Computability and Computable Models

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The intuitive notion of computability was formalized in the XXth century, which strongly affected the development of mathematics and applications, new computational technologies, various aspects of the theory of knowledge, etc. A rigorous mathematical definition of computability and algorithm generated new approaches to understanding a solution to a problem and new mathematical disciplines such as computer science, algorithmical complexity, linear programming, computational modeling and simulation databases and search algorithms, automatical cognition, program languages and semantics, net security, coding theory, cryptography in open systems, hybrid control systems, information systems, etc.

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Computability theory is traditionally understood as a branch of mathematical logic. However, owing to the ubiquitous use of computers and other electronic devices, many aspects usually studied within the framework of computability theory have become actual in numerous various areas even very far from mathematics.

In view of the wide range of applications, the two following directions of the further development of computability theory are of great interest.

- Investigate and determine bounds for the applicability of given computable model and algorithm to an real object existing in reality and processes flowing there.
- Create computability theory over abstract structures which could provides a unique approach to both computational processes in continuous models in reality and their discrete analogs.

In this paper, we discuss the first direction. We review recent important results and formulate more than 30 actual problems and open questions dictated by applications of the theory of computable models.

1. Preliminaries

The theory of constructive and computable models dates back to the works of Fröhlich and Shepherdson [**44**], Mal'tsev [**101**], Rabin [**137**], and Vaught [**152**] in the 1950's. This theory studies algorithmic properties of abstract models by constructing representations of the models on natural numbers and clarify relationships between properties and structural properties of the models. A systematic study of constructive and computable algebraic systems was initiated by Mal'tsev [**100**]. Historically, there were two approaches to the study of computable models.

The first approach is based on the notion of a numbering of the basic set of a model by natural numbers. Model properties are expressed in the binary form of natural numbers and thereby can be handled with computer technologies. In general, instead of numbers, names in some finite alphabet are ascribed to elements of a model. Since an element can possess several names, recognition algorithms are required. Such algorithms must recognize the names of given elements and determine whether certain properties are realized on elements with given names.

The second approach deals with models whose basic sets consist of natural numbers [**101**]. This approach leads to the notion of a recursive (computable) model.

Due to R. Soare and his critical revision of the terminology used in computability theory, the term "computable model" becomes common last years. Indeed, this choice reflects our intuitive impression of computability.

Both approaches were developed simultaneously and are closely connected. In fact, they are equivalent from the mathematical point of view.

In this section, we recall basic facts in model theory, numberings, computability theory which are necessary for discussing current problems in the theory of computable models. The material of this section mainly follows [**40**].

Throughout the paper, we use the standard set-theoretic notation: $P(M)$ is the set of all subsets of a set M, id_M is the identity mapping of a set M, and $\omega = \{0, 1, \ldots\}$. If f is a (partial) mapping, then $\mathrm{Rang}\, f$ ($\mathrm{Dom}\, f$) denotes the range (domain) of f. The symbols "$\Rightarrow$" and "$\Leftrightarrow$" mean the expressions "if ..., then ..." and "...if and only if ...". The expression $a \leftrightharpoons b$ means b is denoted by a. For category theory we refer to [**12**] and [**24**].

1.1. Algebraic structures, models, and theories

The classical theory of models and algebraic systems, founded by Mal'tsev and Tarski, was one of the main directions in mathematical logic where the key results were obtained during the second half of the XXth century.

A *signature* σ of the language of the first-order predicate calculus is the pair consisting of the triple of disjoint sets σ^P, σ^F, σ^C and a mapping μ: $\sigma^P \cup \sigma^F \to \omega^+$, where $\omega^+ \leftrightharpoons \{1, 2, \ldots\}$. If $P \in \sigma^P$ and $\mu(P) = n$, then P is called an *n-ary predicate symbol.* If $f \in \sigma^F$ and $\mu(f) = m$, then f is called an *m-ary functional symbol.* Elements of σ^C are called *constant symbols.* We often write σ in the form $\sigma = \langle P_1^{n_1}, \ldots, P_k^{n_k}; f_1^{m_1}, \ldots, f_s^{m_s}; c_1, \ldots, c_t\rangle$, where the superscripts are the values of μ for the corresponding symbols. The expression $P \in \sigma$ ($f \in \sigma$ or $c \in \sigma$) means that P (f or c) is a predicate (functional or constant) symbol of the signature σ.

The set of all formulas of the language of the first-order predicate calculus of a signature σ is denoted by L_σ (cf. definitions in [**16**] and [**24**]). We write $\Phi(x_1, \ldots, x_n)$ if every free variable of a formula Φ belongs to the set $\{x_1, \ldots, x_n\}$.

An *algebraic structure* (or *model*) $\mathfrak{A}$ of a signature σ is the pair consisting of a nonempty set $|\mathfrak{A}|$, called the *basic set* of $\mathfrak{A}$, and a family of (*basis*) predicates $P^{\mathfrak{A}} \subseteq |\mathfrak{A}|^{\mu(P)}$ ($P \in \sigma^P$), operations $f^{\mathfrak{A}}$: $|\mathfrak{A}|^{\mu(f)} \to |\mathfrak{A}|$ ($f \in \sigma^F$), and constants $c^{\mathfrak{A}} \in |\mathfrak{A}|$ ($c \in \sigma^C$).

For a formula $\Phi(x_1, \ldots, x_n)$ of a signature σ and an algebraic structure $\mathfrak{A}$ of the same signature σ we introduce the notion of the *truth* of Φ in $\mathfrak{A}$ for $x_i \to a_i \in |\mathfrak{A}|$, $i = 1, \ldots, n$. We write $\mathfrak{A} \models \Phi(a_1, \ldots, a_n)$ if Φ is true in $\mathfrak{A}$ on $a_1, \ldots, a_n$. If T is a system of *sentences* (i.e., formulas without free variables), then $\mathfrak{A} \models T$ means $\mathfrak{A} \models \Phi$ for all $\Phi \in T$.

A set T of sentences of a signature σ is called a *theory* if for any sentence Φ and model $\mathfrak{A}$ of the signature σ from $\mathfrak{A} \models T \Rightarrow \mathfrak{A} \models \Phi$ it follows that $\Phi \in T$. Using the notion of the deducibility $\vdash$ in the first-order predicate calculus (cf. [**24**]), we can define a theory T as follows: $T \vdash \Phi \Rightarrow \Phi \in T$. It is clear that for every class K of algebraic structures of a signature σ the set of all sentences Φ such that $\mathfrak{A} \in K \Rightarrow \mathfrak{A} \models \Phi$ is a theory, called the *elementary theory* of K and denoted by $\mathrm{Th}(K)$.

A subset A of a theory T is called a *system of axioms* of T and is denoted by $T = [A]$ if $\mathfrak{A} \models A$ implies $\mathfrak{A} \models T$ for any algebraic structure $\mathfrak{A}$ or, which is the same, $T = \{\Phi \mid \Phi$ is a sentence of the signature σ and $A \vdash \Phi\}$.

A theory T is *consistent* if it differs from the set of all sentences. A theory T is *complete* if it is consistent and $\Phi \in T$ or $\neg\Phi \in T$ for any sentence Φ.

Consider an algebraic structure $\mathfrak{A}$ of a signature σ. Let a nonempty subset $B \subseteq |\mathfrak{A}|$ be closed with respect to the basic operations and constants, i.e., $f^{\mathfrak{A}}(a_1, \dots, a_m) \in B$ for any $a_1, \dots, a_m \in B$, $f^m \in \sigma^F$, and $c^{\mathfrak{A}} \in B$ for any $c \in \sigma^C$. On B, we introduce an algebraic structure of the signature σ and denote it by $\mathfrak{A} \restriction B$. If $\mathfrak{A}_0$ and $\mathfrak{A}_1$ are algebraic structures of the signature σ, $|\mathfrak{A}_0| \subseteq |\mathfrak{A}_1|$, and $\mathfrak{A}_0 = \mathfrak{A}_1 \restriction |\mathfrak{A}_0|$, then $\mathfrak{A}_0$ is called a *substructure* of $\mathfrak{A}_1$ and is denoted by $\mathfrak{A}_0 \leqslant \mathfrak{A}_1$.

A substructure $\mathfrak{A}_0$ of an algebraic structure $\mathfrak{A}_1$ of a signature σ is said to be *elementary* and is denoted by $\mathfrak{A}_0 \preccurlyeq \mathfrak{A}_1$ if $\mathfrak{A}_0 \vDash \Phi(a_1, \dots, a_n) \Leftrightarrow \mathfrak{A}_1 \vDash \Phi(a_1, \dots, a_n)$ for any formula $\Phi(x_1, \dots, x_n)$ of the signature σ and $a_1, \dots, a_n \in |\mathfrak{A}_0|$.

Let $\mathfrak{A}$ and $\mathfrak{B}$ be algebraic structures of a signature σ. A mapping φ: $|\mathfrak{A}| \to |\mathfrak{B}|$ is called a *homomorphism* from $\mathfrak{A}$ into $\mathfrak{B}$ and is denoted by φ: $\mathfrak{A} \to \mathfrak{B}$ if

- $\langle a_1, \dots, a_n \rangle \in P^{\mathfrak{A}} \Rightarrow \langle \varphi a_1, \dots, \varphi a_n \rangle \in P^{\mathfrak{B}}$ for any predicate symbol $P^n \in \sigma$ and $a_1, \dots, a_n \in |\mathfrak{A}|$,
- $\varphi f^{\mathfrak{A}}(a_1, \dots, a_m) = f^{\mathfrak{B}}(\varphi a_1, \dots, \varphi a_m)$ for any functional symbol $f^m \in \sigma$ and $a_1, \dots, a_m \in |\mathfrak{A}|$,
- $\varphi(c^{\mathfrak{A}}) = c^{\mathfrak{B}}$ for any constant symbol $c \in \sigma$.

An equivalence relation η on the basic set $|\mathfrak{A}|$ of an algebraic structure $\mathfrak{A}$ of a signature σ is called a *congruence* on $\mathfrak{A}$ if $\langle f^{\mathfrak{A}}(a_1, \dots, a_m), f^{\mathfrak{A}}(b_1, \dots, b_m) \rangle \in \eta$ for every functional symbol $f^m \in \sigma$ and $\langle a_1, b_1 \rangle, \dots, \langle a_m, b_m \rangle \in \eta$. A congruence η on $\mathfrak{A}$ is said to be *strict* if $\langle a_1, \dots, a_n \rangle \in P^{\mathfrak{A}} \Leftrightarrow \langle b_1, \dots, b_n \rangle \in P^{\mathfrak{A}}$ for any predicate symbol $P^n \in \sigma$ and $\langle a_1, b_1 \rangle, \dots, \langle a_n, b_n \rangle \in \eta$.

If η is a congruence on an algebraic structure $\mathfrak{A}$ of a signature σ, then, on the set $A^* \leftrightharpoons |\mathfrak{A}|/\eta$, we can introduce an algebraic structure $\mathfrak{A}^*$, called a *quotient structure* and denoted by $\mathfrak{A}/\eta$, of the signature σ as follows:

- if $P^n \in \sigma$, then $P^{\mathfrak{A}^*} \leftrightharpoons \{\langle [a_1]_\eta, \ldots, [a_n]_\eta \rangle \mid$ there exist $b_i \in [a_i]_\eta$, $i = 1, \ldots, n$, such that $\langle b_1, \ldots, b_n \rangle \in P^{\mathfrak{A}}\}$,
- if $f^m \in \sigma$ and $[a_1]_\eta, \ldots, [a_m]_\eta \in A^*$, then $f^{\mathfrak{A}^*}([a_1]_\eta, \ldots, [a_m]_\eta) \leftrightharpoons [f^{\mathfrak{A}}(a_1, \ldots, a_m)]_\eta$,
- if c is a constant symbol of σ, then $c^{\mathfrak{A}^*} \leftrightharpoons [c^{\mathfrak{A}}]_\eta$.

Here, $[a]_\eta$ denotes the set of all elements that are η-equivalent to a. The mapping $a \mapsto [a]_\eta$, $a \in |\mathfrak{A}|$, is a homomorphism. If $\varphi\colon \mathfrak{A} \to \mathfrak{B}$ is a homomorphism from $\mathfrak{A}$ into $\mathfrak{B}$, then $\eta_\varphi \leftrightharpoons \{\langle a, b \rangle \mid a, b \in |\mathfrak{A}|,\ \varphi a = \varphi b\}$ is a congruence relation on $\mathfrak{A}$.

If a homomorphism $\varphi\colon \mathfrak{A} \to \mathfrak{B}$ is a one-to-one mapping from $|\mathfrak{A}|$ onto $|\mathfrak{B}|$ and the inverse mapping φ^{-1} is a homomorphism from $\mathfrak{B}$ into $\mathfrak{A}$, then φ is called an *isomorphism* (from $\mathfrak{A}$ into $\mathfrak{B}$). Two algebraic structures $\mathfrak{A}$ and $\mathfrak{B}$ are said to be *isomorphic* ($\mathfrak{A} \simeq \mathfrak{B}$) if there exists an isomorphism $\varphi\colon \mathfrak{A} \to \mathfrak{B}$. If $\mathfrak{A} \leqslant \mathfrak{B}_0$, $\mathfrak{A} \leqslant \mathfrak{B}_1$, and $\varphi\colon \mathfrak{B}_0 \to \mathfrak{B}_1$ is an isomorphism such that $\varphi \restriction |\mathfrak{A}| = \mathrm{id}_{|\mathfrak{A}|}$, then φ is called an $\mathfrak{A}$-*isomorphism.*

For signatures σ and σ' we write $\sigma \subseteq \sigma'$ if every functional (predicate, constant) symbol of σ is a functional (predicate, constant) symbol of σ' with the same arity. If $\sigma \subseteq \sigma'$ and $\mathfrak{A}'$ is an algebraic structure of the signature σ', then we can construct an algebraic structure of the signature σ by "forgetting" the values of symbols of $\sigma' \setminus \sigma$. This structure, denoted by $\mathfrak{A}' \restriction \sigma$, is called the σ-*restriction* of $\mathfrak{A}'$, and $\mathfrak{A}'$ is called the σ'-*enrichment* of $\mathfrak{A}' \restriction \sigma$. We write $\mathfrak{A} \leqslant \mathfrak{A}'$ if $\mathfrak{A} \leqslant \mathfrak{A}' \restriction \sigma$.

Let $\mathfrak{A}$ be an algebraic structure of a signature σ. We extend σ by adding constant symbols $\langle c_a \mid a \in |\mathfrak{A}| \rangle$. We set $\sigma^* \leftrightharpoons \sigma \cup \langle c_a \mid a \in |\mathfrak{A}| \rangle$. Setting $c_a^{\mathfrak{A}^*} \leftrightharpoons a$, we obtain the natural σ^*-enrichment $\mathfrak{A}^*$ of $\mathfrak{A}$. A *diagram* $D(\mathfrak{A})$ of $\mathfrak{A}$ is a set of sentences of the signature σ^* such that every sentence in $D(\mathfrak{A})$ is an atomic formula or the negation of an atomic formula and is true in $\mathfrak{A}^*$. By a *complete diagram* $FD(\mathfrak{A})$ we mean the set of sentences of the signature σ^* that are true in $\mathfrak{A}^*$.

Two algebraic structures $\mathfrak{A}_0$ and $\mathfrak{A}_1$ of a signature σ are *elementarily equivalent* ($\mathfrak{A}_0 \equiv \mathfrak{A}_1$) if $\mathrm{Th}(\mathfrak{A}_0)$ ($\leftrightharpoons \mathrm{Th}(\{\mathfrak{A}_0\})$) $=$ $\mathrm{Th}(\mathfrak{A}_1)$ ($\leftrightharpoons \mathrm{Th}(\{\mathfrak{A}_1\})$) or, in other words, $\mathfrak{A}_0 \models \Phi$ if and only if $\mathfrak{A}_1 \models \Phi$ for any sentence Φ of the signature σ.

If $\sigma' \subseteq \sigma$ and $\mathfrak{A}_0 \equiv \mathfrak{A}_1$, then $\mathfrak{A}_0 \restriction \sigma' \equiv \mathfrak{A}_1 \restriction \sigma'$.

We describe a canonical approach to the study of models without functional symbols. For every m-ary functional symbol $f \in \sigma$ we introduce a new $(m+1)$-ary predicate symbol P_f. Let σ^* be obtained from σ by replacing every functional symbol f with a predicate symbol P_f ($\sigma^{*P} \leftrightharpoons \sigma^P \cup \langle P_f \mid f \in \sigma^F \rangle$, $\sigma^{*F} \leftrightharpoons \varnothing$, $\sigma^{*C} \leftrightharpoons \sigma^C$, $\mu^* \restriction \sigma^P \leftrightharpoons \mu \restriction \sigma^P$, $\mu^*(P_f) \leftrightharpoons \mu(f)+1$). Any algebraic structure $\mathfrak{A}$ of the signature σ can be "transformed" to a model $\mathfrak{A}^*$ of the signature σ^* by setting

$$P_f^{\mathfrak{A}^*} \leftrightharpoons \big\{ \langle a_1, \ldots, a_m, b \rangle \mid a_1, \ldots, a_m, b \in |\mathfrak{A}|,\ f^{\mathfrak{A}}(a_1, \ldots, a_m) = b \big\}$$

for $f^m \in \sigma^F$. It is obvious that $\mathfrak{A}_0 \equiv \mathfrak{A}_1$ if and only if $\mathfrak{A}_0^* \equiv \mathfrak{A}_1^*$.

Therefore, in order to obtain a criterion for the elementary equivalence of two algebraic structures, it suffices to find such a criterion in the case of a finite signature.

We recall some model-theoretic methods of proving the completeness of theories.

Proposition 1.1. *A consistent theory T is complete if and only if there exists a model $\mathfrak{M}$ such that $T = \mathrm{Th}(\mathfrak{M})$.*

Corollary 1.2. *A consistent theory T is complete if and only if any models $\mathfrak{M}_0$ and $\mathfrak{M}_1$ of T are elementarily equivalent, i.e., $\mathrm{Th}(\mathfrak{M}_0) = \mathrm{Th}(\mathfrak{M}_1)$.*

A theory T is *categorical in power* α if any two models of T of power α are isomorphic.

The following assertion is often used in the proof of completeness and decidability.

Proposition 1.3. *If a theory T has no finite models and is categorical in some infinite power, then T is complete.*

A theory T is said to be *model-complete* if for every model $\mathfrak{M}$ of T the theory of the signature $\sigma^* = \sigma \cup \langle c_a \mid a \in |\mathfrak{M}|\rangle$ defined by the system of axioms $T \cup D(\mathfrak{M})$ is complete.

We indicate properties equivalent to the model completeness.

Theorem 1.4. *Let T be a theory. The following assertions are equivalent.*

(1) *The theory T is model complete.*
(2) *Let $\mathcal{M}$ and $\mathcal{N}$ be models of T. If $\mathcal{M}$ is a submodel of $\mathcal{N}$, then $\mathcal{M}$ is an elementary submodel of $\mathcal{N}$.*
(3) *Let $\mathcal{M}$ and $\mathcal{N}$ be models of T with fixed infinite cardinality $\varkappa$. If $\mathcal{M}$ is a submodel of $\mathcal{N}$, then $\mathcal{M}$ is an elementary submodel of $\mathcal{N}$.*
(4) *For any formula $\varphi(\overline{x})$ there exists $\exists$-formula $\psi(\overline{x})$ such that $T \vdash \varphi(\overline{x}) \Leftrightarrow \psi(\overline{x})$.*

Note that a complete theory is not necessarily model-complete and, conversely, a model-complete theory is not necessarily complete. However, there is a canonical method of obtaining a model-complete theory from an arbitrary theory. To demonstrate it, we need the definition of a first-order definable enrichment of a theory of a signature σ for a family of formulas. Let $\Phi(x_1, \dots, x_n)$ be a formula of the signature σ, and let σ' be obtained from σ by adding an n-ary predicate symbol P_Φ. By a *first-order definable enrichment* of a theory T of a signature σ for the formula $\Phi(x_1, \dots, x_n)$ we mean the theory T' of the signature σ' defined by the following system of axioms:

$$T \cup \bigl\{\forall x_1 \dots x_N \bigl(P_\Phi(x_1, \dots, x_N) \longleftrightarrow \Phi(x_1, \dots, x_n)\bigr)\bigr\}.$$

A first-order definable enrichment of a theory for a family of formulas Φ is defined in a similar way. A first-order definable enrichment T' of T is *complete* if it is obtained by adding new predicate symbols for all formulas of the signature σ. If T' is a first-order definable enrichment of T, then T and T' have the same models in the following sense: a model $\mathfrak{M}$ of T admits a unique σ'-enrichment to a model of T'.

Theorem 1.5. *The complete first-order definable enrichment of a theory T is a model-complete theory.*

Models $\mathfrak{M}_0$ and $\mathfrak{M}_1$ of a signature σ are *universally equivalent* if $\mathfrak{M}_0 \models \Phi \Leftrightarrow \mathfrak{M}_1 \models \Phi$ for any universal sentence Φ of the signature σ.

Proposition 1.6. *If any two models of a model-complete theory T are universally equivalent, then T is complete.*

Owing to these assertions, it is reasonable to introduce the following definition. Let T be a consistent theory of a signature σ. A theory $T^* \supseteq T$ of the signature σ is called the *model completion* of T if T^* is a model-complete theory relative to T.

The existence of the model completion of an arbitrary theory is not a trivial question. The following condition is sufficient for existing the model completion of a universally axiomatizable theory T of a finite signature. If $\mathfrak{M}$, $\mathfrak{M}_0$, and $\mathfrak{M}_1$ are models of T and $\varphi_0\colon \mathfrak{M} \to \mathfrak{M}_0$, $\varphi_1\colon \mathfrak{M} \to \mathfrak{M}_1$ are isomorphic embeddings, then there exists a model $\mathfrak{M}^*$ of T and isomorphic embeddings $\psi_0\colon \mathfrak{M}_0 \to \mathfrak{M}^*$, $\psi_1\colon \mathfrak{M}_1 \to \mathfrak{M}^*$ such that $\psi_0\varphi_0 = \psi_1\varphi_1$. In this case, the model completion T^* of T exists and is a complete countably categorical theory admitting the quantifier elimination.

Consider types of models. Let T be a (consistent) theory of a signature σ. Denote by Fr_n the set of all formulas of the signature σ with free variables in $\{x_0, \dots, x_{n-1}\}$, $n \in \omega$. Let $F_n(T)$ be the quotient set of Fr_n by the equivalence relation η_T defined as follows: $\langle \varphi, \psi \rangle \in \eta_T \leftrightharpoons \forall x_0 \dots \forall x_{n-1} \, [(\varphi \to \psi) \ \& \ (\psi \to \varphi)] \in T$ for $\varphi, \psi \in Fr_n$. For $\varphi \in Fr_n$ denote by $[\varphi]$ the element of $F_n(T)$ containing φ. The following natural embeddings hold:

$$S_0(T) \subseteq S_1(T) \subseteq \dots, \quad F_0(T) \subseteq F_1(T) \subseteq \dots$$

Remark. The set $F_n(T)$ can be regarded as a Boolean algebra (cf. [**40**]) if $[\varphi] \sqcup [\psi] \leftrightharpoons [\varphi \vee \psi]$, $[\varphi] \sqcap [\psi] \leftrightharpoons [\varphi \ \& \ \psi]$, $c[\varphi] \leftrightharpoons [\neg \varphi]$, $0 \leftrightharpoons [\forall x \, (x \neq x)]$, and $1 \leftrightharpoons [\exists x \, (x = x)]$.

By an *n-type* of a theory T we mean any maximal T-inconsistent subset $S \subseteq Fr_n$ (i.e., the sentence $\exists x_0 \dots \exists x_{n-1}\Big(\mathop{\&}\limits_{i=1}^{k} \varphi_i\Big)$ belongs to T for any $\varphi_1, \dots, \varphi_k \in S$). If $\mathfrak{M} \models T$ and $a_0, \dots, a_{n-1} \in |\mathfrak{M}|$, then $S \leftrightharpoons \{\varphi \mid \varphi \in Fr_n, \mathfrak{M} \models \varphi(a_0, \dots, a_{n-1})\}$ is an n-type, called the *type* of the n-tuple $\langle a_0, \dots, a_{n-1}\rangle$ of elements of $\mathfrak{M}$. If a type S of T is the type of some n-tuple of elements of $\mathfrak{M}$, we say that S *is realized* in $\mathfrak{M}$. Every n-type of a theory T is realized in some model of T.

An n-type S is *principal* if there exists a formula $\varphi \in S$, called the *complete formula of the type* S, such that S is a unique n-type containing φ. Let $\mathfrak{M}$ be a model of a theory T, and let S be an n-type of T. We say that $\mathfrak{M}$ *omits* the type S if S is not the type of any n-tuple of elements $a_0, \dots, a_{n-1}$ of $|\mathfrak{M}|$. Any principal type is realized in any model, but this is not true for nonprincipal types in view of the omitting type theorem. As is known (cf. [**16**]), if σ is an at most countable signature and $S_0, S_1, \dots$ is a countable family of nonprincipal types of a theory T, then there exists a countable model $\mathfrak{M}$ of T omitting all the types $S_0, S_1, \dots$.

If S is an n-type and $k < n$, then $S \cap S_k(T)$ is a k-type. Suppose that $k < n$, S is a k-type, S' is an n-type, and $S \subseteq S'$.

The type S' is *principal over the type* S if there exists a formula $\varphi \in S'$ such that S' is a unique n-type containing $S \cup \{\varphi\}$.

There is the natural one-to-one correspondence between n-types of a theory T and ultrafilters of Boolean algebras $F_n(T)$: if $U \subseteq F_n(T)$ is an ultrafilter, then $\pi^{-1}(U)$ is an n-type, where $\pi\colon Fr_n \to F_n(T)$ is the natural projection.

Now, we can characterize countably categorical theories.

Theorem 1.7 ([**16**]). *Let T be a complete theory of an at least countable signature. The following assertions are equivalent.*

(a) *T is categorical in countable power.*

(b) *For every $n \in \omega$ the theory T has finitely many n-types.*

(c) *For every $n \in \omega$ the set $F_n(T)$ is finite.*

A model $\mathfrak{M}$ is *homogeneous* if for any $a_0, \dots, a_{n-1}, a_n, b_0, \dots, b_{n-1} \in |\mathfrak{M}|$ such that the types of the n-tuple $\langle a_0, \dots, a_{n-1}\rangle$ and the n-tuple $\langle b_0, \dots, b_{n-1}\rangle$ coincide there exists an element $b \in |\mathfrak{M}|$ such that the types of $\langle a_0, \dots, a_{n-1}, a_n\rangle$ and $\langle b_0, \dots, b_{n-1}, b\rangle$ coincide.

One of the most pleasant properties of homogeneous countable models is presented by the following assertion (cf. the proof in [**16**]).

Proposition 1.8. *Let $\mathfrak{M}_0$ and $\mathfrak{M}_1$ be homogeneous countable models of the same signature. The following assertions are equivalent.*

(a) *The models $\mathfrak{M}_0$ and $\mathfrak{M}_1$ are isomorphic.*

(b) *The same types are realized in $\mathfrak{M}_0$ and $\mathfrak{M}_1$.*

Any prime model is homogeneous. Recall that a model $\mathfrak{M}$ of a theory T is *prime* if every model $\mathfrak{M}'$ of T has an elementary submodel $\mathfrak{M}_0$ isomorphic to $\mathfrak{M}$. Only finite principal types (if they exist) are realized in a prime model. Therefore, a prime model is unique up to an isomorphism.

We formulate an important sufficient existence condition for prime models. A theory T of a signature σ is called a *Henkin theory* if for any sentence of the form $\exists x\, \Phi(x)$ in T there exists a constant c of σ such that $\Phi(c) \in T$.

Proposition 1.9. *Let T be a complete Henkin theory, and let $\mathfrak{M}$ be a model of T. The submodel $\mathfrak{M}_0$ of $\mathfrak{M}$ determined by the set of the values of constants of σ is a prime model of T.*

To formulate an existence criterion for prime models, we need the following definition. A family $\mathfrak{S}$ of types of a theory T of a signature σ is *dense* if the following conditions hold.

- Let $p \in \mathfrak{S}$ be an n-type, and let $k \leqslant n$. Then $q \leftrightharpoons p \cap S_k(T) \in \mathfrak{S}$. If $\tau\colon \{x_0, \dots, x_{n-1}\} \to \{x_0, \dots, x_{n-1}\}$ is a permutation, then $[p]^{x_0,\dots,x_{n-1}}_{\tau x_0,\dots,\tau x_{n-1}} \in \mathfrak{S}$.

- If $p \in \mathfrak{S}$ is an n-type and $\varphi(x_0, \ldots, x_n)$ is a formula of the signature σ such that $\exists x_n\, \varphi \in p$, then there exists an $(n+1)$-type $q \in \mathfrak{S}$ such that $p \cup \{\varphi\} \subseteq q$.

For a model $\mathfrak{M}$ of a theory T we introduce the family $\mathfrak{S}(\mathfrak{M})$ of types realized in $\mathfrak{M}$. The family $\mathfrak{S}(\mathfrak{M})$ is dense.

Using the Henkin construction, we can prove the following assertion.

Proposition 1.10. *If $\mathfrak{S}$ is a dense countable family of types of a complete theory T, then there exists an at most countable model $\mathfrak{M}$ of T such that every type realized in $\mathfrak{M}$ belongs to $\mathfrak{S}$.*

Corollary 1.11. *A complete theory T of an at most countable signature σ has a prime model if and only if the family $\mathfrak{S}_0$ of all principal types of T is dense.*

Remark. The assumption of Corollary 1.11 is equivalent to the condition that every Boolean algebra $F_n(T)$, $n \in \omega$, is atomic.

A saturated model is homogeneous. Denote by $\varkappa$ a cardinal. A model $\mathfrak{M}$ of a signature σ is said to be *$\varkappa$-saturated* if for any subset $X \subseteq |\mathfrak{M}|$ of power less than $\varkappa$, any 1-type of $\mathrm{Th}(\mathfrak{M}, X)$ is realized in the model $\langle \mathfrak{M}, X\rangle$ obtained by the natural enrichment of $\mathfrak{M}$ to a model of the signature $\sigma_X \leftrightharpoons \sigma \cup \langle c_a \mid a \in X\rangle$ ($c_a^{\langle \mathfrak{M}, X\rangle} \leftrightharpoons a$). A countable ω-saturated model $\mathfrak{M}$ of T is called a *countably saturated model* of T.

The following criterion was established in [**16**].

Criterion 1.12 (existence of a countably saturated model)**.** *A theory T has a countably saturated model if and only if the Boolean algebra $F_n(T)$, $n \in \omega$, is superatomic.*

The following assertion describes the family of all types of homogeneous countable models of a complete theory T.

Proposition 1.13. *Let T be a complete theory. Then $\mathfrak{S}$ is a family of all types of T that are realized in some homogeneous countable model of T if and only if $\mathfrak{S}$ is a countable dense family* *of types possessing the following property: if* $p, q \in \mathfrak{S}$ *are* $(n+1)$*-types such that* $p \cap Fr_n = q \cap Fr_n$*, then there exists an* $(n+2)$*-type* $s \in \mathfrak{S}$ *such that* $p \cup [q]^{x_n}_{x_{n+1}} \subseteq s$*.*

1.2. Numberings

The theory of constructive models studies algorithmic properties of algebraic structures. For this purpose, effective representations of constructive models are considered and the study is based on computability theory and the theory of algorithms and computable functions. We refer to the monographs [**103, 140, 146**] for basic methods and details of algorithm theory. In this section, we formulate only the main results which will be used in the following sections. We follow [**140**] in presentations of computable functions.

By a *numbering* of a nonempty set S we mean any mapping ν from $\mathbb{N}$ onto S. Let S_0 and S_1 be nonempty sets such that $S_0 \subseteq S_1$, and let ν_0 and ν_1 be numberings of S_0 and S_1 respectively. We say that the numbering ν_0 is *reduced* to the numbering ν_1 (and write $\nu_0 \leqslant \nu_1$) if there exists a computable function f from $\mathbb{N}$ into $\mathbb{N}$ such that $\nu_0(n) = \nu_1 f(n)$ for any $n \in \mathbb{N}$. Numberings ν_0 and ν_1 are *computably equivalent* or *equivalent* ($\nu_0 \equiv \nu_1$) if $\nu_0 \leqslant \nu_1$ and $\nu_1 \leqslant \nu_0$. In this case, S_0 and S_1 coincide. Numberings ν_0 and ν_1 are *computably isomorphic* ($\nu_0 \underset{\text{rec}}{\simeq} \nu_1$) if there exists a computable permutation f of $\mathbb{N}$ such that $\nu_0(n) = \nu_1 f(n)$ for every n. Note that $\nu_0 \underset{\text{rec}}{\simeq} \nu_1$ implies $\nu_0 \equiv \nu_1$. The converse assertion does not hold in general.

On the class $\mathrm{Num}(S)$ of all numberings of a set S, the relation $\equiv$ is an equivalence relation and the reducibility $\leqslant$ induces a partial order on the equivalence classes by $\equiv$. Let

$$\mathbf{Num}(S) = \langle \mathrm{Num}(S)/{\equiv}, \leqslant \rangle.$$

If ν_0 and ν_1 are numberings of S_0 and S_1 respectively, then the numbering $\nu_0 \oplus \nu_1$ of the union $S_0 \cup S_1$ is defined as follows: $\nu_0 \oplus$

$\nu_1(2n) = \nu_0(n)$ and $\nu_0 \oplus \nu_1(2n+1) = \nu_1(n)$. If ν and μ are numberings of S, then $\nu \oplus \mu$ is a numbering of S and determines the least upper bound of the pair $\nu/{\equiv}$, $\mu/{\equiv}$ in $\mathbf{Num}(S)$. Thus, $\mathbf{Num}(S)$ can be regarded as an upper semilattice.

For a language L and an interpretation $\underline{\mathrm{int}}_L$ of L on a set S ($\underline{\mathrm{int}}_L\colon L \to S$) we say that a numbering ν_0 of a subset $S_0 \subseteq S$ is *computable* relative to $\underline{\mathrm{int}}_L$ if there exists a computable function f from $\mathbb{N}$ into L such that $\nu_0(n) = \underline{\mathrm{int}}_L(f(n))$ for any $n \in \mathbb{N}$. If $\nu_0 \leqslant \nu_1$, where ν_1 is a computable numbering relative to $\underline{\mathrm{int}}_L$, then ν_0 is a computable numbering relative to the same interpretation $\underline{\mathrm{int}}_L$. If ν_0 and ν_1 are computable numberings relative to $\underline{\mathrm{int}}_L$, then the sum $\nu_0 \oplus \nu_1$ is also computable relative to the interpretation $\underline{\mathrm{int}}_L$. Thus, if an equivalence class contains some computable numbering relative to $\underline{\mathrm{int}}_L$, then any numbering of this class is computable relative to the same interpretation $\underline{\mathrm{int}}_L$.

Denote by $\mathbf{R}(S, \underline{\mathrm{int}}_L)$ a submodel of $\mathbf{Num}(S)$ consisting of classes containing numberings computable relative to $\underline{\mathrm{int}}_L$. The upper semilattice $\mathbf{R}(S, \underline{\mathrm{int}}_L)$ is called the *Rogers semilattice* of the class of numberings computable relative to $\underline{\mathrm{int}}_L$. If ν is a numbering of S and Ξ is some class of subsets of $N^{<\infty}$, then $P \subseteq S^k$ is referred to as an Ξ-*set* provided that there exists a set $A \in \Xi$ such that $P \leftrightharpoons \{\langle \theta n_1, \ldots, \theta n)k\rangle \mid \langle n_1, \ldots, n_k\rangle \in A\}$.

Consider a family S of partial computable functions. For L we take the language of Turing machines and for $\underline{\mathrm{int}}_L$ we take the function $\underline{\mathrm{int}}_{\text{p.r.}}(M)$ computable by a Turing machine M. Thus, we arrive at a standard computable numbering of partial computable functions. In this case, we say that the numbering is *computable.* Note that ν is computable relative to $\underline{\mathrm{int}}_{\text{p.r.}}$ if and only if there exists a partial computable function $g(n, x)$ such that $\nu(n)$ and $\lambda x g(n, x)$ coincide for any $n \in \mathbb{N}$.

Consider a family S of computably enumerable sets. For L we take the language of Turing machines and for $\underline{\mathrm{int}}_L$ we take the function $\underline{\mathrm{int}}_{\text{r.e.}}(M) \leftrightharpoons \mathrm{Dom}\,(\underline{\mathrm{int}}_{\text{p.r.}}(M))$, where M is a Turing machine. In this case, we again obtain the standard notion of a computable numbering of computably enumerable sets (cf. [**27**] and [**36**]), i.e., a numbering ν of a family of computably enumerable sets is a computable relative to $\underline{\mathrm{int}}_{\mathrm{r.e.}}$ if and only if the set $\{\langle x, y\rangle \mid y \in \nu(x)\}$ is computably enumerable. A numbering computable relative to $\underline{\mathrm{int}}_{\mathrm{r.e.}}$ is referred to as *computable*. A family S is *computable* if there is a computable numbering of S relative to $\underline{\mathrm{int}}_{\mathrm{r.e.}}$.

Let S be a family of total functions on $\mathbb{N}$. Introduce the topology β_S on S as follows. For basis open sets we take sets of the family

$$B_S = \{V_g \mid g \text{ is a finite part of some function in } S\},$$

where $V_g \risingdotseq \{f \mid f \in S,\ g \subseteq f\}$. The family S is *discrete* if the topology β_S is discrete, i.e., for any $f \in S$ the finite part g of f is such that $V_g = \{f\}$. The family S is *effectively discrete* if there exists a strictly computable sequence of finite sets $g_0, g_1, \dots, g_n, \dots$ such that V_{g_n} contains only one element of S for any n and any element $f \in S$ belongs to some V_{g_n}, $n \in \mathbb{N}$. In this case, we say that the family $\{g_n \mid n \in \mathbb{N}\}$ *distinguishes* S and g_n *distinguishes* $f \supseteq g_n$. Note that an effectively discrete family is discrete. We say that a numbering ν of a set S is *single-valued* (or is a *Friedberg numbering*) if ν is bijective, i.e., $\nu(x) \neq \nu(y)$ for any $x \neq y$ in $\mathbb{N}$. A numbering ν of a set S is *positive* (*negative*) if the set $\eta_\nu = \{\langle x, y\rangle \mid \nu x = \nu y\}$ ($\overline{\eta}_\nu = \{\langle x, y\rangle \mid \nu x \neq \nu y\}$) is computably enumerable. A numbering ν is *solvable* if it is positive and negative, i.e., η_ν is computable. A single-valued numbering is solvable. A numbering ν of a set S is *minimal* if $\nu/\!\equiv$ is a minimal element of $\mathbf{Num}(S)$. Note that single-valued, solvable, and positive numberings are minimal.

One can prove that the Rogers semilattice of a computable nondiscrete family is infinite and, in the case of an effectively discrete family, consists of a single element [**36**]. There exists a discrete family with infinite Rogers semilattice (cf. [**144**]). As was shown in [**87**], the Rogers semilattice of any computable family of computably enumerable sets is infinite or has only one element. Selivanov [**144**] proved that the effective discreteness is not necessary for the Rogers semilattice to have only one element.

1.3. Models and Computability

Computability theory became play an important role in mathematics when the notion of computability was rigorously formulated and was applied by K. Gödel, A. Church, A. Turing, S. Kleene, E. Post, and A. Markov to the decidability of classical mathematical problems and to the proof of the Gödel theorem about the incompleteness of arithmetics.

In the XXth century, computability theory was rapidly developed. On the basis of results and methods of computability theory, new applications of mathematics have been formed, such as computer science, programing technology, automatization of various processes, etc. This can be explained by the fact that the computability approach suggests to represent an information in terms of natural numbers. Here, we briefly describe how numberings can be used for representation of mathematical objects and their structures.

Algorithmic properties of algebraic structures are naturally formulated and solved in numbering theory. Consider numberings of the basic sets of algebraic structures. Based on the standard algorithm theory, we can study the decidability of relations on elements with respect to numberings of such structures.

Consider a signature

$$\sigma = \langle P_0^{n_0}, \ldots, P_k^{n_k}, \ldots; F_0^{m_0}, \ldots, F_s^{m_s}, \ldots; c_0, \ldots, c_n, \ldots \rangle$$

such that there exist partial computable functions $[n]$ and $[m]$ defined as follows: $[n](i) = n_i$, where n_i is the arity of the predicate symbol P_i, and $[m](i) = m_i$, where m_i is the arity of the functional symbol F_i. We also consider the signature

$$\sigma_1 \leftrightharpoons \sigma \cup \langle a_0, a_1, \ldots \rangle$$

obtained from σ by adding constant symbols.

Let L and L_1 be families of all formulas of the first-order predicate calculus with equality (P_0) of the signature σ and σ_1 respectively. By a *Gödel numbering* of L_1 we mean any numbering γ: $L_1 \to \omega$ such that for a given γ-number we can effectively construct a formula with this number and for a given formula of L_1 we can effectively find its γ-number.

Now, define the Gödel numbering of formulas and terms of the signature σ. Note that the functions $[n]$ and $[m]$ from the definition of σ exist if there are countably many predicate symbols or functional symbols. If there are countably many symbols with indices, it is required to recognize effectively the arity by the index.

We fix a set V of variables $v_0, v_1, \ldots, v_n, \ldots$ and introduce the set $\mathrm{Term}_\sigma(V)$ of terms of the signature σ with variables in V and the set $\mathrm{Form}_\sigma(V)$ of formulas in variables of V. The Gödel numbering γ is defined as the mapping γ_σ: $\mathrm{Term}_\sigma(V) \cup \mathrm{Form}_\sigma(V)@ > 1-1 >> \mathbb{N}$ such that we can effectively recognize a number of a formula or a term and obtain some information about the structure of formulas and terms. Then we construct γ by induction on the complexity of formulas. We begin with $\mathrm{Term}_\sigma(V)$:

(1) $\gamma(v_i) = c(0, c(0, i))$,

(2) $\gamma(c_i) = c(0, c(1, i))$ for i such that $c_i \in \sigma$,

(3) if t has the form $F_i(t_1, \ldots, t_{m_i})$, where F_i is an m_i-ary predicate symbol, and $t_1, \ldots, t_{m_i}$ have the Gödel numbers $\gamma(t_1) = l_1, \ldots, \gamma(t_{m_i}) = l_{m_i}$, then $\gamma(t) = c(0, c((i + 2), c^{m_i}(l_1, \ldots, l_{m_i})))$.

It is obvious that the set of numbers of terms is computable. If the number of a term is known, we can recognize variables and their indices, as well as constants and their indices. Furthermore, we can find the index of the operation and the numbers of those subterms from which the term is constructed with the help of the symbol of this operation.

Define γ on the set of formulas as follows:

(1) if t and q are terms and $\gamma(t) = n$, $\gamma(q) = m$, then $\gamma(t = q) \leftrightharpoons c(1, c(0, c(n, m)))$,

(2) if P_i is an n_i-ary predicate symbol and $t_1, \ldots, t_{n_i}$ are terms with Gödel numbers $\gamma(t_1) = l_1, \ldots, \gamma(t_{n_i}) = l_{n_i}$, then $\gamma(P_i(t_1, \ldots, t_{n_i})) \leftrightharpoons c(1, c(1, c(i + 1, c^{n_i}(l_1, \ldots, l_{n_i}))))$ and $\gamma(t_1 = t_2) = c(1, c(1, c(0, c(l_1, l_2))))$,

(3) if φ and ψ are formulas with Gödel numbers $\gamma(\varphi) = n$ and $\gamma(\psi) = m$, then

$$\begin{aligned}
&\gamma((\varphi \mathbin{\&} \psi)) \leftrightharpoons c(1, c(2, c(n, m))),\\
&\gamma((\varphi \vee \psi)) \leftrightharpoons c(1, c(3, c(n, m))),\\
&\gamma((\varphi \to \psi)) \leftrightharpoons c(1, c(4, c(n, m))),\\
&\gamma(\neg\varphi) \leftrightharpoons c(1, c(5, n)),\\
&\gamma((\exists v_i)\varphi) \leftrightharpoons c(1, c(6, c(i, n))),\\
&\gamma((\forall v_i)\varphi) \leftrightharpoons c(1, c(7, c(i, n))).
\end{aligned}$$

By induction on the complexity of formulas, it is easy to show that every formula of $\mathrm{Form}_\sigma(V)$ has a Gödel number. Furthermore, we can recognize whether a given number is the Gödel number of a formula and obtain an information about the structure of this formula, for example, about free variables constants, the form of the formula, the presence of quantifiers, the complexity of the prefix formed by quantifiers, and the numbers of formulas that can be obtained by substitutions.

If the number of a formula is known, we can find the number of the equivalent formula in prenex normal form.

With every subset $S \subseteq L_1$ we associate the set $\gamma(S)$ of all numbers of formulas of S. A set S is said to be *decidable* (*enumerable*) if $\gamma(S)$ is computable (computably enumerable).

Choosing some hierarchy of the complexity of subsets of $\mathbb{N}$ (for example, the arithmetic hierarchy, the analytic hierarchy [**140**], the Ershov hierarchy [**25, 26, 29, 36, 37**], etc.), we say that X belongs to the *complexity class* Δ if $\gamma(X)$ belongs to Δ.

For a given number n we can recognize whether a formula with number n is an axiom of the first-order predicate calculus PC^σ. For a set of numbers we can recognize whether a given formula can be obtained from a finite set of formulas with the corresponding numbers by some of the rules of PC^σ. Hence we can recognize whether a sequence of formulas with given Gödel numbers is a proof in PC^σ. Thus, we arrive at the following assertion.

Proposition 1.14. *If a set of formulas is provable in* PC^σ *from an enumerable set, then it is enumerable.*

Proposition 1.14 implies the following assertion.

Proposition 1.15. *If the set of axioms* A *is enumerable, then the theory* $T_A \leftrightharpoons \{\varphi \mid A \vdash \varphi\}$ *is enumerable.*

We define the principal computable numbering $p_0(\overline{x}_0), \ldots, p_n(\overline{x}_n), \ldots$ of the set of all enumerable partial types consistent with a decidable theory T.

By a *partial type* $p(\overline{x})$ of a theory T we mean the set of formulas in variables of $\overline{x}$ such that the set $p(\overline{x}) \cup T$ is consistent. A numbering $d_0(\overline{x}_0), \ldots, d_n(\overline{x}_n), \ldots$ of partial types of a theory T is *computable* if $\overline{d}_0, \overline{d}_1, \ldots, \overline{d}_n, \ldots$, where $\overline{d}_i = \{n \mid n$ is the Gödel number of a formula in $d_i\}$, is a computable numbering of computably enumerable sets and there exists a computable function v such that $v(n)$ is equal to the number of the tuple $\langle i_1, \ldots, i_{m_n}\rangle$ of indices such that $\overline{x}_n = (v_{i_1}, \ldots, v_{i_{m_n}})$. A numbering $p_0(\overline{x}_0), \ldots, p_n(\overline{x}_n), \ldots$ of partial types of a theory T is *principal* if for every computable numbering $d_0(\overline{x}'_0), \ldots, d_n(\overline{x}'_n), \ldots$ of partial types of T there is a computable function $f(n)$ such that $d_n(\overline{x}'_n) = p_{f(n)}(\overline{x}_{f(n)})$ for any n.

Consider the following sequence of finite sets:

$$\varnothing = p_n^0(\overline{x}_n) \subseteq p_n^1(\overline{x}_n) \subseteq \ldots \subseteq p_n^t(\overline{x}_n) \subseteq \ldots$$

Let $p_n(\overline{x}_n) \leftrightharpoons \bigcup\limits_t p_n^t(\overline{x}_n)$. For n we introduce i and k such that $c(i,k) = n$. We regard i as the number of the ith computably enumerable set W_i and k as the number of the tuple $\langle i_1, \ldots, i_s\rangle$ relative to numberings of all tuples of finite length. We set

$$\begin{aligned} W_i^t \upharpoonright k \leftrightharpoons \{m \in W_i^t \mid\ & m \text{ is the Gödel number} \\ & \text{of a formula in free variables with} \\ & \text{indices in } \{i_1, \ldots, i_s\} \text{ and number } k\}, \end{aligned}$$

$$p_n^t(\overline{x}_n) \leftrightharpoons \{\varphi \mid \varphi \text{ has the Gödel number in } W_{l(n)}^m \upharpoonright r(n)\},$$

where m is the maximal number less than $t+1$ and such that the set

$$T \cup \{\varphi \mid \varphi \text{ has the Gödel number in } W^m_{l(n)} \restriction r(n)\}$$

is consistent. Since T is decidable, the consistency condition is decidable. Therefore, for n and t we can recognize whether a formula belongs to $p^t_n(\overline{x}_n)$ and indicate its number, i.e., we can list formulas in $p^t_n(\overline{x}_n)$. It is obvious that the Gödel numbers of such formulas are less than $t+1$ because of the assumption on W^t_n.

By the definition of p_n on the basis of $W_{l(n)}$ and the possibility to compute exactly the set of free variables in a computable numbering of a family of finite types, as well as the fact that $\{W_n\}_{n\in\mathbb{N}}$ is a principal numbering, we conclude that p_n is a principal numbering. Since the numbering $\{p_n\}_{n\in\mathbb{N}}$ is principal, we obtain the following assertion.

Proposition 1.16. *A family S of partial types of a theory T is computable, i.e., there is a computable numbering $d_0(\overline{x}'_0), \ldots, d_n(\overline{x}'_n), \ldots$ such that $S = \{d_0(\overline{x}'_0), \ldots, d_n(\overline{x}'_n), \ldots\}$ if and only if there exists a computably enumerable set W such that $S = \{p_n(\overline{x}_n) \mid n \in W\}$.*

By a *numbered model* of the signature σ without functional symbols we mean the pair $(\mathfrak{M}, \nu)$, where $\mathfrak{M} = \langle M, P_0, P_1, \ldots\rangle$ is a model of the signature σ and ν is a numbering of the basic set M of the model $\mathfrak{M}$. By a *homomorphism* from a numbered model $(\mathfrak{M}_0, \nu_0)$ into a numbered model $(\mathfrak{M}_1, \nu_1)$ we mean a mapping μ: $M_0 \to M_1$ from the basic set M_0 of the model $\mathfrak{M}_0$ into the basic set M_1 of the model $\mathfrak{M}_1$, i.e., a homomorphism from $\mathfrak{M}_0$ into $\mathfrak{M}_1$ and a morphism from (M_0, ν_0) into (M_1, ν_1).

For a numbered model $(\mathfrak{M}, \nu)$ we can construct a σ_1-enrichment $\mathfrak{M}_\nu$ of $\mathfrak{M}$, i.e., a model of the signature σ_1 whose basic set is the basic set of $\mathfrak{M}$ and predicates of σ in $\mathfrak{M}_\nu$ coincide with the corresponding predicates of $\mathfrak{M}$. Namely, for the value of the constant a_k, $k \in \omega$, we take $\nu k \in M$. We say that $\mathrm{Th}(\mathfrak{M}, \nu)$ is the *elementary theory* of $\mathfrak{M}_\nu$, i.e., the set of all closed formulas of the signature σ_1 that are true in $\mathfrak{M}_\nu$.

A numbered model $(\mathfrak{M},\nu)$ is *constructive* if the set $\overline{D}(\mathfrak{M},\nu) \rightleftharpoons \{\langle k, m_1,\dots,m_{n_k}\rangle \mid \mathfrak{M} \vDash P_k(\nu m_1,\dots,\nu m_{n_k})\}$ is computable.

Let $D(\mathfrak{M},\nu) = \{\varphi(c_{m-1},\dots,c_{m_k} \mid \varphi(x_1,\dots,x_{m_k})$ be a quantifier-free formula, and let $\mathfrak{M} \models \varphi(\nu m_1,\dots,\nu m_k)\}$.

The following special class of constructive models plays an important role in the study of decidable theories. A numbered model $(\mathfrak{M},\nu)$ is *strongly constructive* if $\mathrm{Th}(\mathfrak{M},\nu)$ is a decidable theory. Models admitting strong constructivizations are said to be *decidable.*

The constructibility of a numbered model $(\mathfrak{M},\nu)$ is equivalent to the decidability of the set of quantifier-free formulas in $\mathrm{Th}(\mathfrak{M},\nu)$. Hence every strongly constructive model is constructive.

However, in the case of arbitrary numbered models and algebras, only numberings of algebras with effective operations are of interest. We consider this case in more detail. Let $\sigma = \langle f_0^{m_0}, f_1^{m_1},\dots\rangle$. If the signature σ is infinite, we assume that the function $h\colon n \mapsto m_n$ is computable.

By a *computable numbering* of an algebra $\mathfrak{A} = \langle A, g_0, g_1,\dots\rangle$ of the signature σ we mean a numbering $\nu\colon \omega \to A$ of the basic set of $\mathfrak{A}$ such that there exists a binary computable function G such that $g_n(\nu y_1,\dots,\nu y_{m_n}) = \nu G\big(n, c^{m_n}(y_1,\dots,y_{m_n})\big)$ for any $n \in \omega$ and $y_1,\dots,y_{m_n}$.

The pair $(\mathfrak{A},\nu)$ is referred to as a *computable numbered algebra* if $\nu\colon \omega \to A$ is a numbering of $\mathfrak{A}$. It turns out that any algebra admits a computable numbering.

Theorem 1.17 ([**37**])**.** *Any at most countable algebra $\mathfrak{A}$ admits a computable numbering of this algebra.*

In this case, the complexity of this algebra depends only from numbering equivalence of that numbering.

A computable numbered algebra $(\mathfrak{A},\nu)$ is constructive (i.e., it is a constructive model of the corresponding signature consisting of only functions) if and only if the numbering ν is solvable.

Let $\mathfrak{A} = \langle A; P_0, \ldots, P_n; F_0, \ldots, F_k; c_0, \ldots, c_s \rangle$ be an algebraic structure of a signature σ. If σ is infinite, the functions $i \to m_i$ and $i \to n_i$ are assumed to be computable. A pair $(\mathfrak{A}, \nu)$, where ν is a mapping from $\mathbb{N}$ or from an initial interval of $\mathbb{N}$ to the basic set A of $\mathfrak{A}$, is called a *numbered structure* and ν is called a *numbering* of $\mathfrak{A}$.

Let $\mathfrak{K}$ be a class of subsets and functions on $\mathbb{N}$. For $\mathfrak{K}$ we can take, for example, one of the following classes:

- the class R of computable functions and relations,
- the class R_A of computable relative to A functions and relations,
- the class PRIM of primitive computable functions and relations,
- the class P of relations and functions of the polynomial complexity,
- the class exp of relations and functions of the exponential complexity,
- the corresponding classes $\Delta^0_\alpha(\Sigma^{0,A}_\alpha, \Pi^{0,A}_\alpha)$ of relations and functions of the arithmetic hierarchy relative to A,
- the corresponding classes $\Delta^1_\alpha(\Sigma^{1,A}_\alpha, \Pi^{1,A}_\alpha)$ of relations and functions of the analytic hierarchy (relative to A),
- the corresponding classes $\Delta^{m-1}_\alpha(\Sigma^{m-1,A}_\alpha, \Sigma^{m-1,A}_\alpha)$ of the Ershov hierarchy (relative to A) [**25, 26, 29**].

Let B be a set or a family of sets, and let $\mathfrak{A}$ be an algebraic structure of a signature σ. A numbered structure $(\mathfrak{A}, \nu)$ is said to be B-*positive* if $\eta_\nu \leftrightharpoons \{(n, m) \mid \nu n = \nu m\}$ and $\nu^{-1}(P_i) = \{\langle l_1, \ldots, l_{m_i}\rangle \mid \langle \nu l_1, \ldots, \nu l_{m_i}\rangle \in P_i\}$, $i \leqslant n$, are computably enumerable with respect to a set in B or with respect to the entire set B and there exist B-computable functions f_i, $i \leqslant k$, such that $\nu f_i(l_1, \ldots, l_{n_i}) = F_i(\nu l_1, \ldots, \nu l_{n_i})$ for all $l_1, \ldots, l_{n_i} \in \mathbb{N}$. A B-positive structure $(\mathfrak{A}, \nu)$ is said to be B-*constructive* if η_ν and $\nu^{-1}(P_i)$ are B-computable. If the signature σ is infinite, it is necessary to require the uniform computable numbering [**100**].

To study algorithmic properties of models, we need an algorithm checking the truth of formulas. We define the relative constructibility and strong constructibility. Let $\mathcal{B}$ be a class of subsets of $\mathbb{N}$. Suppose that a quantifier-free formula has no alternating groups of quantifiers and a formula Φ has n alternating groups of quantifiers if the prenex normal form of Φ has n alternating groups of quantifiers. Denote by $\mathfrak{F}_n$ the set of formulas possessing n alternating groups of quantifiers and by $\mathfrak{F}_\omega$ the set of all formulas, called *fragments* (of the language). The sets $\mathfrak{F}_n$ are called *restricted fragments* (of the language). Let $\mathfrak{F}$ be a set of formulas of a signature σ. A numbered structure $(\mathfrak{A}, \nu)$ is said to be *$\mathcal{B}$-$\mathfrak{F}$-constructive* or *$\mathfrak{F}$-constructive relative to $\mathcal{B}$* if the following set belongs to $\mathcal{B}$:

$$\{\langle s, l_1, \dots, l_k\rangle \mid s \text{ is the number of a formula } \Phi(x_1, \dots, x_k) \text{ in } \mathfrak{F} \text{ with } k \text{ free variables and } \mathfrak{A} \models \Phi(\nu l_1, \dots, \nu l_k)\}.$$

It is easy to see that a structure is $\mathfrak{F}_0$-constructive relative to $\mathcal{B}$ if and only if it is $\mathcal{B}$-constructive. For the sake of brevity, we write $\mathfrak{F}$-constructive in the case of the $\mathfrak{F}$-constructibility relative to the class of computable relations and $\mathcal{B}$-constructive in the case $\mathfrak{F} = \mathfrak{F}_0$. If $\mathfrak{F} = \mathfrak{F}_0$ or $\mathcal{B} = \varnothing$, we omit $\mathfrak{F}$ or $\mathcal{B}$ in the notation.

$\mathcal{B}$-$\mathfrak{F}_\omega$-constructive structures are said to be *strongly $\mathcal{B}$-constructive* or *$\mathcal{B}$-ω-constructive*, whereas $\mathcal{B}$-$\mathfrak{F}_n$-constructive structures are referred to as *$\mathcal{B}$-n-constructive* .

We describe the other approach. Let $\mathfrak{A}$ be an algebraic structure of a signature σ such that the basic set A is a subset of $\mathbb{N}$. Then it is reasonable to consider the effectiveness of different relations without any mention of numbers.

An algebraic structure $\mathfrak{A}$ is said to be *$\mathcal{B}$-computable* if the basic predicates and operations of $\mathfrak{A}$ belong to a class $\mathcal{B}$. For many computability classes $\mathcal{B}$ an abstract structure is $\mathcal{B}$-constructivizable if and only if it is isomorphic to a $\mathcal{B}$-computable structure. For a $\mathcal{B}$-constructive structure we can effectively construct a $\mathcal{B}$-computable structure relative to $\mathcal{B}$ provided that we can choose exactly one number in every set of the numbers of elements. We can pass from a $\mathcal{B}$-computable structure to a $\mathcal{B}$-constructive structure by using the $\mathcal{B}$-computable function that enumerates the basic set of the $\mathcal{B}$-computable model. Then we can define the $\mathcal{B}$-constructivization of this structure.

Similarly, for $\mathfrak{F}$-constructive structures relative to $\mathcal{B}$ we can define $\mathcal{B}$-$\mathfrak{F}$-computable models. If a model is isomorphic to a $\mathcal{B}$-$\mathfrak{F}_\omega$-computable model, then it is $\mathcal{B}$-decidable. In the case $B \subseteq \mathbb{N}$, the class of B-computable sets is denoted by $\mathcal{B}(B)$. We say that $\mathcal{B}(B)$-$\mathfrak{F}$-computable ($\mathcal{B}(B)$-$\mathfrak{F}$-constructive) structures are $\mathfrak{F}$-*computable* ($\mathfrak{F}$-*constructive*) relative to B.

We give the most important examples of relatively computable models. Assume that the language is computable and the basic set is a subset of ω. We identify a structure $\mathcal{A}$ with its atomic diagram $D(\mathcal{A})$ and sentences with their Gödel numbers. In this case, we say that $\mathcal{A}$ is *computable* (*arithmetical* or *hyperarithmetical*) if $D(\mathcal{A})$, regarded as a subset of ω, is computable (arithmetical or hyperarithmetical).

We say that a model has *constructivization* or admits a computable (arithmetical or hyperarithmetical) representation if there exists an isomorphic computable (arithmetical or hyperarithmetical) model. If for an abstract model there exists an isomorphic (arithmetical or hyperarithmetical) decidable model, then we say that this model has a decidable representation with respect to the class of (arithmetical or hyperarithmetical) sets.

Let $(\mathfrak{A}, \nu)$ and $(\mathfrak{B}, \mu)$ be numbered models, and let $\varphi\colon \mathfrak{A} \to \mathfrak{B}$ be a homomorphism. We say that φ is C-*computable* if there exists a C-computable function f such that $\varphi\nu = \mu f$, i.e., the following diagram is commutative:

$$\begin{array}{ccc} N & \xrightarrow{f} & N \\ \nu \downarrow & & \downarrow \mu \\ A & \xrightarrow{\varphi} & B \end{array}$$

In this case, the function f represents φ and φ is called a C-*homomorphism.* If there exists a C-computable isomorphism φ from $(\mathfrak{A}, \nu)$ into $(\mathfrak{B}, \mu)$, then $(\mathfrak{B}, \mu)$ is called a C-*extension* of

$(\mathfrak{A}, \nu)$ with respect to φ. If $\mathfrak{A} \subseteq \mathfrak{B}$ and the identity embedding of $\mathfrak{A}$ in $\mathfrak{B}$ is C-computable, then $(\mathfrak{B}, \mu)$ is a C-extension of $(\mathfrak{A}, \nu)$.

Let $(\mathfrak{N}, \nu)$ and $(\mathfrak{N}, \mu)$ be numbered algebraic structures. Recall that numberings ν and μ of $\mathfrak{N}$ are computably equivalent if there exist computable functions f and g such that $\nu = \mu f$ and $\mu = \nu g$. For constructivizations ν and μ it suffices to require the existence of only one computable function f such that $\nu = \mu f$. We note that if there is the continual group of automorphisms of a constructivizable structure $\mathfrak{A}$, then there is the continuum of noncomputable equivalent constructivizations. Thus, for an atomless Boolean algebra we have the continuum of noncomputable equivalent constructivizations, although it has simple algorithmic structure. However, we consider abstract structures up to an isomorphism. The definition of the autoequivalence introduced by Mal'tsev [**101**] turns out to be more suitable in this situation. Two numberings ν and μ of an algebraic structure are *autoequivalent* if they are computably equivalent up to an automorphism, i.e., there exists an automorphism φ of $\mathfrak{A}$ such that $\varphi\nu$ and μ are computably equivalent.

The questions on nonequivalent representations and their classification are important in the study of constructive structures. Within the framework of the above approaches, we can investigate the same properties by choosing a suitable language. In fact, the above approaches are equivalent. To demonstrate this fact, we show that the corresponding categories are equivalent.(for category theory we refer to [**12**] and [**36**]).

We consider the category Num of all numbered models with homomorphisms for morphisms and the category Nat of all models whose basic sets are subsets of $\mathbb{N}$ and morthisms are computable homomorphisms. Let $(\mathfrak{M}, \nu)$ be a numbered model. We define the value of the functor Com on $(\mathfrak{M}, \nu)$ by setting $\mathrm{Com}\,(\mathfrak{M}, \nu) \leftrightharpoons (N_{\mathfrak{M}}, \Sigma)$, where

$$N_{\mathfrak{M}} \leftrightharpoons \{n \mid n \text{ is the least number of } \nu n\},$$

$$P \leftrightharpoons \{\langle n_1, \dots, n_k\rangle \in N_{\mathfrak{M}} \mid \mathfrak{M} \models P(\nu n_1, \dots, \nu n_k)\},$$

$$F(n_1, \dots, n_k) \leftrightharpoons \min\{m \mid F(\nu n_1, \dots, \nu n_k) = \nu m\}.$$

If φ is a homomorphism from $(\mathfrak{M}, \nu)$ into $(\mathfrak{N}, \mu)$, then

$$\mathrm{Com}\,(\varphi) \leftrightharpoons \{\langle n, m\rangle \mid n \in \mathbb{N}_{\mathfrak{M}}, m \in \mathbb{N}_{\mathfrak{N}},\ \text{and } \varphi(\nu n) = \mu m\}.$$

Note that $\mathrm{Com}\,(\varphi)$ is a homomorphism from $\mathrm{Com}\,(\mathfrak{M}, \nu)$ into $\mathrm{Com}\,(\mathfrak{N}, \mu)$.

REMARK. $\mathfrak{M}$ and $\mathrm{Com}\,(\mathfrak{M}, \nu)$ are isomorphic.

REMARK. If φ is an isomorphism, then $\mathrm{Com}\,(\varphi)$ is also an isomorphism.

We define the functor K from Nat into Num by setting $\mathrm{K}\,(\mathfrak{M}) \leftrightharpoons (\mathfrak{M}, \nu)$, where ν is a numbering of $|\mathfrak{M}|$ in ascending order. If $|\mathfrak{M}|$ is finite, then all the numbers which do not appear in this numbering go to the last element (with respect to the numbering of $\mathfrak{M}$). As a result, we obtain a numbering ν which will be denoted by ν_{K}.

REMARK. The functor Com determines an equivalence between the categories Num and Nat.

Consider the subcategory Con^B of Num consisting of B-constructive models with B-computable homomorphisms for morphisms. We also consider the subcategory Com^B of Nat consisting of B-computable models with B-computable homomorphisms for morphisms.

Theorem 1.18. *The restriction* Com^B *of the functor* Com *to the subcategory* Con^B *determines an equivalence between* Con^B *and* Com^B*.*

PROOF. It is easy to verify that if $(\mathfrak{M}, \nu)$ is B-constructive, then the model $\mathrm{Com}\,(\mathfrak{M}, \nu)$ is B-computable. Since the basic sets are B-computable and there exists a B-computable function f such that $\varphi\nu = \mu f$, we conclude that $\mathrm{Com}\,\varphi$ is a partial B-computable function with B-computable graph. The restriction Com^B of Com to Con^B acts from Con^B into Com^B. There exist isomorphisms $\varphi\colon 1_{\mathrm{Con}^B} \to \mathrm{Com\ K}$ and $\psi\colon 1_{\mathrm{Com}^B} \to \mathrm{K\ Com}$ such that $\mathrm{Com}\ \varphi = \psi\,\mathrm{Com}$. $\square$

Corollary 1.19. *Numbered models* $(\mathfrak{N}, \nu)$ *and* $(\mathfrak{M}, \mu)$ *are isomorphic if and only if* $\mathrm{Com}\,(\mathfrak{N}, \nu)$ *and* $\mathrm{Com}\,(\mathfrak{M}, \mu)$ *are isomorphic.*

Corollary 1.20. *B-constructive models* $(\mathfrak{N}, \nu)$ *and* $(\mathfrak{M}, \mu)$ *are B-isomorphic if and only if* $\mathrm{Com}\,(\mathfrak{M}, \mu)$ *is B-isomorphic to* $\mathrm{Com}\,(\mathfrak{N}, \nu)$.

Thus, the study of constructivizations of a model $\mathfrak{M}$, defined up to an autoequivalence, is equivalent to the study of computable models isomorphic up to a computable isomorphism to $\mathfrak{M}$. Consequently, if ν and μ are constructive models of $\mathfrak{M}$, then $\mathrm{Com}\,(\mathfrak{M}, \nu)$ and $\mathrm{Com}\,(\mathfrak{M}, \mu)$ are computable models isomorphic to $\mathfrak{M}$; moreover, ν and μ are autoequivalent if and only if $\mathrm{Com}\,(\mathfrak{M}, \nu)$ and $\mathrm{Com}\,(\mathfrak{M}, \mu)$ are computably isomorphic and for any computable model $\mathfrak{N}$ isomorphic to $\mathfrak{M}$ there exists a constructivization ν of $\mathfrak{M}$ such that $\mathrm{Com}\,(\mathfrak{M}, \nu)$ and $\mathfrak{N}$ are computably isomorphic.

Let $\mathfrak{M} = \langle M, P_0^{n_0}, \ldots, P_k^{n_k}, a_0, \ldots, a_s \rangle$ be a finite model of a finite signature σ without functional symbols, and let ν be a mapping from $[0, n] = \{i \mid 0 \leqslant i \leqslant n\}$ onto M. The pair $(\mathfrak{M}, \nu)$ is called a *finitely numbered* (n-*numbered*) *model.*

With every finite signature $\sigma = \langle P_0^{n_0}, \ldots, P_k^{n_k}, a_0, \ldots, a_s \rangle$ we associate the number $\langle\langle\langle 0, n_0 \rangle, \ldots, \langle k, n_k \rangle\rangle, s\rangle$, $n_i \geqslant 1$. We extend σ by constant symbols $c_0, \ldots, c_n, \ldots$ and define the $(n+1)$-diagram $\mathcal{D}(\mathfrak{M}, \nu)$ of $(\mathfrak{M}, \nu)$ by constructing the enrichment $\mathfrak{M}^n$ of $\mathfrak{M}$ to a model of the signature $\sigma_n = \sigma \cup \{c_0, \ldots, c_n\}$. For this purpose, assume that the value of c_i is equal to νi, $i \leqslant n$, and $\mathcal{D}(\mathfrak{M}, \nu) = \{\varphi \mid \varphi$ is an atomic formula of the signature σ_n without free variables or the negation of such a formula and $\mathfrak{M}^n \vDash \varphi\}$. Let $G\mathcal{D}(\mathfrak{M}, \nu)$ be the set of the Gödel numbers of formulas in $\mathcal{D}(\mathfrak{M}, \nu)$. The number $\langle n, \langle\langle\langle 0, n_0 \rangle, \langle 1, n_1 \rangle, \ldots, \langle k, n_k \rangle\rangle, s\rangle, u\rangle$, where u is the canonical number of the finite set $D_u = G\mathcal{D}(\mathfrak{M}, \nu)$, is called the *Gödel number* of the numbered finite model of the finite signature σ. Such models are called *finitely numbered models* and their numbers are referred to as the *Gödel numbers.*

A numbering of finitely enumerable models possess the following obvious properties.

⟨1⟩ *The set of the Gödel numbers of finitely numbered models is computable.*

⟨2⟩ *For a given Gödel number of a finitely numbered model* $(\mathfrak{M}, \nu)$ *it is possible to compute the number of elements of* $|\mathfrak{M}|$.

⟨3⟩ *For a given Gödel number of a finitely numbered model it is possible to compute how many predicate symbols and constant symbols are contained in* σ *and compute the arity of all predicate symbols.*

⟨4⟩ *For two Gödel numbers of finitely numbered models it is possible to recognize whether these models can be considered in the same signature.*

⟨5⟩ *The set of numbers of finite signatures is computable.*

⟨6⟩ *For numbers* n *and* m *it is possible to recognize whether a finitely numbered model with number* n *is a model of the signature with number* m.

⟨7⟩ *For numbers* n *and* m *it is possible to recognize whether a finitely numbered model with number* n *of the signature* $\sigma = \langle P_0^{n_0}, \ldots, P_k^{n_k}, a_0, \ldots, a_s \rangle$ *has an enrichment to a finitely numbered model of the signature* σ' *with number* m.

A finite n-numbered model $(\mathfrak{M}, \nu)$ is called an *extension* of a k-numbered model $(\mathfrak{N}, \mu)$ if $k \leqslant n$, the models $\mathfrak{M}$ and $\mathfrak{N}$ are of the same signature, and the set $\{\langle \nu(i), \mu(i) \rangle \mid i \leqslant k\}$ is an isomorphic embedding of $\mathfrak{M}$ in $\mathfrak{N}$.

⟨8⟩ *For numbers* n *and* m *it is possible to recognize whether a finitely numbered model with Gödel number* m *is an extension of a finitely numbered model with Gödel number* n.

An n-numbered model $(\mathfrak{M}, \nu)$ of the signature $\sigma = \langle P_0^{n_0}, \ldots, P_k^{n_k}, \ a_0, \ldots, a_s \rangle$ is called an *enriched extension* of a k-numbered model $(\mathfrak{N}, \mu)$ of the signature $\sigma' = (P_0^{m_0}, \ldots, P_{r'}^{m_{r'}}, a_0, \ldots, a_{s'})$ if $k \leqslant n$, $s' \leqslant s$, $r' \leqslant r$ and, for any $0 \leqslant i \leqslant r'$, the arity m_i of the predicate P_i is equal to the arity n_i of the predicate P_i; moreover, $\{\langle \mu(i), \nu(i) \rangle \mid i \leqslant k\}$ is an isomorphism from $\mathfrak{N}$ into $\mathfrak{M} \restriction \sigma$.

⟨9⟩ *For n and m it is possible to recognize whether a model with Gödel number n is an enriched extension of a finitely numbered model with Gödel number m.*

For a numbered model $(\mathfrak{M}, \nu)$ of the signature $\sigma = \langle P_0^{n_0}, \ldots, P_k^{n_k}, \ldots, a_0, \ldots, a_s, \ldots \rangle$ we set $M_n = \{\nu i \mid i \leqslant n\}$, $n \in \mathbb{N}$, and consider $\sigma_n = \langle P_0^{n_0}, \ldots, P_r^{n_r}, a_{i_1}, \ldots, a_{i_k} \rangle$, where $r = n$ if σ contains at least n predicates and r is the number of predicates of σ otherwise. The set $\{i_1, \ldots, i_k\}$ consists of the indices i of constants c_i of σ such that $i \leqslant n$ and the value of c_i belongs to M_n.

Let $\mathfrak{M}_n$ be a submodel of the restriction $\mathfrak{M} \upharpoonright \sigma_n$ with the basic set M_n. Finitely numbered models $(\mathfrak{M}_n, \nu_n)$, where $\nu_n(k) \rightleftharpoons \nu(k)$, $k \leqslant n$, are called *finitely numbered submodels* of $(\mathfrak{M}, \nu)$. Denote by $(e\mathfrak{M}, \nu)$ the set of the Gödel numbers of finitely numbered submodels of $(\mathfrak{M}, \nu)$. The set $W(\mathfrak{M}, \nu)$ is called the *representation* of $(\mathfrak{M}, \nu)$.

Proposition 1.21. *A numbered model $(\mathfrak{M}, \nu)$ is constructive if and only if $W(\mathfrak{M}, \nu)$ is computably enumerable.*

By Proposition 1.21, it is possible to construct a universal computable numbering of all constructive and all computable models of a fixed signature without functional symbols.

The empty model of the empty signature with the empty numbering, as well as n-numbered models, is constructive. For a given set $W(\mathfrak{M}, \nu)$ we define a model $\mathfrak{M}$ and a numbering ν as follows. Let $M_W^0 = \{c_i \mid$ there exists the Gödel number of an n-numbered model in $W(\mathfrak{M}, \nu)$ and $i \leqslant n\}$. Introduce an equivalence relation on M_W^0 as $c_i \sim_W c_j$ if $c_i = c_j$ occurs in the diagram of some n-numbered model with number in $W(\mathfrak{M}, \nu)$. Let M_W be the quotient set $M_W^0/{\sim_W}$. We set $\nu_W(i) = c_i/{\sim_W}$, where $i \in M_W^0$, and $\nu_W(i) = a_j$ for a constant of the signature σ of the model $\mathfrak{M}$ if $c_i = a_j$ occurs in the diagram of some n-numbered model with number in W. We set $P_i(\nu_W n_0, \ldots, \nu_W n_k)$ if $P_i(c_{n_0}, \ldots, c_{n_k})$ occurs in the diagram of some n-numbered model with number in W.

Thus, we obtain a model $\mathfrak{M}_W$ of the same signature as $\mathfrak{M}$. We also define the numbering ν_W. Setting $\varphi(c_i/\sim) \leftrightharpoons \nu i$, we conclude that φ is an isomorphism between $(\mathfrak{M}_W, \nu_W)$ and $(\mathfrak{M}, \nu)$; moreover, φ is the identity mapping on numbers. If $W(\mathfrak{M}, \nu)$ is computable, then $(\mathfrak{M}_W, \nu_W)$ is constructive. The converse assertion is obvious.

Let us consider a finite signature σ without functional symbols and define a numbering $\varkappa^\sigma$ of all constructive models of the signature σ. For this purpose, we consider the principal numbering $\{W_n\}_{n\in\mathbb{N}}$ of all computably enumerable subsets of $\mathbb{N}$. As usual, W_n^t is the part of W_n which was already numbered at the step t. We recall that we enumerate only $x < t$ in W_n^t. For W_n we construct V_n as follows. Let $V_n^0 = \varnothing$. At the step $t+1$, we verify the following conditions:

(a) any element of W_n^{t+1} is the Gödel number of some k-model of the signature σ,

(b) for any $x, y \in W_n^{t+1}$ one of finitely numbered models with the Gödel numbers x and y is an extension of the other.

We set $V_n^{t+1} \leftrightharpoons V_n^t$ if conditions (a) and (b) are not satisfied. Otherwise, we set $V_n^{t+1} \leftrightharpoons V_n^t \cup W_n^{t+1}$. The sequence $\{V_n\}_{n\in\mathbb{N}}$, where $V_n = \cup V_n^t$, is computable. Consequently, there is a computable function ρ such that $V_n = W_{\rho(n)}$ for any n. Furthermore, $W_{\rho(\rho(n))} = W_{\rho(n)}$ for any n.

It is easy to see that every set V_n is computably enumerable and represents some constructive model. By the above results, we can restore the constructive model $\mathfrak{M}_{V_n}$ and constructivization ν_{V_n}.

We set $\varkappa^\sigma(n) \leftrightharpoons (\mathfrak{M}_{V_n}, \nu_{V_n})$ and write $\mathfrak{M}_n^\varkappa$ instead of $\mathfrak{M}_{V_n}$ and $\nu_n^\varkappa$ instead of ν_{V_n}. It is easy to see that $\varkappa^\sigma(n)$ enumerates all constructive models of the signature σ including finitely numbered models and the empty model as well. Assume that σ is infinite and the function $i \to n_i$ is computable, where n_i is the arity of the ith predicate symbol. Arguing as above, we obtain $W(\mathfrak{M}, \nu)$ and the numbering $\varkappa^\sigma$ of all constructive models of the signature σ and finite constructive models of finite parts of σ if, in the construction of V_n, finitely numbered models are models of the initial segments of the signature σ and the enriched extension condition is used instead of the extension condition.

The notion of a computable sequence of constructive models is often used (cf., for example, [**22, 34, 37**]).

Definition 1.22. A sequence of constructive models $(\mathfrak{M}_0, \nu_0), \ldots, (\mathfrak{M}_n, \nu_n), \ldots$ is *computable* if the models are uniformly constructive, i.e., all computable functions of numbers of constants and indices of computable functions defining basic operations and predicates on numbers of elements can be computed for $(\mathfrak{M}_n, \nu_n)$ from computable functions by the number n.

Using the idea of numberings of sets, we can define, up to a recursive isomorphism, a numbering of any class S of constructive models.

Definition 1.23. A numbering ν of models in S is called a *computable numbering of class S* if the sequence $(\mathfrak{M}_0, \nu_0), \ldots, (\mathfrak{M}_n, \nu_n), \ldots$ of constructive models in S is computable, where $(\mathfrak{M}_n, \nu_n)$ is a model with number n in the numbering ν $(\nu(n) = (\mathfrak{M}_n, \nu_n))$ and for any constructive model $(\mathfrak{M}, \mu)$ in S there exists n such that $(\mathfrak{M}, \mu)$ and $(\mathfrak{M}_n, \nu_n)$ are computably isomorphic.

A computable sequence of computable models is defined in a similar way.

Definition 1.24. A sequence $\mathfrak{M}_0, \ldots, \mathfrak{M}_n, \ldots$ of computable models is *computable* if the models are uniformly computable, i.e., all computable functions of constants and the indices of computable functions defining basic operations and predicates are computed for the models $\mathfrak{M}_n$ from computable functions by the number n.

Using again the ideas of numberings of sets, we can define, up to a recursive isomorphism, a numbering of any class S of computable models.

Definition 1.25. A numbering ν of models in S is called a *computable numbering of class S* if the sequence $\mathfrak{M}_0, \ldots, \mathfrak{M}_n, \ldots$ of computable models in S is computable, where $\mathfrak{M}_n$ is a computable model with number n in the numbering ν ($\nu(n) = \mathfrak{M}_n$) and for any computable model $\mathfrak{M}$ in S there exists n such that $\mathfrak{M}$ and $\mathfrak{M}_n$ are computably isomorphic.

Proposition 1.26. *A sequence $(\mathfrak{M}_0, \nu_0), \ldots, (\mathfrak{M}_n, \nu_n), \ldots$ of constructive models is computable if and only if there exist computable functions f and g such that $(\mathfrak{M}_n, \nu_n)$ and $\varkappa(f(n))$ are computably isomorphic for every n and the number $g(n)$ of the computable function $\varkappa_{g(n)}$ defining this computable isomorphism is computed by g from n, i.e., for $\varphi_n(\nu_n(m)) \leftrightharpoons \nu^{\varkappa}_{f(n)}(\varkappa_{g(n)}(m))$, φ_n is an isomorphism from $\mathfrak{M}_n$ onto $\mathfrak{M}^{\varkappa}_{f(n)}$, where $\varkappa_n$ is the universal numbering of all partial computable functions.*

Theorem 1.27. *The sequence $(\mathfrak{M}^{\varkappa}_n, \nu^{\varkappa}_n)$ of constructive models is computable.*

The proof is based on the construction and definition of a computable sequence. Indeed, for $(\mathfrak{M}^{\varkappa}_n, \nu^{\varkappa}_n)$ and n we can find the diagram $V_n = W(\mathfrak{M}^{\varkappa}_n, \nu^{\varkappa}_n)$.

Let $\alpha = \{\langle \mathfrak{M}_n, \nu_n \rangle\}$ and $\beta = \{(\mathfrak{N}_n, \mu_n)\}$ be numberings of numbered models. We say that α is *reduced* to β if there exists a computable function f such that the constructive models $(\mathfrak{M}_n, \nu_n)$ and $(\mathfrak{N}_{f(n)}, \mu_{f(n)})$ are computably isomorphic (in the sense of constructive models). The reduction of α to β is denoted by $\alpha \leqslant \beta$. We say that α is *effectively reduced* to β if there exist computable functions f and g such that for any n the function $\varkappa_{g(n)}$ defines an isomorphism from a constructive model $(\mathfrak{M}_n, \nu_n)$ onto the numbered model $(\mathfrak{N}_{f(n)}, \mu_{f(n)})$, i.e., the mapping $\varphi_n(\nu_n(m)) \leftrightharpoons \mu_{f(n)}(\varkappa_{g(n)}(m))$ is well defined and realizes an isomorphism between $\mathfrak{M}_n$ and $\mathfrak{N}_{f(n)}$.

Similarly, based on general ideas of numbering theory, we can define the reducibility for numbering of computable models.

Let $\alpha = \{\mathfrak{M}_n, n \in \omega\}$ and $\beta = \{\mathfrak{N}_n, n \in \omega\}$ be numberings of models. We say that α is *reduced* to β if there exists a computable function f such that the constructive models $(\mathfrak{M}_n, \nu_n)$ and $(\mathfrak{N}_{f(n)}, \mu_{f(n)})$ are computably isomorphic (in the sense of computable models). The reduction of α to β is denoted by $\alpha \leqslant \beta$. We say that α is *effectively reduced* to β if there exist computable functions f and g such that for any n the function $\varkappa_{g(n)}$ is an isomorphism from a model $\mathfrak{M}_n$ onto the model $\mathfrak{N}_{f(n)}$, i.e., the mapping $\varphi_n(\nu_n(m)) \leftrightharpoons \mu_{f(n)}(\varkappa_{g(n)}(m))$ is well defined and realizes an isomorphism between $\mathfrak{M}_n$ and $\mathfrak{N}_{f(n)}$.

The following assertion follows from definitions.

Proposition 1.28. *If a numbering* $(\mathfrak{M}_n, \nu_n)$*,* $n \in \mathbb{N}$*, of numbered models is effectively reduced to a computable numbering of constructive models* $(\mathfrak{N}_n, \mu_n)$*,* $n \in \mathbb{N}$*, then* $(\mathfrak{M}_n, \nu_n)$*,* $n \in \mathbb{N}$*, is a computable numbering of constructive models.*

From the construction of numberings of constructive models $\varkappa^\sigma(n) \leftrightharpoons (\mathfrak{M}_{V_n}, \nu_{V_n})$ we obtain an important result due to Nurtazin about the existence of a universal computable numbering of constructive models and the existence of universal computable numbering of all computable models from the existence of a functor between categories.

Theorem 1.29 ([**130**])**.** *There exists up to a recursive isomorphism a universal computable numbering of all constructive models of a computable signature without functional symbols i.e., a computable numbering* $(\mathfrak{M}_n, \nu_n)$*,* $n \in \mathbb{N}$*, of constructive models of the fixed structure such that any other computable numbering of constructive models of the same signature is reduced to this numbering.*

Corollary 1.30 ([**130**])**.** *There exists up to recursive isomorphism a universal computable numbering of all computable models of a comuptable signature without functional symbols i.e., a computable numbering* $\mathfrak{M}_n$*,* $n \in \mathbb{N}$*, of computable models of this fixed signature to which is reduced any other computable numbering of computable models of the same signature.*

1.4. Perspective directions in the theory of computable models

We list the most important topics for the future development of the theory of computable models (cf. also [**39**]).

1. One of the main problem is connected with existence of computable representations. In particular, this approach is presented in [**40, 37, 3, 5, 7, 13, 17, 28, 31, 32, 33, 35, 39, 43, 47, 53, 54, 57, 59, 65, 68, 69, 74, 76, 81, 83, 84, 85, 90, 92, 96, 100, 107, 108, 109, 110, 111, 112, 113, 114, 115, 116, 117, 118, 119, 120, 121, 122, 123, 124, 125, 129, 130, 132, 133, 134, 135, 136, 138, 150, 151, 152, 154**].
2. The second approach is connected with the nonuniqueness of computable representations and algorithmic dimension (with some special properties). In particular, it is represented in [**1, 5, 8, 10, 18, 23, 32, 35, 37, 39, 40, 44, 45, 46, 48, 49, 54, 57, 60, 67, 63, 64, 65, 72, 75, 101**].
3. Interesting problems on the classification of computable models relative to structures connected with computable models. [**40, 39**].
4. Computable classes of models in the light of the above two approaches. The computability of families of computable representations and the computability of classes of computable models were studied in [**40, 39, 1, 66, 14, 21, 22, 37, 45, 56, 58, 67, 85, 91, 92, 130**].
5. Another class of problems connected with the classification of algorithmic problems with respect to complexity (cf., for example, [**1, 2, 3, 5, 10, 13, 14, 37, 39, 40, 42, 43, 44, 54, 55, 56, 57, 62, 68, 72, 70, 75, 89, 82, 147, 148**]).
6. There exists a closed connection between definability and complexity. An approach based on this fact was used in many papers, for example, [**1, 5, 6, 10, 13, 14, 38, 39, 40, 42, 43, 58, 62, 66, 71, 70, 75, 77, 89, 127, 99, 147, 148, 154**].

2. Bounds for Computable Models

Bounds for computable models are used for describing various mathematical constructions. We consider bounds for theories of computable models and the complexity of some models. We also examine the structure bounds from the point of view of the complexity of descriptions of computable models in a language with infinite disjunctions and conjunctions.

2.1. Bounds for the theory of computable models

By definition, the theory of a decidable model is decidable. We establish the existence of computable models satisfying a given specification in the language of the first-order predicate calculus.

Theorem 2.1 ([**32, 152**]). *A decidable consistent theory T possesses a decidable model.*

The situation is rather complicated if additional model-theoretic properties are required. Goncharov–Nurtazin and Harrington independently proved the following assertion for prime models.

Theorem 2.2 ([**65, 74**]). *A decidable complete theory T possesses a decidable prime model if and only if there exists an algorithm that for any formula consistent with T produces a principal type of the theory containing this formula.*

Morley proved the existence theorem for saturated models and posed the decidability problem for homogeneous models.

Theorem 2.3 ([**125**]). *A decidable complete theory T possesses a decidable saturated model if and only if the set of all types of T admits a computable numbering.*

Goncharov [**52**] and Peretyat'kin [**136**] independently found the decidability criteria for homogenous models. Goncharov [**53**] constructed an example of a totally transcendental decidable theory without decidable homogeneous models.

Problem 1 (Goncharov)**.** Whether there exists a decidable homogeneous model, determined up to an isomorphism, for an arbitrary decidable theory with countably many countable models?

The situation is quite simple in the case of countably categorical theories.

Definition 2.4. A theory T is *countably categorical* if T has a unique up to an isomorphism countable model.

Within the framework of model theory, countably categorical theories and models of such theories have been well studied. The following assertion is a simple consequence of the effective completeness theorem.

Theorem 2.5. *A countably categorical theory T is decidable if and only if all models of T are decidable, which holds if and only if T has a decidable model.*

Thus, if we are interested in decidable models of countably categorical theories, an answer can be obtained in terms of decidability. However, the situation essentially changes for computable models.

If a theory T possessing a computable model is computable in $\mathbf{0}^{\omega}$, then the degree of the ω–jump of a computable set is 0^{ω}. This bound is sharp because there exists a theory (for example, the theory of $(\omega, +, \times, \leqslant)$) possessing a computable model that is Turing equivalent to $\mathbf{0}^{\omega}$.

Theorem 2.6. *If* $\mathbf{A}$ *is a computable model, then the theory* $\mathrm{Th}\,(\mathbf{A}^*)$ *is* 0^{ω}*–decidable and the theory* $\mathrm{Th}_{\Sigma_{n+1}}(\mathbf{A}^*)$ *is computably enumerable in* 0^n *uniformly with respect to* n*, where* $\mathbf{A}^*$ *is an extension of* $\mathbf{A}$ *by constants.*

Corollary 2.7. *If* $\mathbf{A}$ *is a computable model, then the theory* $\mathrm{Th}\,(\mathbf{A}^*)$ *is* 0^{ω}*–decidable.*

These results suggest the following interesting problem.

Problem 2. Find necessary and sufficient conditions for the existence of computable models.

Our goal is to find natural sufficient conditions for the existence of computable models of given theories and to determine bounds for the complexity under different classifications of the complexity degrees of theories. Consider some special classes of theories.

2.1.1. Computable countably categorical models.

For countably categorical theories the question is trivial: All countable models of a countably categorical theory are decidable if and only if the theory is decidable. Naturally, the situation becomes much more complicated if we require the computability condition.

Problem 3. Characterize countably categorical theories possessing computable models.

Lerman and Schmerl [**96**] presented a sufficient condition for an arithmetic countably categorical theory to have a constructive model. More precisely, they proved that if T is a countably categorical arithmetic theory such that the set of all sentences beginning with the existential quantifier and having $n+1$ groups of quantifiers of the same type (Σ_{n+1}–formulas) is Σ_n^0 for every n, then T has a constructive model.

It would be useful to weaken this condition, say, as follows: "the set of all sentences beginning with the existential quantifier and having $n+1$ groups of quantifiers of the same type (Σ_{n+1}–formulas) is Σ_{n+1}^0 for every n."

Problem 4. Whether a countable model is 1-computable under the Lerman–Schmerl condition?

Knight [**90**] generalized the result to the case of non-arithmetical countably categorical theories. However, none of the mentioned results solves the problem. We do not even know any example of a theory satisfying this sufficient condition for sufficiently large n. Knight conjectured the existence of arithmetical and non-arithmetical countably categorical theories with computable models. An answer to this conjecture is contained in the following assertions which develop general methods for constructing computable models from arithmetical models with preserving some model-theoretical properties.

Theorem 2.8 ([**68**]). *For every $n \geqslant 1$ there exists a countably categorical theory of Turing degree $\mathbf{0}^n$ possessing a computable model.*

Theorem 2.9 ([**43**]). *For every arithmetical Turing degree d there exists a countably categorical theory of Turing degree d possessing a computable model.*

The proof of the following assertion about the existence of a non-arithmetic countably categorical theory with computable models was based on the ideas of [**68**] and [**43**].

Theorem 2.10 (Fokina, Goncharov, Khoussainov). *There exists a countably categorical theory T with a computable model such that the Turing degree of T is non-arithmetical.*

Having an answer to the question in Problem 5 below, it would be possible to obtain a complete description of the Turing degrees of countably categorical theories possessing computable models.

Problem 5. Is it true that for every Turing degree $d \leqslant 0^{(\omega)}$ there exists a countably categorical theory of Turing degree d possessing a computable model?

2.1.2. Computable uncountably categorical models.

Here, we deal only with models of uncountably categorical theories.

Morley proved that a theory is categorical in uncountable power α if and only if the theory is categorical in uncountable power ω_1. Among typical examples of uncountably categorical theories, there are the theory of algebraically closed fields of fixed characteristic, the theory of vector spaces over a fixed countable field, the theory of the structure (ω, S), where S is the successor function on ω. Roughly speaking, all countable models of each of these theories can be listed into an $\omega + 1$ chain so that the first element is the prime model, the last element is the saturated model, and any two models are embedded each other. Apparently, it is one of the main structural properties of the class of models of an uncountably categorical theory.

Baldwin and Lachlan [**9**] showed that all models of an uncountably categorical theory T can be listed in the following chain of elementary embeddings:

$$\text{chain}\,(T) : \mathbf{A}_0 \preccurlyeq \mathbf{A}_1 \preccurlyeq \mathbf{A}_2 \preccurlyeq \ldots \mathbf{A}_\omega,$$

where $\mathbf{A}_0$ is the prime model of T, $\mathbf{A}_\omega$ is the saturated model of T, and every $\mathbf{A}_{i+1}$ is prime over $\mathbf{A}_i$.

Assume that a theory T is decidable. In the general case, the decidability of T does not imply the decidability of all models of T. However, the following important result on decidable models of T was established by Harrington and Khisamiev.

Theorem 2.11 ([**74, 83, 84**]). *Let T be an uncountably categorical theory. Then T is decidable if and only if T has a decidable model, which holds if and only if all models of T admit decidable presentations.*

The situation is similar to that for countably categorical theories. Theorem 2.11 mainly answers to the question about the existence of decidable models of uncountably categorical decidable theories. However, it does not clarify how to build computable models of uncountably categorical theories if the decidability assumption is omitted. Correspondingly, the following problem is actual.

Problem 6. Characterize uncountably categorical theories possessing computable models.

The case of uncountably categorical theories is more complicated. In general, the existence of a computable model of an uncountably categorical theory T does not imply that all models of T admit constructivizations. As was shown by Goncharov [**47**], there exists an uncountably categorical theory T such that only the prime model of T is computable.

Problem 7. Is it true that any countable model of a strongly minimal theory possessing a computable prime model is 0^2–computable?

Problem 8. Whether there exists an ω_1–categorical theory T such that the model $\mathbf{M_0}$ is computable, but any other model $\mathbf{M}_{n+1}$, $i = 2, 3, \ldots$, is not 0^i-computable?

It is remarkable that all known uncountably categorical theories possessing computable models were regarded as computable in the double jump of $\mathbf{0}$ recently. But, at present, some of such theories are not viewed as computable owing to the lowering of the complexity of models preserving the basic model-theoretic properties. New theories with a given arithmetic complexity were constructed by the method suggested in [**68**].

Theorem 2.12 ([**68**])**.** *There exist uncountably categorical theories T_n of Turing degree $\mathbf{0}^n$, $n > 2$, such that all their models admit constructivizations.*

Theorem 2.13 ([**43**])**.** *For every arithmetical Turing degree d there exists an uncountably categorical theory T of Turing degree d such that any countable model of T is isomorphic to a computable model.*

In the case of a special subclass of noncountably categorical theories (for example, strongly minimal theories with trivial pregeometry), an arithmetical bound holds for the complexity of such theories [**61**]. The sharpness of this bound was later proved by Khoussainov, Lempp, and Solomon. We consider this special case in more detail because of a new unexpected property allowing us to derive bounds for the complexity of such theories. A natural question arises: If the above results can be extended to the class of all strongly minimal theories or to some a sufficiently large subclass of such theories?

Definition 2.14. A formula $\varphi(x)$ is a *strongly minimal formula* of a complete theory T if for any model $\mathcal{M}$ of T, elements $\overline{b}$ of $\mathcal{M}$, and a formula $\psi(x, \overline{y})$ one of the sets $\{a|\mathcal{M} \vDash \psi(a, \overline{b})\&\varphi(a)\}$ or $\{a|\mathcal{M} \vDash \neg\psi(a, \overline{b})\&\varphi(a)\}$ is finite.

If $\varphi(x)$ is strongly minimal formula of a complete theory T, then for any model $\mathcal{M}$ of the theory T it is possible to define an operator of $\mathrm{cl}(X)$ from the set $P(\varphi(\mathcal{M}))$ of all subsets $\varphi(\mathcal{M})$ to $P(\varphi(\mathcal{M}))$.

Let $\varphi(\mathcal{M}) \rightleftharpoons \{a|\mathcal{M}\varphi(a)\}$, and let X be a subset of $\varphi(\mathcal{M})$. We set $\mathrm{cl}(X) \rightleftharpoons \{a \mid$ there exists a formula $\theta(x)$ such that $\mathcal{M} \models \theta(a)$ and the set $\theta(\mathcal{M})$ is finite$\}$. Let $\varphi(x)$ be a strongly minimal formula of a complete theory T, and let $\mathcal{M}$ be a model of T. The cardinality of any maximal independent subset Y of the model $\varphi(\mathcal{M})$ is called the *dimension* of the model $\mathcal{M}$ of T and is denoted by $\dim(\mathcal{M})$.

Baldwin and Lachlan [**9**] found a remarkable property of theories categorical in uncountable power, owing to which it becomes possible to clarify globally the structure of all models of such theories.

Theorem 2.15 ([**9**])**.** *Let T be a complete uncountably categorical theory. Then there exists a complete formula $\rho(\overline{z})$ and constants $\overline{c}$ such that $T^* \rightleftharpoons T \cup \{\rho(\overline{c})\}$ is a complete theory (principal expansion of T) and there exists a strongly minimal formula $\varphi(x)$ of the theory T^*.*

Theorem 2.16 ([**9**]). *Let* $\mathcal{M}_1$ *and* $\mathcal{M}_1$ *be models of a complete theory* T *with strongly minimal formula. If* $\dim(\mathcal{M}_1) = \dim(\mathcal{M}_2)$*, then the models* $\mathcal{M}_1$ *and* $\mathcal{M}_1$ *are isomorphic.*

Consider a natural subclass of theories categorical in uncountable power.

Definition 2.17. A theory T is *strongly minimal* if the formula $x = x$ is strongly minimal in T.

Definition 2.18. A model $\mathcal{M}$ is *strongly minimal* if the theory $TH(\mathcal{M})$ is strongly minimal.

Definition 2.19. We say that a strongly minimal theory T has *trivial pregeometry* if for any model $\mathcal{M}$ of T and any subset X of the universe of $\mathcal{M}$ the following equality holds: $\mathrm{cl}(X) = \cup_{a \in X} cl\{a\}$.

Theorem 2.20 ([**61**]). *Let* $\mathcal{M}$ *be a computable strongly minimal theory with trivial pregeometry. Then* $\mathrm{Th}(\mathcal{M})$ *forms a* $\mathbf{0}''$*-computable set of* $\mathcal{L}$*-sentences. Consequently, all countable models of* $\mathrm{Th}(\mathcal{M})$ *are* $\mathbf{0}''$*-decidable and, in particular, are* $\mathbf{0}''$*-computable.*

Theorem 2.21 ([**61**]). *For any strongly minimal theory* T *with trivial pregeometry the elementary diagram* $FD(\mathcal{M})$ *of any model* $\mathcal{M}$ *of* T *is a model complete* $\mathcal{L}_M$*-theory.*

Note that a model of a strongly minimal theory T with trivial pregeometry is not necessarily model complete in the original language (for example, $\langle \omega S \rangle$ is not model complete).

Proof of Theorem 2.21. Consider a model $\mathcal{M}_0$ of T. To simplify the notation, we write T^* instead of $\mathrm{Th}((\mathcal{M}_0)_{M_0})$. Let $\mathcal{L}^*$ be the language of T^* (i.e., $\mathcal{L}^* = \mathcal{L}_{M_0}$). Consider two models $\mathcal{M} \subseteq \mathcal{N}$ of T^* of size $\varkappa$, where $\varkappa > |M_0|$ is fixed. Since $\mathcal{M}$ and $\mathcal{N}$ are models of T^*, we can assume that $\mathcal{M}_0 \preceq \mathcal{M}$ and $\mathcal{M}_0 \preceq \mathcal{N}$. We need to show that $\mathcal{M} \preceq \mathcal{N}$. For this purpose, we use two standard facts, the so-called *non-finite cover property* and *finite satisfiability.*

The non-finite cover property of an uncountably categorical theory means that for all $\mathcal{L}^*$-formulas $\varphi(\overline{x}, \overline{y})$ there is a number k such that for any $\mathcal{M}^* \models T^*$ and $\overline{b}$ in M^* either $\varphi(\overline{b}, \mathcal{M}^*)$ is infinite or has size at most k. The number k depends only on φ and the partition of free variables in $(\overline{x}, \overline{y})$. Thus, one can use the quantifiers $\exists^{<\infty}$ and $\exists^{\infty}$, where $\exists^{<\infty}\overline{y}\varphi(\overline{x}, \overline{y})$ denotes $\exists^{\leqslant k}\overline{y}\varphi(\overline{x}, \overline{y})$ and $\exists^{\infty}\overline{y}\varphi(\overline{x}, \overline{y})$ denotes $\neg\exists^{<\infty}\overline{y}\varphi(\overline{x}, \overline{y})$.

The following assertion is an immediate consequence of the pigeon-hole principle.

Lemma 2.22. *If* $\mathcal{N} \models \exists^{\infty}\overline{y}\varphi(\overline{b}, \overline{y})$ *and* $\mathrm{lg}(\overline{y}) = k + 1$, *then there is a partition of* $\overline{y}$ *in* $w\overline{z}$ *with* $\mathrm{lg}(w) = 1$ *and* $\mathrm{lg}(\overline{z}) = k$ *such that* $\mathcal{N} \models \exists^{\infty}w\exists\overline{z}\varphi(\overline{b}, w, \overline{z})$.

The following general fact, referred to as the *finite satisfiability*, asserts that if $\mathcal{M}_0 \preceq \mathcal{N}$ are models of a stable theory and $\mathcal{N} \models \varphi(\overline{b}, \overline{c})$ for some $\mathcal{L}_{M_0}$-formula and some $\overline{b}, \overline{c}$ in N that are independent (i.e., do not fork over M_0), then there is $\overline{a}$ in M_0 such that $\mathcal{N} \models \varphi(\overline{a}, \overline{c})$. This fact is obvious because, in a stable theory, every complete type over a model is definable. We formulate this assertion in a special case of strongly minimal theories.

Lemma 2.23. *Suppose that* $\mathcal{M}_0 \preceq \mathcal{N}$ *are models of a strongly minimal theory and* $\overline{b}$, $\overline{c}$ *are tuples in* N *such that* $\mathrm{acl}(M_0\overline{b}) \cap \mathrm{acl}(M_0\overline{c}) = M_0$. *If* $\mathcal{N} \models \varphi(\overline{b}, \overline{c})$ *for any* $\mathcal{L}_{M_0}$*-formula* φ, *then there is* $\overline{a}$ *in* M_0 *such that* $\mathcal{N} \models \varphi(\overline{a}, \overline{c})$.

We also need the following notion.

Definition 2.24. An $\mathcal{L}^*$-formula $\varphi(\overline{x})$ is *absolute* if for all $\overline{b}$ in M we have $\mathcal{M} \models \varphi(\overline{b})$ if and only if $\mathcal{N} \models \varphi(\overline{b})$.

To complete the proof of $\mathcal{M} \preceq \mathcal{N}$, it suffices to show that any $\mathcal{L}^*$-formula is absolute. It is obvious that every quantifier-free $\mathcal{L}^*$-formula is absolute and a family of absolute formulas is closed under the Boolean operations. Thus, to obtain the model completeness of T^*, it suffices to show that if an $\mathcal{L}^*$-formula $\varphi(\overline{x}, y)$ is absolute, then $\exists y \varphi(\overline{x}, y)$ is also absolute.

Definition 2.25. An $\mathcal{L}^*$-formula $\varphi(\overline{x}, \overline{y})$ is said to be an (n, m)-*formula* if $\lg(\overline{x}) = n$ and $\lg(\overline{y}) = m$. We identify three interrelated families of statements:

- $A_{n,m}$ is the statement that for all absolute (n, m)-formulas $\varphi(\overline{x}, \overline{y})$ the formula $\exists^{<\infty} \overline{y} \varphi(\overline{x}, \overline{y})$ is absolute,
- $B_{n,m}$ is the statement that for all absolute (n, m)-formulas $\varphi(\overline{x}, \overline{y})$, if $\overline{b} \in M^n$ and $\mathcal{N} \models \exists^{<\infty} \overline{y} \varphi(\overline{b}, \overline{y})$, then $\varphi(\overline{b}, \mathcal{N}) = \varphi(\overline{b}, \mathcal{M})$, i.e., every realization of $\varphi(\overline{b}, \overline{y})$ in N^m is an element of M^m,
- $C_{n,m}$ is the statement that for all absolute (n, m)-formulas $\varphi(\overline{x}, \overline{y})$ the formula $\exists \overline{y} \varphi(\overline{x}, \overline{y})$ is absolute.

By the above arguments, to prove the model completeness of T^*, it suffices to show that $C_{n,1}$ holds for all $n \in \omega$.

It is obvious that each of three statements in Definition 2.25 is preserved if subscripts decrease (for example, $B_{n,m}$ implies $B_{n',m'}$ for all $n' \leqslant n$ and all $m' \leqslant m$).

Lemma 2.26. *The following assertions hold*:

(a) $B_{n,m}$ *implies* $C_{n,m}$ *for all* $n, m \in \omega$,

(b) $B_{n,m}$ *implies* $A_{n,m+1}$ *for all* $n, m \in \omega$,

(c) $B_{1,m}$ (*consequently,* $B_{0,m}$) *holds for all* $m \in \omega$.

Proposition 2.27. $B_{n,m+1}$ *and* $A_{n+1,m}$ *imply* $B_{n+1,m}$ *for all* $n, m \in \omega$.

As was already noted, T^* is model complete if $\mathcal{M} \preceq \mathcal{N}$.

We show that $B_{n,m}$ holds for all $n, m \in \omega$. For this purpose, we show by induction on n that $B_{n,m}$ holds for all n. Note that $B_{1,m}$ holds for all $m \in \omega$. We fix $n \geqslant 1$ and assume that $B_{n,m}$ holds for all m. Let us prove that $B_{n+1,m}$ holds for all m by induction on m. It is obvious that $B_{n+1,0}$ holds. Assume that $B_{n+1,m}$ holds for some m. Then $B_{n,m+2}$ holds by the induction assumption and $A_{n+1,m+1}$ holds since $B_{n+1,m}$ holds. Thus, $B_{n+1,m+1}$ holds by Proposition 2.27, and the induction procedure is complete.

By Lemma 2.26 (a), $C_{n,m}$ holds for all $n, m \in \omega$. In particular, $C_{n,1}$ holds for all $n \in \omega$. This means that the family of absolute $\mathcal{L}^*$-formulas is closed under the existential quantification. As is known, the family of absolute $\mathcal{L}^*$-formulas contains quantifier-free formulas and is closed under Boolean connectives. Hence every $\mathcal{L}^*$-formula is absolute. Thus, $\mathcal{M} \preceq \mathcal{N}$, as required. $\square$

For a structure $\mathcal{M}$ we denote by $\mathrm{Th}_{\forall\exists}(\mathcal{M}_M)$ the set of all $\forall\exists$-sentences $\sigma \in \mathrm{Th}(\mathcal{M}_M)$ (in the language $\mathcal{L}_M$).

Lemma 2.28. *If the elementary diagram of a structure* $\mathcal{M}$ *is model complete, then* $\mathrm{Th}_{\forall\exists}(\mathcal{M}_M)$ *and* $\mathrm{Th}(\mathcal{M}_M)$ *are equivalent* $\mathcal{L}_M$*-theories.*

PROOF. It is obvious that $\mathrm{Th}_{\forall\exists}(\mathcal{M}_M)$ is a subset of $\mathrm{Th}\,(\mathcal{M}_M)$. On the other hand, if $\mathrm{Th}(\mathcal{M}_M)$ is model complete, then it is $\forall\exists$-axiomatizable in the language $\mathcal{L}_M$. But any $\forall\exists$-axiomatization of $\mathrm{Th}(\mathcal{M}_M)$ is a subset of $\mathrm{Th}_{\forall\exists}(\mathcal{M}_M)$. $\square$

It turns out that the model completeness of the elementary diagram of a structure $\mathcal{M}$ is a property of the theory of $\mathcal{M}$. To prove this fact, we introduce the following definition.

Definition 2.29. An existential $\mathcal{L}$-formula $\psi(\overline{x}, \overline{y})$ and an $\forall\exists$-formula of $\mathcal{L}$ form a *linked pair* (for T) if $T \models \exists\overline{y}\theta(\overline{y})$ and $T \models \forall\overline{y}\forall\overline{y}'\forall\overline{x}(\theta(\overline{y}) \wedge \theta(\overline{y}') \wedge \psi(\overline{x}, \overline{y}) \to \psi(\overline{x}, \overline{y}'))$.

Proposition 2.30 ([**67**])**.** *The elementary diagram of an* $\mathcal{L}$-*structure* $\mathcal{M}$ *is model complete if and only if for every* $\mathcal{L}$*-formula* $\varphi(\overline{x})$ *there is a linked pair* (θ, ψ) *such that* $\mathcal{M} \models \exists\overline{y}\theta(\overline{y})$ *and*

$$\mathcal{M} \models \forall\overline{y}(\theta(\overline{y}) \to \forall\overline{x}[\varphi(\overline{x}) \leftrightarrow \psi(\overline{x}, \overline{y})]). \qquad (*)$$

PROOF. Assume that the elementary diagram of $\mathcal{M}$ is model complete. Fix an $\mathcal{L}$-formula $\varphi(\overline{x})$. Since $\mathrm{Th}(\mathcal{M}_M)$ is model complete, there is an existential $\mathcal{L}$-formula $\psi(\overline{x}, \overline{y})$ and a tuple $\overline{b}$ in M

such that $\mathcal{M} \models \delta(\bar{b})$, where $\delta(\overline{y}) := \forall\overline{x}[\varphi(\overline{x}) \leftrightarrow \psi(\overline{x},\overline{y})]$. Hence $\mathrm{Th}_{\forall\exists}(\mathcal{M}_M) \models \delta(\bar{b})$ in view of Lemma 2.28. By compactness, there is an $\forall\exists$-formula $\theta(\overline{y})$ in $\mathcal{L}$ such that $\theta(\bar{b}) \in \mathrm{Th}_{\forall\exists}(\mathcal{M}_M)$ and $\{\theta(\bar{b})\} \models \delta(\bar{b})$. (Without loss of generality, by padding δ, we can assume that any constant symbol appearing in θ also appears in δ.)

Conversely, assume that the right-hand side of (∗) holds. Fix an $\mathcal{L}_M$-formula $\varphi(\overline{x},\overline{a})$, where $\varphi(\overline{x},\overline{z})$ is an $\mathcal{L}$-formula and $\overline{a}$ belongs to M. Choose $\theta(\overline{y})$ and $\psi(\overline{x},\overline{z},\overline{y})$ corresponding to $\varphi(\overline{x},\overline{z})$. Let $\bar{b}$ be any realization of $\theta(\overline{y})$ in M. Then

$$\mathcal{M} \models \forall\overline{x}\forall\overline{z}[\varphi(\overline{x},\overline{z}) \leftrightarrow \psi(\overline{x},\overline{z},\bar{b})].$$

In particular, $\mathcal{M} \models \forall\overline{x}[\varphi(\overline{x},\overline{a}) \leftrightarrow \psi(\overline{x},\overline{a},\bar{b})]$. Thus, every $\mathcal{L}_M$-formula is $\mathrm{Th}(\mathcal{M}_M)$-equivalent to an existential $\mathcal{L}_M$-formula, which implies the model completeness of $\mathrm{Th}(\mathcal{M}_M)$. □

Corollary 2.31 ([**61**]). *If $\mathcal{M}$ and $\mathcal{N}$ are elementarily equivalent $\mathcal{L}$-structures then the elementary diagram of $\mathcal{M}$ is model complete if and only if the elementary diagram of $\mathcal{N}$ is model complete. In particular, if T is a complete theory and the elementary diagram of some model of T is model complete, then the elementary diagram of every model of T is model complete.*

Proposition 2.32 ([**61**]). *Let T be an $\mathcal{L}$-theory such that the elementary diagram of every model of T is model complete. Then T is $\exists\forall\exists$-axiomatizable.*

Proof. Let $\mathcal{M}$ be an arbitrary model of T. Then $\mathrm{Th}_{\forall\exists}(\mathcal{M}_M)$ implies σ. Therefore, there is a conjunction ψ of $\forall\exists$-sentences of $\mathcal{L}_M$ that logically implies σ. Since none of the extra constant symbols in M appears in σ, we can existentially quantify out these constant symbols and obtain a formula of the desired complexity which logically implies σ. □

The following assertion immediately follows from Theorem 2.21 and Proposition 2.32.

Corollary 2.33 ([**61**]). *Every strongly minimal theory with trivial pregeometry is $\exists\forall\exists$-axiomatizable.*

PROOF. By Theorem 2.21, $\mathrm{Th}(\mathcal{M}_M)$ is model complete and, consequently, $\forall\exists$-axiomatizable. Then $\mathrm{Th}_{\forall\exists}(\mathcal{M}_M)$ is a $\mathbf{0}''$-computable set of formulas which axiomatizes $\mathrm{Th}(\mathcal{M}_M)$, and so $\mathrm{Th}(\mathcal{M}_M)$ and its reduct $\mathrm{Th}(\mathcal{M})$ are $\mathbf{0}''$-computable sets of formulas as well. By relativisation theorem due to Harrington [**74**] and Khisamiev [**83**], any countable model of the theory $\mathrm{Th}(\mathcal{M})$ decidable relative to $\mathbf{0}''$, is $\mathbf{0}''$-computable. □

The following question still remains open.

Problem 9. Is the assertion of Corollary 2.33 remains valid for an arbitrary strongly minimal theory?

Recently, Khoussainov, Lempp, and Solomon proved the following result.

Theorem 2.34 ([**86**]). *There exists an uncountably categorical strongly minimal theory T with trivial pregeometry possessing a computable prime model such that all other models has the complexity of Turing degree $\mathbf{0}^2$.*

It is of interest to generalize the result of [**86**].

Problem 10. Whether there are examples of uncountably categorical strongly minimal theories T_n possessing a computable models such that other models have the complexity of Turing degree $\mathbf{0}^{n+3}$, $n \geqslant 0$?

The following conjecture was suggested by S. Lempp.

Conjecture 2.35. *An uncountably categorical theory possessing a computable model is arithmetical.*

Note that the above result of Harrington [**74**] and Khisamiev [**83, 84**] can be relativized to show that if T is uncountably categorical and arithmetic, then all models of T admit arithmetic numbering. If Conjecture 2.35 could be confirmed, this would mean that all models of an uncountably categorical arithmetic theory admit arithmetic numberings. To confute Conjecture 2.35, it suffices to construct a theory with the properties listed in the following problem.

Problem 11. Whether there exists an uncountably categorical theory T with models $\mathbf{A}_0 \preceq \mathbf{A}_1 \preceq \ldots \preceq \mathbf{A}_\omega$ such that $\mathbf{A}_0$ has a constructivization, and every $\mathbf{A}_{i+1}$, $i \in \omega$, has $\mathbf{0}^{i+1}$–constructivization, but not $\mathbf{0}^{i}$–constructivization?

2.1.3. Computable models of Ehrenfeucht theories.

In the case of countably categorical theories, the question about bounds for theories with decidable models is trivial. All countable models of such a theory are decidable if and only if the theory is decidable. The Ehrenfeucht theories are close to the countably categorical theories. Recall that a theory is a called an *Ehrenfeucht theory* if it has finitely many countable models. Naturally, the question is much more complicated if the computablity condition is required.

Peretyat'kin [**135**] proved that a prime model of an Ehrenfeucht theory is decidable. Lachlan constructed the first example of an Ehrenfeucht theory possessing six countable models such that only the prime model is decidable. Later, such examples for any $n \geqslant 3$ were constructed by Peretyat'kin.

However, there are still many open questions concerning theories possessing decidable models. First of all, we recall the well-known Morley problem.

Problem 12 ([**125**])**.** Is it true that any countable model of any Ehrenfeucht theory with computable types is decidable?

The following weakened version of the Morley problem is also of interest.

Problem 13. Is it true that any countable models of any Ehrenfeucht theory with computable types is arithmetical?

Ash and Millar [**7**] proved that all models of hyperarithmetical Ehrenfeucht theories are decidable in hyperarithmetical degrees. Millar [**107**] and Reed [**138**] constructed examples of decidable Ehrenfeucht theories with a given complexity for some nonprincipal type and, consequently, with a given hyperarithmetical complexity of their countably saturated model and some other models. The following problem arises in a natural way.

Problem 14. Is it true that all countable models of any Ehrenfeucht theory with arithmetical types are decidable (computable) relative to some arithmetical Turing degree?

Note that all homogeneous models of an Ehrenfeucht theory with decidable (arithmetic) types are decidable (relative to some arithmetic Turing degree) [**40**]. This fact immediately follows from the decidablity theorem for homogeneous models with countable family of types realized there and countable family of all decidable types of theories of this model [**40**].

The above discussion suggests the following strategy.

Problem 15. Show that all almost homogeneous models of an Ehrenfeucht theory with decidable (arithmetical) types are decidable (relative to some arithmetical type).

The following weaker property is also of interest.

Problem 16. Show that all almost homogeneous models of an Ehrenfeucht theory with decidable (arithmetical) types are computable (relative to some arithmetical type).

Recall that a model is said to be *almost homogeneous* if it is homogeneous in some enrichment by constants for a finite collection of its elements.

It is remarkable that, in all known examples of Ehrenfeucht theories, all countable models are almost homogeneous. The assertion that all countable models of any Ehrenfeucht theory are almost homogeneous (if it is true) could be helpful for resolving the problems. On the other hand, a counterexample could open a door to the negative solution of the Morley problem.

To consider computable models of such theories, we start with the following open question.

Problem 17. Characterize Ehrenfeucht theories possessing computable models.

It is of interest to generalize the result of Lerman and Schmerl [**96**] for countably categorical *arithmetic* theories to the case of Ehrenfeucht theories. The same question can be considered regarding the Knight theorem [**90**] for non-arithmetic Ehrenfeucht theories.

The above results on the complexity for countably categorical theories yield the following assertions.

Corollary 2.36 ([**68**])**.** *For every* $n \geqslant 1$ *there exists am Ehrenfeucht theory of Turing degree* $\mathbf{0}^n$ *that has a computable model.*

Corollary 2.37 ([**43**])**.** *For every arithmetical Turing degree d there exists an Ehrenfeucht theory of Turing degree d that has a computable model.*

We complete this section with the following result asserting the existence of computable models of non-arithmetic countably categorical theories.

Theorem 2.38 (Fokina, Goncharov, Khoussainov)**.** *There exists an Ehrenfeucht theory* T *with a computable model and the Turing degree* T *is non-arithmetical.*

3. Structure Complexity of Computable Models

In this section, we discuss necessary conditions on the structure of computable models from the point of view of the model-theoretic complexity. For this purpose, we choose a language with infinite disjunctions and conjunctions. Then every countable model can be described up to an isomorphism and the number of necessary infinite disjunctions and conjunctions determines the ordinal level of the structure complexity of the model. Using the theory of admissible sets, it is possible to obtain an upper bound for the complexity of a computable model. The sharpness of the bound and the realizability of all less complexities play an important role for describing structural properties of computable models. We present two methods based on the Scott rank and on the Barwise rank.

3.1. Definability of computable models

Recall that the Scott rank is a measure of the model-theoretic complexity. This term came from the Scott isomorphism theorem [**144**].

Theorem 3.1 (the Scott isomorphism theorem)**.** *For every countable structure $\mathcal{A}$ (for a countable language L) there is an $L_{\omega_1\omega}$ sentence whose countable models are isomorphic copies of $\mathcal{A}$.*

To prove this assertion, Scott assigned countable ordinals to tuples in $\mathcal{A}$ and to $\mathcal{A}$ itself. There are several different definitions of the *Scott rank*.

Let $\overline{a}$ and $\overline{b}$ be tuples in $\mathcal{A}$.

- We write $\overline{a} \equiv^0 \overline{b}$ if $\overline{a}$ and $\overline{b}$ satisfy the same quantifier-free formulas.
- Let $\alpha > 0$. We write $\overline{a} \equiv^\alpha \overline{b}$ if for all $\beta < \alpha$ and $\overline{c}$ there exists $\overline{d}$ and for every $\overline{d}$ there exists $\overline{c}$ such that $\overline{a}, \overline{c} \equiv^\beta \overline{b}, \overline{d}$.

Definition 3.2. The *Scott rank of a tuple* $\overline{a}$ in $\mathcal{A}$ is the least β such that for all $\overline{b}$ from $\overline{a} \equiv^\beta \overline{b}$ it follows that $(\mathcal{A}, \overline{a}) \cong (\mathcal{A}, \overline{b})$.

Definition 3.3. The *Scott rank* $\mathrm{SR}\,(\mathcal{A})$ *of* $\mathcal{A}$ is the least ordinal α greater than the ranks of all tuples in $\mathcal{A}$.

Example 3.4. If $\mathcal{A}$ is an ordering of type ω, then $\mathrm{SR}(\mathcal{A}) = 2$. We have $\overline{a} \equiv^0 \overline{b}$ if $\overline{a}$ and $\overline{b}$ are ordered in the same way. We have $\overline{a} \equiv^1 \overline{b}$ if the corresponding intervals (before the first element and between successive elements) are of the same size, and this fact is enough to assure an isomorphism. Hence the tuples have Scott rank 1 and the ordering has Scott rank 2.

3.2. The Kleene notation system $\mathcal{O}$

As in the general algorithm theory, for constructing models of given complexity and estimating the complexity an important role is played by computable ordinals and the Kleene notation system $\mathcal{O}$ (cf. [**140**]) for all computable ordinals. The least ordinal having no notation in the Kleene system is referred to as the *Church–Kleene ordinal* and is denoted by ω_1^{CK}. It is easy to check that it is the least noncomputable ordinal.

Recall that the Kleene notation system consists of a set $\mathcal{O}$ of notations equipped with a partial ordering $<_{\mathcal{O}}$. The ordinal 0 has notation 1. If a is the notation of α, then 2^a is the notation of $\alpha + 1$. Then $a <_{\mathcal{O}} 2^a$, and $b <_{\mathcal{O}} a$ implies $b <_{\mathcal{O}} 2^a$.

Suppose that α is a limit ordinal. If φ_e is a total function providing notations for an increasing sequence of ordinals with limit α, then $3 \cdot 5^e$ is the notation of α. For all n we have $\varphi_e(n) <_{\mathcal{O}} 3 \cdot 5^e$, and $b <_{\mathcal{O}} \varphi_e(n)$ implies $b <_{\mathcal{O}} 3 \cdot 5^e$. The set $\mathcal{O}$ is Π_1^1 complete.

3.3. Computable infinitary formulas

For any notation from the Kleene notation system $\mathcal{O}$ it is possible to introduce infinitary formulas which are used to describe computable structures. Roughly speaking, we will define infinitary formulas on a fixed level where the disjunctions and conjunctions of computable formulas from previous levels are computable. They are essentially the same as the formulas in the least admissible fragment of $L_{\omega_1\omega}$.

We may classify computable infinitary formulas as *computable* Σ_α, or *computable* Π_α, for various computable ordinals α. We have the useful fact that in a computable structure, a relation defined by a computable Σ_α (or computable Π_α) formula will be Σ^0_α (or Π^0_α). To illustrate the expressive power of computable infinitary formulas, we note that there is a natural computable Π_2 sentence characterizing the class of Abelian p-groups. For every computable ordinal α there is a computable $\Pi_{2\alpha}$ formula saying that the height is at least $\omega{\cdot}\alpha$ for an element of an Abelian p-group.

The following theorem presents a well-known useful version of the compactness theorem for computable infinitary formulas.

Theorem 3.5 (the Barwise–Kreisel compactness theorem)**.** *Let* Γ *be a* Π^1_1 *set of computable infinitary sentences. If every* Δ^1_1 *subset of* Γ *has a model, then* Γ *also has a model.*

Theorem 3.5 can be used for obtaining computable structures and special computable sequences of computable structures.

Corollary 3.6. *Let* Γ *be a* Π^1_1 *set of computable infinitary sentences. If every* Δ^1_1 *subset has a computable model, then* Γ *also has a computable model.*

Corollary 3.6 can be applied uniformly to Π^1_1 sets of computable infinitary sentences.

The following two assertions demonstrate the expressive power of computable infinitary formulas.

Corollary 3.7. *If* $\mathcal{A}$ *and* $\mathcal{B}$ *are computable structures satisfying the same computable infinitary sentences, then* $\mathcal{A} \cong \mathcal{B}$.

Corollary 3.8. *Suppose that* $\overline{a}$ *and* $\overline{b}$ *are tuples satisfying the same computable infinitary formulas in a computable structure* $\mathcal{A}$. *Then there is an automorphism of* $\mathcal{A}$ *sending* $\overline{a}$ *to* $\overline{b}$.

Theorem 3.5 and Corollaries 3.6–3.8 are well known and may be found, for example, in [**5**] (cf. also [**66**]).

3.4. Computable rank

Definition 3.9. The *computable rank* $R^c(\mathcal{A})$ of a structure $\mathcal{A}$ is the first ordinal α such that for all tuples $\overline{a}$, $\overline{b}$ in $\mathcal{A}$ of the same length the following holds: if for all $\beta < \alpha$ all computable Π_β formulas true of $\overline{a}$ are also true of $\overline{b}$, then there is an automorphism sending $\overline{a}$ to $\overline{b}$.

By Corollary 3.7, if $\mathcal{A}$ is a hyperarithmetical structure, then $R^c(\mathcal{A}) \leqslant \omega_1^{CK}$. In this case, Definition 3.9 can be formulated as follows: The *computable rank* is the first ordinal α such that for all tuples $\overline{a}$ and $\overline{b}$ of the same length the following holds: if $\overline{a}$ and $\overline{b}$ satisfy the same computable Π_β formulas for $\beta < \alpha$, then they satisfy the same computable Π_α formulas.

Proposition 3.10. *For any computable language L and computable ordinal α (or any notation) there exists a computable infinitary sentence saying that $R^c(\mathcal{A}) \geqslant \alpha$ for an L-structure $\mathcal{A}$.*

Note that the notion of computable rank essentially differs from that of Scott rank. Nevertheless, in the case of a hyperarithmetical structure $\mathcal{A}$, the computable rank is a computable ordinal just as the Scott rank is computable. If $R^c(\mathcal{A})$ is computable, then $\mathcal{A}$ has a computable Scott sentence. The converse assertion is also true.

Proposition 3.11 (J. Millar[1]). *Suppose that $\mathcal{A}$ is a hyperarithmetical and $R^c(\mathcal{A}) = \omega_1^{CK}$. If ψ is a computable infinitary sentence true in $\mathcal{A}$, then ψ is also true in some hyperarithmetical $\mathcal{B} \ncong \mathcal{A}$.*

Sketch of Proof. Let $\mathcal{A}^*$ be an expansion of $\mathcal{A}$ with a predicate R_φ for every computable infinitary formula φ, up to complexity α. Since the rank of $\mathcal{A}$ is not computable, $\mathcal{A}^*$ is not homogeneous. Therefore, there is some tuple $\overline{a}$ realizing a non-principal type in $\mathcal{A}^*$. We produce a hyperarithmetical model $\mathcal{B}^*$ of the

[1]Private communication.

elementary first order theory of $\mathcal{A}^*$ omitting the type of $\overline{a}$ and satisfying ψ. To guarantee that ψ is true, we make sure that for all subformulas $\varphi(\overline{u})$

$$\mathcal{B}^* \models \forall \overline{u}\, [\varphi(\overline{u}) \leftrightarrow R_\varphi(\overline{u})].$$

If $\varphi(\overline{u})$ is the disjunction of $\varphi_i(\overline{u})$, we need to omit the type consisting of $R_\varphi(\overline{u})$ and the formulas $\neg R_{\varphi_i}(\overline{u})$. If $\varphi(\overline{u})$ is the conjunction of $\varphi_i(\overline{u})$, we need to omit the type consisting of $\neg R_\varphi(\overline{u})$ and the formulas $R_{\varphi_i}(\overline{u})$. □

3.5. Rank and isomorphisms

We revise the Scott isomorphism theorem by looking for isomorphisms of bounded complexity.

Definition 3.12. Let α be a computable ordinal. A *formally* Σ^0_α *Scott family* is a c.e. Scott family Φ made up of computable Σ_α formulas, possibly with a fixed tuple of parameters.

Definition 3.13. A computable structure $\mathcal{A}$ is Δ^0_α *categorical* if $\mathcal{A} \cong_{\Delta^0_\alpha} \mathcal{B}$ for every computable copy $\mathcal{B}$.

Theorem 3.14 (Ash, Goncharov)**.** *Suppose that $\mathcal{A}$ is computable. If $\mathcal{A}$ has a formally Σ^0_α Scott family, then it is Δ^0_α categorical. With some added effectiveness on one copy of $\mathcal{A}$, the converse holds.*

This assertion was proved in [**45, 46**] in the computable case and in [**1**] in the general case.

Proposition 3.15. *Let α be a computable ordinal. For a given index of a computable structure $\mathcal{A}$ such that $R^c(\mathcal{A}) = \alpha$ there is an index of a formally $\Sigma^0_{\alpha+2}$ Scott family for $\mathcal{A}$ without parameters.*

Suppose that K is a class of structures such that there is a computable bound on $R^c(\mathcal{A})$ for $\mathcal{A} \in K^c$. Proposition 3.15 asserts that for a given index of $\mathcal{A} \in K^c$ we can find an index of a Scott family consisting of formulas of bounded complexity. Then we can pass to a computable infinitary Scott sentence.

These results yield a bound on the Scott ranks for computable structures [**127**]. There are examples of computable structures having various computable Scott ranks and familiar structures (for example, the Harrison ordering) with Scott rank $\omega_1^{CK} + 1$ [**70**]. Makkai [**99**] constructed a structure of Scott rank ω_1^{CK} which can be made computable [**68**] and simplified it so that it is just a tree [**14**]. As was shown in [**13**], it is possible to construct further computable structures of Scott rank ω_1^{CK} in the classes of undirected graphs, fields of any characteristic, and linear orderings. These results give us interesting examples of computable structures with different complexity of the isomorphism problem for different computable representations.

Proposition 3.16. *Let $\mathcal{A}$ be a computable structure. Then* $\mathrm{SR}\,(\mathcal{A}) \leqslant \omega_1^{CK} + 1$.

The further properties of computable structures are listed in the following assertion.

Proposition 3.17. *Let $\mathcal{A}$ be a computable structure. Then*

(1) $\mathrm{SR}\,(\mathcal{A}) < \omega_1^{CK}$ *if there is a computable ordinal β such that the orbits of all tuples are defined by computable Π_β formulas,*

(2) $\mathrm{SR}\,(\mathcal{A}) = \omega_1^{CK}$ *if the orbits of all tuples are defined by computable infinitary formulas, but there is no computable bound on the complexity of these formulas,*

(3) $\mathrm{SR}\,(\mathcal{A}) = \omega_1^{CK} + 1$ *if there is some tuple whose orbit is not defined by any computable infinitary formula.*

The low Scott rank is associated with simple Scott sentences. Recall that a *Scott sentence* for $\mathcal{A}$ is a sentence whose countable models are just the isomorphic copies of $\mathcal{A}$ (as in the Scott isomorphism theorem).

Theorem 3.18 ([**127, 128**]). *Let $\mathcal{A}$ be a computable structure. Then* $\mathrm{SR}(\mathcal{A})$ *is computable if and only if $\mathcal{A}$ has a computable infinitary Scott sentence.*

Corollary 3.19. *Let Γ be a Π^1_1-set of computable infinitary sentences. If every Δ^1_1-set $\Gamma' \subseteq \Gamma$ has a computable model, then Γ has a computable model.*

Proposition 3.20 ([**139**]). *Suppose that $\mathcal{A}$ is a hyperarithmetical structure. Let Γ be a Π^1_1-set of computable infinitary sentences in a finite expansion of the language of $\mathcal{A}$. Suppose that for each Δ^1_1-set $\Gamma' \subseteq \Gamma$ the structure $\mathcal{A}$ can be expanded to a model of Γ'. Then $\mathcal{A}$ can be expanded to a model of Γ.*

Corollary 3.21. *Let $\mathcal{A}$ be a hyperarithmetical structure. If $\overline{a}$ and $\overline{b}$ are tuples in $\mathcal{A}$ satisfying the same computable infinitary formulas, then there is an automorphism of $\mathcal{A}$ sending $\overline{a}$ to $\overline{b}$.*

Consider three different types of Scott rank for computable models described in Proposition 3.17 that are realized in classical algebras and models.

In the case $\mathrm{SR}(\mathcal{A}) < \omega_1^{CK}$, the following structural property of computable models holds.

Proposition 3.22. *All computable members of the following structures have a computable Scott rank*:

- *well orderings,*
- *superatomic Boolean algebras,*
- *reduced Abelian p-groups.*

An interesting class of models is formed by computable models with $\mathrm{SR}(\mathcal{A}) = \omega_1^{CK} + 1$. There are well-known examples of computable structures of Scott rank $\omega_1^{CK} + 1$. Harrison showed that there is a computable ordering of type $\omega_1^{CK}(1 + \eta)$, called the *Harrison ordering*, which gives rise to some other computable structures with similar properties. The *Harrison Boolean algebra* is the interval algebra of the Harrison ordering. The *Harrison*

Abelian p-group has length ω_1^{CK}, with all infinite Ulm invariants and a divisible part of infinite dimension.

Proposition 3.23. *The Harrison ordering, Harrison Boolean algebra, and Harrison Abelian p-groups have Scott rank* $\omega_1^{CK}+1$.

It was unexpected that there exist models with $\mathrm{SR}(\mathcal{A}) = \omega_1^{CK}$.

In the case of the Scott rank ω_1^{CK}, it is not easy to find computable examples. An arithmetical example was constructed by Makkai.

Theorem 3.24 ([**99**])**.** *There is an arithmetical structure* $\mathcal{A}$ *of rank* ω_1^{CK}.

Models $\mathcal{A}$ of Scott rank $\mathrm{SR}(\mathcal{A}) = \omega_1^{CK}$ will be referred to as *Makkai models.*

In the Makkai example, in contrast to the Harrison ordering, the set of computable infinitary sentences that are true in the structure is $\aleph_0$ categorical. Hence the conjunction of these sentences is a Scott sentence for the structure. The following assertion can be proved on the basis of the results of [**68**] and [**92**].

Theorem 3.25. *There exists a computable structure of Scott rank* ω_1^{CK}.

As was proved in [**14**], there exists a computable tree of Scott rank ω_1^{CK}. This construction may be employed in other situations. The authors of [**14**] used the idea to take trees as Knight–Millar Trees and add a homogeneity property. In more detail, let $\mathcal{T}$ be a subtree of $\omega^{<\omega}$. We have a top node $\varnothing$. We will define the *tree rank* for $\sigma \in \mathcal{T}$ and then for $\mathcal{T}$. Below, we use the notation $rk(\sigma)$, $rk(\mathcal{T})$.

- $rk(\sigma) = 0$ if σ is terminal.
- For $\alpha > 0$, $rk(\sigma) = \alpha$ if all successors of σ have ordinal rank, and α is the first ordinal greater than these ordinals.

- $rk(\sigma) = \infty$ if σ does not have ordinal rank.

  We set $rk(\mathcal{T}) = rk(\varnothing)$.

  REMARK. $rk(\sigma) = \infty$ if and only if σ extends to a path.

If $\mathcal{T}$ is a tree, we denote by $\mathcal{T}_n$ the set of elements at level n in the tree, i.e., $\mathcal{T}_n = \mathcal{T} \cap \omega^n$.

Definition 3.26. A tree $\mathcal{T}$ is *thin* if for all n the set of ordinal ranks of elements of $\mathcal{T}_n$ has order type at most $\omega \cdot n$.

This definition is used as follows. If $\mathcal{T}$ is a computable thin tree, then for every n there is a computable α_n such that for all $\sigma \in \mathcal{T}_n$ from $rk(\sigma) \geqslant \alpha_n$ it follows that $rk(\sigma) = \infty$.

Theorem 3.27 ([**92**])**.** *The following assertions hold.*

(1) *There exists a computable thin tree $\mathcal{T}$ with a path but no hyperarithmetical path.*

(2) *If $\mathcal{T}$ is a computable thin tree with a path but no hyperarithmetical path, then $\mathcal{A}(\mathcal{T})$ is a computable structure of Scott rank ω_1^{CK}.*

A computable tree of Scott rank ω_1^{CK} was constructed in [**14**]. This tree satisfies some conditions from [**92**] and the following homogeneity property.

Definition 3.28. A tree $\mathcal{T}$ is *rank-homogeneous* if for all n the following conditions are satisfied:

- for all $\sigma \in \mathcal{T}_n$ and computable α, if there exists $\tau \in \mathcal{T}_{n+1}$ such that $rk(\tau) = \alpha < rk(\sigma)$, then σ has infinitely many successors σ' with $rk(\sigma') = \alpha$,
- for all $\sigma \in \mathcal{T}_n$, if $rk(\sigma) = \infty$, then σ has infinitely many successors σ' with $rk(\sigma') = \infty$.

REMARK. Suppose that $\mathcal{T}$ and $\mathcal{T}'$ are rank-homogeneous trees and for all n there is an element in T_n of rank $\alpha \in \mathrm{Ord} \cup \{\infty\}$ if and only if there is an element in T_n' of rank α. Then $\mathcal{T} \cong \mathcal{T}'$.

In [**14**], the construction of a tree of Scott rank ω_1^{CK} is based on the following result.

Theorem 3.29 ([**92**]). *The following assertions hold.*

(1) *There is a computable thin rank-homogeneous tree $\mathcal{T}$ such that $rk(\mathcal{T}) = \infty$ but $\mathcal{T}$ has no hyperarithmetical path.*

(2) *If $\mathcal{T}$ is a computable thin rank-homogeneous tree such that $rk(\mathcal{T}) = \infty$ but $\mathcal{T}$ has no hyperarithmetical path, then* $\mathrm{SR}\,(\mathcal{T}) = \omega_1^{CK}$.

As in the case of group-trees, the computable infinitary theory is $\aleph_0$ categorical for the trees considered in [**14**]. But, unlike group-trees, there are many nontrivial hyperarithmetical automorphisms. It is possible to produce a tree as above, with the property of strong computable approximability [**14**].

Definition 3.30. A structure $\mathcal{A}$ is *strongly computably approximable* if for any Σ_1^1 set S there exists a uniformly computable sequence $(\mathcal{C}_n)_{n\in\omega}$ such that $n \in S$ if and only if $\mathcal{C}_n \cong \mathcal{A}$. The structures $\mathcal{C}_n$ with $n \notin S$ are said to be *approximating.*

For example, it is well known that the Harrison ordering is strongly computably approximable by computable well orderings.

Theorem 3.31 ([**14**]). *There is a computable tree $\mathcal{T}$ of Scott rank ω_1^{CK} such that $\mathcal{T}$ is strongly computably approximable. Moreover, the approximating structures are trees of computable Scott rank.*

Using these trees, it is possible to construct many new examples of Makkai models.

Theorem 3.32 ([**13**]). *Each of the following classes contains computable structures of Scott rank ω_1^{CK} :*

- *undirected graphs,*
- *linear orderings,*
- *Boolean algebras,*
- *fields of any characteristic.*

Thus, it is of interest to clarify how to determine the nonconstructive Scott rank on a computable model from its computable representation.

Problem 18. What is the complexity of the index set of Makkai models in universal computable numberings of computable models of a fixed signature?

Problem 19. What is the complexity of the index set of computable models of Scott rank $\omega_1^{CK}+1$ in universal computable numberings of computable models of a fixed signature?

3.5.1. Barwise rank.

Recall the definition of the *quantifier rank* of a formula (we assume that the implication $\Rightarrow$ is expressed in terms of $\neg$ and $\wedge$ and thereby it does not occur directly in the formulas under consideration):

$$qr(\varphi) = \begin{cases} 0 & \text{if } \varphi \text{ is quantifier-free;} \\ qr(\psi) & \text{if } \varphi \text{ is } \neg\psi; \\ qr(\psi)+1 & \text{if } \varphi \text{ is } \exists v\psi \text{ or } \forall v\psi; \\ sup\{qr(\psi) \mid \psi \in \Phi\} & \text{if } \varphi \text{ is } \bigwedge\Phi \text{ or } \bigvee\Phi. \end{cases}$$

Show that for computable models we have $\mathrm{SR}\,(\mathcal{A}) \leqslant \omega_1^{CK}$ for the complexity of Barwise rank. Let α be an ordinal.

Models $\mathfrak{M}$ and $\mathfrak{N}$ are *α-equivalent* ($\mathfrak{M} \equiv^\alpha \mathfrak{N}$) if they satisfy the same sentences with quantifier rank at most α. Two tuples $\overline{a}, \overline{b} \in \mathfrak{M}^{<\omega}$ are *α-equivalent* ($\overline{a} \equiv^\alpha \overline{b}$) if they satisfy the same formulas with quantifier rank at most α.

We say that a tuple $\overline{a} \in \mathfrak{M}^{<\omega}$ has *quantifier rank* α in $\mathfrak{M}$ if $(\overline{a} \equiv^\alpha \overline{b} \Rightarrow \overline{a} \equiv \overline{b})$ for all tuples $\overline{b} \in \mathfrak{M}^{<\omega}$.

The *Barwise rank* $\mathrm{br}\,(\mathfrak{M})$ of a model $\mathfrak{M}$ is the minimal ordinal α such that $(\overline{a} \equiv^\alpha \overline{b} \Rightarrow \overline{a} \equiv^{\alpha+1} \overline{b})$ for all $\overline{a}, \overline{b} \in \mathfrak{M}^{<\omega}$.

As is known, the Barwise rank of a model $\mathfrak{M} \in \mathbb{HYP}_\omega$ does not exceed ω_1^{CK}.

The following assertion concerning the existence of hyperarithmetical isomorphisms for different computable representations of models shows a close connection between this problem and the $\Pi_1^1\Pi$ definability of relations on computable models.

Theorem 3.33 ([**71**]). *Let $\mathfrak{M}$ be a hyperarithmetical model. The following assertions are equivalent.*

(1) *There exist tuples $\overline{a}, \overline{b} \in \mathfrak{M}^{<\omega}$ such that $\langle \mathfrak{M}, \overline{a} \rangle \cong \langle \mathfrak{M}, \overline{b} \rangle$, but $\langle \mathfrak{M}, \overline{a} \rangle \ncong_h \langle \mathfrak{M}, \overline{b} \rangle$.*

(2) *There is a tuple $\overline{a} \in \mathfrak{M}^{<\omega}$ such that there exists an infinite family $(\bar{a}_i)_{i<\omega}$ of tuples in $\mathfrak{M}^{<\omega}$ with the following properties:*
   (a) $\langle \mathfrak{M}, \overline{a} \rangle \cong \langle \mathfrak{M}, \bar{a}_i \rangle$ *for all* $i < \omega$,
   (b) $\langle \mathfrak{M}, \overline{a}_i \rangle \ncong_h \langle \mathfrak{M}, \overline{a}_j \rangle$ *for all* $i < j < \omega$.

(3) *The Barwise rank of $\mathfrak{M}$ is equal to ω_1^{CK}.*

(4) $I_{\mathfrak{M}} \notin \Pi_1^1$, *where* $I_{\mathfrak{M}} = \{\langle \overline{a}, \overline{b} \rangle \in \mathfrak{M}^{<\omega} \times \mathfrak{M}^{<\omega} \mid \overline{a} \cong \overline{b}\}$.

Assume that there exist two isomorphic hyperarithmetical models $\mathfrak{M}$ and $\mathfrak{N}$ that are not hyperarithmetically isomorphic.

Problem 20. Is it true that there exists a computable sequence of hyperarithmetical models $\mathfrak{M}_n$, $n \in \omega$, such that every $\mathfrak{M}_n$ is isomorphic, but not hyperarithmetically isomorphic to $\mathfrak{M}$?

3.5.2. Intrinsically Π_1^1 relations.

In view of Theorem 3.33, it is important to have a description for relations with Π_1^1 complexity in computable hyperarithmetical models. The first results on analytic complexity were obtained by Soskov [**147, 148**].

Proposition 3.34 ([**148**]). *Suppose that $\mathcal{A}$ is computable and R is a Δ_1^1 relation invariant under automorphisms of $\mathcal{A}$. Then R is definable in $\mathcal{A}$ by a computable infinitary formula without parameters.*

Corollary 3.35. *For a computable structure $\mathcal{A}$ and a relation R on $\mathcal{A}$ the following assertions are equivalent*:

(1) *R is intrinsically Δ_1^1 on $\mathcal{A}$,*

(2) *R is relatively intrinsically Δ_1^1 on $\mathcal{A}$,*

(3) *R is definable in $\mathcal{A}$ by a computable infinitary formula with finitely many parameters.*

Definition 3.36. A relation R on $\mathcal{A}$ is *formally* Π_1^1 *on* $\mathcal{A}$ if it is defined in $\mathcal{A}$ by the Π_1^1 disjunction of computable infinitary formulas with finitely many parameters.

We formulate the result of [**147**] in the following form.

Proposition 3.37. *For a computable* (*hyperarithmetical*) *structure $\mathcal{A}$ and a relation R on $\mathcal{A}$ the following assertions are equivalent*:

(1) *R is relatively intrinsically Π_1^1 on $\mathcal{A}$,*

(2) *R is formally Π_1^1 on $\mathcal{A}$.*

Theorem 3.38 ([**70**])**.** *Suppose that $\mathcal{A}$ is a computable structure and R is a relation on $\mathcal{A}$ such that it is Π_1^1 and is invariant under automorphisms of $\mathcal{A}$. Then R is formally Π_1^1. Moreover, it is possible to define it without parameters.*

Corollary 3.39 ([**70**])**.** *For a computable structure $\mathcal{A}$ and a relation R the following assertions are equivalent*:

(1) *R is intrinsically Π_1^1 on $\mathcal{A}$,*

(2) *R is relatively intrinsically Π_1^1 on $\mathcal{A}$,*

(3) *R is formally Π_1^1 on $\mathcal{A}$.*

We say that a relation is *properly* Π_1^1 if it is Π_1^1, but not Σ_1^1.

Corollary 3.40 ([**70**])**.** *If a relation R on a computable structure $\mathcal{A}$ is invariant and properly Π_1^1, then the image of R in any computable copy is also properly Π_1^1.*

There are several examples of computable structures with intrinsically Π_1^1 relations.

Example 3.41. The *Harrison ordering* is a computable ordering of type $\omega_1^{CK}(1+\eta)$. The existence of such an ordering was proved by Harrison who showed that for any computable tree $T \subseteq \omega^{<\omega}$ such that T has paths, but no hyperarithmetical paths, the Kleene–Brouwer ordering on T is a computable ordering of type $\omega_1^{CK}(1+\eta)+\alpha$ with some computable ordinal α.

Let $\mathcal{A}$ be the Harrison ordering, and let R be the initial segment of type ω_1^{CK}. This set is intrinsically Π_1^1 since it is defined by the disjunction of computable infinitary formulas saying that the interval to the left of x has order type β for computable ordinals β.

Example 3.42. The *Harrison Boolean algebra* is the interval algebra of the Harrison ordering.

Let $\mathcal{A}$ be the Harrison Boolean algebra, and let R be the set of superatomic elements containing in some of the Frechet ideals. This set is intrinsically Π_1^1 since it is defined by the disjunction of computable infinitary formulas saying that x is a finite join of α-atoms, where α is a computable ordinal.

Example 3.43. Recall that a countable Abelian p-group $\mathcal{G}$ is determined up to an isomorphism by its Ulm sequence $(u_\alpha(\mathcal{G}))_{\alpha<\lambda(\mathcal{G})}$ and the dimension of the divisible part. The *Harrison p-group* is a computable Abelian p-group $\mathcal{G}$ such that $\lambda(\mathcal{G}) = \omega_1^{CK}$, $u_{\mathcal{G}}(\alpha) = \infty$ for all $\alpha < \omega_1^{CK}$ and the divisible part D has infinite dimension.

By a *Harrison group* we mean the Harrison p-group for some p. Let $\mathcal{A}$ be a Harrison group, and let R be the set of elements with computable ordinal height, the complement of the divisible part. Then R is intrinsically Π_1^1 on $\mathcal{A}$ since it is defined by the disjunction of computable infinitary formulas saying that x has height α, where α is a computable ordinal.

Theorem 3.44. *For the Harrison groups, Harrison Boolean algebra, and Harrison ordering there are computable representations without hyperarithmetical isomorphisms.*

Problem 21. Characterize Π^1_1 relations for other classes of analytic hierarchy.

4. Isomorphism Problem

In this section, we consider isomorphisms of constructive and computable models. Some of the results described below are taken from [**67**].

4.1. Isomorphisms of countably categorical models

Owing to the fundamental concept of a computable isomorphism, it is possible to recognize whether or not two constructivizations of a model have the same computability–theoretic properties.

Definition 4.1. Constructive algebraic systems $(\mathbf{A}, \nu)$ and $(\mathbf{A}, \mu)$ are *computably isomorphic* if there exists an automorphism α of $\mathbf{A}$ and a computable function f such that $\alpha\nu(n) = \mu(f(n))$ for all $n \in \omega$. In this case, ν and μ are said to be *autoequivalent.*

A similar definition can be introduced for computable models.

Definition 4.2. Let $\mathcal{A}$ be a computable structure. We say that $\mathcal{A}$ is *computably categorical* if for all computable $\mathcal{B} \cong \mathcal{A}$ there is a computable isomorphism from $\mathcal{A}$ onto $\mathcal{B}$.

Computably isomorphic structures cannot be distinguished in terms of computability–theoretic properties of definable relations. This means that for any definable relation R in $\mathbf{A}$ (or even for R invariant under automorphisms of $\mathbf{A}$) the Turing degrees of R

under the constructivizations ν and μ are equivalent, i.e., $\nu^{-1}(R)$ and $\mu^{-1}(R)$ have the same Turing degree. In addition, if ν and μ are bijections, then $\nu^{-1}(R)$ and $\mu^{-1}(R)$ are computably invariant. Within the study of computable isomorphisms, the following important notion was introduced by Goncharov.

Definition 4.3. The *dimension* $\dim(\mathbf{A})$ of an algebraic system $\mathbf{A}$ is the maximal number of its nonautoequivalent constructivizations of $\mathbf{A}$.

It is easy to see that the algebraic dimension can be expressed in terms of computable models. Namely, the dimension of an algebraic system $\mathbf{A}$ is equivalent to the maximal number of computable models that are not computably isomorphic each other, but they are isomorphic to $\mathbf{A}$. Informally, if we know the dimension of an algebraic system $\mathbf{A}$, we know the number of effective realizations of $\mathbf{A}$. The dimension of an algebraic system $\mathbf{A}$ can be represented in computability–theoretic terms as the number of computable isomorphism types of $\mathbf{A}$. Thus, if $\dim \mathbf{A} = 1$, the algebraic system $\mathbf{A}$ has exactly one effective realization. We single out algebraic systems of dimension 1.

Definition 4.4 ([**100**])**.** An algebraic system $\mathbf{A}$ is said to be *autostable* if $\dim(\mathbf{A}) = 1$ and *strongly autostable* if all strong constructivizations of $\mathbf{A}$ are autoequivalent.

The notion of an effectively infinite algebraic system, introduced by Goncharov, is used in the study of computable isomorphisms. A sequence $(\mathbf{A}_0, \nu_0), (\mathbf{A}_1, \nu_1), \ldots$ of constructive models is *effective* if the set $\{(i, \varphi) | \varphi \in AD_{\nu_i}(\mathbf{A}_i)\}$ is uniformly computable.

Definition 4.5. An algebraic system $\mathbf{A}$ is said to be *effectively infinite* if there is an algorithm such that, applying it to any index of an effective sequence of constructive systems $(\mathbf{A}, \nu_0), (\mathbf{A}, \nu_1), \ldots$, we obtain a constructive model $(\mathbf{A}, \nu)$ such that $(\mathbf{A}, \nu)$ is not computably isomorphic to $(\mathbf{A}, \nu_i)$ for any $i \in \omega$.

Thus, an effectively infinite algebraic system $\mathbf{A}$ has infinite dimension.

The following characterization of strongly autostable algebraic systems was one of the first important results of the theory of autostable models.

Theorem 4.6 ([**129**]). *A strongly constructive algebraic system* $(\mathbf{A}, \nu)$ *is strongly autostable if and only if there exists finitely many elements* $a_0, \ldots, a_n \in A$ *such that*

(1) *the set of all complete formulas of the theory* T *of the algebraic system* $(\mathbf{A}, a_0, \ldots, a_n)$ *is computable,*

(2) *the algebraic system* $(\mathbf{A}, a_0, \ldots, a_n)$ *is the prime model of the theory* T*.*

Furthermore, if $(\mathbf{A}, \nu)$ is not strongly autostable, then there exists an algorithm such that, applying it to any index of an effective sequence of strongly constructive systems $(\mathbf{A}, \nu_0), (\mathbf{A}, \nu_1), \ldots$, we obtain a strongly constructive algebraic system $(\mathbf{A}, \nu)$ such that $(\mathbf{A}, \nu)$ is not computably isomorphic to $(\mathbf{A}, \nu_i)$ for all $i \in \omega$. Thus, the dimension of a strongly constructive algebraic system that is not strongly autostable is infinite.

Similar questions are considered for other classes of algebraic structures, for example, linearly ordered sets, Boolean algebras, Abelian groups, rings, groups, partially ordered sets, fields, vector spaces, etc. The first results were obtained for linearly ordered sets, Boolean algebras, and torsion-free Abelian groups.

Together with the result of Nurtazin [**130**], the following theorem provides a characterization of all strongly autostable countably categorical models.

The known Ryll–Nardzewski theorem (cf. [**16**]) characterizes countably categorical theories in terms of types. It asserts that a theory T is countably categorical if and only if for every n the number of n-types of T is finite. This theorem suggests to introduce a Ryll–Nardzewski type function type_T associating with every $n \geqslant 1$ the number of n–types of T. For a decidable theory T the function type_T is a Δ_2^0–function.

Theorem 4.7. *A strongly constructive model* $(\mathbf{A}, \nu)$ *of a countably categorical theory* T *is strongly autostable if and only if the type function* type_T *is computable.*

Corollary 4.8. *Let* $\mathbf{A}$ *be a model of a countably categorical theory* T *that admits the effective elimination of quantifiers. Then the following assertions are equivalent.*

(1) *The dimension of* $\mathbf{A}$ *is* 1*.*

(2) *There exists a finite sequence* $a_0, \ldots, a_n$ *of elements of* $\mathbf{A}$ *such that* $(\mathbf{A}, a_0, \ldots, a_n)$ *is the prime model of the theory* T' *of* $(\mathbf{A}, a_0, \ldots, a_n)$ *and the set of atoms of* T' *is computable.*

(3) *The type function* type_T *is computable.*

A natural question arises: What can be said about the computability–theoretic complexity of type_T if T is decidable? An answer is contained in the following assertion proved independently by Venning [**153**].

Theorem 4.9. *For any c.e. degree* $\mathbf{x}$ *there exists a decidable countably categorical theory* T *such that* type_T *has degree* x*.*

Note that there exists a strongly autostable, but not autostable countably categorical model.

At the first glance, it seems that, if type_T of a countably categorical theory is not computable, the dimension of the model of T is greater than 1. However, there exists a counterexample that can be obtained from the following result due to Khoussainov, Lempp, and Solomon.

Theorem 4.10 ([**86**])**.** *There exists a countably categorical theory* T *such that the type function* type_T *is not computable, whereas the model of* T *is autostable.*

If a countably categorical theory T has a computable model, then the type function of T is computable in $\mathbf{O}^\omega$. Together with the above results, this remark leads to the following open question.

Problem 22. Whether there exists a countably categorical theory T such that the type function type_T is not arithmetical, whereas T has a constructive autostable model?

Note that the results concerning the construction of nonautostable algebraic systems of finite dimension do not control the model–theoretic properties of structures. For example, all the structures constructed in [**48, 49, 40, 18, 85**] have theories without prime models. Moreover, all known countably categorical models have dimensions equal to either 1 or ω. So, it is reasonable to put the following questions.

Problem 23. Whether a countably categorical model is effectively infinite if it is not autostable?

Problem 24. Assume that a countably categorical theory T has a computable model. Is it true that the model of T is not autostable if T is computable in $\mathbf{0}^n$ and type_T is not computable in $\mathbf{0}^n$?

4.2. Isomorphisms of uncountably categorical models

Consider the algebraic system (ω, S). The theory T of (ω, S) is uncountably categorical. The isomorphism type of a model $\mathbf{A}$ of T is determined by the number of its components. The saturated model of T has infinitely many components. All nonsaturated models of T are autostable. One can prove that the saturated model of T is not autostable; moreover, it is effectively infinite.

Let V be a vector space over an infinite computable field F. Then the theory T of V (in the language consisting of $+$ for vector addition and unary operation f, $f \in F$, for multiplication by f) is uncountably categorical. As is known, the isomorphism type of a model $\mathbf{A}$ of T is characterized by the dimension of $\mathbf{A}$.

The saturated model of T has infinite dimension. As above, every finite dimensional vector space over F is autostable, the saturated model of T is not autostable and; moreover, is effectively infinite.

Theorem 4.11. *Let T be the theory of algebraically closed fields of a fixed characteristic. Then a model $\mathbf{A}$ of T is autostable if and only if it has a finite transcendence degree over its prime field.*

In all these examples, all the theories are decidable and admit the elimination of quantifiers; moreover, non-saturated models are autostable. At the same time, there exists a decidable uncountably categorical theory T admitting the elimination of quantifiers such that the prime model of T is not autostable.

Let T be a decidable uncountably categorical theory with strongly autostable prime model.

Problem 25. Is it true that every nonsaturated model of T is strongly autostable?

Conjecture 4.12. *There exists an uncountably categorical theory such that the countably saturated model is autostable.*

Problem 26. Is it true that any field with infinite basis is not autostable?

Without the requirement of decidability of an uncountably categorical theory, the situation becomes much more complicated. No results are known for computable isomorphisms and dimensions of computable models of uncountably categorical theories. For example, we do not know the spectra of dimensions of uncountably categorical models. Recall that all models of an uncountably categorical theory T can be listed in the $\omega + 1$ chain of models chain (T): $\mathbf{A}_0 \preceq \mathbf{A}_1 \preceq \mathbf{A}_2 \preceq \ldots \preceq \mathbf{A}_\omega$, where $\mathbf{A}_i$ is the prime model over $\bar{\mathbf{A}}_i$ and $\mathbf{A}_\omega$ is the saturated model.

Problem 27. Let $\mathbf{A}_i$ be a model of an uncountably categorical theory T in chain (T). What sufficient and necessary conditions for $\mathbf{A}_i$ to be autostable?

In Problem 27, it is also of interest to control the dimension of uncountably categorical models. In particular, the following open question can be suggested.

Problem 28. Whether there exists an uncountably categorical nonautostable model of finite dimension?

As was already mentioned, Goncharov constructed a nonautostable algebraic system of finite dimension. Thus, it is reasonable to formulate the following problem.

Problem 29. Whether it is possible to construct an algebraic system of finite dimension greater than 1 whose theories belong to some class of well–studied theories, for example, countable or uncountably categorical theories, Erenfeucht theories, etc.

The following problem is of general character.

Problem 30. Characterize uncountably categorical models of dimension 1.

There are known examples of computable structures of computable Scott rank. At the same time, there are known structures (for example, the Harrison ordering) of Scott rank $\omega_1^{CK}+1$. Makkai [**99**] constructed a structure of Scott rank ω_1^{CK} which can be made computable [**68**].““ Then he simplified it in such a way that it becomes just a tree [**14**]. As was shown in [**13**], there are other computable structures of Scott rank ω_1^{CK} among undirected graphs, fields of any characteristic, and linear orderings. The new examples share a strong approximability property with the Harrison ordering and the tree in [**14**]. These results provide us with examples of computable structures with different complexity of the isomorphism problem for different computable representations.

4.3. Computable categoricity

Let $\mathcal{A}$ be a computable structure. We say that $\mathcal{A}$ is *computably categorical* if for all computable $\mathcal{B} \cong \mathcal{A}$ there is a computable isomorphism from $\mathcal{A}$ onto $\mathcal{B}$. Similarly, $\mathcal{A}$ is Δ^0_α *categorical* if for all computable $\mathcal{B} \cong \mathcal{A}$ there is a Δ^0_α isomorphism. We say that $\mathcal{A}$ is *relatively computably categorical* if for all $\mathcal{B} \cong \mathcal{A}$ there is an isomorphism that is computable relative to $\mathcal{B}$, and we say that $\mathcal{A}$ is *relatively* Δ^0_α *categorical* if for all $\mathcal{B} \cong \mathcal{A}$ there is a $\Delta^0_\alpha(\mathcal{B})$ isomorphism.

Definition 4.13. A *Scott family for* $\mathcal{A}$ is a set Φ of formulas with a fixed tuple of parameters $\overline{c}$ in $\mathcal{A}$ such that

- every tuple in $\mathcal{A}$ satisfies some $\varphi \in \Phi$,
- if $\overline{a}$, $\overline{b}$ are tuples in $\mathcal{A}$ satisfying the same formula $\varphi \in \Phi$, then there is an automorphism of $\mathcal{A}$ sending $\overline{a}$ to $\overline{b}$.

A *formally c.e. Scott family* is a c.e. Scott family made up of finitary existential formulas.

A *formally* Σ^0_α *Scott family* is a Σ^0_α Scott family made up of "computable Σ_α" formulas.

Proposition 4.14. *For a structure* $\mathcal{A}$ *the set* $\{\overline{a} : \mathcal{A} \models \varphi(\overline{a})\}$ *is* $\Sigma^0_\alpha(\mathcal{A})$ *if* φ *is computable* Σ_α*, and* $\Pi^0_\alpha(\mathcal{A})$ *if* φ *is computable* Π_α*. Moreover, this assertion remains valid with all imaginable uniformity over structures and formulas.*

It is easy to see that if $\mathcal{A}$ has a formally c.e. Scott family, then it is relatively computably categorical, so it is computably categorical. More generally, if $\mathcal{A}$ has a formally Σ^0_α Scott family, then it is relatively Δ^0_α categorical and, consequently, Δ^0_α categorical.

Goncharov showed that, under some additional effectiveness conditions (on a single copy), if $\mathcal{A}$ is computably categorical, then it has a formally c.e. Scott family.

Ash showed that, under some effectiveness conditions (on a single copy), if $\mathcal{A}$ is Δ^0_α categorical, then it has a formally Σ^0_α Scott family.

For the relative notions, we do not have the effectiveness conditions. The following assertion was proved in [**6**] and [**17**].

Proposition 4.15. *A computable structure $\mathcal{A}$ is relatively Δ^0_α categorical if and only if it has a formally Σ^0_α Scott family. In particular, $\mathcal{A}$ is relatively computably categorical if and only if it has a formally c.e. Scott family.*

4.4. Basic results in numbering theory

We present some basic results in numbering theory [**72**] and applications to computable models. For $\mathcal{S} \subseteq P(\omega)$ a *numbering* is a binary relation ν such that $\mathcal{S} = \{\nu(i) : i \in \omega\}$, where $\nu(i) = \{x : (i, x) \in \nu\}$. A numbering ν of $\mathcal{S}$ is called a *Friedberg numbering* if it is a bijection in the sense that $i \neq j$ implies $\nu(i) \neq \nu(j)$.

Suppose that ν and μ are two numberings of a family $\mathcal{S}$. We write $\nu \leqslant \mu$ if there is a computable function f such that $\nu(i) = \mu(f(i))$ for all i, i.e., we can effectively pass from a ν-index to a μ-index for the same set. We say that ν and μ are *computably equivalent* if $\mu \leqslant \nu$ and $\nu \leqslant \mu$. Note that if μ and ν are Friedberg numberings of $\mathcal{S}$, then $\mu \leqslant \nu$ implies $\nu \leqslant \mu$.

Definition 4.16. A family $\mathcal{S} \subseteq P(\omega)$ is *discrete* if for every $A \in \mathcal{S}$ there exists $\sigma \in 2^{<\omega}$ such that for all $B \in \mathcal{S}$ the following holds: $\sigma \subseteq \chi_B$ if and only if $B = A$.

Definition 4.17. A family is *effectively discrete* if there is a c.e. set $E \subseteq 2^{<\omega}$ such that

(a) for every $A \in \mathcal{S}$ there is $\sigma \in E$ such that $\sigma \subseteq \chi_A$,

(b) for all $\sigma \in E$ and $A, B \in \mathcal{S}$ from $\sigma \subseteq \chi_A, \chi_B$ it follows that $A = B$.

Proposition 4.18 ([**144**]). *There exists a unique up to a computable equivalence family $\mathcal{S} \subseteq P(\omega)$ with computable Friedberg numbering such that it is discrete, but not effectively discrete.*

Proposition 4.19 ([**50**]). *For every finite $n \geqslant 1$ there is a family of sets with just n computable Friedberg numberings determined up to a computable equivalence.*

Proposition 4.20 ([**154, 145**]). *There is a family $\mathcal{S} \subseteq P(\omega)$ with numberings in all noncomputable degrees but not a computable numbering.*

The numbering results of Selivanov, Goncharov, and Wehner can be relativized. In [**40, 72**], one can find a general method of constructing a model from any computable family of c.e. sets with computable numberings. Owing to this method, problems in the theory of computable models are reduced to some problems in numbering theory.

Let $\mathcal{S}$ be a family of sets. For every $A \in \mathcal{S}$ we can construct a *daisy graph* $\mathcal{G}_A$ such that

(a) $\mathcal{G}(\mathcal{S})$ is a rigid graph,

(b) if $\mathcal{S}$ has a unique computable Friedberg numbering, then $\mathcal{G}(\mathcal{S})$ is computably categorical,

(c) if $\mathcal{S}$ has just n computable Friedberg numberings determined up to a computable equivalence, then $\mathcal{G}(\mathcal{S})$ has computable dimension n,

(d) if $\mathcal{S}$ is discrete, then every element of $\mathcal{G}(\mathcal{S})$ has a finitary existential definition without parameters,

(e) if $\mathcal{S}$ has a computable Friedberg numbering, and is discrete but not effectively discrete, then $\mathcal{G}(\mathcal{S})$ does not have a formally c.e. defining family.

For lifting the basic results of Goncharov, Manasse, Slaman, and Wehner, we formulate them in the following form.

Proposition 4.21 ([**46, 40**]). *There is a rigid graph structure $\mathcal{G}$ that is computably categorical without a formally c.e. defining family.*

Proposition 4.22 ([**126**]). *There is a computable structure $\mathcal{A}$ with a relation R that is intrinsically c.e. but not relatively intrinsically c.e.*

Consider the cardinal sum of disjoint computable copies of the graph structure $\mathcal{G}$ from Proposition 4.21. Let R be a unique isomorphism.

Proposition 4.23 ([**48, 49, 40, 39**]). *For every finite n there is a rigid graph structure $\mathcal{G}$ with computable dimension n.*

Proposition 4.24 ([**145, 154**]). *There is a structure $\mathcal{A}$ with copies in just the noncomputable degrees.*

A coding of a Δ^0_α structure in a computable structure was suggested in [**72**] to preserve some complexity of algorithmic properties.

To lift the basic results of Goncharov and Manasse, we relativize by producing a Δ^0_α graph. To pass to a computable structure, we use a pair of structures for coding the arrow relation.

For a graph $\mathcal{G}$, a pair of structures $\mathcal{B}_1$, $\mathcal{B}_2$, and a relational language we set $\mathcal{G}^* = (G \cup U, G, U, Q, \ldots)$, where G is the basic set of $\mathcal{G}$, G and U are disjoint, Q is a ternary relation assigning to every pair $a, b \in G$ an infinite set $U_{(a,b)}$, the sets $U_{(a,b)}$ form a partition of U, every relation in the notation "..." is the union of the bounds to $U_{(a,b)}$, and for every pair $a, b \in G$

$$(U_{(a,b)}, \ldots) \cong \begin{cases} \mathcal{B}_1 & \text{if } \mathcal{G} \models a \to b, \\ \mathcal{B}_2 & \text{otherwise.} \end{cases}$$

Theorem 4.25 ([**72**]). *Suppose that $\mathcal{G}$ is a graph structure and $\mathcal{G}^*$ is constructed from $\mathcal{G}$, $\mathcal{B}_i$ in the same way as above. In this case, $\mathcal{G}$ has a Δ^0_α copy if and only if $\mathcal{G}^*$ has a computable copy. More generally, for any X the structure $\mathcal{G}$ has a $\Delta^0_\alpha(X)$ copy if and only if $\mathcal{G}^*$ has an X-computable copy. In addition,*

(a) *if $\mathcal{G}$ has a unique up to a Δ^0_α isomorphism Δ^0_α copy, then $\mathcal{G}^*$ is Δ^0_α categorical,*

(b) *if $\mathcal{G}$ has just n Δ^0_α copies, determined up to a Δ^0_α isomorphism, then $\mathcal{G}^*$ has Δ^0_α dimension n,*

(c) *if $\mathcal{G}$ does not have a Σ^0_α Scott family made up of finitary existential formulas, then $\mathcal{G}^*$ does not have a formally Σ^0_α Scott family.*

The following construction allows us to reduce the above consideration to graph structure and other algebraic structures.

Theorem 4.26 ([**58**]). *Suppose that $\mathfrak{M}$ is a countable structure of a signature σ such that the arity of all predicate and functional symbols in σ is bounded by a number k. There exists a partial ordering (graph) $\mathfrak{M}^*$ with the following properties: The model $\mathfrak{M}$ has a computable copy if and only if $\mathfrak{M}^*$ has a computable copy. More generally, for any X the model $\mathfrak{M}$ has an X-computable copy if and only if $\mathfrak{M}^*$ has an X-computable copy. In addition,*

(a) *if $\mathfrak{M}$ is Δ^0_α categorical, then $\mathfrak{M}^*$ is Δ^0_α categorical,*

(b) *if $\mathfrak{M}$ has Δ^0_α dimension n, then $\mathfrak{M}^*$ has Δ^0_α dimension n,*

(c) *if $\mathfrak{M}$ does not have a formally Σ^0_α Scott family, then $\mathfrak{M}^*$ does not have a formally Σ^0_α Scott family.*

The proof is based on the following constructions [**48**] of categories of computable algebraic systems. Consider a computable signature $\sigma = \langle P_0^{n_0}, P_1^{n_1}, \dots, P_k^{n_k}, \dots\rangle$ such that σ is countable or finite. Denote by Mod^σ the category whose objects are models of the signature σ and morphisms are their isomorphisms. Introduce the subcategory $\mathrm{Mod}^\sigma_{\mathrm{com}}$ of Mod^σ. It is easy to see that a model $\mathfrak{M}'$ is computable if it is computably isomorphic to a computable model $\mathfrak{M}$ with computable basic set $|\mathfrak{M}|$ which is a computable subset of some sets of words of finite alphabet and the set $\{\langle i, m_1, \dots, m_{n_i}\rangle / \mathfrak{M} \models P_i(m_1, \dots, m_{n_i})\}$ is computable. If $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are computable models, then the isomorphism

$\varphi : \mathfrak{M}_1 \underset{\text{onto}}{\to} \mathfrak{M}_2$ is computable provided that φ is partially computable. The objects of $\text{Mod}^{\sigma}_{\text{com}}$ are computable models of the signature σ and the morphisms are computable isomorphisms. If $\mathfrak{M}$ is a model of the signature σ, then $\text{Mod}^{\sigma}_{\text{com}}(\mathfrak{M})$ is the complete subcategory of $\text{Mod}^{\sigma}_{\text{com}}$ whose objects are computable models isomorphic to $\mathfrak{M}$. If K_0 is a subcategory of K and F is a function from K into K_1, then denote by $F \upharpoonright K_0$ the restriction of F to K_0. A signature σ is *bounded* if there exists k such that $m_i \leqslant k$ for every i.

Proposition 4.27 ([**58**]). *For an arbitrary bounded signature σ there exists a finite signature σ_0 and a completely univalent functor F_1 from* Mod^{σ} *to* Mod^{σ_0} *such that the following assertions hold.*

(i) *$F_1 \upharpoonright \text{Mod}^{\sigma}_{\text{com}}$ is a completely univalent functor from $\text{Mod}^{\sigma}_{\text{com}}$ to $\text{Mod}^{\sigma_0}_{\text{com}}$.*

(ii) *For an arbitrary model $\mathfrak{M}$ of the signature σ the functor $F_1 \upharpoonright \text{Mod}^{\sigma}_{\text{com}}(\mathfrak{M})$ realizes an equivalence of the categories $\text{Mod}^{\sigma}_{\text{com}}(\mathfrak{M})$ and $\text{Mod}^{\sigma_0}_{\text{com}}(F_1(\mathfrak{M}))$. In addition,*

  (a) *if $\mathfrak{M}$ is Δ^0_α categorical, then $F_1(\mathfrak{M})$ is Δ^0_α categorical,*

  (b) *if $\mathfrak{M}$ has Δ^0_α dimension n, then $F_1(\mathfrak{M})$ has Δ^0_α dimension n,*

  (c) *if $\mathfrak{M}$ does not have a formally Σ^0_α Scott family, then $F_1(\mathfrak{M})$ does not have a formally Σ^0_α Scott family.*

PROOF. Let $\sigma = \langle P_0^{n_0}, P_1^{n_1}, \ldots, P_k^{n_k}, \ldots\rangle$. Suppose that the set $\langle n_i | i \in \mathfrak{N}\rangle$ is bounded by k. For every $k \leqslant K$ we consider all predicates $P_{i^k{}_0}, P_{i^k{}_1}, \ldots, P_{i^k{}_l}, \ldots,\ l \in N'_k$, from σ of arity k, where N'_k is equal to N or is an initial segment of N. We set $\sigma_0 = \{=, P_0^1, P_1^2, \ldots, P_k^{k+1}, \ldots, P_k^{k+1}, A^{12}, \lhd\}$ and define a functor F_1 on objects of Mod^{σ}. Let $\mathfrak{M}$ be an arbitrary model of the signature σ. If M is the basic set of $\mathfrak{M}$, then for the basic set of $\mathfrak{M}_0 \rightleftharpoons F_1(\mathfrak{M})$ we take $M_0 = M \cup \{a_0, a_1, \ldots, a_n, \ldots\}$, where $\{a_0, a_1, \ldots, a_n, \ldots\} \cap M = \varnothing$ and $a_i \neq a_j$ for $i \neq j$.

Introduce predicates as follows:

1) $A_{\mathfrak{M}_0} \leftrightharpoons \{a_0, a_1, \ldots, a_n, \ldots\}$,

2) $x \lhd y$ if $x = a_n$ and $y = a_{n+1}$ for some n,

3) $(x_0, x_1, \ldots, x_s) \in (P_s)_{\mathfrak{M}_0}$ if $x_0 = a_d$, $\bigwedge\limits_{j=1}^{i} x_j \in M$ and $\mathfrak{M} \models P^s_{i^s d}(x_1, \ldots, x_s)$.

It is easy to see that $\mathfrak{M}_0$ is a computable model if $\mathfrak{M}$ is computable. If $\mathfrak{M}$ and $\mathfrak{M}^0$ are objects of the category $\mathrm{Mod}^{\,0}$ and φ is an isomorphism of $\mathfrak{M}$ onto $\mathfrak{M}^0$, then we define $F(\mathfrak{M}, \mathfrak{M}^0)(\varphi)$. We define it only in the case where the basic sets of both models are subsets of N. The remaining cases are treated in a similar way. Thus,

$$[F(\mathfrak{M}, \mathfrak{M}^0)(\varphi)](x) \leftrightharpoons \begin{cases} x & \text{if } x \in \{a_0, a_1, \ldots, a_n, \ldots\}, \\ \varphi(x) & \text{if } x \in M\}. \end{cases}$$

It is clear that $F_1(\mathfrak{M}, \mathfrak{M}^0)(\varphi)$ is a computable isomorphism relative to a Turing degree a if φ is a computable isomorphism relative to a.

To prove that F_1 from K_1 into K_2 is completely univalent, it suffices to show that $F_*(A, B) : \mathrm{Hom}\,(A, B) \longrightarrow \mathrm{Hom}\,(F_*(A), F_*(B))$ is a bijection for every pair A, B of objects of K_1. Thus, F_1 and $F_1 \restriction \mathrm{Mod}^{\,\sigma}_{\,\mathrm{com}}(\mathfrak{M})$ are completely univalent functors. We can prove that $F_1 \restriction \mathrm{Mod}^{\,\sigma}_{\,\mathrm{com}}(\mathfrak{M})$ realizes an equivalence by showing that for every object $\mathfrak{M}'$ of the category $\mathrm{Mod}^{\,\sigma_0}_{\,\mathrm{com}}(F_1(\mathfrak{M}))$ there exists an object $\mathfrak{M}'_0$ of the category $\mathrm{Mod}_{\,\mathrm{com}}(F_1)(\mathfrak{M})$ such that $\mathfrak{M}'$ and $F_1(\mathfrak{M}'_0)$ are isomorphic in the category $\mathrm{Mod}_{\,\mathrm{com}}(F_1(\mathfrak{M}))$. Let $\mathfrak{M}' \in \mathrm{Mod}^{\,\sigma}_{\,\mathrm{com}}(F_1)(\mathfrak{M})$. The case of a finite model is trivial.

Let $\mathfrak{M}$ be an infinite model. We can consider a computable function $f : N \xrightarrow[\text{onto}]{1-1} M' \backslash A_{\mathfrak{M}'}$. Since $\mathfrak{M}'$ is a computable model, it follows that $N' \setminus A_{\mathfrak{M}'}$ is computable and the function exists. Let a be an element of $A_{\mathfrak{M}'}$ that does not have a $\lhd$–predecessor. Let us now define predicates of the signature σ on N: $(n_1, \ldots, n_{m_k}) \in P^{m_k}_{i_k}$ if and only if $(l, f(n_1), \ldots, f(n_{m_i})) \in P^{m_k+1}_{m_k}$, where the elements $l_0, l_1, l_2, \ldots, l_k$ are such that $l_i \lhd l_{i+1}$ for $0 \leqslant i < k$, $a = l_0$, and

$l_k = l$. It is easy to see that such a model $\mathfrak{M}''$ of the signature σ is computable. We show that $F(\mathfrak{M}'')$ is computably isomorphic to $\mathfrak{M}'$. For this purpose, consider a function g defined as follows:

$$g(m) = \begin{cases} f(m) & \text{if } m \epsilon N, \\ l & \text{if } m = a_k \text{ and there exist } l_0, l_1, \ldots, l_k \text{ such} \\ & \text{that } \mathfrak{M}' \models \bigwedge_{i=0}^{k-1} l_i \lhd l_{i+1} \text{ and } l_0 = a \& l_k = l. \end{cases}$$

It is clear that g is an isomorphism and a computable function.

Properties (a)–(c) of th Scott families can be derived from the definability of basic predicates and their negations by $\exists$–formulas. □

Proposition 4.28 ([**58**]). *For an arbitrary finite signature σ_0 there exist a signature σ_1 consisting of a single predicate symbol P and a completely univalent functor F_2 from* Mod^{σ_0} *into* Mod^{σ_1} *such that the following assertions hold.*

(i) $F_2 \restriction \mathrm{Mod}^{\sigma_0}_{\mathrm{com}}$ *is a completely univalent functor from* $\mathrm{Mod}^{\sigma_0}_{\mathrm{com}}$ *into* $\mathrm{Mod}^{\sigma_1}_{\mathrm{com}}$.

(ii) *For every model $\mathfrak{M}$ of the signature σ_0 the functor $F_2 \restriction \mathrm{Mod}^{\sigma_0}_{\mathrm{com}}(\mathfrak{M})$ realizes an equivalence of the categories* $\mathrm{Mod}^{\sigma_0}_{\mathrm{com}}(\mathfrak{M})$ *and* $\mathrm{Mod}^{\sigma_1}_{\mathrm{com}}(F_2(\mathfrak{M}))$. *In addition,*

   (a) *if $\mathfrak{M}$ is Δ^0_α categorical, then $F_2(\mathfrak{M})$ is Δ^0_α categorical,*

   (b) *if $\mathfrak{M}$ has Δ^0_α dimension n, then $F_2(\mathfrak{M})$ has Δ^0_α dimension n,*

   (c) *if $\mathfrak{M}$ does not have a formally Σ^0_α Scott family, then $F_2(\mathfrak{M})$ does not have a formally Σ^0_α Scott family.*

Proof. Our goal is to define the functor F_2. Let $\sigma_0 = \langle P_0^{n_0}, P_1^{n_1}, \ldots, P_k^{n_k} \rangle$. Suppose that $\mathfrak{M}$ is a model of the finite signature σ_0. Consider a predicate symbol P of arity $n = \sum_{i=0}^{k} n_i$ and a signature $\sigma_1 = \langle P^n \rangle$. We begin by defining F_2 on the objects of Mod^{σ_0}.

Let $\mathfrak{M}$ be a model of the signature σ_0 with the basic set M. For the basic set M_0 of the model $F_2(\mathfrak{M})$ we take $\{\infty\} \cup M$. We define P on M_0 as follows: $\langle x_1, \ldots, x_n\rangle \in P$ if and only if one of the following conditions is satisfied:

(a) $x_1 = x_2 = \ldots = x_n = 0$,

(b) there exist $i \leqslant k$ and $y_1, \ldots, y_{n_i}$ such that $x_j = 0$ and $y_j = x_{j+m_i}$ for any j such that $1 \leqslant j \leqslant n_i$ and $\mathfrak{M} \models P_i(y_1, \ldots, y_{n_i})$.

But $x_j = 0$ for any j such that $1 \leqslant j \leqslant m_i$ or $m_i + n_i + 1 \leqslant j \leqslant n$. We put $m_0 = 0$ and $m_i = \sum\limits_{l=0}^{i-1} n_l$ for $i > 1$.

If $\mathfrak{M}$ is a computable model, the model $F_2(\mathfrak{M})$ is also computable. Let $\mathfrak{M}$ and $\mathfrak{N}$ be two models of the signature σ_0. We define a mapping $F_2(\mathfrak{M}, \mathfrak{N}) : \mathrm{Hom}\,(\mathfrak{M}, \mathfrak{N}) \to \mathrm{Hom}\,(F_2(\mathfrak{M}), F_2(\mathfrak{N}))$ as follows:

$$[F_2(\mathfrak{M}, \mathfrak{N})(\varphi)](x) \leftrightharpoons \begin{cases} \infty & \text{if } x = \infty, \\ \varphi(x) & \text{if } x \neq \infty. \end{cases}$$

It is easy to see that $F_2(\mathfrak{M}, \mathfrak{N})$ is an isomorphism if φ is an isomorphism, and it is computable if φ is computable. The remaining assertions can be proved in the same way as in Proposition 4.27. The additional properties (a)–(c) can be proved by induction. □

4.5. Categories of graphs and partial orders

Consider a signature σ^* consisting of a single binary predicate Q. The category Mod^{σ^*} is called the *category of graphs* and is denoted by Graph. Denote by Ord the complete subcategory of Mod^{σ^*} whose objects are the models $\langle M, Q\rangle$, where Q is a partial order on M.

Proposition 4.29 ([**58**]). *For every signature σ_1 consisting of a single predicate of arity $n \geqslant 3$ there exists a completely univalent functor F_3 from* Mod^{σ_1} *into* Graph *with binary predicate R such that the following assertions hold.*

(i) *$F_3 \upharpoonright \mathrm{Mod}^{\sigma_1}_{\mathrm{com}}$ is a completely univalent functor from $\mathrm{Mod}^{\sigma_1}_{\mathrm{com}}$ into $\mathrm{Graph}_{\mathrm{com}} = \mathrm{Mod}^{\sigma^*}_{\mathrm{com}}$.*

(ii) *For every model $\mathfrak{M} \in \mathrm{Ob}_{\mathrm{Mod}^{\sigma_1}}$ the functor $F_3 \upharpoonright \mathrm{Mod}^{\sigma_1}_{\mathrm{com}}(\mathfrak{M})$ realizes an equivalence between the categories $\mathrm{Mod}^{\sigma_1}_{\mathrm{com}}(\mathfrak{M})$ and $\mathrm{Mod}^{\sigma^*}_{\mathrm{com}}(F_3(\mathfrak{M}))$. In addition,*

  (a) *if $\mathfrak{M}$ is Δ^0_α categorical, then $F_3(\mathfrak{M})$ is Δ^0_α categorical,*

  (b) *if $\mathfrak{M}$ has Δ^0_α dimension n, then $F_3(\mathfrak{M})$ has Δ^0_α dimension n,*

  (c) *if $\mathfrak{M}$ does not have a formally Σ^0_α Scott family, then $F_3(\mathfrak{M})$ does not have a formally Σ^0_α Scott family.*

Proof. (i) We construct directly the functor F_3 from Mod^{σ_1} into Graph. Let $\langle M, P\rangle$ be a model of the signature σ_1, where P is a predicate of arity n. Consider $I = \{0, 1, \ldots, n\}$ and $M' = I \times M^n \cup M$. For the basic set $|F_3(\mathfrak{M})|$ we take the set $M_0 \leftrightharpoons M' \cup \{a_0,$ a_1, a_2, b_0, b_1, b_2, c_0, c_1, c_2, c_3, c_4, c_5, c_6, c_7, $c_8\}$. Suppose that all the elements in $\{a_0, a_1, a_2, b_0, b_1, b_2, c_0, c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8\}$ are different and new. Fix a_0, a_1, a_2, b_0, b_1, b_2, c_0, c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8. These elements will be referred to as basic elements for the definability of F_3 on $\mathfrak{M}$. We define a predicate R on M_0 as follows. Let $x, y, \in M_0$. We set $\langle x, y\rangle \in R$ if one of the following conditions is satisfied:

(a) $x = a_i \& y = c_j$, and $1 \leqslant i \leqslant 3$ and $(i = 0 \& j \in \{0, 1\}) \vee (i = 1 \& j \in \{2, 3, 4\}) \vee (i = 2 \& j \in \{5, 6, 7, 8\})$,

(b) $x = c_j \& y = b_i$, and $1 \leqslant i \leqslant 3$ and $(i = 0 \& j \in \{0, 1\}) \vee (i = 1 \& j \in \{2, 3, 4\}) \vee (i = 2 \& j \in \{5, 6, 7, 8\})$,

(c) $x \in M \& y \in I \times M^n \& y = \langle i, x_1, \ldots, x_n\rangle \& x = x_i$ and $n \geqslant i \geqslant 1$,

(d) $x, y \in I \times M^n \& x = \langle i, x_1, \dots, x_n \rangle \& y = \langle i+1, x_1, \dots, x_n \rangle$ and $i \geqslant 1$,

(e) $x = a_1 \& y = \langle 0, y_1, \dots, y_n \rangle \in I \times M^n \& \mathfrak{M} \not\models P(y_1, \dots, y_n)$,

(f) $x = a_0 \& y = \langle 0, y_1, \dots, y_n \rangle \in I \times M^n \& \mathfrak{M} \models P(y_1, \dots, y_n)$,

(g) $x = a_2 \& y \in M$.

Thus, we constructed a graph on M_0. Let $\mathfrak{M}$ and $\mathfrak{M}^0$ be two models of the signature σ_1, and let φ be an isomorphism from $\mathfrak{M}$ onto $\mathfrak{M}^0$. We set

$$[F'(\mathfrak{M}, \mathfrak{M}^0)(\varphi)](x) \rightleftharpoons \begin{cases} \varphi(x), & \text{if } x \in M, \\ \langle i, \varphi(x_1), \dots, \varphi(x_n) \rangle & \text{if } x = \langle i, x_1, \dots, x_n \rangle \in I \times M^n, \\ x & \text{otherwise.} \end{cases}$$

Successively considering all the cases, we can show that $F_3'(\mathfrak{M}, \mathfrak{M}^0)(\varphi)$ is an isomorphism; moreover, it is computable if φ is computable.

It remains to prove that the functor F_3 is completely univalent. Let Ψ be an isomorphism from $F_3(\mathfrak{M})$ onto $F_3(\mathfrak{M}^0)$. Then the restriction of Ψ to the definable by an existential formula subset M in $F_3(\mathfrak{M})$, equal to $\{2n \mid n \in N\}$, induces an isomorphism Ψ_0 between the models $\mathfrak{M}$ and $\mathfrak{M}^0$. Since all the elements of $\langle M_0, P \rangle$ are of the type $\langle i, x_1, \dots, x_n \rangle$ and are definable over elements of M by existential formulas, it is easy to show that $F_3(\mathfrak{M}, \mathfrak{M}^0)(\Psi_0) = \Psi$.

(ii) Consider a model $\mathfrak{M}$ of the signature σ_1 and $\mathfrak{M}' \in \mathrm{Mod}_{\mathrm{com}}^{\sigma^*}(F_3(\mathfrak{M}))$. Since elements among a_0, a_1, a_2, b_0, b_1, b_2, c_0, c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8 are definable by existential formulas over elements in $F_3(\mathfrak{M})$, we select them in $\mathfrak{M}'$. Suppose that these elements are the following: a_0^0, a_1^0, a_2^0, b_0^0, b_1^0, b_2^0, c_0^0, c_1^0, c_2^0, c_3^0, c_4^0, c_5^0, c_6^0, c_7^0, c_8^0. Choosing elements connected with a_2^0 by the basic binary predicate, we obtain exactly the definable set X_0 which is isomorphic to M in $F_3(\mathfrak{M})$.

Define the predicate P^n on X_0 as follows:

$\langle x_1, \ldots, x_n\rangle \in P^n \Leftrightarrow \mathfrak{M}' \models (\exists y_1, \ldots, y_n)(y_1 R \ldots R y_n$
$\&\big(\bigwedge_{1\leqslant i\leqslant j\leqslant n} x_i R y_j\big) \& a_0' R y_1)$

It is easy to see that

$\langle x_1, \ldots, x_n\rangle \notin P^n \Leftrightarrow \mathfrak{M}' \models (\exists y_1 \ldots y_n)(y_1 P^2 \ldots P^2 y_n$
$\&(\bigwedge_{1\leqslant i\leqslant j\leqslant n} x_i P^2 y_j) \& a_1' P^2 y_1)$.

Therefore, $\langle X_0, P^n\rangle$ is a computable model of the signature σ_1. A direct verification shows that the model of $F'(\langle X_0, P\rangle)$ is computably isomorphic to $\mathfrak{M}'$.

It remains to prove the additional properties (a)–(c). All the elements of $M' = I \times M^n \cup \{a_0, a_1, a_2, b_0, b_1, b_2, c_0, c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8\}$ are definable in $F_3(\mathfrak{M})$ over elements of M by existential formulas from the computable set of these formulas. Thus, we can construct a formally Σ^0_α Scott family for $F_3(\mathfrak{M})$ from the formally Σ^0_α Scott family for $\mathfrak{M}$. If we have a formally Σ^0_α Scott family for the model $F_3(\mathfrak{M})$, we can see that $F_3(\mathfrak{M})$ is Δ–definable in $\mathfrak{M}$ with the basic set $M \bigcup \bigcup_{i=1}^{n+1} M^{n+i}/\Theta_i \bigcup \bigcup_{i=1}^{15} M^{2n+1+i}/\Delta_i$. Here, we put $\langle X, Y\rangle \in \Theta_i$ if $X = \langle x_1, \ldots, x_{n+i}\rangle$, $Y = \langle y_1, \ldots, y_{n+i}\rangle$ and $x_j = y_j$ for any $1 \leqslant j \leqslant n$. For the other equivalence relation we put $\langle X, Y\rangle \in \Delta_i$ for any elements X, Y of M^{2n+1+i}. Since this model is Δ-definable, we can define a formally Σ^0_α Scott family for $\mathfrak{M}$. □

Proposition 4.30 ([**58**]). *For every signature σ_1 consisting of a single binary predicate R there exists a completely univalent functor F_4 from* Mod$^{\sigma_1}$, *into* Ord *such that the following assertions hold.*

(i) *$F_4 \restriction \mathrm{Mod}^{\sigma_1}_{\mathrm{com}}$ is a completely univalent functor from $\mathrm{Mod}^{\sigma_1}_{\mathrm{com}}$ into $\mathrm{Mod}^{\sigma^*}_{\mathrm{com}}$.*

(ii) *For every model $\mathfrak{M} \in \mathrm{Ob}_{\mathrm{Mod}^{\sigma_1}}$ the functor $F_4 \restriction \mathrm{Mod}^{\sigma_1}_{\mathrm{com}}(\mathfrak{M})$ realizes an equivalence between the categories $\mathrm{Mod}^{\sigma_1}_{\mathrm{com}}(\mathfrak{M})$ and $\mathrm{Mod}^{\sigma^*}_{\mathrm{com}}(F_4(\mathfrak{M}))$. In addition,*

(a) *if $\mathfrak{M}$ is Δ^0_α categorical, then $F_4(\mathfrak{M})$ is Δ^0_α categorical,*

(b) *if* $\mathfrak{M}$ *has* Δ^0_α *dimension* n, *then* $F_4(\mathfrak{M})$ *has* Δ^0_α *dimension* n,

(c) *if* $\mathfrak{M}$ *does not have a formally* Σ^0_α *Scott family, then* $F_4(\mathfrak{M})$ *does not have a formally* Σ^0_α *Scott family.*

PROOF. We construct the functor F_4 from $\mathrm{Mod}^{\{R\}}$ into Ord satisfying the requirement conditions. Let $\mathfrak{M} = \langle M, R\rangle$ be a model with a single binary predicate R. We define the partially ordered set $\langle M_0, \leqslant\rangle$, where the basic set M_0 is the image of M under the functor F_4. Then we set $M_0 = M \cup M^2 \times \{0,1\} \cup \{a_1, a_2, a_3, a_4, a_5\} \cup \{b_1, \ldots, b_7, b_8\}$, where elements of the set $\{a_1, a_2, a_3, a_4, a_5\} \cup \{b_1, \ldots, b_7, b_8\}$ are new.

Introduce a partial order $\leqslant$ on M_0 such that its transitive closure is the desired partial order on M_0:

1) $a_1 \leqslant a_2, a_2 \leqslant a_4, a_2 \leqslant a_3, a_4 \leqslant a_5,$

2) $b_1 \leqslant b_2, b_2 \leqslant b_3, b_3 \leqslant b_4, b_4 \leqslant b_5, b_5 \leqslant b_6, b_5 \leqslant b_7, b_7 \leqslant b_8,$

3) if $x_1, x_2 \in M$ and $x_1 \neq x_2$, then $\langle\langle x_1, x_2\rangle, 0\rangle \leqslant x_1$, and $\langle\langle x_1, x_2\rangle, i\rangle \leqslant x_2$ for $i \in \{0,1\}$,

4) if $x_1 \neq x_2 \in M$ and $\mathfrak{M} \models P(x_1, x_2)$, then $a_5 \leqslant \langle\langle x_1, x_2\rangle, 0\rangle$,

5) if $x_1 \neq x_2 \in M$ and $\mathfrak{M} \not\models P(x_1, x_2)$, then $a_3 \leqslant \langle\langle x_1, x_2\rangle, 0\rangle$,

6) if $x_1 \in M$ and $\mathfrak{M} \models P(x_1, x_1)$, then $b_6 \leqslant \langle\langle x_1, x_2\rangle, 0\rangle$,

7) if $x_1 \in M$ and $\mathfrak{M} \not\models P(x_1, x_1)$, then $b_8 \leqslant \langle\langle x_1, x_2\rangle, 0\rangle$.

We define $F_4(\mathfrak{M}, \mathfrak{M}')$ on isomorphisms φ in the same way as in the case of the functor F_3. The proof of the properties of this functor is similar to that in Proposition 4.29. □

Using the idea of the proof of Proposition 4.29, it is easy to construct a functor from the category of an arbitrary signature into the category of a bounded signature.

Proposition 4.31 ([**58**]). *For every signature Σ there exists a bounded signature Σ_0 and a completely univalent functor F_6 from* Mod^{Σ} *into* Mod^{Σ_0} *such that the following assertions hold.*

(i) $F_5 \restriction \mathrm{Mod}^{\Sigma}_{\mathrm{com}}$ *is a completely univalent functor from* $\mathrm{Mod}^{\Sigma}_{\mathrm{com}}$ *into* $\mathrm{Mod}^{\Sigma_0}_{\mathrm{com}}$.

(ii) *For every model $\mathfrak{M}$ of the signature Σ, the functor* $F_5 \restriction \mathrm{Mod}^{\Sigma}_{\mathrm{com}}(\mathfrak{M})$ *realizes an equivalence of the categories* $\mathrm{Mod}^{\Sigma}_{\mathrm{com}}(\mathfrak{M})$ *and* $\mathrm{Mod}^{\Sigma_0}_{\mathrm{com}}(F_5(\mathfrak{M}))$. *In addition,*

(a) *if $\mathfrak{M}$ is Δ^0_α categorical, then $F_5(\mathfrak{M})$ is Δ^0_α categorical,*

(b) *if $\mathfrak{M}$ has Δ^0_α dimension n, then $F_5(\mathfrak{M})$ has Δ^0_α dimension n,*

(c) *if $\mathfrak{M}$ does not have a formally Σ^0_α Scott family, then $F_5(\mathfrak{M})$ does not have a formally Σ^0_α Scott family.*

Proof. Consider a new signature σ^*. We put in σ^* all predicates from σ with arity $n \leqslant 2$. If a predicate symbol P_n has arity $m_n \geqslant 3$, then we add three new predicate symbols in σ^*: the binary predicate symbol R_n and two unary predicate symbols A_n and B_n. We also add one new unary predicate symbol U. Then we consider the impoverishment $\mathfrak{M}_n$ of the model $\mathfrak{M}$ of the signature $\Sigma_n = \langle P_n^{m_n} \rangle$ for every $m_n \geqslant 3$. We consider a model $\mathfrak{L}_n$ with $M \subseteq |\mathfrak{L}_n|$ that is isomorphic to the model $F_3(\mathfrak{M}_n)$ from Proposition 4.29 with isomorphism φ_n from $F_3(\mathfrak{M}_n)$ on this model $\mathfrak{L}_n$ such that for any $m \in M$ we have $\varphi_(m) = m$, but $|\mathfrak{L}_n| \cap |\mathfrak{L}_k| = M$ for any $n \neq k$. The basic set $|F_5(\mathfrak{M})|$ of the model $F_5(\mathfrak{M})$ is $\bigcup_n |\mathfrak{L}_n|$. We set $U = M$. Define a predicate symbol P from σ with arity $n \leqslant 2$ as the interpretation of this predicate in $\mathfrak{M}$. Now, define the remaining symbols of the signature σ^*. Let R_n on $|F_5(\mathfrak{M})|$ be equal to the binary predicate from $\mathfrak{L}_n$. But A_n is the set $|\mathfrak{L}_n| \setminus M$ and B_n is the set $\{\varphi_(a_0), \varphi_(a_1), \varphi_(a_2), \varphi_(b_0), \varphi_(b_1), \varphi_(b_2), \varphi_(c_0), \varphi_(c_1), \varphi_(c_2), \varphi_(c_3), \varphi_(c_4), \varphi_(c_5), \varphi_(c_6), \varphi_(c_7), \varphi_(c_8)\}$, where $\{a_0, a_1, a_2, b_0, b_1, b_2, c_0, c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8\}$ is the set of basic elements for the definability of F_3 on $\mathfrak{M}_n$.

Thus, we get the desired functor F_5 on the objects of the category. Using the construction of Proposition 4.29, we can define it on the morphisms of our category. The proof of the remaining assertions is similar to that of Proposition 4.29. $\square$

REMARK. If the signature contains functional symbols, we can pass to a new signature with predicates for graphs of this functions.

Thus, we proved the following assertion.

Theorem 4.32 ([**58**]). *For every signature* Σ *there exist a signature* Σ_0 *containing only one binary predicate* R *and a completely univalent functor* F *from* Mod^{Σ} *to* Mod^{Σ_0} *such that the following assertions hold.*

(i) $F \restriction \mathrm{Mod}^{\Sigma}_{\mathrm{com}}$ *is a completely univalent functor from* $\mathrm{Mod}^{\Sigma}_{\mathrm{com}}$ *to* $\mathrm{Mod}^{\Sigma_0}_{\mathrm{com}}$.

(ii) *For every model* $\mathfrak{M}$ *of the signature* Σ *the functor* $F \restriction \mathrm{Mod}^{\Sigma}_{\mathrm{com}}(\mathfrak{M})$ *realizes an equivalence of the categories* $\mathrm{Mod}^{\Sigma}_{\mathrm{com}}(\mathfrak{M})$ *and* $\mathrm{Mod}^{\Sigma_0}_{\mathrm{com}}(F_5(\mathfrak{M}))$. *In addition,*

  (a) *if* $\mathfrak{M}$ *is* Δ^0_α *categorical, then* $F(\mathfrak{M})$ *is* Δ^0_α *categorical,*

  (b) *if* $\mathfrak{M}$ *has* Δ^0_α *dimension* n*, then* $F(\mathfrak{M})$ *has* Δ^0_α *dimension* n*,*

  (c) *if* $\mathfrak{M}$ *does not have a formally* Σ^0_α *Scott family, then* $F(\mathfrak{M})$ *does not have a formally* Σ^0_α *Scott family.*

By Theorem 4.32, it suffices to consider only problems connected to computable equivalence and self-equivalence on partially ordered sets or graphs since there are no essential difficulties arise in the case of a more complicated signature.

The above results lead to the following assertion.

Theorem 4.33 (Goncharov–Tusupov, [**58**]). *Suppose that* $\mathcal{G}$ *is a graph structure and the partial ordering (graph)* $\Delta(\mathcal{G})$ *is constructed from* $\mathcal{G}$, $\mathcal{B}_i$ *in the same way as in Theorems* 4.25 *and* 4.26. *Then* $\mathcal{G}$ *has a* Δ^0_α *copy if and only if* $\Delta(\mathcal{G})$ *has a computable* *copy. In general, for any X the structure $\mathcal{G}$ has a $\Delta^0_\alpha(X)$ copy if and only if $\Delta(\mathcal{G})$ has an X-computable copy. In addition,*

(a) *if $\mathcal{G}$ has a unique up to a Δ^0_α isomorphism Δ^0_α copy, then $\Delta(\mathcal{G})$ is Δ^0_α categorical,*

(b) *if $\mathcal{G}$ has just n Δ^0_α copies, determined up to a Δ^0_α isomorphism, then $\Delta(\mathcal{G})$ has Δ^0_α dimension n,*

(c) *if $\mathcal{G}$ does not have a Σ^0_α-computable Scott family made up of finitary existential formulas, then $\Delta(\mathcal{G})$ does not have a formally Σ^0_α–Scott family.*

4.6. Lift of basic results

The following assertion lifts the result of Goncharov about computably categorical structures that are not relatively computably categorical.

Theorem 4.34 ([**72**])**.** *For every computable successor ordinal α there is a structure that is Δ^0_α categorical, but not relatively Δ^0_α categorical* (*and does not have a Σ^0_α–Scott family*)*.*

Corollary 4.35 (Goncharov–Tusupov, [**58**])**.** *For every computable successor ordinal α there is a partial ordering* (*graph*) *that is Δ^0_α categorical, but not relatively Δ^0_α categorical* (*and does not have a Σ^0_α–Scott family*)*.*

The following assertion lifts the result of Manasse [**106**] about relations that are intrinsically c.e., but not relatively intrinsically c.e.

Theorem 4.36 ([**72**])**.** *For every computable successor ordinal α there is a computable structure with a relation that is intrinsically Σ^0_α, but not relatively intrinsically Σ^0_α.*

Corollary 4.37 (Goncharov–Tusupov, [**58**])**.** *For every computable successor ordinal α there is a computable partial ordering* *(graph) with a relation that is intrinsically Σ^0_α but not relatively intrinsically Σ^0_α.*

The following assertion lifts the result of Goncharov about structures with finite computable dimension.

Theorem 4.38 ([**72**]). *For any computable successor ordinal α and a finite number n there is a computable structure with Δ^0_α dimension n.*

Corollary 4.39 (Goncharov–Tusupov, [**58**]). *For any computable successor ordinal α and a finite number n there is a computable partial ordering (graph) with Δ^0_α dimension n.*

The following assertion lifts the result of Slaman and Wehner.

Theorem 4.40 ([**72**]). *For every computable successor ordinal α there is a structure with copies in just the degrees of sets X such that $\Delta^0_\alpha(X)$ is not Δ^0_α. In particular, for every finite n there is a structure with copies in just the non-low$_n$ degrees.*

Corollary 4.41 (Goncharov–Tusupov, [**58**]). *For every computable successor ordinal α there is a partial ordering (graph) with copies in just the degrees of sets X such that $\Delta^0_\alpha(X)$ is not Δ^0_α. In particular, for every finite n there is a structure with copies in just the non-low$_n$ degrees.*

Based on examples of computable graphs and the construction of [**77**], one can construct many other algebraic structures with the same properties as in Theorems 4.34, 4.36, 4.38, 4.40.

5. Classes of Computable Models and Index Sets

In the study of computable models, it is important to consider not only individual models, but also classes of models defined by certain properties and to find relationships between the definability problems and algorithmic complexity expressed in terms of their index sets. As was shown by Nurtazin, [**130**], for a predicate signature there exists a computable numbering of all computable models of a given signature which is universal computable numbering of this class and is unique up to a recursive permutation. This fact provides us with a good tool for studying the algorithmic complexity of different classes of models of this signature.

One of questions in this direction is to express the complexity of definability of a class of mathematical structures in terms of the complexity of definability of the corresponding index sets in a universal numbering of all computable models of a given signature. This question is close to the investigations of Goncharov and Knight [**66**] on the structural properties of classes of computable models.

5.1. Computable classification or structure theorem

If K is a class, we denote by K^c the set of computable members of K. A *computable characterization* for K should separate computable members of K from other structures that either are outside K or belong to K, but are not computable. A computable classification (or a structure theorem) should describe up to an isomorphism (or up to a some other equivalence relation) every member of K^c, in terms of relatively simple invariants. On the other hand, a computable non-structure theorem should assert the absence of a computable structure theorem.

We consider three different approaches from [**66**]. Each of them gives a "correct" answer in the case of vector spaces over Q and linear orderings. Under each of these three approaches, both classes have computable characterization and there is a computable classification for vector spaces, but not for linear orderings.

In the first approach, K has a computable characterization if K^c is the set of computable models of some "computable" infinitary sentence. There is a computable classification for K if there is a computable bound on the "ranks" of elements of K^c.

In the second approach, K has a computable characterization if the set $I(K)$ of computable indices for elements of K^c is hyperarithmetical. There is a computable classification for K if the set $E(K)$ of pairs of indices corresponding to isomorphic structures is hyperarithmetical. (We also consider computable isomorphisms or Δ^0_α isomorphisms.)

In the third approach, K has a computable characterization if there is a hyperarithmetical list (an *enumeration*) of elements of K^c representing all isomorphism types. A computable classification theorem holds for K if there is an enumeration such that every computable isomorphism type is represented only once. (Again, we consider computable isomorphisms or Δ^0_α isomorphisms.)

Uncountable and countable structures are of great interest in model theory, The compactness theorem is a central result, so it is natural to use elementary first order formulas. In model theory, classes are normally characterized by elementary first order theories. In computable structure theory, we are interested in computable structures. Within the framework of computable structure theory, the compactness theorem does not play an essential role since it does not yield computable structures. If the compactness is established, we can deal with such classes as the Abelian p-groups which are not characterized by the elementary first order theory.

5.1.1. First approach.

We discuss characterization and classification in the following sense.

Computable characterization. There is a computable infinitary sentence whose computable models are just elements of K^c.

Computable classification. In addition to a computable infinitary sentence characterizing the computable members of K, there is a computable bound on "ranks" of elements of K^c.

We beging with a definition of rank and then indicate how it is connected with the complexity of isomorphisms. Then we consider applications of the characterization and classification statements to some well-known classes of structures.

Let us clarify how the above computable characterization and classification statements can be applied to some well-known classes of structures.

Computable characterization. Linear orderings, Boolean algebras, and equivalence structures can be characterized by a single elementary first order sentence. Vector spaces over Q and algebraically closed fields of a given characteristic can be characterized by either an infinite set of elementary first order sentences or a single computable Π_2 sentence. The class of Abelian p-groups is not characterized by any set of elementary first order sentences, but it is characterized by a single computable Π_2 sentence.

Some classes, for example, well orderings, superatomic Boolean algebras, and reduced Abelian p-groups cannot be characterized by a computable infinitary sentence. In fact, they cannot be characterized by any $L_{\omega_1\omega}$ sentence. The case of well orderings was considered by Lopez–Escobar [**98**].

Computable classification. For vector spaces over Q and algebraically closed fields of a given characteristic the computable rank is 1; we have the elimination of quantifiers. For equivalence structures the rank is at most 3.

The following assertion is well known.

Proposition 5.1 ([**66**]). *There is no computable bound on the ranks for the following classes* K :

(a) *linear orderings,*

(b) *Boolean algebras,*

(c) *Abelian p-groups,*

(d) *structures for language with at least one binary relation symbol.*

Each of the classes listed in Proposition 5.1 contains a structure of a noncomputble rank. The following assertion shows that it is a general fact.

Proposition 5.2 ([**66**]). *Let K^c be a set of computable models of a computable infinitary sentence ψ. If there is no computable bound on $R^c(\mathcal{A})$, for $\mathcal{A}$ in K^c, then there exists $\mathcal{A}$ in K^c such that $R^c(\mathcal{A}) = \omega_1^{CK}$.*

5.1.2. Second approach.

We consider the characterization and classification in terms of indices.

Definition 5.3. The *computable index* of a structure $\mathcal{A}$ is a number e such that $D(\mathcal{A}) = W_e$. The *index set* $I(K)$ of a class K is the set of computable indices of elements of K^c.

We assume that $\mathcal{A}_e$ is a structure with computable index e. The *isomorphism problem* for a class K is stated as follows:

$$E(K) = \{(a, b) : a, b \in I(K) \ \&\ \mathcal{A}_a \cong \mathcal{A}_b\}.$$

We write $\mathcal{A} \cong_{\Delta^0_\alpha} \mathcal{B}$ if $\mathcal{A}$ and $\mathcal{B}$ are isomorphic by a Δ^0_α isomorphism. The Δ^0_α *isomorphism problem* is stated as follows:

$$E_{\Delta^0_\alpha}(K) = \{(a, b) : a, b \in I(K) \ \&\ \mathcal{A}_a \cong_{\Delta^0_\alpha} \mathcal{A}_b\}.$$

Computable characterization: $I(K)$ is hyperarithmetical.

Computable classification: $E(K)$ is hyperarithmetical.

If we consider Δ^0_α isomorphisms, the classification result means that $E_{\Delta^0_\alpha}(K)$ is hyperarithmetical.

For many classes the index set is at a low level in the hyperarithmetical hierarchy.

Proposition 5.4 ([**66**]). *$I(K)$ is Π^0_2 for the following classes K :*

(a) *linear orderings,*

(b) *Boolean algebras,*

(c) *Abelian p-groups,*

(d) *equivalence structures,*

(e) *vector spaces over Q,*

(f) *structures for a fixed computable language.*

For some well-known classes, the index set is not hyperarithmetical.

Proposition 5.5 ([**140**]). *$I(K)$ is Π^1_1 complete for the following classes K:*

(a) *well orderings,*

(b) *superatomic Boolean algebras,*

(c) *reduced Abelian p-groups.*

We refer to [**139**] or [**5**] for the proof of (a).

We turn to the isomorphism problems. If $I(K)$ is hyperarithmetical, then $E(K)$ is at least Σ^1_1. For vector spaces over Q and algebraically closed fields of a given characteristic the isomorphism problem is at a low level of the hyperarithmetical hierarchy.

Proposition 5.6 (Calvert). *$E(K)$ is Π^0_3 complete for the following classes K:*

(a) *vector spaces over Q (or other infinite computable field),*

(b) *algebraically closed fields of a given characteristic.*

Below, we list several classes for which the isomorphism problem is Σ^1_1 complete (maximum complexity). These results are firmly established in folklore and are seemed to be known since the 1960's. However, I am not able to say exactly who was the first who proved them. In [**44**], there are related results in descriptive set theory concerning the Borel completeness of the isomorphism problem for various classes of structures with a fixed countable basic set. Note that the arguments in [**44**] can serve as the proof of the assertions formulated below.

Let $E(K)$ be Σ^1_1 complete for the following classes K:

(a) Abelian p-groups,

(b) trees,

(c) Boolean algebras,

(d) linear orderings,

(e) arbitrary structures for language with at least one binary relation symbol.

5.2. Special isomorphisms

We considered the set $I(K)$ with the equivalence relation $E(K)$. Now, we replace $E(K)$ with a computable isomorphism.

Proposition 5.7. *If $I(K)$ is Δ^0_3, then $E_{\Delta^0_1}(K)$ is at least Σ^0_3.*

Theorem 5.8 ([**66**]). *$E_{\Delta^0_1}(K)$ is Σ^0_3 complete (maximum complexity) for the following classes K :*

(a) *linear orderings,*

(b) *arbitrary structures for language with at least one binary relation symbol,*

(c) *Boolean algebras,*

(d) *Abelian p-groups,*

(e) *equivalence structures.*

We consider Δ^0_α isomorphisms instead of computable ismorphisms and generalize Proposition 5.7 and Theorem 5.8.

Proposition 5.9 ([**66**]). *If $I(K)$ is $\Delta^0_{\alpha+2}$, then $E_{\Delta^0_\alpha}(K)$ is $\Sigma^0_{\alpha+2}$.*

Theorem 5.10. *Let* $\alpha > 1$ *be computable. Then* $E_{\Delta^0_\alpha}(K)$ *is* $\Sigma^0_{\alpha+2}$ *complete (maximum complexity) for the following classes* K :

(a) *linear orderings,*

(b) *arbitrary structures for a computable language with at least one binary relation symbol,*

(c) *Boolean algebras,*

(d) *Abelian* p*-groups.*

The following assertion about linear orderings is very useful. The construction is based on the method from the Ash metatheorem [**5, 1**]. This method has many applications and, possibly, can serve as a metaconstruction for new computable models.

Theorem 5.11 ([**66**])**.** *There is a fixed computable linear ordering* $\mathcal{B}$ *such that for any* $\Sigma^0_{\alpha+2}$ *set* S *there is a uniformly computable sequence of linear orderings* $(\mathcal{C}_n)_{n\in\omega}$ *such that*

$$\begin{cases} \mathcal{C}_n \cong_{\Delta^0_\alpha} \mathcal{B} & \textit{if } n \in S, \\ \mathcal{C}_n \ncong \mathcal{B} & \textit{otherwise.} \end{cases}$$

To prove this theorem, we need the following lemma.

Lemma 5.12. *If* $\mathcal{A}$ *is a* Δ^0_α *ordering, then there is a computable* $\mathcal{B} \cong \omega^\alpha \cdot \mathcal{A}$ *with a* Δ^0_α *function sending every element of* $\mathcal{A}$ *to the first element of the corresponding copy of* ω^α *in* $\mathcal{B}$ *and there is a* Δ^0_α *procedure associating with every* $b \in \mathcal{B}$ *the position of* b *in the copy of* ω^α*. Moreover, it is possible to pass effectively from a* Δ^0_α *index for* $\mathcal{A}$ *to a computable index for* $\mathcal{B}$*,* Δ^0_α *indices for the rest.*

Proof. There are known related results (cf., for example, [**4, 5**]), but it seems that none of these results provides us with the desired assertion. Namely, the mentioned results can yield a Δ^0_3 embedding of a Δ^0_3 ordering $\mathcal{A}$ in a computable ordering of type

$\omega\mathcal{A}$, but not a Δ_2^0 embedding of a Δ_2^0 ordering $\mathcal{A}$ in a computable ordering of type $\omega\mathcal{A}$.

We use the metatheorem of Ash [**1**]. However, the general formulation is too large and restrict ourselves with some definitions and verify one nontrivial condition.

We define an α-system $(L, U, \widehat{\ell}, P, E, (\leqslant_\beta)_{\beta<\alpha})$ and a Δ_α^0 instruction function q such that $E(\pi)$ is the diagram of the desired $\mathcal{C}$, whereas π yields the rest. Without loss of generality, we assume that $\mathcal{A}$ has the first element. Suppose that the basic set A of $\mathcal{A}$ is an infinite computable set of constants and the first element in the ordering is also the first constant. Let U be the set of linear orderings on initial segments of A, including the first element. For every $u \in U$ we denote by $\mathcal{O}_u$ an ordering of type $\omega^\alpha u$. Assume that the following assertions hold.

(i) If $u \subseteq v$, then $\mathcal{O}_u \subseteq \mathcal{O}_v$.

(ii) The orderings $\mathcal{O}_u$ are computable uniformly in u and it is possible to determine effectively the Cantor normal form of intervals.

Let B be an infinite computable set of constants. Suppose that L consists of pairs (u, f), where $u \in U$ and f is a finite one-to-one function from B to $\mathcal{O}_u$. Let $\widehat{\ell} = (u, \varnothing)$, where u consists of only the first element. If $\ell = (u, f)$, we denote by $E(\ell)$ the set of atomic sentences and the negations of atomic sentences $\varphi(\overline{b})$ involving constants $\overline{b}$ from dom(f) such that f makes $\varphi(\overline{b})$ true in $\mathcal{O}_u$. If $\ell = (u, f)$ and $\ell' = (v, g)$, we assume that $\ell \leqslant_0 \ell'$ if $g \circ f^{-1}$ preserves order. Suppose that $\ell \leqslant_\beta \ell'$ if it preserves order and sends elements of a single copy of ω^β to elements at the corresponding positions, also in the single copy of ω^β. Let $\ell \subseteq \ell'$ if $u \subseteq v$ and $f \subseteq g$.

Denote by P the set of finite alternating sequences $\widehat{\ell} u_1 \ell_1 u_2 \ell_2 \ldots$, where

(1) $u_n \in U$ is an ordering on the first $n+1$ constants in A,

(2) $u_n \subseteq u_{n+1}$,

(3) $\ell_n \subseteq \ell_{n+1}$,

(4) if $\ell_n = (u, f)$, then $u = u_n$, $\operatorname{dom}(f)$ includes the first n elements of B, and $\operatorname{ran}(f)$ includes the first n elements of $\mathcal{O}_k$, for all $k \leqslant n$,

Thus, we defined the ingredients of the α-system. As usual, conditions (1)–(3) are trivially satisfied. Relative to condition (4), we suppose that $\sigma\ell^0 u \in P$, where $\ell^0 \leqslant_{\beta_0} \ell^1 \leqslant_{\beta_1} \ldots \leqslant_{\beta_{k-1}} \ell^k$ and $\alpha > \beta_0 > \beta_1 > \ldots > \beta_{k-1} > \beta_k$. We set $\ell_m = (u_m, f_m)$. Thus, we find $\ell'_m \supseteq \ell_m$ such that $\ell'_k = \ell_k$ and $\ell'_{m+1} \leqslant_{\beta_{m+1}} \ell'_m$. On the top, we have $\ell'_0 \supseteq \ell_0$. We set $\ell'_0 = (u_0, f)$. We have $u \supseteq u_0$. Let $\ell = (u, g)$, where $g \supseteq f$ includes suitable elements in the domain and range so that $\sigma\ell^0 u\ell \in P$. This ℓ is what we need to verify condition (4).

Thus, we have an α-system. Define a Δ^0_α instruction function q such that if $\sigma = \widehat{\ell} u_1 \ell_1 \ldots \ell_n$ is an element of P of length $2n+1$. Then $q(\sigma)$ is the substructure of $\mathcal{A}$ whose basic set consists of the first $n+1$ constants. Now, we can use the Ash metatheorem. We find a Δ^0_α run $\pi = \widehat{\ell} u_1 \ell_1 u_2 \ell_2 \ldots$ of (P, q) (π is a path through the tree P with u_n chosen by the instruction function q) such that $E(\pi) = \cup_n E(\ell_n)$ is c.e.

We set $\ell_n = (u_n, f_n)$. Then $\cup_n f_n$ is a one-to-one function from B onto $\mathcal{A}' = \cup_n \mathcal{O}_{u_n}$, where $\mathcal{A}'$ is a copy of $\omega^\alpha \mathcal{A}$. Let F be the inverse, and let $\mathcal{B}$ be the copy of $\mathcal{A}'$ induced on B by F. Then $D(\mathcal{B}) = E(\pi)$, so that $\mathcal{B}$ is computable. Now, F and $\mathcal{A}'$ are Δ^0_α. For a given $a \in \mathcal{A}$ we can use Δ^0_α to find the first element of the corresponding copy of ω^α in $\mathcal{A}'$. Similarly, for a given $b \in \mathcal{B}$ we can find $F^{-1}(b)$. Since we know the position of $F^{-1}(b)$ in its copy of ω^α, we also know the position of a. This completes the proof of Lemma 5.12. □

PROOF OF THEOREM 5.11. Relativizing the above lemma to Δ^0_α, we obtain a fixed computable ordering $\mathcal{B}^*$ of type ω^2 such that for any $\Sigma^0_{\alpha+2}$ set S there is a computable sequence of indices for Δ^0_α orderings $(\mathcal{C}^*_n)_{n\in\omega}$ such that $\mathcal{C}^*_n \cong_{\Delta^0_\alpha} \mathcal{B}^*$ for $n \in S$ and $\mathcal{C}^*_n \not\cong \mathcal{B}^*$ in the opposite case.

Let $\mathcal{B}$ and $\mathcal{C}_n$ be obtained from $\mathcal{B}^*$ and $\mathcal{C}_n^*$ as in Lemma 5.12. The sequence $(\mathcal{C}_n^*)_{n\in\omega}$ is uniformly computable. It is easy to see that if $n \in S$, then $\mathcal{C}_n^*$ has order type $\omega^{\alpha+2}$, whereas if $n \notin S$, then $\mathcal{C}_n$ has type $\omega^{\alpha+1}$. We are interested in Δ^0_α isomorphisms.

Claim 5.13. *If $n \in S$, then $\mathcal{C}_n^* \cong_{\Delta^0_\alpha} \mathcal{B}^*$ (with no uniformity).*

Proof. There is a Δ^0_α isomorphism from $\mathcal{C}_n$ onto $\mathcal{B}$. There is a Δ^0_α procedure that can be applied to $\mathcal{C}_n$ and $\mathcal{B}$ for determining the first element of every copy of ω and the successor relation on these elements. Consequently, there is a Δ^0_α procedure that can be applied to $\mathcal{C}_n^*$ and $\mathcal{B}^*$ for determining the Cantor normal form for the interval preceding every element. Therefore, there is a Δ^0_α isomorphism from $\mathcal{C}_n^*$ onto $\mathcal{B}^*$. This proves the claim. □

The proof of Theorem 5.11 is complete. □

Let $E_{\Delta^1_1}(K)$ be the set of pairs (a, b) such that $a, b \in I(K)$ and there is a hyperarithmetical isomorphism between $\mathcal{A}_a$ and $\mathcal{A}_b$. We might think of the statement that $E_{\Delta^1_1}(K)$ is hyperarithmetical as an alternative classification statement. If $I(K)$ is hyperarithmetical, the sets $E_{\Delta^0_\alpha}(K)$ are hyperarithmetical for all computable ordinals α. In the cases where we can show that $E(K)$ is hyperarithmetical, it is because there is a bound on ranks and $E(K)$ is equal to one of these sets.

Proposition 5.14 ([**66**]). *If $E_{\Delta^1_1}(K)$ is hyperarithmetical, then it is equal to $E_{\Delta^0_\alpha}(K)$ for some computable ordinal α.*

To prove this assertion, we can use the Barwise–Kreisel compactness. By assumptions, $I(K)$ is hyperarithmetical. We form a hyperarithmetical structure including all of the structures from K^c, their indices, and the relation $E_{\Delta^1_1}(K)$. Then we produce a hyperarithmetical set of computable infinitary sentences, say new constants a, b a pair of indices in the relation $E_{\Delta^1_1}(K)$, and there is no Δ^0_α isomorphism between the corresponding structures. Since we cannot satisfy the whole set, we try to get a suitable bound on the complexity of isomorphisms.

5.2.1. Third approach.

Here, we discuss characterization and classification statements involving lists (enumerations). In particular, a list is often taken for classification. Consider the classification of finite simple groups. A good list means that isomorphism types (other equivalence classes) are represented. It is natural to require that no isomorphism type (or equivalence class) appears twice.

Definition 5.15. An *enumeration of* $K^c/_{\cong}$ is a sequence $(\mathcal{A}_n)_{n\in\omega}$ representing every isomorphism type in K^c. An *enumeration of* $K^c/_{\cong_{\Delta^0_\alpha}}$ is a sequence representing every equivalence class in K^c under a Δ^0_α isomorphism.

Definition 5.16. A *Friedberg enumeration* of $K^c/_{\cong}$ or $K^c/_{\cong_{\Delta^0_\alpha}}$ is an enumeration such that every isomorphism type or every equivalence class under Δ^0_α isomorphism is represented only once.

Definition 5.17. An enumeration is *computable* (a Δ^0_α *enumeration*) if there is a computable (Δ^0_α-) sequence of computable indices for the structures.

Computable characterization. K has a computable characterization if there is a hyperarithmetical enumeration of $K^c/_{\cong}$ (other equivalence can be substituted for an isomorphism).

Computable classification. K has a computable classification if there is a hyperarithmetical Friedberg enumeration of $K^c/_{\cong}$ (other equivalence can be substituted for an isomorphism).

A computable enumeration $(\mathcal{A}_n)_{n\in\omega}$ of K^c is *universal* up to an isomorphism if for a given computable index for $\mathcal{B} \in K^c$ there exists n such that $\mathcal{B} \cong \mathcal{A}_n$. An enumeration is *principal* if for any other enumeration $(\mathcal{B}_n)_{n\in\omega}$ up to an isomorphism there is a computable function f such that $\mathcal{B}_n \cong \mathcal{A}_{f(n)}$. It is clear that a universal enumeration is principal.

5.2.2. Computable enumerations.

The following result of Nurtazin [**130**] yields the existence condition for computable enumerations of $K^c/_{\cong}$.

Theorem 5.18 ([**130**]). *Suppose that K is a class of structures such that for some $\mathcal{U} \in K^c$ and every $\mathcal{A} \in K^c$ there is a computable embedding of $\mathcal{A}$ into $\mathcal{U}$ and every c.e. subset W of $\mathcal{U}$ generates a unique structure $\mathcal{B} \subseteq \mathcal{U}$ in K. Then there is a computable enumeration of $K^c/_{\cong}$ determined up to an isomorphism. If for a given index for $\mathcal{A}$ there is an index for a computable embedding of $\mathcal{A}$ into $\mathcal{U}$, then there exists a computable universal enumeration of $K^c/_{\cong}$.*

Corollary 5.19 ([**66**]). *A computable universal enumeration of $K^c/_{\cong}$ exists for each of the following classes K :*

(a) *linear orderings,*

(b) *Boolean algebras,*

(c) *equivalence structures,*

(d) *Abelian p-groups (not necessarily reduced),*

(e) *algebraic fields of characteristic p,*

(f) *structures for a fixed computable relational language.*

In case (f), a universal model $\mathcal{U}$ can be obtained as the union of a chain of finite structures, where, at every stage, new elements are added in order to satisfy all possible open types over the set of "old" elements. The structure $\mathcal{U}$ is computably categorical, and its theory is $\aleph_0$ categorical.

Further, we can obtain the conclusion of Nurtazin's theorem without the assumption of computable embeddings.

Proposition 5.20. *If K is the class of vector spaces over Q, then there is a computable enumeration of $K^c/_{\cong}$. In fact, there is a principal enumeration.*

5.2.3. Existence of Friedberg enumerations.

For some classes with simple invariant it is easy to produce computable Friedberg enumerations.

Proposition 5.21 ([**66**]). *There is a computable Friedberg enumeration of $K^c/_{\cong}$ for the following classes K :*

(a) *vector spaces over Q,*

(b) *algebraically closed fields of a given characteristic,*

(c) *well orderings of type less than a fixed computable ordinal α.*

For computable equivalence structures there are natural invariants, but they are not so simple as the above examples. We suspect that there is no computable Friedberg enumeration up to an isomorphism. We have the following result.

Theorem 5.22 ([**66**]). *If K is the class of equivalence structures with infinitely many infinite classes, then $K^c/_{\cong}$ has a computable Friedberg enumeration.*

A direct proof of the nonexistence of a computable Friedberg enumeration is apparently a rather difficult question. Suppose that there exists a computable bound on the ranks of elements of K^c and there exists a computable Friedberg enumeration $(\mathcal{C}_n)_{n\in\omega}$ of $K/_{\cong}$. To obtain a contradiction, we try to find a computable $\mathcal{A} \in K$ satisfying the condition $\mathcal{A} \ncong \mathcal{C}_n$ for all n. It is difficult to work out a suitable strategy even if we restrict ourselves to only one of these conditions, for some n. The following assertions clarify the difficulties we meet in this way. The assumptions of the first assertion are the same as in Nurtazin's theorem.

Theorem 5.23 ([**66**]). *Suppose that there is a $\mathcal{U} \in K^c$ such that*

(1) *for every index e for a structure $\mathcal{A} \in K^c$ it is possible to find an index for a computable embedding of $\mathcal{A}$ into $\mathcal{U}$,*

(2) *every c.e. set $W \subseteq \mathcal{U}$ generates a unique substructure $\mathcal{B} \subseteq \mathcal{U}$ in K.*

Then there is no partial computable function f such that for any index e of $\mathcal{A} \in K^c$, $f(e)$ is an index of some $\mathcal{B} \in K^c$ such that $\mathcal{B} \not\cong \mathcal{A}$.

Corollary 5.24 ([**66**])**.** *For the following classes K there is no effective procedure such that for a given index for a computable $\mathcal{A}$ in K it yields an index for a computable $\mathcal{B}$ in K such that $\mathcal{A} \not\cong \mathcal{B}$:*

(a) *linear orderings,*

(b) *Boolean algebras,*

(c) *equivalence structures,*

(d) *arbitrary structures for a computable relational language.*

Thus, we clarified some of the difficulties arising in the attempts to obtain a direct proof of the nonexistence of Friedberg enumerations. Therefore, we use some results on the complexity of the isomorphism problems.

Proposition 5.25 ([**66**])**.** *Suppose that $I(K)$ is hyperarithmetical and $E(K)$ is properly Σ^1_1. Then there is no hyperarithmetical Friedberg enumeration of $K^c/_{\cong}$.*

Corollary 5.26 ([**66**])**.** *There is no hyperarithmetical Friedberg enumeration of $K^c/_{\cong}$ for the following classes K :*

(a) *linear orderings,*

(b) *Boolean algebras,*

(c) *Abelian p-groups,*

(d) *structures for a computable language with at least one binary relation symbol.*

Some of these results are known for isomorphisms of a fixed complexity. In particular, one of the results concerns the class K of vector spaces over Q. Note that the computable members of K are isomorphic only if they are Δ^0_2 isomorphic. As we seen, $K^c/_{\cong}$ has a computable principal enumeration and a computable Friedberg enumeration.

Proposition 5.27 ([**66**]). *If K is a class of vector spaces over Q, then there is no computable enumeration of $K^c/_{\Delta_1^0}$.*

Like Proposition 5.25, the following assertion concerns the nonexistence of Friedberg enumerations of various classes, up to a Δ_α^0 isomorphism.

Proposition 5.28 ([**66**]). *Suppose that $I(K)$ is $\Delta_{\alpha+2}^0$ and the Δ_α^0 isomorphism problem for K is properly $\Sigma_{\alpha+2}^0$. Then there is no $\Delta_{\alpha+2}^0$ Friedberg enumeration of $K^c/_{\cong_{\Delta_\alpha^0}}$.*

Corollary 5.29 ([**66**]). *Let K be one of the following classes:*

(a) *linear orderings,*

(b) *structures for a computable relational language with at least one binary relation symbol,*

(c) *Boolean algebras,*

(d) *Abelian p-groups.*

Then K^c has no Δ_3^0 Friedberg enumeration up to a computable isomorphism and for a computable ordinal α, K^c has no $\Delta_{\alpha+2}^0$ Friedberg enumeration up to a Δ_α^0 isomorphism.

We write $K^c/_{\cong_{\Delta_1^1}}$ for the set of equivalence classes of elements of K^c under a hyperarithmetical isomorphism. The assertion that $K^c/_{\cong_{\Delta_1^1}}$ has a hyperarithmetical Friedberg enumeration is an alternative classification statement.

Proposition 5.30 ([**66**]). *If $(\mathcal{A}_n)_{n\in\omega}$ is a hyperarithmetical enumeration of K^c determined up to a Δ_1^1-isomorphism then it is an enumeration of K^c determined up to a Δ_α^0 isomorphism for some computable ordinal α.*

Using the Barwise–Kreisel compactness, we obtain a Π_1^1 set of computable infinitary sentences describing a structure $\mathcal{B}$, an index e, and a function F such that $\mathcal{B}$ is computable, F is an isomorphism from $\mathcal{A}_e$ onto $\mathcal{B}$, and there is no Δ_α^0 isomorphism from $\mathcal{A}_e$ onto $\mathcal{B}$ for any computable ordinal α. If such an α does not exist, then every Δ^1_1 subset is satisfied. Hence we obtain a model of the whole set, which leads to a contradiction.

5.2.4. Relationship between three approaches.

We present the relationship between the basic characterization statements in the following form:

I. K^c is the set of computable models of a computable infinitary sentence
$\Downarrow \Uparrow$
II. $I(K)$ is hyperarithmetical
$\Downarrow \not\Uparrow$
III. $K^c/_{\cong}$ has a hyperarithmetical enumeration

It is easy to see that I $\Rightarrow$ II $\Rightarrow$ III. The result below asserts that II $\Rightarrow$ I.

Theorem 5.31 ([**66**]). *Suppose that K is a class of structures closed under an isomorphism and $I(K)$ is hyperarithmetical. Then there is a computable infinitary sentence for which K^c is the class of computable models.*

The following result asserts that III $\not\Rightarrow$ II.

Proposition 5.32 ([**66**]). *Let K consist of copies of $\omega_1^{CK}(1+\eta)$ and the linear orderings of rank at most ω. Then $K^c/_{\cong}$ has a hyperarithmetical Friedberg enumeration. However, $I(K)$ is not hyperarithmetical.*

In the first classification statement, we added the corresponding characterization statement. A computable bound on the ranks of elements of K^c, without a sentence whose computable models are these elements, does not tell us much. The remaining classification statements imply the corresponding classification statements. We summarize the relations among the basic classification statements as follows.

I. There is a computable bound on the ranks of elements of K^c, in addition to a computable infinitary sentence whose computable models are these structures
$\Downarrow \; \Uparrow ?$
II. $E(K)$ is hyperarithmetical
$\Downarrow \; \not\Uparrow$
III. $K^c/_{\cong}$ has a hyperarithmetical Friedberg enumeration

It is easy to see that I $\Rightarrow$ II $\Rightarrow$ III. By Proposition 5.32, III $\not\Rightarrow$ II. For a class K in this proposition, $K^c/_{\cong}$ has a hyperarithmetical Friedberg enumeration, but $I(K)$ is not hyperarithmetical. Hence $E(K)$ cannot be hyperarithmetical. Under the assumption that there is a computable infinitary sentence ψ whose computable models are elements of K^c, it is not known whether III $\Rightarrow$ II.

We formulate a partial result concerning the implication II $\Rightarrow$ I or III $\Rightarrow$ I.

Theorem 5.33 ([**66**]). *Suppose that $K^c/_{\cong}$ has a hyperarithmetical Friedberg enumeration. Then there is computable ordinal α such that for $\mathcal{A}$, $\mathcal{B}$ in K^c, if every computable Π_α sentence true in $\mathcal{A}$ is true in $\mathcal{B}$, then $\mathcal{A} \cong \mathcal{B}$.*

The first approach to the characterization problem is natural from the mathematical point of view. Known classes of structures (for example, groups and fields) are described by using axioms. The second approach, involving index sets, seems to be far from common practice in mathematics. Nevertheless, the characterization statements for the first and second approaches are equivalent. The third approach to the classification problems which yields a list without repetition of invariants is very natural from the mathematical point of view. As we seen, the classification statements obtained by the second and third approaches are not equivalent, although there are relations between them. For some classes with nice invariants (for example, vector spaces over the rational numbers) we can give a computable Friedberg enumeration. However, in the majority of cases, where we established the nonexistence of computable Friedberg enumerations, the proof was indirect and used the complexity of the isomorphism problem.

It is very important to determine precisely the complexity of the isomorphism problem for various classes. Having a classification, it is reasonable to look for the least computable ordinal α such that $E(K) = E_{\Delta^0_\alpha}(K)$.

Problem 31. Whether there is an example of a class K for which the isomorphism problem is properly at some level, Σ^1_1, Π^0_3, etc., but is not complete at this level?

In all cases where we located $E(K)$ properly at some level of complexity (by proving that it is Σ^1_1, but not Δ^1_1 or by proving that it is Π^0_3, but not Δ^0_3), it turned out to be complete at that level. Problem 31 is related to the long-standing challenge of finding a "natural" example of a c.e. set such that it is neither computable nor complete.

The following problem concerns a special case of the missing implication II $\Rightarrow$ I for the classification problems, where K consists of copies of a single computable structure $\mathcal{A}$. In this case, $E(K)$ is essentially the same as $I(\mathcal{A})$.

Problem 32. Whether $R^c(\mathcal{A})$ is computable provided that $I(\mathcal{A})$ is hyperarithmetical?

Definition 5.34. Let $\mathcal{A}$ be a computable structure such that $R^c(\mathcal{A}) = \omega_1^{CK}$. We say that $\mathcal{A}$ is *computably approximable* if every computable infinitary sentence true in $\mathcal{A}$ is also true in some computable $\mathcal{B} \not\cong \mathcal{A}$.

The known examples of computable structures of noncomputable rank (for example, the Harrison ordering) are computably approximable. This can be explained by the fact that they were obtained from a family of computable approximations by using the Barwise–Kreisel compactness or by some other similar methods.

Problem 33. Let $\mathcal{A}$ be computable, and let $R^c(\mathcal{A}) = \omega_1^{CK}$.

(a) Whether $\mathcal{A}$ is computably approximable?

(b) Whether any true computable infinitary sentence in $\mathcal{A}$ is also true in some computable structure $\mathcal{B}$ of computable rank?

By Proposition 3.11, if $\mathcal{A}$ is computable and $R^c(\mathcal{A}) = \omega_1^{CK}$, then any computable infinitary sentence true in $\mathcal{A}$ is also true in some *hyperarithmetical* $\mathcal{B} \ncong \mathcal{A}$.

REMARK. Problems 32 and 33 (a) are equivalent. If Problem 33 (a) has a negative answer confirmed by a computable structure $\mathcal{A}$, then Problem 32 has a negative answer confirmed by the same structure $\mathcal{A}$. If Problem 33 (a) has a positive answer, then we can use the Barwise–Kreisel compactness to show that Problem 32 has a positive answer.

Problem 34. Let K be a class of equivalence structures. Whether a computable Friedberg enumeration of K^c exists up to an isomorphism?

5.3. Definability and index sets of natural classes of computable models

As was already mentioned, the study of the complexity of index sets for computable models is important for understanding structural properties, classifications of models, and complexity level of classifications. On the other hand, if there exists a universal enumeration of computable classes of models of a given structure numbering, we can compare classes by the complexity of their description and choose the most adequate description corresponding to their real algorithmic complexity.

One of the goals of the theory of computable models is to characterize the complexity of classes of autostable models of finite or infinite algorithmic dimension with the Scott family of $\exists$ formulas in a finite enrichment by constants. This question is also of interest for a certain level of arithmetic hierarchy and its extension by notations of constructive ordinals, and the interaction of the complexity of the definition of these classes of models. Related topics are the complexity of index sets of computable models of a given Scott rank; in particular, the case of nonconstructive Scott ranks models of Scott rank ω_1^{CK} and $\omega_1^{\mathrm{CK}}+1$.

Another cycle of problems is connected with the complexity of finding computable models with theories of a given type and, in particular, the case of a theory categorical in uncountable power, a finitely axiomatizable theory, an Ehrenfeucht theory, a theory without prime model, a theory with countably many countable models, an ω-stable theory, a stable theory, a theory with countably many types, a decidable theory, an elementary theory of a given complexity, a theory with a given complexity of decidability of computable models with respect to Turing degrees, a theory with one computable model, a strongly minimal theory, etc. It is of interest to clarify whether the Turing degree of a theory from the above list is universal in the corresponding hierarchy class of complexity.

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