Region-Based Theory of Space: Algebras of Regions, Representation Theory, and Logics

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Preface

In this paper, we present recent results in the region-based theory of space that concern algebras of regions, the corresponding topological and discrete models, and representation theory. We also discuss applications to Qualitative Spatial Reasoning (QSR), an actively developing branch of AI and Knowledge Representation (KR). In particular, we show how new results in some practically motivated areas of QSR and KR can be obtained by combining methods from such established classical disciplines as Boolean algebras, topology and logic.

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The paper is organized as follows. Section 1 is a historical excursion into the region-based theory of space. We discuss the "pointless approach" to this theory, whose roots can be found in some philosophical ideas of de Laguna [**13**] and Whitehead [**64**]. We show connections of region-based theory of space with mereology (the theory of *part–whole relations*) and applications to QSR. In Section 2, we consider algebras of regions known as contact algebras. We study topological and discrete point-based models of contact algebras, and discuss different definitions of a point depending on the choice of axioms. We also consider representation theorems establishing a correspondence between the chosen axiomatizations and the required point-based models. In Section 3, we deal with a class of spatial logics. Some of them are related to the well-known system of Region Connection Calculus (RCC). In that section, we obtain completeness and decidability results by using representation theorems.

Some of the most important statements and new results are supplied with brief proofs. Standard definitions and facts from Boolean algebra can be found in [**52**], from topology in [**24**], from proximity spaces in [**42**], and from modal logic in [**7, 8**].

1. Historical Excursion into the Region-Based Theory of Space

One of the oldest theories of space is classical Euclidean geometry. It can be regarded as a *point-based* theory in the sense that the notion of a point is basic, whereas all other geometrical figures are defined as sets of points. The same can be said about topology considered as a more abstract kind of geometry. In general, by a point we mean the simplest spatial entity without dimension and internal structure. However, this notion is too abstract to have an adequate analog in reality, in contrast to many geometrical figures for which we can find their images in nature. The following idea then arises: to develop an alternative theory of space where the basic notion is not a point, but some other objects that are more closely related to the real world, for example, solid bodies. As basic relations between solids we could take, for instance, "one solid is part of another solid," "two solids overlap," or "one solid touches another solid," etc. This point of view is close to the ideas of some abstract philosophical disciplines such as *ontology*, the theory of "Existent," and especially *mereology* understood as a theory of "part-whole" relations. One of the founders of mereology was Leśnewski [**38**], who developed it as part of an ambitious and nonorthodox programme of constructing new foundations of mathematics. But due to Tarski [**58**], the mathematical content of mereology can be clearly presented in terms of complete Boolean algebras (cf. also [**33**] for such a presentation, and [**53**] for some other systems of mereology). The only difference between mereology and complete Boolean algebras is that Boolean algebras have an analog of the empty set (zero element), whereas mereology excludes such a zero individual.

The pointless approach does not mean that points are not considered at all. The notion of a point is necessary for a pointless theory of space to be equivalent in some sense to the classical point-based theory. But, in this case, points must be defined in terms of new primitive notions. This idea, as well as the necessity to use mereology for constructing a pointless theory of space, was expressed in the philosophical paper "*Point, line and surface as sets of solids*" [**13**] by de Laguna in 1922 and in the famous book "*Process and Reality*" [**64**] by Whitehead in 1929.

De Laguna considered a ternary relation between solids, "*x connects y with z*," and defined a point, a line, and a surface via certain collections of solids. Whitehead developed this idea and simplified the ternary connection relation to the following binary relation: "*x is connected with y*," which he called the *connection relation.* Here we use the term *contact relation.* Whitehead called solids *regions*, which later gave the name *region-based theory of space.*

As a primitive relation Whitehead took the contact relation between regions. He also introduced mereological relations such as *part-of*, *overlap* and some new relations called *external connection*, *tangential inclusion*, and *nontangential inclusion*. From the intuitive point of view, two regions are *in contact* if they have a common point. However, according to Whitehead, this property cannot be taken as a definition because a point is not defined and points must be defined by means of regions and the contact relation.

In [**64**] Whitehead listed explicitly a large number of assumptions and definitions about regions and the contact relation, and illustrated some of them by pictures. He did not make any attempt to reduce the number of his assumptions to a logical minimum. To define the notion of point, he introduced quite complicated notions of *geometrical element* and the relation of *incidence* between geometrical elements (see Definitions 13 and 15 in [**64**]). Then the definition of a point (Definition 16) sounds as follows: "A geometrical element is called a *point* when there is no geometrical element incident with it." Whitehead pointed out an analogy of his definition with the first definition of Euclid's *Elements*: "A point is that of which there is no part." This analogy shows that some mereological foundations of "pointless" geometry have their roots even in the old Euclid's Elements. Whitehead's final goal was to approach the Euclidean notions of a *straight line* and of *plane* in a similar way. Note that Whitehead's pointless theory of space is quite vague, and it is still a problem to extract a readable axiomatization and present it in a standard mathematical format (we refer the reader to the nice survey of pointless geometry by Gerla [**31**]). However, the idea to define points via regions is quite remarkable. Something similar can be found in Boolean algebras which can be considered as pointless analogs of sets. In Stone's representation theory of Boolean algebras [**55**] (1937) points in a given Boolean algebra are identified with ultrafilters, sets of elements of the algebra. So de Laguna–Whitehead's ideas of pointless approach to the theory of space could be regarded as early predecessors of the representation theory of Boolean algebras.

For further references we summarize here some formal properties of the contact relation and some other Whitehead's spatial relations between regions. We write aCb for "region a is in a contact with region b."

(W1) $(\forall a)(aCa)$,

(W2) $(\forall a, b)(aCb \rightarrow bCa)$,

(W3) $a = b$ if, and only if, $(\forall c)(aCc \leftrightarrow bCc)$,

(W4) a is included in b ($a \leqslant b$) if, and only if, $(\forall c)(aCc \rightarrow bCc)$,

(W5) a and b overlaps (aOb) if, and only if, $(\exists c)(c \leqslant a$ and $c \leqslant b)$,

(W6) a is externally connected with b ($aC^{\text{ext}}b$) if, and only if, aCb and not aOb,

(W7) a is tangentially included in b ($a \leqslant^{\circ} b$) if, and only if, $a \leqslant b$ and $(\exists c)(cC^{\text{ext}}a$ and $cC^{\text{ext}}b)$,

(W8) a is non-tangentially included in b ($a \ll b$) if, and only if, $a \leqslant b$ and not $a \leqslant^{\circ} b$.

Axiom (W3), known as the *axiom of extensionality* of contact, is very important. It can be proved that it is equivalent to axiom (W4), which says that part-of relation in Whitehead's system is definable by means of contact.

Another, much simpler, pointless reconstruction of Euclidean geometry was given by Tarski [**57**] in 1927. He called his system *Geometry of solids.* Geometry of solids is an extension of Leśnewski's mereology with the primitive notion of sphere. To define points, Tarski first introduced the relation of two spheres being concentric, and then points were identified with certain sets of concentric spheres. A simplified version of Tarski's system can be found in [**4**], where similar approaches are also discussed.

Another attempt to build a pointless theory of space was made by Grzegorczyk [**34**] in 1960. Independently from de Laguna [**13**] and Whitehead [**64**], Grzegorczyk developed a system that was close to Whitehead's system.

As primitives he took the relations of part-of and separation, which, in fact, is the complement of the Whitehead contact relation. Grzegorczyk's results were presented in [**6**], where the notion of contact was used instead of separation. According to [**6**], Grzegorczyk's pointless geometry $(R, \leqslant, C)$ is given by the following axioms:

(G0) $(R, \leqslant)$ is a *mereological field*, i.e., a complete Boolean algebra with deleted zero element.
(G1) C is a reflexive relation in R,
(G2) C is a symmetric relation in R,
(G3) C is monotone with respect to $\leqslant$ in the sense that we have: $a \leqslant b \rightarrow (\forall c \in R)(aCc \rightarrow bCc)$.

Then the relation of non-tangential inclusion $\ll$ is defined in the same way as by Whitehead (see axiom (W8) above). A set p of regions is called a *representative of a point* if the following conditions are satisfied:

(1) p has no minimum and is totally ordered by the relation $\ll$,
(2) given two regions u and v, if we have uOc and vOc, for every $c \in p$, then uCv.

A filter P in R is called a *point* if it is generated by a representative of a point. We say that P *belongs to a region* a if a is a member of P.

Then two additional axioms are introduced:

(G4) every region has at least one point,
(G5) if aCb then there is a point P such that a and b overlap with every member of P.

Denote by $\mathbb{P}$ the set of all points of $(R, \leqslant, C)$ and by $\pi(r)$ the set of all points of a region r.

Grzegorczyk proved the following two important theorems.

Theorem 1. *Let (X, τ) be a Hausdorff topological space, and let R be a family of nonempty regular open sets of (X, τ). For any $a, b \in R$, we set aCb if, and only if,* $\mathrm{Cl}(a) \cap \mathrm{Cl}(b) \neq \varnothing$. *Then $(R, \subseteq, C)$ satisfies* (G0)–(G3). *If every point of X is the* *intersection of a decreasing (with respect to* $\ll$*) family of open sets, then axioms* (G4) *and* (G5) *are also satisfied.*

Theorem 2. *Suppose that* $(R, \leqslant, C)$ *satisfies* (G0)–(G5)*. Let* τ *be a topology in* $\mathbb{P}$ *generated by the set* $\{\pi(r) : r \in R\}$*. Then* $\{\pi(r) : r \in R\}$ *coincides with the set of all nonempty regular open sets of* $(\mathbb{P}, \tau)$*, and* π *is an isomorphism.*

As was noted in [**6**], the implication in axiom (G3) can be replaced with equivalence, which eliminates the part-of relation from the primitives. This means that, as in the case of Whitehead, the system can be based on the unique primitive C.

Theorems 1 and 2 show that there is an equivalence between the point-based and pointless theories of space. Theorem 1 also shows the importance of regular (open or closed) sets in topological spaces as models of regions. In fact, Theorem 2 is the first representation theorem of a special system of region-based theory of space which is an extension of mereology with the primitive of Whitehead's contact relation. Since the models of such extended mereologies are topological, some authors prefer to call them *mereotopologies* or *region-based topologies.*

An interesting comparison between the notions of a point used by Whitehead [**64**] and Grzegorcyk [**34**] was given by Biacino and Gerla in [**6**]. They proved that these definitions are equivalent in some sense if the relation of non-tangential inclusion $\ll$ satisfies the following additional axiom:

(G6) if $a \ll b$, then $a \ll c \ll b$ for some region $c \in R$.

Using the complement a^*, we can equivalently express axiom (G6) in terms of C:

(G6′) if $a\overline{C}b$, then $a\overline{C}c$ and $c^*\overline{C}b$ for some $c \in R$.

This axiom is referred to as the *normality axiom*, since it is satisfied by regular open (closed) sets in a Hausdorff space provided that the space is normal.

One unpleasant feature of Grzegorczyk's system is that it includes axioms containing the second-order definition of a point and, consequently, it is not a first-order system.

It is of interest to note that by accepting the normality axiom (G6) one can obtain first-order axiomatizations of pointless theory of space. This was done independently by several authors: [**63, 62, 61, 15**]. The first to do this was de Vries [**63**] (1962) in his thesis "*Compact Spaces and Compactifications.*" This work, independent from Whitehead [**64**] and Grzegorczyk [**34**], was completely unknown to the community of authors interested in the region-based theory of space. Thus, de Vries is mentioned neither in Gerla's survey of pointless geometry [**31**], nor in later papers on region-based theory of space.

Note that axiom (G6) is well known among specialists in the theory of Proximity spaces. Proximity spaces are abstract spaces [**42**] with the proximity relation $A\delta B$ between subsets satisfying almost all axioms for the contact relation C. They can also be axiomatized using the relation $A \ll B$ definable by δ in the same way as $\ll$ is definable by C. By analogy with the axioms of proximity spaces based on the relation $\ll$, de Vries considered Boolean algebras $(B, 0, 1, ., +, *, \ll)$ with the additional relation $\ll$, called *compingent algebras*, which satisfy the following first-order axioms:

(P0) $(B, 0, 1, . , +, *)$ is a Boolean algebra with * as the Boolean complement,

(P1) $0 \ll 0$,

(P2) $a \ll b$ implies $a \leqslant b$,

(P3) $a \leqslant a' \ll b$ implies $a \ll b$,

(P4) $a \ll b$ and $c \ll d$ imply $a.c \ll b.d$,

(P5) $a \ll b$ implies $b^* \ll a^*$,

(P6) $a \ll b \neq 0$ implies $\exists c \neq 0$ with $a \ll c \ll b$.

Note that axiom (P6) can be replaced by two axioms:

(P6′) $a \ll b$ implies $\exists c$ with $a \ll c \ll b$ (which is just the normality axiom), and

(P7) if $b \neq 0$ then $\exists a \neq 0$ with $a \ll b$.

Observe that axioms (P1)–(P5), (P6′) are algebraic analogs of the axioms of Efremovič's proximity spaces [**25**] (cf. also [**42**]). We will see in Section 2 that axiom (P7) is equivalent to Whitehead's extensionality axiom for the contact relation.

Using the well-known techniques from the proximity spaces and Smirnov's theory of compactifications, de Vries proved that each compingent algebra is isomorphic to a subalgebra of the algebra of regular open sets of a compact Hausdorff space with the compingent relation on regular open sets defined as follows: $a \ll b$ if, and only if, $\mathrm{Cl}\,(a) \subseteq b$. The points defined by de Vries, called *compingent filters*, are just lattice analogs of the *ends*, special filters used in proximity theory. In fact, de Vries established a one-to-one correspondence between complete compingent algebras and compact Hausdorff spaces. Similar results were obtained also by Fedorčuk [**26**].

Another, more general than de Vries–Fedorčuk's, first-order axiomatization of a region-based theory of space was given by Roeper [**49**] in 1997. His theory corresponds to the point-based theory of locally compact Hausdorff spaces, and his approach is a skillful combination of de Vries–Fedorčuk's methods and Leader's compactification theory of local proximity (cf. [**37**], [**42**]). Roeper's axiomatization is based, like Leader's notion of local proximity, on two primitive spatial relations: the contact and the unary relation of *limitedness*. An attempt to give a different formulation of the same theory using only one primitive relation, called *interior parthood*, was made by Mormann [**41**] (see also [**61**]).

We continue our historical excursion into the region-based theory of space by mentioning the contribution made by Clarke [**10, 11**]. Clarke noted that his system should be understood as a formalization of the ideas of Whitehead [**64**]. Clarke's system (R, C) is based on a unique primitive relation C of contact satisfying Whitehead's axioms and definitions (W1)–(W8). Clarke assumed also the so-called *fusion axiom*:

If A is a nonempty subset of R, then there exists $a \in R$ (called a *fusion* of A) such that $C(a) = \bigcup\{C(x) : x \in A\}$, where $C(x) = \{y \in R : xCy\}$.

Points in Clarke's system are identified with certain subsets P of R satisfying some closure conditions. He needed also the following axiom, containing a definable notion of point:

If aCb, then there exists a point P such that $a, b \in P$.

Biacino and Gerla [**5**] studied this system in detail and proved that (R, C) is equivalent to a complete Boolean algebra with zero element removed (mereological field). It follows from this fact that the contact C coincides with the overlap O, which is not satisfactory. Another unsatisfactory feature is that the system has an axiom containing the second-order notion of a point and, consequently it is not a first-order one. Nevertheless, Clarke's system had a remarkable impact on some research areas in AI for which the pointless approach to the theory of space was important. One such area is the so-called *Qualitative Spatial Reasoning* (QSR). It is related to a new generation of information systems dealing with geographical information and known as Geographical Information Systems (GIS). It has been recognized that reasoning techniques in GIS using *quantitative* methods of classical theory of space are not efficient and tractable. This motivated researchers in these areas to look for new, *qualitative* models of space. Similar problems have appeared in robotics, computer vision, natural language semantics related to a commonsense spatial vocabulary, etc. Models of space based on mereology proved to fit well into the problems of QSR, and this made region-based theory of space important for AI and computer science (see [**48**]). Several attempts to build systems similar to that of Clarke have been made within the QSR community. One of the most important and popular systems is *Region Connection Calculus* (RCC), proposed by Randel, Cui and Cohn [**48**] in 1992. Now RCC is in the center of an intensive research in the realm of QSR, and one of the most active is Cohn's group at the University of Leeds. A comprehensive overview of the QSR research and related work was given by Cohn and Hazarika [**12**] (2001). Recent collections of papers on QSR are the special issues of Fundamenta Informaticae (2001) edited by I. Düntsch [**17**] and the Journal of Applied Non-Classical Logics (2002) edited by Balbiani [**1**].

Stell [**54**] and Düntsch et al. [**20**] presented an equivalent version of RCC based on Boolean algebras satisfying all axioms for contact given by Whitehead plus an additional axiom of *connectedness* forcing topological models to be connected spaces. So connected regular spaces form a correct semantics for RCC. A representation theorem for RCC in a class of more general spaces, called *weakly regular*, was proved by Dünch and Winter [**21**] in 2005. A representation theorem for a variety of related systems was proved in [**15**] (2006).

The main point-based models of the region-based theory of space considered in QSR are the contact algebras of regular open or regular closed sets in certain topological spaces. Since topology aims to formalize some *continuous*, *indiscrete* features of space we may call this kind of models continuous or indiscrete. More special models of regions generated by *polygonal regions* were considered by Pratt and Schoop [**46, 47**]. It has been pointed out by several authors that continuous models are not so convenient in computer modelling of space, and a modified and generalized region-based theory of space, admitting discrete models, is required. One solution was proposed by Galton [**29, 30**]. Instead of topological spaces, Galton proposes to consider the so-called *adjacency spaces*. An adjacency space is a relational system of the form (W, R), where W is a nonempty universe whose elements are called *cells* and R is a binary relation between cells, called an adjacency relation. Galton defines regions to be arbitrary sets of cells, and the contact relation between regions is defined by taking aCb if, and only if, $\exists x \in a, \exists y \in b$ with xRy. This definition relates Galton's adjacency spaces to the Kripke semantics of modal logic [**7**] which makes it possible to use methods from modal logic for studying discrete region-based theories of space [**3**]. Pointless formulations of Galton's theory of discrete spaces and the corresponding representation theory was given in [**19**]. It was shown in [**14**] that the algebras corresponding to discrete spaces have also standard topological representations in which regions are represented by regular closed or open sets. In this way both kinds of models of region-based theory of space—discrete and indiscrete—can be considered in a unified way. This unified approach is presented in more detail in Section 2.

To conclude this historical excursion, we mention that, in the realm of QSR, different kinds of logical systems for reasoning about space have been developed and their computational properties have been studied. Some authors advocated logical systems based on first-order languages (cf., for example, [**43, 45**]). One of the practical motivations for dealing with first-order systems of region-based theories of space is that this makes it possible to employ first-order provers for some applications. Using some results of Grzegorczyk [**32**], one can show, however, that most of these systems are undecidable. That is why weaker, quantifier-free systems with better computational properties have been designed. Examples are the system RCC-8 introduced by Egenhofer and Franzosa [**23**] and its extension with Boolean terms introduced by Wolter and Zakharyaschev [**65**]. Completeness theorems and decidability results for these and other related to RCC quantifier-free systems with respect to their topological and discrete semantics are given in [**3**]. For more information on these logics see Section 3 below. A decidable system with predicates of component-counting was presented by Pratt-Hartmann [**44**]. Dynamic Logics for discrete region-based theory of space have been studied in [**2**]. Modal logics with Kripke frames based on the RCC-8 relations have been introduced by Lutz and Wolter [**40**]. For various combinations of spatial and temporal logics see Gabelaia et al. [**28**] and Konchakov et al. [**36**].

This section does not cover all aspects of the region-based theory of space. We have only concentrated on pointless approaches similar to those of de Laguna and Whitehead. Of course, this is not the only way to look at the region-based theory of space: an alternative one is described, for example, by Pratt-Hartmann [**43, 45**]. Another alternative is given by Schoop [**50**] who motivates the idea of taking both regions and points as primitives. We hope that the survey above presents the region-based theory of space as an active and developing area. Started from some very abstract philosophical ideas of de Laguna and Whitehead, it has reached its flourishing stage, with a clear mathematical theory and multiple applications in practically oriented areas of QSR, GIS and KR.

2. Algebras of Regions, Models, and Representation Theory

2.1. Contact algebras

Following [**15**], by a *contact algebra* we mean any system $\underline{B} = (B, C) = (B, 0, 1, ., +, *, C)$, where $(B, 0, 1, ., +, *)$ is a nondegenerate Boolean algebra, $*$ denotes the complement, and C is a binary relation in B, called a *contact*, such that

(C1) if xCy, then $x, y \neq 0$,

(C2) $xC(y+z)$ if and only if xCy or xCz,

(C3) if xCy, then yCx,

(C4) if $x.y \neq 0$, then xCy.

Elements of B are called *regions*. The negation of C is denoted by $\overline{C}$. The relation $\ll$ of *nontangential inclusion* is defined as follows: $x \ll y$ if and only if $x\overline{C}y^*$. We say that $\underline{B}$ is *complete* if B is complete.

Axiom (C2) implies the monotonicity of C with respect to $\leqslant$:

(Mono) if aCb and $a \leqslant a'$ and $b \leqslant b'$, then $a'Cb'$.

A contact algebra can be equivalently defined in terms of $\ll$ (cf. the axioms of de Vries in Section 1):

($\ll 1$) $1 \ll 1$,

($\ll 2$) if $x \ll y$, then $x \leqslant y$,

($\ll 3$) if $x \leqslant y \ll z \leqslant t$, then $x \ll t$,

($\ll 4$) if $x \ll y$, then $y^* \ll x^*$,

($\ll 5$) if $x \ll y$ and $x \ll z$, then $x \ll y.z$.

Axioms (C1)–(C4) are Boolean versions of the axioms of basic proximity spaces (known as Čech proximity spaces, cf. [**9, 56**]). Note that the main intended models of contact algebras are not basic proximity spaces, but some other models of topological nature that can be constructed in the following way.

Example 2.1.1. (1) *Contact algebra of regular closed sets.* Let (X, τ) be a topological space with closure $\mathrm{Cl}(a)$ and interior operations $\mathrm{Int}(a)$. A subset a of X is *regular closed* if $a = \mathrm{Cl}(\mathrm{Int}(a))$. The set of all regular closed subsets of (X, τ) is denoted by $\mathrm{RC}(X, \tau)$ or $\mathrm{RC}(X)$. As is known, the regular closed sets with operations $a + b = a \cup b$, $a.b = \mathrm{Cl}(\mathrm{Int}(a \cap b))$, $a^* = \mathrm{Cl}(X \setminus a) = \mathrm{Cl}(-a)$, $0 = \varnothing$, and $1 = X$ form a Boolean algebra. Moreover, if we consider the infinite join operation $\sum_{i \in I} a_i = \mathrm{Cl}(\bigcup_{i \in I} a_i)$, then the Boolean algebra $\mathrm{RC}(X)$ is complete. The contact is defined as follows: $a\, C_X\, b$ if and only if $a \cap b \neq \varnothing$. It satisfies axioms (C1)–(C4). This contact is called the *standard contact for regular closed sets* and the corresponding contact algebra is called the *standard contact algebra of regular closed sets.* The nontangential inclusion is defined as follows: $a \ll b$ if and only if $a \subseteq \mathrm{Int}(b)$.

(2) *Contact algebra of regular open sets.* A subset a of (X, τ) such that $a = \mathrm{Int}(\mathrm{Cl}(a))$ is called a *regular open set.* The set of all regular open subsets of (X, τ) is denoted by $\mathrm{RO}(X, \tau)$ or $\mathrm{RO}(X)$. The Boolean operations and contact in $\mathrm{RO}(X)$ are defined as follows: $a + b = \mathrm{Int}(\mathrm{Cl}(a \cup b))$, $a.b = a \cap b$, $a^* = \mathrm{Int}(X \setminus a) = \mathrm{Int}(-a)$, $0 = \varnothing$, $1 = X$, and aC_Xb if and only if $\mathrm{Cl}(a) \cap \mathrm{Cl}(b) \neq \varnothing$ (consequently, $a \ll b$ if and only if $\mathrm{Cl}(a) \subseteq b$). Then $(\mathrm{RO}(X), C_X)$ is a contact algebra and it is complete relative to the infinite meet $\prod_{i \in I} a_i = \mathrm{Int}(\bigcap_{i \in I} a_i)$. In this case, C_X is called the *standard contact for regular open sets* and the corresponding contact algebra is called the *standard contact algebra of regular open sets.*

Note that $(\mathrm{RO}(X), C_X)$ and $(\mathrm{RC}(X), C_X)$ are isomorphic contact algebras. The corresponding isomorphism f is defined as $f(a) = \mathrm{Cl}(a)$ for every $a \in \mathrm{RO}(X)$. This fact explains, why we will consider only models with regular closed sets.

In Section 2.5, we will establish the existence of topological models of contact algebras related to proximity spaces, where elements of the algebra are regular closed (open) sets, but the contact is not standard, unlike these examples.

Note that the Boolean part in the definition of a contact algebra incorporates the mereological component of the notion. Although the zero element is not traditionally accepted in mereology, we consider the zero element, which makes the definition more suitable for our considerations.

For a Boolean algebra we introduce the following basic mereological relations between regions:

part-of relation $a \leqslant b$ is the lattice ordering of B,

overlap aOb if and only if $a.b \neq 0$.

This definition of an overlap agrees with that introduced by Whitehead: $\exists c \in B \setminus \{0\} : c \leqslant a$ and $c \leqslant b$. Indeed, it suffices to take $c = a.b \neq 0$.

Another mereological relation is the following:

dual overlap $a\check{\mathrm{O}}b$ if and only if a^*Ob^*

or, equivalently:

$a\check{\mathrm{O}}b$ if and only if $a + b \neq 1$.

It is natural to find a general definition of a "mereological relation" and one possibility to do this is to identify them with all Boolean relations definable by open formulas in the first-order theory of Boolean algebras. Having such a definition, we can obtain finitely many mereological relations of given arity, so that for $n = 2$ there are exactly 30 such relations and each of them can be defined by an open first-order formula in terms of $\leqslant$, O, and $\check{\mathrm{O}}$.

Using the notions of contact, overlap, and nontangential inclusion, it is possible to introduce the so-called RCC-8 basic mereotopological relations between two nonzero regions:

RCC −8 *relations*

- disconnected DC(a, b): $a\overline{C}b$,
- external contact EC(a, b): aCb and $a\overline{O}b$,
- partial overlap PO(a, b): aOb and $a \not\leqslant b$ and $b \not\leqslant a$,
- tangential proper part TPP(a, b): $a \leqslant b$ and $a \not\ll b$ and $b \not\leqslant a$,
- tangential proper part^{-1} TPP$^{-1}(a, b)$: $b \leqslant a$ and $b \not\ll a$ and $a \not\leqslant b$,
- nontangential proper part NTPP(a, b): $a \ll b$ and $a \neq b$,
- nontangential proper part^{-1} NTPP$^{-1}(a, b)$: $b \ll a$ and $a \neq b$,
- equal EQ(a, b): $a = b$.

It is easy to see that these relations are pairwise disjoint and exhaustive. Pure topological definitions, introduced by Egenhofer and Franzosa [**23**] and sometimes referred to as *Egenhofer–Franzosa relations*, were studied by many authors (cf. Wolter and Zakharyaschev [**65**] for complexity and Lutz and Wolter [**40**] for more references).

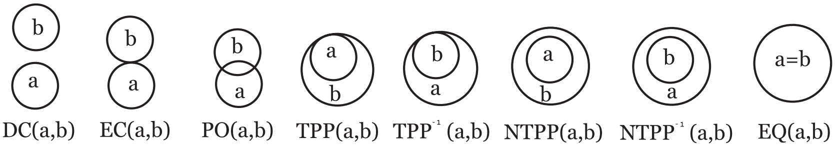

FIGURE 1

In the language of contact algebras, we can define some other mereotopological relations, for example, the one-place predicate Con(a): "the region a is *connected* or a is a *one-piece region*" which is formally expressed as follows:

Con(a) if and only if $(\forall b, c)(b \neq 0$ and $c \neq 0$ and $b + c = a \rightarrow bCc)$.

In the case of Con(1), the contact algebra is said to be *connected*. The negation of Con(a) is denoted by $\overline{\text{Con}(a)}$. From an intuitive point of view, $\overline{\mathrm{Con}(a)}$ says that the region a is the sum of at least two disconnected nonzero regions. We can consider a more general predicate by assuming that $c^{\geqslant n}(a)$ is the sum of n pairwise disconnected nonzero regions $b_1, \ldots, b_n$ or, formally:

$c^{\geqslant n}(a)$ if and only if $(\exists b_1 \ldots b_n)(a = b_1 + \ldots + b_n$ and $(\forall i = 1 \ldots n)(b_i \neq 0)$ and $(\forall i \neq j, i, j = 1, \ldots, n)(b_i \overline{C} b_j))$.

It is obvious that $\overline{Con}(a)$ is equivalent to $c^{\geqslant 2}(a)$. The computational complexity of $c^{\geqslant n}$, called the *component counting*, and part-of relation is studied by Pratt-Hartmann [**43**].

Another interesting mereotopological relation considered by Gabelaia et al [**28**] is the following n-ary contact $C_n(a_1, \ldots, a_n)$ with the standard meaning in the contact algebra of regular closed sets:

$C_n(a_1, \ldots, a_n)$ if and only if $a_1 \cap \ldots \cap a_n \neq \varnothing$.

We do not know whether this relation is definable in the language of contact algebras by a first-order formula. In Section 2.3, we will give a definition using a second-order formula.

2.2. Extensions of contact algebras by adding new axioms

Consider contact algebras satisfying some of the following axioms:

(Con) if $a \neq 0$ and $a \neq 1$, then aCa^* *connectedness*

(Ext) if $a \neq 1$, then $\exists b \neq 0$ such that $a\overline{C}b$ *extensionality*

(Nor) if $a \ll b$, then $\exists c$ such that $a \ll c \ll b$ *normality*

A contact algebra satisfying axiom (Con) ((Ext) or (Nor)) is said to be *connected* (*extensional* or *normal*).

Contact algebras satisfying axioms (Con) and (Ext) were introduced by Stell in [**54**] under the name *Boolean contact algebras* and were considered as an equivalent formulation of the system

RCC [**48**]. Stell proved that (Ext) is equivalent (under axioms (C1)–(C4)) to each of the following axioms:

(Ext′) $a \leqslant b$ if and only if $(\forall c \in B)(aCc \to bCc)$,

(Ext″) $a = b$ if and only if $(\forall c \in B)(aCc \to bCc)$,

(Ext‴) $(\forall b \neq 0)(\exists a \neq 0)(a \ll b)$.

Note that (Ext′) is just Whitehead's definition of the part-of relation and (Ext″) is Whitehead's axiom of extensionality.

Contact algebras satisfying (Nor) and (Ext) were first studied by de Vries [**63**] and Fedorčuk [**26**]. Independently, such algebras were introduced in [**62, 61**], where the authors noted the connection with proximity theory and the possibility to use proximity theory for proving topological and proximity representation theorems for contact algebras.

We recall some topological notions.

A topological space X is said to be

- *semiregular* if it has a base $\mathbb{B}$ of regular closed sets; namely, every closed set is the intersection of elements of $\mathbb{B}$,
- *normal* if every pair of closed disjoint sets can be separated by a pair of open sets,
- $\varkappa$*-normal* (cf. [**51**]) if every pair of regular closed disjoint sets can be separated by a pair of open sets,
- *extensional* if $\mathrm{RC}(X)$ satisfies axiom (Ext),
- *weakly regular* (cf. [**21**]) if it is semiregular and for every nonempty open set a there exits a nonempty open set b such that $\mathrm{Cl}(a) \subseteq b$,
- *connected* if it cannot be represented as the sum of two disjoint nonempty open sets,
- a T_0*-space* if for every two different points $x \neq y$ there exists an open set that contains one of them and does not contain the other,
- a T_1*-space* if every one-point set $\{x\}$ is a closed set,
- a *Hausdorff space* (or a T_2*-space*) if every two different points can be separated by a pair of disjoint open sets,

- a *compact space* if it satisfies the following condition: if $\{A_i : i \in I\}$ is a nonempty family of closed sets of X such that for every finite subset $J \subseteq I$ we have $\bigcap\{A_i : i \in J\} \neq \varnothing$, then $\bigcap\{A_i : i \in I\} \neq \varnothing$.

Lemma 2.2.1. *The following assertions hold.*

(1) *Let X be semiregular. Then X is weakly regular if and only if* $\mathrm{RC}(X)$ *satisfies* (Ext) [**21**].
(2) *X is $\varkappa$-normal if and only if* $\mathrm{RC}(X)$ *satisfies* (Nor) [**21**].
(3) *X is connected if and only if* $\mathrm{RC}(X)$ *satisfies axiom* (Con) [**5, 21**].
(4) *If X is a compact Hausdorff space, then* $\mathrm{RO}(X)$ (*consequently,* $\mathrm{RC}(X)$) *satisfies* (Ext) *and* (Nor) [**63**].
(5) *If X is a normal Hausdorff space, then* $\mathrm{RO}(X)$ *satisfies* (Nor) [**6**].

Note that axiom (Con) is equivalent to the axiom

(Con′) if $a \neq 0$, $b \neq 0$, and $a + b = 1$, then aCb.

Similarly, (Nor) is equivalent to the axiom

(Nor′) if $a\overline{C}b$, then $(\exists a'b')(a\overline{C}a'$ and $b\overline{C}b'$ and $a' + b' = 1)$.

Below, we consider embedding theorems for contact algebras regarded as contact subalgebras of the contact algebras of regular closed sets in some topological spaces. It is important to know the conditions under which an algebra satisfies some of axioms (Con), (Ext), and (Nor) if and only if its subalgebra satisfies the same axioms.

A contact subalgebra B_1 of B_2 is said to be *dense* if

(Dense) $(\forall a_2 \in B_2)(a_2 \neq 0 \to (\exists a_1 \in B_1)(a_1 \neq 0$ and $a_1 \leqslant a_2))$

and *co-dense* if

(Co-dense) $(\forall a_2 \in B_2)(a_2 \neq 1 \to (\exists a_1 \in B_1)(a_1 \neq 1$ and $a_2 \leqslant a_1))$.

It is easy to see that (Dense) is equivalent to (Co-dense).

We say that B_1 is a *C-separable* subalgebra of B_2 if

$$(C\text{-separation}) \quad (\forall a_2 b_2 \in B_2)(a_2 \overline{C} b_2 \to (\exists a_1 b_1 \in B_1)(a_2 \leqslant a_1 \text{ and } b_2 \leqslant b_1 \text{ and } a_1 \overline{C} b_1)).$$

If h is an embedding of B_1 regarded as a contact subalgebra of B_2, then h is a *dense embedding* provided that $h(B_1)$ is a dense subalgebra of B_2. We say that h is a *C-separable* if $h(B_1)$ is a C-separable subalgebra of B_2.

The following assertion is important.

Theorem 2.2.2. *Let B_1 be a C-separable contact subalgebra of B_2. Then the following assertions hold.*

(1) *B_1 satisfies* (Con) *if and only if B_2 satisfies* (Con).
(2) *Let B_1 be a dense subalgebra of B_2. Then B_1 satisfies* (Ext) *if and only if B_2 satisfies* (Ext).
(3) *B_1 satisfies* (Nor) *if and only if B_2 satisfies* (Nor).

Proof. We prove assertion (3) taking (Nor′) instead of (Nor).

($\to$) Let B_1 satisfies (Nor′), and let $a_2 \overline{C} b_2$ for $a_2, b_2 \in B_2$. By (C-separation), there exist a_1 and b_1 in B_1 (consequently, in B_2) such that $a_2 \leqslant a_1$, $b_2 \leqslant b_1$, and $a_1 \overline{C} b_1$. By (Nor), there exist $a_1^{'}$ and $b_1^{'}$ in B_1 (consequently, in B_2) such that $a_1^{'} + b_1^{'} = 1$, $a_1 \overline{C} a_1^{'}$, and $b_1 \overline{C} b_1^{'}$. Since C is monotone and symmetric, we have $a_2 \overline{C} a_1^{'}$ and $b_2 \overline{C} b_1^{'}$, which shows that B_2 satisfies (Nor′).

($\leftarrow$) Let B_2 satisfy (Nor′), and let $a_1 \overline{C} b_1$. for a_1 and b_1 in B_1 (consequently, in B_2). By (Nor′), there exist $a_2^{'}, b_2^{'} \in B_2$ such that $a_2^{'} + b_2^{'} = 1$, $a_1 \overline{C} a_2^{'}$, and $b_1 \overline{C} b_2^{'}$. By (C-separation), if $a_1 \overline{C} a_2^{'}$, then there exist $c_1, d_1 \in B_1$ such that $a_1 \leqslant c_1$, $a_2^{'} \leqslant d_1$ and $c_1 \overline{C} d_1$. Similarly, by (C-separation), $b_1 \overline{C} b_2^{'}$ implies that there exist $e_1, f_1 \in B_1$ such that $b_1 \leqslant e_1$, $b_2^{'} \leqslant f_1$,, and $e_1 \overline{C} f_1$. Therefore, $d_1 + f_1 = 1$, $a_1 \overline{C} d_1$, and $b_1 \overline{C} f_1$, which shows that B_1 satisfies (Nor′). □

The following assertion is well known.

Proposition 2.2.3 ([**52**]). *If h is a dense embedding of a Boolean algebra B_1 in a Boolean algebra B_2 and B_1 is complete, then h is a complete isomorphism of B_1 onto B_2.*

2.3. Points in contact algebras and topological representation theorems. A simple case

We begin by discussing how to define canonically points in contact algebras. Then we discuss how to introduce canonically a topology in the set of points. Finally, we show that regions in the algebra can be identified with regular closed sets in the topological space by an appropriate canonical isomorphism. This procedure is not unique. Choosing different axioms of contact algebra, we obtain different kinds of points and thereby different canonical constructions implying different kinds of topological spaces. This shows that the notion of a point is not unique and points of a more complicated structure can provide better topological spaces. We illustrate this fact by considering the simplest notion of a point. A more complicated notion of a point and the corresponding canonical constructions will be considered in Section 2.4. We mainly follow [**15**], However, the presented construction is new and leads to stronger results. Therefore, we give proofs.

Let X be a topological space, and let $x \in X$ be a point. The set $P_x = \{a \in \mathrm{RC}(X) : x \in a\}$ satisfies the following conditions:

(1) $X \in P_x$,

(2) $a \cup b \in P_x$ if and only if $a \in P_x$ or $b \in P_x$.

(3) If $a, b \in P_x$, then aCb.

The set P_x is a collection of regions. If the space is at least T_0, then $x \neq y$ implies $P_x \neq P_y$. Another interesting property of P_x is that if regions a and b are in a contact, then there exists P_x such that $a, b \in P_x$. Thus, the sets P_x react like points. This fact can be used to identify points with sets P_x. There are no points in contact algebras, but, instead of points, we can consider collections of regions satisfying (1)–(3). The situation is similar to that in the representation theory of Boolean algebras (cf. [**55**]), where abstract points in a Boolean algebra are associated with ultrafilters, collections of elements of the algebra. Sets satisfying (1)–(3) are similar to ultrafilters and were considered in the theory of proximity spaces, where they were called *clans* (cf. [**56**]). For contact algebras clans were used in [**61, 21, 15**]. A clan is defined as follows.

Let $\underline{B} = (B, C)$ be a contact algebra. A set $\Gamma \subseteq B$ of regions is called a *clan* (in $\underline{B}$) if it satisfies the following conditions:

(Clan 1) $1 \in \Gamma$,

(Clan 2) $a + b \in \Gamma$ if and only if $a \in \Gamma$ or $b \in \Gamma$,

(Clan 3) If $a, b \in \Gamma$, then aCb.

Clans in $\mathrm{RC}(X)$ in the form P_x are called *point clans.* A clan is said to be *maximal* if it is maximal with respect to inclusion. By the Zorn lemma, every clan is contained in a maximal clan. Denote by $\mathrm{CLANS}(\underline{B})$ ($\mathrm{MaxCLANS}(\underline{B})$) the set of all clans (maximal clans) in $\underline{B}$. For brevity, we write CLANS and MaxCLANS if a contact algebra $\underline{B}$ is fixed. Thus, we have two candidates for points: CLANS and MaxCLANS. In this section, we consider only CLANS.

We show how to construct a clan. First of all, note that every ultrafilter in B satisfies (Clan 1) and (Clan 2) and also (Clan3) by (C4), which means that it is a clan. Another construction is as follows. For two filters F and G in B we define: $F\rho G$ if and only if $F \times G \subseteq C$. It is easy to see that the relation ρ is reflexive and symmetric. Let Σ be a nonempty set of maximal filters of B such that for any $F, G \in \Sigma$ we have $F\rho G$. Then the union of all elements of Σ is a clan and every clan can be obtained by such a construction (cf. [**15**]).

The following assertion is a simple consequence of the Zorn lemma.

Lemma 2.3.1 ([**19, 15**]). *The following assertions hold.*

(1) *If F and G are filters and $F\rho G$, then there exist maximal filters $F' \supseteq F$ and $G' \supseteq G$ such that $F'\rho G'$.*

(2) *aCb if and only if there exist maximal filters F and G such that $F\rho G$, $a \in F$, and $b \in G$.*

The following assertion characterizes contacts and part-of in terms of clans.

Lemma 2.3.2 ([**15**]). *The following assertions hold.*

(1) *aCb if and only if $(\exists\Gamma \in \mathrm{CLANS}\,(B))(a, b \in \Gamma)$.*

(2) *$a \leqslant b$ if and only if $(\forall\Gamma \in \mathrm{CLANS}\,(B))(a \in \Gamma \to b \in \Gamma)$.*

(3) *$a = 1$ if and only if $(\forall\Gamma \in \mathrm{CLANS}\,(B))(a \in \Gamma)$.*

We explain the idea of the proof of (1). If aCb, then for the filters $F' = \{a' : a \leqslant a'\}$ and $G' = \{b' : b \leqslant b\}$ we have $F'\rho G'$. By Lemma 2.3.1, F' and G'. can be extended to maximal filters F and G such that $F\rho G$. Then the clan $\Gamma = F \cup G$ contains both a and b. The converse implication follows from the properties of clans. Assertions (2) and (3) are proved in a standard Boolean way because ultrafilters are clans.

For $a \in B$ we introduce the Stone-like mapping $h(a) = \{\Gamma \in \mathrm{CLANS}\,(B) : a \in \Gamma\}$.

From Lemma 2.3.2 and the properties of clans we obtain the following assertion.

Lemma 2.3.3 ([**15**]). *The following assertions hold.*

(1) *$h(a + b) = h(a) \cup h(b)$, $h(0) = \varnothing$, and $h(1) = \mathrm{CLANS}\,(B)$.*

(2) *$a \leqslant b$ if and only if $h(a) \subseteq h(b)$.*

(3) *$a = 1$ if and only if $h(a) = \mathrm{CLANS}\,(B)$.*

(4) *aCb if and only if $h(a) \cap h(b) \neq \varnothing$.*

Our next goal is to turn the set $X = \mathrm{CLANS}$ into a topological space and to establish a representation theorem. For this purpose, as in the Stone representation theory for Boolean algebras, we define a topology τ taking $\{h(a) : a \in B\}$ for the base of closed sets and considering h as the required embedding. We expect that h will embed the contact algebra $\underline{B}$ into the contact algebra $\mathrm{RC}(X)$. Proposition 2.3.4 shows that in a sense regular closed sets cannot be excluded. Recall that the reduct $(B, 0, 1, +)$ of a Boolean algebra $(B, 0, 1, +, ., *)$ is a Boolean algebra, called the *upper semi-lattice* of B, and it generates the same ordering relation $\leqslant$ as in B.

Proposition 2.3.4. *Suppose that X is a topological space, $\underline{B} = (B, 0, 1, +, ., *)$ is a Boolean algebra, and h is an embedding of the upper semi-lattice $(B, 1, +)$ in the upper semi-lattice of closed sets of X such that the set $\{h(a) : a \in B\}$ is a base of closed sets of X. Then the following assertions hold*:

(1) $h(a^*) = \mathrm{Cl}(-h(a))$,
(2) *for every $a \in B$, $h(a)$ is a regular closed set in X and, consequently, X is a semiregular space,*
(3) *h is an embedding in* $\mathrm{RC}(X)$.

Proof. (1) Consider an arbitrary point $x \in X$. Assertion (1) follows from the sequence of equivalences

$$\begin{aligned} x \in \mathrm{Cl}(-h(a)) &\Leftrightarrow (\forall b \in B)(-h(a) \subseteq h(b) \to x \in h(b)) \\ &\Leftrightarrow (\forall b \in B)(h(a) \cup h(b) = X \to x \in h(b)), \\ &\Leftrightarrow (\forall b \in B)(a + b = 1 \to x \in h(b)), \\ &\Leftrightarrow (\forall b \in B)(a^* \leqslant b \to x \in h(b)), \\ &\Leftrightarrow (\forall b \in B)(h(a^*) \subseteq h(b) \to x \in h(b)) \Leftrightarrow x \in h(a^*) \end{aligned}$$

since $\mathrm{Cl}(-h(a))$ is the intersection of all elements in the base containing $-h(a)$. Here, we repeatedly used the assumption that h is an embedding preserving 1, +, and $\leqslant$.

(2) Applying (1) twice, we find

$$\begin{aligned} x \in h(a) &\Leftrightarrow x \in h(a^{**}) \\ &\Leftrightarrow x \in \mathrm{Cl}(-\mathrm{Cl}(-h(a))) \\ &\Leftrightarrow x \in \mathrm{Cl}(\mathrm{Int}(h(a))), \end{aligned}$$

which shows that for every $a \in B$, $h(a)$ is a regular closed set and, consequently, X is a semiregular space.

(3) This assertion follows from (2), (1), and the assumption that h preserves $+$ and 1. □

Combining Lemmas 2.3.2, 2.3.3, and 2.3.4, we obtain the following assertion.

Lemma 2.3.5. *h is an embedding of $\underline{B}$ in* RC(X) *with* $X =$ CLANS $(\underline{B})$.

Properties of X = CLANS are presented by the following assertion.

Lemma 2.3.6. *The space* $X =$ CLANS $(\underline{B})$ *is semiregular, possesses the T_0 property, and is compact.*

Proof. The space X is semiregular since it has the base of regular closed sets.

To prove the T_0 property, we suppose that Γ and Δ are two different points of X. Since Γ and Δ are clans, one of them, say Γ, is not included in the other, Δ. Then there is $a \in \Gamma$ such that $a \notin \Delta$. Hence the open set $-h(a)$ contains Δ and not Γ.

To prove the compactness of X, it suffices to prove the following. Let I be a nonempty set of indices, and let $A = \bigcap\{h(a) : a \in I\}$. If for every finite set $I_0 \subseteq I$ we have $\bigcap\{h(a) : a \in I_0\} \neq \varnothing$, then $A \neq \varnothing$. Indeed, the condition that $\bigcap\{h(a) : a \in I_0\} \neq \varnothing$ for all finite subsets I_0 of I guarantees the existence of an ultrafilter U such that $\{h(a) : a \in I\} \subseteq U$. It is easy to see that the set $\Gamma = \{a : h(a) \in U\}$ is a clan. Hence for every $a \in I$

$$a \in I \to h(a) \in U \to a \in \Gamma \to \Gamma \in h(a).$$

Thus, $\Gamma \in A$ and, consequently, $A \neq \varnothing$. □

We show how the additional axioms (Con), (Ext), and (Nor) affect the properties of the canonical space $X =$ CLANS $(\underline{B})$.

Let A be a regular closed set in the canonical space X. The set $F_A = \{a \in B : A \subseteq h(a)\}$ is called the *canonical filter* of A.

Lemma 2.3.7. *The canonical filter F_A possesses the following properties*:

(1) F_A *is a filter,*
(2) $(\forall\Gamma \in X)(\Gamma \in A$ *if and only if* $F_A \subseteq \Gamma)$,
(2) *If* $A \neq X$, *then there is* $a \in B$ *such that* $a \neq 1$ *and* $A \subseteq h(a)$,
(4) $F_A \times F_B \subseteq C$ *if and only if* $A \cap B \neq \varnothing$,
(5) $A \cap B = \varnothing$ *if and only if* $(\exists a, b \in B)(A \subseteq h(a)$ *and* $B \subseteq h(b)$ *and* $a\overline{C}b)$.

Proof. (1) This assertion is a direct consequence of the definition of F_A and Lemma 2.3.3.

(2) Since A is a closed set and the set of all $h(a)$ is a closed base for the topology of X, for any clan Γ

$$\begin{aligned}\Gamma \in A &\Leftrightarrow (\forall a \in B)(A \subseteq h(a) \rightarrow a \in \Gamma)\\ &\Leftrightarrow (\forall a \in B)(a \in F_A \rightarrow a \in \Gamma)\\ &\Leftrightarrow F_A \subseteq \Gamma.\end{aligned}$$

(3) Let $A \neq X$. Then there is a clan Γ such that $\Gamma \notin A$. By (2), $F_A \not\subseteq \Gamma$ and, consequently, there is $a \in B$ such that $a \in F_A$ and $a \notin \Gamma$. Hence $A \subseteq h(a)$, and $a \neq 1$ by Lemma 2.3.3.

(4) $(\leftarrow)$ Assume that there is a clan $\Gamma \in A$ such that $\Gamma \in B$. Then $F_A \subseteq \Gamma$ and $F_B \subseteq \Gamma$. Consequently, $(\forall a, b \in B)(A \subseteq h(a)$ and $B \subseteq h(b) \rightarrow a, b \in \Gamma)$. Hence $(\forall a, b \in B)(A \subseteq h(a)$ and $B \subseteq h(b) \rightarrow aCb)$, which yields $F_A \times F_B \subseteq C$.

$(\rightarrow)$ Let $F_A \times F_B \subseteq C$. By Lemma 2.3.1, there exist maximal filters F_1 and F_2 such that $F_A \subseteq F_1$, $F_B \subseteq F_2$, and $F_1 \rho F_2$, i.e., $F_1 \times F_2 \subseteq C$. Then $\Gamma = F_1 \cup F_2$ is a clan and $F_A \subseteq \Gamma$, $F_B \subseteq \Gamma$. By (2), $\Gamma \in A$, $\Gamma \in B$ and, consequently, $A \cap B \neq \varnothing$.

(5) This assertion is equivalent to (4). □

Corollary 2.3.8. *h is a dense C-separable embedding of $\underline{B}$ in* $\mathrm{RC}(X)$ *with* $X = \mathrm{CLANS}(\underline{B})$.

Proof. The assertion immediately follows from Lemma 2.3.7, (3), (4). □

The above results yield the following

Theorem 2.3.9 (representation of contact algebras)**.** *Let $\underline{B} = (BC)$ be a contact algebra. Then there exists a compact semiregular* T_0*-space* (X, τ) *and a dense* C*-separable embedding* h *of* $\underline{B}$ *in the contact algebra of regular closed sets* $\mathrm{RC}(X)$*. Moreover,*

(1) $\underline{B}$ *satisfies* (Con) *if and only if* X *is connected,*

(2) $\underline{B}$ *satisfies* (Ext) *if and only if* X *is weakly regular,*

(3) $\underline{B}$ *satisfies* (Nor) *if and only if* X *is* $\varkappa$*-normal,*

(4) *if* $\underline{B}$ *is a complete algebra, then* h *is an isomorphism between* $\underline{B}$ *and the complete contact algebra* $\mathrm{RC}(X)$*.*

Proof. Assertions (1)–(4) follow from Lemmas 2.2.2, 2.2.1, and 2.2.3. □

A similar assertion was proved in [**15**] with the compactness of X replaced with a stronger notion of *C-semiregularity* (a semiregular T_0-space is C-*semiregular* if every clan in $\mathrm{RC}(X)$ is a point clan). Note that any C-semiregular space is compact, but there are compact semiregular spaces that are not C-semiregular.

Based on the definition of a point in a contact algebra, we can give a second-order definition of the n-ary contact:

$C_n(a_1, \ldots, a_n)$ if and only if there exists a clan Γ such that $\{a_1, \ldots, a_n\} \subseteq \Gamma$.

Using this definition and Theorem 2.3.9, we find

$C_n(a_1, \ldots, a_n)$ if and only if $h(a_1) \cap \ldots \cap h(a_n) \neq \varnothing$,

which shows that the above definition agrees with the notion of the standard topological n-ary contact.

2.4. Another topological representation of contact algebras

Under additional assumptions, contact algebras can be represented in better topological spaces, T_1 or T_2. If contact algebras satisfy axiom (Ext), we can prove a representation theorem for compact weakly regular T_1-spaces with maximal clans instead of points. By axiom (Ext), it is possible to repeat all the arguments of Section 2.3 to obtain a representation result similar to Theorem 2.3.9, but X should be replaced with T_1 in view of the maximality of clans.

Theorem 2.4.1 (representation of extensional contact algebras)**.** *Let* $\underline{B} = (BC)$ *be a contact algebra satisfying axiom* (Ext)*. Then there exists a compact weakly regular* T_1*-space* (X, τ) *and a dense* C*-separable embedding* h *of* $\underline{B}$ *in the contact algebra of regular closed sets* $\mathrm{RC}(X)$*. Moreover,*

(1) $\underline{B}$ *satisfies* (Con) *if and only if* X *is connected,*
(2) $\underline{B}$ *satisfies* (Nor) *if and only if* X *is* $\varkappa$*-normal,*
(3) *if* $\underline{B}$ *is a complete algebra, then* h *is an isomorphism between* $\underline{B}$ *and* $\mathrm{RC}(X)$*.*

This theorem covers the case of the RCC system. Similar assertions were proved by Düntsch and Winter in [**21**] (without compactness) for RCC system and by Dimov and Vakarelov in [**15**], where the compactness was replaced with the stronger condition of *CM-semiregularity*

For contact algebras satisfying both axioms (Ext) and (Nor) the representation theorem can be improved.

Theorem 2.4.2 (representation of extensional normal contact algebras, [**61, 15**])**.** *Let* $\underline{B} = (BC)$ *be a contact algebra satisfying both axioms* (Ext) *and* (Nor)*. Then there exists a compact Hausdorff space* (X, τ) *and a dense embedding* h *of* $\underline{B}$ *in the contact algebra of regular closed sets* $\mathrm{RC}(X)$*. Moreover,*

(1) $\underline{B}$ *satisfies* (Con) *if and only if* X *is connected,*
(2) *if* $\underline{B}$ *is a complete algebra, then* h *is an isomorphism between* $\underline{B}$ *and* $\mathrm{RC}(X)$*.*

An assertion similar to Theorem 2.4.2 was first proved by de Vries [**63**] for $\mathrm{RO}(X)$ instead of $\mathrm{RC}(X)$.

To prove Theorem 2.4.2, we introduce another kind of points. A subset Γ of $\underline{B}$ is called a *cluster* if it is a clan such that

(Cluster) if aCb for every $b \in \Gamma$, then $a \in \Gamma$.

Any cluster is a maximal clan. However, to prove the existence of clusters in $\underline{B}$, we need axioms (Ext) and (Nor). Clusters were used in proximity theory for obtaining compactification theorems for topological spaces (cf. [**42**]).

For representing contact algebras in some special topological spaces (for example, *regular* spaces), other (not necessarily first-order) axioms can be required. The role of points in such algebras is played by clusters of special kind, called *co-ends*. Formally, a *co-end* Γ is a cluster such that for every $a \notin \Gamma$ there exists $b \notin \Gamma$ such that $a \ll b$.

Contact algebras representable in $\mathrm{RC}(X)$ with a regular space X satisfy the following regularity axiom:

(Reg) if aCb, then there exists a co-end Γ containing a and b.

We refer to [**15**] for details.

Note that (Reg) is not a first-order axiom because it contains the second-order notion of a co-end. It is not known if there is a first-order axiom equivalent to (Reg). The following general question can be posed: For a given class Σ of topological spaces find axioms providing representation of algebras in $\mathrm{RC}(X)$ with $X \in \Sigma$.

The above representation theorems are of embedding type, i.e., they state that a contact algebra $\underline{B}$ can be embedded in the contact algebra of regular closed sets $\mathrm{RC}(X)$ of some topological space X. Such representations do not exclude the case where non-isomorphic contact algebras are embedded in the contact algebra of the same space X. Moreover, X_1 and X_2 can be nonhomeomorphic, whereas $\mathrm{RC}(X_1)$ and $\mathrm{RC}(X_2)$ are isomorphic. To establish a one-to-one correspondence between contact algebras (up to an isomorphism) and topological spaces (up to a homeomorphism), we require the completeness of contact algebras. Then for X we take the so-called C-*semiregular space*, i.e., a semiregular space X such that every clan in $\mathrm{RC}(X)$ is a point clan. Representations theorems for complete contact algebras satisfying some axioms like (Con), (Ext), and (Nor) can be found in [**15**].

2.5. Models of contact algebras in proximity spaces

Proposition 2.3.4 motivates the following observation: In order for a topological representation h of contact algebras to generate a topology, h must be an embedding of the Boolean part of the contact algebra in the Boolean algebra $\mathrm{RC}(X)$ with some semiregular space X. However, Proposition 2.3.4 does not guarantee that the contact relation C in $\mathrm{RC}(X)$ is defined in the standard way, i.e., $aCb \Leftrightarrow a \cap b \neq \varnothing$.

In this section, we demonstrate topological models for contact algebras, where elements of the Boolean algebra are regular closed sets of some topological space, whereas the relation aCb is not the standard topological contact. To construct such examples, we use *proximity spaces* introduced by Efremovič in [**25**] (cf. also [**42**]) and known as *Efremovič proximity spaces* or simply E-proximity spaces.

An *Efremovič proximity space* is a system (X, δ), where X is a nonempty set and δ is a binary relation, called *proximity relation*, on subsets of X such that the following axioms are satisfied:

(E1) if $A\delta B$, then $A, B \neq \varnothing$,

(E2) $A\delta(B \cup C)$ if and only if $A\delta B$ or $A\delta C$,

(E3) if $A\delta B$, then $B\delta A$,

(E4) if $A \cap B \neq \varnothing$, then $A\delta B$,

(E5) if $A\overline{\delta}B$, then there exists C such that $A\overline{\delta}C$ and $(X \setminus C)\overline{\delta}B$.

A proximity space (X, δ) is said to be *separated* if it satisfies the following condition:

$$\text{if } x, y \in X, \text{ then } \{x\}\delta\{y\} \text{ implies } x = y$$

.

Spaces satisfying only axioms (E1)–(E4) were considered by Čech [**9**]. Other generalizations of E-proximity spaces can be found in [**42**].

The relation $\ll$ in a Čech proximity space is defined as

$$A \ll B \text{ if and only if } A\overline{\delta}(X \smallsetminus B).$$

If $A \ll B$, then B is called a *δ-neighborhood* of A. It is obvious that the relations δ and $\ll$ are interdefinable and the axioms of Čhech proximity space can be expressed in terms of $\ll$ as follows:

($\ll$ 1) $X \ll X$,

($\ll$ 2) if $A \ll B$, then $A \leqslant B$,

($\ll$ 3) if $A \leqslant B \ll C \leqslant D$ then $A \ll D$,

($\ll$ 4) if $A \ll B$, then $(X \smallsetminus B) \ll (X \smallsetminus A)$,

($\ll$ 5) if $A \ll B$ and $A \ll C$, then $A \ll B \cap C$.

In terms of $\ll$, axiom (E5) takes the form

($\ll$ 6) if $A \ll B$, then for some C: $A \ll C \ll B$.

Note that axioms (E1)–(E4) are the same as axioms of contact algebras (C1)–(C4); moreover, axiom (E5) or ($\ll$ 6) is the same as axiom (Nor). Owing to this fact, it is possible to use proximity spaces for constructing models of contact algebras.

A standard example of E-proximity space comes from metric spaces (X, d). Using the distance $d(A, B) = \inf \{d(a, b) : a \in A, b \in B\}$ between two sets A and B of a metric space, we define the *proximity relation*

$$A\delta B \text{ if and only if } d(A, B) = 0.$$

In this case, all the axioms of E-proximity space are satisfied.

A relational kind of proximity spaces is considered in [**60**]: for a given relational system (X, R), where $X \neq \varnothing$ and R is a binary relation in X, the relation δ_R on subsets of X is defined as

$$A\delta_R B \text{ if and only if } (\exists x \in A)(\exists y \in B)(xRy).$$

In this case, axiom (E1) and the right and left implications in axiom (E2) are satisfied by any R, axiom (E3) is satisfied if R

is symmetric, axiom (E4) is satisfied if R is reflexive, and axiom (E5) is satisfied if R is transitive relation.

For an example of a Čech proximity space we can consider a system (X, R) with a reflexive symmetric relation R, and for an example of an E-proximity space we can take a system (X, R), where R is an equivalence relation. These examples will be used for presenting discrete models of contact algebras in the following section.

Now, we use E-proximity spaces to construct topological models of contact algebras with a nonstandard proximity model of contact relation.

Every Čech proximity space (X, δ) defines a topology in X in the following way. Let $\mathrm{Cl}(A) = \{x \in X : \{x\}\delta A\}$. Then Cl is a Kuratowski closure operator defining a topology in X. The following assertion shows how Cl and Int are connected with the relation $\ll$.

Lemma 2.5.1 ([**42**]). *The following assertions hold*:

(1) $A \ll B$ *implies* $\mathrm{Cl}(A) \ll B$*,*

(2) $A \ll B$ *implies* $A \ll \mathrm{Int}(B)$*.*

Having a topology in a proximity space X, we can consider the set of regular closed subsets of X with respect to this topology. Consider the Boolean algebra $\mathrm{RC}(X, \delta)$ of regular closed sets with respect to the introduced topology in (X, δ). Since axioms (C1)–(C4) are the same as the axioms of proximity space, we conclude that $\mathrm{RC}(X, \delta)$ is a contact algebra. We show that axioms (Nor) and (Ext) are also satisfied. Axiom (Nor) follows from the following stronger version of axiom $(\ll 6)$:

$(\ll 6')$ if $A \ll B$, then $A \ll C \ll B$ for some regular closed set C.

Indeed, let $A \ll B$. By axiom $(\ll 6)$, there is a subset D (not necessarily regular and closed) such that $A \ll D \ll B$. By Lemma 2.5.1,

$$A \ll \mathrm{Int}(D) \subseteq \mathrm{Cl}(\mathrm{Int}(D)) \subseteq \mathrm{Cl}(D) \ll B.$$

Hence $A \ll \mathrm{Cl}(\mathrm{Int}(D)) \ll B$. Then $C = \mathrm{Cl}(\mathrm{Int}(D))$ is the required regular closed subset.

To verify axiom (Ext), assume that $A \neq \varnothing$ is a regular closed set. Then there is a point $x \in \mathrm{Int}(A)$ and, consequently, $\{x\} \ll A$. By ($\ll 6'$), we get a regular closed set B such that $\{x\} \ll B \ll A$ and, consequently, $B \neq \varnothing$ and $B \ll A$. Thus, axiom (Ext) is satisfied.

The above arguments lead to the following assertion.

Theorem 2.5.2 ([**61**]). *Let* (X, δ) *be an* E*-proximity space, and let* $\mathrm{RC}(X, \delta)$ *be the Boolean algebra of regular closed sets in* (X, δ)*. Then* $(\mathrm{RC}(X, \delta), \delta)$ *is a contact algebra with contact* δ *satisfying axioms* (Nor) *and* (Ext)*.*

Note that the proximity contact defined in Theorem 2.5.2 is not necessarily the standard topological contact for regular closed sets. For example, consider the metric space of rational numbers and the corresponding proximity space. For the regular closed sets $A = \{x : 0 \leqslant x^2 \leqslant 2\}$ and $B = \{x : 2 \leqslant x^2 \leqslant 4\}$ we have $d(A, B) = 0$, which implies $A\delta B$. But these sets are not in the relation of the topological contact because $A \cap B = \varnothing$. If we consider the same sets over the real numbers, then both proximity and topological contacts hold. In the further consideration, we have exactly $A \cap B = \{\sqrt{2}\}$. The reason is that the space of the rational numbers does not have points (in this case, the point $\sqrt{2}$) enough to describe the standard contact, but this can be done with the help for the proximity contact. Thus, the proximity contact is more suitable for describing the real picture between regular closed sets. Generalizing the notion of an Efremovič proximity space in different ways, we can obtain models with proximity-like contacts for other contact algebras (some examples are contained in [**16**]).

2.6. Contact algebras with predicate of boundedness

In this section, we extend the language of contact algebras by introducing the predicate of boundedness. To explain this notion at the intuitive level, we consider the real line R. A regular closed set a in R is bounded if it is contained in a closed interval $[x, y]$ of R. A generalization to the space R^n is obvious: closed spheres should be taken instead of $[x, y]$.

The notion of boundedness was used in topology by Hu [**35**] and in proximity spaces by Leader [**37**] (cf. also [**42**]).

The *boundedness* is defined as a class $\mathcal{B}$ of subsets of a space X such that

(B1) $\varnothing \in \mathcal{B}$,

(B2) if $B \in \mathcal{B}$ and $A \subseteq B$, then $A \in \mathcal{B}$,

(B3) if $A, B \in \mathcal{B}$, then $A \cup B \in \mathcal{B}$.

From the formal point of view, it is a fixed ideal of sets in X. The boundedness predicates related to the topology of X are of great interest. For example, in R^n, the set of bounded regular closed regions coincides with the set of compact regular closed regions. Note that R^n is a locally compact Hausdorff space (recall that a topological space X is *locally compact* if for every point $x \in X$ there is a compact regular closed set a such that $x \in \mathrm{Int}(a)$). Therefore, the above definition can be taken for a topological definition of boundedness in locally compact spaces. Based on this definition of boundedness, Leader [**37**] introduced *local proximity spaces* by adding the following axioms to axioms (B1)–(B3) in the definition of Čech proximity spaces:

(B4) if $A\delta B$, then $\exists C \in \mathcal{B}$ such that $C \subseteq B$ and $A\delta C$,

(C) if $A \in \mathcal{B}$ and $A \ll C$, then $\exists C \in \mathcal{B}$ such that $A \ll C \ll B$ $(A \ll B \Leftrightarrow A\overline{\delta} - B)$.

Note that axiom (C) is equivalent to the conjunction of the axiom

(B5) if $A \in \mathcal{B}$, then $\exists B \in \mathcal{B}$ such that $A \ll B$

and the Efremovič axiom

(E5) if $A \ll B$, then $\exists C$ such that $A \ll C \ll B$.

A typical example of a local proximity space is any locally compact Hausdorff spaces X with $A\delta B$ defined as $\mathrm{Cl}(A) \cap \mathrm{Cl}(B) \neq \varnothing$ and $A \in \mathcal{B}$ if and only if $\mathrm{Cl}(A)$ is a compact subset of X. Leader used this example to develop the local compactification theory for local proximity spaces.

Using an analogy between contact algebras and proximity spaces (cf. Section 2.5), we introduce a *local contact algebra* as a system $\underline{B} = (B, 0, 1, +, ., *, C, \mathcal{B})$, where $(B, 0, 1, +, ., *, C)$ is a contact algebra and $\mathcal{B}$ is a subset of B satisfying axioms similar to (B1)–(B5) and denoted in the same way:

(B1) $0 \in \mathcal{B}$,

(B2) if $b \in \mathcal{B}$ and $a \leqslant b$, then $a \in \mathcal{B}$,

(B3) if $a, b \in \mathcal{B}$, then $a + b \in \mathcal{B}$,

(B4) if aCb, then $\exists c \in \mathcal{B}$ such that $c \leqslant b$ and aCc,

(B5) if $a \in \mathcal{B}$, then $\exists b \in \mathcal{B}$ such that $a \ll b$ $(a \ll b \Leftrightarrow a\overline{C}b^*)$.

We say that $\underline{B}$ is *connected* (*extensional* or *normal*) if it satisfies axiom (Con) ((Ext) or (Nor)).

Standard examples of local contact algebras can be obtained from a locally compact space X: the contact algebra $\mathrm{RC}(X)$ and the set of bounded regions $\mathcal{B}(X)$ coinciding with the compact regular closed sets in X.

In mereotopology, the notion of boundedness was first used by Roeper in [**49**], where it was referred to as the *limitedness*. The *region-based topology* introduced by Roeger is equivalent to the local contact algebras satisfying axioms (Ext) and (Nor). The axioms of Roeper are (C1)–(C4), (B1)–(B4), and the following:

(R) if $a \in \mathcal{B}$, $b \neq 0$ and $a \ll b$, then $\exists c \in \mathcal{B}$ such that $c \neq 0$ and

$a \ll c \lll b$.

Note that axiom (R) is close to Leader's axiom (C) and de Vries' axiom (P6). Roeper did not make any reference to their works, and it is quite impressive that he independently worked out some ideas and methods of proximity theory. For example, his definition of a point as a *coincidence set* is the same as a bounded cluster introduced by Leader. Roeper gave an elegant proof of the fact that every complete contact algebra satisfying axioms (Ext) and (Nor) (the *complete region-based topology* in the terminology of Roeper) is isomorphic to the local contact algebra $\mathrm{RC}(X)$ of regular closed sets of a Hausdorff locally compact space X and that there is a one-to-one correspondence between region-based topologies (up to an isomorphism) and Hausdorff locally compact spaces (up to a homeomorphism). Another proof of the Roeper theorem is contained in [**61**], where the Leader compactification theorem is generalized to local proximity spaces.

The goal of this section is to expand the Roeper embedding theorem to the case of local contact algebras under additional axioms (Con), (Ext), and (Nor).

Theorem 2.6.1 (representation of local contact algebras). *Let* $\underline{B} = (B, C, \mathcal{B})$ *be a local contact algebra. Then there exists a locally compact semiregular* T_0*-space* (X, τ) *and a dense* C*-separable embedding* h *of* $\underline{B}$ *in the local contact algebra of regular closed sets* $\mathrm{RC}(X)$. *Moreover,*

(1) $\underline{B}$ *satisfies* (Con) *if and only if* X *is connected,*

(2) $\underline{B}$ *satisfies* (Ext) *if and only if* X *is weakly regular,*

(3) $\underline{B}$ *satisfies* (Nor) *if and only if* X *is* $\varkappa$*-normal,*

(4) *if* $\underline{B}$ *is a complete algebra, then* h *is an isomorphism between* $\underline{B}$ *and the complete local contact algebra* $\mathrm{RC}(X)$.

The proof of this theorem is similar to that of Theorem 2.3.9, but, instead of ultrafilters and clans, *bounded ultrafilters and bounded clans* (ultrafilters and clans possessing bounded regions) and the constructions from Section 2.3 are used.

A stronger result (similar to Theorem 2.4.1) for locally compact weakly regular T_1-spaces can be obtained under the assumption that local algebras satisfy axiom (Ext). In this case, the role of points is played by *bounded maximal clans*. If both axioms (Ext) and (Nor) are assumed, locally compact Hausdorff spaces are obtained (as was established by Roeper). If, in addition, axiom (Con) is satisfied, we obtain a connected space.

As was noted by Roeper [**49**], if axiom (R) (equivalently, axioms (Ext) and (Nor)) is not assumed, it is impossible to establish a one-to-one correspondence between local contact algebras and locally compact spaces.

2.7. Algebras of regions based on non-Boolean lattices

It is reasonable to weaken the Boolean part of a contact algebra since it constitutes the mereological basis for the contact algebra, but the basic mereological relations (part-of, overlap, and underlap) admit equivalent definitions in terms of the lattice operations. Another reason is to examine how much the lattice properties affect properties of the mereological relations. We give two examples. The relations overlap and underlap are extensional in the following sense:

(Ext-O) $\quad a = b$ if and only if $(\forall c)(aOc \leftrightarrow bOc)$,

(Ext-U) $\quad a = b$ if and only if $(\forall c)(aUc \leftrightarrow bUc)$.

If we restrict the Boolean part to a distributive lattice with 0 and 1, then (Ext-O) and (Ext-U) are not necessarily valid. In the case of a distributive lattice, (Ext-O) is equivalent to the following stronger condition:

(Ext-O′) $\quad a \leqslant b$ if and only if $(\forall c)(aOc \rightarrow bOc)$.

Similarly, (Ext-U) is equivalent to the following:

(Ext-U′) $\quad a \leqslant b$ if and only if $(\forall c)(bUc \rightarrow aUc)$.

For more details we refer to [**22**] and [**18**]).

The following lemma illustrates the importance of the extensionality principles for the representability results in Boolean contact algebras of regular closed sets.

Lemma 2.7.1 ([**18**]). *Suppose that X is a topological space, $L = (L, 0, 1, +, .)$ is a lattice, and h is an embedding of the upper semi-lattice $(L, 0, 1, +)$ in the lattice $C(X)$ of closed sets of X. Let $\mathbf{B} = \{h(a) : a \in L\}$ be the closed base of a topology for X.*

(1) *The following conditions are equivalent*:
  (a) *L is U-extensional,*
  (b) $\mathbf{B} \subseteq \mathrm{RC}(X)$,
  (c) *$h(a.b) = \mathrm{Cl}(\mathrm{Int}(h(a) \cap h(b)))$ for all $a, b \in L$,*
  (d) *h is the dual dense embedding of L in* $\mathrm{RC}(X)$.
(2) *If some of conditions* (a)–(d) *in* (1) *are satisfied, then*
  (a) *L is an U-extensional distributive lattice,*
  (b) *X is a semiregular space.*

Lemma 2.7.1 shows that in order for a lattice L to be embedded in $\mathrm{RC}(X)$ so that the image of L to form a basis of closed sets for X, the lattice L should be distributive and U-extensional with semiregular topology.

Theorem 2.7.2 below shows that such representability results can also take place for distributive U-extensional lattices with axioms (C1)–(C4).

Theorem 2.7.2 (topological representation of U-extensional distributive contact lattices, [**18**]). *Let $D = (D, 0, 1, +, ., C)$ be an U-extensional distributive contact lattice. Then there exists a semiregular T_0-space X and the dual dense embedding h of D in* $\mathrm{RC}(X)$ *such that $\{h(a) : a \in D\}$ is a basis of closed sets for X.*

Note that the embedding of a distributive contact lattice in $\mathrm{RC}(X)$ is possible even if the lattice is not necessarily U-extensional provided that we omit the condition that $\{h(a) : a \in D\}$ generates the topology of X. This shows that this assumption has a lattice equivalent in the form of the U-extensionality of the lattice.

We can further weaken the mereological part of contact algebras. In particular, examples with nondistributive lattices were considered in [**22**]. The question is: What can be regarded as a nice point-based representation? For a candidate for topological modelling of nondistributive contact lattices we can consider regions in some bi-topological spaces. Let (X, τ_1, τ_2) be a space with two different topologies τ_1 and τ_2. A set $a \subseteq X$ is called a *mixed regular closed set* if $a = \mathrm{Cl}_1(\mathrm{Int}_2(a))$. We set $0 = \varnothing$, $1 = X$, $a + b = a \cup b$, $a.b = \mathrm{Cl}_1(\mathrm{Int}_2(a \cap b))$, and aCb if and only if $a \cap b \neq \varnothing$. Then such mixed regions form a (not necessarily distributive) lattice and C satisfies axioms (C1)– (C4). By a result of Urquhart [**59**], any lattice can be embedded in such a special lattice. The problem is that such a representation does not hold for the contact relation. The question of finding a satisfactory model and representation theory for nondistributive contact lattices remains still open.

The further generalization is to drop the entire lattice part and consider only some mereotopological relations with suitable axioms that are valid in the standard Boolean model. For example, as we can see for contact algebras, all the RCC-8 relations are definable and their definition uses only O, C, $\leqslant$. and $\ll$. Thus, it is of interest to find the complete set of axioms for O, C, $\leqslant$, and $\ll$. The author does not know whether there are results in this direction.

2.8. Precontact algebras and discrete spaces

In this section, we describe discrete nontopological models of contact algebras. We begin with a general definition.

A *precontact algebra* is a system $\underline{B} = (B, C) = (B, 0, 1, +, ., *, C)$, where $(B, 0, 1, +, ., *)$ is a Boolean algebra and C is a binary relation, called a *precontact*, satisfying the following axioms:

(C1) if aCb, then $a, b \neq 0$,

(C2′) $aC(b + c)$ if and only if aCb or aCc, (C2$''$) $(a+b)Cc$ if and only if aCc or bCc.

Note that $\underline{B}$ is a contact algebra if it satisfies axioms (C3) (i.e., $aCb \to bCa$) and (C4) (i.e., $a.b \neq 0 \to aCb$).

Precontact algebras were considered in [**14**] and in [**19**], where they are called *proximity algebras*.

We give nontopological examples of precontact and contact algebras using the notion of an *adjacency space* introduced by Galton [**29, 30**]. An *adjacency space* is a relational system (X, R), where $X \neq \varnothing$ is a set whose elements are called *cells* and R is a binary *adjacency relation* on cells. By a *region* in (X, R) we mean any subset of X. We say that two regions $a, b \subseteq X$ are in the *adjacency contact* C_R and write aC_Rb if $(\exists x \in a)(\exists y \in b)(xRy)$. Such a binary relation was used in [**60**] for defining relational proximity spaces, but its interpretation there was different from Galton's one. Galton assumed that R is reflexive symmetric, whereas R is arbitrary in [**19**]. An intuitive example of an adjacency space, adopted from Galton, is a chess-board table with cells – squares such that two squares are adjacent if they have a common point. This is an example of a reflexive symmetric adjacency relation. However, there are also several nonreflexive nonsymmetric adjacency relations, for example: "a to be next on the left of b" or "a to be on the top of b," etc., which motivates the choice of an arbitrary binary relation of R in [**19**].

It is easy to prove the following assertion.

Lemma 2.8.1 ([**19**]). *Let (X, R) be an adjacency space, and let $\underline{B}(X) = (B(X), C_R)$, where $B(X)$ is the Boolean algebra of subsets of X and C_R is the adjacency contact. Then the following assertions hold*:

(1) $\underline{B}(X) = (B(X), C_R)$ *is a precontact algebra,*
(2) $\underline{B}$ *satisfies* (C3) *if and only if R is symmetric,*
(3) $\underline{B}$ *satisfies* (C4) *if and only if R is reflexive,*
(4) $\underline{B}$ *is a contact algebra if R is reflexive and symmetric,*
(5) $\underline{B}$ *satisfies* (Nor) *if and only if R is transitive,*

(6) $\underline{B}$ *satisfies* (Con) *if and only if* R *is connected in the sense of graphs, i.e. if* $x \neq y$ *then there is an* R*-path from* x *to* y.

With every precontact algebra we can associate a *canonical adjacency space* $X(\underline{B}) = (X(B), R_B)$ taking the set of all ultrafilters of $\underline{B}$ ("points" of $\underline{B}$) for $X(B)$ and setting for two ultrafilters F and G

$$FR_BG \Leftrightarrow F \times G \subseteq C \Leftrightarrow F\rho G,$$

where the relation ρ was introduced in Section 2.3. The mapping $h(a) = \{F \in X(B) : a \in F\}$ is the Stone embedding of $\underline{B}$ in the Boolean algebra of subsets of $X(B)$. By Lemma 2.3.1, h preserves the relation of precontact. Thus, the following representation result holds.

Theorem 2.8.2 (representation of precontact algebras in adjacency spaces, [**19**]). *Suppose that* $\underline{B}$ *is a precontact algebra,* $(X(B), R_B)$ *is the canonical adjacency space, and* $B(X(B))$ *is the precontact algebra over the canonical space. Then the following assertions hold*:

(1) h *is an embedding of* $\underline{B}$ *in* $B(X(B))$,

(2) $\underline{B}$ *satisfies* (C3) *if and only if* $B(X(B))$ *satisfies* (C3),

(3) $\underline{B}$ *satisfies* (C4) *if and only if* $B(X(B))$ *satisfies* (C4),

(4) $\underline{b}$ *is a precontact algebra if and only if* $B(X(B))$ *is a contact algebra,*

(5) $\underline{B}$ *satisfies* (Nor) *if and only if* $B(X(B))$ *satisfies* (Nor).

As was noted in [**19**], the canonical adjacency space of a connected contact algebra is not in general a connected adjacency space. That is why Theorem 2.8.2 does not cover the case of connected contact algebras, unlike topological representation theorems.

Theorem 2.8.2 gives examples of nontopological discrete representations of contact algebras and normal contact algebras. This fact is remarkable because this means that a contact algebra has two essentially different representations: a discrete representation in a reflexive symmetric adjacency space and the other in a topological space. The points of the discrete representation are ultrafilters and the contact is realized by a binary adjacency relation between ultrafilters, whereas points in the topological representation are clans, i.e., special collections of ultrafilters. Considering both representations in the same space, we see that every region a have two representations: the first, $h_{ultrafilters}(a)$ containing only ultrafilters and the second, $h_{clans}(a)$ containing $h_{ultrafilters}(a)$ and including some clans. Note that $h_{clans}(a)$ is a regular closed set with a boundary containing only clans and the ultrafilters are included only in $\mathrm{Int}(h_{clans}(a)$. These representations reminiscent to consider ultrafilter-points as analogs of *atoms*, and clan-points can be regarded as analogs of *molecules*. Respectively, the representation theory is, in a sense, some kind of establishing certain atomistic micro-structure of the space, in which different kinds of points constitute the microlevels of the regions. Note that this interpretation is quite disputable and arise serious philosophical questions about the atomicity of space. More about this discussion in the realm of the top-level ontology and mereology can be found, for example, in [**53**].

At the first glance, topological modelling of precontact algebras is not possible because the standard topological contact satisfies additional axioms (C3) and (C4). However, for an arbitrary precontact algebra we can define an additional relation of contact $C^{\#}$ as follows: $aC^{\#}b$ if and only if aCb or bCa or $a.b \neq 0$. It is obvious that $C^{\#}$ satisfies all the axioms of contact algebra. Hence we can look for topological models of precontact algebras withe lements represented by regular closed sets of a topological space X and the contact $C^{\#}$ is represented as the standard topological contact. This is possible to be done but in topological structures of a more complicated nature, containing a binary relation R between some points of the space. We refer to [**14**] for definitions and topological representation theorems of precontact algebras in more detail.

3. Region-Based Propositional Modal Logics of Space

In this section, we present a language for propositional, quantifier-free logics of the region-based theory of space. The consideration of a quantifier-free language is mainly motivated by the necessity to obtain decidable fragments of some well-known systems of region-based theory of space related to RCC. We present three kinds of semantics:

- *algebraic semantics* based on algebras of regions,
- *topological semantics* based on contact algebras of some classes of topological spaces,
- *Kripke-type semantics* based on Kripke structures regarded as adjacency spaces.

The main tools in the proof of completeness theorems are the representation theorems for contact and precontact algebras from Section 2. We use a language similar to that of *relative modal logic* introduced by von Wright [**66**], which motivates us to call the considered logics *region-based propositional modal logics of space* (RPMLS). Another motivation is that Kripke-type semantics is very closed to the Kripke semantics in modal logic. Moreover, almost all known techniques of modal logic (in particular, modal definability, filtration, canonical-model constructions, etc.) used for proving completeness theorems can be transferred to our case with slight modifications. In addition, the language has a direct translation into the minimal modal logic K + universal modality, which also motivates our choice. However, the "modal" qualification of our logical language is not obligatory and it can be considered as a quantifier-free version of some first-order language. Note that the introduced language is a simplified version of the language of RCC-8 with Boolean terms, used by Wolter and Zakharyaschev [**65**].

The material of this section is mainly based on [**3**].

3.1. Syntax and semantics of RPMLS

Syntax

The *language* $\mathbf{L}(\mathbf{C}, \leqslant)$ of *region-based propositional modal logics of space* (RPMLS) consists of

- a denumerable set Var of *Boolean variables*,
- *Boolean operations:*: . (Boolean meet), + (Boolean join), $*$ (Boolean complement), and 0, 1 (Boolean constants),
- *propositional connectives*: $\neg, \wedge, \vee, \Rightarrow, \Leftrightarrow$, and propositional constants $\top$ and $\bot$,
- *modal connectives*: $\leqslant$ (part-of) and C (contact).

The set of *Boolean terms* $\mathbf{B}$ is defined in a standard way: from Boolean atoms and Boolean constants by means of Boolean operations.

Atomic formulas are formulas of the form $a \leqslant b$ and aCb, where a and b are Boolean terms.

Complex formulas (or simply *formulas*) are defined in a standard way from atomic formulas and propositional constants $\bot$ and $\top$ by means of propositional connectives.

Abbreviations:

$a = b \stackrel{\text{def}}{=} (a \leqslant b) \wedge (b \leqslant a)$,

$a \neq b = \stackrel{\text{def}}{=} \neg(a = b)$,

$a\overline{C}b \stackrel{\text{def}}{=} \neg(aCb)$,

$aOb \stackrel{\text{def}}{=} a.b \neq 0$ (overlap),

$a \ll b \stackrel{\text{def}}{=} a\overline{C}b^*$ (nontangential inclusion).

Substitution. Let α be a Boolean term or a formula, and let $p_1, \ldots, p_n$ be a list of different Boolean variables. We write $\alpha(p_1, \ldots, p_n)$ to indicate that $p_1, \ldots, p_n$ can occur in α.

If $b_1, \ldots, b_n$ are Boolean terms, then $\alpha(b_1, \ldots, b_n)$ or, more precisely $\alpha(p_1/b_1, \ldots, p_n/b_n)$ means the simultaneous substitution of $b_1, \ldots, b_n$ for $p_1, \ldots, p_n$. The formula $\alpha(b_1, \ldots, b_n)$ is called a *substitutional instance of* α. If we consider $p_1, \dots, p_n$ as meta variables for Boolean terms, then $\alpha(p_1, \dots, p_n)$ is called a "schema." Schemes are usually understood as schemes of axioms of some axiomatic systems.

Let $A = A(q_1, \dots, q_n)$ be a formula of the propositional calculus built up by different propositional variables $q_1, \dots, q_n$ and the propositional connectives $\neg, \wedge, \vee, \Rightarrow, \Leftrightarrow, \bot$, and $\top$. Let $\alpha_1, \dots, \alpha_n$ be formulas of our language. Then $A(\alpha_1, \dots, \alpha_n)$ or, more precisely, $A(q_1/\alpha_1, \dots, q_n/\alpha_n)$ is called the *substitutional instance of the propositional formula* A.

Semantics

First of all, we introduce an algebraic semantics of the language $\mathbf{L}(\leqslant, C)$. Let $\underline{B} = (B, 0, 1, ., +, *, C)$ be a precontact algebra. A mapping v from Var into B is called a *valuation*. It is extended to arbitrary Boolean terms by induction in a standard way: $v(a.b) = v(a).v(b)$, $v(a+b) = v(a) + v(b)$, $v(a^*) = v(a)^*$, $v(0) = 0$, and $v(1) = 1$.

A pair $M = (\underline{B}, v)$, where $\underline{B}$ is a precontact algebra and v is a valuation in B, is called an *algebraic model* or an *interpretation* in $\underline{B}$. The truth of a formula α in $(\underline{B}, v)$, in symbols $(\underline{B}, v) \models \alpha$, is defined inductively as follows:

$(\underline{B}, v) \models a \leqslant b$ if and only if $v(a) \leqslant v(b)$,

$(\underline{B}, v) \models aCb$ if and only if $v(a) C v(b)$,

$(\underline{B}, v) \models \alpha \wedge \beta$ if and only if $(\underline{B}, v) \models \alpha$ and $(\underline{B}, v) \models \beta$,

$(\underline{B}, v) \models \alpha \vee \beta$ if and only if $(\underline{B}, v) \models \alpha$ or $(\underline{B}, v) \models \beta$,

$(\underline{B}, v) \models \neg\alpha$ if and only if $(\underline{B}, v) \not\models \alpha$.

We say that $\mathcal{M}$ is a *model of a formula* α if $\mathcal{M} \models \alpha$ and $\mathcal{M}$ is a *model of the set of formulas* A if $\mathcal{M}$ is a model of all members of A.

We say that α is *true in a precontact algebra* $\underline{B}$ if α is true in all interpretations in $\underline{B}$. If Σ is a class of precontact algebras, α is said to be *true in* Σ if α is true in all members of Σ. The set of all formulas true in Σ is called the *logic* of Σ and is denoted by $\mathcal{L}(\Sigma)$. This is a semantic definition of logic.

Let Σ be a class of topological spaces. The *topological semantics* of $\mathbf{L}(\mathbf{C}, \leqslant)$ in Σ consists of interpretations in contact algebras $\mathrm{RC}(X)$ of regular closed sets of topological spaces $X \in \Sigma$. Pairs (X, v), where X is a topological space and v is a valuation in $\mathrm{RC}(X)$, are referred to as *topological model* or *topological interpretation*. If α is true in $\mathrm{RC}(X)$, we write "α is true in X" for brevity.

Let Σ be a class of relational systems (X, R) considered as *adjacency spaces* (cf. Section 2.8). The *Kripke semantics* of $\mathbf{L}(C, \leqslant)$ in Σ consists of interpretations in precontact algebras over structures $(W, R) \in \Sigma$. As in modal logic, structures of the form (W, R) are called *frames* (*Kripke frames* or *Kripke structures*) and the Kripke semantics is called *relational semantics*. Triples (X, R, v), where v is a valuation in the precontact algebra over (W, R), is called a *Kripke model* or a *Kripke interpretation*. If α is true in the precontact algebra $(B(X), C_R)$ over the frame (X, R), we write α is true in (X, R) for brevity. The class of all frames is denoted by Σ_{all}. Note that the truth of a formula aCb in the Kripke model (X, R, v) can be expressed in the equivalent way as follows:

> $(X, R, v) \models aCb$ if and only if $(\exists x, y \in X)(xRy$ and $x \in v(a)$ and $y \in v(b))$ if and only if $v(a) C_R v(b)$ if and only if $v(a) \cap \langle R \rangle v(b) \neq \varnothing$, where $\langle R \rangle v(b) = \{x \in X : (\exists y \in X)(xRy$ and $y \in v(b))\}$.

A translation into modal logic K with universal modality

Owing to the relational semantics, we can define a translation τ of our language into the modal logic K_U with standard modalities, denoted by $[R]A$ or $\langle R \rangle A$, and universal modalities, denoted by $[U]A$ or $\langle U \rangle A$. The modalities $[R]A$ and $\langle R \rangle A$ are interpreted by the relation R in the modal frames, whereas $[U]A$ and $\langle U \rangle A$ are interpreted by the universal relation $U = W \times W$ in the frames (W, R). The formal definition of τ is the following.

- *For Boolean terms*: If p is a Boolean variable, then $\tau p = p$ considered as a propositional variable in K_U, $\tau a^* = \neg\tau a$, $\tau(a + b) = \tau a \vee \tau b$, $\tau(a.b) = \tau a \wedge \tau b$, $\tau 0 = \bot$, and $\tau 1 = \top$.
- *For atomic formulas*: $\tau(a \leqslant b) = [U](\tau a \Rightarrow \tau b)$, and $\tau(aCb) = \langle U\rangle(\tau a \wedge \langle R\rangle\tau b)$.
- *For compound formulas*: $\tau\neg A = \neg\tau A$, $\tau(A \wedge B) = \tau A \wedge \tau B$, $\tau(A \vee B) = \tau A \vee \tau B$, $\tau(A \Rightarrow B) = \tau A \Rightarrow \tau B$, and $\tau(A \Leftrightarrow B) = \tau A \Leftrightarrow \tau B$.

The following assertion is easily proved by induction.

Lemma 3.1.1 (on translation, [**3**])**.** *Let $F = (W, R)$ be a frame. Then for any formula A the following is true*: $F \models A$ *in the sense of* RPMLS *if and only if* $F \models \tau A$ *in the sense of the modal logic* K_U*.*

If we consider only reflexive symmetric frames corresponding to the adjacency representation of contact algebras, then the above-introduced translation is in the logic KTB + universal modality (T is the code of the reflexivity axiom $[R]p \Rightarrow p$ and B is the code of the symmetry axiom $p \Rightarrow [R]\langle R\rangle p$ from modal logic).

3.2. Modal definability and undefinability in Kripke semantics

Modal definability

The modal definability of a class of frames by a formula is defined in the same way as the global modal definability in modal logic. Namely, we say that a class Σ of frames is *modally definable by a formula* α if for every frame $\mathcal{F} = (X, R)$

$$\mathcal{F} \in \Sigma \text{ if and only if } \mathcal{F} \models \alpha.$$

If Σ is defined by a first-order formula F, then we say that F is *modally definable by* α or F *is a first-order equivalent of* α.

Lemma 3.2.1 (modal definability: first-order examples, [**3**])**.** *Let $\mathcal{F} = (W, R)$ be a frame, and let p, q be Boolean variables. Then the following equivalencies hold*:

(1) [nonemptiness of R]
$R \neq \emptyset \;\Leftrightarrow\; \mathcal{F} \models 1C1$,

(2) [right seriality of R]
$(\forall x \in W)(\exists y \in W)(xRy) \;\Leftrightarrow\; \mathcal{F} \models (p \neq 0 \Rightarrow pC1)$,

(3) [left seriality of R]
$(\forall y \in W)(\exists x \in W)(xRy) \;\Leftrightarrow\; \mathcal{F} \models (p \neq 0 \Rightarrow 1Cp)$,

(4) [weak seriality of R]
$(\forall x \in W)(\exists y \in W)(xRy \vee yRx) \;\Leftrightarrow\; \mathcal{F} \models (p \neq 0 \Rightarrow 1Cp \vee pC1)$,

(5) [reflexivity of R]
$(\forall x \in W)(xRx) \;\Leftrightarrow\; \mathcal{F} \models (p \neq 0 \Rightarrow pCp)$,

(6) [symmetry of R]
$(\forall x, y \in W)(xRy \rightarrow yRx) \;\Leftrightarrow\; \mathcal{F} \models (pCq \Rightarrow qCp)$,

(7) [definability of overlap]
$(\forall x, y \in W)(xRy \leftrightarrow x = y) \;\Leftrightarrow\; \mathcal{F} \models (pCq \Leftrightarrow p.q \neq 0)$,

(8) [universality of R]
$(\forall x, y \in W)(xRy) \;\Leftrightarrow\; \mathcal{F} \models (a \neq 0 \wedge b \neq 0 \Rightarrow aCb)$

Note that the first-order conditions in (1), (3), (4), and (8) are not modally definable in the classical modal language. Below we will show that there are examples of definable first-order conditions in modal logic that are not modally definable in our language. For an example the transitivity condition R can be considered.

Since the reflexive symmetric frames are important for our purposes, we denote Σ_{ref} (Σ_{sym} or $\Sigma_{\mathrm{ref,sym}}$) for the class of all reflexive (symmetric or reflexive and symmetric) frames and Σ_{e} for the class of all equivalence relations. For the corresponding formulas which modally define these properties we use the notation

(Ref) $p \neq 0 \Rightarrow pCp$,

(Sym) $pCq \Rightarrow qCp$.

A relation R (or a frame (W, R)) is said to be *connected* if for all $x \neq y \in W$ there exists an R-path from x to y.

Let $n > 0$ be a natural number. A relation R (or $\mathcal{F}$), regarded as a graph, is said to be *n-colorable* if it is an n-colorable graph (i.e. all points can be colored by the colors from a given set of n colors in such a way that any two points connected by R have different colors).

Lemma 3.2.2 (modal definability: second-order examples, [**3**]). *The following assertions hold for a frame* $\mathcal{F} = (W, R)$:

(1) [connectedness of R]
$\mathcal{F}$ is connected if and only if
$\mathcal{F} \models (p \neq 0 \wedge p \neq 1 \Rightarrow pCp^*)$.

(2) [non-n-colorability of R]
$\mathcal{F}$ is not n-colorable if and only if
$\mathcal{F} \models (\bigvee_{i=1,\dots,n} p_i = 1 \wedge \bigwedge_{i \neq j, i,j=1,\dots,n} p_i \sqcap p_j = 0$
$\Rightarrow \bigvee_{i=1,\dots,n} (p_i C p_i))$.

Modal undefinability

For obtaining examples of the *modal undefinability* results, the following simple assertion is very useful.

Lemma 3.2.3 (modal undefinability criterion). *Let Σ and Σ' be two classes of frames such that $\Sigma \subseteq \Sigma'$, $\Sigma \neq \Sigma'$ and they determine the same logics, $\mathcal{L}(\Sigma) = \mathcal{L}(\Sigma')$. Then the class Σ is not modally definable.*

To use this criterion, we need to show that different classes of frames can determine the same logics. In the case of the classical modal language, the notion of a p-morphism was used for such a purpose. We introduce a similar notion adapted for the language of RPMLS.

Let $\mathcal{F} = (W, R)$ and $\mathcal{F}' = (W', R')$ be two frames. A surjective function f from W to W' is called a *p-morphism* from $\mathcal{F}$ to $\mathcal{F}'$ if for any $x, y \in W$ and $x', y' \in W'$ the following conditions are satisfied:

(P1) if xRy, then $f(x)R'f(y)$,

(P2) if $x'R'y'$, then $(\exists x, y \in W)(x' = f(x), y' = f(y), xRy)$.

If v is a valuation in W and v' is a valuation in W', then f is a p-morphism from (W, R, v) to (W', R', v') provided that for any Boolean variable p and $x \in W$ we have

$$x \in v(p) \text{ if and only if } f(x) \in v'(p).$$

The following assertion can be proved in the same way as the corresponding analog in modal logic.

Lemma 3.2.4 (p-morphism, [**3**])**.** *Let f be a p-morphism from a model $\mathcal{M}$ to a model $\mathcal{M}'$. Then for any formula φ*

$$\mathcal{M} \models \varphi \textit{ if and only if } \mathcal{M}' \models \varphi.$$

Lemma 3.2.5 ([**3**])**.** *The following assertions hold.*

(1) *The logic $\mathcal{L}(\Sigma_{\mathrm{ref,sym}})$ of all reflexive symmetric frames coincides with the logic $\mathcal{L}(\Sigma_{\mathrm{e}})$ of all equivalence relations.*

(2) *The class Σ_{e} is not modally definable.*

Idea of the proof. (1) Let $\mathcal{F} = (W, R)$ be a reflexive symmetric frame, and let $R_0 = \{\{x, y\} : xRy\}$. Define $W' = \{(x, \alpha) : x \in \alpha \text{ and } \alpha \in R_0\}$, $(x, \alpha)R'(y, \beta)$ if and only if $\alpha = \beta$. Let $f(x, \alpha) = x$. It is obvious that R' is an equivalence relation in W' and f is a p-morphism from the frame (W', R') to the frame (W, R). Consequently, the logics $\mathcal{L}(\Sigma_{\mathrm{e}})$ and $\mathcal{L}(\Sigma_{\mathrm{ref,sym}})$ coincide.

(2) By the criterion of modal undefinability (Lemma 3.2.3), the class Σ_{e} is not modally definable. □

Similarly, it is possible to prove that the first-order condition of transitivity alone is not modally definable.

Lemmas 3.2.1 and 3.2.5 show that RPMLS and the classical modal language are essentially different from the point of view of the modal definability.

3.3. Axiomatizations and completeness theorems

Axiomatization

We first introduce the axiomatic system $\mathbb{L}^{\text{precont}}_{\text{min}}$ for the minimal logic of all precontact algebras. It is a Hilbert-type axiomatic system consisting of axioms and inference rules.

Axioms of $\mathbb{L}^{\text{precont}}_{\text{min}}$

I. The complete set of axiom schemes of classical propositional logic (or all formulas which are substitution instances of tautologies of classical propositional logic),

II. The set of axiom schemes for Boolean algebra in terms of the part-of $\leqslant$ (a, b, and c are arbitrary Boolean terms):

$$a \leqslant a,\ (a \leqslant b) \wedge (b \leqslant c) \Rightarrow (a \leqslant c),\ 0 \leqslant a,\ a \leqslant 1,$$
$$(c \leqslant a.b) \Leftrightarrow (c \leqslant a) \wedge (c \leqslant b),\ (a+b \leqslant c) \Leftrightarrow (a \leqslant c) \wedge (b \leqslant c),$$
$$(a.(b+c)) \leqslant (a.b) + (a.c),$$
$$(c.a \leqslant 0) \Leftrightarrow (c \leqslant a^{*}),\ a^{**} \leqslant a.$$

III. The set of axiom schemes for the precontact C:

(C1) $(aCb) \Rightarrow (a \neq 0) \wedge (b \neq 0)$,

(C2) $(aC(b+c)) \Leftrightarrow (aCb) \vee (aCc),\ ((b+c)Ca) \Leftrightarrow (bCa) \vee (cCa)$.

Inference rule of $\mathbb{L}^{\text{precont}}_{\text{min}}$. Modus ponens: A and $A \Rightarrow B$ imply B.

The notion of a *proof* in $\mathbb{L}^{\text{precont}}_{\text{min}}$ is standard. All provable formulas are called *theorems* of $\mathbb{L}^{\text{precont}}_{\text{min}}$. It is easy to see that the set of theorems of $\mathbb{L}^{\text{precont}}_{\text{min}}$ is closed under the *substitution rule*:

> if $\alpha(p_1 \dots, p_n)$ is a theorem of $\mathbb{L}^{\text{precont}}_{\text{min}}$ and $p_1, \dots, p_n$ is a sequence of different Boolean variables, then for any Boolean terms $b_1, \dots, b_n$, the formula $\alpha(b_1, \dots, b_n)$ is a theorem of $\mathbb{L}^{\text{precont}}_{\text{min}}$.

We consider extensions of $\mathbb{L}^{\text{precont}}_{\text{min}}$ by new axioms, for example, by some of the formulas from Lemma 3.2.1 considered as modal schemes (the variables p and q are arbitrary modal terms). The minimal logic of all contact algebras $\mathbb{L}^{\text{cont}}_{\text{min}}$ is an extension of $\mathbb{L}^{\text{precont}}_{\text{min}}$ by the axiom schemes

(C3) $aCb \Rightarrow bCa$,

(C4) $a.b \neq 0 \Rightarrow aCb$.

Let $\mathbb{L}$ be an extension of $\mathbb{L}^{\text{precont}}_{\text{min}}$ by a set of arbitrary axiom schemes Ax. Denote it by $\mathbb{L}^{\text{precont}}_{\text{min}} + Ax$ and call the *axiomatic extension* of $\mathbb{L}^{\text{precont}}_{\text{min}}$. Similar notions are introduced for extensions of $\mathbb{L}^{\text{cont}}_{\text{min}}$.

We also consider extensions $\mathbb{L}^{\text{precont}}_{\text{min}} + R$ of $\mathbb{L}^{\text{precont}}_{\text{min}}$ by an additional inference rule R. In this paper, we are interested only in some special rules, so a general definition of an inference rule is omitted. On the other hand, we assume that any set of rules determines proofs and theorems in the standard sense. We identify $\mathbb{L}$ with the set of its theorems and call it also a *logic*. Hereinafter, $\mathbb{L}$ is an arbitrary logic considered as an extension of $\mathbb{L}^{\text{precont}}_{\text{min}}$.

Canonical models

Let $\mathbb{L}$ be an arbitrary extension of $\mathbb{L}^{\text{precont}}_{\text{min}}$. A set Γ of formulas is called an $\mathbb{L}$-*theory* or a *theory* if it contains all theorems of $\mathbb{L}$ and is closed under the rule

(MP) if A and $A \Rightarrow B$ are in Γ, then B in Γ.

For example, the set of all theorems of $\mathbb{L}$ is a theory; moreover, it is the smallest theory. A theory Γ is said to be *consistent* if $\bot \notin \Gamma$ and *maximal* if it is consistent and $\Gamma \subseteq \Delta$ implies $\Gamma = \Delta$ for any consistent theory Δ. Maximal theories are also referred to as *maximal consistent sets.*

Some well-known properties of theories are listed in the following assertion.

Lemma 3.3.1. *The following assertions hold.*

(1) *Let* Γ *be a theory, and let* α *be a formula. Then the set* $\Gamma + \alpha = \{\beta : \alpha \Rightarrow \beta \in \Gamma\}$ *is the smallest theory containing* Γ *and* α*. The set* $\Gamma + \alpha$ *is inconsistent if and only if* $\neg\alpha \in \Gamma$.

(2) *The following conditions are equivalent for any theory* Γ :
  (a) Γ *is maximal,*
  (b) *for any formula* α, $\neg\alpha \in \Gamma$ *if and only if* $\alpha \notin \Gamma$,
  (c) *for any formulas* α *and* β, $\alpha \vee \beta \in \Gamma$ *if and only if* $\alpha \in \Gamma$ *or* $\beta \in \Gamma$.

(3) *Any consistent theory can be extended to a maximal theory* (*the Lindenbaum lemma*).

The following assertion presents a semantical construction of maximal theories in $\mathbb{L}_{\min}^{\text{precont}}$.

Lemma 3.3.2. *Let* $\mathcal{M}$ *be a model. Then the set of formulas* $\Gamma = \{\alpha : \mathcal{M} \models \alpha\}$ *is a maximal* $\mathbb{L}_{\min}^{\text{precont}}$*-theory. If* $\mathcal{M}$ *is a model over contact algebra, then* Γ *is a maximal* $\mathbb{L}_{\min}^{\text{cont}}$*-theory.*

A set of formulas A is *consistent* in $\mathbb{L}$ if A is contained in an $\mathbb{L}$-consistent theory and, consequently, A is contained in a maximal $\mathbb{L}$-theory in view of the Lindenbaum lemma.

Let S be a maximal theory in $\mathbb{L}$. Based on the Lindenbaum-algebra construction, we construct in a canonical way a precontact algebra associated with S. In the set of Boolean terms $\mathbf{B}$, we introduce the *equivalence relation*: $a \equiv b$ if and only if $a = b \in S$. Since $\equiv$ is a congruence relation depending on S, it is possible to consider equivalence classes of Boolean terms $|a| = \{b : a \equiv b\}$ and to define the canonical precontact algebra $\underline{B}_S$ over S by setting $|a|.|b| = |a.b|$, $|a| + |b| = |a + b|$, $|a|^* = |a^*|$, $|a| \leqslant |b|$ if and only if $a \leqslant b \in S$, and $|a|C|b|$ if and only if $aCb \in S$.

Using the axioms of logic, we can prove that $\underline{B}_S$ is a precontact algebra and, if $\mathbb{L}$ is an extension of $\mathbb{L}_{\min}^{\text{cont}}$, $\underline{B}(S)$ is a contact algebra.

We define a canonical valuation for Boolean variables putting $v_S(p) = |p|$. Then the pair $M_S = (\underline{B}_S, v_S)$ is called a *canonical model over* S. We have $v_S(a) = |a|$ for any Boolean term a. With S we can canonically associate the *canonical frame* $F_S =$

(W_S, R_S) of S by taking for F_S the canonical adjacency space of the canonical precontact algebra $\underline{B}_S$ (cf. Section 2.8). If $\mathbb{L}$ is an extension of $\mathbb{L}^{\mathrm{cont}}_{\mathrm{min}}$, with S we can associate the *canonical topological space* X_S by taking for X_S the canonical topological space corresponding to the contact algebra $\underline{B}_S$ (cf. Section 2.3).

The following assertion is proved in a standard way.

Lemma 3.3.3. *Let $\mathbb{L}$ be a logic. Then the following two conditions are satisfied by any formula α :*

(1) *α is a theorem of $\mathbb{L}$,*

(2) *α is true in all canonical models M_S of $\mathbb{L}$.*

Now, we can state a completeness theorem for the minimal logics $\mathbb{L}^{\mathrm{precont}}_{\mathrm{min}}$ and $\mathbb{L}^{\mathrm{cont}}_{\mathrm{min}}$.

Theorem 3.3.4 (completeness of $\mathbb{L}^{\mathrm{precont}}_{\mathrm{min}}$, [**3**])**.** *The following conditions are equivalent for any formula α:*

(1) *α is a theorem of $\mathbb{L}^{\mathrm{precont}}_{\mathrm{min}}$,*

(2) *α is true in all precontact algebras,*

(3) *α is true in all Kripke frames.*

Proof. The implications (1) → (2) → (3) are obvious.

(2) → (1) Let α be true in all precontact algebras. Then α is true in all canonical models of $\mathbb{L}^{\mathrm{precont}}_{\mathrm{min}}$ and α is a theorem of $\mathbb{L}^{\mathrm{precont}}_{\mathrm{min}}$ in view of Lemma 3.3.3.

(3) → (1) Suppose that α is not a theorem of $\mathbb{L}^{\mathrm{precont}}_{\mathrm{min}}$. Then there is a canonical model $M_S = (\underline{B}_S, v_S)$ such that $M_S \not\models \alpha$. By the representation theorem for precontact algebras in adjacency spaces (cf. Theorem 2.8.2), there exists a frame (X, R) and an embedding h of the canonical precontact algebra $\underline{B}_S$ in the precontact algebra $B(X)$ over the frame (X, R). Define the valuation $v(p) = v_S(h(|p|))$. Then $(X, R, v) \not\models \alpha$, which means that α is not true in the Kripke frame (X, R). The proof is complete. □

Theorem 3.3.5 (completeness of $\mathbb{L}^{\mathrm{cont}}_{\mathrm{min}}$, [**3**])**.** *The following conditions are equivalent for any formula α :*

(1) *α is a theorem of $\mathbb{L}^{\mathrm{cont}}_{\mathrm{min}}$,*

(2) *α is true in all contact algebras,*
(3) *α is true in all reflexive symmetric Kripke frames,*
(4) *α is true in all topological spaces,*
(5) *α is true in all compact and semiregular* T_0*-spaces.*

PROOF. The implications (1) → (2) → (3) and (2) → (4) → (5) are obvious. The implications (2) → (1) and (3) → (1) are proved in the same way as in Theorem 3.3.4 with the help of Theorem 2.8.2. The implication (5) → (1) is proved in the same way as the implication (3) → (1) in Theorem 3.3.4 with the help of Theorem 2.3.9. □

Theorem 3.3.6 (completeness of $\mathbb{L}^{\mathrm{cont}}_{\mathrm{min}}$ + (Con), [**3**])**.** *The following conditions are equivalent for any formula α :*

(1) *α is a theorem of* $\mathbb{L}^{\mathrm{cont}}_{\mathrm{min}}$ + (Con),
(2) *α is true in all connected contact algebras,*
(3) *α is true in all connected topological spaces.*

Note that Theorem 3.3.6 does not assert the completeness with respect to Kripke semantics. This fact will be proved by another method in the following section.

Theorems 3.3.5 and 3.3.6 present weak completeness statements. The strong statements are also valid (cf. [**3**]). For the logic $\mathbb{L}^{\mathrm{cont}}_{\mathrm{min}}$ if can be formulated as follows.

Theorem 3.3.7 (strong completeness of $\mathbb{L}^{\mathrm{cont}}_{\mathrm{min}}$)**.** *The following conditions are equivalent for any set A of formulas*:

(1) *A is consistent in* $\mathbb{L}^{\mathrm{cont}}_{\mathrm{min}}$,
(2) *A has an algebraic model,*
(3) *A has a Kripke model,*
(4) *A has a topological model.*

Theorem 3.3.5 asserts that the logic $\mathbb{L}^{\mathrm{cont}}_{\mathrm{min}}$ is complete with respect to both topological and discrete semantics (semantics with respect to Kripke frames).

3.4. Filtration with respect to Kripke semantics and small canonical models

Filtration

Let Φ be a finite set of formulas closed under subformulas. Denote by Γ_Φ the smallest set of Boolean terms satisfying the following conditions:

- if $aCb \in \Phi$, then $a, b \in \Gamma_\Phi$,
- if $a \leqslant b \in \Phi$, then $a, b \in \Gamma_\Phi$,
- Γ_Φ is closed under subterms of its members,
- Γ_Φ contains 0, 1 and is closed under Boolean combinations of its members.

Note that Γ_Φ is infinite, but it is logically finite in the sense that there is a finite subset Γ_Φ^0 of Γ_Φ such that any term in Γ_Φ is Boolean equivalent to an element of Γ_Φ^0. If n is the number of Boolean variables occurring in the formulas from Φ, then the cardinality of Γ_Φ^0 is equal to 2^{2^n}. Denote by Φ' the set of all formulas containing only Boolean terms in Γ_Φ. It is obvious that $\Phi \subseteq \Phi'$ and Φ' is infinite, but logically finite.

Let $\mathcal{M} = (W, R, v)$ be a model. We define the *equivalence relation* $\equiv$ in W (depending on $\mathcal{M}$ and Φ) in the same way as in the definition of a filtration in modal logic:

- $x \equiv y$ if and only if $(\forall a \in \Gamma_\Phi)(x \in v(a) \leftrightarrow y \in v(a))$,
- for $x \in W$ define $|x| = \{y \in W : x \equiv y\}$ and set $W' = \{|x| : x \in W\}$,
- for $|x|, |y| \in W'$ define: $|x|R'|y|$ if and only if $(\exists x' \equiv x)(\exists y' \equiv y)(x'Ry')$,
- for any Boolean variable $p \in \Gamma_\Phi$ define $v'(p) = \{|x| : x \in v(p)\}$.

The model $\mathcal{M}' = (W', R', v')$ is called a *filtration of the model* $\mathcal{M}$ through Φ. Similarly, the frame (W', R') is called a *filtration of the frame* (W, R). The valuation v' is called the *canonical valuation* of the filtration.

Note that the above defined filtration coincides with the so-called minimal filtration in the classical modal language.

Lemma 3.4.1 (filtration, [**3**]).

(1) *The set* W' *is finite and the cardinality of* W' *is less than or equal to* n*, where* n *is the cardinality of* Γ^0_Φ*.*
(2) *For any* $x, y \in W$*,* xRy *implies* $|x|R'|y|$*.*
(3) *For any Boolean term* $a \in \Gamma_\Phi$ *and* $x \in W$*,* $x \in v(a)$ *if and only if* $|x| \in v'(a)$*.*
(4) *For every formula* $\psi \in \Phi'$*,* $\mathcal{M} \models \psi$ *if and only if* $\mathcal{M}' \models \psi$*.*

Now, we describe a construction which is also used in filtration theory in modal logic. Let (W', R', v') be a filtration of (W, R, v) through Φ, and let w be the new valuation in the filtrated frame (W', R'). Then for each Boolean variable p in Γ_Φ we define a Boolean term $b_{w(p)}$ obtained as a Boolean combination of terms in Γ_Φ as follows. For every $y \in W$ we set $b_{|y|} = \bigwedge\{b : b \in \Gamma^0_\Phi \text{ and } |y| \in v'(b)\}$. Then define $b_{w(p)} = \bigvee\{b_{|y|} : |y| \in w(p)\}$.

Using $b_{w(p)}$, we define a new valuation w' in (W, R) for variables from Γ_Φ as follows: $w'(p) = \{x \in W : x \in v(b_{w(p)})\}$.

The valuation w defines also the substitution Sub_w for variables from Γ_Φ as follows: $\mathrm{Sub}_w(p) = b_{w(p)}$ and then extended inductively for Boolean terms from Γ_Φ and formulas from Φ.

Then the following stronger version of the filtration lemma holds. There are no analog of this lemma in the classical modal logic.

Lemma 3.4.2 (strong filtration, [**3**]). *Let* w *be a valuation in* $F' = (W', R')$*, and let* w' *be the corresponding valuation in* (W, R) *defined by* $b_{w(p)}$*. Then for a term* $a \in \Gamma_\Phi$ *and a formula* $\psi \in \Phi'$ *the following assertions hold for any* $x \in W$ *:*

(1) $x \in v(\mathrm{Sub}_w(a))$ *if and only if* $x \in w'(a)$ *if and only if* $|x| \in w(a)$*,*
(2) $F, v \models \mathrm{Sub}_w(\psi)$ *if and only if* $F, w' \models \psi$ *if and only if* $F', w \models \psi$*.*

If w coincides with the canonical valuation v', then w' acts as v, i.e., for all $a \in \Gamma_\Phi$ and $\psi \in \Phi'$

(3) *$x \in w'(a)$ if and only if $x \in v(a)$,*
(4) *$F, w' \models \mathrm{Sub}_{v'}(\psi)$ if and only if $F, v \models \psi$.*

Let Σ be a class of frames. We say that Σ (or the logic $\mathcal{L}(\Sigma)$) *admits a filtration* if for any formula φ there is a finite set of formulas Φ, closed under subformulas and containing φ, such that the filtrated frame F' through Φ of any frame F in Σ belongs to Σ.

Remark 3.4.3. Suppose that Σ admits a filtration and Σ^{fin} is the class of all finite frames in Σ. Then the logics $\mathcal{L}(\Sigma)$ and $\mathcal{L}(\Sigma^{fin})$ coincide. Thus, $\mathcal{L}(\Sigma)$ possesses the finite model property.

We say that a class of frames Σ is *determined* if there exists a set of formulas A such that Σ coincides with the class of all frames in which the formulas from A are true. In this case, Σ will denoted by Σ_A.

Theorem 3.4.4 ([**3**]). *Every determined class of frames admits a filtration.*

Proof. Let Σ_A be a class of frames determined by a set of formulas A. Suppose that $F = (W, R) \in \Sigma_A$, φ is a formula, Φ is the set of all subformulas of φ, and $F' = (W', R', v')$ is a filtration of the model (F, v) through Φ. We show that F' belongs to Σ_A, i.e., all formulas in A are true in F'. Assume the opposite, i.e. there exist a formula $\psi \in A$ and a valuation w in F' such that $F', w \not\models \psi$. Let w' be the valuation determined by w in (W, R). By Lemma 3.4.2, $F, w' \not\models \psi$. Consequently ψ is not true in F and, consequently, ψ is not true in Σ_A. We arrive at a contradiction. □

Small canonical models

Let $\mathbb{L}$ be a consistent extension of $\mathbb{L}^{\mathrm{precont}}_{\mathrm{min}}$, and let S be a maximal theory in $\mathbb{L}$. Consider the canonical frame F_S and the canonical model $M_S = (F_S, v_S)$ for S. Let $M'_S = (F'_S, v'_S)$ be any filtration of M. Then M'_S is called a *small canonical model* of $\mathbb{L}$ and F'_S is called a *small canonical frame* of $\mathbb{L}$.

Lemma 3.4.5 (small canonical frame, [**3**]). *Let $\mathbb{L}$ be a consistent extension of $\mathbb{L}^{\text{precont}}_{\text{min}}$, let $A = \text{Th}(\mathbb{L})$ be the set of all theorems of $\mathbb{L}$, and let $\Sigma_{\mathbb{L}} = \Sigma_A$ be the set of frames determined by A. Then $\Sigma_{\mathbb{L}}$ contains all small canonical frames of $\mathbb{L}$.*

Proof. Let $M'_S = (W'_S, R'_S, v'_S)$ be a small canonical model related to the maximal consistent theory S. It suffices to prove that all formulas in A are true in the small canonical frame $F'_S = (W'_S, R'_S)$. Assume the opposite, i.e., for some $\psi \in A$ and valuation w in (W'_S, R'_S) we have $(W'_S, R'_S, w) \not\models \psi$. By Lemma 3.4.2, $(W_S, R_S, v_S) \not\models \text{Sub}_w(\psi)$. Hence $\text{Sub}_w(\psi) \notin S$. However, $\text{Sub}_w(\psi)$ is a substitution instance of a theorem ψ of $\mathbb{L}$. Therefore, it belongs to the maximal theory S. We arrive at a contradiction. □

Weak completeness theorems for extensions of $\mathbb{L}^{\text{precont}}_{\text{min}}$

Theorem 3.4.6 (weak completeness and the finite model property of all consistent extensions of $\mathbb{L}^{\text{precont}}_{\text{min}}$, [**3**]). *Let $\mathbb{L}$ be a consistent extension of $\mathbb{L}_{\text{min}}$, and let $\Sigma_{\mathbb{L}}$ be the class of frames determined by the set $\text{Th}(\mathbb{L})$ of all theorems of $\mathbb{L}$. Then the following conditions are equivalent for any formula φ :*

(1) *φ is a theorem of $\mathbb{L}$,*

(2) *φ is true in $\Sigma_{\mathbb{L}}$,*

(3) *φ is true in $\Sigma^{fin}_{\mathbb{L}}$.*

The proof is based on Lemmas 3.4.1 and 3.4.5.

Although Theorem 3.4.6 is not too informative concerning the frames of $\mathbb{L}$, but it asserts that there are no incomplete logics for the relational semantics under consideration and that all consistent logics are characterized by their finite frames. These facts have no analogs in the classical modal logic. The following assertion gives more information about axiomatic extensions of $\mathbb{L}_{\min}^{\text{precont}}$.

Theorem 3.4.7 (weak completeness and the finite model property of all axiomatic extensions of $\mathbb{L}_{\min}$, [**3**]). *Suppose that A is a set of formulas, Σ_A is the class of all frames determined by A, and Σ_A^{fin} is the class of all finite frames in Σ_A. Let $\mathbb{L}$ be an extension of $\mathbb{L}_{\min}$ with formulas from A for additional axiom schemes. Then for any formula φ the following conditions are equivalent*:

(1) *φ is a theorem of $\mathbb{L}$,*

(2) *φ is true in Σ_A^{fin},*

(3) *φ is true Σ_A.*

Hence $\mathbb{L}$ possesses the finite model property and is decidable if A is finite.

Corollary 3.4.8 ([**3**]). *The logics $\mathbb{L}_{\min}^{\text{cont}}$ and $\mathbb{L}_{\min}^{\text{cont}}$* + (Con) *are complete in the class of their finite frames and. consequently, are decidable.*

3.5. Logics related to RCC

According to Stell's formulation, the RCC system is equivalent to the contact algebras satisfying axioms (Ext) and (Con). In a sense, all extensions of the notion of contact algebras with axioms (Ext), (Con), and (Nor) are related to RCC as follows: RCC+(Nor) is an extension with good properties, whereas any other extension is a subsystem of RCC+(Nor). Considering all these eight types of contact algebras as first-order region-based theories of space, we introduce the following abbreviations:

WRCC – weak RCC based on axioms (C1)–(C4),

WRCC_{Con} – weak connected RCC=WRCC+(Con),

WRCC_{Ext} – weak extensional RCC=WRCC+(Ext),

WRCC_{Nor} – weak normal RCC=WRCC+(Nor),

$\mathrm{WRCC}_{\mathrm{Con,Nor}}$ – weak connected normal RCC=WRCC+(Con)+(Nor),

$\mathrm{WRCC}_{\mathrm{Ext,Nor}}$ – weak extensional normal RCC=WRCC+(Ext)+(Nor),

RCC – WRCC+(Ext)+(Con),

$\mathrm{RCC}_{\mathrm{Nor}}$ – normal RCC=RCC+(Nor).

The goal of this section is to introduce propositional logics based on the language $\mathbf{L}(\leqslant, \mathbf{C})$ corresponding to each of these first-order systems. For propositional systems we put the letter "P" before the abbreviation of the corresponding first-order system. Two propositional systems were already introduced: PWRCC – $\mathbb{L}^{\mathrm{cont}}_{\mathrm{min}}$ and $\mathrm{PWRCC}_{\mathrm{Con}}$ – $\mathbb{L}^{\mathrm{cont}}_{\mathrm{min}}$ + (Con). It is obvious that these systems are propositional (quantifier-free) analogs of WRCC and $\mathrm{WRCC}_{\mathrm{Con}}$ because all the axioms of the first-order systems WRCC and $\mathrm{WRCC}_{\mathrm{Con}}$ are universal formulas of the same form as quantifier-free axioms in PWRCC and $\mathrm{PWRCC}_{\mathrm{Con}}$. However, there are no analogs of axioms (Ext) and (Nor) in our language because they are not universal sentences. We imitate them by some inference rules analogous to the quantifier rules in the first-order logic. The rules have the same impact on the canonical contact algebras as the corresponding first-order axioms and will be used in the proof of the strong completeness theorem of the required logic with respect to the topological semantics which are suggested by the topological representation theorems for the corresponding contact algebras.

For an analog of axiom (Nor) we introduce the following *rule of normality*:

NOR $\quad \dfrac{\alpha \Rightarrow (aCp \vee p^{*}Cb)}{\alpha \Rightarrow aCb}$, where p is a Boolean variable that does not occur in a, b, and α.

For an analog of the first-order axiom (Ext) we introduce the *rule of extensionality*

EXT $\dfrac{\alpha \Rightarrow (p = 0 \vee aCp)}{\alpha \Rightarrow (a = 1)}$, where p is a Boolean variable that does not occur in a and α.

These rules are similar to the irreflexivity rule introduced by Gabbay [**27**] in the context of the classical modal logic. In the language under consideration, these rules were introduced in [**3**].

Taking into account the correspondences between the first-order RCC-like systems and the propositional systems, we present the diagram of extensions of $\mathbb{L}^{\text{cont}}_{\text{min}}$, where the logics are identified with the sets of additional axioms and rules.

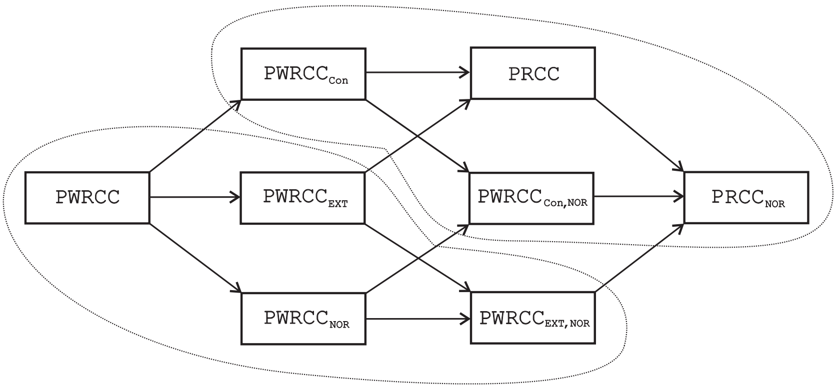

FIGURE 2

Consider the logics in this diagram. All logics satisfy the assumptions of Theorem 3.4.6 and, consequently, they are complete in certain classes of finite frames and possess the finite model property. Moreover, they are decidable because each of them has a finite set of axioms. The logics are strongly complete with respect to their intended topological semantics (cf. the following section). Concerning the weak completeness theorem, we see that the additional rules can be eliminated and thereby these rules do not affect the sets of the theorems. Thus, these eight logics collapse to the following two logics: PWRCC and $\text{PWRCC}_{\text{Con}}$. Below we give some information about these two logics and their relationships with other systems in the literature. Then we discuss the admissibility of the introduced rules to PWRCC and $\text{PWRCC}_{\text{con}}$.

PWRCC

As was already mentioned, PWRCC, propositional weak RCC, is $\mathbb{L}^{\text{cont}}_{\text{min}}$. By Theorems 3.3.5 and 3.3.7, PWRCC is weakly and strongly complete in the class of all topological spaces (and in the smaller class of all semiregular and compact T_0-spaces) and in the class of all reflexive symmetric Kripke frames considered as adjacency spaces. Thus, PWRCC, is complete with respect to both topological and discrete semantics. By Corollary 3.4.8, PWRCC has the finite model property and, consequently, is decidable. As was noted in Section 2.1, all the RCC-8 relations are definable in our language by means of quantifier-free definitions. Therefore, we can use the same definitions in the language of propositional logics. This fact, together with the topological part of the completeness theorem, shows that the system PWRCC is equivalent to the system BRCC-8 (RCC-8 with Boolean terms), introduced by Wolter and Zakharyaschev [**65**], which can be interpreted in all topological spaces. Thus, PWRCC can be considered as an axiomatization of BRCC-8 with several completeness theorems. Wolter and Zakharyaschev [**65**] proved that the satisfiability problem for BRCC-8 is NP-complete. Respectively, the same assertion holds for PWRCC.

$\text{PWRCC}_{\text{Con}}$

$\text{PWRCC}_{\text{Con}}$, propositional weak connected RCC, is an extension of PWRCC by the connectedness axiom

(Con) $\quad a \neq 0 \wedge a \neq 1 \Rightarrow aCa^*$

which defines the second-order connectedness property in frames. By Theorem 3.3.6, $\text{PWRCC}_{\text{Con}}$ is weakly and strongly complete in the class of all connected spaces and in the smaller class of all connected semiregular compact T_0-spaces. By Corollary 3.4.8,

$\mathrm{PWRCC_{Con}}$ is weakly complete in the class of all finite connected reflexive symmetric frames and, consequently, has the finite model property and is decidable. Thus, $\mathrm{PWRCC_{Con}}$ is weakly complete with respect to both topological and discrete semantics.

As in the case of PWRCC, we can conclude that $\mathrm{PWRCC_{Con}}$ is equivalent to the logic BRCC-8 introduced by Wolter and Zakharyaschev [**65**] who studied this logic in the class of all connected topological spaces. They proved that the satisfiability problem is PSPACE-complete. Their result implies a similar assertion for $\mathrm{PWRCC_{Con}}$.

The completeness of $\mathrm{PWRCC_{Con}}$ with respect to the class of connected reflexive symmetric adjacency spaces shows that the logic is equivalent to the logic GRCC (generalized region connection calculus) introduced semantically by Li and Ying [**39**] as a discrete version of RCC. Thus, $\mathrm{PWRCC_{Con}}$ can be also understood as a complete axiomatization of GRCC.

BRCC-8-like systems were studied only with respect to their intended topological semantics. Since the modal logic S4 corresponds to the topological interpretation of the classical modal language, systems like BRCC-8 were also treated by means of a translation into the modal logic S4 + universal modality. Taking into account that PWRCC is complete in the class of all reflexive symmetric frames, we can conclude that the exact translation of these systems is in KTB + universal modality, which shows that the weaker modal logic KTB also has a spatial meaning. This translation is not used in this paper because for our purpose it is easier to exploit directly the relational semantics of the language of RPMLS.

Admissibility of EXT and NOR

Lemma 3.5.1 (admissibility of EXT in the logics PWRCC and $\mathrm{PWRCC_{Con}}$, [**3**])**.** *The set of theorems of the logics* PWRCC *and* $\mathrm{PWRCC_{Con}}$ *are closed with respect to the rule* EXT *and, consequently,* EXT *is an admissible rule in* PWRCC *and* $\mathrm{PWRCC_{Con}}$*.*

Proof. We use the completeness of the logics PWRCC and $\text{PWRCC}_{\text{Con}}$ with respect to the Kripke semantics and the p-morphism techniques in Section 3.2.

Denote by L the logic PWRCC. We use the completeness of L in the class $\Sigma_{\text{ref,sym}}$ of all reflexive symmetric frames. We prove that if $L \vdash \varphi \Rightarrow (p = 0 \vee aCp)$, where p does not occur in a and in φ, then $L \vdash \varphi \Rightarrow (a = 1)$. Assume the contrary: $L \nvdash \varphi \Rightarrow (a = 1)$. By the completeness theorem, we can choose a reflexive symmetric frame $\mathcal{F} = (W, R)$ and a valuation V over $\mathcal{F}$ such that $\mathfrak{M} \vDash \varphi$ and $\mathfrak{M} \nvDash a = 1$, where $\mathfrak{M} = (\mathcal{F}, V)$. Thus, $W \setminus V(a) \neq \varnothing$. Consider two cases.

Case (a). $W \setminus \langle R\rangle(V(a)) \neq \varnothing$.

Case (b). $W \setminus \langle R\rangle(V(a)) = \varnothing$.

Here, $\langle R\rangle(V(a)) = \{x \in W : (\exists y \in V(a))(xRy)\}$.

In case (a), we choose a Boolean variable p that does not occur in a and φ and a valuation V' coinciding with V for Boolean variables different from p: $V'(p) = W \setminus \langle R\rangle(V(a))$. It is clear that $\mathfrak{M}' \vDash \varphi$ and $\mathfrak{M}' \vDash \neg(p = 0 \vee aCp)$, where $\mathfrak{M}' = (\mathcal{F}, V')$. Hence $L \nvdash \varphi \Rightarrow (p = 0 \vee aCp)$.

In case (b), we choose $w_1 \in \langle R\rangle(V(a))$ and $w_0 \notin W$. We set $W_1 = W \cup \{w_0\}$, $R_1 = R \cup \{(w_0, w_0), (w_0, w_1), (w_1, w_0)\}$, $f(w) = w$ if $w \neq w_0$, $f(w_0) = w_1$, and $V_1(q) = f^{-1}(V(q))$. It is easy to verify that $\mathfrak{M}$ is the p-morphic image of $\mathfrak{M}_1 = ((W_1, R_1), V_1)$ under f, (W_1, R_1) is a reflexive symmetric frame (consequently, it verifies the theorems of L). Thus, we can apply case (a) to $\mathfrak{M}_1$, which completes the proof for PWRCC.

For $\text{PWRCC}_{\text{Con}}$ we proceed in the same way. The only difference is that we start with a connected reflexive symmetric frame (W, R). It is easy to see that the above construction of the p-morphic pre-image (W', R') preserves the connectedness property. □

Lemma 3.5.2 (admissibility of NOR in the logics PWRCC and $\text{PWRCC}_{\text{Con}}$, [**3**])**.** *The set of theorems in the logics* PWRCC *and* $\mathrm{PWRCC}_{\mathrm{Con}}$ *are closed under the rule* NOR *and, consequently,* NOR *is an admissible rule in* PWRCC *and* $\mathrm{PWRCC}_{\mathrm{Con}}$.

The proof is similar to that of Lemma 3.5.1.

Corollary 3.5.3 ([**3**])**.** *The following assertions hold.*

(1) PWRCC*,* $\mathrm{PWRCC}_{\mathrm{EXT}}$*,* $\mathrm{PWRCC}_{\mathrm{NOR}}$*, and* $\mathrm{PWRCC}_{\mathrm{EXT,NOR}}$ *have the same set of theorems.*

(2) $\mathrm{PWRCC}_{\mathrm{Con}}$*,* PRCC*,* $\mathrm{PWRCC}_{\mathrm{CON,NOR}}$*, and* $\mathrm{PRCC}_{\mathrm{NOR}}$ *have the same set of theorems.*

(3) *All the eight logics are weakly complete in the corresponding class of frames determined by their axioms, have the finite model property, and, consequently, are decidable. The satisfiability problem for the logics in* (1) *is* NP*-complete and for the logics in* (2) *is* PSPACE*-complete.*

3.6. Strong completeness theorems for RCC-like logics

The purpose of this section is to illustrate how to work with additional rules of type NOR and EXT and how to modify the canonical-model construction in the presence of such rules. For an example we consider the logic $\mathrm{PWRCC}_{\mathrm{NOR}}$ which is an extension of the logic PWRCC with the rule NOR. The material of this section mainly follows [**3**]. Some important results are supplied with sketches of proofs.

$\mathrm{PWRCC}_{\mathrm{NOR}}$

We establish the strong completeness of topological semantics of $\mathrm{PWRCC}_{\mathrm{NOR}}$. The proof is a modification of the Henkin construction for the first-order logic.

For the canonical construction we must modify the notion of a theory in order to reflect the role of the rule NOR in the deduction. In this rule, the Boolean variable p plays a special role like bounded variables in quantifier logics, and this will be incorporated in the new notion of a theory. First of all, we need some preparations.

If A is a formula or a set of formulas, then $\mathrm{Var}\,(A)$ denotes the set of Boolean variables occurring in the members of A. By $\mathrm{Th}\,(\mathrm{Var})$ we denote the set of all theorems of $\mathrm{PWRCC_{NOR}}$ constructed form the set of all Boolean variables Var of our language. Sometimes, we need to extend the set of Boolean variables to a set Var'. In this case, $\mathrm{Th}\,(\mathrm{Var}')$ will denote the set of theorems constructed from Var'. Note that $\mathrm{Th}\,(\mathrm{Var})$ and $\mathrm{Th}\,(\mathrm{Var}')$ are not too different because theorems in $\mathrm{Th}\,(\mathrm{Var}')$ are versions of theorems in $\mathrm{Th}\,(\mathrm{Var})$. However, it is more convenient to consider them separately.

A pair $T = (\mathcal{V}, \Gamma)$ is called a NOR-*theory* if $\mathcal{V}$ is a set of Boolean variables and Γ is a set of formulas satisfying the following conditions:

(1) all theorems of $\mathrm{PWRCC_{NOR}}$ belong to Γ,

(2) if $\alpha, \alpha \Rightarrow \beta \in \Gamma$, then $\beta \in \Gamma$,

(3) if $\alpha \Rightarrow aCp \vee p^*Cb \in \Gamma$ for some Boolean variable $p \notin \mathcal{V} \cup \mathrm{Var}\,(\alpha \Rightarrow aCb)$, then $\alpha \Rightarrow aCb \in \Gamma$.

The variables in $\mathcal{V}$ are called the *free variables of* T and the members of Γ are called *formulas* of T. We will also write $T = (T_1, T_2)$, where T_1 is the set of free variables of T and T_2 is the set of formulas of T. We say that a formula α *belongs to* T and write $\alpha \in T$ if $\alpha \in T_2$. By (1) and (2), T_2 is a theory. We say that T is *consistent* if $\bot \notin T_2$ or, equivalently, if T_2 is a consistent theory.

A set A of formulas is said to be NOR-*consistent* if there is a consistent NOR-theory T such that $A \subseteq T_2$.

A theory T is called a *good* NOR-*theory* if out of T_1 there are infinitely many Boolean variables.

For example, $(\varnothing, \mathrm{Th}\,(\mathrm{Var}))$ is a good NOR-theory. If Γ is a consistent theory, then the pair $T = (\mathrm{Var}, \Gamma)$ is a consistent NOR-theory because T is trivially closed under the rule NOR (out of Var there are no variables). But T is not a good theory.

We say that T is *included* in T' and write $T \subseteq T'$ if $T_i \subseteq T'_i$, $i = 1, 2$. A theory T is a *complete* NOR-*theory* if it is a consistent NOR-theory and for any formula α we have either $\alpha \in T_2$ or $\neg\alpha \in T_2$. A theory T is called a *rich* NOR-*theory* if for any formula β of the form $\alpha \Rightarrow aCb$

if $\beta \notin T_2$, then $\alpha \Rightarrow aCp \vee p^*Cb \notin T_2$ for some Boolean variable p.

Our next goal is to show that every consistent good NOR-theory can be extended to a complete rich NOR-theory. For this purpose, we formulate and prove several lemmas.

Let Γ be a set of formulas, and let α be a formula. Denote $\Gamma + \alpha = \{\beta : \alpha \Rightarrow \beta \in \Gamma\}$. Let T be an NOR-theory, and let α be a formula. Denote $T \oplus \alpha = (T_1 \cup \mathrm{Var}\,(\alpha), T_2 + \alpha)$.

The following preliminary assertion is used in the proof of the Lindenbaum lemma (Lemma 3.6.2 below).

Lemma 3.6.1. *Let T be a good* NOR-*theory, and let α be a formula. Then*

(1) *$T \oplus \alpha$ is a good* NOR-*theory containing T, and $\alpha \in T_2$,*
(2) *$T \oplus \alpha$ is inconsistent if and only if $\neg\alpha \in T_2$,*
(3) *if for some β of the form $\neg(\alpha \Rightarrow aCb)$ the theory $T \oplus \beta$ is consistent, then there is a Boolean variable $p \notin T_1 \cup \mathrm{Var}\,(\beta)$ such that $(T \oplus \beta) \oplus \neg(\alpha \Rightarrow aCp \vee p^*Cb)$ is consistent.*

Lemma 3.6.2 (Lindenbaum lemma for NOR-theories)**.** *Every good consistent* NOR-*theory $T = (\mathcal{V}, \Gamma)$ can be extended to a complete rich* NOR-*theory $T' = (\mathcal{V}', \Gamma')$.*

Proof. Let $T = (\mathcal{V}, \Gamma)$ be a consistent good NOR-theory, and let $\alpha_1, \alpha_2 \ldots$ be an enumeration of all formulas. Introduce an increasing sequence of consistent good NOR-theories $T_n = (\mathcal{V}_n, \Gamma_n)$, $n = 1, 2, \ldots$, by induction. Let $T_1 = T$. Assume that $T_1, \ldots, T_n$ are already defined. To define T_{n+1}, we consider several cases.

Case 1. $T_n = (\mathcal{V}_n, \Gamma_n)$ is consistent.

(a) α_n is not of the form $\neg(\alpha \Rightarrow aCb))$. In this case, we put $T_{n+1} = T_n \oplus \alpha_n$. By Lemma 3.6.1, it is a good NOR-theory.

(b) α_n is of the form $\neg(\alpha \Rightarrow aCb))$. By Lemma 3.6.1(3), there exists a Boolean variable $p \notin \mathcal{V}_n \cup \mathrm{Var}\,(\alpha_n)$ such that $(T_n \oplus \alpha_n) \oplus \neg(\alpha \Rightarrow aCp \vee p^*Cb)$ is a consistent good NOR-theory. Let p be the first NOR-theory possessing this property. In this case, we put $T_{n+1} = (T_n + \alpha_n) \oplus \neg(A \Rightarrow aCp \vee p^*Cb)$.

Case 2. $T_n + \alpha_n$ is not consistent. Then we put $T_{n+1} = T_n$. In this case, $\neg\alpha_n \in \Gamma_n$. Define $\Gamma' = \bigcup_{n=1}^{\infty} \Gamma_n$ and $\mathcal{V}' = \bigcup_{n=1}^{\infty} \mathcal{V}_n$. Then $T' = (\mathcal{V}', \Gamma')$ is the required NOR-theory. □

Note that Lindenbaum lemma can be applied only to good NOR-theories, whereas the "goodness"' property is not essential for consistency because it depends on the amount of Boolean variables in our language. However, this fact is not too important, because the language can be extended. The following assertion clarifies this case.

Lemma 3.6.3 (conservativeness). *Let* $T = (\mathcal{V}, \Gamma)$ *be a consistent* NOR-*theory in the language based on the set* Var *of Boolean variables, and let* Var$'$ *be an extension of* Var *by a denumerable set of Boolean variables. Then there exists a consistent good* NOR-*theory* $U = (\mathcal{W}, \Delta)$ *in the language with* Var$'$ *such that* $\Gamma \subseteq \Delta$.

Idea of the proof. Let $\Delta = \{\gamma : (\exists\beta \in \Gamma)(\beta \Rightarrow \delta \in \mathrm{Th}\,(\mathrm{Var}'))\}$, and let $U = (\mathcal{V}, \Delta)$. Then U is a consistent good theory in the language with Var$'$ such that $\Gamma \subseteq \Delta$. □

Corollary 3.6.4. *The following assertions hold.*

(1) *A set of formulas is* NOR-*consistent if it is contained in a good consistent* NOR-*theory in an extension of the language by a countable infinite set of new Boolean variables.*

(2) *A set of formulas is* NOR-*consistent if it is contained in a complete rich* NOR-*theory in an extension of the language by a countable infinite set of new Boolean variables.*

The canonical construction uses complete rich theories $T = (\mathcal{V}, S)$. For canonical models we use the second component S. All other constructions are the same as in Section 3.3.

The following assertion shows the influence of the rule NOR on the canonical contact algebras: all of them are normal.

Lemma 3.6.5. *Let $T = (\mathcal{V}, S)$ be a complete rich* NOR-*theory, and let $(\underline{B}_S, v_S)$ be the canonical model corresponding to S. Then $\underline{B}_S$ satisfies axiom* (Nor).

Proof. Let $T = (\mathcal{V}, S)$ be a complete rich theory, and let $(\underline{B}_S, v_S)$ be the corresponding canonical model. Suppose that $|a|\overline{C}|b|$. $aCb \notin S$. Since S is rich, there exists a Boolean variable p such that $aCp \vee p^*Cb \notin S$ and, consequently, $aCp \notin S$ and $p^*Cb \notin S$. Then $|a|\overline{C}|p|$ and $|p|^*\overline{C}|b|$, which proves that $\underline{B}_S$ is a normal contact algebra. □

We are ready to prove the main result of this section.

Theorem 3.6.6 (strong completeness of $\mathrm{PWRCC}_{\mathrm{NOR}}$, [**3**]). *Let A be a set of formulas. Then the following conditions are equivalent*:

(1) *A is* NOR-*consistent,*
(2) *A has an algebraic model in the class of normal contact algebras,*
(3) *A has a model in the class of all $\varkappa$-normal semiregular spaces,*
(4) *A has a model in the class of compact semiregular T_0 $\varkappa$-normal spaces.*

Proof. The implications (4) → (3) → (2) are obvious. To prove the implication (2) → (1), we assume that A has an algebraic model $\mathcal{M} = (\underline{B}, v)$ in the class of normal contact algebras. Let $\Gamma = \{\alpha : \mathcal{M} \models \alpha\}$. It is easy to show that Γ is a consistent theory containing A. Thus, $T = (VAR, \Gamma)$ is a consistent (but not good) NOR-theory containing A. Hence A is NOR-consistent.

(1) → (2) Assume that A is NOR-consistent. By Corollary 3.6.4, A is contained in some complete rich NOR-theory $T = (\mathcal{V}, S)$

in a possible extension of the language by a countable set of new Boolean variables. By Lemma 3.6.5, the canonical contact algebra $\underline{B}_S$ is normal. Hence the canonical model $(\underline{B}_S, v_S)$ is a model of A.

(2) → (4) This implication holds in view of the topological representation theorem (Theorem 2.3.9). □

Corollary 3.6.7 (weak completeness of $\mathrm{PWRCC_{NOR}}$ and PWRCC)**.** *Let* $\mathbb{L}$ *be any of the logics* $\mathrm{PWRCC_{NOR}}$ *and* PWRCC*. Then the following conditions are equivalent for any formula* α :

(1) α *is a theorem of* $\mathbb{L}$*,*

(2) α *is true in all normal contact algebras,*

(3) α *is true in all semiregular* $\varkappa$*-normal spaces,*

(4) α *is true in all compact semiregular* $\varkappa$*-normal* $\mathrm{T_0}$*-spaces.*

PROOF. The required assertion is valid for $\mathrm{PWRCC_{NOR}}$ by Theorem 3.6.6 and for PWRCC by the fact that NOR is an admissible rule in PWRCC. □

Strong completeness theorem for PRCC-like logics

The proof of the completeness of $\mathrm{PRCC_{NOR}}$ can be repeated for any logic in Fig. 2. The canonical construction depends on the choice of a rule, NOR or EXT. For example, if both rules are assumed, then the theories are closed under these rules. If these rules are not assumed either, the notion of a theory becomes standard. We formulate a completeness theorem in a uniform way for all the logics in Fig. 2. For the sake of simplicity, we consider only the algebraic semantics. Using representation theorems for contact algebras, the completeness theorem can be further generalized to some topological spaces.

Theorem 3.6.8 (strong completeness of PWRCC-like logics)**.** *Let* $\mathbb{L}$ *be any of the logics in Fig. 2, and let* A *be a set of formulas. Then the following conditions are equivalent:*

(1) A *is consistent in* $\mathbb{L}$*,*

(2) *A has an algebraic model in the class of contact algebras corresponding to* $\mathbb{L}$.

We formulate the strongest topological completeness theorem only for $\mathrm{PWRCC}_{\mathrm{NOR,EXT}}$ and $\mathrm{PRCC}_{\mathrm{NOR}}$.

Theorem 3.6.9 (strong topological completeness of the logics $\mathrm{PWRCC}_{\mathrm{NOR,EXT}}$ and $\mathrm{PRCC}_{\mathrm{NOR}}$).

I. *The following conditions are equivalent for any set of formulas* A :
   (1) A *is consistent in* $\mathrm{PWRCC}_{\mathrm{NOR,EXT}}$,
   (2) A *has a model in the class of all compact Hausdorff spaces,*

II. *If, in addition, axiom* (Con) *is satisfied, then the corresponding spaces are connected.*

Corollary 3.6.10 (weak topological completeness theorem for PWRCC and $\mathrm{PWRCC}_{\mathrm{NOR,EXT}}$). *Let* $\mathbb{L}$ *be any of the systems* PWRCC *or* $\mathrm{PWRCC}_{\mathrm{NOR,EXT}}$. *Then the following conditions are equivalent for any formula* α :

(1) α *is a theorem of* $\mathbb{L}$,
(2) α *is true in all compact Hausdorff spaces.*

Note that Corollary 3.6.10 yields a stronger completeness result for PWRCC than Corollary 3.6.7. The following assertion states a similar result for $\mathrm{PWRCC}_{\mathrm{Con}}$.

Corollary 3.6.11 (weak topological completeness theorem for PWRCC and $\mathrm{PWRCC}_{\mathrm{NOR,EXT}}$). *Let* $\mathbb{L}$ *be any of the systems* PWRCC *or* $\mathrm{PWRCC}_{\mathrm{NOR,EXT}}$. *Then the following conditions are equivalent for any formula* α :

(1) α *is a theorem of* $\mathbb{L}$,
(2) α *is true in all compact connected Hausdorff spaces.*

3.7. Extending the language with new primitives

We consider some extensions of the language $\mathbf{L}(\leqslant, C)$ by new primitives: boundedness and connectedness of regions.

The *Boundedness* is a primitive one-place predicate in bounded contact algebras. Some of the boundedness axioms are not universal sentences, but, fortunately, they can be replaced with additional inference rules for the corresponding axiomatic system.

The *connectedness* is a definable one-place predicate in contact algebras with quantifiers. Thus, such a predicate must be taken for the connectedness predicate in a quantifier-free language; moreover, both sides of the equivalence in the definition must be imitated by suitable inference rules.

These examples show that some additional rules are very useful in the axiomatic characterizations of the predicates under consideration. Further we discuss the complete axiomatization of the predicates of connectedness and boundedness.

Connectedness

The predicate of connectedness $\mathrm{Con}(a)$ was introduced in Section 2.1 as follows:

(#) $\mathrm{Con}(a)$ if and only if $(\forall b, c)(b \neq 0$ and $c \neq 0$ and $b + c = a \rightarrow bCc)$.

We extend the language $\mathbf{L}(\leqslant, C)$ by the predicate Con. We can also extend the notion of an *atomic formula* by setting that $\mathrm{Con}(a)$ is an atomic formula for any Boolean term a. The desired topological semantics for $\mathrm{Con}(a)$ is as follows. If (X, v) is a topological model, then

$(X, v) \models \mathrm{Con}(a)$ if and only if $v(a)$ is a connected regular closed set of $\mathrm{RC}(X)$.

We can also define the relational semantics in Kripke structures. We give a complete axiomatization of Con with respect to the topological semantics. The implication "⇒" in (#) suggests the following axiom:

(Connect) $\mathrm{Con}(a) \wedge p \neq 0 \wedge q \neq 0 \wedge a = p + q \Rightarrow pCq.$

The implication "⇐" in (#) suggests the following inference rule:

$$\text{CONNECT} \quad \frac{\alpha \wedge p \neq 0 \wedge q \neq 0 \wedge p + q = a \Rightarrow pCq}{\alpha \Rightarrow \mathrm{Con}(a)},$$

where p and q are Boolean variables not occurring in a and α.

The *axiomatic system* PWRCC-Connect, for $\mathrm{Con}(a)$ is an extension of the axiomatic system for PWRCC extended by axiom (Connect) and the inference rule CONNECT.

The following formula is an example of a nontrivial theorem of PWRCC-Connect:

$\mathrm{Con}(a) \wedge \mathrm{Con}(b) \wedge aCb \Rightarrow \mathrm{Con}(a + b).$

The canonical-model-construction for PWRCC-Connect can be done in the same way as for $\mathrm{PWRCC}_{\mathrm{NOR}}$. The following lemma shows how axiom (Connect) and the inference rule CONNECT affect the canonical contact algebra.

Lemma 3.7.1. *Let* $(\underline{B}_S, v_S)$ *be a canonical model of* PWRCC-Connect. *Then for any* $|a| \in B_S$

$\mathrm{Con}(|a|)$ *if and only if* $(\forall |p|, |q| \in B_S)(|p| \neq |0|$ *and* $|q| \neq |0|$ *and* $|a| = |p| + |q| \to |p|C|q|)$.

Lemma 3.7.1 and the topological representation theorems for contact algebras lead to the following completeness result.

Theorem 3.7.2 (topological strong completeness of PWRCC-Connect)**.** *The following conditions are equivalent for any set of formulas A of* PWRCC-Connect:

(1) *A is a consistent set in* PWRCC-Connect,
(2) *A has an algebraic model in the class of all contact algebras with the definable predicate* $\mathrm{Con}(a)$,

(3) *A has a model in the class of all topological spaces,*
(4) *A has a model in the class of all semiregular compact* T_0-*spaces.*

A similar completeness result can be established for the extensions of PWRCC-Connect by axiom (Con) and the rules EXT and NOR. However, the question about the completeness with respect to Kripke models and decidability is still open. It is of interest to clarify relationships between the contact C and connectedness in special classes of contact algebras. Pratt-Hartmann [**43**] shows that in some natural contact algebras C is definable by Con in some special sense. A natural candidate for C in terms of Con can be contact algebras satisfying the condition

(C-connect) if aCb, then $(\exists a', b')(a' \leqslant a$ and $b' \leqslant b$ and $\mathrm{Con}(a')$ and $\mathrm{Con}(b')$ and $a'Cb')$.

This condition asserts that a contact between two regions is realized between their connected parts. If (C-connect) is satisfied, we obtain the following equivalence defining C in terms of Con:

aCb if and only if $(\exists a' \neq 0, b' \neq 0)(a' \leqslant a$ and $b' \leqslant b$ and $\mathrm{Con}(a')$ and $\mathrm{Con}(b')$ and $\mathrm{Con}(a+b))$.

Boundedness

To define the quantifier-free logic of boundedness, we extend the language $\mathbf{L}(\leqslant, C)$ by a one-place predicate B with the obvious extension of the notion of a formula. The algebraic semantics in local contact algebras (with the boundedness predicate $\mathcal{B}$) was introduced in Section 2.6. We recall the boundedness axioms:

(B1) $0 \in \mathcal{B}$,

(B2) if $b \in \mathcal{B}$ and $a \leqslant b$, then $a \in \mathcal{B}$,

(B3) if $a, b \in \mathcal{B}$, then $a + b \in \mathcal{B}$,

(B4) if aCb, then $\exists c \in \mathcal{B}$ such that $c \leqslant b$ and aCc, (B5) if $a \in \mathcal{B}$, then $\exists b \in \mathcal{B}$ such that $a \ll b$, $(a \ll b \Leftrightarrow a\overline{C}b^*)$.

Axioms (B1)–(B3) are universal sentences and have direct translation in the language $\mathbf{L}(\leqslant, C, B)$ by the following formulas (denoted by the same symbols):

(B1) $B(0)$,

(B2) $B(b) \wedge a \leqslant b \Rightarrow B(a)$,

(B3) $B(a) \wedge B(b) \Rightarrow B(a+b)$.

Axioms (B4) and (B5) are not universal sentences and should be replaced with the following inference rules:

BOUND-1 $$\frac{\alpha \Rightarrow (B(p) \wedge p \leqslant b \Rightarrow a\overline{C}p)}{\alpha \Rightarrow a\overline{C}b},$$

where p is a Boolean variable not occurring in a, b, α,

BOUND-2 $$\frac{\alpha \Rightarrow a \ll p}{\alpha \Rightarrow \neg B(a)},$$

where p is a Boolean variable not occurring in a and α.

Denote by PWRCC-Bound the extension of PWRCC in the language $\mathbf{L}(\leqslant, C, B)$ by axioms (B1)–(B3) and the inference rules BOUND-1 and BOUND-2. Then we can introduce canonical models for PWRCC-Bound.

Lemma 3.7.3. *Assume that* $(\underline{B}_S, v_S)$ *is a canonical model with* $B(|a|)$ *if and only if* $B(a) \in S$*. Then* $\underline{B}_S$ *is a local contact algebra.*

Lemma 3.7.3 and the corresponding topological representation theorems for local contact algebras lead to the following completeness result.

Theorem 3.7.4 (strong topological completeness of PWRCC-Bound)**.** *The following assertions hold.*

(1) *A is a consistent set in* PWRCC-Bound*,*

(2) *A has an algebraic model in the class of all local contact algebras,*

(3) *A has a model in the class of all locally compact topological spaces,*

(4) *A has a model in the class of all locally compact semiregular* T_0*-spaces.*

Similar completeness theorems can be obtained for extensions of PWRCC-Bound by axioms (Con), (Ext), and (Nor). The question about the decidability of such systems is still open.

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