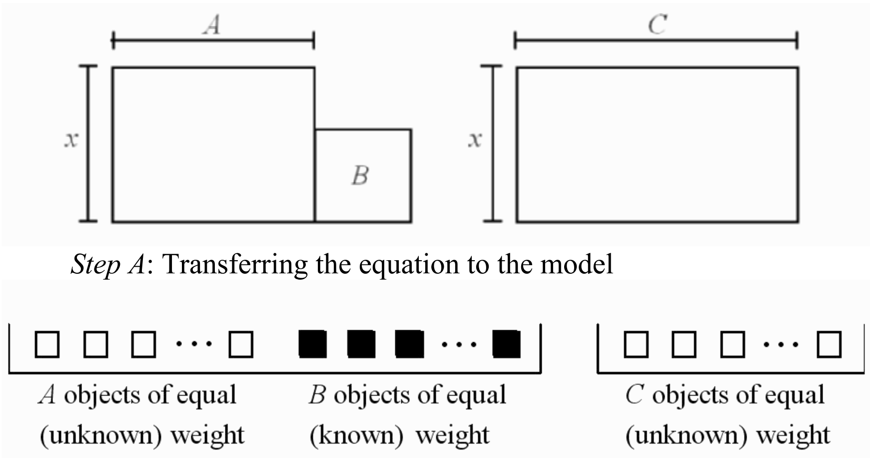


Figure 7.3

Step B. Setting in motion of known actions in the more concrete language stratum in order to decode the problem situation.

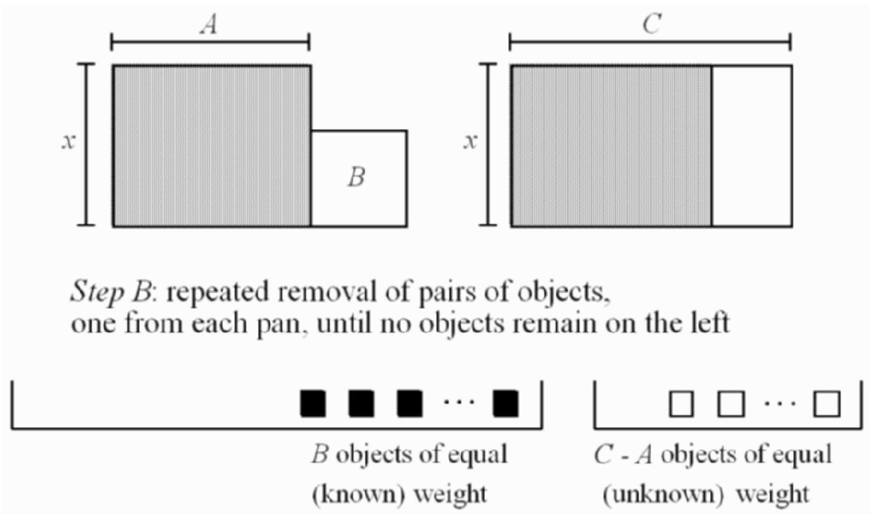


Figure 7.4

Step C. Performance of concrete actions.

Step D. Decoding of the problem situation until a solution is reached: a solution described in another Text[c].

Comment 1. The hypothesis assumes that the learner is competent to perform actions of this kind, and use their properties, and that this will enable him to solve the problem situation presented by the Text^c in the more concrete language stratum MSS_c.

3.2. Second episode

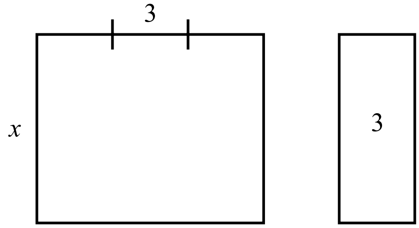
Step E. Translation into a Text^a, the solution of the problem situation described in Text^c.

Text^c is translated into the new equation, $(C - A)\,x = B$, which is the new Text^a.

Step F. Solving the equation $(C - A)\,x = B$.

Previous teaching makes it possible to solve this arithmetic equation. Of course, things are not always so simple, because the learners may find themselves with an intermediate problem situation (in a more concrete stratum), either in a new Text^a or in a new Text^c.

An example of an intermediate problem situation is presented in Figure 7.5.



reduced equation $3x = 3$

Figure 7.5

When a situation like this appears, sometimes it is necessary to perform steps B, C, and D again.

3.3. Third episode

The teaching strategy of *repetition and practice* now appears in the teaching sequence, with more cases to be solved; but, for example, with increasingly larger numbers in the equation $Ax + B = Cx$, $C > A$.

Step G. Steps C, D, E, and F are performed once again, giving rise to steps H and I.

Step H. A process of abbreviation tending to set aside many of the meanings that appeared in the Texts^c, aiming to arrive at a syntactic level of proceeding.

Step I. Production of intermediate personal codes to represent the actions performed and the intermediate results (see Section 1).

Comment 2. An intermediate language stratum has been introduced, and the meanings in it come from the personal syntactic rules just used.

Until steps H and I, all the actions performed on Text^c are *dependent on the sense of the (concrete) context*. Until these steps we have only created a didactic device for solving equations of the variety $Ax + B = Cx$, $C > A$. What will happen if another Text^a of a different variety is presented to the learner, for example, the new Text^a $8x + 5 = 3x + 15$?

3.4. Fourth episode

Step J. Recognition that the problem situation is a new one and cannot be reduced to a reading made with the recently created intermediate language strata, which are more abstract than the original concrete language stratum.

Step K. A new process of learning (by discovery) is performed with the same teaching strategy, and thereby it is possible to set in motion actions A, B, C, … to J.

But new factors appear:

Step L. Step H is performed in a shorter time.

Comment 3. In step I, the intermediate codes are refined in the direction of possessing meanings which are freer of dependence on the sense of the concrete context.

Step L is a negation of part of the meanings that come from the language stratum in which the text in question is described. *Different situations that were irreducible to one another* can *now* be interpreted in the same way, and the syntactic rules constructed in I are now applied to the new Texts[c]. Only *now* are the two texts recognized as being of the same kind of problem situation, and as a result the same solving process is set in motion.

3.5. Fifth episode

Step M. There is a return to step K with new types of problem situations. For example, with a new Text[a] such as $8x - 3 = 5x + 6$.

Step N. Steps A, B, … L are set in motion once again.

Step O. Operations are created in a new language stratum in which the senses are no longer dependent on the concrete context, giving new meanings to the new, more abstract concepts.

Comment 4. With the strategy of repetition and practice a new, more abstract language stratum is achieved in which it is possible to model (translate) more abstract situations, i.e., families of problems that previously were irreducible to one another are recognized as equal from the viewpoint of the solving processes set in motion by using the new MSSs.

With the process in its totality —the five episodes— a collection of stratified MSSs with interrelated codes has been created, and they make it possible to produce texts whose decoding must make reference to several of those strata: the transformation of the texts will use actions and concepts whose properties are described in one or more of the strata.

This is an example of what is described theoretically in Chapter 8, in terms that we will now advance here.

Two texts, T and T', both products of the use of an MSS, L, are called transversal when the user cannot create T in the same way as in the decoding of T' – that is, if T is not reducible to T' by the use of L (remember Comment 3). What really happens is that the learner can produce T and T' but cannot recognize the two decoding processes as a product of the use of the same actions, procedures and concepts of the various strata of L.

If we now have another stratified MSS, M, in which T and T'can be coded and the production of both can be described by means of the same actions, procedures, and concepts in M, the meaning of which has as referents the actions, procedures, and concepts used in the decoding of T and T' in the strata of L, then we will say that M is a more abstract stratified MSS than L for T and T' (remember Step O).

In order to accomplish this, the objects, actions, procedures, and concepts used in M have lost part of their semantic-pragmatic meaning: they are more abstract.

4. PROPORTIONAL VARIATION, THALES' THEOREM. AN EXPERIMENTAL STUDY

The Theorem of Similarity, also known as Thales' Theorem, represents a didactic cut for the acquisition of competences in the use of important mathematical concepts that range from the first notions of some models of rational numbers to the properties of continuous linear variation, and from the introduction of linear functions and their algebraic representation to their use in geometric representation in trigonometry and the beginnings of infinitesimal calculus.

This section describes an experimental study in which the main aim was to explore what competences are necessary to understand and use Thales' Theorem, what natural obstructions present themselves, and what importance all this has for the expansion of rational numbers to a stratified MSS in which the numeric signs have as referents both the fractions that are used in the MSS of elementary arithmetic and the geometric signs that we call ratios between continuous magnitudes. From these results one can clearly see that until a user has a correct interpretation of all the concepts involved in the theorem of similarity, brought together in the strata of a new MSS, he cannot call on stable notions with which to operate and establish relations of order, to use in the same way as he does with the more primitive MSSs used in the representations of the rational numbers, introduced earlier with the competent use of the MSS of elementary arithmetic.

In order to observe the didactic cut just mentioned, an experimental design was put into operation, in which the teaching was controlled for a period of two consecutive school years, in the first and second years of secondary school in Mexico (ages 12 to 14). In this experiment, the teaching sequences that were used to teach two cohorts of 30 students at the Centro Escolar Hermanos Revueltas in Mexico City were controlled. A diagnostic test was designed with three axes: manipulative skills, competence in the solving of linear equations, and a final axis concerning the concepts and equations connected with the understanding of the geometric situation in which a staircase with a straight rake or incline is studied, where the central and culminating point is the use of the notion of the slope of a straight line (the incline, in the concrete case referred to here). As an example, we then present the seventh of the nine units that make up the teaching model used.

4.1. Unit 7 in the teaching model

a) Drawing straight staircases

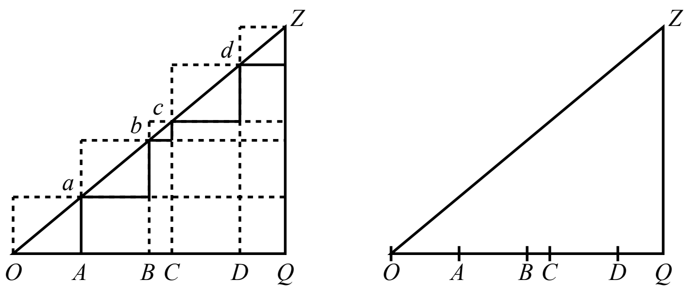


Figure 7.6

In the triangle on the left, color the line that represents the incline of a straight staircase red. This line is $\overleftrightarrow{OZ}$. You can extend it beyond *O* and *Z* to show that you are indicating the line and not just a piece of it, which we will call a *segment* of it.

Fill in the gaps:

$$\overleftrightarrow{OZ} = \overleftrightarrow{Oa} = \overleftrightarrow{Ob} = \quad = \quad = \quad = \quad = \overleftrightarrow{dZ}$$

Color the line $\overleftrightarrow{ZQ}$, which represents the wall, blue. Color the line that represents the floor green. This line is:

$$\overleftrightarrow{OQ} = \overleftrightarrow{BC} = \quad = \quad = \quad = \quad = \quad = \overleftrightarrow{AQ}$$

In the triangle on the right, first draw lines perpendicular to the line $\overleftrightarrow{OQ}$ at points *A*, *B*, *C*, and *D* (use a set square).

Then draw lines perpendicular to the line $\overleftrightarrow{QZ}$ that pass through the points where the lines you have just drawn cut the incline $\overleftrightarrow{OZ}$. Finish drawing the steps so that you have a staircase like the one in the drawing on the left.

Note that in this way you can make steps with whatever depth of tread you want. The depth of tread that you choose determines the rise (the height of the step).

Now in your notebook construct two right-angled triangles, with one of the perpendicular sides in each triangle measuring the same as the distance between point *O* and point *Q* in the previous figure. Draw this side horizontally.

In the first of these triangles, draw the other perpendicular side *shorter* than the distance between *Q* and *Z*.

In the second triangle, draw the other perpendicular side *longer* than the distance between *Q* and *Z*.

In each of the triangles, on the side $\overleftrightarrow{OQ}$ mark points *A*, *B*, *C*, and *D* at the same distance from O as the same points in the figure at the beginning of this unit.

Construct steps with treads that are the distance from *O* to *A*, the distance from *A* to *B*, etc.

Verify that, in each triangle, all the steps give the same result for:

$$\text{slope} = \frac{\text{rise}}{\text{tread}}.$$

Which is greater, the slope of the hypotenuse of the first triangle or the slope of the hypotenuse of the second triangle?

Choose a step in one of the triangles. Find the corresponding step in the other triangle. In which of them is the rise greater?

Do the same with the other steps.

You can see that the corresponding steps in the two triangles have the same tread, because that is how we constructed them.

But the slope of the hypotenuse of the first triangle is less than the slope of the hypotenuse of the second triangle. That is why the rise of the steps constructed on the hypotenuse of the first triangle is less than the rise of the corresponding steps on the hypotenuse of the second triangle.

b) Drawing steps with a given rise

In this right-angled triangle we have started with points on the line that represents the wall. Two lines have been drawn perpendicular to $\overleftrightarrow{ZQ}$. Now draw a line perpendicular to $\overleftrightarrow{OQ}$, passing through *m'*.

With a set square it is very easy. Try not to go beyond the point where you cut the line $\overline{nn'}$. You have just drawn a step with a rise equal to the distance between *n* and *m*. Now draw a step with a rise equal to the distance between *n* and *q*; and another one, with a rise equal to the distance between *q* and *Z*, and one more, with a rise equal to the distance between *l* and *m*.

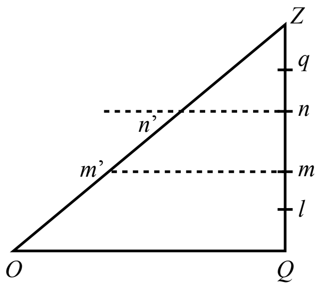


Figure 7.7

If we ask you to find a step in which

rise = dist (Q,l),

do you understand? Once again we are using more symbols. We have already told you that we will be using more and more symbols.

It will always be the same; if you understand properly the first time that we use a symbol, afterwards you just have to remember what the symbol represented.

In this case we have simply abbreviated the words "with a rise equal to the distance between *Q* and *l*" and in their place we have put:

rise = dist (Q,l)

Do you think it's difficult to indicate the step with

rise = dist (m,n)?

It shouldn't be, because it's the first one you drew.

In your notebook, draw two right-angled triangles with one of the perpendicular sides more or less horizontal, because it is going to represent the floor of a staircase whose incline is the hypotenuse; the other perpendicular side will represent the wall.

Put the same letters as before, *O*, *Q*, and *Z*, at the corners of each triangle.

In the first triangle, on the horizontal line, which is where the *O* and the *Q* should be, mark five more letters, *E*, *F*, *G*, *H*, and *I*. Construct steps such that:

1. *tread = dist (O,E)*
2. *tread = dist (E,F)*
3. *tread = dist (F,G)*, etc.

In the second triangle, put the letters *m*, *n*, *p*, *q*, and *r* on the vertical line, which is where you put the letters *Q* and *Z*. Construct steps such that:

1. *rise = dist (m,n)*
2. *rise = dist (p,q)*, etc.

Verify that each of these examples gives the same result for:

$$\text{slope} = \frac{\text{vertical travel}}{\text{horizontal travel}} = \frac{\text{rise}}{\text{tread}}.$$

On this right-angled triangle draw two more perpendicular lines to construct a step which, when you climb it, will take you from the height marked by point *m* to the height marked by point *n*. Then put three more points on the hypotenuse (the incline) and draw perpendicular lines so as to make steps which, when you climb them, will take you from each point to the next.

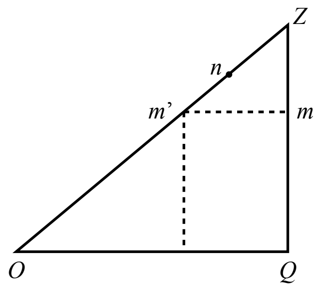


Figure 7.8

In this example, once again verify that you get the same result for:

$$\text{slope} = \frac{\text{vertical travel}}{\text{horizontal travel}}.$$

4.2. Description of the study

The first diagnostic test was given at the end of the first year of secondary school, when the students were 13 years old. The population observed was classified on the basis of the three axes described in Section 2.5 of Chapter 3, with a view to carrying out case studies by means of videotaped interviews. Two students were selected from each class. The first part of the interviews was used to confirm the diagnostic test and check that the case actually corresponded to what was expected.

The next stage consisted in observing the difficulties that appeared when the students were trying to use the notion of the slope of a straight line. The clinical interview also consisted in the provision of a teaching sequence in which the students analyzed various situations presented in terms of the construction and use of straight staircases. Of the various concepts, ideas, and actions that were employed, some favored the use of a new MSS stratum, whereas others proved to be genuine obstructions to the possibility of using the new signs competently in order to set processes in motion to find the correct solution of the problem situations presented to the students or considered by them in their processes of analysis.

The following year, further interviews were conducted with the cases that seemed most interesting, in order to observe what types of obstruction were still present and which ones had undergone some kind of development to make it possible to acquire the competences required by the formal competence component of the local theoretical model (LTM) that was being used to codify, analyze, and interpret the behavior of the students when they confronted the problem situations that they were given in order to acquire the organization of those competences, as envisaged by the teaching sequence of the LTM.

For these interviews, therefore, problem situations were designed that were of the same kind as those that had been used the previous year. An important difference that appeared was the observation, in various cases, of the possibility of giving general explanations about why the slope of the incline remained constant despite the fact that it was calculated (in arithmetic terms) at different points on the staircase.

An even more important advance appeared with students whom the diagnosis had indicated as being more capable in their previous history of using the competences in the three axes of the diagnostic test. This advance was achieved by using the elementary arithmetic MSS with the addition of the geometric signs that had to do with comparison of magnitudes, the properties of the variation of a slope, and other geometric notions such as angle of inclination, measurement of segments, and elementary operations with them;

in other words, by the use of a new stratified MSS, with which it was possible to give intrinsic explanations (i.e., within the new MSS) of the stability of the slope of the line, giving an account of the *senses* that the learner gave to the series of processes of analysis of the problem situations that he set in motion each time that he reached a correct solution. In short, this description of sense structured the steps that could be taken in the concrete examples presented by the teaching sequence in order to produce a proof of Thales' Theorem when the magnitudes involved are commensurable. Thus, in the concrete cases in which the steps of a staircase appeared, it was possible to measure all the pertinent magnitudes with a unit of the decimal metric system, for example centimeters.

4.3. A few particular observations

We start this section by defining the terms "Egyptian model" and "Greek model" of the rational numbers, which we are going to use here. In broad outline, the Egyptian model uses an MSS stratum in which the rational numbers are conceived as equal parts of discrete or continuous wholes, but in which fractions represent the relationship between the part and the whole that contains it. This relationship is transferred to other wholes with their corresponding parts through what is called equivalence of fractions, understood as an arithmetic proportion.

With this model, the second series of interviews in the case studies included a part of the observation that was intended to measure competence in the use of the unit of measurement and the relationship between two magnitudes when they have been measured, even though the units of measurement may be different.

The Greek model is an MSS stratum developed on the basis of the usual language for designating linear magnitudes, the notion of order between them, the comparison between two magnitudes in which the notion of ratio is introduced, and finally the variation of one magnitude with respect to another. Thus, the language stratum in which one can describe a situation that illustrates the notion of the slope of a straight line presupposes a competent use of an MSS (with a strong geometric component) that contains operations between ratios and in which the concept of geometric proportionality has been mastered to the point at which ratios are expressed as stable "objects." Its use permits the introduction of continuous linear functions and the start of their use as new objects with which it will be necessary to learn to perform new operations.

4.3.1. First observation

The natural but wrong reply to question Q, which we mentioned in Section 2.2, was given by all the students: when comparing the ratio between the height climbed and the distance traveled forward between two points, *A* and *B*, on a straight staircase, all the students said that the ratio at the higher point was greater, because both the height climbed and the distance travelled forward were greater.

This blockage (see below, cognitive tendencies 5 and 6), in which the order between linear magnitudes was transferred to the order between ratios, had to be corrected, using the teaching model, by the arithmetic reading of the problem situation presented, using the Egyptian model. This correction was achieved only by some of the students, all belonging to the upper stratum of the diagnostic classification mentioned earlier.

What is more, the error was transferred when the same question was asked but with one of the magnitudes increasing while the others remained constant, but this happened only when the Greek model was used, while the same question in the Egyptian model was answered correctly. This is an example of a circumstance that indicates that the arithmetic reading and the geometric reading of the situation are irreducible to one another.

4.3.2. Second observation

When the steps of a straight staircase constructed on an incline are introduced, the question in geometric language can be translated into a question about the relationship between the measurement of the height climbed (taking the rise of the steps as the unit) and the measurement of the distance travelled forward (taking the tread as the unit). Thus, for each point A, when a step finishes,

$$\frac{AA'}{OA'} \cong \frac{No.(A) \times rise}{No.(A) \times tread},$$

where No.(*A*) is the number of steps from *O* to *A*.

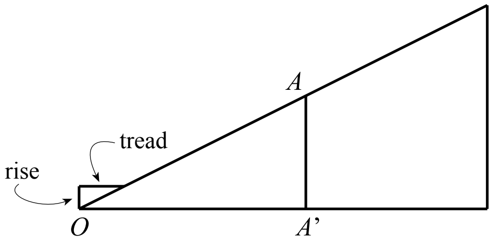


Figure 7.9

This new reading of the situation did not succeed in becoming pertinent in order to obtain a correct response to question Q, and the blockage persisted, even when the student had just discovered that the fractions that appeared in the arithmetic analysis of the situation were equivalent. Even when they had these results before their eyes, all the students repeatedly answered that $\frac{BB'}{OB'}$ was greater because $\overline{BB'}$ and $\overline{OB'}$ were greater.

4.3.3. Third observation

Those who corrected the mistakes made in the teaching situation of the interview did so with the introduction of the measurement of the rise and the tread, answering that the rational numbers $\frac{BB'}{OB'}$ and $\frac{AA'}{OA'}$ were both reducible to the same decimal number by dividing the numerator by the denominator.

4.3.4. Fourth observation

When they realized that they had predicted something (in the geometric reading) that the arithmetic reading contradicted, the students of the upper stratum corrected their mistakes. They tended to give general geometric explanations, declaring that the step was the same at each point on the staircase because the incline was straight and "this straight line continues in the same way."

Once again we must point out that, although the same notation is used in the Greek model as in the Egyptian model (namely a/b) to designate the rational numbers that appear in the two language strata, these objects are not the same, because their meanings are so different that if the same facts are described in the two language strata, the affirmation in one of them contradicts what is asserted in the other.

4.3.5. Fifth observation

The students who took part in the second interview were about 14 years old and were familiar with the teaching situation —the analysis of the situation of the straight staircase, which they had done during the first year of instruction, was intended to make them capable of producing texts and solving the following arithmetic equations:

$$A \pm \square = B,\ \square \pm A = B,\ A/\square = B/C, \text{ and } \square/A = B/C,$$

where A, B, C, and D are any rational numbers.

At the same time, they also knew that the relationship $\frac{AA'}{OA'}$ in the incline was constant, and that it was called the slope of the line. Despite their knowledge of the statements of Thales' Theorem in this particular situation, the students continued with the same kind of blockage during the geometric reading. We can say, therefore, that familiarity with Thales' Theorem is not pertinent for developing the competence required in order to answer question Q.

4.3.6. Sixth observation

Some of the learners of this age who were in the middle stratum according to the classification of the diagnostic test succeeded in achieving the desired correction, not by giving a general geometric explanation but by interpreting the situation through the arithmetic reading. Essentially, the interpretation of the straight line as the incline of a staircase enabled them, given points A and B, to construct a staircase with equal steps in which A and B defined steps.

Here, the commensurability of the magnitudes involved was used, as they were always measured in units of the decimal metric system and it was always possible to imagine the steps as being of the size of the smallest unit that could be used. Thus, it is possible to convert $\frac{OA}{OA'}$ and $\frac{OB}{OB'}$ to rational numbers whose numerators are a multiple of the measurement of the tread and whose denominators are the same multiple, but this time of the rise.

This interpretation and the reading of question Q in this context is an intuitive proof in concrete terms of Thales' Theorem in the case described.

4.3.7. Seventh observation

Part of the study indicates that proportional variation with discrete magnitudes is a necessary precursor in order to arrive at conclusions such as those described in the sixth observation.

The recognition of the continuous linear variation represented by the equation $\frac{OA}{AA'} = k$ when A varies requires an interpretation of the slope of the straight line such as is made in the sixth observation.

The possibility of representing lines using the Cartesian plane with algebraic expressions (linear equations) requires mastery of continuous linear variation. This is essential for an understanding of the possibility of having two related continuous variables, such as the ones that are described by equations like $y = kx$ (the x and y that vary in direct proportion), and the fact that continuous linear variation can therefore be represented as straight lines, by translation to analytic geometry.

4.4. The distinction between meaning and sense exemplified again

Here we anticipate what appears in Section 3.1 of Chapter 8, because for the students involved in this study sense had to be provided in the new MSS by the use of new signs in the ways that were required by each of the steps in the process of analysis and solution, as we said in that section.

In this study, students who were not yet competent in the use of the new MSS, corresponding to other strata of the diagnostic classification, were able to accomplish the linking of the various steps required for the solution, because they remembered the sequence presented by the teaching model, but most of the time they did so *without sense*. It was only when they had the senses with which they were provided by the concrete proof of Thales' theorem that the concept of the slope of a straight line acquired stability. Then they were able to set in motion the processes of abstraction in this study too.

5. COGNITIVE TENDENCIES REVISITED

At the beginning of this chapter, we made a list of cognitive tendencies that were observed when a learner was in the process of making himself competent in the use of a more abstract MSS than those used in the Teaching Model. We will make use of the list again in Chapter 8 when we provide an example to illustrate the dialectic process between the meaning of certain concepts and the sense of the processes used to acquire a competent use of those concepts.

Here we use the same list of cognitive tendencies and relate them to the various steps and episodes that we described in Section 3 of this chapter when we analyzed the interviews described in Chapters 4 and 6. We will also add what we obtained in the study concerning the notion of the slope of a straight line, which we have just described in Section 4.

5.1. The presence of a process of abbreviation of concrete texts in order to be able to produce new rules of syntax

Consider Episode 1, and especially Episode 4 and Comment 3. See also Chapter 4 and Step H.

5.2. The production of intermediate senses

See Comment 2, the analysis of Step L made in Comment 3, and the process described in Step O. See also Section 3.4 of Chapter 8.

5.3. The return to more concrete situations when an analysis situation presents itself

This fact is always present in most of the actions of mathematical thinking and has been reported in many other studies (Filloy and Rojano 1984, 1985a, 1985b, 1989). In the description that appears earlier, it can be seen in Step F and especially in Step J, where there is a return to using part of the concrete model that had already been discarded in previous steps. The return to a more concrete situation is also observed in Step M, and in Section 3.4 of Chapter 8.

5.4. The impossibility of setting in motion operations that could be performed a few moments before

See the Introduction (Chapter 1), which contains a description of behavior of this kind when trying to solve the equation $Ax = B$. In Section 4 of this chapter the operation of fractions is inhibited by the presence of mistaken spontaneous readings of a geometric nature concerning the notions of ratio and proportion of magnitudes. See also Step F, in which, when arithmetic equations ($Ax = B$), the manipulative skills of which were totally mastered by the whole population, were presented in the sequence of steps described earlier, most of the students lost their great operational ability to solve such equations.

5.5. Getting stuck in readings made in language levels that will not allow the problem situation to be solved

See once again, in the reverse of multiplication syndrome, the observation about the performance of 12- to 13-year-old students when trying to solve problem situations based on the solution of the equation $Ax = B$, which we have already discussed in the Introduction (Chapter 1). In Section 4 of this chapter there is also an example of getting stuck in the mistaken geometric reading about the order of magnitude between ratios of magnitudes. Also, in Section 3 of this chapter this behavior can be found in Steps F and I, especially as a result of what is stated in Comment 2 about the dependence of sense on the concrete context in which the learner gets stuck. See also Section 3.4 of Chapter 8.

5.6. The articulation of mistaken generalizations

See what is stated in the Chapter 1.

5.7. The presence of calling mechanisms that cause the learner to get stuck in setting in motion mistaken solving processes

In many cases, some learners cannot properly resolve what is described in Step F as a result of this behavior. For example, as we mentioned in the

Presentation chapter, when they try to find the side of a rectangle for which they know the area and the measurement of the base by using trial and error instead of using the operation of division. Many of the phenomena corresponding to cognitive tendency 9 are also due to this behavior.

5.8. The presence of inhibiting mechanisms

See what is stated in the Presentation chapter and in Section 3.4 of Chapter 8.

5.9. The effect of obstructions derived from semantics on syntax and viceversa

See what is stated in the Presentation chapter and in Section 3.4 of Chapter 8.

5.10. The generation of syntactic errors due to the production of intermediate personal codes in order to produce senses for intermediate concrete actions

Consider Step I and Comment 3. In the Introduction (Chapter 1) and chapter 4 there is also a description of this cognitive tendency, in which one can see how the production of personal codes can generate mistakes of syntax.

5.11. The need to produce senses for increasingly abstract networks of actions until they become operations

All the steps in the clinical interview combine to provide an example of this assertion. See also Section 3.4 of Chapter 8.

SUMMARY

This chapter summarizes the two preceding chapters and advances toward the notion of *sense* within a *mathematical sign system*. We begin by illustrating the ideas through the example of a proof of the Thales Theorem. Here we shall explain what the *Greek model* and the *Egyptian model* consist of for rational numbers. We then continue to explore the relationship of abstraction processes with the theoretical notions of *meaning* and *sense*, both for the case of solving linear equations and of proportional variation and the Thales Theorem. The presentation of the foregoing cases is based on the protocols of clinical interviews described by episodes and in terms of theoretical nature such as the *mathematical sign system*, *language strata* and *mathematical texts*. We finally recap the material presented in the chapter and identify a series of cognitive tendencies in the interview episodes, which is presented in the form of a table of correspondences (between episodes and specific cognitive trends).

In the next chapter we further exemplify these cognitive tendencies in the context of research on a more advanced level of the use of algebraic sign systems. We advance also in the theoretical explanation of the role of the competence model in the description of the teaching model as a sequence of texts, and in the distinction between meaning and sense.

ENDNOTES

[1] See an earlier discussion on this issue In Filloy (1993b).

CHAPTER 8

MATHEMATICAL SIGN SYSTEMS. MEANING AND SENSE

OVERVIEW

The concepts of meaning and sense are central here, as in any semiotic treatment of algebraic language. We use these concepts to analyze the relationship between the theoretical approach and the commitment to the transformation of mathematical practices in the classroom. In other words, meaning and sense are related to the comparison between models of formal competence and teaching models. Once again we use cases developed to illustrate this relationship: the case of Thales' Theorem, presented in Chapter 7, and a new one introduced in this chapter, the case of the methods for solving systems of two equations with two unknowns, in which it is necessary to master algebraic substitution and the comparison of algebraic expressions. In this chapter we attach importance not only to teaching models but also to the role of the teacher as an active agent in the processes of communication and signification in the math classroom.

1. INTRODUCTION

Theoretical developments in research on mathematics education need to go hand in hand with working with teachers. We now explore one way in which this synergy can be achieved.

1.1. Research and working with teachers

On the basis of experimental results that are already known, work groups set up by teachers and researchers conceive new ideas for curriculum design and development, while at the same time new problems emerge that deserve the attention of future experimental research of the kind described in Chapter 3. Experience shows that total mastery by the teachers of the theoretical framework proposed is not necessary for experimental research of this kind to be rich in results that are useful for them and for dissemination of the results among other teachers to be possible.

It is fortunate that this is the case, because otherwise we would be in an impasse in which the teacher, to be able to make use of the research outcomes, would have to become an expert in the bodies of knowledge that provide the basis for the theoretical frameworks (psychology, semiotics, etc.), and would also have to master the processing and interpretation of the data, which among other things would include advanced techniques for data analysis or processing of the observations carried out. This can be expected of only a minority of teachers in the existing systems of education, and therefore the research results would lack a concrete referent where they could be applied —they would be discourses devoid of content and totally ideologized.

Nevertheless, we are certainly faced with the paradox of trying to find theoretical frameworks that are increasingly well supported by precise notions, described in increasingly sophisticated metalanguages, with increasingly powerful methodologies for processing information; but at the same time also trying to transform the teaching practices that are used in classrooms in the existing education systems.

Finally, to be able to use, in all its richness and variety, what worldwide research has achieved in recent years it is necessary to create a new field in curriculum design and development (to be used in the controlled teaching mentioned in diagram B, presented in Chapter 3), belonging to this new branch of knowledge which aspires, on a theoretical level, to use all the forms of knowledge that can be brought into play, but which, on the other hand, finds its full justification in the possibility of the transformation of the systems of education. In Chapter 10 we propose that this should be developed in the future.

In Chapter 10 we explore paths by means of which research on mathematics education can progress in the practice of transforming the existing systems of education, which makes its theoretical output meaningful. We now analyze experimental research, the practice most commonly used for impacting on the reproduction of theoretical knowledge and the expansion of its gnoseo-territory within the episteme, but not forgetting to mention that to this practice we must add others that are necessary to it and consubstantial with it: publishing output, research groups, conferences, seminars, symposiums, etc.

Reference to diagrams A and B in Chapter 3 will enable us to shorten the analysis. The researcher has a meeting with the teachers to discuss a problem area (box 1 in diagram A) that is of interest to all. The team's experience is used to put forward a preliminary plan that defines the lines of work required so that an initial analysis of the problems can be carried out (box 2 in diagram A); after some time this preliminary plan should become a project. At that point the definitive design should already be clear (box 5 in diagram A) and trials should have been made of the instruments of observation, classification, and measurement and the curriculum development that will be used during the study of a system of controlled teaching (diagram B).

Once a plan exists in which the problem area has been analyzed by structuring an ad hoc local theoretical model, the steps in diagram B are organized to disprove or add knowledge that will make it possible to verify the theses of the local theoretical model (LTM), and to refine and expand it to allow future studies, as shown in the final box in diagram B.

A report on the observations is produced and a new LTM is designed, and writings are prepared in which all the theses, empirical results, and their interpretation are published in the form of articles or books.

1.2. Proof and the formal model

We begin this section with an analysis of how certain results attributed to the Greek mathematical philosopher Thales may have been proved.[1]

1.2.1. What Thales may have been able to prove

Thales is the first person who is credited with having discovered various mathematical results for himself. Among them are the following geometric results:

[1] A diameter bisects a circle.
[2] The angles at the base of an isosceles triangle are equal.
[3] The opposite angles at the vertex where two lines intersect are equal.
[4] Two triangles are congruent if they have one equal side and two equal angles.
[5] The angle inscribed in a semicircle is a right angle. (This result was already known to the Babylonians some 1400 years previously. Later we will refer to it as Theorem C.)
[6] The Fundamental Theorem of Geometric Proportionality for commensurable magnitudes, which the pedagogical tradition of mathematics calls “Thales’ Theorem.” We have presented a version of this theorem in Chapter 7.

Some of these results must have been known considerably earlier, and of some it is simply said that they were stated by Thales. The important thing here is the belief that Thales used logical reasoning in order to show that these assertions are true, and he did not do so on the basis of intuition, experimentation, and repeated verification, as had been done until then. In any case, what is certainly true is that, whether Thales did so or not, a little later the most ancient Pythagoreans developed mathematics in a deductive manner.

We now analyze proof in mathematics, through a discussion about the possibility that Thales did actually prove that the angle in a semicircle is a right angle (result [5] above).

If Thales had known that the angle in a semicircle is a right angle, he could have proved Theorem T: "the sum of the three internal angles of any right-angled triangle is equal to two right angles".

Suppose that *BC* is the diameter of a semicircle, *O* is the center and *A* a point on the semicircumference (Figure 8.1).

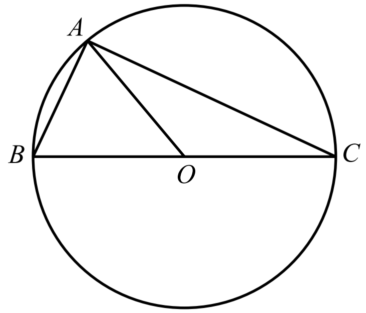


Figure 8.1

Our reasoning is as follows:

Thales knows [5], and therefore

angle *BAC* is a right angle.

If we draw *OA*, two isosceles triangles are formed, *OAB* and *OAC* (in both of which two of the sides are equal to the radius).

Thales knows [2], and therefore

the angles at the base of these triangles are equal.

In other words,

angle *OCA* = angle *OAC* and angle *OBA* = angle *BAO* [A]

Therefore the sum of angles *OCA* and *OBA* is equal to the sum of angles *OAC* and *BAO*, and by using [5] Thales concludes that this sum is a right angle [C].

By using [A] and [C] Thales can conclude that

the sum of the three angles of the triangle *ABC* is equal to two right angles.

This we will call Theorem [T].

It is also easy to see that any triangle can be divided into two right-angled triangles, as Figure 8.2 shows, and hence we can conclude that the sum of the internal angles of any triangle is equal to two right angles [T].

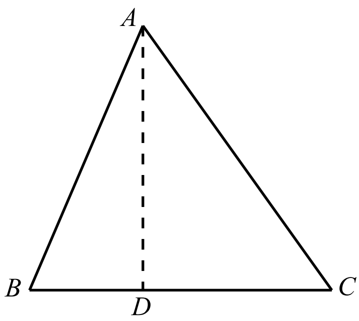


Figure 8.2

If we analyze what we have just written, we see that, on the basis of knowing [2] and [5], Thales could have concluded another geometric result, [C], by means of logical arguments, and he could then have obtained [T] by the same kind of argument. This is an example of what deduction in mathematics means.

There is something that immediately leaps to the eye as a result of this analysis: [A] and [5] could have been obtained by means of experimentation, that is, they could have been verified as valid in repeated observations made with triangles that had been drawn or found in nature. In fact, it is possible that Thales obtained them in this way. However, [C] or [T] are established by means of deduction, by the use of certain simple arguments, certain processes of reasoning that we describe as logical, which apparently have nothing to do with direct experimentation performed by using our senses. Yet when the ancient Greeks took [C] or [T] and experimented with real triangles, they actually came to the conclusion that [C] and [T] were true: in any triangle that they drew, the correctness of [C] and [T] was verified (by means of their experimental procedures).

1.2.2. Which of the two propositions came first, [C] or [T]?

Let's go back to our discussion about the possibility that Thales may have proved that the angle subtended by a diameter is a right angle [5]. We have seen that this might have led him to conclude, on the basis of simple reasoning, that the sum of the angles inside a triangle is equal to two right angles [T]. But if we look at the proof of [5] in Euclid's *Elements*, we find that Euclid used [T] in order to demonstrate it.

Euclid knows [T]:

$$\text{angle } ABC + \text{angle } BCA + \text{angle } CAB = 2 \text{ right angles.}$$

Euclid knows [A]:

$$\text{angle } OCA = \text{angle } OAC \text{ and angle } OBA = \text{angle } BAO.$$

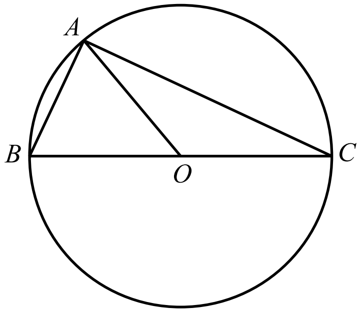


Figure 8.3

And as

$$\text{angle } BAC = \text{angle } BAO + \text{angle } OAC,$$

he concludes that

$$\text{angle } BAC = \text{angle } ABC + \text{angle } BCA,$$

which, by substituting in [T], gives

$$2 \text{ times angle } BAC = 2 \text{ right angles},$$

in other words,

$$\text{angle } BAC = 1 \text{ right angle}.$$

This is a case of the equivalence of two results. If we know [5], we can prove [T]. If we know [T], we can prove [5] (in both cases by using logical reasoning and other results such as [A], for example).

1.2.3. Proof in mathematics

Bearing in mind what we have just written, we teach students to construct proofs in Euclidean geometry by using various complementary proof techniques. But, even though we might consider them as different proofs in our day-to-day work, in Wittgenstein's conception, which we introduced in Chapter 5, what we have here, strictly speaking, is the same proof, but dressed up in two different ways. What matters is the *grammar* of the proof and not its style.

The important thing is that, from the viewpoint of the internal relations between the concepts, essentially it is the same proof. That is, in terms of the problem posed in the previous section, the two proofs are the same in the sense that, whether we follow one route or the other, in either case we will be using the same rules. To conclude, a mathematical proof articulates the internal network of relations that make a mathematical proposition a rule of syntax.

The essence of a mathematical proof, according to Wittgenstein, is that in itself it produces a grammatical (logical) rule. Thus the expression, the result, our conviction of its establishment, is the fact that we adopt it as a rule. It is worth repeating that what matters here is the internal relations that are established between various concepts with the acceptance of the new result: "the proof changes the grammar of our language, changes our concepts. It makes new connexions, and it creates the concept of these connexions." (Wittgenstein, 1956, III, 31).

1.3. The formal competence model and the teaching model

When one is implementing a curriculum for use in controlled teaching (start of diagram B, Chapter 3) in an experimental study, the sequence of texts that constitute the teaching model (as established in Chapter 5) is outlined – in order thus to be able to produce sense – by the formal competence model that has been adopted so as to be able to carry out the observation. Remember that the formal model provides the observer with a more abstract MSS which encompasses all the MSSs used in the process observed (Chapter 2, Section 2.5). The sense produced by the sequence of texts in the teaching model (1) changes our language (making us competent in the use of a more abstract MSS); (2) changes our concepts; (3) creates new connections; and (4) creates the concept of those new connections.

Let's give an example to make this clearer. We will use an LTM to observe the use of an MSS that, as a stratum, forms part of almost any MSS. We are referring to competence in the use of natural numbers and their basic operations.

There is a well-known controversy between those who declare that order is constructed from cardinality and those who maintain the opposite. Underlying this are the various positions on the matter in the philosophy of mathematics and in the foundations of mathematics.

The construction by Von Neumann[2] starts with a definition of natural number that sets out from the properties of order, which enables him subsequently to call on the properties of finite induction and thence define the notion of counting, cardinality, and the operations of arithmetic. A local theoretical model designed in order to observe interactions and contrasts in competences in the use of the notions just indicated would require an observer who has competences in the use of an MSS with which he could decode exchanges of messages at the point when someone (a teacher) with a good level of use (competent performance) of numeration and its operations tried to get a learner to attain that level of competence.

Note that the teaching model to be used in the LTM proposed for the carrying out of the experimental observation will determine the formal approach that is useful: one along the lines of Frege, Von Neumann, Zermelo, Cantor, Dedekind, Poincaré, Couturat, Peano, Hilbert or perhaps someone else.[3]

In order to make the foregoing paragraph clearer, let's outline the steps that would have to be covered by controlled teaching that aimed to introduce natural numbers, counting and the elementary operations, and in which one wished to use the notion of order (between numbers) as early as possible:

1. Introduction of the first numbers $0 = \emptyset$, $1 = \{\emptyset\}$, $2 = \{\emptyset, \{\emptyset\}\}$, $3 = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$, etc.
2. Definition of $\in$-ordered set.
3. Definition of natural number, taking those of [1] as examples.
4. Introduction of the principle of finite induction.

5. Definition of counting and various properties.
6. Introduction of the operation of addition and its properties.
7. Introduction of the operation of multiplication.

We will not continue here with the more concrete version that could be used to work on the sequence of texts: our aim is to speak of algebra and various ways of teaching it with the use of concrete models, such as the one introduced in Chapter 4 for teaching how to solve linear equations, which is studied in the following chapter.

A further example appears in Chapter 7, in which we discuss a teaching model that uses straight staircases whose organizing principle is the Fundamental Theorem of Geometric Proportionality, which teaching tradition attributes to Thales and the statement of which is: "If a straight line is parallel to one of the sides of a triangle, then it cuts the other two sides in proportional segments, and, inversely, if a straight line divides two of the sides of a triangle into proportional parts, then it is parallel to the third side."

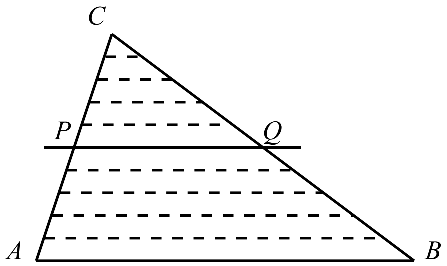


Figure 8.4

2. SIGNIFICATION AND COMMUNICATION

2.1. Sources of meaning of MSSs

The notion of MSS that is used to interpret observations in mathematics education must be broad enough to accomplish the tasks enumerated in the preceding chapters, and it must be accompanied by a notion of the meaning of sign that covers both the formal meaning of mathematics and its pragmatic meaning.

Furthermore, the notion of MSS that is used must be efficient enough to deal with a theory of production of MSSs in which one is working with the systems of intermediate signs that the learner uses in teaching/learning processes. During these processes it is necessary to correct the use of those intermediate MSSs so that by the end of the teaching process the learner will be competent in the desired MSS, which is the educational aim of any teaching model.

Being idiosyncratic, some of the intermediate MSSs cannot be considered as MSSs owing to the personal character of the codes invented by the learner, which would not enable him to use that sign system in a broad process of communication because of the lack of an agreed social convention about the code. But as we are also dealing with observation of these processes of mathematical thinking, we must be prepared to study those sign systems and interpret the learner's personal codes in order to discover the obstructions that are created by the tension of dealing with the various MSSs available to the user when he is trying to become competent in the use of a new MSS and achieve a good performance in terms of its socially determined pragmatic meaning.

Any explanatory LTM must deal with four types of sources of meaning (see Kaput [1987, 1989] with respect to the first three):

1. Of transformations within an MSS without reference to another MSS.
2. Of translations across different MSSs.
3. Of translations between MSSs and non-mathematical sign systems — such as natural language or texts produced with visual images— and the sign systems used by those observed during teaching/learning processes that enable us to observe the cognitive processes of the learner and, from those psychological results, propose new hypotheses for an educational mathematics analysis of the teaching models involved in the experimental design of the LTM being studied.
4. With the consolidation, simplification, generalization and correction of actions, procedures, and concepts of the intermediate MSSs created during the development of the teaching sequences proposed by the teaching component of the LTM being studied, these intermediate MSSs evolve toward a new, more abstract MSS in which there will be new actions, procedures, and concepts that will have as referents all the actions, procedures, and concepts belonging to the intermediate MSSs for use in new processes of signification. If the aims of the teach ing model are achieved, the new stage will have a higher level of organization and will represent a corresponding new stage in the cognitive development of the learner.

2.2. Meaning and sense

In the processes described in the previous section, there is a need not only to give meaning to the signs of the MSS, but also to produce sense for the new expressions and the operations required in order to use them. One way of producing sense for them comes through the *process of verification*: for example, in order to give a new meaning to the term equality in algebraic equations, seeing them as being the equations in which it is possible to perform a series of operations in such a way as to obtain a value for the unknown, and, when this value is substituted in the left side of the equation and the operations indicated are performed and the same is done in the other side of the equation, the results coincide.

As an example, see the behavior of Mt with item 26 in Subseries I-g, which is described in Chapter 4.

2.3. The production of MSSs

Whereas the first three sources of sign-functors (translations, in Kaput's terminology) represent means of dealing with primitive expressions and ways of combining them (see Kaput, 1987, 1989), the fourth represents processes of abstraction and generalization by means of which compound signs can be named and manipulated as units and subsequently used in processes of signification to solve the new problem-solving situations with which the learner is confronted in the teaching sequences of the teaching component of the LTM. If, as in this case, we have to deal with learning/teaching processes, there is no way of avoiding these processes of abstraction and generalization as our main focus of observation, and we need a theory of the production of MSSs in which an abstraction-functor relates the various strata of intermediate MSSs (used during the development of the teaching sequences) to the final, more abstract MSS (the aim of the LTM being studied). Subsequently, an analysis in mathematics education would interpret that evidence in order to propose new hypotheses that will have to be observed by means of the appropriate methodological procedures, by designing a new LTM.

2.4. Signification and communication

To interpret observations in mathematics education it is necessary to develop the theoretical possibility of a unified approach for phenomena of mathematical signification and communication. Let's make a preliminary exploration of the theoretical possibilities with a view to raising a few queries.

We will give the name of general mathematical semiotic theory to the approach in question, capable of explaining sign-functors in terms of categories that underlie the sign systems correlated by one or more codes. This design can contain at least two different fields: (1) a theory of codes for MSSs, and (2) a theory of production of MSSs. In all this one must take into account a broad range of phenomena such as the common use of MSSs, the evolution of their codes, different kinds of communicative conduct in mathematical interaction,[4] the use of MSSs for models of things or states of the world or some other theoretical sign systems, such as those used in biology, physics, etc.

These notions enable us to distinguish an MSS from other sign systems, and to start to develop a notion of sign-functor that can be explained within the same theory of codes. This allows a distinction between "signification" and "communication": in principle, a semiotics of signification involves a theory of MSS codes, whereas a semiotics of mathematical communication involves a theory of the production of MSSs. The theory of codes developed for these purposes will take into account rules of competence, formation of texts, and contextual and circumstantial disambiguation, and will propose a semantics that, within its own framework, solves the problems posed by a pragmatics of the use of MSSs. In Chapter 10 we propose various hypotheses for studying the use of the communication component of an LTM (see also the use that we give in the example at the end of this chapter).

Our semiotic theory has the capability of offering appropriate formal definitions of any kind of mathematical sign-functor, whether or not they have been described or coded by the user. Thus, the typology of modes of production of MSSs aims to propose categories capable of describing even those situations in which sign-functors that are not totally coded are present (conventionally considered by the teaching model) when they are being constructed through teaching processes.

2.5. Mathematical texts

We have to face the fact that an MSS (with its corresponding code) exists when there is a socially conventionalized possibility of generating functional signs (by the use of a sign-functor), whether the domain of the functions is discrete units called mathematical signs or vast portions of discourse (which in Chapter 5 we called mathematical texts), in which a mixed concatenation of signs is produced. In this concatenation process signs are used that come from different sign systems (including natural language and the learner's personal sign system which we mentioned earlier), assuming that the functional correlation of the textual space (Chapter 5) has previously been posited by a social convention – even in ephemeral cases, such as the didactic sign systems that appear during the intermediate steps in the sequences of certain teaching models (balance scales, sets of objects, spreadsheets, Logo environments, diagrams, etc.), which will subsequently have to be set aside, in the future use of the MSS, by the same process of abstraction.

On the other hand, there will be a process of communication when one uses the possibilities provided by an MSS for the physical production of expressions for many practical purposes –in particular, the meaningful manipulation of objects is a mathematical text.

These processes of performance require processes of signification, the rules (discursive competence) of which must be taken into account by the cognitive (theoretical) component of the production of mathematical signs when they have already been coded, because, as we mentioned in Chapter 5, we are going to be interested in observing how new competences are acquired by the user with the expansion of those intermediate MSSs to other new ones that contain them.

In Chapter 5 we gave a definition of a teaching model as a sequence of problem situations, a sequence of mathematical texts T_n, the production and decoding of which by the learner eventually enables him to interpret all the texts T_n in a more abstract MSS. We are now going to examine the relations between meanings in an MSS and the sense provided by the sequences of texts T_n at the point when a more abstract MSS is produced. In Section 4 of this chapter we provide a succinct presentation of the results of an empirical study that makes use of the cognitive tendencies mentioned in Chapter 7, and also the production of sense, by the introduction of two different methods of solution, which lead to two different meanings of the equals sign when one is learning for the first time to solve systems of two equations and the corresponding word problems.

3. MEANING AND SENSE

3.1. The sense of a succession of texts

In a new MSS, sense is provided by the use of new signs in the ways in which they are required by each of the steps in the process of analysis and solution, in view of the whole sign system associated by the concatenation of actions set in motion by the process of solving the various problem situations which were previously considered irreducible to one another. Now these situations, thanks to the use of the new MSS, are solved with processes that are established as being the same, that is, processes that are transferred from the solution of one problem to another, converting what was previously a diversity of problems into what can now be called a family of problems, *all* the members of which can be solved by using the *same process*.

Learners who are not yet competent in the use of the new MSS may perform the linking of the various steps required by the solution because they remember the sequence proposed by a teaching model, but generally they do so *without sense*, and at any small obstruction or variation in the problem situation they revert to using propositions that they had previously recognized as mistaken. Only when they are in possession of the senses that the sequence of texts T_n (provided by the teaching model) gives them, only then will the new concepts acquire stability. These senses (of use) provide the new MSS with more abstract signs, because such signs have as referents signs of a greater quantity of MSS strata, related to one another.

3.2. Teaching models and stratified MSSs

What we use for thinking mathematically and for communicating what we think to others is a collection of stratified MSSs whose codes are interrelated in such a way that it is possible to produce texts for the decoding of which it is necessary to make use of several of those strata. Moreover, the production of text0s uses actions, procedures, and concepts whose properties are described in one or another of the strata.

Two texts, T and T', both products of the use of an MSS, L, will be called transversal when the user cannot create T in the same way as in the decoding of T' – that is, if T is not reducible to T' by the use of L. What really happens is that the learner can produce T and T' but cannot recognize the two decodings as a product of the use of the same actions, procedures, and concepts of the various strata of L.

If we now have another stratified MSS, M, in which T and T' can be coded and the production of both can be described by means of the same actions, procedures, and concepts in M, the meaning of which has as referents the actions, procedures, and concepts used in the decoding of T and T' in the strata of L, then we will say that M is a more abstract stratified MSS than L for T and T'.

In order to accomplish this, the objects, actions, procedures, and concepts used in M have lost part of their semantic-pragmatic meaning: they are more abstract.

This brief description of how we can define the abstraction-functor enabled us in Chapter 5 to give a definition of a Teaching Model as a set of sequences of mathematical texts T_n whose production and decoding by the learner eventually enables him to interpret all the texts T_n in a more abstract MSS whose code makes it possible to decode the texts T_n as messages with a socially well-established mathematical code, the one posited by the educational aims of the teaching model.

The analysis of how the competences necessary for developing these processes of decoding the text sequences T_n are acquired by the learner in order to become a competent user of the MSS F (i.e., the MSS described in the formal competence component) is part of an educational mathematics study of the teaching model that must take into account the cognitive processes described by means of the cognitive component of the LTM being studied. We now give an example of an empirical study that, although in abbreviated form, we think will help to make the distinction between meaning and sense clearer.

4. TWO MEANINGS OF THE EQUALS SIGN AND THE SENSES OF THE METHODS OF COMPARISON AND SUBSTITUTION

4.1. The methods of comparison and substitution and the equals sign

In this section we analyze the meanings of the equals sign generated by the production of the sense of the methods of algebraic comparison and substitution when solving problems with two unknowns and systems of equations. To do so we will use parts of an experiment carried out with 13- to 14-year-old children, providing a brief description of the components of its LTM. The introduction of these methods was effected in the teaching model by means of a process of extension of the syntax and significations just learned for the solving of problems of linear equations with one unknown. By this process some users were able to give sense to the methods and they thus generated the new meanings required.

In Chapter 7 we used the notions of *meaning* and *sense* for the analysis of the processes of learning and of the creation of rules that make it possible to coordinate the actions carried out in the solving of problems with one unknown by means of concrete models (see also Filloy, 1991; Filloy and Rojano, 2001; Filloy, Rojano and Solares, 2002). We now make use of these notions to study the transition from the representation and manipulation of one unknown to the representation and manipulation of an unknown that is given in terms of another unknown. This transition actually corresponds to a *didactic cut* (see Chapters 4 and 6). The new representation of what is not known is used in the methods of comparison and substitution in such a way as to make it possible to reduce a problem with two unknowns to a problem with just one unknown, and then the syntax learned previously can be applied to solve linear equations with one unknown.

In the special case of system S_1, the method of *comparison* consists in comparing two sequences of operations on one of the unknowns and on the given values, which make it possible to calculate the value of the other unknown. Consequently, two ways of calculating the value of one of the unknowns are equated.

$$S_1 : \begin{cases} y = 12 - x \\ 5x - 6 = y \end{cases}$$

In the case of system S_2, the method of *algebraic substitution* consists in introducing into the first equation the y in the second equation given by the sequence of operations, which makes it possible to find its value. In this way, one sequence of operations is substituted into another sequence of operations.

$$S_2 : \begin{cases} x + y = 12 \\ 5x - 6 = y \end{cases}$$

The study was carried out with students aged 13 or 14. For them, the *sense* of these methods was given by the concatenation of all the actions performed. Indeed, at the beginning of the learning process these sequences of actions were not yet endowed with sense. Both the increase in the syntactic complexity of the relations between the data and the unknowns and the variations in the numeric domains of the data or the solutions could obstruct the applications of the methods and the spontaneous solving strategies. At that point, a reading based on the more concrete strata of the new MSS did not allow the student to identify the variations in the problem situation as members of the same family of problems. Only by the acquisition of the sense provided by the sequence of mathematical texts in the teaching model could they be identified as members of the same family of problems, capable of being solved by the same process or sequence of actions. Then the new notions and the new notion of equality attained stability. Earlier results on the meanings of equality are described in Matz (1982), Kieran (1981), Kieran and Sfard (1999), and Drouhard (1992).

In Chapter 4, Section 5.3.2, we have seen how one of the students interviewed in another study generated what is described in the literature as the "algebraic" meaning of the equals sign, by solving the equation $10x - 18 = 4x + 6$.

> Mt: […] if I take the value of x and do this operation [pointing to $10x - 18$], I get a result. That result has to be the same as this [pointing to $4x + 6$].

4.2. An LTM for the methods of algebraic comparison and substitution

The study was carried out with 12 children at the Centro Escolar Hermanos Revueltas in Mexico City during the academic year 1987–88. Clinical interviews were conducted and videotaped. The children had already received instruction in pre-algebra and had been introduced to elementary algebra through the solving of linear equations with one unknown and word problems connected with those equations, but the teaching had not yet introduced them to the systematic use of open algebraic expressions or to systems of linear equations.

Diagrams A and B in Chapter 3 describe the development of the study. We now give a brief description of the components of competence, teaching, and cognitive processes in the LTM. In Section 4.4.3 we mention the communication component.

4.2.1. The formal competence model

We created the formal competence component in the LTM from the syntax model for simple algebraic expressions and equations developed by Kirshner (1987) and completed by Drouhard (1992). We also incorporated the elements of semantics proposed by Drouhard (1992) for studying the significations of algebraic writings. However, although those studies on algebraic syntax and semantics produced important results for teaching, such as Drouhard's definition of *formal automaton* for learners who concentrate their attention on the rules that have to be applied (*sens* in Drouhard's sense) and not on the truth value of the results obtained (*dénotation* in Drouhard's sense), they did not include in the analysis the spontaneous use that learners make of the elements of algebraic language that they already possess in order to solve the new problems.

This formal competence model enabled us to study the syntactic complexity of the methods of algebraic substitution and comparison for the solving of systems of equations. We will not expound that study here but will simply say that from that study it transpired that the method of substitution is more complex than that of comparison.

4.2.2. Teaching model

On the basis of the analysis carried out in the formal competence component we adopted the following *teaching route* for the introduction of the methods, setting out from the competences previously acquired for the solution of linear equations with one unknown: (1) Reduction of the system of two equations with two unknowns to one equation with one unknown by applying algebraic comparison or substitution. (2) Solution of the equation with one unknown by applying the syntax previously learned. (3) Substitution of the numeric value found in either of the two equations. (4) Solution of the equation obtained by applying the syntax previously learned.

4.2.3. Cognitive processes model

For the definition of the cognitive processes model we used the list of *cognitive tendencies* presented in Chapter 7. The ones that were particularly important in the present study are numbered as follows in that chapter: (2) the production of intermediate senses; (3) the return to more concrete situations when an analysis situation presents itself; (5) readings made at language levels that will not allow the problem situation to be solved; (8) the presence of inhibiting mechanisms; (9) the effect of obstructions derived from semantics on syntax and viceversa; (11) the need to produce senses for increasingly abstract networks of actions so that they become operations.

4.3. Items in the interview

The list of items is divided into two sections: word problems and purely syntactic tasks. The list of items presented to each student was allowed to vary, depending on the cognitive tendencies encountered during the interview.

Nine word problems were presented. Here are three examples:

The sum of two numbers is 90. If one of them is added to 16 and then multiplied by 5, the result is 360. What are the numbers?
The difference between two numbers is 27. We know that 7 times the smaller number plus 30 gives 12 times the same number (the smaller one). What are the numbers?
The perimeter of a rectangle is 5 times the width. The length is 12 meters. What is the width?

The syntax questions were:

S.1	$x + 2 = 4$ $x + y = 8$	S.8	$x + y = 60$ $3x = 171$	S.14	$4 \times (3 - x) = 4$ $x + y = 13$
S.2	$x + y = 10$ $x - y = 4$	S.9	$2 \times (x + 6) = 84$ $x + y = 104$	S.15	$x - y = 1$ $x + y = 5$
S.3	$x + y = 9$ $2x + 3y = 23$	S.10	$4 \times (x - 8) = 72$ $x + y = 17$	S.16	$y - 6 = 3x + 20$ $5y - 4x = 64$
S.4	$x + y = 12$ $5x - 6 = y$	S.11	$3x + 4 = 22$ $4x + 2y = 34$	S.17	$3x + 8y = 84$ $8x + 3y = 59$
S.5	$x + 33 = 48$ $x + y = 73$	S.12	$3 \times (8 + x) = 6$ $2x + y = 23$	S.18	$4x - 3 = y$ $6x = y - 7$
S.6	$14 + x = 37$ $4 - y = 28$	S.13	$2 \times (3 - x) = 6$ $4x + 3y = 12$	S.19	$3x - 2 = y$ $5x = y + 8$
S.7	$45 - x = 17$ $x + y = 41$				

4.4. The empirical study. Observations

We now present a few extracts from the analysis of the interviews in two cases from the upper stratum, Mn and Mt, and one case from the middle stratum, L. From these cases it is possible to describe different ways of giving meaning to the new algebraic objects, the ones obtained by the production of the sense of the methods of comparison and substitution.

4.4.1. Trial and error

This appeared as a spontaneous solving strategy in all the cases analyzed and it was connected with the spontaneous readings that the learners made of the first systems of equations with two unknowns. This is the trial–and–error strategy used by L (cognitive tendencies 2 and 3).

S L 3 $x + y = 9$ $2x + 3y = 23$	L writes 0 + 9 1 + 8 2 + 7 3 + 6 4 + 5 5 + 4 He crosses out the last line. Then he points to each line, starting with the first. He stops at 4 + 5 and writes: 4 5 8 15 23 L: The numbers are 4 and 5. I: Four and five? L: Yes. The first thing I did was to find the possible sums that come to 9, and then, by trial and error, 2 times 1 is 2 [pointing to 1 + 8] … and see if they add up.	L interprets this pair of equations as being "linked," i.e., equations in which the x and the y have the same value. L writes down the various ways of obtaining 9 by adding two positive integers, then he performs the operations on the unknowns indicated in the second equation until he finds the values of the unknowns with which he can obtain 23.

As we will see later, these spontaneous readings and strategies may become obstructions for the learning of general methods of solution such as comparison and substitution (cognitive tendencies 5, 8, 9).

4.4.2. The difficulties in producing sense for comparison and substitution and for the meaning of the equals sign

There are two obstructions in the application of the method of algebraic comparison: the reading of the objects (unknowns and data) and of the operations in the context of positive whole numbers, and the lack of knowledge necessary for establishing the new equality, that is, algebraic equivalence (tendencies 2, 5, 8, 9).

S Mt 18 $4x - 3 = y$ $6x = y - 7$	Mt wants to obtain the value of y. She transforms the system into: $4x - 3 = y$ $6x + 7 = y$ but she does not compare the two expressions, and instead she tries to find the solution among the positive whole numbers by trial and error. Mt: Here [pointing to $4x - 3 = y$] it says that four times x minus three is equal to y, and here [pointing to $6x + 7 = y$] six x … plus seven! … is equal to y. This [pointing to the y in $6x + 7 = y$] must be bigger than this [pointing to the y in $4x - 3 = y$].	Mt is capable of solving equations with one unknown, regardless of the numerical domains of the numbers in the operations and the solutions, and the complexity of the algebraic structure of the equations. Also, she applies comparison in the case of systems of equations derived from word problems in which the same unknown is isolated in the two equations and the solutions are positive integers.

The difficulties encountered in producing sense for the method of algebraic substitution have to do with the readings given to the representation of the unknown in terms of the unknown on various levels of abstraction, and with the *inhibition against using* algebraic substitution. Moreover, as was to be expected from the analysis performed with the formal competence model, we find that the meaning given to the equals sign when it is used to equate two sequences of operations that make it possible to calculate the same value is different from the meaning given to it when algebraic substitution is applied.

4.4.3. Different levels of abstraction: the case of names

The communication model enables us to establish the difference in the readings given by the interviewer and the student. Indeed, when a competent user applies the method of algebraic substitution he makes use of the equivalence of two expressions knowing the sense of the method, knowing that it will lead him to find a unique value for x and y, if there is one. For example, in the system S_2 presented in Section 4.1 a competent user considers the expressions $4x - 3$ and y as being equivalent, for him they are representations or *names* of the same object: the unknown y. But the spontaneous readings made by learners concentrate on the sequence of operations performed in order to calculate the value of the unknown y, as they say when they are asked in the interview. This difference in the meanings attributed to algebraic expressions is present in any process of teaching algebra where the teacher is a competent user and the students are learners, generating difficulties such as those reported here (tendencies 2 and 5).

4.4.4. The criss-cross method

Mn invented a method of solution that, in combination with trial and error, enabled him to solve all the systems of equations that were presented to him. His method of solution, which he called "criss-crosses," consisted in adding the left side of one of the equations to the right side of the other, simplifying by elimination of terms, and, if possible, isolating one unknown in the simplified equation. In some cases, with the criss-cross method one can obtain equations with one unknown. The application of this method maintains the equivalence of the different sides, and also the value of the unknowns. Mn added not only numbers or identical expressions but also complete algebraic expressions.

Here is an example of how Mn solved a system of equations using a combination of the criss-cross method with his strategy of trial and error.

S Mn 17 $3x + 8y = 84$ $8x + 3y = 59$	Mn: I get $8x + 3y + 84 = 3x + 8y + 59$ by mixing the two together. What I get is that if I take this [pointing to the left side of the second equation] and add this [pointing to the right side of the first equation] it will give me the same as if I take this [pointing to the left side of the first equation] and add this [pointing to the right side of the second equation] because the two are equivalent. If I eliminate $3x$ from here [pointing to $8x + 3y + 84 = 3x + 8y + 59$] … well, and at the same time I eliminate $3y$ while I'm at it, I get $5x + 84 = 5y + 59$ … which gives me $5x + 25 = 5y$, from which I deduce … so now, if I divide it all by 5, I get $y = x + 5$. So I deduce that x is 5 less than y … Now let's see. What's the equivalence? How much is x and how much is y. For example, if I make x equal to 4, for the sake of argument … no it can't be 4 because it just can't. If, well, x is 4 and so y is obviously 9, then I get [pointing to $3x + 8y = 84$] 12 [multiplying 3 by 4 in his head] plus 9 times 8, which comes to 72. 12 + 72 = 84, which is right. Now I'll check the other one [pointing to $8x + 3y = 59$], which says that $8x$, which is 32, plus 27 [multiplying 3 by 9 in his head] equals 59. I check it and I see that they're both correct, so I deduce that $x = 4$ and $y = 9$.

Mn has no difficulty with the numerical domains of the equations and systems that he has to solve. His strategies of canceling and simplifying are strongly linked to the extension of the meaning of the equals sign established in an equation. His strategy of trial and error, which in this case is reduced to a minimum, is based on his ability to perform numeric calculations mentally, his mechanisms for anticipating the numeric values that have to be obtained, and his coordination of the actions carried out (cognitive tendency 11).

4.4.5. Inhibition against the use of substitution

The design of the interview was focused on getting the students to use the method of substitution, and therefore, toward the end of the interview, they were provided with one of the unknowns isolated on one side of the equation. However, although Mn had sufficient syntactic competence to generate a comparison method of his own, the criss-cross method, he kept away from the possibility of using the method of substitution (cognitive tendencies 5 and 8).

SUMMARY

The contents of this chapter are eminently of a theoretical nature. The chapter deals with emphasizing the role of the *formal competency model* (one of the components of a *local theoretical model*) in the description of the teaching model as a *succession of texts*. The foregoing is illustrated through a local theoretical model for competency in the use of natural numbers and their basic operations. Under the heading of *signification and communication*, we deal with the topics of *source of meaning* for a *mathematical sign system*; *meaning and sense*; and production of *mathematical sign systems*. Finally in order to make the difference between *meaning* and *sense* more clear, we resort to the clinical study of operating unknowns at a second level of representing unknowns (the context of solving two linear equation systems with two unknowns) in which the *meanings* of the equal sign are related to the *senses* of the comparison and substitution solving methods.

In the next chapter all the theoretical notions developed in the book are applied to the study of algebraic problem solving.

ENDNOTES

[1] No texts by Thales have been conserved but only indirect accounts of his works. Our interpretation of what Thales knew is based on Heath (1921).
[2] See Von Neumann (1923) and compare with Zermelo (1909). An introduction for the use of middle school teachers can be found in Hamilton and Landin (1961). A discussion of a philosophical nature can be found in Benacerraf, "What numbers could not be," included in Benacerraf and Putman, eds. (1983), pp. 272-294.
[3] For other definitions of "natural number" or, more generally, "ordinal number," see Bachmann (1955) and Isbell (1960).
[4] An example of classification of communicative exchange behavior can be seen in Brousseau (1996).

CHAPTER 9

SOLVING ARITHMETIC-ALGEBRAIC PROBLEMS

OVERVIEW

The chapter begins with an Introduction in which we shall refer to the work of other authors who broach different aspects of arithmetic-algebraic problem solving. Those aspects include the arithmetic or algebraic nature of the problems; the relation of that nature to the underlying structure of the word problem and to the processes of translating the problem into a mathematical sign system; and the entrenched nature of arithmetic-type reasoning that may eventually inhibit implementation of algebraic solution strategies or methods.

Afterwards we refer to classical methods, such as the Cartesian Method, and non-traditional methods for solving problems in order to discuss cognitive aspects such as that of the problem of transference, the competent use of the logic-semiotic outline, the strata of mathematical sign systems used as representations, and the use of primitive methods and their relationship with the use of memory.

The sense of the text of a word problem with the use of the *Method of Successive Analytic Inferences* (MSAI) is determined by the logical numeric structure presented in the problem situation. This thesis permeates all others to be discussed subsequently.

We present also results from empirical studies concerning the competencies that are necessary for the use of four teaching models based on four methods for solving arithmetic algebraic word problems: the *Method of Successive Analytical Inferences* (MSAI), the *Analytic Method of Successive Explorations* (AMSE), the *Spreadsheets Method* (SM), and the *Cartesian Method* (CM). We stress the need to be competent in increasingly abstract and general uses of the representations required to attain full competency in the algebraic method par excellence, which here is called the CM, contrasting it with the competencies required by the other three methods, which are rooted more in arithmetic. These methods are related to competency in usage of different *language strata* of the *algebra sign system* and to the appearance of cognitive tendencies within this context of solving word problems.

1. INTRODUCTION

The topic of arithmetic-algebraic problem solving has been extensively studied, both in the curricular realm (Bell, 1996) and with regard to the cognitive and change of focus demands represented by the activity for students in their transition from arithmetic to algebra (Bednarz and Janvier, 1996; Puig and Cerdán, 1990; Filloy, Rojano and Rubio, 2001). The researchers who have dealt with these problems have at the same time had to face the difficulty of specifying the differences between arithmetic and algebraic problems. The foregoing has led to discussions as to whether or not it is possible to make a dichotomic classification of this type because the elements that make up a word problem are apparently insufficient for its characterization. It seems that it is the relationship of those elements with the solution strategies put into play by the problem solvers that defines the arithmetic or algebraic nature of the entire activity.

One of the studies in algebra dealing with problem/strategy relations is that undertaken by N. Bednarz, L. Radford, B. Janvier, and A. Lapage (1989). Their findings show the influence wielded by by the structural factors of a problem in the solution strategies applied by pre-algebraic students. In their work, the latter researchers use an analytical framework for the problem types, in which they consider the "relational claculation" proposed by Vergnaud (1982) on the one hand, and the result of analyzing the problems that correspond to the arithmetics and algebra sections in textbooks, on the other. The results of their empirical work, developed within the foregoing framework, suggest the existence of differences between the "relational calculation" upon which an arithmetic mode of thought is based and the "relational calculation" upon which an algebraic mode of thought is based. That is to say, the mode of thought –be it arithmetic or algebraic– appears to be determined by the type of "relational calculation" that underlies the problem structure. And it is in this sense that the authors give themselves leave to speak of "algebraic" problems and of "arithmetic" problems (Bednarz and Janvier, 1996).

L. Puig and F. Cerdán refer to the nature of word problem solution, analyzing the processes of translating the text expressed in natural language into an expression through which the problem can be solved. Depending on whether the translation process leads to an expression that only involves givens or an expression that involves an unknown in the chain of operations (equation), the problem solution is said to be of an arithmetic or of an algebraic nature, respectively. The latter authors resort to two general methods in order to analyze said translation processes: the analysis and synthesis method and the Cartesian Method (Puig and Cerdán, 1990). Later in this same chapter, we expound upon what those methods consist of and how they are used to classify the forms of translation.

However in studying the transition toward algebraic thought within in the field of problem solving, the issue of deep-seated attachment to arithmetic modes of solution also arises as an unavoidable topic. One of those modes is that of proportional reasoning. Studies such as that carried out by L. Verschaffel et al. (2000) demonstrate that students 12 years of age and older tend to apply proportionality in an over-generalised manner, for instance to non-linear cases or to cases that require purely algebraic procedures. What is more, an extrapolation of those methods has been found to be present in students beyond the secondary level in their attempts to solve probability or differential calculus problems (van Dooren et al., 2003). In this chapter, we analyze the relations between arithmetic and algebraic problems solving methods, specifically taking students' tendency to remain anchored to an arithmetic mode of thought into consideration. The classification of problems is focused on the Family of Problems idea and the evolution toward mastery of the Cartesian (or algebraic) method is analyzed by way of progressive symbolization usage and of overcoming the difficulties that arise from a series of cognitive tendencies that act as obstructors of algebraic-type reasoning.

For consistency as regards the theoretical elements presented in previous chapters, we use the notion of a "more abstract" or "less abstract" mathematical sign system (MSS) and to that of intermediate strata of MSS in order to undertake our analysis of the evolution toward the Cartesian Method.

2. THE SOLUTION OF PROBLEM SITUATIONS IN ALGEBRA. COGNITIVE ASPECTS

To approach this issue we shall consider three classic methods for solving problems, and shall return to them later, when we describe an empirical study on a specific use of these methods in the solution of arithmetic/algebraic problems.

1) What we call the Method of Successive Analytic Inferences (MSAI), which is in fact the Classic Analytic Method for solving problems. In this method, the statements of the problems are conceived as descriptions of "real situations" or "possible states of the world," and consequently these texts are transformed by means of analytic sentences, i.e., using "facts" that are valid in "any possible world." These analytic sentences constitute logical inferences that act as descriptions of transformations of the "possible situations" until the solver comes to one that is recognized as the solution of the problem.

2) What we call the Analytic Method of Successive Explorations (AMSE). In this solving method the solver uses explorations with particular data to set in motion the analysis of the problem and thereby its solution.
3) What we call the Cartesian Method (CM), which is the usual approach to problem solving in current algebra texts. In this method, some of the unknown elements in the text are represented by means of expressions belonging to a more abstract MSS, and the text of the problem is then translated to a series of relations expressed in that MSS that lead to one or various texts, the decoding of which, via a regression in the translation to the original MSS, produces the solution of the problem.

We are interested in describing what kind of difficulties, obstacles, and facilities are produced by the use of each of these three methods for solving the word problems that appear in textbooks. But we are particularly interested in what kind of competences are generated by the use of the AMSE, in order that the user may come to be competent in the use of the CM, and which of the competences generated by the MSAI are necessary for competent use of the CM, given that the teaching objective is competence in the use of the CM.

Some experts and many beginners, when they use an MSS, spontaneously resort to the use of particular values and operate with them in order to explore and thus solve certain problems, since the use of particular data and their operation spontaneously provides meanings in a more concrete MSS to the relations that are immersed in a problem, and in many cases this produces more possibilities that the logical analysis may be set in motion. With the use of a more abstract MSS it is hard to capture the sense of the symbolic representations, as they are more abstract, and therefore it is hard to find strategies for solving the problem.

To solve more complex problems it is necessary to advance in the competence to make logical analyses of problem situations. But to be able to set in motion the analytic reasoning required for problem solving, it is necessary that certain obstructers should not be present, and that there should not be uncertainty about the tactics that need to be used to solve the problem, and in order to progress in all this it is necessary to advance in intermediate tactics immersed in the uses of the strata of the intermediate MSSs that are being used. We will explore these matters in what follows.

2.1. Competent use and cognitive tendencies

Competent use of the CM for solving problems implies an evolution in the use of symbolization, in which a competent user can eventually make sense of a symbolic representation of problems that is detached from the particular concrete examples given in the teaching process, thus creating Families of Problems, the members of which are problems identified by a particular scheme of solution. The sense of the CM will arrive when the user becomes aware that by using it he is going to be able to solve such Families of Problems. The sense of the CM for solving problems is not achieved by exemplifying it separated out, with example after example disarticulated, as is encouraged by the usual traditional teaching. The integral conception of the method requires the confidence of the user that the general application of its steps necessarily leads to the solution of such Families of Problems.

A competent use prevents the user from lapsing into certain cognitive tendencies that obstruct the possibility of making appropriate use of the CM to solve problems. Examples of this would be (1) the presence of calling mechanisms that lead to the setting in motion of incorrect solving processes, for example, if, in the solution of a problem, a type of mathematical text appears that the user does not know how to decode; (2) the presence of obstructions derived from semantics that affect syntax and viceversa, for example, when solving problems and endowing signs with meanings, which predisposes the user to a good use of syntax; and (3) the presence of inhibitory mechanisms, for example, when the values of certain data are changed in a problem from a Family of Problems that has already been solved (see latter in this chapter).

2.2. Mastery of intermediate tactics and cognitive tendencies

Mastery of intermediate tactics must contribute to the development of positive cognitive tendencies that present themselves in the processes for learning more abstract concepts, such as (1) the return to more concrete situations when an analysis situation presents itself, analysis being a necessary part for advancing in competence with the CM, or (2) the presence of a process of abbreviation of a concrete text in order to be able to produce new rules of syntax, for example, in problem solving when one is operating with the particular values assigned to the unknown in a problem in each exploration of the AMSE, and one then gradually operates on the abstract text with the rules of the more abstract MSS, no longer making reference to the concrete situation.

Symbolic representations of problems in the CM make the use of the working memory more efficient. When the student succeeds in creating relations between given values and unknowns the information is integrated, making more complex chunks of information. At the point when the student succeeds in creating these relations, the use of syntax avoids the need to burden the working memory with semantic descriptions bound up with the statement of the problems.

2.3. The problem of transference

With respect to the difference in solving problems that exists between an expert and a novice, some researchers establish that a competent user has preformed mental schemes that enable him or her to recognize a problem from the very first words, and when he recognizes it he realizes what kind of strategy has to be followed to solve it. Other researchers state that the formation of schemes enables users to classify problems on the basis of general principles, ignoring superficial aspects, in a process in which the outline of the problem that is obtained is brought into agreement with the mental scheme that is stored in the user's long-term memory.

From this it would seem that it is as a result of such schemes that a way of working forward is set in motion, in which what is produced is a synthesis of the problem rather than an analysis. However, although this may happen in many problem situations, it would not be of much help in the explanation of more complex processes that might enable us to say why some individuals can transfer the solution that appears in one kind of problem to another that has not been dealt with, i.e., the transfer of the use of a method from one MSS to another.

2.4. Competent use of the logical/semiotic outline

The most competent individuals in formal terms generally use the CM to solve certain kinds of problems that are presented to them. However, when they are solving some problems they first go through a brief phase of reflection, in which they themselves evaluate whether they are able to anticipate the steps of the solution, i.e., in which they make a logical/semiotic outline of the situation that includes, among other things, clarification or identification of what is "unknown" and discrimination of the central relations involved in the problem, for this purpose using an MSS stratum that often is not really the sign system required by the CM but a more concrete MSS stratum, for example the MSS of the MSAI or a stratum of the sort of MSSs that are used in the explorations of the AMSE.

To produce this outline one can set out from the given values and from there arrive at the value of the unknown, or else one can make a logical analysis that involves the establishment of relations in which one operates with the unknown, either in a particular form, as in the AMSE, or else with the unknown being represented directly by the MSS of the CM.

2.5. The pertinent use of certain intermediate strata

Formal competence in problem solving is not necessarily due to the formation of a large number of mental schemes referring to types of problems. In other words, although it may be possible to identify someone as being competent in problem solving because he or she uses strata of the MSS of the CM to solve them, apparently making automatic use of previously acquired mental schemes concerning the solving of different families of problems, if one wishes to make a better characterization of formal competence in problem solving one must consider a user's progress in the ability to make a logical/semiotic analysis of problem situations.

This means that a competent user of a more abstract MSS really is competent if he is also competent in other, more concrete MSS strata that enable him to have a greater possibility of setting in motion the logical analysis of a problem situation, tackling it by using MSS strata that are not necessarily the most abstract, but using the MSS stratum that enables him to understand the problem and thereby set in motion a logical analysis of it.

2.6. The logical/semiotic outline, the MSS strata used as representation

By the use of certain strata of the MSS required by the CM, users generate intermediate senses linked only to those levels: this enables them to simplify the solution of some Families of Problems. Once these senses are mastered, the use of this new sign system, solely with these levels, brings about the simplification of certain problems (see, for example, in Krutetskii (1976) the case of problems of the "chickens and rabbits" variety, the statement of which is in Section 2.1.2 of Chapter 6). Thus, by teaching a method such as the AMSE one is trying to make ad hoc use of intermediate strata, which can be identified among the more concrete strata required by the CM in order to simplify the analysis of the problem (although the more abstract strata also appear). The aim is to generate senses progressively for such representations which will be implemented by the use of the CM. Each Family of Problems determines the levels of representation —MSS strata— required for its solution.

2.7. The level of representation and the use of memory

To solve a problem such as the problem of the chickens and the rabbits, for example, with primitive methods, a high level of competence may be required. Consequently, the natural tendency is to use the method of trial and error, trying to find a way round the series of consecutive analytic inferences required by the arithmetic logical analysis of the situation. These inferences require representations that permit an analysis, which in turn demands a more advanced use of the sign systems involved. In other words, mathematics and natural language become interwoven and are set to work, and then competences are needed in order to produce logical/semiotic outlines that will make the solving strategy meaningful. What makes this analysis and logical/semiotic outline complicated is the fact that for some problems intense use of working memory is required, and this implies training that only expert problem solvers possess.

2.8. The use of primitive methods and the use of memory

When primitive methods are used, what is generated is not a unique representation of a certain style, but rather the representation changes with each Family of Problems. Moreover, with the use of a new MSS more advanced methods are used as a means for writing, arranging, and working, and the representation is produced using *canonic forms*. This constitutes part of the sense of the use of such an MSS. When a primitive method is used, representations must be invented for each Family of Problems, and this will call for a certain competent use of working memory in order to go on representing the solving actions proposed in the logical/semiotic outline, subsequently leaving new marks and indicators —or new chunks in the memory— by means of which the previous results can be grouped together and not left drifting. Other more advanced methods require the students to learn how to leave marks that progressively release units of memory, thus enabling the user to make use of these units in setting in motion the analysis and solution of the problem.

Intermediate representations arrange the information in chunks of more complex organization, even though it may not be possible to distinguish this from the signs produced by the user. Thus, during the interviews some students reached a representation of the problem in which they very probably made calculations —for example, by means of a calculator or a computer— and in the end they simply wrote down the numeric solution of the problem.

2.9. Personal codes

An important aspect to be considered is the use of personal notation (codes) to indicate the actions already carried out and the actions yet to be performed on the elements of the solving process. This suggests the existence of a stage prior to the operational stage. Obstructions also appear in this stage, imposed by these personal notations when the complexity of the situation increases, generating what are later considered to be *natural mistakes of syntax* in subsequent studies: the inappropriate use of equals signs or their absence, the forgetting of certain terms, etc.

2.10. Problem solving and syntax

Empirical evidence can be found to show that the process of analyzing a typical problem situation, expressed in natural language, leads to the appearance of phenomena of reading of the situation that inhibit the setting in motion of algorithms that a few moments before were carried out immediately and correctly. Thus, in the presence of an expression written in the usual algebraic language of a first-degree equation, the student is unable to decode it as such and is therefore unable to use the brilliant operational abilities that the same student had exhibited a few moments before with the same equation. Examples of problem situations can be quoted —in those parts in which translations are made from ordinary language to an MSS— that show the existence of a tension between the interpretation of the expression (decoding of the text) given by a reading that comes from a context belonging to the MSS, and the practices of mechanisation of operations (syntax), inhibiting the necessary reading given by the semantic interpretation that the concrete situation gives it in the word problem. Once again, a syntactic reading inhibits the reading of the concrete context in which the problem is situated, not allowing these expressions to be given an interpretation that will make it possible to go on with the correct solving strategy (which will lead to the solution), and which would include that part of the translation as one of its tactics.

2.11. Mechanization and practice

It is at this point in the discussion that some of the theoretical preoccupations of Thorndike (1923) and their implications in teaching acquire a new presence, because of the peremptory need for the automation of certain operations that come from the decoding of a concrete problem situation (problems of ages, mixtures, alloys, coins, work, etc.), since neither the sense of the algorithms required, nor the semantic interpretation in terms of the contexts in which the operations have been performed, nor the mechanisms of anticipation —especially inhibitory mechanisms— must obstruct the setting in motion of a solving strategy. moreover, when this strategy is transferred to the short-term memory, it is necessary that the length of time that it may remain there should not obstruct the possibility of considering all the intermediate tactics required for the solution proposed and should allow the concatenation of all the tactics —before all the steps necessary for the achievement of these partial goals have been carried out— to be performed in that part of the short-term memory, which it is difficult to keep activated for such a long time. it could be said that this ability to make considerable quantities of information remain for a long time, so as to be able to move out of that part of the memory and bring in important new information, is generally hard to find among middle school students, because it calls for substantial resources that typical teaching does not provide. in this situation, mechanisation as a result of intense practice permits optimum use of the expressions and operations customary in the mss, and thus it breaks away from the anticipatory mechanisms that inhibit the setting in motion of the necessary solving strategies.

3. SOLVING ARITHMETIC-ALGEBRAIC PROBLEMS

The sense of the text of a word problem as it is understood in Section 4 of this chapter —with the use of the Method of Successive Analytic Inferences (MSAI)— is determined by the logico-numeric structure presented in the problem situation. This thesis permeates all the others that will be discussed later.

In this chapter we present experimental results concerning the competences that are necessary for the use of four teaching models based on four methods for solving arithmetic-algebraic word problems: the Method of Successive Analytic Inferences (MSAI), the Analytic Method of Successive

Explorations (AMSE), the Spreadsheet Method (SM), and the Cartesian Method (CM). Definitions of these methods, with the exception of the SM, have already appeared in Section 2.

We will stress the need to be competent in increasingly abstract and general uses of the representations required to attain full competence in the algebraic method par excellence, which here is called the CM, contrasting it with the competences required by the other three methods, which are more rooted in arithmetic: the MSAI (see Section 4), the AMSE, and the SM (see Section 5). In Chapter 6 we analyzed the strengths and weaknesses of introducing the Cartesian Method with students who have just acquired competence in solving equations using as a teaching model the abstraction of operations via the concrete model that we gave in Chapters 4 and 6.

3.1. The "Solving Arithmetic-Algebraic Problems" project

It can be said that from the first stage, in which an exploratory study was carried out, the work was done with a set of results from which empirical results were obtained, and these were then converted into theses that in turn were put to the test in the experimental study in the final stage of the project. Thus the original theses not only evolved but were also gradually extended and modified when they were used as reference elements in the new study, serving as (1) an instrument of analysis of the exploratory questionnaires used to characterize the students who were selected for the clinical interviews, (2) an instrument of analysis of the performance of the students in the teaching sequence, and (3) an instrument of analysis of the performance of the students in the clinical interviews. Furthermore, in this use of the theses as one of the tools for interpretation in the experimental study other theses emerged, which were then put to the test in the final parts of the experimental study, as also were the clinical interviews in the case study —see Filloy and Rojano (2001), in which one of these cases is presented.

The theses that appear in this chapter may be considered as the empirical results that emerged from the final research, and as such they can be put to the test by means of other experimental studies, or else by the observations that emerge from teaching practice. This may make it possible to advance them, or to modify or even reject them.

The series of theses presented may have interrelations and similarities in some aspects, but our intention is to present them just as they were used as an instrument for interpretation in the research. It must be clearly understood that some of these theses were gradually refined, others evolved, having been enriched, made more precise and even modified during the study, and others simply come from the results that were obtained in the final phase of the research, that is, from the analysis of clinical interviews.

It must also be said that these theses have drawn both implicitly and explicitly on ideas from various fields of knowledge, such as history, epistemology, teaching, psychology, mathematics, etc. However, these suppositions were obtained both from the interaction with the students in the classroom when solving problems and from the interpretation of the results of their execution via the teaching model proposed. Let's begin by stating, in the following section, some theses concerning the behavior observable in middle school students when they solve arithmetic-algebraic word problems.

3.2. Some preliminary observations

3.2.1. A cognitive tendency: resistance to producing sense for an algebraic representation when one is in a numeric context

A cognitive tendency that is observed in a considerable part of the student population of this age consists in a resistance to producing sense for an algebraic representation when one is in a dynamic of numeric solving. For example, when one proposes systematic use of the AMSE and the SM after showing their virtues for solving certain problems by a procedure of trial and error, some students come to understand the operations that they have performed only after obtaining the numeric solution of the problem and establishing the equation that represents it.

This tendency to obtain an equation only from an equality, in order to follow the requirements of the teaching process, leads to a situation in which the representation of an unknown quantity or magnitude in the problem is only used as a label because, when it is used, the solution of the problem has already been obtained, and at best the students relate the value found by other means to the letter that appears in the equation. In these cases, the letters that are used in the equation obtained have the status of a name, and these letters are not associated with the supposed numeric values that were used to find the representation of the solution of the problem.

3.2.2. Concerning the natural tendency to use numeric values to explore problems

When using the language of algebra, some expert students and many beginners resort spontaneously to the use of numeric values (and arithmetic operations) to explore and thus solve certain algebraic word problems, since the use of numbers and arithmetic operations spontaneously gives meaning to the relations that are immersed in a problem, which in many cases increases the possibilities of being able to set in motion the logical analysis of the problem. With algebraic language it is more difficult to capture the sense of symbolic representations because they are more abstract, and therefore it is more difficult to find strategies to solve the problem.

3.2.3. The relationship between competence to make a logical analysis and mastery of intermediate tactics

In order to solve more complex problems it is necessary to advance in the competence to make a logical analysis of problem situations. However, to be able to set in motion the analytic reasonings required for solving problems, it is necessary that there should be no obstructers such as the following:

a) When they make a logical analysis of a problem, some students do not accept the operativity of the unknowns; in other words, when they try to make the analysis, they tend to use or give values to the unknowns and not to manipulate them as such, even in problems with situations that can be performed with concrete objects. When they come to make the analysis, the students cannot follow the train of thought that carries out concrete actions (which, separately, they accept without any difficulty), because when they think of something unknown, such as a number of children, they cannot follow the logical implications that derive from it.
b) There is great difficulty in being able to represent one unknown in terms of another, even among students who have overcome the difficulty expressed in the previous point.
c) There is uncertainty about the tactics that have to be used when solving the problem.

In turn, in order that these obstructers should not appear it is necessary to advance in the use of intermediate strategies immersed in the uses of:

a) algebraic expressions,
b) proportionality,
c) percentages,
d) multiplication within the schemes

$$\square \times A = B;\ A \times B = \square;\ A \times \square = B,$$

e) negative numbers.

Only a competent use of all this can prevent the user from lapsing into cognitive tendencies such as tendencies 7, 8, and 9, which obstruct the possibility of making use of the CM to solve word problems, as we indicated in Section 2.

3.2.4. One way of observing the complexity of a problem is through the difficulty that a user has in inventing a problem of the same family

One way of observing the complexity of a problem consists in analyzing the difficulties that are produced when one invents problems similar to a problem that has been solved previously, because varying the data allows one to observe whether the student perceives that the problems are the same from a logical viewpoint and that the difficulty depends only on finding relations between data and unknowns.

Similarly, the complexity of the relations of the problem can be observed when one invents problems similar to one that has been solved by setting out from the solution, that is, knowing the value of the unknown or unknowns – setting out from assigning a value to the unknown when one invents a similar problem is not a natural tendency in users. This process tests the establishment of the relations made previously and opens up the way to recognition of the family of problems because of the need to create the data of an analogous problem. The creation of problems similar to one solved previously tests the forms of mental representation or comprehension that were used when analyzing the original problem.

3.2.5. For a user to be competent in a more abstract MSS, he must also be competent in other, more concrete MSSs

For a student to become a competent user of the mathematical sign system of algebra (MSS_{al}), which, formally, is the most abstract sign system in our study, it is necessary that he or she should be competent in other, less abstract sign systems, such as the mathematical sign system of arithmetic (MSS_a), which is used in the MSAI, and the sign system in between these two, (MSS_i), which is used by the AMSE and the SM.

3.2.6. The sense of the CM is related both to the capability of going back to more concrete MSSs and also to the aptitude for recognizing the algebraic expressions used to solve the problem as expressions that involve unknowns

To give a full sense of use to the CM for solving algebraic word problems it is necessary that the (competent) user should have the capability of going back to sign systems with a greater semantic load; for example, to the intermediary mathematical sign system (MSS_i) associated with the AMSE and the SM, or to the most concrete mathematical sign system, that of arithmetic (MSS_a).

The sense of the CM in problem solving requires that users should recognize the algebraic expressions used in the solution of the problem as expressions that involve unknowns. It can be said that there is a competent use of expressions with unknowns when it makes sense to perform operations between the unknown and the data of the problem. In steps prior to competent use of the CM, the pragmatics of the more concrete sign systems leads to using the letters as variables, passing through a stage in which the letters are only used as names and representations of generalized numbers, and a subsequent stage in which they are used only for representing what is unknown in the problem. These last two stages, both clearly distinct, are predecessors of the use of letters as unknowns and using algebraic expressions as relations between magnitudes, in particular as functional relations.

3.3. Four teaching models

The use of the four teaching models associated with the MSAI, the AMSE, the SM, and the CM comes after observations made in the classroom over several years: when students with similar characteristics (students with knowledge of elementary algebra, from official secondary schools) begin subsequent levels of study they generally show a tendency to tackle word problems by means of the mathematical sign system of arithmetic (MSS_a).

The solution of problems by using the language of algebra causes great difficulties, even with problems equivalent to others that have been solved previously. The students cannot establish correct meanings for the algebraic relations of word problems. An attempt was made to find an answer to this situation by proposing teaching models associated with methods for solving word problems by a numeric approach: this not only helps to solve the problems but also, especially, aids the student to produce meanings.

This teaching proposal was formulated with the intention of facilitating the setting in motion of the analysis of word problems and making it possible to link the students' pre-algebraic tendencies for tackling problems with the learning of the model socially demanded, which is the CM, that is, the model in which problems are represented and solved by means of the language of algebra. For this purpose we took into account four mathematical sign systems (with their signs, way of operating on the unknown, strategies, actions, ideas, etc.): the MSS_a of arithmetic, linked with the MSAI; the MSS_{al} of algebra, related to the CM; the MSS_i corresponding to the intermediary sign system between these two, that is, the one associated with the AMSE; and, finally, the MSS that pertains to spreadsheets.

It seemed to us important to use the four teaching models, starting out from the theoretical analysis made in the first phase of the project and the empirical observations made up to that point. One aspect that should be pointed out has to do with the different uses that students make of the unknown when they are trying to represent and solve arithmetic-algebraic word problems, and also the difficulties that those uses generate.

The earlier empirical results indicated that students such as those in the study (teenagers 13 to 16 yeas of age, with prior knowledge of arithmetic and algebra) generally showed serious obstacles that hampered the setting in motion of the logical analysis of various families of arithmetic-algebraic word problems when using an algebraic expression with the use of a letter, usually *x*, to represent the unknown. Moreover, in general there was also a certain inability to make use of the mathematical sign system of algebra (MSS_{al}) and, therefore, to represent and solve word problems by means of the teaching model associated with the CM.

The students also had great difficulty in making the arithmetic logical analysis of many kinds of problem, and, consequently, in solving them with the MSAI. This classic arithmetic method requires great competence both in the use of the MSS_a and in making an analysis of the situations presented in the problems, especially in those whose text involves assertions that are not expressed in terms of unknowns, so that the analysis is then made with reasonings that involve unknown magnitudes or quantities.

See Section 4 for a more detailed presentation of the MSAI teaching model and the problems in its use, where it is shown that situations such as those mentioned make logical analysis of them more complex, especially if the MSS_a of arithmetic is used. This is even so in simple problems like those in Section 4: to solve them by applying the MSAI calls for a much greater capability of logical analysis of the situation than if one tackles it with numeric explorations as in the AMSE or the SM, and even greater if one approaches it with the CM.

Moreover, the teaching models associated with the AMSE and the SM contain elements that facilitate the setting in motion of the logical analysis of certain kinds of problem. As they use hypothetical numeric values for the unknown, arithmetic operations are performed between them and the data, and as these operations have a greater semantic load than algebraic relations, they make it more likely that the user may be able to produce meanings for these operations in accordance with the conditions of the problem. This last point is a key factor in problems in which it is complex to set the logical analysis in motion using the MSS_a of arithmetic.

One of the aims that we propose in Section 5 is to observe whether, with the use of the teaching model based on the AMSE and the SM, the user can: (1) start to relinquish the use of the mathematical sign system of arithmetic with which the MSAI is associated; (2) begin to break away from the arithmetic use that is made of the unknown by solving problems and operating with it by reasoning and making inferences with a representation of one's own; and (3) moving on to a use of the unknown in which one operates with a representation of one's own, through the solution of families of problems, for an understanding of which, and therefore for the setting in motion of their analysis, one requires the use of hypothetical numeric values for the unknown in natural form, knowing beforehand that in problems of this kind it is more complex to set their analysis in motion by the use of the MSS_a of arithmetic associated with the MSAI than by the use of the other two methods.

To sum up, the use of the four teaching models based on the MSAI, the AMSE, the CM, and the SM for solving arithmetic-algebraic word problems takes account of the fact that, in order to solve arithmetic-algebraic word problems with the CM and, in general, to achieve competent use of the mathematical sign system of algebra (MSS_{al}), it is necessary to consider the competences, adjustments, and limitations that other, more concrete mathematical sign systems, in this case the MSS_a of arithmetic and those associated with the AMSE and the SM, may impose on the more abstract sign system that it is socially desired to teach, that of algebra, which one usually wishes to use as a method for solving problems in algebraic form —a method identified in this work as the Cartesian Method.

3.4. The Cartesian Method

It is worth mentioning that any of the indicative procedures that are usually proposed in teaching or in textbooks for solving word problems by translating them to the MSS_{al} of algebra take into account, in some way, what we have called the Cartesian Method.

The reason for calling the method Cartesian is that part of Descartes's *Regulæ ad directionem ingenii (Rules for the direction of the mind)*[1] can be interpreted as an examination of the nature of the work of translating an arithmetic-algebraic word problem to the MSS_{al} of algebra and its solution in that MSS. This is how it was understood by Polya, who, in the chapter "The Cartesian Pattern" in his book *Mathematical Discovery*, rewrote the pertinent Cartesian rules in such a way that they could be seen as problem solving principles that use the MSS_{al} of algebra. Polya's paraphrase of Descartes's rules is as follows:

(1) First, having well understood the problem, reduce it to the determination of certain unknown quantities (Rules XIII–XVI).[2]
[…]
(2) Survey the problem in the most natural way, taking it as solved and visualizing in suitable order all the relations that must hold between the unknowns and the data according to the condition (Rule XVII).[3]
[…]
(3) Detach a part of the condition according to which you can express the same quantity in two different ways and so obtain an equation between the unknowns. Eventually you should split the condition into as many parts, and so obtain a system of as many equations, as there are unknowns (Rule XIX).[4]
[…]
(4) Reduce the system of equations to one equation (Rule XXI).[5] (Polya, 1966, pp. 27–28)

In Puig and Cerdán (1990) the process of solving arithmetic-algebraic word problems modeled by the analysis and synthesis method is compared with the process modeled by the Cartesian method. In the process modeled by the analysis and synthesis method one works from the unknown in the problem and concludes when one does not come to further unknown quantities (auxiliary unknowns) but rather known quantities (data of the problem), that is, when the unknown has been reduced to data. The product of the analysis is then a set of relations between the quantities of the problem linked in such a way that they can be represented in the form of a tree that leads from the unknown to the data of the problem. The synthesis then consists in making one's way through this diagram in the opposite direction, from the data to the unknown, performing the corresponding arithmetic operations or, if one wishes, writing the arithmetic expression to solve the problem. Therefore, when the analysis and synthesis method is used for solving problems of this kind and leads to their solution, it does so in the MSS_a of arithmetic.

To illustrate this we now present the statement of a problem, the representation of its analysis in a diagram, and the arithmetic expression that results from the synthesis.

The suit cloth problem

Four pieces of cloth, each 50 meters long, are going to be used to make 20 suits, each of which needs 3 meters of cloth. The rest of the cloth will be used to make coats. If each coat needs 4 meters, how many coats can be made?

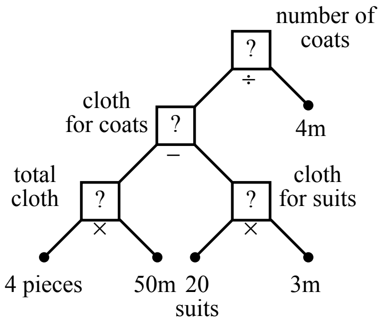


Figure 9.1

Problems like this can also be solved with the use of the MSS of algebra, by translating the statement into an equation and then solving it. Thus, in the problem just stated, it can be considered, when following the third of the rules rewritten by Polya, that the quantity that can be expressed in two different ways is the "total cloth," so that one writes the equation $4x + 20\cdot 3 = 4\cdot 50$, the solution of which is precisely the arithmetic expression given by the synthesis above. What makes the (indicated) solution of this equation coincide with this arithmetic expression is the fact that the sequence of operations indicated in the equation can be inverted, with the inverse operations affecting only known quantities. In other words, one can go through the set of quantities and relations expressed in this equation by proceeding from the unknown to data, as one does in analysis and synthesis. Observe that this equation is in fact one of those that we have called "arithmetic" equations.

It is not hard to realize that the equations that we have called "algebraic," that is, those in which the unknown appears on both sides of the equation, cannot be inverted in the same way, because it is necessary to operate on the unknown in order to solve them. So that one cannot go through the set of quantities and relations expressed in such an equation by proceeding from the unknown to the data as one does in analysis and synthesis.

Let us take as an example an equation such as $\frac{x}{217} + 171 = \frac{x}{198}$. If we try to trace the path of the analysis from the unknown, using the relations between quantities that are expressed in this equation, this does not reduce the unknown to data, and instead one returns to the unknown when one uses the relation that corresponds to its second appearance. We will show this by using a word problem in one of whose solutions this equation appears. Thus we will be able to name the quantities and relations expressed in the equation in accordance with their meanings in the context of the story that the statement of the problem tells.

The problem comes from Kalmykova (1975) and has already been used in Puig and Cerdán (1988) and Puig (1996). We will call it "the hay problem."

The hay problem

A collective farm assumed that some hay stockpiled for cattle would last for 198 days, but the hay lasted for 217 days since it was of the highest quality and they used 171 kg less per day than they thought they would. How much hay had been prepared on the farm? (Kalmykova, 1975, p. 90)

If this problem can be translated into the equation $\frac{x}{217}+171=\frac{x}{198}$, it is because, from the story that the statement tells, we have extracted the known quantities "days planned," D_p (198), "actual days," D_a (217) and "daily reduction in consumption of hay," C_r (171), the unknown quantities "planned daily consumption," C_p, "actual daily consumption," C_a, and "hay stockpiled," T, and the relations between these quantities $D_p \times C_p = T$, $D_a \times C_a = T$ and $C_a + C_r = C_p$.

However, the use of these quantities and relations in the analysis of the unknown leads to one of the following two diagrams, which cannot end with data because once again the unknown appears, so that the analysis cannot conclude in such a way that the solution of the problem is an arithmetic expression obtained by synthesis.

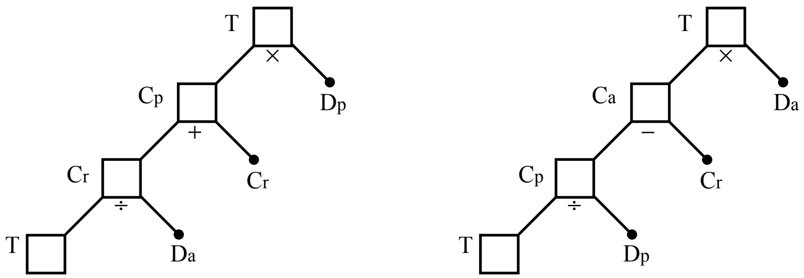


Figure 9.2

The diagrams in Figure 9.2 correspond to the two possible attempts to isolate each of the occurrences of x by inverting the operations indicated in the equation, which lead to $\left(\frac{x}{217}+171\right)198=x$ and $x=\left(\frac{x}{198}+171\right)217$, but not to x equals an arithmetic expression. In fact, with such equations it is not sufficient to invert the operations indicated in order to isolate the unknown, but rather it also necessary to operate on the unknown.

If, instead of solving the hay problem with the CM (and therefore the MSS_{al} of algebra), we had tried to solve it with the analysis and synthesis method (and the MSS_a of arithmetic) and the relations $D_p \times C_p = T$, $D_a \times C_a = T$ and $C_a + C_r = C_p$ had been established in the analysis, then it would not have been possible to solve the problem because this analysis does not allow us to reduce the unknown to data. The diagrams in Figure 9.2 show this clearly.

The fundamental difference between the analysis of the statement in the analysis and synthesis method and in the CM lies in the fact that the logic-semiotic outline that one makes when one uses the CM anticipates the use of the MSS_a of algebra. This entails not only the use of letters to designate the quantities that are determined in the analysis but also new meanings for arithmetic operations and relations, particularly the equals sign, which belong to that MSS_{al}. Consequently, when making this analysis the known and unknown quantities are considered in the same way —Descartes himself indicated that the whole art of the method lay in this.[6] In contrast, in the analysis and synthesis method the analysis is developed by situating oneself in the unknown of the problem and considering on what data one would have to operate in order to obtain it, and in the logic-semiotic outline one does not contemplate the possibility of operating other than on known quantities.

The diagrams shown so far, which reflect a solution modeled by the analysis and synthesis method, are not suitable for giving an account of the analytic reading in the CM. There is a different kind of diagram, which *is* suitable, however —one in the form of a graph that we have adapted from Fridman (1990) and that Cerdán (in preparation) studies and uses. These graphs represent the analytic reading of the statement of an arithmetic-algebraic word problem characteristic of the CM because their vertices represent quantities and their edges represent relations between quantities, so that the graph shows the network of relations between quantities that has been determined in this analytic reading. Moreover, the vertices corresponding to the data of the problem are represented by black circles (which we will call "dark vertices"), and the vertices corresponding to the unknown quantities (the unknowns of the problem or auxiliary unknowns) are represented by unfilled squares (which we will call "light vertices"). As the four basic arithmetic operations are binary, the corresponding relations are ternary, so that in the most common arithmetic-algebraic problems the edges have three vertices.[7]

Thus, the analytic reading of the hay problem (taken as a textual space) which produces the known quantities D_p (198), D_a (217), and C_r (171), the unknown quantities C_p, C_a , and T, and the relations between these quantities

$D_p \times C_p = T$, $D_a \times C_a = T$ and $C_a + C_r = C_p$ (a new text) is represented by the graph in Figure 9.3.

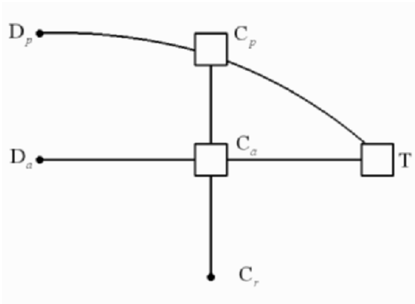


Figure 9.3

The analytic reading of the suit cloth problem in which the quantities and relations that we represented previously by a diagram are determined can also be represented by the graph in Figure 9.4. In it the known quantities are "number of pieces of cloth," N_p (4), "number of suits," N_s (20), "cloth per coat," C_c (4 m), "cloth per piece." C_p (50 m) and "cloth per suit," C_s (3 m); the unknown quantities are "number of coats," N_c, "cloth for the coats," T_c, "cloth for the pieces or total cloth," T_p, and "cloth for the suits," T_s, and the relations are $N_i \times G_i = T_i$, $i \in \{c, p, s\}$, $T_p = T_c + T_s$.

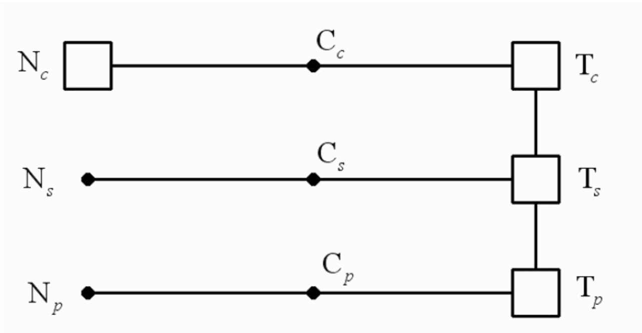


Figure 9.4

In these graphs it is also clear why one network of relations makes it possible to obtain the solution of the problem by using the MSS_a of arithmetic and the other does not. In fact, in order to avoid having to operate on unknown quantities it is necessary that on one ternary edge two of the vertices should be dark (should be known quantities), for then the light vertex can be converted into a dark vertex (the unknown quantity can be calculated from known quantities) by performing the corresponding arithmetic operation. The unknown of the problem can be obtained from the data as long as there is a way of progressively converting light vertices into dark vertices until one arrives at the unknown. In the graph corresponding to the analytic reading of the suit cloth problem this path exists and it coincides with the path described by the analysis and synthesis diagram in Figure 9.1. In the diagram of the reading of the hay problem the path cannot exist because there is no edge that has two dark vertices. Consequently, it is consistent with the terminology introduced earlier to describe as "arithmetic" those graphs that share with the graph in Figure 9.4 the property that we have explained, and as "algebraic" those that do not have it (such as the one in Figure 9.3), just as is done by Cerdán (in preparation).

These graphs represent the analytic reading of the statements of the problems when they are solved by means of the CM, but this analytic reading is only the first step in the method. To obtain the equation $4x + 20{\cdot}3 = 4{\cdot}50$ or the equation $\frac{x}{217} + 171 = \frac{x}{198}$ it is necessary to complete three further steps.

The second step consists in choosing a quantity (or several quantities) which one designates with a letter (or several different letters).

The third step consists in writing algebraic expressions to designate the other quantities, using the letter (or letters) introduced in the second step and the relations found in the analytic reading made in the first step.

The fourth step consists in writing an equation (or as many independent equations as the number of letters introduced in the second step) based on the observation that two (non-equivalent) algebraic expressions written in the third step designate the same quantity.

In Figures 9.5, 9.6, and 9.7 we show how these steps can also be represented in graphs.

Second step

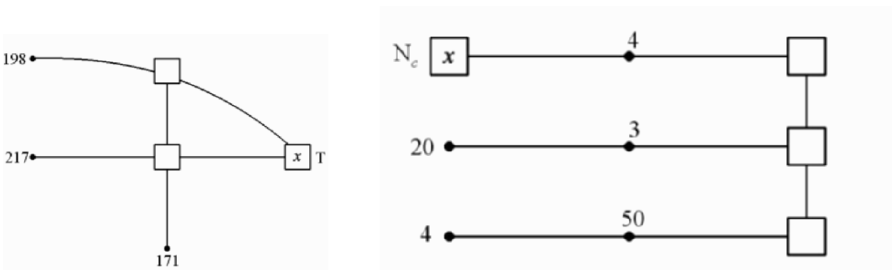


Figure 9.5

Third step

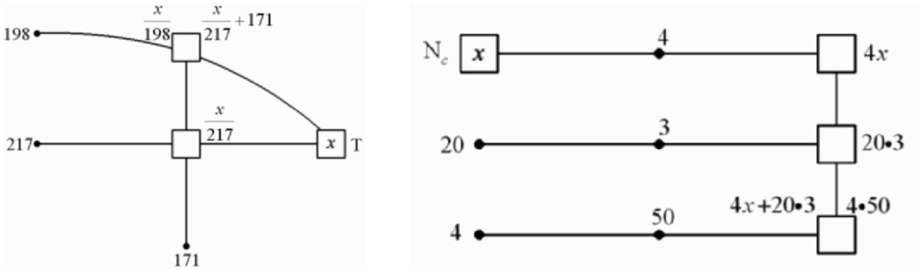


Figure 9.6

Fourth step

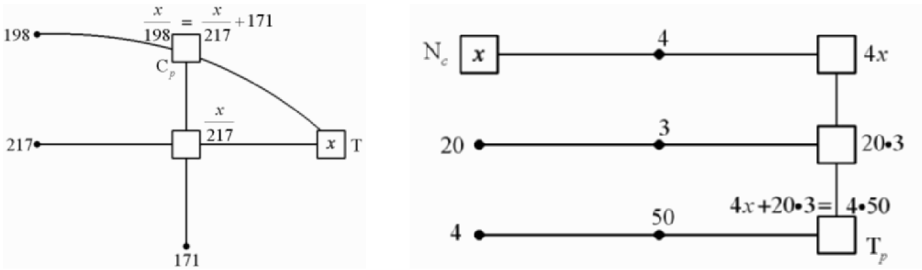


Figure 9.7

Observe that the equation obtained is arithmetic if the graph is arithmetic, and algebraic if it is algebraic. We have also seen that the corresponding analysis and synthesis diagrams either produce an arithmetic solution or they do not. However, this does not mean that we can describe the corresponding problems as "arithmetic" (the suit cloth problem) and "algebraic" (the hay problem). In fact, in the case of the hay problem what is not arithmetic is a solving process carried out by the analysis and synthesis method (represented by the analysis and synthesis diagram) and an analytic reading that constitutes the first step of the CM (represented by the graph); but it is possible that there may be another solving process or another analytic reading that determines another network of relations between quantities that *is* arithmetic. Such is indeed the case: if the analytic reading determines not only the quantities determined previously but also the unknown quantities "additional days," D_m, "consumption on the additional days," C_{Dm} and "total saving," S_t, and the new relations $D_p \times C_r = S_t$, $D_m \times C_a = C_{Dm}$, D_p $D_m = D_a$ and $C_{Dm} = S_t$, the corresponding graph is arithmetic, as can be seen in Figure 9.8, and the unknown is determined by means of the arithmetic expression $\left(\frac{198\cdot171}{217-198}+171\right)\cdot198$.

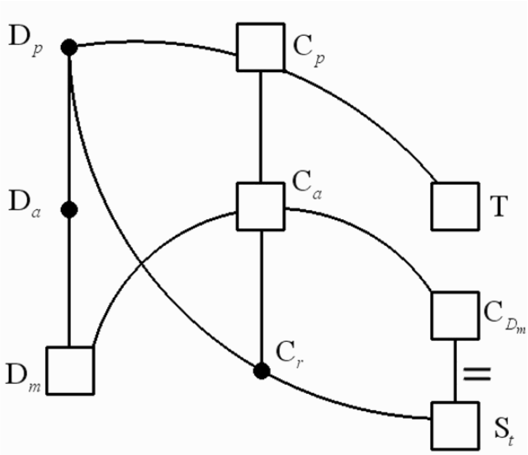


Figure 9.8

What can be described as arithmetic or algebraic, therefore, is the solving process (represented by the analysis and synthesis diagram), the analytic reading (represented by the graph), or the equation that translates the statement, but not the problem.

The splitting of the CM into steps, as presented, describes the competent behaviour of the ideal subject. Only in this sense is it possible that each step begins with the completion of the previous one. In fact, there are obvious connections between the termination of one step and the commencement of the next. For example, the writing of algebraic expressions (step two) is complete precisely when two expressions have been written that designate the same quantity, which in turn makes step four possible. What this splitting into steps clearly shows is that the CM is the algebraic method par excellence, because each of the steps makes sense only with the use of the MSS_{al} of algebra.

3.5. Spreadsheets used to solve word problems

As we saw at the beginning, some of the theses presented indicate the virtues of using numeric values to explain the solving of word problems. When a method such as the AMSE is put into practice in a context such as that of the computer spreadsheet, the search for the value of the unknown is done only with the use of numeric values. In other words, unlike what happens with the AMSE, in the SM one does not make an explicit formulation of the equation.

When one observes students using the SM to represent and solve word problems, the spreadsheet medium influences their preliminary solving strategies, but it can also be said that earlier experience in solving problems has an impact on the strategies used in the SM.

When this method is applied, once the problem has been expounded in the MSS of the spreadsheet the students have at their disposal a means with which to explore the possible solving strategies.

We will now indicate some of the observations that we have gathered concerning what happens when students use the SM.

1) Most students do not think spontaneously in terms of an algebraic experience when they first work in an environment such as the one provided by the SM.
2) The SM stimulates students to stop focusing on a specific example and move on to considering a general relation.
3) The SM also stimulates students to accept working with an unknown. The use of one cell in the spreadsheet to represent the unknown is established, and by using the mouse they can then express the various relations stated in the problem in terms of the cell used in the first place.
4) After using the SM there is a greater awareness of the relations between the unknowns, and between the unknowns and the data of the problem.
5) Before a sequence of sessions in the use of the SM one can observe an evolution toward a more general algebraic method consisting in proceeding from the unknown to the given.
6) In the SM one can see an integration of various solving strategies, such as the refinement of the whole and part strategy and trial and error.

4. THE METHOD OF SUCCESSIVE ANALYTIC INFERENCES

4.1. An example of the use of the MSAI

We have already described the MSAI in Section 2. We indicated there that the use of the MSAI for solving arithmetic-algebraic problems presents itself as a product of logical inferences which act as descriptions of the transformations of the possible situations of the problem until one comes to one that is recognized as the solution of the problem. We will illustrate this kind of inference with the problem that we call “the typist problem”:

The typist problem

A typist has to type 1200 pages in a certain number of days. If she types 40 more pages a day, she will finish the work 8 days sooner. How many pages a day does she type and how many days is she expected to take to finish the work?

Solution

If □ is the number of pages that the typist normally types, the number of pages not typed in the 8 days will be 8 times □, which will have to be made up by typing 40 extra pages a day. The days that the typist will work will be 8 less than the number of days that she would take to do the work normally, which we can calculate by dividing 1200 by □. From all this we obtain the following equalities:

$$8\square = 40\left(\frac{1200}{\square} - 8\right)$$

$$\square = 5\left(\frac{1200}{\square} - 8\right) = \frac{6000}{\square} - 40$$

Multiplying both sides by the quantity □ we obtain the equality

$$\square^2 = 6000 - 40\square$$

which is the same as

$$\square^2 + 40\square = 6000\,.$$

In order to calculate the quantity we use the seventh proposition in the first book of Jordanus de Nemore’s *De Numeris Datis*, written in the early 13th century:

If one divides a number into two parts, one of which has been given, and the product of the one that has not been given by itself and by the one that has been given is a given number, then the divided number will have been given.[8]

In Nemore's book each proposition has three parts: the first is the statement, in which he affirms that if some numbers (or ratios) have been given, and certain relations between them have also been given, then other numbers (or ratios) have also been given; the second part is the proof, in which he makes transformations of the numbers (or ratios) and the relations which either show that the numbers are in fact given or else convert them into the numbers and relations of the hypothesis of some previous proposition; and the third part is the calculation of an example with concrete numbers, which therefore has the value of an algorithm.

If we use the algorithm that Nemore presents in the third part of proposition I-7, we will solve the problem in the following manner:

One of the parts is 40, and the other part squared and by 40 is 6000. Double 6000 and double it again, giving 24000. Add to this the square of 40, which is 1600, making 25600, the square root of which is 160. From this take 40 and halve the result, giving 60. This is the unknown part (in our case, ☐). So that the number divided is 40 + 60 = 100.[9]

Verification

If the typist does 60 pages a day, she would do 1200 in 1200 ÷ 60 = 20 days. If she does 40 more pages a day, she will do 100, so that she will take 1200 ÷ 100 = 12 days, which is 8 less than 20.

4.2. Difficulties in the use of the MSAI

4.2.1. The tendency not to admit the possibility of making inferences about something that is unknown

There are middle school students who do not admit the possibility of making inferences about something that is unknown. A case of a similar nature is that of other students who simply avoid operating on the unknown. In other words, some students show resistance to bringing into play operations on the representation of something unknown.

4.2.2. Lack of knowledge of concepts as an obstructer

When a user solves a problem by means of the MSAI, this enables him to recognize that there is a logical structure in the problem. As a result of this recognition he may become aware that concepts that he has not mastered are involved in the logical relations of the problem, and this may become an obstructer for recognizing and generating problems of the same family as the one that he has just solved. On the other hand, if the student is competent in the concepts that are involved in the implications obtained from the logical outline of the problem, then he or she is capable of representing the unknown and making use of that representation, even if the unknown parts vary. This happens, for example, if one passes from using a directly proportional relation to using several directly proportional relations in the same problem.

4.2.3. Families of problems determine their level of representation.

Each family of problems determines the levels of representation that their solution requires. For example, when a student is really competent in a more abstract MSS and is presented with a problem of mixtures (see the mixtures problem that we present in Section 4.3.2), the solution may lead naturally to the use of the MSS_a of arithmetic. However, this kind of problem is usually solved algebraically.

4.2.4. The use of trial and error to avoid the difficulty of the inferences of the MSAI

To be able to solve certain families of problems with the MSAI an expert level of competence is required, and therefore there is a natural tendency to use trial and error —for example, to get away from the series of successive analytic inferences that logical analysis of the situation requires.

With the use of trial and error it is actually possible to simplify the difficulty of the inferences of the MSAI. This is due to the fact that there are problems in which, in order to tackle them with the MSAI, the series of successive inferences required in order to make the analysis of the situation calls for representations that involve competence in more advanced uses of the arithmetic sign system —the more complicated problems require a greater mastery of the codes that relate syntax and semantics, both in natural language and in the MSS_a of arithmetic, and also in their pragmatics, that is, in the uses that permit crossing between the two sign systems.

4.2.5. The need for intensive use of memory as an obstructer

A factor that complicates the establishment in the MSAI of the logical outline in the MSS_a of arithmetic is the fact that some problems require an intensive use of working memory, and this implies a training that only expert solvers have had. Moreover, when one uses the MSAI to solve word problems one must invent the representations problem by problem, and this calls for a certain capability of using working memory in order to represent the actions proposed in the logical outline of the solution and leave new markers and indicators —or new groupings in the memory— for preliminary results, and so not leave them isolated or forgotten.

4.2.6. The singularity of the representation of each problem in the MSAI as opposed to representation using canonic forms in the CM

When one uses the MSAI, one does not generate one sole representation of a certain style, but rather the representation changes for each problem, or at any rate for families of problems; on the other hand, when one uses the MSS_{al} of algebra or the CM, one always uses the representation provided by certain expressions which belong to that MSS_{al}, and those representations are reduced to canonic forms in order to solve them.

4.3. Advances with the MSAI

4.3.1. Modification of the natural tendency to tackle arithmetic-algebraic problems by means of arithmetic, and its relation to the representation of the unknown.

The natural tendency to tackle arithmetic-algebraic problems by means of arithmetic weakens when one tries to solve certain families of problems that are difficult to solve with the MSAI. When variations in the value of what is unknown are brought into play, it is possible to propose families that will require the student to use representations of a different kind, in which unknown quantities have to be represented so that inferences can then be made with them (see the problem in Section 4.3.2). In the end, with the Cartesian Method it will be necessary to operate on the representation of what is unknown in the problem.

The needs of representation generate new senses, which bring the possibility of making more abstract uses of the MSSs used to make the representation of the problem on the basis of the outline of the solution. The essential difference between the traditional introduction to solving problems with algebra and these preliminary approaches, such as the MSAI, lies in the fact that, in the solution of the problems (1) the unknown is represented, but one does not operate on it; (2) inferences are made that refer to the representation of the unknown; (3) if one operates on something, one always operates on data; (4) if one speak of unknowns, one does so in terms of the results of the operations that are performed on the data.

4.3.2. On the processes of abstraction and generalization

As more complex families of problems are solved, the sign systems used gradually become more abstract. These processes of generalization and abstraction operate on the families of problems, either by finding common elements —which we will call "generalization" —or by making negations in part of the members of the family —in which case we will speak of a process of abstraction.

Thus, for example, a mechanism to explain why mixture problems are more difficult than problems of other families can be found in observation of the need to break away from the use of only inferring from the representation of something unknown in order to be able to use the representation in which the unknown parts also vary.

As an example of what we have just said, we will use the MSAI to solve a mixture problem.

The mixture problem

A man wants to change the mixture of water and antifreeze in the radiator of his car, which contains 20% antifreeze. He has discovered that the best mixture is one that contains 50% antifreeze, so he has to remove a certain quantity of the mixture in the radiator and then add antifreeze until it represents 50% of the mixture. The radiator has a capacity of 30 liters. What quantity of mixture must he replace?

Solution

The mixture in the car initially contains 6 L of antifreeze and 24 L of water. It is necessary to remove 9 L of water so that only 15 L remains. In order to do this we note that any quantity of mixture is always 80% water and the rest is antifreeze. We have to remove a quantity of mixture such that 80% of it is 9 L, that is, $9 \div 0.8 = 11.25$ L.

Note that the MSAI allows one to envisage a family of problems in which the radiator is of any capacity and the initial mixture can be of any proportion. Here, in solving the problem we vary both the water and the antifreeze, both of which are initially unknown quantities.

4.3.3. With time, the MSAI requires representations similar to those of the AMSE and the SM

When this happens, representations are established that show a distancing from the use of the MSS_a of arithmetic that is used in the MSAI. In other words, in the AMSE and the SM one brings into use a representation of what is not known with a view to operating on the unknown that represents it, whereas in the MSAI what is not known is only represented and inferences are made that speak of that representation, but one never operates on the representation of what is not known. This is one of the greatest differences between the use of what is not known in the MSAI and the use that one seeks to provide in the intermediate $MSSs_i$ that are used in the AMSE and the SM.

4.3.4. The use of numeric trial and error in the arithmetic MSS stratum can enable the user to correct a faulty analysis made with the MSAI

4.3.5. The succinctness of the use of the MSAI

So that the reader may recognize the power of a solution obtained with the MSAI, we are going to solve the following problem, also solved with the AMSE in Filloy, Rojano, and Rubio (2001), which we will call the "teacher problem."

The teacher problem

A teacher at Kinder has 120 chocolates and 192 toffees. She is going to distribute them fairly among the students. If each student receives 3 more toffees than chocolates, how many students are there?

Solution

We begin by giving one chocolate and one toffee to each student. As there are fewer chocolates, they run out before the toffees, so that now only toffees remain to be given out: the 192 – 120 = 72 that remain.

Now we give them out, knowing that each student receives 3 of them; therefore there are 72 ÷ 3 = 24 students.

Verification

Each student receives 192 ÷ 24 = 8 toffees and 120 ÷ 24 = 5 chocolates, i.e., three fewer.

The reader can compare this solution with the one given in Filloy, Rojano, and Rubio (2001) using the AMSE, and will be able to see the succinctness of the solution just given, and also appreciate that from this solution one can generate a family of problems similar to that of the teacher. One has only to vary the number of toffees, chocolates, etc. One can also then see the kind of restrictions that have to be made so that these quantities produce a real problem, that is, a problem that has a solution.

5. TOWARD THE CM VIA THE MSAI, THE AMSE AND THE SM

In this section we are going to present the two other methods that are more deeply rooted in arithmetic: the Analytic Method of Successive Explorations (AMSE) and the Spreadsheet Method (SM). In Filloy, Rojano, and Rubio (2001) we gave examples of how they are used with students. Here we will give a series of reasons that make these two methods, together with the MSAI presented in the last section, suitable precursors before trying to get middle school students to become competent in the method traditionally used, which we have here called the Cartesian Method (CM).

It could be said that, hitherto, making students competent in the use of the CM has been the only aim indicated in traditional algebra courses in the chapters that talk about solving word problems. In Filloy and Rojano (2001) we showed the difficulties of introducing the CM when one has just taught how to solve equations. The results presented in that work are gratifying, although it presents only one case to analyze the difficulties and the successes.

We will now present results that endorse the appropriateness of using the MSAI, the AMSE, and the SM as vehicles for achieving the competences that the CM requires.

5.1. The AMSE and the SM as a bridge to unite syntactic and semantic development

The teaching models based on the AMSE and the SM serve as a bridge to unite syntactic and semantic development through the production of meanings for arithmetic-algebraic operations in the transition from the use of the notion of variable to that of unknown.

5.2. The MSAI, the AMSE, and the SM serve as precursors for creating the meanings of algebraic relationships

The meanings of arithmetic operations, their properties, and their results, as used in the MSAI, the AMSE, and the SM, serve as precursors for creating the meanings of the algebraic relations established in the use of expressions with unknowns and data, and even the meanings of the more complex expressions involving unknowns of a word problem that are presented in the use of the CM. The meanings of the arithmetic operations within strata of the MSS_a of arithmetic serve as precursors of more abstract representations —algebraic, for example. However, correctly signifying arithmetic operations, their properties and their results with the MSAI, the AMSE, and the SM in order to create meaning for the algebraic relations of the CM also implies the need to make competent use of them.

5.3. The AMSE and the SM encourage different algebraic interpretations of the word problem

The algebraic interpretations encouraged by the AMSE and the SM do not generally represent the relations between data and unknown in the order in which they appear in the statement of the problem, something that does usually happen in the teaching sequences with which the CM is illustrated.

Indeed, this freedom in interpretation is based on the student's natural tendency to manipulate only one unknown in problems that may involve the manipulation of two or more, in contrast to the classic teaching strategies, which are generally versions of the CM, tending to use two unknowns in the solution of such problems.

5.4. Dimensional analysis of equations serves as an element of control

Making a dimensional analysis of the equations obtained from numeric relations that are established between quantities involved in a word problem helps one to understand the notion of relation between quantities and magnitudes that emerge from the word problem, and therefore serves as an element of control of the representation of the problem and as a means for producing sense for the notion of equivalence between algebraic expressions.

In general, however, it serves to create senses that lead to the notion of equivalence between algebraic expressions that involve the use of unknowns as the common element in two algebraic relations, this being expressed in an equation.

5.5. For some problems the MSAI is more efficient than the AMSE or the SM

To embark on a plan to find a solution via the CM or the AMSE is not always the best path to the solution. An easier solving strategy may be one that is set in motion on the basis of a direct logic-arithmetic analysis, as in the MSAI (see, for example, the teacher problem and the mixture problem, presented in Sections 4.3.5 and 4.3.2). In the problems in which this happens, the intermediate character of the MSAI is not seen.

5.6. The relationship between representations in the CM and the efficient use of working memory

Symbolic representations of problems in the CM make the use of the working memory more efficient. When the student succeeds in making relations between data and unknowns he combines the information, making more complex packets of information. At the point when the student succeeds in making these relations, the use of syntax obviates the need to burden the working memory with semantic descriptions bound up with the posing of the problems.

5.7. The competent use of the CM and its relation to the various uses of algebraic expressions

To simplify the more complex problems of arithmetic and medium level algebra one requires a competent use of the MSS_{al} of algebra and therefore of the CM. Part of the order of complexity of families of arithmetic-algebraic word problems comes from the difficulties presented by their logic-arithmetic analysis.

To understand progress in the competent use of the CM one must:

a) Explore the tensions that exist between the uses of the concepts of name, representation of a generalized number, representation of what is not known, unknown, variable, and relation. To do so, one must understand what happens with the difficulty of a problem and of the solution of the equation that represents it when the data of the problem are varied.
b) Analyze the relation between the complexity of a family of problems and the development of algebraic syntax and semantics: the operation of negatives, use of rational numbers, simplification of algebraic expressions, solution of equations, etc. This has to do with the clarification of what we might call the various uses of the algebraic expressions indicated in (a).

5.8. The logical outline, the analysis of the problem, and other competences of students

In the first stage of the AMSE and the SM (and probably of any method), which consists in the reading and understanding of the text of the problem, it is necessary to make a logical outline of the problem situation. This outline involves, among other things, a logical mental representation of the problem that contains the basic information of the problem situation and that identifies the relations that are central for the possibility of setting any solving strategy in motion. However, having an understanding or overall logic-mental representation of the problem is not enough to enable one to set the AMSE and the CM in motion; one must also, as part of the logical analysis of the problem, have developed competences to:

1) Make a breakdown of the principal question of a problem that is given generically. In the AMSE and the SM, this becomes stage 1 of the teaching model based on this method: explanation of each of the unknowns of the problem.
2) Split up the problem in such a way that if there is an implicit unknown it is made explicit, and even becomes the principal unknown (ability to change the unknown).
3) Create new unknowns, based on the problem situation, with which one can set solving strategies in motion.
4) Represent relations between the various unknowns.

5) Identify representations of relations to find an element common to one or more of these relations.
6) Represent this identification via an equation.
7) Use algebraic procedures for the solving of equations as a tactic for the search for the unknown in the problem situation.

All of these competences are an important and necessary part of the sense of the Cartesian method.

5.9. The AMSE and the SM use special markers in their representations to release units of memory that allow the progressive setting in motion of the analysis

The AMSE and the SM require the student to learn to leave markers that release units of memory, enabling him or her to use them in the progressive setting in motion of the analysis and subsequently the solution of the problem. These markers can enable him or her to recover the senses of the relations that are established between the data and the unknown quantities.

Some students do not create enough markers, so that in their system of representation only a few of the equations that they propose are correct. These intermediate representations group the information into packets that have a complex organization, although this cannot be distinguished in the notation produced by these students.

5.10. The solution of some problems depends on whether the logical outline establishes a suitable representation

Proposing a more abstract representation is not sufficient to solve some problems. There are problems whose solution depends more on whether the representation established by the logical outline is suitable than on whether it is more abstract.

If one is using arithmetic methods, it is also not sufficient to be capable of retaining everything that one produces in the working memory. In certain families of problems some solvers try to get closer and closer to the result, yet by this path it is very difficult for them to find an equality as they get progressively closer.

In this case the problem is not a matter of not having records of the calculations; the difficulty lies in the fact that one is not taking the logical outline of the problem as a basis for seeking an equation as a representation of what is happening in the problem. Consequently, in order to progress one will require either a more abstract representation or, at least, a more articulated representation of the problem solving process, that is, a representation in which what one seeks is to be in possession of a process in which one carries out a series of stages that enable one gradually to clarify the relations between the data, so that some of these mutual relations can then be identified: in other words, a representation in which the aim is not to find the numeric solution of the problem but to establish the linear equation that models it.

5.11. Some abbreviations that use natural language are related to the production of mistaken representation in the MSSs

Some abbreviations that make use of natural language, referring to arithmetic relations established in the process of solving a problem or in the logical outline of it, possibly combined with the limitations of working memory, encourage non-competent users to produce mistaken representations, both when they use the MSS_a of arithmetic and when they use the MSS_{al} of algebra.

5.12. In some contexts one finds a cognitive tendency to make transfers (mistaken or otherwise) from one problem to another as a result of immediate recognition.

When one presents a problem after another problem with a statement that speaks of similar things but is not of the same family, there is a natural tendency to bring into play automatic processes that are based on a mistaken recognition of known forms or schemes in the statement, with the result that the user produces generalizations that lead him or her to represent the problem in the same way as the preceding one.

This tendency is related to the reading of problems not as the kind of texts that arithmetic-algebraic word problems are, but as narrative texts. The use of this kind of reading to replace the reading that constitutes the logical analysis of the problem situation may lead to errors in the representation of the problem.

5.13. The articulation of mistaken generalizations

When the student gets stuck in readings based on the use of certain parts of the arithmetic-algebraic language that do not enable him or her to solve the problem situation, he tends to get round the difficulty by the device of extending a rule to other contexts where its application does not make sense.

The context may play the part of an obstructer or encourager of an incorrect coding of a representation of a concept in which the user is trying to acquire formal competence through the teaching process. The cognitive tendency of getting stuck in readings made within an MSS prevents the setting in motion of a solving process by means of a different MSS stratum.

SUMMARY

In this chapter we provide the results of an empirical study concerning the competences that are necessary for the use of four methods for solving arithmetic-algebraic word problems. The methods in question are the Method of Successive Analytic Inferences, the Analytic Method of Successive Explorations, the Spreadsheet Method, and the Cartesian Method. Emphasis is placed on the need to be competent in increasingly general and more abstract uses of representations required for mastery of the algebraic method par excellence, the Cartesian Method, and the competences of the Cartesian Method are contrasted with those required for mastery of the other three methods, which are more deeply rooted in arithmetic.

The next chapter concludes the book by describing ways to further study educational algebra.

ENDNOTES

[1] The canonic edition of the works of Descartes is the one by Charles Adam and Paul Tannery, *Œuvres de Descartes*, volume X of which includes the original Latin of the rules. The original posthumous edition is Descartes (1701), and the first French translation is contained in the eleventh volume of the edition by Victor Cousin, *Œuvres de Descartes*, which was published in 1826.

[2] Although Polya says that this sentence paraphrases four of Descartes's rules, rule XIII really contains all that it paraphrases: "Quand nous comprenons parfaitement une question, il faut la dégager de toute conception superflue, la réduire au plus simple, la subdiviser le plus possible au moyen de l'énumération." (Descartes, 1826, p. 284). Previously (rule VII) Descartes has already stated the importance of "énumération", which he defines as "la recherche attentive et exacte de tout ce qui a rapport à la question proposée. […] cette recherche doit être telle que nous puissions conclure avec certitude que nous n'avons rien mis à tort" (Descartes, 1826, p. 235). Rule XIV speaks of the understanding of "l'étendue réelle des corps" and says that the preceding rule also applies to it. Rules XV and XVI are advice for the mind to pay attention to the essential and for the memory not to weary itself with what may be necessary but does not require the attention of the mind. Rule XV recommends drawing figures to keep the mind attentive: "Souvent il est bon de tracer ces figures, et de les montrer aux sens externes, pour tenir plus facilement notre esprit attentif." (Descartes, 1826, p. 313). Rule XVI recommends not using complete figures, but mere jottings in order to unburden the memory, when the attention of the mind is not needed: "Quant à ce qui n'exige pas l'attention de l'esprit, quoique nécessaire pour la conclusion, il vaut mieux le désigner par de courtes notes que par des figures entières. Par ce moyen la mémoire ne pourra nous faire défaut, et cependant la pensée ne sera pas distraite, pour le retenir, des autres opérations auxquelles elle est occupée" (Descartes, 1826, p. 313).

[3] "Il faut parcourir directement la difficulté proposée, en faisant abstraction de ce que quelques uns de ses termes sont connus et les autres inconnus, et en suivant, par la marche véritable, la mutuelle dépendance des unes et des autres" (Descartes, 1826, p. 319).

[4] "C'est par cette méthode qu'il faut chercher autant de grandeurs exprimées de deux manières différentes que nous supposons connus de termes inconnus, pour parcourir directement la difficulté; car, par ce moyen, nous aurons autant de comparaisons entre deux choses égales" (Descartes, 1826, p. 328).

[5] "S'il y a plusieurs équations de cette espèce, il faudra les réduire toutes à une seule, savoir à celle dont les termes occuperont le plus petit nombre de degrés, dans la série des grandeurs en proportion continue, selon laquelle ces termes eux-mêmes doivent être disposés" (Descartes, 1826, p. 329).

[6] "[…] tout l'art en ce lieu doit consister à pouvoir, en supposant connu ce qui ne l'est pas, nous munir d'un moyen facile et direct de recherche même dans les difficultés les plus embarrassées. […] Si […] nous les mettions, quoique inconnues, au nombre des choses connues, pour en déduire, graduellement et par la vraie route, le connu même comme s'il étoit inconnu, nous remplirons tout ce que cette règle exige" (Descartes, 1826, pp. 320–321).

[7] In fact, Fridman (1990) considers only trinomial graphs, that is, graphs with all the edges having three vertices. However, in order to give an account of all the arithmetic-algebraic problems that are set in primary and secondary school it is necessary to consider other kinds of edges: with four vertices (e.g., for relations of proportionality), with two vertices (e.g., for the relation of equality between two quantities), and others (e.g., to give an account of the relations corresponding to the operations of raising to powers and extracting roots –see Nassar, 2001).

[8] This book by Jordanus de Nemore has been published by Barnabas Hughes in Latin, with an English translation and a transcription into the language of modern algebra (Hughes, 1981).

Here we give a different translation of Nemore's propositions. The reasons for this are set out in Puig (1994). The Latin text of the statement of proposition I-7 is as follows (Hughes, 1981, p. 59):
Si dividatur numerus in duo, quorum alterum tantum datum, ex non dato autem in se et in datum provenerit numerus datus, erit et numerus qui divisus fuerat datus.
[9] Our translation is not literal, and in it we have replaced the numbers of Nemore's example with those of the typist problem. The Latin text is given below (Hughes, 1981, p. 59):
Huius operatio est verbi gratia. Sit vi unum dividentium, et ex reliquo in se et in vi fiant xl quorum duplum id est lxxx duplicentur, et erunt clx, quibus addatur quadratum vi hoc est xxxvi, et fient cxcvi, cuius radix est xiiii, de quo sublatis vi et reliquo mediato fient iiii, qui est reliquum. Eritque totus divisus x, coniunctis iiii et vi.

CHAPTER 10

WIDENING PERSPECTIVES

OVERVIEW

In this chapter we put forward points of interest from the perspective of future research, in seven sections: (1) The history of algebraic ideas. (2) The dialectics of syntax and semantics in the study of the generation of errors and in new points of interest concerning the representation of the unknown, based on something that is also unknown. This leads us to analyze the usual treatments for teaching systems of equations and to propose new ways of teaching them based on the use of trial and error, canceling, comparison, and substitution. (3) The study of the new problems posed by information and communication technology (ICT) by analyzing the new roles in the classroom or in the communication between learner and teacher. (4) An analysis of the MSS of Jordanus de Nemore's *De Numeris Datis*, which makes it possible (a) to propose teaching models for second-degree equations; (b) to study the reason for the greater difficulty in the use of factorization compared with the learning of algebraic identities; and (c) to study the difficulties in using an algorithm as a subroutine in another more complex algorithm. (5) Early algebra, a rich new field for research with a huge past of research results. (6) Investigation of theoretical questions that involve the use of research into the instruction imparted by teachers. And (7), in the interests of (6), the need to develop the theoretical aspects of the communication model for the classroom using ICT.

1. HISTORICAL ANALYSIS OF ALGEBRAIC IDEAS

In Chapter 3 we showed the part played by the analysis of the history of algebra in our research, and consequently what kind of historical analysis is of interest to us.

Without repeating all that was said there, our use of history has two fundamental features. On the one hand, it is concerned with an analysis of algebraic ideas. As a result, there is very little interest for us in, for example, questions of dating or of priority in developing the concepts of algebra. What we are basically interested in is identifying the algebraic ideas that are brought into play in a specific text and the evolution of those ideas, which can be seen by comparing texts; in this context we can consider historical texts as cognitions and analyze them as we analyze the performance of pupils, whose productions also constitute mathematical texts (provided that one uses a notion of text such as the one we presented in Chapter 5).

The second feature creates a close bond between historical research and research into mathematics education, which allows us to state that our historical research belongs to research into mathematics education, and is characterized by a two-way movement between historical texts and school systems:

1) The problematics of the teaching and learning of algebra is what determines which texts must be sought out in history and what questions should be addressed to them.
2) The examination of historical texts leads to (a) considering new items that have to form part of the model of competence, (b) having new ways of understanding the performance of pupils and, therefore, of developing the model of cognition, and, lastly, (c) developing teaching models. With all these things teaching experiments are organized and the performance of students is analyzed.
3) Attention is redirected to the historical texts in order to question them once again, now using the results obtained with students, that is, the results derived from the performance of students when all that has been extracted from the analysis of algebraic ideas is incorporated into the teaching model and the analysis of the teaching and learning processes.
4) And so on, repeatedly.

1.1. Current and future research

In this section we indicate some of the directions in which we are turning to history as a result of issues present in the current problematics of research in educational algebra, and another one is mentioned especially in Section 3.

The use of spreadsheets to teach how to solve problems that students were traditionally taught to solve by using some version of the Cartesian method, whether with the aim of serving as an intermediary for the teaching of the Cartesian method or with a view to replacing it with a new method, poses the question of the ways of naming unknown quantities.

In fact, in a spreadsheet one can refer to unknown quantities by assigning one or more cells for one or more unknown quantities, like the way in which one assigns one or more letters to one or more unknown quantities in the second step of the Cartesian Method, and it is advisable to place the cell beneath a cell in which the name of the quantity is written in the vernacular, either complete or else abbreviated in some way. The relationships between the quantities can then be represented as operations expressed in spreadsheet language. In that language the cells are designated by a matrix code consisting of a letter and a number, e.g., B3, which indicates the column and row of the cell, and the cell name then becomes the name of the corresponding quantity in the spreadsheet language. The cell name can be written explicitly using the keyboard or generated by clicking on it with the mouse.

When the spreadsheet is used to solve word problems its language has considerably more complexity than what we have just described, but this brief outline shows that the way in which the unknown is named in this language is different from that of the language of modern school algebra. This has consequences for the use of the spreadsheet in teaching how to solve problems and in the performance of pupils when they are taught in this way. It is therefore worth turning now to historical texts in which the language of modern school algebra had not yet been developed, in order to see how unknown quantities are named in them and what effects the way of naming them has on the way of representing the relationships between quantities that are translated from the statement of the problem and the relationships generated in the course of the calculations.

Therefore, both in the case of the use of the spreadsheet and in that of the teaching of problem solving by using the Cartesian Method, at some point it is necessary to teach the use of more than one letter (or cell) to represent unknown quantities, together with the corresponding way of handling expressions in the corresponding languages. As we pointed out in Chapter 3 and in Puig and Rojano (2004), most of the languages of algebra prior to Viète were incapable of this, and yet problems were solved that we would now naturally represent by using more than one letter. It would be interesting, therefore, to turn to the historical texts once again in order to examine the ways in which this was done.

Finally, the use of graphic calculators, with the possibility of collecting data with calculator-based ranger (CBR) or calculator-based laboratory (CBL) sensors, to teach the idea of family of functions as a means for organizing phenomena through the modeling of real situations poses the problems of the establishment of canonical forms in which the parameters express properties of each family of functions and, therefore, of the situations that are modeled. Bound up with this is the development of a calculation, that is, a set of algebraic transformations, which makes it possible to reduce the expressions obtained by modeling the real situation to one of the canonical forms. The history of the idea of canonical form (and that of calculation or algebraic transformations that is bound up with it) thus acquires a new perspective which is worth studying, in addition to the aspect studied in Chapter 2 and in Puig and Rojano (2004).

2. COGNITIVE TENDENCIES AND THE INTERACTION BETWEEN SEMANTICS AND ALGEBRAIC SYNTAX IN THE PRODUCTION OF SYNTACTIC ERRORS

The literature on algebraic errors in the learning of algebra concentrates mainly on their syntactic component (Matz, 1982; Kirshner, 1987; Drouhard, 1992). There are not many books like Booth (1984) and Bell (1996) that place this problematic component in a more general context such as problem solving. In this section we analyze the interaction between semantics and algebraic syntax as a source of syntactic errors when the interaction takes place in a teaching process that uses concrete models. We defend the view that an analysis of this kind provides a perspective that enables us to give explanations that are different from the usual explanations for the presence of certain typical errors of algebraic syntax.

Undertaking a semantic introduction of new algebraic concepts, objects and operations involves selecting a concrete situation (i.e., a situation which is familiar to the learner in some context) in which such objects and operations can be modeled. With this focus it is possible to make use of previous knowledge in order to achieve the acquisition of new knowledge. This is one of the guiding principles of modeling, the strengths and weaknesses of which appear as soon as a specific model is brought into operation (see Chapter 5, for example).

2.1. Different tendencies

In Chapter 4 we introduced the syntactic/semantic opposition with respect to cognition.

The antagonism of these two tendencies (Vt's and Ma's) was evident from mere observation of their respective interviews. However, from a comparative analysis of them, there are a couple of points concerning aspects common to both cases that deserve to be emphasized. One can see, on the one hand, that despite the antagonism just mentioned there is a common tendency to abbreviate the process (with the pupils going their own way to perform the abbreviation in each case); and, on the other hand, a certain number of obstacles and errors are generated during these abbreviation processes that are also common, and that can be considered as typical of the subsequent syntactic handling of symbolic algebra. In one case (Vt), the tendency to abbreviation consists in trying to ease the operations performed in the model, but remaining in it. To do this it is necessary to pay attention to the actions (translation, comparison, etc.) that are performed repeatedly. This reflection leads, in turn, to a process of abbreviation of those actions. It is precisely through this abbreviation that some parts of the concrete model are lost: on the one hand, the “base-line” (the linear dimension that corresponds to the unknown), and, on the other, the area condition of the constant term and the operational handling of it. This leads to a tendency to perform the addition of x with the terms of degree zero, resulting in an aberrant operation between terms of different degrees.

2.2. Syntactic errors

The generation of the same kind of syntactic errors in the two cases that we have just mentioned is not accidental and can be explained by an analysis of the general level, as we did in Chapter 6. When one teaches with models there is a danger that what constitutes the main virtue of any concrete model (i.e., the fact that it seeks the support of previous knowledge) may become the main obstruction for the acquisition of new knowledge. In the cases of the pupils interviewed, who were allowed to develop the use of the geometric model on their own, what happened was that the component of the model which tended to abbreviate, and therefore to conceal, the operation with the unknown persisted in both cases. In cases like Vt’s, pupils who possess a strong semantic tendency allow this to happen because the automation of the actions in the model weakens the presence of the unknown throughout the whole procedure. In cases like Ma’s, this tendency is due to the effects of the creation of personal codes, created to record intermediate states of the equation proposed originally. The corrections in each case are of a local nature and in accordance with the tendency of each pupil. Thus, when there is an inclination to remain in the model, the correction of the syntactic aberration mentioned earlier is performed in the model itself, because only semantic models can make such an aberration evident. In the case of the syntactic tendency, however, the correction is normally performed together with other events in the syntax, specifically through an essential modification of the notions of equation and unknown.

2.3. New studies needed

By way of conclusion, the interaction between semantics and algebraic syntax, which is presented throughout the processes of abstraction of the operations performed with algebraic objects (which have been endowed with meaning and sense in the context of a concrete model) when learning the language of algebra, is modulated by the tendencies of the individual and by features of the specific model which is being used. Nevertheless, there are some aspects of this interaction that remain constant when there is a change in the tendency of the individual or the type of model. These essential aspects of the relationship between semantics and algebraic syntax in turn reflect essential aspects of another interaction that appears between the two basic components of the model, namely, the reduction to the concrete and the relinquishing of the semantics of the concrete. The transfer of the problem of semantics versus algebraic syntax to the level of actions with the model enables us to close the breach that exists between these two domains of algebra. The analysis of the interaction between semantics and syntax on this new level points to the need to intervene with the teaching model at key moments in the early days of the use of algebraic language.

This dialectics of semantics and syntax and the theoretical description of the relationships between the deep and superficial forms of the syntax of the MSS of algebra, i.e., the generative and transformational aspects of the description of grammar (cf. Kirschner, 1987; Drouhard, 1992), may be linked to the explanation of why mistakes are made when, in order to follow a rule, it is necessary to use one or more rules that previously were used competently.

Future research will clarify this matter. Moreover, in this context it is possible to continue with the study of the solution of systems of equations when solving problems, as we indicated at the end of Chapter 8.

3. JORDANUS NEMORARIUS'S *DE NUMERIS DATIS* AS AN MSS. THE CONSTRUCTION OF A TEACHING MODEL FOR THE SECOND-DEGREE EQUATION AND THE INTRODUCTION OF CERTAIN ALGEBRAIC IDENTITIES

In this book we have proposed a certain kind of reading of the classic texts of the history of mathematics (see Chapter 3 and Section 1 in this chapter).

From that viewpoint we have studied the MSS of Jordanus de Nemore's book *De Numeris Datis* (cf. Puig, 1994, where there is also a detailed description of the propositions of Book I of that work and three propositions from Book IV which are equivalent to al-Khwârizmî's three compound canonical forms), seeking in it the characteristics of the mathematical sign system (MSS) —or mathematical sign systems— in which it is written, the way in which that language configures the objects that one can speak of with it, the problems that it poses and seeks to solve, and the operativity that the MSS has over the objects expressed in it. As we have already indicated in Chapter 6, we conceive the construction of symbolic algebra as the final identification, within a single language stratum, of earlier language strata that are irreducible from one stratum to another until the more abstract language has been developed. From this perspective, the interest in turning to a 13th-century text such as Nemorarius's work, which is prior to Viète's establishment of the language of symbolic algebra, lies in the possibility of taking it as a monument and describing one of the MSSs that are seen retrospectively, from the viewpoint of symbolic algebra, as being less abstract. This interest attaches to research into the didactics of mathematics as soon as one conceives that what students do when they learn symbolic algebra and are taught in educational systems can also[1] be described in terms of the use of MSSs —some of them idiosyncratic— which have to culminate in the competent use of the more abstract MSS of symbolic algebra —or, at least, that is the aim of educational systems.

We will now present an analysis of proposition I-7,[2] which shows how the results of the analysis of *De Numeris Datis* set out in Puig (1994) could be used to construct a teaching model (which includes the use of new technologies) to teach how to solve the second-degree equation, and to study the difference in the difficulty of the use of algebraic identities, on the one hand, and the factorization of algebraic expressions, on the other. To give some idea of the construction of such a teaching model we will also reproduce that proposition from *De Numeris Datis*, interpreted in a figure.

The statement of proposition I-7 is the following:

If we divide a number into two parts, one of which has been given, and the product of the other by itself and by the one that has been given is a given number, the divided number will also have been given.

This proposition is important in the general organization of Nemore's book, as many other propositions reduce to it. On the other hand, if it is translated into the MSS of modern algebra the statement reduces to the equation $x^2 + ux = v$, which is the first of al-Khwârizmî's canonical forms.

This translation is obtained if the analysis of the statement determines the following quantities:

> the number that is divided,
> the part given,
> the part not given,
> the other quantity given (the part not given by itself and by the other given part),

and one then decides to represent "the part not given" with the letter *x*, and the data with the letters *u*, "the part given," and *v*, "the other quantity given," and one constructs the expressions $x + u$, for "the number that is divided," and $x (x + u)$, for the other given quantity, so that one can construct the equation $x (x + u) = v$ (equivalent to al-Khwârizmî's first canonical form).

However, the argument that Jordanus de Nemore develops to prove this proposition goes as follows (the division into parts is ours):

> [1] Let the number be divided into *a* and *b*, *b* given
> [2] Let *a* by itself and by *b*, that is, by all of *ab*, be *d*, given
> [3] Let us also add *c* to *ab*, and let *c* be equal to *a*
> [4] Thus, all of *abc* is divided into *ab* and *c*
> [5] So that *ab* by *c* is *d*, given
> [6] and the difference between *ab* and *c* is *b*, given
> [7] *abc* and *c* will be given, and equally *a* and *ab*.

To understand this argument one must know that Nemore has already proved in proposition I-5 that "if a number is divided into two parts, the product and difference of which have been given, the two parts will also have been given", so that what Nemore does is to reinterpret the quantities of the statement of I-7 so that the situation is that of I-5. Table 10.1 shows the correspondence between the letters that Nemore uses in the argument, the quantities in the statement that they represent, the meaning that he gives them so as to be able to use I-5, and the corresponding algebraic expressions in the MSS of modern algebra, and Figure 10.1 gives a representation of them all as lengths and areas.

Table 10.1

The letters in Nemore's MSS	Meaning in the statement	Meaning in the interpretation	Expression in the MSS of modern school algebra
a	the part not given	the smaller part (*s*)	*X*
b	the part given	the difference (*d*)	*U*
ab	the number that is divided	the larger part (*l*)	$x + u$
c	a number equal to *a*	the smaller part (*s*)	*X*
d	the part not given by itself and by the part given: the other given quantity	the product (*p*)	$x (x + u) = v$
abc		the total (*t*)	$x^2 + ux = v$

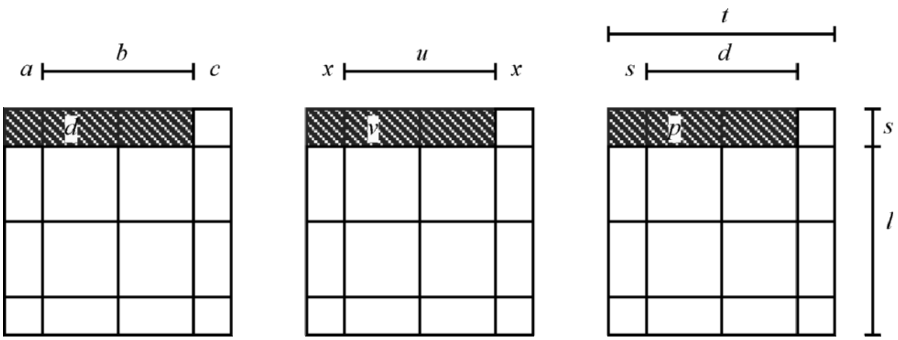


Figure 10.1

The divisions of the square in Figure 10.1 are selected in such a way that they easily represent the smaller part and the larger part into which a number has been divided (or, alternatively, the sum of two numbers), the difference of the parts, their product, the square of each of them, etc. Then the algebraic identities and the Babylonian procedures of cutting and pasting to solve quadratic problems are represented in this figure by shading parts of it and using the equalities between the subdivisions of the square, so that it can be used in the study of the teaching of these aspects of school algebra with a concrete model. By way of example, in Figure 10.2 we show the algebraic identity "sum by difference equals difference of squares."

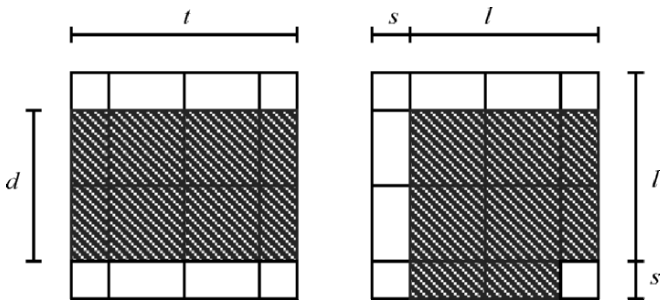


Figure 10.2

4. NEW USES OF TECHNOLOGY IN THE CLASSROOM AND THE COMMUNICATION MODEL

The powerful resources that are available nowadays, from calculators to computers, offer new ways of teaching mathematics. The numeric strategies and visual focuses that they provide represent a challenge to symbolic algebra as a means for obtaining the desired competences in solving problems. These numeric and visual resources enable us to design teaching activities by which students work with questions of a mathematical nature without having previously had any formal introduction to the mathematics involved in them.

Much stress has recently been laid on the use of computerised environments, but little light has been cast (from a theoretical viewpoint) on how this use has led to a new organization of classrooms. The result is undoubtedly beneficial, in view of the obsoleteness of teaching practices. And so it seems relevant that we should concentrate our discussion on the organization of the classroom environment.

A reflection on the relationship between theory and practice in the context of innovative approaches in the teaching and learning of algebra leads one to take into account both real classroom practice and curriculum contents such as those that result from the study of new proposals.

It is worth concentrating attention on the changes that have taken place in school classroom practice with the introduction of innovative techniques, whether or not they are based on the use of information and communication technology. It would be interesting to study the changes in the role of the teacher, the role of the students, and the role of the environments in which the teaching activities take place.

The continuation of studies such as the ones that we presented in Chapter 9 (about problem solving) and those that are connected with the processes of generalisation in algebra is bound up with the use of ICT. Much research has already been done on how the environments of technology can affect the learning and teaching of mathematics.

Specifically for the case of algebra, it is known that environments such as LOGO and spreadsheets allow the design of student activities for learning to express and manipulate the general. For example, it is known that young students are capable of perceiving the regularity of a pattern in sequences of numbers or figures, but that they find great difficulty in expressing the regularity with an algebraic formula, and it is even more difficult for them to manipulate such a formula (if they know it) in order to analyze the sequence, make predictions, generate new terms, etc. By using a spreadsheet the students can try to reproduce a given sequence, first numerically and then by introducing a general formula to generate the sequence and checking whether the formula is correct. This not only gives them the possibility of expressing the general in a language similar to that of algebra but also allows them to explore and discover properties of the sequence dynamically. That is to say, in this environment the transition to the expression and manipulation of the general can take place in an exploratory and experimental manner. In a similar way the regularities in a figure can be expressed in a LOGO program and the students can analyze them by using the program; and when they are the ones who write the program, they take control of processes of generalization that can be of great complexity.

In another area, symbolic manipulators (CAS) and even the spreadsheet have been used to help pupils to solve algebraic word problems, probably one of the most difficult items to teach in the curriculum. An argument widely accepted for the use of CAS is that the pupil concentrates on the phases of the analysis of the text of the problem and its translation into algebraic code without having to fight with the syntactic manipulation of the equation (or system of equations) to find the solution. Finding the solution is the responsibility of the automated processes of the symbolic manipulator. On the other hand, the spreadsheet has been applied to help students to organize the information of the text of a problem in cells and to develop formulas that express functional relationships between unknown and data and between unknowns (when there are two or more). The solution of the problem can then be found by means of the numerical variation of one of the unknowns. Thus the spreadsheet method also emphasizes the process of analysis of the text of the problem, but here the development of the formulas (in spreadsheet syntax) is done with an arbitrary initial numeric assignation to the value of one of the unknowns, which serves as an independent variable in the search for the solution. That is, unlike CASs, in the spreadsheet method the stage of posing and solving the problem is not done with the language of algebra, but it is done with a strong numeric support, which is favored by the predominance of arithmetic thinking in adolescent pupils.

The examples just given illustrate only a few of the uses of computational environments or technology that have been studied, proposed, and implemented for the teaching of algebra in some educational systems. However, the question of ICT in the teaching of algebra is far from being an exhausted topic in the field of research. The dynamic quality and immediate visual feedback of computer methods, so highly praised since they began to be used in education, do in fact play a fundamental part in the cases to which we have referred. But there is also widespread recognition of the fact that equally fundamental is the possibility of taking control of that dynamism by working with executable expressions, written and manipulated with a code (that of the software) which is very similar to algebraic code, but that is not, or does not behave, completely like symbolic algebra. This naturally raises the question of whether, in the long term, the use of these tools will tend to replace the manipulative aspects of algebra in teaching, or whether the intention is that the students' experiences with technological tools may eventually link up with the syntax of the algebra of pencil and paper.

These seemingly simple questions really give rise to fundamental considerations: on the one hand, concerning the importance of whether or not, in teaching, one conserves certain levels of competence in the use of the basic language of mathematics and science, and, on the other, concerning matters of situated learning, of the possibility of transferring knowledge and competence from one medium to another. These queries, subject to the specificity of the theoretical formulations that have been given in the preceding chapters, give rise to items in a research agenda, focusing on matters of interaction between languages, or rather of transition between mathematical sign systems of different levels of abstraction, inside and outside computer environments. This last aspect would include analysis, with this semiotic perspective, of the studies currently in progress, carried out with environments of simulation and graphing, where one is working with phenomena of variation and where the absence of the analytical representations of the functional relationships is deliberate.

5. EARLY ALGEBRA

For the presentation of this field we consider the following question to be of capital importance: What forms does syntactic competence adopt in early algebra? Has it been expelled from early algebra or has it only been hidden? In Section 5.3 we will give a reply that points to the need, when researching in this field, to take both the syntactic competences and the cognitive tendencies generated into account.

5.1 Generalization

To set bounds to this point of view, we will initially confine ourselves to one of the subsidiary areas most studied currently, generalization, given its importance in the implementation of curriculum developments for the youngest learners, for whom the aim is to start to lay the foundations of what will ultimately be developed in algebra in later years. See Ginsburg, Inoue and Seo (1999); Usiskin (1999); Cuevas and Yeatts (2001); Friel, Rachlin and Doyle (2001); Greenes, Cavanagh, Dacey, Findell and Small (2001); Malara and Navarra (2003).

5.2. Implementation of the teaching model

We can analyze the kind of tasks connected with generalization that have recently preoccupied the research community with seven examples taken from research papers (presented in strictly alphabetical order): (1) Iwasaki and Yamaguchi (1997), (2) Lee (1996), (3) Mason (1996), (4) Radford (2000b), (5) Stacey (1989), (6) Zazkis and Liljeddahl (2001) and (7) Zazkis and Liljeddahl (2002).

In general, in these articles we find:

1) Exchange of messages with mathematical texts (T_n), in an attempt to get the learners to acquire the desired competences. Usually these articles show a good performance on the part of the learners presented in parts of the series of texts T_n and the probable appearance of personal codes in the performances.
2) Some learners are successful in producing sense for the series of texts T_n and they thereby become competent users of the language game proposed by the series T_n (the Process of Generalization).
3) In all cases, what has been said in (1) and (2) implies the use of a stratum of the MSS in which syntactic competences in arithmetic and the rudiments of algebra are necessary.
4) Point (3) forces the appearance in the users of cognitive tendencies that have been well known for some time: those described in (a) Vergnaud (1981), (b) Carpenter, Moser and Romberg (eds.) (1982), (c) Fuson (1998), (d) Kieran (1981), (e) Booth (1984), (f) Figueras, Filloy and Valdemoros (1985, 1986), (g) Filloy and Lema (1996), (h) Filloy and Rojano (1989) (reverse of multiplication syndrome, polysemy of *x*, very many cognitive tendencies indicated in this book), (i) Filloy and Rubio (1991, 1993a, 1993b), (j) Filloy, Rojano and Rubio (2001), etc.

5.3. Grammatical change

To sum up what has been said: the reorganization of the new conceptual field developed on the basis of achieving the necessary competences in the use of the language set involves a change in the logical syntax of the new stratum of the MSS.

5.4. Research needed

We therefore have to put forward new local theoretical models (LTMs) to be used with these K–8 students to study the processes of abstraction and visualization characteristic of the use of ICT: electronic blackboards, symbolic calculators, specific software (spreadsheets, Cabri, SimCalc, etc.), programming languages (LOGO, the language of electronic calculators, Visual Basic, etc.), optical readers, videotaped interviews, and so on.

The cognition components of these new LTMs will have to take into account all the cognitive tendencies detected in the past, which were presented summarily in point (4) of 5.2.

6. RESULTS OF RECENT RESEARCH INTO PROBLEMS OF LEARNING ALGEBRA USED AS THE CORE FOR IN-SERVICE COURSES IN THE TEACHING OF MATHEMATICS

The need to apply a theoretical focus to the problem of teaching mathematics began to become evident in the middle of the last century. From the outset, this new awareness drew the attention of groups of mathematicians, educators, psychologists, and epistemologists, stimulating the introduction of new curricula at all levels of the educational system. This attitude resulted in the appearance of many areas for investigation that had not been studied previously and which posed unexpected problems.

The changes in the mathematics curriculum made it essential for teachers to have some knowledge that fit in with the new ideas about the teaching of mathematics. The need for researchers in this field also emerged.

6.1. Some topics for discussion

The development of a theoretical focus in research has led to the posing of questions such as the following:

1) What is the role of a theory in the didactics of mathematics?
2) How can a teacher obtain some benefit from learning theories of the didactics of mathematics or research methods used in the didactics of mathematics?

3) To what extent are research results relevant for conditions in the real world? To what extent are they useful for teachers in their day-to-day activities in the classroom?

This first group of questions arises as a result of the development of theoretical perspectives that, because of their complexity, are not easily accessible for teachers. Moreover, there seems to be a view that “didactic objectivity” can exist independently of the real conditions in which the processes of teaching and learning take place in current school systems. What, then, is the point of studying “didactics”? Of what use is the discourse in which its concepts are expressed? Could one sensibly imagine a course of study on “didactics” that did not aim to change the real practices that characterise the teaching and learning of mathematics?

It is certainly significant in this respect that five of our studies on algebra which are presented in this book were developed with the assistance of teachers and pupils in the educational systems in the countries where the authors live.

We ought to point out that in each of those studies the attitudes of the teachers involved underwent a transformation as the experimental work advanced. It is appropriate, therefore, to contrast the use of theories in the design of experiments for research and even the interpretation of the results with the intermediate products of the experiment as it progresses. In the experiments in the studies just mentioned, the teachers were involved directly as a fundamental part of the experiment and they produced a great abundance of ideas about the immediate results.

It is worth asking oneself how profound and penetrating an idea a teacher needs to have of the fundamentals of a didactic theory or research methodology to be influenced by the results of the experiment in the direction of changing his basic attitudes toward his work and getting the corresponding improvement from it. The examples in our studies suggest that a complete understanding of the theory used is not essential for the experiment to produce a wealth of useful data for the teachers who take part in the experiment, or for those data to be provided to other teachers. If it were essential we would be in a no-win situation, as it would be necessary, before a teacher could use the results of the study, for him to become an expert in the theoretical fields on which it was based: psychology, linguistics, artificial intelligence, epistemology, the history of ideas, etc., and also on the processing and interpretation of data, including advanced techniques of data analysis or methods for processing observations, etc.

6.2 Research needed

The following questions should be the focus of future research:

1) Why and how can many of the immediate results of research in educational mathematics be expressed in the direct discourse that teachers use in their ordinary speech?
2) Why are those results perceived as new information, previously unknown to the teachers who take part in the experiments and in the discussions held at the end of each experiment with other teachers not involved in them?
3) Why is this information perceived by the teachers as decisive for a new way of seeing their problems and changing what they previously thought was normal (in their activities in day-to-day teaching)?
4) Why is it that, as a result of a practical experiment, previous practices of teaching are seen by the teachers as inconsistent with the immediate results produced by the study that is being carried out?

In order to do research to answer questions such as these we will have to separate the theoretical framework that supports semiotic research in educational algebra (with teachers involved in it) from the theoretical framework on the basis of which the teachers play an active part in designing the research. To achieve this we will have to develop our ideas about the communication model. In the following section we put forward some ideas on which to work in the future. It is also necessary for what is proposed in Sections 3 and 5 of this chapter.

7. OBSERVATION IN THE CLASSROOM. A SEMIOTIC PERSPECTIVE

The only occasion on which the communication model has been used in this book is in the final example in Chapter 8. It is worth reproducing the conclusion that we reached in point 4.4.3, entitled “Different levels of abstraction: the case of names”: “The communication model enables us to establish the difference in the readings given by the interviewer and the student […] This difference in the meanings attributed to algebraic expressions is present in any process of teaching algebra where the teacher is a competent user and the students are learners, generating difficulties such as those reported here (tendencies 2 and 5).”

The words just quoted could be the most general thesis that guides the design of any communication model. In this book, as the observations reported concentrated on the results of the case studies, concluding with clinical interviews, the communication model has not been used much, and consequently it has not been introduced much; we have merely mentioned a few general considerations concerning it in Chapter 8. But if one is trying to analyze situations that occur in the classroom, or any situation in which there is a teacher (human or an interactive electronic teaching model) and learners, it is necessary to develop its theoretical aspects further.

CODA

Chapter 10 brings the book to a close, proposing a research agenda that includes questions and topics that arise from the discussions and analyses presented in the preceding chapters, from the perspective of local theoretical models, mathematical sign systems, and historical analysis. To start with, we propose going back to history to study the evolution of certain algebraic ideas, analyzing historical texts as cognitions in the same way that we analyze students' productions, which in turn constitute mathematical texts in the sense used in Chapter 5. We also express an interest in making a deeper study of operating with the unknown, on a second level of representation, that is, when it is expressed in terms of another unknown (a typical situation in the solution of systems of equations with two or more unknowns) and in exploring new methods for solving systems of two equations with two unknowns, going beyond the classical methods of substitution and comparison. Also in the field of teaching, it is of interest to explore teaching models for solving the quadratic equation, based on analysis of Jordanus de Nemore's work *De Numeris Datis*. We also propose the need for a greater theoretical development with respect to models of communication in the classroom for the case of the teaching of algebra. In particular, it is of interest to investigate how the use of ICT affects patterns of communication in the classroom (the communication component in local theoretical models). This last point leads naturally to the theme of using the results of research in teacher training. Finally, the research trend which has acquired importance in recent years concerning the early introduction to algebra is also put forward, with questions such as: What forms does syntactic competence adopt in early algebra? Is this competence avoided in early algebra or is it present implicitly? We invite the reader to engage with these questions from the theoretical perspective expounded throughout this book, in which the identification of “the algebraic” in these early approaches is closely connected with explicit handling of the mathematical sign system of symbolic algebra.

ENDNOTES

[1] The MSSs on which the more abstract MSS is erected, present in history and in the history of each individual, cannot be the same; nor, therefore, can the paths toward the more abstract MSS. In the case of algebra, one need only take account of the fact that modern school arithmetic is not written in the vernacular but in an MSS imbued with signs and even rules of syntax that come from the MSS of symbolic algebra and which have come down from it to arithmetic.

[2] This proposition has already appeared in Chapter 9. There we used the third part of the proposition, the problem with which the algorithm obtained for the solution is exemplified, whereas here we will analyze the second part, which develops the proof of the proposition.

CHAPTER 11

REFERENCES. A DEEP SEA OF LUMINESCENT IDEAS

Abel, N. H. (1881). *Œuvres Complètes*. Tome I. Christiana: Grondahl & Son. [Reedición Sceaux: Éditions Jacques Gabay, 1992]

Alarcón, J., Figueras, O., Filloy, E., Gispert, M., Lema, S., Parra, B. M., Recio, J., Rojano, T. and Zubieta, G. (1981–1982). *Matemáticas 100 Horas, 1 y 2*. México, Bogotá, Caracas, Santiago, San Juan y Panamá: Fondo Educativo Interamericano.

al-Tûsî, S. (1986). *ŒuvresMmathématiques, Algèbre et Géométrie au XIIe siècle*. Tomes I et II. Texte établi et traduit par Roshdi Rashed. Paris: Les Belles Lettres.

Anbouba, A. (1978). L'algèbre arabe aux IXe et Xe siècles. Aperçu général. *Journal for the History of Arabic Science*, *2*, 66–100.

Bachmann, H. (1955). *Transfinite Zahlen*. Berlin: Springer Verlag.

Bednarz, N. and Janvier, B. (1996). Emergence and development of algebra as a problem-solving tool: Continuities and discontinuities with arithmetic. In N. Bednarz, C. Kieran, and L. Lee (Eds.), *Approaches to Algebra: Perspectives for Research and Teaching* (pp. 115–136). Dordrecht, The Netherlands: Kluwer Academic.

Bednarz, N., Radford, L, Janvier, B., and Lapage, A. (1992). Arithmetical and algebraic thinking in problem solving. In W. Geeslin and K. Graham (Eds.), *Proceedings of the 16th International Conference for the Psychology of Mathematics Education* (Vol. 1, pp. 65–72). Durham, NH: PME Program Committee.

Bell, A. W. (1996). Problem-solving approaches to algebra: Two aspects. In N. Bednarz, C. Kieran, and L. Lee (Eds.), *Approaches to Algebra. Perspectives for Research and Teaching* (pp. 167–186). Dordrecht/Boston/London: Kluwer Academic Publishers.

Benacerraf, P. (1983). What numbers could not be. In P. Benacerraf and H. Putman (Eds.), *Philosophy of Mathematics. Selected Readings* (pp. 272–294). Cambridge, UK: Cambridge University Press.

Boncompagni, B. (Ed.) (1857). *Scritti di Leonardo Pisano Matematico del Secolo Decimoterzo. I. Il Liber Abbaci di Leonardo Pisano*. Roma: Tipografia delle Scienze Matematiche e Fisiche.

Boohan, R. (1994). Interpreting the world with numbers: An introduction to quantitative modelling. In H. Mellar, J. Bliss, R. Boohan, J. Ogborn, and C. Tampsett (Eds.), *Learning with Artificial Worlds* (pp. 49–58). London: The Falmer Press.

Booth, L. (1984). *Algebra: Children's Strategies and Errors.* Windsor: NFER-Nelson.

Bortolotti, E. (Ed.) (1966). *R. Bombelli. L'Algebra*. A cura di U. Forti e E. Bortolotti. Milano: Feltrinelli.

Bourbaki, N. (1966). *Éléments de Mathématique. Fascicule XVII. Théorie des Ensembles. Chapitre 1 et 2. Description de la Mathématique Formelle. Théorie des Ensembles*. Paris: Hermann.

Boyer, C. B. (1959). *The History of the Calculus*. New York: John Wiley.

Boyer, C. B. (1968). *A History of Mathematics*. New York: John Wiley.

Brousseau, G. (1996). Fondements et méthodes de la didactique des mathématiques. *Recherches en Didactique des Mathématiques*, *7*, 33–115.

Brousseau, G. (1997). *Theory of Didactical Situations in Mathematics*. Dordrecht/Boston/London: Kluwer Academic Publishers.

Brown, T. (2001). *Mathematics Education and Language. Interpreting Hermeneutics and Post-Structuralism*, revised second edition. Dordrecht/Boston/London: Kluwer Academic Publishers.

Carpenter, T., Moser, J. and Romberg, T. (Eds.) (1982). *Addition and Subtraction: A Cognitive Perspective*. Hillsdale, NJ: Lawrence Erlbaum.

Cerdán, F. (in preparation). *Problemas verbales aritmético-algebraicos. El método de análisis y síntesis y el método cartesiano*. Tesis doctoral. Universidad de Valencia.

Clagett, M. (1959). *The Science of Mechanics in the Middle Ages*. Madison, Milwaukee and London: The University of Wisconsin Press.

Clagett, M. (1968). *Nicole Oresme and the Medieval Geometry of Qualities and Motions*. Madison, Milwaukee and London: The University of Wisconsin Press.

Colebrooke, H. T. (ed. and trans.) (1817). *Algebra with Arithmetic and Mensuration from the Sanscrit. Brahmegupta and Bháscara*. London: John Murray.

Cuevas, G. J. and Yeatts, K. (2001). *Navigating Through Algebra in Grades 3–5*. Reston, VA: National Council of Teachers of Mathematics.

Chevallard, Y. (1983). Le passage de l'arithmétique à l'algébrique dans l'enseignement des mathématiques au Collège. *Petit X*, *5*, 51–94.

Descartes, R. (1701). *Opuscula Posthuma Physica et Mathematica*. Amsterdam: Typographia P. and Blaev J.

Descartes, R. (1826). *Règles pour la Direction de l'Esprit*. In *Œuvres de Descartes*, publiées par Victor Cousin. Tome onzième. Paris: Chez F. G. Levrault, libraire.

Descartes, R. (1996). *Regulæ ad Directionem Ingeni.* In *Œuvres de Descartes*. Tome X. Édition de Charles Adam et Paul Tannery. Paris: Librairie Philosophique J. Vrin.

Drouhard, J. P. (1992). *Les Écritures Symboliques de l'Algèbre Élémentaire*. Thèse de doctorat, Université Dénis Diderot, Paris 7.

Duval, R. (1995). *Sémiosis et Pensée Humaine. Registres sémiotiques et apprentissage intellectuels*. Berne: Peter Lang.

Eco, U. (1984). *Semiotics and Philosophy of Language*. London: The Macmillan Press.

Edwards, C. (1979). *The Historical Development of the Calculus*. New York: Springer Verlag.

Figueras, O., Filloy, E. and Valdemoros, M. (1985). The development of spatial imagination abilities and contextualisation atrategies: Models sased on the teaching of fractions. In L. Streefland (Ed.), *Proceedings of the Ninth International Conference for the Psychology of Mathematics Education* (Vol. 1, pp. 328–333). Noordwijkerhout, The Netherlands.

Figueras, O., Filloy, E. and Valdemoros, M. (1986). Some considerations on the graphic representation of fractions and their interpretation. In G. Lapan and E. Ruhama (Eds.), *Proceedings of the Eighth Annual Meeting of the PME-NA* (pp. 78–83). East Lansing, Michigan.

Filloy, E. (1980). Álgebra del nivel medio y análisis epistemológico: de Bombelli a Vieta. *Actas del V Congreso Nacional de Profesores*. México.

Filloy, E. (1990). PME algebra research. A working perspective. In G. Booker et al. (Eds.), *Proceedings of the Fourteenth Annual Conference for the Psychology of Mathematics Education* (Vol. 1, pp. PII 1–PII 33). Oaxtepec, Morelos, México.

Filloy, E. (1991). Cognitive tendencies and abstraction processes in algebra learning. In F. Furinghetti (Ed.), *Proceedings of the Fifteenth Annual Conference for the Psychology of Mathematics Education* (Vol. 2, pp. 48–55). Assisi, Italia.

Filloy, E. (1993a). El libro de los cuadrados de Leonardo de Pisa. In E. Filloy, L. Puig, and T. Rojano (Eds.), *Memorias del Tercer Simposio Internacional sobre Investigación en Educación Matemática. Historia de las Ideas Algebraicas* (pp. 11–30). México, DF: CINVESTAV/PNFAPM.

Filloy, E. (1993b) Tendencias cognitivas y procesos de abstracción en el aprendizaje del álgebra y de la geometría. *Enseñanza de las ciencias*, *11*(2), 160–166.

Filloy, E. and Lema, S. (1996). El Teorema de Thales: significado y sentido en un sistema matemático de signos. In F. Hitt (ed.), *Investigaciones en Matemática Educativa* (pp. 55–75). México, DF: Grupo Editorial Iberoamérica.

Filloy, E. and Rojano, T. (1983). *Movimiento de la Incógnita, Terminación de la Aritmética?* México: SME.

Filloy, E. and Rojano, T. (1984). From an arithmetical to an algebraic thought (a clinical study with 12–13 year olds). In J. Moser (ed.), *Proceedings of the Sixth Annual Meeting for the Psychology of Mathematics Education, North American Chapter* (pp. 51–56). Madison, WI, USA.

Filloy, E. and Rojano, T. (1985a). Obstructions to the acquisition of elemental algebraic concepts and teaching strategies. In L. Streefland (ed.), *Proceedings of the Ninth Annual Conference for the Psychology of Mathematics Education* (pp. 154–158). Utrecht, Holanda.

Filloy, E. and Rojano, T. (1985b). Operating the unknown and models of teaching (a clinical study with 12-13 year olds with a high proficiency in Pre-Algebra). In S. K. Damarin and M. Shelton (eds.), *Proceedings of the Seventh Annual Meeting for the Psychology of Mathematics Education, North American Chapter* (pp. 75–79). Columbus, OH.

Filloy, E. and Rojano, T. (1989). Solving equations: The transition from arithmetic to algebra. *For the Learning of Mathematics*, *9*(2), 19–25.

Filloy, E. and Rojano, T. (2001). Algebraic syntax and word problems solution: First steps. In M. van den Heuvel-Panhuizen (ed.), *Proceedings of the Twenty-fifth Annual Conference for the Psychology of Mathematics Education* (Vol. 2, pp. 417–424). Utrecht, The Netherlands.

Filloy, E. and Rubio, G. (1991). Unknown and variable in analytical methods for solving word arithmetic/algebraic problems. In R. G. Underhill (ed.), *Proceedings of the Thirteenth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 64–69). Blacksburg, VA.

Filloy, E. and Rubio, G. (1993a). Didactic models, cognition and competence in the solution of arithmetic and algebra word problems. In I. Hirabayashi, N. Nohda, K. Shigematsu, and F. Lin (eds.), *Proceedings of the Seventeenth International Conference for the Psychology of Mathematics Education* (Vol. 1, pp. 154–161).Tsukuba, Ibaraki, Japan.

Filloy, E. and Rubio, G. (1993b). Family of arithmetical and algebraic word problems, and the tensions between the different uses of algebraic expressions. In J. R. Becker and B. J. Pence (eds.), *Proceedings of the Fifteenth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 142–148). Pacific Grove, CA.

Filloy, E. and Sutherland, R. (1996). Designing curricula for teaching and learning algebra. In A. Bishop, K. Clements, C. Keitel, J. Kilpatrick and C. Laborde (eds.), *International Handbook of Mathematics Education* (Vol. 1, pp. 139–160). Dordrecht/Boston/London: Kluwer Academic Publishers.

Filloy, E., Rojano, T. and Rubio, G. (2001). Propositions concerning the resolution of arithmetical-algebraic problems. In R. Sutherland, T. Rojano, A. Bell, and R. Lins (eds.), *Perspectives on School Algebra* (pp. 155–176). Dordrecht/Boston/London: Kluwer Academic Publishers.

Filloy, E., Rojano, T. and Solares, A. (2002). Cognitive tendencies: The interaction between semantics and algebraic syntax in the production of syntactic errors. In A. Cockburn and E. Nardi (eds.), *Proceedings of the Twenty-sixth Annual Conference for the Psychology of Mathematics Education* (Vol. 4, pp. 129–136). Norwich, UK.

Filloy, E., Rojano, T. and Solares, A. (2004). Arithmetic/algebraic problem-solving and the representation of two unknown quantities. In M. Johnsen Høines, and A. B. Fuglestad (Eds.), *Proceedings of the Twenty-eighth Annual Conference for the Psychology of Mathematics Education* (Vol. 2, pp. 391–398). Bergen, Norway.

Freudenthal, H. (1973). *Mathematics as an Educational Task*. Dordrecht: Reidel.

Freudenthal, H. (1983). *Didactical Phenomenology of Mathematical Structures*. Dordrecht: Reidel.

Fridman, L. M. (1990). Los grafos trinomiales como metalenguaje de los problemas. *Matemáticas. Revista del Departamento de Matemáticas de la Universidad de Sonora*, *17-18*, 51–59.

Friel, S., Rachlin, S. and Doyle, D. (2001). *Navigating Through Algebra in Grades 6–8*. Reston, VA: National Council of Teachers of Mathematics.

Fuson, K. (1998). *Children's Counting and Concept of Number*. New York: Springer Verlag.

Galois, É. (1846). Œuvres mathématiques d'Évariste Galois. *Le Journal de Liouville*, *XI*, 381–444. [Reedición *Œuvres Mathématiques, Publiées en 1846 dans le Journal de Liouville, Suivies d'une étude par Sophus Lie*. Sceaux: Éditions Jacques Gabay, 1989.]

Ginsburg, H., Inoue, N. and Seo, K. (1999). Young children doing mathematics: Observations of everyday sctivities. In J. V. Copley (ed.), *Mathematics in the Early Years* (pp. 88–99). Reston, Va.: National Council of Teachers of Mathematics, and Washington, D.C.: National Association for the Education of Young Children.

Glassner, J.-J. (2000). *Écrire à Sumer. L'Invention du Cunéiforme*. Paris: Seuil.

Grant, E. (1969). *Nicole Oresme, De Proportionibus and Pauca Respicientes.* Madison, Milwaukee and London: The University of Wisconsin Press.

Grant, E. (1971). *Nicole Oresme and the Kinematics of Circular Motions*. Madison, Milwaukee and London: The University of Wisconsin Press.

Greenes, C., Cavanagh, M., Dacey, L., Findell, C. and Small, M. (2001). *Navigating Through Algebra in Prekindergarten-Grade 2*. Reston, VA: National Council of Teachers of Mathematics.

Hamilton, N. and Landin, J. (1961). *Set Theory and The Structure of Arithmetic*. Boston: Allyn and Bacon, Inc.

Heath, Th. (1921). *A History of Greek Mathematics*. 2 vols. Oxford: Clarendon Press. [Reprinted in New York: Dover, 1981.]

Høyrup, J. (1985). Varieties of mathematical discourse in pre-modern socio-cultural contexts: Mesopotamia, Greece, and the Latin Middle Ages. *Science and Society*, *XLIX*, 4–41.

Høyrup, J. (1986). Al-Khwârizmî, Ibn Turk, and the Liber Mensurationum: On the origins of Islamic algebra. *ERDEM*, *2*, 445–484.

Høyrup, J. (1987). The formation of 'Islamic Mathematics'. Sources and conditions. *Science in Context*, *1*, 281–329.

Høyrup, J. (1991). 'Oxford' and 'Cremona': On the relations between two versions of al-Khwârizmî's Algebra. *Filosofi og videnskabsteori på Roskilde Universitetcenter*. 3. Række: Preprint og Reprints nr. 1.

Høyrup, J. (1999a). The founding of Italian vernacular algebra. *Filosofi og videnskabsteori på Roskilde Universitetcenter*. 3. Række: Preprint og Reprints 1999 nr. 2.

Høyrup, J. (1999b). VAT. LAT. 4826. Jacopo da Firenze, Tractatus algorismi. Preliminari transcription of the manuscript, with occasional commentaries. *Filosofi og videnskabsteori på Roskilde Universitetcenter. 3. Række: Preprint og Reprints* 1999 nr. 3.

Høyrup, J. (2002a). *Lengths, Widths, Surfaces. A Portrait of Old Babylonian Algebra and Its Kin*. New York: Springer-Verlag.

Høyrup, J. (2002b). Pre-Modern "Algebra": A concise survey of that which was shaped into the technique and discipline we know, *Quaderni di Ricerca in Didattica G.R.I.M.*, 11, http://math.unipa.it/~grim/quaderno11.htm.

Hughes, B. (1981). *Jordanus de Nemore. De Numeris Datis,* Berkeley, CA: University of California Press.

Hughes, B. (1986). Gerard of Cremona's translation of al-Khwârizmî's al-jabr: A critical edition. *Mediaeval Studies*, *48*, 211–263.

Ifrah, G. (1994). *Histoire Universelle des Chiffres. L'Intelligence des Hommes Racontée par les Nombres et le Calcul*. Paris: Robert Laffont.

Isbell, J. R. (1960). A definition of ordinal numbers. *American Mathematical Monthly, 67*(1), 51–52.

Iwasaki, H. and Yamaguchi, T. (1997). The cognitive and symbolic analysis of the generalization process: the comparison of algebraic signs with geometric figures. In E. Pehkonen (ed.), *Proceedings of the Twenty First Annual Conference for the Psychology of Mathematics Education* (Vol. 3, pp. 105–112). Lahti, Finland.

Jones, C. V. (1978). *On the Concept of One as a Number*. Doctoral dissertation, University of Toronto.

Kalmykova, Z. I. (1975). Processes of analysis and synthesis in the solution of arithmetic problems. In J. Kilpatrick, I. Wirszup, E. G. Begle, and J. W. Wilson, J. (eds.), *Soviet Studies in the Psychology of Learning and Teaching Mathematics. Vol. XI. Analysis and Synthesis as Problem Solving Methods* (pp. 1–171). Stanford, CA: NCTM.

Kaput, J. J. (1987). Representations systems and mathematics. In C. Janvier (ed.), *Problems of Representation in the Teaching and Learning of Mathematics* (pp. 19–26). Hillsdale, NJ: Lawrence Erlbaum.

Kaput, J. J. (1989). Linking representations in the symbol systems of algebra. In S. Wagner and C. Kieran (Eds.), *Research Issues in the Learning and Teaching of Algebra* (Vol. 4, pp. 167–194). Hillsdale, NJ/Reston, VA: Lawrence Erlbaum Associates/National Council of Teachers of Mathematics

Kasir, D. S. (1931). *The Algebra of Omar Khayyam*. New York, NY: Bureau of Publications Teacher College of Columbia University.

Kieran, C. (1980). The interpretation of the equal sign: Symbol for an equivalence relation vs. an operator symbol. In R. Karplus (ed.), *Proceedings of the Fourth Conference of the International Group for the Psychology of Mathematics Education* (pp. 163–169). Berkeley, CA, University of California,.

Kieran, C. (1981). Concepts associated with the equality symbol. *Educational Studies in Mathematics*, *12*, 317–326.

Kieran, C. and Filloy, E. (1989). El aprendizaje del álgebra escolar desde una perspectiva psicológica. *Enseñanza de las Ciencias*, *7*, 229–240.

Kieran, C. and Sfard, A. (1999). Seeing through symbols: The case of equivalent expressions. *Focus on Learning Problems in Mathematics*, *21*(1), 1–17.

Kirshner, D. (1987). *The Grammar of Symbolic Elementary Algebra.* Doctoral dissertation, University of British Columbia, Vancouver.

Klein, J. (1968). *Greek Mathematical Thought and the Origins of Algebra*. Cambridge, MA: MIT Press. [Reprinted in New York: Dover, 1992.]

Krutetskii, V. D. (1976). *The Psychology of Mathematical Abilities in School Children*. Chicago: The University of Chicago Press.

Kuhn, T. S. (1962). *The Structure of Scientific Revolutions*. Chicago: The University of Chicago Press.

Lagrange, J. L. de. (1899). *Œuvres*. Publiées par les soins de J.-A. Serret. Tome troisième. Paris: Gauthier-Villars. [Reprinted in Hildesheim and New York: Georg Olms Verlag, 1973.]

Lakatos, I. (1976). *Proofs and Refutations*. Cambridge, UK: Cambridge University Press.

Lee, L. (1996). An initiation into algebraic culture through generalization activities. In N. Bednarz, C. Kieran and L. Lee (eds.), *Approaches to Algebra. Perspectives for Research and Teaching* (pp. 87–106). Dordrecht/Boston/London: Kluwer Academic Publishers.

Levey, M. (ed.) (1966). *The Algebra of Abû Kâmil, in a Commentary by Mordecai Finzi.* Hebrew text and translation, and commentary. Madison, WI: The University of Wisconsin Press.

Malara, N. and Navarra, G. (2003). *ArAl Projec. Atrithmetic Pathways Towards Favouring Pre-algebraic Thinking*. Bologna: Pitagora Editrice.

Marre, A. (1880). Notice sur Nicolas Chuquet et son Triparty en la science des nombres. *Bulletino di Bibliografia e Storia delle Scienze Matematice e Fisice*, *13*, 555—659 and 693–814.
Mason, J. (1996). Expressing generality and roots of algebra. In N. Bednarz, C. Kieran and L. Lee (Eds.), *Approaches to Algebra. Perspectives for Research and Teaching* (pp. 65–86). Dordrecht/Boston/London: Kluwer Academic Publishers.
Matz, M. (1982). Towards a process model for high school algebra errors. In D. Seeman and J. S. Brown (eds.), *Intelligent Tutoring Systems* (pp. 25–50). New York: Academic Press.
Mazzinghi, M. A. di. (1967). *Trattato di Fioretti.* (Arrighi, G. Ed.) Pisa: Domus Galileana.
Nassar, A. (2001). *El Efecto de Enseñar Algunas Estrategias de Resolución de Problemas en la Actuación de los Alumnos del Nivel 3º de Secundaria al Resolver Problemas Verbales Algebraicos en Gaza (Palestina)*, Tesis doctoral. Universitat de València.
Nemirovsky, R. (1996). Mathematical narratives, modeling and algebra. In N. Bednarz, C. Kieran and L. Lee (eds.), *Approaches to Algebra. Perspectives for Research and Teaching* (pp. 197–220). Dordrecht/Boston/London: Kluwer Academic Publishers.
Neugebauer, O. (1969). *The Exact Sciences in Antiquity*. New York: Dover.
Peirce, C. S. (1931–58). *Collected Papers of Charles Sanders Peirce*. Edited by Charles Hartshorne and Paul Weiss (vols. 1–6) and by Arthur Burks (vols. 7–8). Cambridge, MA: The Belknap Press of Harvard University Press.
Peirce, C. S. (1982-). *Writings of Charles S. Peirce: A Chronological Edition*, 6 vols. to date. Edited by Edward C. Moore, Max Fisch, Christian J. W. Kloesel et al. Bloomington: Indiana University Press.
Pimm, D. (1987). *Speaking Mathematically. Comunication in Mathematics Classrooms*. London: Routledge.
Polya, G. (1957). *How to Solve It.* 2nd. edition. Princeton, NJ: Princeton University Press.
Polya, G. (1966). *Mathematical Discovery. 2 vols*. New York: John Wiley and Sons.
Puig, L. (1994). El *De Numeris Datis* de Jordanus Nemorarius como sistema matemático de signos. *Mathesis, 10*(1), 47–92.
Puig, L. (1996). *Elementos de Resolución de Problemas.* Granada: Comares, col. Mathema.
Puig, L. (1997). Análisis fenomenológico. In L. Rico (Ed.), *La Educación Matemática en la Enseñanza Secundaria* (pp. 61–94). Barcelona: ICE/Horsori.
Puig, L. (1998). Componentes de una historia del álgebra. El texto de al-Khwârizmî restaurado. In F. Hitt (Ed.), *Investigaciones en Matemática Educativa II* (pp. 109–131). México, DF: Grupo Editorial Iberoamérica.
Puig, L. and Cerdán, F. (1988). *Problemas Aritméticos Escolares*. Madrid: Ed. Síntesis.
Puig, L. and Cerdán, F. (1990). Acerca del carácter aritmético o algebraico de los problemas verbales. In E. Filloy, and T. Rojano (Eds.), *Memorias del Segundo Simposio Internacional sobre Investigación en Educación Matemática* (pp. 35–48). Cuernavaca, Morelos: PNFAPM.
Puig, L. and Rojano, T. (2004). The history of algebra in mathematics education. In K. Stacey, H. Chick, and M. Kendal (eds.), *The Teaching and Learning of Algebra: The 12th ICMI study* (pp. 189–224). Boston/Dordrecht/New York/London: Kluwer Academic Publishers.
Radford, L. (2000a). The historical origins of algebraic thinking. In R. Sutherland, T. Rojano, A. Bell, and R. Lins (Eds.), *Perspectives on School Algebra* (pp. 13–36). Dordrecht/Boston/London: Kluwer Academic Publishers.
Radford, L. (2000b). Signs and meanings in students' emergent algebraic thinking: A semiotic analysis, *Educational Studies in Mathematics*, *42*(3), 237–268.
Radford, L. (2003). On the epistemological limits of language. Mathematical knowledge and social practice in the Renaissance. *Educational Studies in Mathematics*, *52*(2), 123–150.

Radford, L. (2004). The Cultural-Epistomological Conditions of the Emergence of Algebraic Symbolism. Paper presented at the *2004 History and Pedagogy of Mathematics Conference*, Uppsala, Sweden.

Rashed, R. (ed.) (1984). *Diophante. Tome III. Les Arithmétiques. Livre IV,* et *Tome IV, Livres V, VI et VII.* Texte de la traduction arabe de Qustâ ibn Lûqâ établi et traduit par Roshdi Rashed. Paris: Les Belles Lettres.

Rashed, R. and Vahebzadeh, B. (1999). *Al-Khayyâm mathématicien*. Paris: Librairie Scientifique et Technique Albert Blanchard.

Rojano, T. (1985). De la artimética al álgebra: un estudio clínico con niños de 12 a 13 años de edad. Unpublished Doctoral dissertation, Centro de Investigación y de Estudios Avanzados del IPN, D.F., México.

Rosen, F. (1831). *The Algebra of Mohammed Ben Musa.* London: Oriental Translation Fund.

Rotman, B. (1988). Toward a semiotics of mathematics. *Semiotica, 72*, 1–35.

Rotman, B. (1993). *Ad Infinitum… The Ghost in Turing's Machine.* Stanford, CA: Stanford University Press.

Rotman, B. (2000). *Mathematics as Sign. Writing, Imagining, Counting*. Stanford, CA: Stanford University Press.

Russell, B. (1973). Is mathematics purely linguistic? In D. Lackey (Ed.), *Essays in Analysis* by Bertrand Russell (pp. 295–306). London: George Allen and Unwin Ltd.

Schmandt-Besserat, D. (1992). *Before Writing. I. From Counting to Cuneiform*. Austin: University of Texas Press.

Sesiano, J. (ed.) (1982). *Books IV to VII of Diophantus' Arithmetica in the Arabic Translation attributed to Qustâ ibn Lûqâ*. Berlin, Heidelberg, New York: Springer Verlag.

Sesiano, J. (1993). La version latine médiévale de l'Algèbre d'Abû Kâmil. In M. Folkerts, and J. P. Hogendijk (Eds.), *Vestigia Mathematica. Studies in Medieval and Early Modern Mathematics in Honour of H. L. L. Busard* (pp. 315–452). Amsterdam and Atlanta, GA: Editions Rodopi B. V.

Sigler, L. E. (1987). *Leonardo Pisano Fibonacci. The Book of Squares*. An annotated translation into modern English by L. E. Sigler. Orlando, FL: Academic Press.

Sigler, L. E. (2002). *Fibonacci's Liber Abaci. A Translation into Modern English of Leonardo Pisano's Book of Calculation*. Translated by Laurence E. Sigler. New York/Berlin/ Heidelberg: Springer Verlag.

Stacey, K. (1989). Finding and using patterns in linear generalising problems. *Educational Studies in Mathematics*, *42*(3), 379–402.

Stevin, S. (1634). *Les Œuvres Mathématiques de Simon Stevin de Bruges*, edited by Girard. Leyden: Elzevirs.

Szabó, Á. (1977). *Les débuts des mathématiques grecques.* Paris: Vrin.

Taisbak, C. M. (2003). *Euclid's Data. The Importance of Being Given*. Copenhagen: Museum Tusculanum Press.

Talens, J. and Company, J. M. (1984). The textual space: On the notion of text. *The Journal of the Midwest Modern Language Association*, *17*(2), 24–36.

Tannery, P. (Ed.) (1893). *Diophanti Alexandrini Opera Omnia cum Graecis Commentariis*. Edidit et latine interpretatus est Paulus Tannery. 2 vols. Stuttgart: B. G. Teubner. [Reprinted in 1974.]

Thorndike, E. L. et al. (1923). *The Psychology of Algebra*. New York: The Macmillan Company.

Usiskin, Z. (1999). Conceptions of school algebra and uses of variables. In B. Moses (ed.), *Algebraic Thinking, Grades K-12: Readings from NCTM's School-Based Journals and Other Publications* (pp. 7–13). Reston, VA: National Council of Teachers of Mathematics.

Vallejo, J. M. (1841). *Tratado Elemental de Matemáticas, Escrito de Orden de S. M. para Uso de los Caballeros Seminaristas del Seminario de Nobles de Madrid y Demás Casas de*

Educación del Reino. Cuarta Edición Corregida y Considerablemente Aumentada. Tomo I. Parte primera, que contiene la Aritmética y Álgebra. Madrid: Imprenta Garrasayaza.

Van Der Waerden, B. L. (1954). *Science Awakening I: Egyptian, Babylonian & Greek Mathematics*. Groningen: Noordhoff.

Van Dooren, W., De Bock, D., Depaepe, F., Janssens, D. and Verschaffel, L. (2003). The illusion of linearity: Expanding the evidence towards probabilistic reasoning. *Educational Studies in Mathematics*, 53, 113–138.

Van Egmond, W. (1980). *Practical mathematics in the Italian renaissance. (A Catalog of Italian Abacus Manuscripts and Printed Books to 1600)*. Firenze, Italy: Annali dell'Instituto e Museo di. Storia della Scienza, fascicolo 1, Instituto e Museo di Storia della Scienza.

Van Schooten, F. (1646). *Francisci Vietæ Opera Mathematica*. Lugduni Batavorum: Ex Officinâ Bonaventuræ & Abrahami Elzeviriorum.

Ver Eecke, P. (1952). *Leonard de Pise. Le Livre des Nombres Carrés*. Traduit pour la première fois du latin médiéval en Français, avec une introduction et des notes par Paul Ver Eecke. Paris: Albert Blanchard.

Ver Eecke, P. (1959). *Diophante d'Alexandrie. Les Six Livres d'Arithmétiques et le Livre des Nombres Polygones.* Traduit du grec, avec une introduction et des notes par Paul Ver Eecke. Paris: Albert Blanchard.

Ver Eecke, P. (1973). *Leonardo de Pisa. El Libro de los Números Cuadrados*. Traducción de la versión francesa de Paul Ver Ecke de Pastora Sofía Nogués Acuna. Buenos Aires: Editorial Universitaria de Buenos Aires.

Vergnaud, G. (1981). *L'Enfant la Mathématique et la Réalité*. Berne: Peter Lang.

Vergnaud, G. (1982). A classification of cognitive tasks and operations of thought involved in addition and subtraction problems. In T. P. Carpenter, J. M. Moser, and T. A. Romberg (eds.), *Addition and Subtraction: A Cognitive Perspective* (pp. 39–59). Hillsdale, NJ: Lawrence Erlbaum.

Verschaffel, L., Greer, B. and De Corte, E. (2000). *Making Sense of Word Problems*. Lisse: Swets & Zeitlinger.

Von Neumann, J. (1923). Zur Einführung der transfiniten Zahlen. *Acta Litterarum ac Scientiarum Regiae Universitatis Hungaricae Francisco-Josephine, Sectio Scientiarum Mathematicarum*, *1*, 199–208. [Translated as "On the Introduction of Transfinite Numbers" by J. Van Heijenoort. In J. van Heijenoort (Ed.), (1967). *From Frege to Gödel. A Source Book in Mathematical Logic, 1879-1931* (pp. 346–354). Cambridge, MA: Harvard University Press.]

Vuillemin, J. (1962). *La Philosophie de l'Algèbre*. Paris: Presses Universitaires de France.

Witmer, T. R. (ed.) (1983). *François Viète. The Analytic Art.* Kent, OH: The Kent State University Press.

Wittgenstein, L. (1956). *Remarks on the Foundations of Mathematics*. Edited by G. H. von Wright, R. Rhees, and G. E. M. Anscombe. Oxford: Basil Blackwell. [Wittgenstein, L. (1984). *Bemerkungen über die Grundlagen der Mathematik*. Herausgegeben von G. H. von Wright, R. Rhees, and G. E. M. Anscombe. Suhrkamp:Frankfurt am Main.]

Zazkis, R. and Liljeddahl, P. (2001). Exploring multiplicative and additive structure of arithmetic sequences. In M. van den Heuvel-Panhuizen (Ed.), *Proceedings of the Twenty Fifth Annual Conference for the Psychology of Mathematics Education* (Vol. 4, pp. 439–446). Utrecht, The Netherlands.

Zazkis, R. and Liljeddahl, P. (2002). Generalization of patterns: The tension between algebraic thinking and algebraic notation. *Educational Studies in Mathematics*, *49*(3), 379–402.

Zermelo, E. (1909). Sur les ensembles finis et le principe de l'induction complète. *Acta Mathematica*, *32*, 185–193.

AUTHOR INDEX

SUBJECT INDEX

Mathematics Education Library

Managing Editor: A.J. Bishop, Melbourne, Australia

1. H. Freudenthal: *Didactical Phenomenology of Mathematical Structures.* 1983
ISBN 90-277-1535-1 HB; 90-277-2261-7 PB
2. B. Christiansen, A. G. Howson and M. Otte (eds.): Perspectives on Mathematics Education. Papers submitted by Members of the Bacomet Group. 1986
ISBN 90-277-1929-2 HB; 90-277-2118-1 PB
3. A. Treffers: Three Dimensions. A Model of Goal and Theory Description in Mathematics Instruction TheWiskobas Project. 1987 ISBN 90-277-2165-3
4. S. Mellin-Olsen: The Politics of Mathematics Education. 1987
ISBN 90-277-2350-8
5. E. Fischbein: Intuition in Science and Mathematics. An Educational Approach. 1987 ISBN 90-277-2506-3
6. A.J. Bishop: Mathematical Enculturation. A Cultural Perspective on Mathematics Education.1988 ISBN 90-277-2646-9 HB; 1991 0-7923-1270-8 PB
7. E. von Glasersfeld (ed.): Radical Constructivism in Mathematics Education. 1991 ISBN 0-7923-1257-0
8. L. Streefland: Fractions in Realistic Mathematics Education. A Paradigm of Developmental Research. 1991 ISBN 0-7923-1282-1
9. H. Freudenthal: Revisiting Mathematics Education. China Lectures. 1991
ISBN 0-7923-1299-6
10. A.J. Bishop, S. Mellin-Olsen and J. van Dormolen (eds.): Mathematical Knowledge: Its Growth Through Teaching. 1991 ISBN 0-7923-1344-5
11. D. Tall (ed.): Advanced Mathematical Thinking. 1991 ISBN 0-7923-1456-5
12. R. Kapadia and M. Borovcnik (eds.): Chance Encounters: Probability in Education. 1991 ISBN 0-7923-1474-3
13. R. Biehler, R.W. Scholz, R. Sträßer and B. Winkelmann (eds.): Didactics of Mathematics as a Scientific Discipline. 1994 ISBN 0-7923-2613-X
14. S. Lerman (ed.): Cultural Perspectives on the Mathematics Classroom. 1994
ISBN 0-7923-2931-7
15. O. Skovsmose: Towards a Philosophy of Critical Mathematics Education. 1994
ISBN 0-7923-2932-5
16. H. Mansfield, N.A. Pateman and N. Bednarz (eds.): Mathematics for Tomorrow's Young Children. International Perspectives on Curriculum. 1996
ISBN 0-7923-3998-3
17. R. Noss and C. Hoyles:Windows on Mathematical Meanings. Learning Cultures and Computers. 1996 ISBN 0-7923-4073-6 HB; 0-7923-4074-4 PB
18. N. Bednarz, C. Kieran and L. Lee (eds.): Approaches to Algebra. Perspectives for Research and Teaching.1996 ISBN 0-7923-4145-7 HB; 0-7923-4168-6, PB
19. G. Brousseau: Theory of Didactical Situations in Mathematics. Didactique des Mathématiques 1970-1990. Edited and translated by N. Balacheff, M. Cooper, R. Sutherland and V. Warfield. 1997 ISBN 0-7923-4526-6
20. T. Brown: Mathematics Education and Language. Interpreting Hermeneutics and Post-Structuralism. 1997 ISBN 0-7923-4554-1 HB
Second Revised Edition. 2001 ISBN 0-7923-6969-6 PB

Mathematics Education Library

21. D. Coben, J. O'Donoghue and G.E. FitzSimons (eds.): Perspectives on Adults Learning Mathematics. Research and Practice. 2000 ISBN 0-7923-6415-5
22. R. Sutherland, T. Rojano, A. Bell and R. Lins (eds.): Perspectives on School Algebra. 2000 ISBN 0-7923-6462-7
23. J.-L. Dorier (ed.): On the Teaching of Linear Algebra. 2000 ISBN 0-7923-6539-9
24. A. Bessot and J. Ridgway (eds.): Education for Mathematics in the Workplace. 2000 ISBN 0-7923-6663-8
25. D. Clarke (ed.): Perspectives on Practice and Meaning in Mathematics and Science Classrooms. 2001 ISBN 0-7923-6938-6 HB; 0-7923-6939-4 PB
26. J. Adler: Teaching Mathematics in Multilingual Classrooms. 2001 ISBN 0-7923-7079-1 HB; 0-7923-7080-5 PB
27. G. de Abreu, A.J. Bishop and N.C. Presmeg (eds.): Transitions Between Contexts of Mathematical Practices. 2001 ISBN 0-7923-7185-2
28. G.E. FitzSimons:What Counts as Mathematics? Technologies of Power in Adult and Vocational Education. 2002 ISBN 1-4020-0668-3
29. H.Alrø andO. Skovsmose:Dialogue and Learning in Mathematics Education. Intention, Reflection,Critique.2002 ISBN 1-4020-0998-4 HB;1-4020-1927-0 PB
30. K. Gravemeijer, R. Lehrer, B. van Oers and L. Verschaffel (eds.): Symbolizing, Modeling and Tool Use in Mathematics Education. 2002 ISBN 1-4020-1032-X
31. G.C. Leder, E. Pehkonen and G. T rner (eds.): Beliefs: A Hidden Variable in Mathematics Education? 2002 ISBN 1-4020-1057-5 HB; 1-4020-1058-3 PB
32. R. Vithal: In Search of a Pedagogy of Conflict and Dialogue for Mathematics Education. 2003 ISBN 1-4020-1504-6
33. H.W. Heymann: Why Teach Mathematics? A Focus on General Education. 2003 Translated by Thomas LaPresti ISBN 1-4020-1786-3
34. L. Burton: Mathematicians as Enquirers: Learning about Learning Mathematics. 2004 ISBN 1-4020-7853-6 HB; 1-4020-7859-5 PB
35. P. Valero, R. Zevenbergen (eds.): Researching the Socio-Political Dimensions of Mathematics Education: Issues of Power in Theory and Methodology. 2004 ISBN 1-4020-7906-0
36. D. Guin, K. Ruthven, L. Trouche (eds.) The Didactical Challenge of Symbolic Calculators: Turning a Computational Device into a Mathematical Instrument 2005 ISBN 0-387-23158-7
37. J. Kilpatrick, C. Hoyles, O. Skovsmose (eds. in collaboration with Paola Valero) Meaning in Mathematics Education. 2005 ISBN 0-387-24039-X
38. H. Steinbring: The Construction of New Mathematical Knowledge in Classroom Interaction: An Epistemological Perspective. 2005. ISBN 0-387-24251-1
39. M.Borba, M. Villarreal: Humans-with-Media and the Reorganization of Mathematical Thinking: Information and Communication Technologies, Modeling, Visualization and Experimentation. 2005 ISBN 0-387-24263-5 HB, ISBN 0-387-32821-1 PB
40. G. Jones (ed): Exploring Probability in School: Challenges for Teaching and Learning. 2005 ISBN 0-387-24529-4
41. D. DeBock, W. Van Dooren, D. Janssens, and L. Verschaffel: The Illusion of Linearity: From Analysis to Improvement. 2007 ISBN 978-0-387-71082-2
42. K. Francois and J. P. Van Bendegem: Philosophical Dimensions in Mathematics Education. 2007 ISBN 978-0-387-71571-1
43. E. Filloy, L. Puig, and T. Rojano: Educational Algebra: A Theoretical and Empirical Approach. 2007. ISBN 978-0-387-71253-6