Chapter 4
Applications of Nonlocally Related PDE Systems

4.1 Introduction

In Chapter 3, it was shown how one can systematically construct a set (*tree*) of PDE systems nonlocally related to a given PDE system. In particular, local conservation laws of a PDE system lead to augmented nonlocally related (potential) systems that explicitly include nonlocal (potential) variables. Moreover, further nonlocally related PDE systems (nonlocally related subsystems) arise when one or more dependent variables (including dependent variable(s) arising after a point transformation that involves an interchange of dependent and independent variable(s)) are excluded from a PDE system or its potential systems, through differential relations. In Section 3.5, an algorithm for the construction of an extended tree of nonlocally related systems was outlined. In particular, n local conservation laws of a given PDE system lead to a tree of up to $2^n - 1$ nonlocally related potential systems. A tree is further extended by considering subsystems of both the given PDE system and its nonlocally related potential systems as well as by considering potential systems arising from conservation laws (whose multipliers have an essential dependence on potential variables) of its nonlocally related potential systems.

Nonlocally related systems in such extended trees are important for applications since they are constructed systematically and each solution of any PDE system in such a tree yields a solution of any other PDE system in the tree, including the given PDE system. More importantly, there is not a one-to-one mapping between solutions of such nonlocally related systems. Consequently, the usefulness of standard methods of analysis, especially coordinate independent methods, can be enhanced when directly applied to different nonlocally related PDE systems. In particular, a method of analysis could be successful in achieving results when applied directly to a nonlocally related system in a tree even if it is unsuccessful in achieving results when directly applied to the given PDE system. Furthermore, from the simplicity

G.W. Bluman et al., *Applications of Symmetry Methods to Partial Differential Equations*, Applied Mathematical Sciences 168, DOI 10.1007/978-0-387-68028-6_4,

of the construction of the mappings that relate PDE systems in an extended tree, it is usually simple to transfer results achieved for a PDE system in such a tree to other PDE systems in the tree, including the given PDE system.

Applications that naturally can arise from the use of such nonlocally related systems include:

(1) The construction of nonlocal conservation laws of a given PDE system that arise as local conservation laws of nonlocally related PDE systems
This application was illustrated in Chapter 3 in the construction of nonlocally related PDE systems arising from local conservation laws of potential systems that in themselves arose from local conservation laws of the given PDE system. Such local conservation laws of potential systems can yield nonlocal conservation laws of the given PDE system, i.e., conservation laws whose fluxes and/or densities have an essential dependence on potential variables. Furthermore, such local conservation laws of potential systems may actually yield further local conservation laws of the given PDE system that had not been previously determined due to lack of completeness in the direct calculation of its local conservation laws.

(2) The construction of nonlocal symmetries of a given PDE system
In this chapter it is shown that point symmetries of a PDE system in a tree of nonlocally related systems can systematically yield nonlocal symmetries of a given PDE system.

A symmetry of a PDE system is defined topologically as a mapping (deformation) of its solution manifold into itself. From this point of view, essentially every PDE system has symmetries. The problem is to find such symmetries and to find those that have applications. In particular, to find explicit symmetries it is necessary to calculate them in some fixed coordinate system. Moreover, such calculations are simple to perform and the resulting symmetries are directly applicable if obtained through a direct application of Lie's algorithm, which yields only local symmetries of a PDE system. The infinitesimals of local symmetries depend at most on a finite number of derivatives of the dependent variables of the PDE system. However, such local symmetries constitute at most a small subset of the total set of symmetries of a PDE system.

In this chapter, it is shown that additional (nonlocal) symmetries of a given PDE system can be found by a direct application of Lie's algorithm to PDE systems in a tree of nonlocally related systems. For the computation of such nonlocal symmetries, it turns out that both nonlocally related potential systems and subsystems can separately yield new symmetries (contrary to the situation in the computation of nonlocal conservation laws, where all local conservation laws of a subsystem are included in the local conservation laws of a PDE system yielding the subsystem).

A point symmetry of a potential system yields a nonlocal symmetry of a given PDE system if at least one of its infinitesimal generator components for the dependent and independent variables of the given PDE system has an essential dependence on a potential (nonlocal) variable. On the other hand, in the case of a nonlocally related subsystem, in order to isolate a nonlocal symmetry of the given PDE system that arises from a point symmetry of the subsystem, one has to compare the local symmetries of both the given PDE system and the nonlocally related subsystem to determine whether a point symmetry of the subsystem yields a nonlocal symmetry of the given PDE system.

For a given PDE system that includes arbitrary constitutive functions and/or parameters, one is interested in the classification of its local and nonlocal symmetries with respect to such functions and/or parameters. In order to do this, one can classify the local symmetries (with respect to such functions and/or parameters) of PDE systems in a tree of nonlocally related systems constructed for a given PDE system. In this chapter, nonlinear diffusion equations, nonlinear wave equations and the equations of planar gas dynamics are considered as illustrative examples for such classifications. Comparisons are made of the point symmetries of various nonlocally related PDE systems in their respective trees to determine the point symmetries yielding nonlocal symmetries of particular systems in trees.

(3) The construction of solutions of a given PDE system that arise from symmetry reductions due to nonlocal symmetries but do not arise as invariant solutions from symmetry reductions due to point symmetries

For a given PDE system, an important application of nonlocal symmetries that arise from point symmetries of a nonlocally related system in a tree results from the construction of the corresponding invariant solutions of the nonlocally related system. In particular, such solutions are especially interesting when the corresponding solutions of the given PDE system are not invariant solutions that can be constructed from the point symmetries of the given PDE system. This application is considered in the next chapter [Section 5.2.3].

In Section 5.2.3, such solutions are constructed for the linear wave equation with a variable wave speed $c(x)$. It is shown that a potential system of such a linear wave equation has point symmetries that are nonlocal symmetries of the linear wave equation for an interesting special form of the constitutive function $c(x)$ corresponding to wave propagation in two-layered media with smooth transitions. These symmetries yield a countable infinite set of invariant solutions for initial value problems. Moreover, this set of solutions is complete and can be used to obtain Fourier series solutions for initial value problems with arbitrary piecewise smooth data in the infinite space domain.

A second example yields physical solutions for the Lagrange system of the planar gas dynamics equations that arise as invariant solutions obtained from nonlocal symmetries that are point symmetries of nonlocally related systems.

(4) The construction of non-invertible mappings relating PDEs
In Chapter 2, two important mapping problems were considered systematically: the invertible mapping of a given nonlinear PDE system to some linear PDE system (in terms of the point/contact symmetries or local conservation law multipliers of the nonlinear PDE system) and the invertible mapping of a given linear PDE with variable coefficients to a linear PDE with constant coefficients (in terms of the point symmetries of abelian type of the linear PDE with variable coefficients). Here these results are extended systematically to include non-invertible mappings.

Firstly, if a nonlocally related PDE system in a tree can be linearized by a point transformation whereas the given PDE system cannot be linearized by a point (contact) transformation, then one obtains a non-invertible mapping of the given PDE system to some linear system. Such non-invertible mappings arise from computing the point symmetries or local conservation law multipliers of a nonlocally related PDE system.

Secondly, suppose a given linear PDE system with variable coefficients cannot be mapped invertibly to a linear PDE system with constant coefficients. It turns out that for any given linear PDE system, it is straightforward to construct an infinite number of potential systems since any solution of the adjoint system of a given linear PDE system yields a set of conservation law multipliers. If one of the corresponding potential systems can be invertibly mapped into a constant coefficient linear PDE system, then as a consequence the given linear PDE is mapped non-invertibly to a constant coefficient linear PDE system. Such non-invertible mappings are constructed for linear parabolic equations with variable coefficients and lead to a significant extension of the classes of linear parabolic equations that can be mapped into the heat equation beyond those found in Section 2.5.1.

The results presented in this chapter have appeared in Bluman & Kumei [(1987), (1988), (1989)], Akhatov, Gazizov & Ibragimov (1991), Ames, Lohner & Adams (1981), Bluman & Cheviakov (2007), Kingston & Sophocleous (2001), Bluman, Temuerchaolu & Sahadevan (2005), Bluman, Cheviakov & Ivanova (2006), and Bluman & Shtelen [(1996a), (2004)].

4.2 Nonlocal Symmetries

Local symmetries of a nonlocally related system can yield explicit symmetries (*nonlocal symmetries*) of a given system of PDEs that do not arise as local symmetries by a direct application of Lie's algorithm to the given system. In particular, such nonlocal symmetries arise as local symmetries of nonlocally related systems with infinitesimal generators having an essential dependence on nonlocal potential variables in the case of nonlocally related systems that are not subsystems. It is shown that this significantly enhances the applicability of symmetry methods.

A symmetry of a system of differential equations is defined topologically as any transformation of its solution manifold into itself. Hence, symmetry transformations are not restricted to local transformations arising from infinitesimal generators whose coefficients are functions of the given system's independent and dependent variables and their derivatives to some finite order. Through many examples, it is demonstrated that local symmetries do not include all calculable (as well as useful) symmetries of a given PDE system.

Suppose a system of PDEs $\mathbf{R}\{x,t\,;u\}$ has a potential system (k-plet) $\mathbf{S}\{x,t\,;u,v\}$ that is invariant under the one-parameter (ϵ) Lie group of point transformations

$$\begin{aligned} x^* &= x + \epsilon\xi_S(x,t,u,v) + O(\epsilon^2), \\ t^* &= t + \epsilon\tau_S(x,t,u,v) + O(\epsilon^2), \\ u^* &= u + \epsilon\eta_S(x,t,u,v) + O(\epsilon^2), \\ v^* &= v + \epsilon\zeta_S(x,t,u,v) + O(\epsilon^2), \end{aligned} \tag{4.1}$$

with corresponding infinitesimal generator

$$\mathrm{X} = \xi_S^i(x,t,u,v)\frac{\partial}{\partial x^i} + \eta_S^\mu(x,t,u,v)\frac{\partial}{\partial u^\mu} + \zeta_S^p(x,t,u,v)\frac{\partial}{\partial v^p}; \tag{4.2}$$

ξ_S^i, $i = 1,2$, are the infinitesimals corresponding to the independent variables $(x^1,x^2) = (x,t)$, η_S^μ are the infinitesimals corresponding to the dependent variables u^μ of $\mathbf{R}\{x,t\,;u\}$, $\mu = 1,\ldots,m$, and ζ_S^p are the infinitesimals corresponding to the potential variables v^p, $p = 1,\ldots,k$ of the k-plet potential system $\mathbf{S}\{x,t\,;u,v\}$.

The point symmetry (4.1) maps any solution of $\mathbf{S}\{x,t\,;u,v\}$ to a solution of $\mathbf{S}\{x,t\,;u,v\}$, and hence through projection, induces a mapping of any solution of $\mathbf{R}\{x,t\,;u\}$ to a solution of $\mathbf{R}\{x,t\,;u\}$. Thus (4.1) yields a symmetry of $\mathbf{R}\{x,t\,;u\}$. However, if the infinitesimals $(\xi_S(x,t,u,v), \eta_S(x,t,u,v))$ do not depend explicitly on the nonlocal potential variables v, i.e.,

$$\frac{\partial\xi_S^i}{\partial v} \equiv 0, \ \frac{\partial\eta_S^\mu}{\partial v} \equiv 0, \quad i = 1,2; \quad \mu = 1,\ldots,m, \tag{4.3}$$

then (4.1) only yields a point symmetry of $\mathbf{R}\{x,t\,;u\}$, in terms of the infinitesimal generator given by

$$\mathrm{X} = \xi_S^i(x,t,u)\frac{\partial}{\partial x^i} + \eta_S^\mu(x,t,u)\frac{\partial}{\partial u^\mu}. \tag{4.4}$$

On the other hand, if the infinitesimals $(\xi_S(x,t,u,v), \eta_S(x,t,u,v))$ have an essential dependence on v, then the point symmetry (4.1) defines a nonlocal symmetry of $\mathbf{R}\{x,t\,;u\}$, since the potential variables v are nonlocal variables. This leads to the following definition and the proof of the subsequent theorem.

Definition 4.2.1. The point symmetry (4.1) of the potential system $\mathbf{S}\{x,t;u,v\}$ defines a *potential symmetry* of a PDE system $\mathbf{R}\{x,t\,;u\}$ if and only if the infinitesimals $(\xi_S(x,t,u,v), \tau_S(x,t,u,v), \eta_S(x,t,u,v))$ depend explicitly on one or more components of v.

Theorem 4.2.1. *A potential symmetry of* $\mathbf{R}\{x,t\,;u\}$ *is a nonlocal symmetry of* $\mathbf{R}\{x,t\,;u\}$.

Nonlocal symmetries of PDE systems can arise as potential symmetries (i.e., point symmetries of singlet or k-plet potential systems), as well as symmetries of nonlocally related subsystems, as discussed below. Related to this, it is important to note that a local symmetry of $\mathbf{R}\{x,t\,;u\}$ could yield a nonlocal symmetry of $\mathbf{S}\{x,t\,;u,v\}$. [By construction, $\mathbf{R}\{x,t\,;u\}$ is an obvious nonlocally related subsystem of $\mathbf{S}\{x,t\,;u,v\}$.]

Suppose $\mathbf{R}\{x,t\,;u\}$ is a given PDE system with m dependent variables, and $\underline{\mathbf{R}}\{x,t\,;u^{\mu_1},\ldots,u^{\mu_{m-p}}\}$ is a subsystem with $m-p$ dependent variables that is obtained by excluding p dependent variables u^α in $\mathbf{R}\{x,t\,;u\}$. Consider the problem of comparing the local symmetries of $\mathbf{R}\{x,t\,;u\}$ with those of its subsystem $\underline{\mathbf{R}}\{x,t\,;u^{\mu_1},\ldots,u^{\mu_{m-p}}\}$.

If the subsystem $\underline{\mathbf{R}}\{x,t\,;u^{\mu_1},\ldots,u^{\mu_{m-p}}\}$ is locally related to $\mathbf{R}\{x,t\,;u\}$ (in the sense of Theorem 3.2.2), then there is a one-to-one correspondence between solutions of the two systems. Consequently, the following theorem holds.

Theorem 4.2.2. *A local symmetry of a locally related subsystem* $\underline{\mathbf{R}}\{x,t;u^{\mu_1},\ldots,u^{\mu_{m-p}}\}$ *of a PDE system* $\mathbf{R}\{x,t\,;u\}$ *is a projection of some corresponding local symmetry of* $\mathbf{R}\{x,t\,;u\}$ *onto the space of variables of* $\underline{\mathbf{R}}\{x,t;u^{\mu_1},\ldots,u^{\mu_{m-p}}\}$.

Note that a *point* symmetry of a PDE system $\mathbf{R}\{x,t\,;u\}$ could project onto a *point* or *contact* (or, more generally, higher-order (*local*)) symmetry of a locally related subsystem $\underline{\mathbf{R}}\{x,t\,;u^{\mu_1},\ldots,u^{\mu_{m-p}}\}$.

The situation is different for a nonlocally related subsystem. Here, there is not a one-to-one correspondence between the solutions of a given PDE system and a nonlocally related subsystem. In particular, numerous examples exist where a local symmetry X of a nonlocally related subsystem $\underline{\mathbf{R}}\{x,t\,;u^{\mu_1},\ldots,u^{\mu_{m-p}}\}$ does not correspond to any local symmetry of $\mathbf{R}\{x,t\,;u\}$, and conversely, a local symmetry Y of $\mathbf{R}\{x,t\,;u\}$ does not correspond to a local symmetry of $\underline{\mathbf{R}}\{x,t\,;u^{\mu_1},\ldots,u^{\mu_{m-p}}\}$. For the rest of this chapter we only consider point symmetries of PDE systems. Correspondingly, one can modify the statements in this paragraph through replacing "local symmetry" by "point symmetry".

Summarizing the above discussion, one can isolate three different types of nonlocal symmetries that can be sought for a given PDE system $\mathbf{R}\{x,t\,;u\}$.

1. Nonlocal symmetries arising from point symmetry analysis of nonlocally related subsystems of $\mathbf{R}\{x,t\,;u\}$ obtained by excluding one or more of its dependent variables. [Recall that such nonlocally related subsystems could also arise through exclusion of a dependent variable that arises after an interchange of one or more independent and dependent variables of $\mathbf{R}\{x,t\,;u\}$.]
2. Nonlocal symmetries (potential symmetries) that arise as point symmetries of potential systems (including k-plet potential systems) of $\mathbf{R}\{x,t\,;u\}$.
3. Nonlocal symmetries that arise as point symmetries of nonlocally related subsystems of potential systems of $\mathbf{R}\{x,t\,;u\}$.

More generally, such nonlocal symmetries of $\mathbf{R}\{x,t\,;u\}$ can arise from seeking local symmetries of any PDE system in an extended tree of nonlocally related systems that includes $\mathbf{R}\{x,t\,;u\}$.

Among all such nonlocal symmetries of a PDE system $\mathbf{R}\{x,t\,;u\}$, the ones that explicitly involve nonlocal variables (Type 2 and, in part, Type 3) are easier to distinguish. In the case of finding Type 1 (and the remaining ones of Type 3) nonlocal symmetries of a PDE system $\mathbf{R}\{x,t\,;u\}$, in order to isolate nonlocal symmetries arising from a subsystem whose infinitesimal components for (x,t,u) do not involve nonlocal variables, one must find all point symmetries of $\mathbf{R}\{x,t\,;u\}$, and then see if a point symmetry of a considered nonlocally related system is included in the complete point symmetry analysis of $\mathbf{R}\{x,t\,;u\}$.

It often turns out, as is illustrated by several examples, that a given system $\mathbf{R}\{x,t\,;u\}$ with an arbitrary constitutive function(s) can have nonlocal symmetries for special forms of the constitutive function(s), arising as point symmetries of one or more systems in an extended tree of nonlocally related systems.

4.2.1 Nonlocal symmetries of a nonlinear diffusion equation

As a first example, consider a symmetry classification problem for the nonlinear diffusion equation $\mathbf{U}\{x,t\,;u\}$ given by

$$u_t - (K(u)u_x)_x = 0, \tag{4.5}$$

with an arbitrary constitutive function $K(u) = L'(u)$ [Bluman & Kumei (1989); Akhatov, Gazizov & Ibragimov (1991)]. All computations below are presented modulo the group of equivalence transformations of the class of PDEs (4.5), given by

$$\begin{gathered} \widetilde{t} = a_4 t + a_1, \quad \widetilde{x} = a_5 x + a_2, \quad \widetilde{u} = a_6 u + a_3, \\ \widetilde{K}(\widetilde{u}) = \frac{a_5^2}{a_4} K(u), \quad \widetilde{L}(\widetilde{u}) = \frac{a_5^2 a_6}{a_4} K(u) + a_7, \end{gathered} \tag{4.6}$$

where $a_1, \ldots, a_7$ are arbitrary constants with $a_4 a_5 a_6 \neq 0$.

An extended tree $\mathcal{T}_4$ of nonlocally related PDE systems for the nonlinear diffusion equation $\mathbf{U}\{x, t\,; u\}$ (4.5), holding for an arbitrary $K(u)$, was constructed in Section 3.5.2, and shown in Figure 3.3. This tree contains nine nonlocally related PDE systems that have equivalence transformations similar to those in (4.6) [Exercise 4.2.1]. It is also important to note that some of the systems within the tree $\mathcal{T}_4$, namely, $\mathbf{UVA}\{x, t\,; u, v, \alpha\}$ (3.73), $\mathbf{UV}\{x, t\,; u, v\}$ (3.19), $\mathbf{V}\{u, t\,; v\}$ (3.22), $\mathbf{X}\{u, v\,; x\}$ [Exercise 3.3.3] have an additional (projective) equivalence transformation

$$\begin{aligned} &\widetilde{t} = t, \quad \widetilde{x} = x - bv, \quad \widetilde{u} = \frac{u}{1 - bu}, \quad \widetilde{K}(\widetilde{u}) = (1 + b\widetilde{u})^{-2} K\left(\frac{\widetilde{u}}{1 + b\widetilde{u}}\right), \\ &\widetilde{L}(\widetilde{u}) = L\left(\frac{\widetilde{u}}{1 + b\widetilde{u}}\right); \end{aligned} \tag{4.7}$$

whereas the remaining nonlocally related systems, $\mathbf{UA}\{x, t\,; u, \alpha\}$ (3.73), $\mathbf{U}\{x, t\,; u\}$ (4.5), and $\mathbf{A}\{x, u\,; \alpha\}$ (3.38) do not have the equivalence transformation (4.7). [It is a nonlocal transformation of these systems!]

Before seeking nonlocal symmetries of $\mathbf{U}\{x, t\,; u\}$ (4.5), we present its point symmetry classification [Table 4.1] [Ovsiannikov (1959)]. One can show that no contact symmetries arise for any form of $K(u)$.

Table 4.1 Local (point) symmetries of the nonlinear diffusion equation $\mathbf{U}\{x, t\,; u\}$ (4.5)

$K(u)$	#	Point Symmetries
Arbitrary	3	$X_1 = \frac{\partial}{\partial x},\ X_2 = \frac{\partial}{\partial t},\ X_3 = x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t}.$
u^ν	4	$X_1,\ X_2,\ X_3,\ X_4 = x\frac{\partial}{\partial x} + \frac{2}{\nu}u\frac{\partial}{\partial u}.$
e^u	4	$X_1,\ X_2,\ X_3,\ X_5 = x\frac{\partial}{\partial x} + 2\frac{\partial}{\partial u}.$
$u^{-4/3}$	5	$X_1,\ X_2,\ X_3,\ X_4\left(\nu = -\frac{4}{3}\right),\ X_6 = x^2\frac{\partial}{\partial x} - 3xu\frac{\partial}{\partial u}.$

In principle, nonlocal symmetries of the nonlinear diffusion equation $\mathbf{U}\{x,t\,;u\}$ (4.5) can arise from any nonlocally related system within the tree $\mathcal{T}_4$ given by (3.75). In Table 4.2, we present the point symmetry classification of the two singlet potential systems $\mathbf{UV}\{x,t\,;u,v\}$ (3.19) and $\mathbf{UA}\{x,t\,;u,\alpha\}$ (3.32).

In comparison with Table 4.1, it is obvious that the point symmetry classification of the singlet potential system $\mathbf{UA}\{x,t\,;u,\alpha\}$ (3.32) yields no nonlocal symmetries of the nonlinear diffusion equation $\mathbf{U}\{x,t\,;u\}$ (4.5). On the other hand, the point symmetry classification of the singlet potential system $\mathbf{UV}\{x,t\,;u,v\}$ (3.19) yields potential symmetries of $\mathbf{U}\{x,t\,;u\}$.

In particular, when $K(u)=u^{-2}$, the system $\mathbf{UV}\{x,t\,;u,v\}$ has an infinite number of point symmetries that lead to the linearization of the system $\mathbf{UV}\{x,t\,;u,v\}$ by a point transformation [Section 2.4]; when $K(u)=e^{\lambda\tan^{-1}u}/(u^2+1)$ (corresponding to $L(u)=\lambda^{-1}e^{\lambda\tan^{-1}u}$), the system $\mathbf{UV}\{x,t\,;u,v\}$ has the point symmetry Y_9 that is obviously a nonlocal symmetry of the nonlinear diffusion equation $\mathbf{U}\{x,t\,;u\}$.

For all other distinguished cases, the point symmetry classification of $\mathbf{UV}\{x,t\,;u,v\}$ (3.19) is greatly simplified through use of the equivalence transformation (4.7). This readily leads to an additional point symmetry of $\mathbf{UV}\{x,t\,;u,v\}$ for

$$K(u)=u^{\nu}(1+bu)^{-(\nu+2)},\quad K(u)=\frac{1}{(1+bu)^2}e^{u/(1+bu)},$$

and for

$$K(u)=\frac{1}{u^2+(1+bu)^2}\exp\left(\lambda\tan^{-1}\frac{u}{1+bu}\right).$$

These additional point symmetries of $\mathbf{UV}\{x,t\,;u,v\}$ are obviously nonlocal symmetries of $\mathbf{U}\{x,t\,;u\}$. Note that since $\widetilde{K}(\widetilde{u})=\widetilde{u}^{-2}$ when $K(u)=u^{-2}$, no additional symmetries arise in the linearization case. The symmetry classification of system $\mathbf{UV}\{x,t\,;u,v\}$ first appeared in a different form in Bluman, Kumei & Reid (1988). This paper did not make use of the important simplifying equivalence transformation (4.7).

The symmetry classification of the couplet potential system $\mathbf{UVA}\{x,t;u,v,\alpha\}$ (3.73) is presented in Table 4.3.

Compared to the situation for the singlet potential system $\mathbf{UV}\{x,t\,;u,v\}$ (3.19), the couplet $\mathbf{UVA}\{x,t\,;u,v,\alpha\}$ (3.73) contains three additional distinguished cases: $K(u)=u^{-2/3}$, $K(u)=u^{-4/3}(1+bu)^{-2/3}$, and $K(u)=u^{-2/3}(1+bu)^{-4/3}$, with the respective point symmetries Z_9, Z_{15} and Z_{16}, which are nonlocal symmetries of all other PDE systems in the tree.

One can show that the point symmetry classification of each of the three remaining nonlocally related subsystems $\mathbf{A}\{x,u\,;\alpha\}$, $\mathbf{V}\{u,t\,;v\}$ and

Table 4.2 Point symmetries of singlet potential systems of the nonlinear diffusion equation (4.5)

$K(u)$	**UV**$\{x,t;u,v\}$		**UA**$\{x,t;u,\alpha\}$	
	#	Point Symmetries	#	Point Symmetries
Arbitrary	4	$\mathrm{Y}_1=\mathrm{X}_1,\quad \mathrm{Y}_2=\mathrm{X}_2,$ $\mathrm{Y}_3=\mathrm{X}_3+v\frac{\partial}{\partial v},\quad \mathrm{Y}_4=\frac{\partial}{\partial v}.$	3	$\hat{\mathrm{Y}}_1=\mathrm{X}_2,$ $\hat{\mathrm{Y}}_2=\mathrm{X}_3+2\alpha\frac{\partial}{\partial \alpha},$ $\hat{\mathrm{Y}}_3=\frac{\partial}{\partial \alpha}.$
u^ν	5	$\mathrm{Y}_1,\mathrm{Y}_2,\mathrm{Y}_3,\mathrm{Y}_4,$ $\mathrm{Y}_5=\mathrm{X}_4+\left(1+\frac{2}{\nu}\right)v\frac{\partial}{\partial v}.$	4	$\hat{\mathrm{Y}}_1,\hat{\mathrm{Y}}_2,\hat{\mathrm{Y}}_3,\hat{\mathrm{Y}}_4=\mathrm{X}_4$ $+2\left(1+\frac{1}{\nu}\right)\alpha\frac{\partial}{\partial \alpha}.$
e^u	5	$\mathrm{Y}_1,\mathrm{Y}_2,\mathrm{Y}_3,\mathrm{Y}_4,$ $\mathrm{Y}_6=\mathrm{X}_5+(2x+v)\frac{\partial}{\partial v}.$	4	$\hat{\mathrm{Y}}_1,\hat{\mathrm{Y}}_2,\hat{\mathrm{Y}}_3,\hat{\mathrm{Y}}_5=\mathrm{X}_5$ $+\left(x^2+2\alpha\right)\frac{\partial}{\partial \alpha}.$
$u^{-4/3}$	5	$\mathrm{Y}_1,\mathrm{Y}_2,\mathrm{Y}_3,\mathrm{Y}_4,\mathrm{Y}_5\ (\nu=-4/3).$	5	$\hat{\mathrm{Y}}_1,\hat{\mathrm{Y}}_2,\hat{\mathrm{Y}}_3,\hat{\mathrm{Y}}_4,$ $\hat{\mathrm{Y}}_6=\mathrm{X}_6.$
u^{-2}	∞	$\mathrm{Y}_1,\mathrm{Y}_2,\mathrm{Y}_3,\mathrm{Y}_4,\mathrm{Y}_5\ (\nu=-2),$ $\mathrm{Y}_7=-xv\frac{\partial}{\partial x}+(xu+v)u\frac{\partial}{\partial u}+2t\frac{\partial}{\partial v},$ $\mathrm{Y}_8=-x(2t+v^2)\frac{\partial}{\partial x}+4t^2\frac{\partial}{\partial t}$ $+u(6t+2xuv+v^2)\frac{\partial}{\partial u}+4tv\frac{\partial}{\partial v},$ $\mathrm{Y}_\infty=F^1(v,t)\frac{\partial}{\partial x}-u^2F^2(v,t)\frac{\partial}{\partial u},$ $(F^1(v,t),F^2(v,t))$ is an arbitrary solution of the linear system $F^1_t=F^2_v,\ F^1_v=F^2.$	4	$\hat{\mathrm{Y}}_1,\hat{\mathrm{Y}}_2,\hat{\mathrm{Y}}_3,\hat{\mathrm{Y}}_4.$
$(u^2+1)^{-1}$ $\times e^{\lambda\tan^{-1}u}$	5	$\mathrm{Y}_1,\mathrm{Y}_2,\mathrm{Y}_3,\mathrm{Y}_4,$ $\mathrm{Y}_9=v\frac{\partial}{\partial x}+\lambda t\frac{\partial}{\partial t}$ $-(u^2+1)\frac{\partial}{\partial u}-x\frac{\partial}{\partial v}.$	3	$\hat{\mathrm{Y}}_1,\hat{\mathrm{Y}}_2,\hat{\mathrm{Y}}_3.$
$u^\nu(1+bu)^{-(\nu+2)}$	5	$\mathrm{Y}_1,\mathrm{Y}_2,\mathrm{Y}_3,\mathrm{Y}_4,$ $\mathrm{Y}_{10}=bv\frac{\partial}{\partial x}+\nu t\frac{\partial}{\partial t}$ $-(1+bu)u\frac{\partial}{\partial u}-v\frac{\partial}{\partial v}.$	3	$\hat{\mathrm{Y}}_1,\hat{\mathrm{Y}}_2,\hat{\mathrm{Y}}_3.$
$(1+bu)^{-2}$ $\times e^{u/(1+bu)}$	5	$\mathrm{Y}_1,\mathrm{Y}_2,\mathrm{Y}_3,\mathrm{Y}_4,$ $\mathrm{Y}_{11}=b(2x+bv)\frac{\partial}{\partial x}$ $+(1+2b)t\frac{\partial}{\partial t}$ $-(1+bu)^2\frac{\partial}{\partial u}-x\frac{\partial}{\partial v}.$	3	$\hat{\mathrm{Y}}_1,\hat{\mathrm{Y}}_2,\hat{\mathrm{Y}}_3.$
$(u^2+(1+bu)^2)^{-1}$ $\times\exp\Big(\lambda$ $\times\tan^{-1}\frac{u}{1+bu}\Big)$	5	$\mathrm{Y}_1,\mathrm{Y}_2,\mathrm{Y}_3,\mathrm{Y}_4,$ $\mathrm{Y}_{12}=(2bx+(b^2+1)v)\frac{\partial}{\partial x}$ $+(\lambda+2b)t\frac{\partial}{\partial t}$ $-((1+bu)^2+u^2)\frac{\partial}{\partial u}-x\frac{\partial}{\partial v}.$	3	$\hat{\mathrm{Y}}_1,\hat{\mathrm{Y}}_2,\hat{\mathrm{Y}}_3.$

Table 4.3 Symmetries of the couplet potential system $\mathbf{UVA}\{x,t\,;u,v,\alpha\}$ (3.73) of the nonlinear diffusion equation (4.5)

$K(u)$	#	Point Symmetries
Arbitrary	5	$Z_1 = X_1 + v\frac{\partial}{\partial\alpha},\quad Z_2 = X_2,\quad Z_3 = \hat{Y}_3,\quad Z_4 = Y_4,$ $Z_5 = X_3 + v\frac{\partial}{\partial v} + 2\alpha\frac{\partial}{\partial\alpha}.$
u^ν	6	$Z_1, Z_2, Z_3, Z_4, Z_5,\ Z_6 = X_4 + \left(1+\frac{2}{\nu}\right)v\frac{\partial}{\partial v} + 2\left(1+\frac{1}{\nu}\right)\alpha\frac{\partial}{\partial\alpha}.$
e^u	6	$Z_1, Z_2, Z_3, Z_4, Z_5,\ Z_7 = X_5 + (2x+v)\frac{\partial}{\partial v} + \left(x^2+2\alpha\right)\frac{\partial}{\partial\alpha}.$
$u^{-4/3}$	7	$Z_1, Z_2, Z_3, Z_4, Z_5, Z_6\ (\nu=-4/3),\ Z_8 = X_6 - \alpha\frac{\partial}{\partial v}.$
$u^{-2/3}$	7	$Z_1, Z_2, Z_3, Z_4, Z_5, Z_6\ (\nu=-2/3),$ $Z_9 = (xv-\alpha)\frac{\partial}{\partial x} - 3uv\frac{\partial}{\partial u} - v^2\frac{\partial}{\partial v} - v\alpha\frac{\partial}{\partial\alpha}.$
u^{-2}	∞	$Z_1, Z_2, Z_3, Z_4, Z_5, Z_6\ (\nu=-2),$ $Z_{10} = -(xv+\alpha)\frac{\partial}{\partial x} + (2xu+v)u\frac{\partial}{\partial u} + 2t\frac{\partial}{\partial v} - v\alpha\frac{\partial}{\partial\alpha},$ $Z_{11} = -(6xt+xv^2+2va)\frac{\partial}{\partial x} + 4t^2\frac{\partial}{\partial t}$ $+u(10t+2u(2xv+a)+v^2)\frac{\partial}{\partial u} + 4tv\frac{\partial}{\partial v} - (2t+v^2)\alpha\frac{\partial}{\partial\alpha},$ $Z_\infty = F^1(v,t)\frac{\partial}{\partial x} - u^2F^2(v,t)\frac{\partial}{\partial u} + F^3(v,t)\frac{\partial}{\partial\alpha},$ $(F^1(v,t), F^2(v,t), F^3(v,t))$ is an arbitrary solution of the linear system $F^3_v = F^1,\ F^3_t = F_2,\ F^1_v = F_2.$
$(u^2+1)^{-1}e^{\lambda\tan^{-1}u}$	6	$Z_1, Z_2, Z_3, Z_4, Z_5,\ Z_{12} = Y_9 + \frac{v^2-x^2}{2}\frac{\partial}{\partial\alpha}.$
$u^\nu(1+bu)^{-(\nu+2)}$	6	$Z_1, Z_2, Z_3, Z_4, Z_5,$ $Z_{13} = bv\frac{\partial}{\partial x} + \nu t\frac{\partial}{\partial t} - (1+bu)u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v} + (\frac{bv^2}{2}-\alpha)\frac{\partial}{\partial\alpha}.$
$(1+bu)^{-2}$ $\times e^{u/(1+bu)}$	6	$Z_1, Z_2, Z_3, Z_4, Z_5,$ $Z_{14} = b(2x+bv)\frac{\partial}{\partial x} + (1+2b)t\frac{\partial}{\partial t} - (1+bu)^2\frac{\partial}{\partial u}$ $-x\frac{\partial}{\partial v} + (\frac{b^2v^2-x^2}{2}+2b\alpha)\frac{\partial}{\partial\alpha}.$
$u^{-4/3}(1+bu)^{-2/3}$	7	$Z_1, Z_2, Z_3, Z_4, Z_5, Z_{13}\ (\nu=-4/3),$ $Z_{15} = (3b^2v^2+2b(2xv+\alpha)+2x^2)\frac{\partial}{\partial x}$ $-6(x+bv)(1+bu)u\frac{\partial}{\partial u} - (bv^2+2\alpha)\frac{\partial}{\partial v} + (b^2v^+2b\alpha)v\frac{\partial}{\partial\alpha}.$
$u^{-2/3}(1+bu)^{-4/3}$	7	$Z_1, Z_2, Z_3, Z_4, Z_5, Z_{13}\ (\nu=-2/3),$ $Z_{16} = (3bv^2+2(xv-\alpha))\frac{\partial}{\partial x} - 6(1+bu)uv\frac{\partial}{\partial u}$ $-2v^2\frac{\partial}{\partial v} + (bv^2-2\alpha)v\frac{\partial}{\partial\alpha}.$
$\frac{1}{u^2+(1+bu)^2}\times$ $\exp\left(\lambda\tan^{-1}\frac{u}{1+bu}\right)$	6	$Z_1, Z_2, Z_3, Z_4, Z_5,\ Z_{17} = (2bx+(b^2+1)v)\frac{\partial}{\partial x} + (\lambda+2b)t\frac{\partial}{\partial t}$ $-((1+bu)^2+u^2)\frac{\partial}{\partial u} - x\frac{\partial}{\partial v} + (\frac{(b^2+1)v^2-x^2}{2}+2b\alpha)\frac{\partial}{\partial\alpha}.$

$\mathbf{X}\{u, v\,; x\}$ yields no new nonlocal symmetries of the nonlinear diffusion equation $\mathbf{U}\{x, t\,; u\}$ [Exercise 4.2.1].

Thus in this particular example, the point symmetry classification of the "grand" couplet potential system $\mathbf{UVA}\{x, t\,; u, v, \alpha\}$ yields all point symmetries of each of the other PDE systems in the tree $\mathcal{T}_4$.

4.2.2 Nonlocal symmetries of a nonlinear wave equation

As a second example, consider a symmetry classification problem for the nonlinear wave equation $\mathbf{U}\{x, t\,; u\}$ given by

$$u_{tt} = (c^2(u)u_x)_x, \tag{4.8}$$

with an arbitrary constitutive function $c(u)$ [Ames, Lohner & Adams (1981); Bluman & Kumei [(1987), (1988)]; Bluman & Cheviakov (2007)].

The group of equivalence transformations of $\mathbf{U}\{x, t\,; u\}$ (4.8) is given by

$$\widetilde{x} = a_1 x + a_4, \quad \widetilde{t} = a_2 t + a_5, \quad \widetilde{u} = a_3 u + a_6, \quad \widetilde{c}(\widetilde{u}) = a_1 a_2^{-1} c(u), \tag{4.9}$$

where $a_1, \ldots, a_6$ are arbitrary constants with $a_1 a_2 a_3 \neq 0$. The point symmetry classification of the nonlinear wave equation $\mathbf{U}\{x, t\,; u\}$ (4.8) [Ames, Lohner & Adams (1981)] is presented in Table 4.4 (modulo the equivalence transformations (4.9)).

Table 4.4 Point symmetries of the nonlinear wave equation $\mathbf{U}\{x, t\,; u\}$ (4.8)

$c(u)$	#	Point Symmetries
Arbitrary	3	$\mathrm{X}_1 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x},\ \mathrm{X}_2 = \frac{\partial}{\partial t},\ \mathrm{X}_3 = \frac{\partial}{\partial x}.$
u^{ν}	4	$\mathrm{X}_1, \mathrm{X}_2, \mathrm{X}_3, \mathrm{X}_4 = \nu x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u}.$
e^u	4	$\mathrm{X}_1, \mathrm{X}_2, \mathrm{X}_3, \mathrm{X}_5 = x\frac{\partial}{\partial x} + \frac{\partial}{\partial u}.$
u^{-2}	5	$\mathrm{X}_1, \mathrm{X}_2, \mathrm{X}_3, \mathrm{X}_4\ (\nu = -2),\ \mathrm{X}_6 = t^2\frac{\partial}{\partial t} + tu\frac{\partial}{\partial u}.$
$u^{-2/3}$	5	$\mathrm{X}_1, \mathrm{X}_2, \mathrm{X}_3, \mathrm{X}_4\ (\nu = -2/3),\ \mathrm{X}_7 = x^2\frac{\partial}{\partial x} - 3xu\frac{\partial}{\partial u}.$

An extended tree $\mathcal{T}_d$ of nonlocally related PDE systems for the nonlinear wave equation $\mathbf{U}\{x, t\,; u\}$ (4.8), holding for an arbitrary wave speed $c(u)$, was constructed in Section 3.5.3, and exhibited in Figure 3.4.

We now classify nonlocal symmetries of the nonlinear wave equation $\mathbf{U}\{x, t\,; u\}$ (4.8) arising as point symmetries of any of the seven singlet potential systems **UA**$\{x,t\,;u,\alpha\}$, **UB**$\{x,t\,;u,\beta\}$, **UV**$\{x,t\,;u,v\}$, **UW**$\{x,t\,;u,w\}$, **TP**$\{u,v\,;t,p\}$, **TQ**$\{u,v\,;t,q\}$ and **TR**$\{u,v\,;t,r\}$ given by (3.81)–(3.84) and (3.92)–(3.94), respectively, or as point symmetries of the two nonlocally related subsystems **X**$\{u,v\,;x\}$ (3.86) and **T**$\{u,v\,;t\}$ (3.87) [Bluman & Cheviakov (2007); references therein].

In Tables 4.5a,b, for each of the nine above-mentioned nonlocally related systems, the situations are summarized where nonlocal symmetries arise for the nonlinear wave equation **U**$\{x,t\,;u\}$ (4.8) from point symmetries of any of these nine systems. The results are given modulo the equivalence transformations (4.9).

Table 4.5 (a) Cases for which nonlocal symmetries of the nonlinear wave equation **U**$\{x,t\,;u\}$ (4.8) arise

System	Potential(s)	Condition on $c(u)$	Symmetries; Remarks
UA (3.83)	α	No special cases	Nonlocal symmetries do not arise.
UB (3.84)	β	$c(u)=u^{-2/3}$	Linearizable by a point transformation.
		$\frac{F''(u)}{(F'(u))^2}=\frac{4F(u)+C_1}{(F(u)+C_2)^2+C_3}$, $(F(u)=\int c^2(u)du,\quad C_1,C_2,C_3=\text{const})$	One nonlocal symmetry.
UV (3.81)	v	Arbitrary	Infinite number of nonlocal symmetries; there exists an invertible mapping to linear system **XT** (3.85) (hodograph transformation).
		$\left[\frac{c'(u)}{c^3(u)}\left(\frac{c(u)}{c'(u)}\right)''\right]'=0$	One or two additional nonlocal symmetries.
UW (3.82)	w	$c(u)=(u+B)^{-2}$	Linearizable by a point transformation.
		$\frac{c'(u)}{c(u)}=-\frac{2u+C_1}{u^2+C_2}\quad(C_1,C_2=\text{const})$	One nonlocal symmetry.

The nonlocal symmetries for the cases listed in Tables 4.5a,b arise as follows.

(1) The potential system **UB**$\{x,t\,;u,\beta\}$
The potential system **UB**$\{x,t\,;u,\beta\}$ (3.84), i.e.,

Table 4.5 (b) Cases for which nonlocal symmetries of the nonlinear wave equation $\mathbf{U}\{x,t\,;u\}$ (4.8) arise

System	Potential(s)	Condition on $c(u)$	Symmetries; Remarks
TP (3.92)	v, p	$\frac{-(2uc^2+u^2cc')c'''+2u^2c(c'')^2}{c^3(uc'+2c)^2} + \frac{-(4c^2+u^2(c')^2-8ucc')c''+6(c')^2(c-uc')}{c^3(uc'+2c)^2} = \lambda^2, \quad \lambda = \text{const}$	One or two nonlocal symmetries.
		$c(u) = u^{-2}$	Infinite number of nonlocal symmetries; there exists a point mapping to a linear system with constant coefficients.
TQ (3.93)	v, q	$c(u) = u^{-2/3}; \quad c(u) = u^{-2}$	Two nonlocal symmetries.
TR (3.94)	v, r	$\frac{ucc''+c'(c-uc')}{(uc'+2c)^2} = \gamma^2 = \text{const}$	Two nonlocal symmetries.
X (3.86)	v	$\frac{(-2cc''+5(c')^2)c^2c''''+3c^3(c''')^2+16c^2(c'')^3}{c^3(2cc''-5(c')^2)^2} + \frac{-24c^2c'''c''c'+12c(c'c'')^2-10(c')^4c''}{c^3(2cc''-5(c')^2)^2} = \sigma^2, \quad \sigma = \text{const}$	One or two nonlocal symmetries.
T (3.87)	v	$(\alpha' + H\alpha)' = \sigma^2\alpha c^2(u), \quad \sigma = \text{const.}$ $(H = c'(u)/c(u), \quad \alpha^2 = (H^2 - 2H')^{-1})$	One or two nonlocal symmetries.
		$c(u) = u^{-2}$	Infinite number of nonlocal symmetries; there exists an invertible mapping to a linear system with constant coefficients.

$$\begin{aligned} \beta_x &= xu_t, \\ \beta_t &= xc^2(u)u_x - \int c^2(u)du, \end{aligned}$$

has the group of equivalence transformations

$$\begin{aligned} &\widetilde{x} = a_1 x, \quad \widetilde{t} = a_2 t + a_4, \quad \widetilde{u} = a_3 u + a_5, \\ &\widetilde{b} = a_1^2 a_2^{-1} a_3 b - a_2 a_7 t + a_6, \quad \widetilde{F}(\widetilde{u}) = a_1^2 a_2^{-2} a_3 F(u) + a_7, \end{aligned} \tag{4.10}$$

where $F(u) = \int c^2(u)du$; $a_1, \ldots, a_7$ are arbitrary constants with $a_1a_2a_3 \neq 0$.

For an arbitrary wave speed $c(u)$, the system $\mathbf{UB}\{x,t\,;u,\beta\}$ has three point symmetries given by

$$\mathrm{Y}_1 = \frac{\partial}{\partial t}, \quad \mathrm{Y}_2 = \frac{\partial}{\partial \beta}, \quad \mathrm{Y}_3 = x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} + \beta\frac{\partial}{\partial \beta}.$$

These point symmetries project onto point symmetries of the nonlinear wave equation $\mathbf{U}\{x, t\,; u\}$ (4.8).

If the wave speed $c(u)$ satisfies the ODE

$$\frac{F''(u)}{(F'(u))^2} = \frac{4F(u) + C_1}{(F(u) + C_2)^2 + C_3}, \tag{4.11}$$

where $F(u) = \int c^2(u)du$ and C_1, C_2, C_3 are arbitrary constants, then the system $\mathbf{UB}\{x, t\,; u, \beta\}$ has an additional point symmetry

$$\begin{aligned}\mathrm{Y}_4 = \left(F(u) + \tfrac{1}{2}C_1\right)x\frac{\partial}{\partial x} + \beta\frac{\partial}{\partial t} + \frac{(F(u) + C_2)^2 + C_3}{F'(u)}\frac{\partial}{\partial u}\\ + \left(2C_2\beta - (C_2^2 + C_3)t\right)\frac{\partial}{\partial \beta},\end{aligned}$$

which is a nonlocal symmetry of the nonlinear wave equation $\mathbf{U}\{x, t\,; u\}$ (4.8).

For $c(u) = u^{-2/3}$, the potential system $\mathbf{UB}\{x, t\,; u, \beta\}$ has an infinite number of point symmetries that lead to the linearization of the potential system $\mathbf{UB}\{x, t\,; u, \beta\}$ by a point transformation, and thus a linearization of the nonlinear wave equation $\mathbf{U}\{x, t\,; u\}$ (4.8) by a nonlocal transformation [Exercise 4.2.3].

(2) The potential system $\mathbf{UV}\{x, t\,; u, v\}$
The potential system $\mathbf{UV}\{x, t\,; u, v\}$ (3.81), i.e.,

$$\begin{aligned} v_x &= u_t, \\ v_t &= c^2(u)\, u_x, \end{aligned}$$

has the group of equivalence transformations

$$\begin{aligned} \widetilde{x} &= a_1 x + a_4 v + a_5, \quad \widetilde{t} = a_2 t + a_1^{-1} a_2 a_4 u + a_6, \\ \widetilde{u} &= a_3 u + a_7 t + a_8, \quad \widetilde{v} = a_1 a_2^{-1} a_3 v + a_1 a_2^{-1} a_7 x + a_9, \\ \widetilde{c}(\widetilde{u}) &= a_1 a_2^{-1} c(u), \end{aligned} \tag{4.12}$$

where $a_1, \ldots, a_9$ are arbitrary constants with $a_1 a_2 a_3 \neq 0$.

The nonlinear PDE system $\mathbf{UV}\{x, t\,; u, v\}$ is locally related to the linear PDE system $\mathbf{XT}\{u, v\,; x, t\}$ (3.85) through an interchange of dependent and independent variables in terms of the hodograph transformation $x = x(u, v),\ t = t(u, v)$. Hence these two systems have the same point symmetries. In particular, the infinite number of point symmetries of the PDE system $\mathbf{XT}\{u, v\,; x, t\}$, due to its linearity, yields an *infinite number of nonlocal symmetries* of the nonlinear wave equation $\mathbf{U}\{x, t\,; u\}$.

The point symmetries of the PDE system $\mathbf{UV}\{x,t\,;u,v\}$ are summarized in Table 4.6.

For an arbitrary wave speed $c(u)$, in addition to the infinite number of point symmetries arising from the linearity of $\mathbf{XT}\{u,v\,;x,t\}$, the system $\mathbf{UV}\{x,t\,;u,v\}$ has four additional point symmetries that project onto the three point symmetries of the nonlinear wave equation $\mathbf{U}\{x,t\,;u\}$ (4.8) [Table 4.4]. Further point symmetries arise when $c(u)$ satisfies the ODE

$$\frac{c'(u)}{c^3(u)}\left(\frac{c(u)}{c'(u)}\right)'' = \lambda^2 = \text{const.} \tag{4.13}$$

For several classes of wave speeds $c(u)$ satisfying (4.13), these point symmetries yield nonlocal symmetries of $\mathbf{U}\{x,t\,;u\}$.

Table 4.6 Point symmetries of the potential system $\mathbf{UV}\{x,t\,;u,v\}$ of the nonlinear wave equation $\mathbf{U}\{x,t\,;u\}$ (4.8)

$c(u)$	#	Point Symmetries
Arbitrary	∞	Infinite number of point symmetries following from the linearity of the invertibly related system $\mathbf{XT}\{u,v\,;x,t\}$.
Arbitrary	4	$\mathrm{W}_1 = \frac{\partial}{\partial t},\ \mathrm{W}_2 = \frac{\partial}{\partial x},\ \mathrm{W}_3 = \frac{\partial}{\partial v},\ \mathrm{W}_4 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}.$
$u^\nu \quad (\nu \neq 0, -1)$	6	$\mathrm{W}_1,\ \mathrm{W}_2,\ \mathrm{W}_3,\ \mathrm{W}_4,$ $\mathrm{W}_5 = \nu t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u} - (1+\nu)v\frac{\partial}{\partial v},$ $\mathrm{W}_6 = -((2\nu+1)tv + xu)\frac{\partial}{\partial t} - (tu^{1+2\nu} + xv)\frac{\partial}{\partial x}$ $+2uv\frac{\partial}{\partial u} + \left[(1+\nu)v^2 + \frac{u^{2+2\nu}}{1+\nu}\right]\frac{\partial}{\partial v}.$
e^u	6	$\mathrm{W}_1,\ \mathrm{W}_2,\ \mathrm{W}_3,\ \mathrm{W}_4,\quad \mathrm{W}_7 = x\frac{\partial}{\partial x} + \frac{\partial}{\partial u} + v\frac{\partial}{\partial v},$ $\mathrm{W}_8 = -(2vt + x)\frac{\partial}{\partial t} - 2e^u t\frac{\partial}{\partial x} + 4v\frac{\partial}{\partial u}$ $+(4e^u + v^2)\frac{\partial}{\partial v}.$
u^{-1}	6	$\mathrm{W}_1,\ \mathrm{W}_2,\ \mathrm{W}_3,\ \mathrm{W}_4,\ \mathrm{W}_5\ (\nu = -1),$ $\mathrm{W}_9 = (tv - xu)\frac{\partial}{\partial t} - (tu^{-1} + xv)\frac{\partial}{\partial x} + 2uv\frac{\partial}{\partial u}$ $+2\log u\frac{\partial}{\partial v}.$
$c(u)$ satisfies (a), (b) or (c): (a) $c' = c^2\nu^{-1}\sinh(\nu\log c)$ (b) $c' = c^2\nu^{-1}\sin(\nu\log c)$ (c) $c' = c^2\nu^{-1}\cosh(\nu\log c)$	6	$\mathrm{W}_1,\ \mathrm{W}_2,\ \mathrm{W}_3,\ \mathrm{W}_4,$ $\mathrm{W}_{10,11} = e^{\pm v}\Big\{((2+\Gamma')t \pm \Gamma x)\frac{\partial}{\partial t}$ $+(\Gamma' x \pm c^2\Gamma t)\frac{\partial}{\partial x} - 2\Gamma\frac{\partial}{\partial u} \mp 2(\Gamma'+1)\frac{\partial}{\partial v}\Big\},$ where $\Gamma = c/c'$.

The point symmetries $\mathrm{W}_6, \mathrm{W}_8, \mathrm{W}_9, \mathrm{W}_{10}, \mathrm{W}_{11}$ of the potential system $\mathbf{UV}\{x,t\,;u,v\}$ correspond to nonlocal symmetries of the nonlinear wave equation $\mathbf{U}\{x,t\,;u\}$ (4.8).

(3) The potential system $\mathbf{UW}\{x,t\,;u,w\}$
The potential system $\mathbf{UW}\{x,t\,;u,w\}$ (3.82), i.e.,

$$\begin{aligned} w_x &= tu_t - u, \\ w_t &= tc^2(u)u_x, \end{aligned}$$

has the group of equivalence transformations that includes the transformations

$$\begin{aligned} &\widetilde{x} = a_1 x + a_4, \quad \widetilde{t} = a_2 t, \quad \widetilde{u} = a_3 u + a_6 t + a_7, \\ &\widetilde{w} = a_1 a_3 w - a_1 a_7 x + a_5, \quad \widetilde{c}(\widetilde{u}) = a_1 a_2^{-1} c(u), \end{aligned} \tag{4.14}$$

where $a_1, \ldots, a_7$ are arbitrary constants with $a_1 a_2 a_3 \neq 0$, and the projective transformation

$$\begin{aligned} &\widetilde{x} = x - bw, \quad \widetilde{t} = \frac{t}{1+bu}, \quad \widetilde{u} = \frac{u}{1+bu}, \quad \widetilde{w} = w, \\ &\widetilde{c}(\widetilde{u}) = (1+b\widetilde{u})^{-2}\, c\Big(\frac{\widetilde{u}}{1+b\widetilde{u}}\Big). \end{aligned} \tag{4.15}$$

For an arbitrary $c(u)$, the potential system $\mathbf{UW}\{x,t\,;u,w\}$ has the point symmetries

$$Z_1 = \frac{\partial}{\partial x}, \quad Z_2 = \frac{\partial}{\partial w}, \quad Z_3 = x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} + w\frac{\partial}{\partial w}.$$

These point symmetries project onto point symmetries of $\mathbf{U}\{x,t\,;u\}$.

If the wave speed $c(u)$ satisfies the ODE

$$\frac{c'(u)}{c(u)} = -\frac{2u + C_1}{u^2 + C_2}, \tag{4.16}$$

where C_1, C_2 are arbitrary constants, then the potential system $\mathbf{UW}\{x,t;\, u,w\}$ has an additional point symmetry

$$Z_4 = w\frac{\partial}{\partial x} + (u + C_1)t\frac{\partial}{\partial t} + (u^2 + C_2)\frac{\partial}{\partial u} - C_2 x\frac{\partial}{\partial w},$$

which is obviously a nonlocal symmetry of $\mathbf{U}\{x,t\,;u\}$.

The general solution of (4.16) is found to be as follows:

$$
\begin{aligned}
C_2 = \omega^2 > 0: \quad & c(u) = \frac{c_0}{u^2+\omega^2} \exp\left\{ -\frac{C_1}{\omega} \tan^{-1} \frac{u}{\omega} \right\}; \\
C_2 = -\omega^2 < 0: \, & c(u) = \frac{c_0}{u^2-\omega^2} \left| \frac{u+\omega}{u-\omega} \right|^{C_1/2\omega}; \\
C_2 = 0: \quad & c(u) = \frac{c_0}{u^2} e^{C_1/u}.
\end{aligned}
\tag{4.17}
$$

In (4.17), c_0 is an arbitrary constant of integration.

For $c(u) = (u+B)^{-2}$, where B is an arbitrary constant, the system $\mathbf{UW}\{x, t\,; u, w\}$ has an infinite number of point symmetries. One can show that here $\mathbf{UW}\{x, t\,; u, w\}$ is linearizable by a point transformation, and thus the nonlinear wave equation $\mathbf{U}\{x, t\,; u\}$ is linearizable by a nonlocal transformation [Exercise 4.2.2].

(4) The potential system $\mathbf{TP}\{u, v\,; t, p\}$

The potential system $\mathbf{TP}\{u, v\,; t, p\}$ (3.92), i.e.,

$$
\begin{aligned}
p_v &= u t_u - t, \\
p_u &= u c^2(u) t_v,
\end{aligned}
$$

has the group of equivalence transformations that includes the transformations

$$
\begin{aligned}
&\widetilde{u} = a_1 u, \quad \widetilde{v} = a_2 v + a_4, \quad \widetilde{t} = a_2^{-1} a_3 t + a_6 + a_7 u, \\
&\widetilde{p} = a_3 p + a_5 - a_2 a_6 v, \quad \widetilde{c}(\widetilde{u}) = a_1^{-1} a_2 c(u),
\end{aligned}
\tag{4.18}
$$

where $a_1, \ldots, a_7$ are arbitrary constants with $a_1 a_2 a_3 \neq 0$, and the projective transformation

$$
\begin{aligned}
&\widetilde{u} = \frac{u}{1+bu}, \quad \widetilde{v} = v, \quad \widetilde{t} = \frac{t}{1+bu}, \quad \widetilde{p} = p, \\
&\widetilde{c}(\widetilde{u}) = (1+b\widetilde{u})^{-2}\, c\left(\frac{\widetilde{u}}{1+b\widetilde{u}}\right)
\end{aligned}
\tag{4.19}
$$

similar to (4.15).

The point symmetry classification of the linear PDE system $\mathbf{TP}\{u, v\,; t, p\}$ (modulo its obvious infinite number of point symmetries due to its linearity) is as follows.

Case 1. For an arbitrary wave speed $c(u)$, the system $\mathbf{TP}\{u, v\,; t, p\}$ has the three point symmetries

$$
\mathrm{L}_1 = \frac{\partial}{\partial v}, \quad \mathrm{L}_2 = t\frac{\partial}{\partial t} + p\frac{\partial}{\partial p}, \quad \mathrm{L}_3 = \frac{\partial}{\partial p},
$$

that project onto point symmetries of the nonlinear wave equation $\mathbf{U}\{x, t\,; u\}$ (4.8).

Case 2. For $c(u) = u^{-2}$, the system $\mathbf{TP}\{u, v\,; t, p\}$ has an infinite number of point symmetries that are related to point symmetries of the system $\mathbf{UV}\{x, t\,; u, v\}$ with $c(u) = \text{const}$, since here the system $\mathbf{TP}\{u, v\,; t, p\}$ is mapped by the point transformation $y = -1/u, \gamma = t/u$ into the system with constant coefficients given by

$$\begin{aligned} p_v - \gamma_y &= 0, \\ p_y - \gamma_v &= 0. \end{aligned}$$

Remark 4.2.1. Note that the PDE system $\mathbf{TP}\{u, v\,; t, p\}$ is obviously not invariant under the translations $u \to u+B$, and thus it *does not* have an infinite number of point symmetries when $c(u) = (u+B)^{-2}$. However, by taking a linear combination of potential systems $\mathbf{TP}\{u, v\,; t, p\}$ and $\mathbf{XT}\{u, v\,; x, t\}$ (3.85) with weights 1 and B, and denoting a "combination" potential variable by $z = p+Bx$, one obtains a potential system $\mathbf{TZ}\{u, v\,; t, z\}$ which *does* have an infinite number of point symmetries when $c(u) = (u+B)^{-2}$

Case 3. For $c(u) \neq u^{-2}$, and $c(u)$ satisfying the ODE

$$\begin{aligned} &\frac{-(2uc^2 + u^2cc')c''' + 2u^2c(c'')^2 - (4c^2 + u^2(c')^2 - 8ucc')c''}{c^3(uc' + 2c)^2} \\ &\quad + \frac{6(c')^2(c - uc')}{c^3(uc' + 2c)^2} = \lambda^2, \end{aligned} \tag{4.20}$$

with λ a real or imaginary constant, the system $\mathbf{TP}\{u, v\,; t, p\}$ has additional point symmetries as follows.

Case 3a. When $\lambda \neq 0$ in (4.20), two additional point symmetries are given by

$$\begin{aligned} \mathrm{L}_{4,\,5} = e^{\pm\lambda v} \Bigg\{ &\Bigg[\pm \frac{\lambda^2 u^2 c}{2(2c + uc')} t \\ &- \left(\lambda \frac{c + uc'}{2c + uc'} - \frac{u^2(cc'' - 3(c')^2) - 4ucc' - 2c^2}{(2c + uc')^2} \right) p \Bigg] \frac{\partial}{\partial p} \\ &\pm \left[\frac{\lambda^2 c}{2(2c + uc')} p + \left(\frac{u^2(cc'' - 3(c')^2) - 4ucc' - 2c^2}{2(2c + uc')^2} \right) t \right] \frac{\partial}{\partial t} \\ &- \frac{\lambda uc}{2c + uc'} \frac{\partial}{\partial u} \pm \left[\frac{u^2(cc'' - 2(c')^2) - 2ucc' - 2c^2}{(2c + uc')^2} \right] \frac{\partial}{\partial v} \Bigg\} \end{aligned}$$

which yield *nonlocal symmetries* of the nonlinear wave equation $\mathbf{U}\{x, t\,; u\}$.

Case 3b. When $\lambda = 0$ in (4.20), the general solution of ODE (4.20) involves three distinguished classes given by

$$c(u) = Au^{\nu}(u+B)^{-2-\nu}; \tag{4.21}$$

$$c(u) = Au^{\nu}; \tag{4.22}$$

$$c(u) = Au^{-2}e^{B/u}; \tag{4.23}$$

A, B, ν are nonzero constants with $\nu \neq -2$.

From the equivalence transformations (4.18), it follows that a system $\mathbf{TP}\{u, v\,; t, p\}$ with wave speed (4.21) is invertibly equivalent to a system $\mathbf{TP}\{u, v\,; t, p\}$ with wave speed (4.22). Hence one only considers the non-equivalent cases (4.22), (4.23) (modulo the equivalence transformations (4.18)).

Case 3b(1). For wave speeds $c(u) = u^{\nu}$ with $\nu \neq -1$, the system $\mathbf{TP}\{u, v\,; t, p\}$ has two additional point symmetries given by

$$\begin{aligned}
\mathrm{L}_6 &= \nu t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u} - (1+\nu)v\frac{\partial}{\partial v} - p\frac{\partial}{\partial p},\\
\mathrm{L}_7 &= -((2\nu+1)tv + p)\frac{\partial}{\partial t} + 2uv\frac{\partial}{\partial u}\\
&\quad + \left[(1+\nu)v^2 + \frac{u^{2+2\nu}}{1+\nu}\right]\frac{\partial}{\partial v} + (tu^{2+2\nu} - vp)\frac{\partial}{\partial p}.
\end{aligned}$$

Note that the symmetry L_7 is nonlocal for $\mathbf{U}\{x, t\,; u\}$ but local for the system $\mathbf{UV}\{x, t\,; u, v\}$; the symmetries L_6 and L_7 correspond to the symmetries W_5 and W_6, respectively, in Table 4.6.

Case 3b(2). For $c(u) = u^{-1}$, the system $\mathbf{TP}\{u, v\,; t, p\}$ again has two additional point symmetries given by

$$\mathrm{L}_6 \quad (\nu = -1), \quad \mathrm{L}_8 = (tv - p)\frac{\partial}{\partial t} + 2uv\frac{\partial}{\partial u} + 2\log u\frac{\partial}{\partial v} - (t - pv)\frac{\partial}{\partial p}.$$

The point symmetry L_8 is nonlocal for $\mathbf{U}\{x, t\,; u\}$ but local for the system $\mathbf{UV}\{x, t\,; u, v\}$. These symmetries correspond to W_5 ($\nu = -1$) and W_7, respectively, in Table 4.6.

Case 3b(3). For $c(u) = u^{-2}e^{1/u}$, the system $\mathbf{TP}\{u, v\,; t, p\}$ has two additional point symmetries given by

$$\begin{aligned}
\mathrm{L}_9 &= (pu - 2tv(u+1))\frac{\partial}{\partial t} - 2u^2 v\frac{\partial}{\partial u} + (u^2 + e^{2/u})\frac{\partial}{\partial v} + t\frac{e^{2/u}}{u}\frac{\partial}{\partial p},\\
\mathrm{L}_{10} &= t(u+1)\frac{\partial}{\partial t} + u^2\frac{\partial}{\partial u} - v\frac{\partial}{\partial v}.
\end{aligned}$$

The symmetries L_9 and L_{10} are nonlocal for both $\mathbf{U}\{x, t\,; u\}$ *and* the system $\mathbf{UV}\{x, t\,; u, v\}$.

(5) The potential system $\mathbf{TQ}\{u, v\,; t, q\}$
The potential system $\mathbf{TQ}\{u, v\,; t, q\}$ (3.93), i.e.,

$$q_v = vt_u,$$
$$q_u = c^2(u)(vt_v - t),$$

has the group of equivalence transformations

$$\begin{aligned} \widetilde{u} &= a_1 u + a_4, \quad \widetilde{v} = a_2 v, \quad \widetilde{q} = a_3 q + a_5 + \tfrac{1}{3} a_2^2 a_7 v^3, \\ \widetilde{t} &= a_1 a_2^{-2} a_3 t + a_2 a_6 v + a_1 a_7 uv, \quad \widetilde{c}(\widetilde{u}) = a_1^{-1} a_2 c(u), \end{aligned} \tag{4.24}$$

where $a_1, \ldots, a_8$ are arbitrary constants with $a_1 a_2 a_3 \neq 0$.

The point symmetry classification of the linear potential system $\mathbf{TQ}\{u, v;$ $t, q\}$ is given in Table 4.7.

Table 4.7 Point symmetries of the potential system $\mathbf{TQ}\{u, v\,; t, q\}$ (3.93)

$c(u)$	#	Point Symmetries
Arbitrary	∞	Infinite number of point symmetries following from the linearity.
Arbitrary	2	$\mathrm{M}_1 = \frac{\partial}{\partial q},\ \mathrm{M}_2 = t\frac{\partial}{\partial t} + q\frac{\partial}{\partial q}.$
u^ν	3	$\mathrm{M}_1,\ \mathrm{M}_2,\ \mathrm{M}_3 = (2\nu+1)t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u} - (\nu+1)v\frac{\partial}{\partial v}.$
e^u	3	$\mathrm{M}_1,\ \mathrm{M}_2,\ \mathrm{M}_4 = 2t\frac{\partial}{\partial t} - \frac{\partial}{\partial u} - v\frac{\partial}{\partial v}.$
u^{-2}	5	$\mathrm{M}_1,\ \mathrm{M}_2,\ \mathrm{M}_3,\ \mathrm{M}_5 = \frac{u^2}{u^2v^2-1}\left[t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}\right],$ $\mathrm{M}_6 = \frac{1}{u^2}\left[(4u^3q - 5tv^2u^2 - 3t)\frac{\partial}{\partial t} - (3u^2v^2+1)u\frac{\partial}{\partial u} + (u^2v^2+3)v\frac{\partial}{\partial v} + \frac{2}{u}(2tv^2 + (u^2v^2+1)uq)\frac{\partial}{\partial q}\right].$
$u^{-2/3}$	5	$\mathrm{M}_1,\ \mathrm{M}_2,\ \mathrm{M}_3,\ \mathrm{M}_7,\ \mathrm{M}_8$ [Exercise 4.2.4]

For $c(u) = u^{-2}$ or $c(u) = u^{-2/3}$, the system $\mathbf{TQ}\{u, v\,; t, q\}$ has five point symmetries; the symmetries $(\mathrm{M}_5, \mathrm{M}_6)$ and $(\mathrm{M}_7, \mathrm{M}_8)$, respectively, yield *nonlocal symmetries* of the nonlinear wave equation $\mathbf{U}\{x, t\,; u\}$ (4.8).

(6) The potential system $\mathbf{TR}\{u, v\,; t, r\}$
The potential system $\mathbf{TR}\{u, v\,; t, r\}$ (3.94), i.e.,

$$r_v = v(ut_u - t),$$
$$r_u = uc^2(u)(vt_v - t),$$

has the group of equivalence transformations that includes the transformations

$$\begin{aligned} &\widetilde{u} = a_1 u, \quad \widetilde{v} = a_2 v, \quad \widetilde{r} = a_3 r + a_4 - \tfrac{1}{3} a_2^2 a_6 v^3, \\ &\widetilde{t} = a_2^{-2} a_3 t + a_5 uv + a_6 v^*, \quad \widetilde{c}(\widetilde{u}) = a_1^{-1} a_2 c(u), \end{aligned} \tag{4.25}$$

where $a_1, \ldots, a_6$ are arbitrary constants with $a_1 a_2 a_3 \neq 0$, and the projective transformation

$$\begin{aligned} &\widetilde{u} = \frac{u}{1+bu}, \quad \widetilde{v} = v, \quad \widetilde{t} = \frac{t}{1+bu}, \quad \widetilde{r} = r, \\ &\widetilde{c}(\widetilde{u}) = (1+b\widetilde{u})^{-2}\, c\Big(\frac{\widetilde{u}}{1+b\widetilde{u}}\Big). \end{aligned} \tag{4.26}$$

It follows that for the system $\mathbf{TR}\{u, v\,; t, r\}$, the wave speeds

$$c(u) = u^{\nu} \ \text{ and } \ c(u) = u^{\nu}(u+B)^{-2-\nu} \quad (\nu \neq -2), \tag{4.27}$$

are equivalent. In particular, the following wave speeds are equivalent.

1. $c(u) = u^{-1}$ and $c(u) = u^{-1}(Au+B)^{-1}$.
2. $c(u) = u^{-4/3}$ and $c(u) = u^{-4/3}(Au+B)^{-2/3}$.
3. $c(u) = 1$ and $c(u) = (Au+B)^{-2}$.

A, B are nonzero constants. The wave speed $c(u) = u^{-2}$ yields invariance under the equivalence transformation (4.27).

The point symmetry classification of the linear potential system $\mathbf{TR}\{u, v; t, r\}$ (modulo its equivalence transformations (4.25), (4.26)) is given in Table 4.8.

Note that the system $\mathbf{TR}\{u, v\,; t, r\}$ has an additional point symmetry when $c(u)$ satisfies the ODE

$$\frac{ucc'' + c'(c - uc')}{(uc' + 2c)^2} = \gamma^2 = \text{const.} \tag{4.28}$$

The general solution of the ODE (4.28) for $\gamma \neq 0$ (modulo the equivalence transformations (4.25), (4.26)) consists of two families of solutions: (a) $c(u) = u^{\nu}$ ($\nu = \text{const}$) and (b) $c(u) = u^{-2}e^{1/u}$. For $c(u)$ satisfying the ODE (4.28) with $\gamma = 0$, i.e., $c(u) = u^{-4/3}$ (modulo the equivalence transformations (4.25), (4.26)), the system $\mathbf{TR}\{u, v\,; t, r\}$ has two additional point symmetries.

Comparing Tables 4.4 and 4.8, one observes that the point symmetries $\mathrm{N}_4, \ldots, \mathrm{N}_{10}$ of the potential system $\mathbf{TR}\{u, v\,; t, r\}$ yield nonlocal symmetries of the nonlinear wave equation $\mathbf{U}\{x, t\,; u\}$ (4.8). Of course, when $c(u) = 1$, $\mathbf{U}\{x, t\,; u\}$ is linear and $\mathbf{TR}\{u, v\,; t, r\}$ is a nonlinear system.

Note that at a first glance the symmetries N_4 and N_7 of the potential system $\mathbf{TR}\{u, v\,; t, r\}$ seem to project onto point symmetries of the nonlinear wave equation $\mathbf{U}\{x, t\,; u\}$. But since x is a nonlocal variable for the potential system $\mathbf{TR}\{u, v\,; t, r\}$, and the symmetry generators N_4 and N_7 do not con-

Table 4.8 Point symmetries of the potential system $\mathbf{TR}\{u, v\,; t, r\}$ (3.94)

$c(u)$	#	Point Symmetries
Arbitrary	∞	Infinite number of point symmetries following from the linearity.
Arbitrary	2	$\mathrm{N}_1 = \frac{\partial}{\partial r},\ \mathrm{N}_2 = t\frac{\partial}{\partial t} + r\frac{\partial}{\partial r}.$
$u^{\nu},\ \ \nu \neq -2$	3	$\mathrm{N}_1,\ \mathrm{N}_2,\ \mathrm{N}_3 = 2(\nu+1)t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u} - (\nu+1)v\frac{\partial}{\partial v}.$
$u^{-2}e^{1/u}$	3	$\mathrm{N}_1,\ \mathrm{N}_2,\ \mathrm{N}_4 = (u+1)t\frac{\partial}{\partial t} + u^2\frac{\partial}{\partial u} - v\frac{\partial}{\partial v} - r\frac{\partial}{\partial r}.$
$u^{-4/3}$	4	$\mathrm{N}_1,\ \mathrm{N}_2,\ \mathrm{N}_5,\ \mathrm{N}_6$ [Exercise 4.2.4.]
u^{-2}	5	$\mathrm{N}_1,\ \mathrm{N}_2,\ \mathrm{N}_3\ (\nu=-2),\ \mathrm{N}_7 = tu\frac{\partial}{\partial t} + u^2\frac{\partial}{\partial u},$ $\mathrm{N}_8 = \frac{1}{u}\left[(tu^2v^2 + 2t - u^2 r)\frac{\partial}{\partial t} + 2v\frac{\partial}{\partial v} + (tv^2 + r)\frac{\partial}{\partial r}\right]$ $-(1+u^2v^2)\frac{\partial}{\partial u}.$
1	5	$\mathrm{N}_1,\ \mathrm{N}_2,\ \mathrm{N}_3\ (\nu=0),\ \mathrm{N}_9 = \frac{1}{u^2-v^2}\left(u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v}\right),$ $\mathrm{N}_{10} = 2[t(u^2+v^2)+2r]\frac{\partial}{\partial t} - u(u^2+3v^2)\frac{\partial}{\partial u} - v(3u^2+v^2)\frac{\partial}{\partial v}$ $+2[2tu^2v^2 - r(u^2+v^2)]\frac{\partial}{\partial r}.$

tain an explicit x-component, it turns out that the actual transformation of x is nonlocal under the actions of both N_4 and N_7.

(7) The nonlocally related subsystem $\mathbf{X}\{u, v\,; x\}$
The linear wave equation $\mathbf{X}\{u, v\,; x\}$ (3.86), i.e.,

$$x_{vv} = (c^{-2}(u)x_u)_u,$$

has the group of equivalence transformations

$$\begin{aligned} \widetilde{u} &= a_1 u + a_4, \quad \widetilde{v} = a_2 v + a_5, \quad \widetilde{x} = a_3 x + a_6 v + a_7, \\ \widetilde{c}(\widetilde{u}) &= a_1^{-1} a_2 c(u), \end{aligned} \tag{4.29}$$

where $a_1, \ldots, a_7$ are arbitrary constants with $a_1 a_2 a_3 \neq 0$.

For the wave speed $c(u) = u^{-2/3}$, the PDE $\mathbf{X}\{u, v\,; x\}$ has an infinite number of point symmetries [Exercise 4.2.5] (in addition to those due to its linearity), which suggests that it can be mapped by a point transformation into a constant coefficient linear PDE.

The PDE $\mathbf{X}\{u, v\,; x\}$ has two additional point symmetries when $c(u)$ satisfies the ODE

$$\begin{aligned} &\frac{(-2cc'' + 5(c')^2)c^2 c'''' + 3c^3(c''')^2 + 16c^2(c'')^3 - 24c^2 c''' c'' c'}{c^3(2cc'' - 5(c')^2)^2} \\ &\quad + \frac{12c(c'c'')^2 - 10(c')^4 c''}{c^3(2cc'' - 5(c')^2)^2} = \sigma^2 = \text{const.} \end{aligned} \tag{4.30}$$

The point symmetries of the PDE $\mathbf{X}\{u, v\,; x\}$ are summarized in Table 4.9.

Table 4.9 Point symmetries of the PDE $\mathbf{X}\{u, v\,; x\}$ (3.86)

$c(u)$	#	Point Symmetries
Arbitrary	∞	Infinite number of point symmetries following from the linearity.
Arbitrary	3	$\mathrm{J}_1 = x\frac{\partial}{\partial x},\ \mathrm{J}_2 = \frac{\partial}{\partial v},\ \mathrm{J}_3 = \frac{\partial}{\partial x}.$
$u^{-2/3}$	∞	Exercise 4.2.5
(4.30) ($\sigma \neq 0$)	5	$\mathrm{J}_1,\ \mathrm{J}_2,\ \mathrm{J}_3,\ \mathrm{J}_{4,5} = e^{\pm\sigma v}\Big\{\frac{1}{2}xFH\frac{\partial}{\partial x} + F(v)\frac{\partial}{\partial u}$ $\pm\sigma^{-1}[F' + FH]\frac{\partial}{\partial v}\Big\}.$
(4.30) ($\sigma = 0$)	5	$\mathrm{J}_1,\ \mathrm{J}_2,\ \mathrm{J}_3,\ \mathrm{J}_6 = v\left\{\frac{1}{2}xFH\frac{\partial}{\partial x} + F\frac{\partial}{\partial u}\right\}$ $+\Big\{K\frac{v^2}{2} + \int c^2 F du\Big\}\frac{\partial}{\partial v},$ $\mathrm{J}_7 = \frac{1}{2}xFH\frac{\partial}{\partial x} + F\frac{\partial}{\partial u} + Kv\frac{\partial}{\partial v}.$
Particular case (a) for $\sigma = 0$: $c(u) = u^{\nu}$ ($\nu = \text{const}$)	5	$\mathrm{J}_6^{(a)} = \nu(\nu+1)xv\frac{\partial}{\partial x} + 2(\nu+1)uv\frac{\partial}{\partial u}$ $+[u^{2\nu+2} + v^2(\nu+1)^2]\frac{\partial}{\partial v},$ $\mathrm{J}_7^{(a)} = u\frac{\partial}{\partial u} + (\nu+1)v\frac{\partial}{\partial v}.$
Particular case (b) for $\sigma = 0$: $c(u) = e^{u}$	5	$\mathrm{J}_6^{(b)} = xv\frac{\partial}{\partial x} + 2v\frac{\partial}{\partial u} + [e^{2u} + v^2]\frac{\partial}{\partial v},$ $\mathrm{J}_7^{(b)} = \frac{\partial}{\partial u} + v\frac{\partial}{\partial v}.$

In Table 4.9, $F(u) = (3H^2(u) - 2H'(u))^{-1/2}, \quad H(u) = c'(u)/c(u)$.

From the symmetry commutator relations, one can show that

$$(F' + HF)^2 - (\sigma c(u)F)^2 = K^2 = \text{const},$$

and hence for $\sigma = 0$, $F' + HF = K = \text{const}$ [Bluman & Kumei (1987)].

Comparing Tables 4.4 and 4.9, one observes that the symmetries $\mathrm{J}_4, \mathrm{J}_5$ and J_6 yield *nonlocal symmetries* of $\mathbf{U}\{x, t\,; u\}$ (4.8); the symmetry J_7 yields a nonlocal symmetry of $\mathbf{U}\{x, t\,; u\}$ except for the two listed particular cases. For the case $c(u) = u^{-2/3}$, the PDE $\mathbf{X}\{u, v\,; x\}$ has an infinite number of point symmetries that are nonlocal symmetries of the nonlinear wave equation $\mathbf{U}\{x, t\,; u\}$.

(8) The nonlocally related subsystem $\mathbf{T}\{u, v\,; t\}$

The linear wave equation $\mathbf{T}\{u, v\,; t\}$ (3.87), i.e.,

$$t_{uu} = c^2(u)t_{vv},$$

has the group of equivalence transformations that includes the transformations

$$\widetilde{u} = a_1 u + a_4, \quad \widetilde{v} = a_2 v + a_5,$$
$$\widetilde{t} = a_3 t + a_6 + a_7 u + a_8 v + a_9 uv, \quad \widetilde{c}(\widetilde{u}) = a_1^{-1} a_2 c(u), \tag{4.31}$$

and the projective transformation

$$\widetilde{u} = \frac{u}{1+bu}, \quad \widetilde{v} = v, \quad \widetilde{t} = \frac{t}{1+bu}, \quad \widetilde{c}(\widetilde{u}) = (1+b\widetilde{u})^{-2}\, c\left(\frac{\widetilde{u}}{1+b\widetilde{u}}\right), \tag{4.32}$$

where $a_1, \ldots, a_9$ and b are arbitrary constants with $a_1 a_2 a_3 \neq 0$.

The point symmetry classification of the PDE $\mathbf{T}\{u, v\,; t\}$, modulo the equivalence transformations (4.31), (4.32), is given in Table 4.10.

Comparing with the point symmetry classification of the nonlinear wave equation $\mathbf{U}\{x, t\,; u\}$ [Table 4.4], one observes that the symmetries K_5, K_7, K_8, K_9, K_{10}, K_{11} and K_{12} yield *nonlocal symmetries* of the nonlinear wave equation $\mathbf{U}\{x, t\,; u\}$ (4.8).

Table 4.10 Point symmetries of the PDE $\mathbf{T}\{u, v\,; t\}$ (3.87) nonlocally related to the nonlinear wave equation $\mathbf{U}\{x, t\,; u\}$ (3.76)

$c(u)$	#	Symmetries
Arbitrary	∞	Infinite number of point symmetries following from the linearity.
Arbitrary	3	$\mathrm{K}_1 = t\frac{\partial}{\partial t},\ \mathrm{K}_2 = \frac{\partial}{\partial v},\ \mathrm{K}_3 = \frac{\partial}{\partial t}$.
$u^\nu,\quad \nu \neq 0, -1, -2$	5	$\mathrm{K}_1,\ \mathrm{K}_2,\ \mathrm{K}_3,\ \mathrm{K}_4 = u\frac{\partial}{\partial u} + (1+C)v\frac{\partial}{\partial v}$, $\mathrm{K}_5 = -\frac{1}{2}Ctv\frac{\partial}{\partial t} + uv\frac{\partial}{\partial u} + \left[\frac{u^{2+2C}}{1+C} + \frac{1}{2}(1+C)v^2\right]\frac{\partial}{\partial v}$.
e^u	5	$\mathrm{K}_1,\ \mathrm{K}_2,\ \mathrm{K}_3,\ \mathrm{K}_6 = \frac{\partial}{\partial u} + v\frac{\partial}{\partial v}$, $\mathrm{K}_7 = -\frac{1}{2}tv\frac{\partial}{\partial t} + v\frac{\partial}{\partial u} + \frac{1}{2}\left[e^{2u} + v^2\right]\frac{\partial}{\partial v}$.
u^{-1}	5	$\mathrm{K}_1,\ \mathrm{K}_2,\ \mathrm{K}_3,\ \mathrm{K}_4\ (C=-1)$, $\mathrm{K}_8 = \frac{1}{2}tv\frac{\partial}{\partial t} + uv\frac{\partial}{\partial u} + (\log u)\frac{\partial}{\partial v}$.
u^{-2}	∞	Infinite number of nonlocal symmetries; there exists a point transformation into a linear PDE with constant coefficients [Exercise 4.2.6].
$\left[(Bu^2+C) \times \exp\{A\int (Bu^2+C)^{-1}du\}\right]^{-1}$ $(A, B, C = \text{const})$	5	$\mathrm{K}_1,\ \mathrm{K}_2,\ \mathrm{K}_3$, $\mathrm{K}_9 = \frac{1}{2}t(A+2Bu)\frac{\partial}{\partial t} + (Bu^2+C)\frac{\partial}{\partial u} - Av\frac{\partial}{\partial v}$, $\mathrm{K}_{10} = \frac{1}{2}t(A+2Bu)v\frac{\partial}{\partial t} + (Bu^2+C)v\frac{\partial}{\partial u} + \left[-\frac{1}{2}Av^2 + \int c^2(u)(Bu^2+C)du\right]\frac{\partial}{\partial v}$.
$c(u)$ satisfies $(\alpha' + H\alpha)' = \sigma^2 c^2(u)\alpha$, where $\sigma = \text{const} \neq 0$, $H(u) = c'(u)/c(u)$, $\alpha^2(u) = (H^2(u) - 2H'(u))^{-1}$	5	$\mathrm{K}_1,\ \mathrm{K}_2,\ \mathrm{K}_3$, $\mathrm{K}_{11,12} = e^{\pm\sigma v}\left[-\frac{1}{2}t\alpha H\frac{\partial}{\partial t} + \alpha\frac{\partial}{\partial u} \pm\sigma^{-1}(\alpha' + H\alpha)\frac{\partial}{\partial v}\right]$.

4.2.3 Classification of nonlocal symmetries of nonlinear telegraph equations arising from point symmetries of potential systems

Consider the nonlinear telegraph (NLT) equation $\mathbf{U}\{x, t\,; u\}$ given by

$$u_{tt} - (F(u)u_x)_x - (G(u))_x = 0. \tag{4.33}$$

The complete point symmetry classification of the PDE (3.60) with respect to the constitutive functions $F(u)$ and $G(u)$ [Kingston & Sophocleous (2001)], modulo the equivalence transformations (3.61), is presented in Table 4.11.

Table 4.11 Point symmetries of the nonlinear telegraph equation $\mathbf{U}\{x, t\,; u\}$ (3.60)

$F(u)$	$G(u)$	#	Point Symmetries
Arbitrary	Arbitrary	2	$\mathrm{X}_1 = \frac{\partial}{\partial t},\ \mathrm{X}_2 = \frac{\partial}{\partial x}$.
$e^{(\alpha+1)u}$	e^u	3	$\mathrm{X}_1, \mathrm{X}_2, \mathrm{X}_3 = (\alpha-1)t\frac{\partial}{\partial t} + 2\alpha x\frac{\partial}{\partial x} + 2u\frac{\partial}{\partial u}$.
u^α	$u^{\alpha+\beta+1}$	3	$\mathrm{X}_1, \mathrm{X}_2, \mathrm{X}_4 = (\alpha+2\beta)t\frac{\partial}{\partial t} + 2\beta x\frac{\partial}{\partial x} - 2u\frac{\partial}{\partial u}$.
u^{-2}	u^{-1}	4	$\mathrm{X}_1, \mathrm{X}_2, \mathrm{X}_4, \mathrm{X}_5 = e^x\frac{\partial}{\partial x} - ue^x\frac{\partial}{\partial u}$.
u^α	$\ln u$	3	$\mathrm{X}_1, \mathrm{X}_2, \mathrm{X}_6 = (\alpha+2)t\frac{\partial}{\partial t} + 2(\alpha+1)x\frac{\partial}{\partial x} + 2u\frac{\partial}{\partial u}$.
$e^{\alpha u}$	u	3	$\mathrm{X}_1, \mathrm{X}_2, \mathrm{X}_7 = \alpha t\frac{\partial}{\partial t} + 2\alpha x\frac{\partial}{\partial x} + 2\frac{\partial}{\partial u}$.
u^{-4}	u^{-3}	4	$\mathrm{X}_1, \mathrm{X}_2, \mathrm{X}_4, \mathrm{X}_8 = t^2\frac{\partial}{\partial t} + ut\frac{\partial}{\partial u}$.

The complete point symmetry classification of the nonlocally related potential system $\mathbf{UV^1}\{x, t\,; u, v^1\}$ (3.62), i.e.,

$$\begin{aligned} v^1_x &= u_t, \\ v^1_t &= F(u)u_x + G(u), \end{aligned}$$

yielding nonlocal symmetries of the NLT equation (4.33), is presented in Table 4.12 for $G'(u) \neq 0$ [Bluman, Temuerchaolu & Sahadevan (2005)]. Part of this classification appears in Reid (1991b).

Observe that the point symmetries of the potential system $\mathbf{UV^1}\{x, t\,; u, v^1\}$ yield one nonlocal symmetry of the NLT equation $\mathbf{U}\{x, t\,; u\}$ for eight classes of constitutive functions. In the cases $F(u) = u^{-2}, G(u) = u^{-1}$, and $F(u)$ arbitrary, $G(u) = \text{const}$, the potential system $\mathbf{UV^1}\{x, t\,; u, v^1\}$ (3.62) is linearizable by a point transformation, and thus the corresponding NLT equation $\mathbf{U}\{x, t\,; u\}$ (4.33) is linearizable by a nonlocal transformation.

4.2.4 Nonlocal symmetries of nonlinear telegraph equations with power law nonlinearities

In this section, local conservation laws of the nonlinear telegraph equation $\mathbf{U}\{x, t\,; u\}$ (4.33) [Section 3.4.3] are used to construct extended trees of nonlocally related PDE systems for the three cases that arise. For the special situation of power law nonlinearities, $F(u) = u^\alpha$, $G(u) = u^\beta$, nonlocal sym-

Table 4.12 Point symmetries of the potential system $\mathbf{UV^1}\{x,t\,;u,v^1\}$ (3.62) that yield nonlocal symmetries of the NLT equation $\mathbf{U}\{x,t\,;u\}$ (4.33)

$F(u)$	$G(u)$	#	Point Symmetries Yielding Nonlocal Symmetries
Arbitrary	const	∞	Infinite number of symmetries; there exists a point mapping of the potential system $\mathbf{UV^1}\{x,t\,;u,v^1\}$ (3.62) to a linear system [Exercise 3.4.6].
u^{-2}	u^{-1}		
$\pm\frac{4u^{2\alpha+\beta-1}}{(u^{2\alpha}\pm1)^2}$	$\frac{(u^{2\alpha}\mp1)}{(u^{2\alpha}\pm1)}$	1	$\mathrm{Y}_1=[(\beta+1)t+2\alpha v]\frac{\partial}{\partial t}+2[\beta x+\alpha\int F(u)du]\frac{\partial}{\partial x}$ $+2u\frac{\partial}{\partial u}+[2\alpha t+(\beta+1)v]\frac{\partial}{\partial v}$.
$u^{\beta-1}\sec^2(\alpha\ln u)$	$\tan(\alpha\ln u)$	1	$\mathrm{Y}_2=[(\beta+1)t-2\alpha v]\frac{\partial}{\partial t}+2[\beta x-\alpha\int F(u)du]\frac{\partial}{\partial x}$ $+2u\frac{\partial}{\partial u}+[2\alpha t+(\beta+1)v]\frac{\partial}{\partial v}$.
$-u^{\beta-1}(\ln u)^{-2}$	$(\ln u)^{-1}$	1	$\mathrm{Y}_3=[(\beta+1)t+2v]\frac{\partial}{\partial t}+2[\beta x+\int F(u)du]\frac{\partial}{\partial x}$ $+2u\frac{\partial}{\partial u}+(\beta+1)v\frac{\partial}{\partial v}$.
$e^{2\beta u}\sec^2 u$	$\tan u$	1	$\mathrm{Y}_4=(\beta t-v)\frac{\partial}{\partial t}+2[\beta x-\int F(u)du]\frac{\partial}{\partial x}+\frac{\partial}{\partial u}$ $+(t+\beta v)\frac{\partial}{\partial v}$.
$e^{2\beta u}\mathrm{sech}^2 u$	$\tanh u$	1	$\mathrm{Y}_5=(\beta t+v)\frac{\partial}{\partial t}+2[\beta x+\int F(u)du]\frac{\partial}{\partial x}+\frac{\partial}{\partial u}$ $+(t+\beta v)\frac{\partial}{\partial v}$.
$-e^{2\beta u}\mathrm{csch}^2 u$	$\coth u$		
$-u^{-2}e^{2\beta u}$	u^{-1}	1	$\mathrm{Y}_6=(\beta t+v)\frac{\partial}{\partial t}+2[\beta x+\int F(u)du]\frac{\partial}{\partial x}$ $+\frac{\partial}{\partial u}+\beta v\frac{\partial}{\partial v}$.

metries are classified that arise as point symmetries of nonlocally related PDE systems within these extended trees.

(1) Trees of nonlocally related systems for the NLT equation

The extended tree construction procedure [Section 3.5] is applied to the NLT equation $\mathbf{U}\{x,t\,;u\}$, through use of the local conservation laws obtained in Section 3.4.3. Note that the exclusion of dependent variables leads only to locally related subsystems. [Here there is no consideration of nonlocally related subsystems arising from interchanges of independent and dependent variables.] Three cases arise.

Case (a): Arbitrary $F(u),G(u)$. Here, the NLT equation (4.33) has two local conservation laws. The corresponding extended tree $\mathcal{T}_a$ consists of $2^2=4$ PDE systems.

- The NLT equation $\mathbf{U}\{x,t\,;u\}$ (4.33).
- Two singlet potential systems $\mathbf{UV^1}\{x,t\,;u,v^1\}$ (3.62) and $\mathbf{UV^2}\{x,t\,;u,v^2\}$ (3.63).
- One couplet $\mathbf{UV^1V^2}\{x,t\,;u,v^1,v^2\}$ [(3.62), (3.63)].

Case (b): $G'(u) = F(u),\ F(u)$ arbitrary. Here, the NLT equation (4.33) has four local conservation laws. The corresponding extended tree $\mathcal{T}_b$ consists of 16 PDE systems.

- The NLT equation $\mathbf{U}\{x, t\,; u\}$ (4.33).
- Four singlet potential systems $\mathbf{UV^1}\{x, t\,; u, v^1\}$ (3.62), $\mathbf{UV^2}\{x, t\,; u, v^2\}$ (3.63), $\mathbf{UB^3}\{x, t\,; u, b^3\}$ (3.64) and $\mathbf{UB^4}\{x, t\,; u, b^4\}$ (3.65).
- Six couplets $\mathbf{UV^1V^2}\{x, t\,; u, v^1, v^2\}$ [(3.62), (3.63)], $\mathbf{UV^1B^3}\{x, t\,; u, v^1, b^3\}$ [(3.62), (3.64)], $\mathbf{UV^1B^4}\{x, t\,; u, v^1, b^4\}$ [(3.62), (3.65)], $\mathbf{UV^2B^3}\{x, t; u, v^2, b^3\}$ [(3.63), (3.64)], $\mathbf{UV^2B^4}\{x, t\,; u, v^2, b^4\}$ [(3.63), (3.65)] and $\mathbf{UB^3B^4}\{x, t\,; u, b^3, b^4\}$ [(3.64), (3.65)].
- Four triplets $\mathbf{UV^1V^2B^3}\{x, t\,; u, v^1, v^2, b^3\}$, $\mathbf{UV^1V^2B^4}\{x, t\,; u, v^1, v^2, b^4\}$, $\mathbf{UV^1B^3B^4}\{x, t\,; u, v^1, b^3, b^4\}$ and $\mathbf{UV^2B^3B^4}\{x, t\,; u, v^2, b^3, b^4\}$, given by the unions (3.62)–(3.64), [(3.62), (3.63), (3.65)], [(3.62), (3.64), (3.65)] and (3.63)–(3.65), respectively.
- One quadruplet $\mathbf{UV^1V^2B^3B^4}\{x, t\,; u, v^1, v^2, b^3, b^4\}$ (3.62)–(3.65), involving all four potentials.

Case (c): $G(u) = u,\ F(u)$ arbitrary. Here the NLT equation (4.33) again has four local conservation laws. The corresponding extended tree $\mathcal{T}_c$ of nonlocally related PDE systems consists of 16 PDE systems.

- The NLT equation $\mathbf{U}\{x, t\,; u\}$ (4.33).
- Four singlet potential systems $\mathbf{UV^1}\{x, t\,; u, v^1\}$ (3.62), $\mathbf{UV^2}\{x, t\,; u, v^2\}$ (3.63), $\mathbf{UC^3}\{x, t\,; u, c^3\}$ (3.66) and $\mathbf{UC^4}\{x, t\,; u, c^4\}$ (3.67).
- Six couplets $\mathbf{UV^1V^2}\{x, t\,; u, v^1, v^2\}$ [(3.62), (3.63)], $\mathbf{UV^1C^3}\{x, t\,; u, v^1, c^3\}$ [(3.62), (3.66)], $\mathbf{UV^1C^4}\{x, t\,; u, v^1, c^4\}$ [(3.62), (3.67)], $\mathbf{UV^2C^3}\{x, t; u, v^2, c^3\}$[(3.63), (3.66)], $\mathbf{UV^2C^4}\{x, t\,; u, v^2, c^4\}$ [(3.63), (3.67)] and $\mathbf{UC^3C^4}\{x, t; u, c^3, c^4\}$ [(3.66), (3.67)].
- Four triplets $\mathbf{UV^1V^2C^3}\{x, t\,; u, v^1, v^2, c^3\}$, $\mathbf{UV^1V^2C^4}\{x, t\,; u, v^1, v^2, c^4\}$, $\mathbf{UV^1C^3C^4}\{x, t\,; u, v^1, c^3, c^4\}$, and $\mathbf{UV^2C^3C^4}\{x, t\,; u, v^2, c^3, c^4\}$, given by the unions [(3.62), (3.63), (3.66)], [(3.62), (3.63), (3.67)], [(3.62), (3.66), (3.67)] and [(3.63), (3.66), (3.67)], respectively.
- One quadruplet $\mathbf{UV^1V^2C^3C^4}\{x, t\,; u, v^1, v^2, c^3, c^4\}$ [(3.62), (3.63), (3.66), (3.67)], involving all four potential variables.

(2) Symmetries of the NLT equation and nonlocally related systems for power law nonlinearities

Case (a): $F(u) = u^\alpha, G(u) = u^\beta;\ \alpha, \beta \neq 0$. The classification of the point symmetries of the four PDE systems within the tree $\mathcal{T}_a$ is presented in Table 4.13.

From the form of the point symmetries listed in Table 4.13, it follows that no nonlocal symmetries are obtained for the systems $\mathbf{U}\{x, t\,; u\}$ (4.33) and

Table 4.13 Point symmetries of the NLT equation (4.33) and nonlocally related systems in the general power law nonlinearity case (a): $F(u) = u^\alpha$, $G(u) = u^\beta$ $(\alpha, \beta \neq 0)$

System	#	Point Symmetries
$\mathbf{UV^1V^2}$ $\mathbf{UV^1}, \mathbf{UV^2}$, $\mathbf{U}$	5	$X_1 = (\alpha - \beta + 1)x\frac{\partial}{\partial x} + (\frac{\alpha}{2} - \beta + 1)t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u}$ $+\frac{\alpha+2}{2}v^1\frac{\partial}{\partial v^1} + (\alpha - \beta + 2)v^2\frac{\partial}{\partial v^2}$, $X_2 = \frac{\partial}{\partial x},\ X_3 = \frac{\partial}{\partial t} + v^1\frac{\partial}{\partial v^2},\ X_4 = \frac{\partial}{\partial v^1},\ X_5 = \frac{\partial}{\partial v^2}$.

$\mathbf{UV^1}\{x, t\,; u, v^1\}$ (3.62). The infinitesimal generator X_3 yields a nonlocal symmetry of the system $\mathbf{UV^2}\{x, t\,; u, v^2\}$ (3.63) (i.e., the system $\mathbf{UV^2}\{x, t\,; u, v^2\}$ is not invariant under translations in t) and a point symmetry of the other systems. All other infinitesimal generators define point symmetries of all systems in Table 4.13.

Case (b): $G'(u) = F(u)$, i.e., $F(u) = (\alpha+1)u^\alpha$, $G(u) = u^{\alpha+1}$, $\alpha \neq 0, -1, -2$. From the equivalence relation (3.61), this case is equivalent to the situation when $F(u) = u^\alpha$, $G(u) = u^{\alpha+1}$. The point symmetry classifications of the 16 PDE systems within the tree $\mathcal{T}_b$ are presented in Table 4.14.

Table 4.14 Point symmetries of the potential NLT systems for case (b): $F(u) = (\alpha+1)u^\alpha$, $G(u) = u^{\alpha+1}$ $(\alpha \neq 0, -1, -2)$

System	$F(u)$	$G(u)$	#	Point Symmetries
$\mathbf{UV^1V^2B^3B^4}$, $\mathbf{UV^1V^2B^3}$, $\mathbf{UV^1V^2B^4}$, $\mathbf{UV^1B^3B^4}$, $\mathbf{UV^2B^3B^4}$,	$(\alpha+1)u^\alpha$	$u^{\alpha+1}$	7	$Y_1 = -\frac{\alpha}{2}t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u} + v^2\frac{\partial}{\partial v^2} + \frac{\alpha+2}{2}v^1\frac{\partial}{\partial v^1}$ $+\frac{\alpha+2}{2}b^3\frac{\partial}{\partial b^3} + b^4\frac{\partial}{\partial b^4}$, $Y_2 = \frac{\partial}{\partial x} + b^3\frac{\partial}{\partial b^3} + b^4\frac{\partial}{\partial b^4}$, $Y_3 = \frac{\partial}{\partial t} + b^3\frac{\partial}{\partial b^4} + v^1\frac{\partial}{\partial v^2},\ Y_4 = \frac{\partial}{\partial v^1}$, $Y_5 = \frac{\partial}{\partial v^2},\quad Y_6 = \frac{\partial}{\partial b^3},\quad Y_7 = \frac{\partial}{\partial b^4}$.
$\mathbf{UV^1V^2}, \mathbf{UV^1B^3}$, $\mathbf{UV^1B^4}, \mathbf{UV^2B^3}$, $\mathbf{UV^2B^4}, \mathbf{UB^3B^4}$, $\mathbf{UV^1}, \mathbf{UV^2}$, $\mathbf{UB^3}, \mathbf{UB^4}$, $\mathbf{U}$	$-3u^{-4}$	u^{-3}	8	$Y_1,\ Y_2,\ Y_3,\ Y_4,\ Y_5,\ Y_6,\ Y_7$, $Y_8 = t^2\frac{\partial}{\partial t} + tu\frac{\partial}{\partial u} - v^2\frac{\partial}{\partial v^1} - b^4\frac{\partial}{\partial b^3}$.
$\mathbf{UV^1V^2}$	$3u^2$	u^3	8	$Y_1,\ Y_2,\ Y_3,\ Y_4,\ Y_5,\ Y_6,\ Y_7$, $Y_9 = 3v^1\frac{\partial}{\partial x} + (tv^1 - v^2 + 3u)\frac{\partial}{\partial t} - uv^1\frac{\partial}{\partial u}$ $-(v^1)^2\frac{\partial}{\partial v^1} - v^1v^2\frac{\partial}{\partial v^2}$.

The case $\alpha = -2$ is not included in Table 4.14 since here the system $\mathbf{UV^1}\{x, t\,; u, v^1\}$ is linearizable by a point transformation [Bluman & Kumei (1989)] [Section 4.2.3].

From Table 4.14, it follows that for the case when $F(u) = 3u^2, G(u) = u^3$, the potential system $\mathbf{UV^1V^2}\{x, t\,; u, v^1, v^2\}$ [(3.62), (3.63)] has the point symmetry Y_9 which yields a nonlocal symmetry of the NLT equation $\mathbf{U}\{x, t;$ $u\}$ (4.33). Moreover, this is the only case yielding a nonlocal symmetry of the NLT equation $\mathbf{U}\{x, t\,; u\}$.

Note that the infinitesimal generator Y_3 yields a nonlocal symmetry of each of the systems $\mathbf{UV^1V^2B^4}\{x, t\,; u, v^1, v^2, b^4\}$ [(3.62), (3.63), (3.65)], $\mathbf{UV^2B^3B^4}\{x, t; u, v^2, b^3, b^4\}$(3.63)–(3.65), $\mathbf{UV^1B^4}\{x, t; u, v^1, b^4\}$[(3.62), (3.65)], $\mathbf{UV^2B^3}\{x, t\,; u, v^2, b^3\}$ [(3.63), (3.64)], $\mathbf{UV^2B^4}\{x, t\,; u, v^2, b^4\}$ [(3.63), (3.65)], $\mathbf{UV^2}\{x, t\,; u, v^2\}$ (3.63) and $\mathbf{UB^4}\{x, t\,; u, b^4\}$ (3.65), and a point symmetry of the other nine systems; the infinitesimal generator Y_8 yields a nonlocal symmetry of the systems $\mathbf{UV^1V^2B^3}\{x, t\,; u, v^1, v^2, b^3\}$ (3.62) – (3.64), $\mathbf{UV^1B^3B^4}\{x, t; u, v^1, b^3, b^4\}$ [(3.62), (3.64), (3.65)], $\mathbf{UV^1B^3}\{x, t\,; u, v^1, b^3\}$ [(3.62), (3.64)], $\mathbf{UV^1B^4}\{x, t\,; u, v^1, b^4\}$ [(3.62), (3.65)], $\mathbf{UV^2B^3}\{x, t\,; u, v^2, b^3\}$ [(3.63), (3.64)], $\mathbf{UV^1}\{x, t\,; u, v^1\}$ (3.62) and $\mathbf{UB^3}\{x, t\,; u, b^3\}$ (3.64), and a point symmetry of the other nine systems; the infinitesimal generator Y_9 yields a point symmetry of the system $\mathbf{UV^1V^2}\{x, t\,; u, v^1, v^2\}$ [(3.62), (3.63)] and a nonlocal symmetry of the other 15 listed nonlocally related systems.

Case (c): $F(u) = u^\alpha$, $G(u) = u$ $(\alpha \neq 0)$. The corresponding classification of the point symmetries is found in Table 4.15. The linear case $\alpha = 0$ is not considered. The entries in Table 4.15 for the triplets $\mathbf{UV^1C^3C^4}\{x, t\,; u, v^1, c^3, c^4\}$ [(3.62), (3.66), (3.67)], $\mathbf{UV^2C^3C^4}\{x, t\,; u, v^2, c^3, c^4\}$ [(3.63), (3.66), (3.67)], and the couplets $\mathbf{V^1C^4}\{x, t\,; u, v^1, c^4\}$ [(3.62), (3.67)], $\mathbf{UC^3C^4}\{x, t\,; u, c^3, c^4\}$ [(3.66), (3.67)] are missing since they are not known.

From the form of the known point symmetries listed in Table 4.15, it follows that no nonlocal symmetries arise for the systems $\mathbf{U}$ (4.33) and $\mathbf{UV}^1$ (3.62); the infinitesimal generator Z_2 yields a nonlocal symmetry of the systems $\mathbf{UV}^2\mathbf{C}^3$ [(3.63), (3.66)], $\mathbf{UC}^3$ (3.66) and $\mathbf{UC}^4$ (3.67), and a point symmetry of the other listed systems; the infinitesimal generator Z_3 yields a nonlocal symmetry of the systems $\mathbf{UV}^1\mathbf{V}^2\mathbf{C}^4$ [(3.62), (3.63), (3.67)], $\mathbf{UV}^1\mathbf{C}^3$ [(3.62), (3.66)], $\mathbf{UV}^2\mathbf{C}^3$ [(3.63), (3.66)], $\mathbf{UV}^2\mathbf{C}^4$ [(3.63), (3.67)], $\mathbf{UV}^2$ (3.63), $\mathbf{UC}^3$ (3.66) and $\mathbf{UC}^4$ (3.67), and a point symmetry of the other listed systems. All other infinitesimal generators yield point symmetries of each of the systems listed in Table 4.15.

Table 4.15 Point symmetries of the potential NLT systems for case (c): $F(u) = u^\alpha$, $G(u) = u$ $(\alpha \neq 0)$

System	Case	#	Point Symmetries
$\mathbf{UV^1V^2C^3C^4}$ $\mathbf{UV^1V^2C^3}$ $\mathbf{UV^1V^2C^4}$ $\mathbf{UV^1V^2}$, $\mathbf{UV^1C^3}$, $\mathbf{UV^2C^3}$, $\mathbf{UV^2C^4}$, $\mathbf{UV^1}, \mathbf{UV^2}$, $\mathbf{UC^3}, \mathbf{UC^4}$, $\mathbf{U}$	$\alpha \neq -1$	7	$Z_1 = \frac{\alpha}{2}t\frac{\partial}{\partial t} + \alpha x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u} + \frac{\alpha+2}{2}v_1\frac{\partial}{\partial v_1}$ $+v_2(a+1)\frac{\partial}{\partial v_2} + \frac{3\alpha+2}{2}c^3\frac{\partial}{\partial c^3} + (2\alpha+1)c^4\frac{\partial}{\partial c^4}$, $Z_2 = \frac{\partial}{\partial x} + v_1\frac{\partial}{\partial c^3} + v_2\frac{\partial}{\partial c^4}$, $Z_3 = \frac{\partial}{\partial t} + v_1\frac{\partial}{\partial v_2} - v_2\frac{\partial}{\partial c^3} + c^3\frac{\partial}{\partial c^4}$, $Z_4 = \frac{\partial}{\partial v_1}$, $Z_5 = \frac{\partial}{\partial v_2}$, $Z_6 = \frac{\partial}{\partial c^3}$, $Z_7 = \frac{\partial}{\partial c^4}$.
	$\alpha = -1$	8	$Z_2,\ Z_3,\ Z_4,\ Z_5,\ Z_6,\ Z_7$, $Z_8 = -\frac{1}{2}t\frac{\partial}{\partial t} - x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u} + \frac{1}{2}v_1\frac{\partial}{\partial v_1}$ $-\left(t + \frac{1}{2}c^3\right)\frac{\partial}{\partial c^3} - \left(\frac{t^2}{2} + c^4\right)\frac{\partial}{\partial c^4}$.
$\mathbf{UV^1C^3C^4}$, $\mathbf{UV^2C^3C^4}$ $\mathbf{UV^1C^4}, \mathbf{UC^3C^4}$			?

4.2.5 Nonlocal symmetries of the planar gas dynamics equations

In Section 3.5.4, an extended tree $\mathcal{T}_b$ of nonlocally related PDE systems was constructed for the planar gas dynamics equations. One should do a point symmetry classification for each PDE system in the tree $\mathcal{T}_b$ with respect to the constitutive function $B(p, q)$. In this section, it is shown that in many cases a point symmetry of one system in the tree yields a nonlocal symmetry of one or more other systems.

(1) A comparison of point symmetries of three nonlocally related PGD systems

In Table 4.16, for several representative classes of constitutive functions $B(p, q)$, there is a comparison of the point symmetries of three nonlocally related PGD systems: the Euler system $\mathbf{E}\{x, t\,; v, p, \rho\}$ (3.39), the Lagrange system $\mathbf{L}\{y, s\,; v, p, q\}$ (3.42), and the potential system $\mathbf{EA}^1\{x, t\,; v, p, \rho, \alpha^1\}$ (3.40) of the Euler system. [For a full classification, see Akhatov, Gazizov & Ibragimov (1991).]

Observe that the symmetry X_7 is local for the systems $\mathbf{E}\{x, t\,; v, p, \rho\}$ and $\mathbf{EA^1}\{x, t\,; v, p, \rho, \alpha^1\}$ but yields a nonlocal symmetry of the system $\mathbf{L}\{y, s\,; v, p, q\}$; the symmetries Z_7, Z_8 and Z_Θ are local for $\mathbf{L}\{y, s\,; v, p, q\}$ but yield nonlocal symmetries of the systems $\mathbf{E}\{x, t\,; v, p, \rho\}$ and $\mathbf{EA^1}\{x, t; v, p, \rho, \alpha^1\}$; the symmetries Y_8, Y_{10}, Y_{11} and Y_{12} are local for the systems $\mathbf{EA^1}\{x, t; v, p, \rho, \alpha^1\}$ and $\mathbf{L}\{y, s\,; v, p, q\}$ but yield nonlocal symmetries of the sys-

Table 4.16 A comparison of point symmetries of the PGD systems $\mathbf{E}\{x, t\,; v, p, \rho\}$, $\mathbf{EA}^1\{x, t\,; v, p, \rho, \alpha^1\}$ and $\mathbf{L}\{y, s\,; v, p, q\}$

$B(p,q)$	Point Symmetries		
	$\mathbf{E}\{x, t\,; v, p, \rho\}$	$\mathbf{EA}^1\{x, t\,; v, p, \rho, \alpha^1\}$	$\mathbf{L}\{y, s\,; v, p, q\}$
Arbitrary	$\mathrm{X}_1 = \frac{\partial}{\partial x}$, $\mathrm{X}_2 = \frac{\partial}{\partial t}$, $\mathrm{X}_3 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}$, $\mathrm{X}_4 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial v}$.	$\mathrm{Y}_1 = \frac{\partial}{\partial x}$, $\mathrm{Y}_2 = \frac{\partial}{\partial t}$, $\mathrm{Y}_3 = \mathrm{X}_3 + \alpha^1\frac{\partial}{\partial \alpha^1}$, $\mathrm{Y}_4 = \mathrm{X}_4$, $\mathrm{Y}_5 = \frac{\partial}{\partial \alpha^1}$.	$\mathrm{Z}_1 = \frac{\partial}{\partial s}$, $\mathrm{Z}_2 = s\frac{\partial}{\partial s} + y\frac{\partial}{\partial y}$, $\mathrm{Z}_3 = \frac{\partial}{\partial v}$, $\mathrm{Z}_4 = \frac{\partial}{\partial y}$.
$3p/q$	$\mathrm{X}_1, \mathrm{X}_2, \mathrm{X}_3, \mathrm{X}_4$, $\mathrm{X}_5 = x\frac{\partial}{\partial x} + v\frac{\partial}{\partial v}$ $-2\rho\frac{\partial}{\partial \rho}$, $\mathrm{X}_6 = p\frac{\partial}{\partial p} + \rho\frac{\partial}{\partial \rho}$, $\mathrm{X}_7 = t^2\frac{\partial}{\partial t} + tx\frac{\partial}{\partial x}$ $+ (x - tv)\frac{\partial}{\partial v}$ $- 3tp\frac{\partial}{\partial p} - t\rho\frac{\partial}{\partial \rho}$.	$\mathrm{Y}_1, \mathrm{Y}_2, \mathrm{Y}_3, \mathrm{Y}_4, \mathrm{Y}_5$, $\mathrm{Y}_6 = \mathrm{X}_5 - \alpha^1\frac{\partial}{\partial \alpha^1}$, $\mathrm{Y}_7 = \mathrm{X}_6 + \alpha^1\frac{\partial}{\partial \alpha^1}$, $\mathrm{Y}_8 = \mathrm{X}_7$.	$\mathrm{Z}_1, \mathrm{Z}_2, \mathrm{Z}_3, \mathrm{Z}_4$, $\mathrm{Z}_5 = -y\frac{\partial}{\partial y} + v\frac{\partial}{\partial v}$ $+2q\frac{\partial}{\partial q}$, $\mathrm{Z}_6 = y\frac{\partial}{\partial y} + p\frac{\partial}{\partial p}$ $-q\frac{\partial}{\partial q}$. Nonlocal
$-p/q$	$\mathrm{X}_1, \mathrm{X}_2, \mathrm{X}_3, \mathrm{X}_4$, $\mathrm{X}_5, \mathrm{X}_6$. Nonlocal Nonlocal	$\mathrm{Y}_1, \mathrm{Y}_2, \mathrm{Y}_3, \mathrm{Y}_4$, $\mathrm{Y}_5, \mathrm{Y}_6, \mathrm{Y}_7$, Nonlocal Nonlocal	$\mathrm{Z}_1, \mathrm{Z}_2, \mathrm{Z}_3, \mathrm{Z}_4$, $\mathrm{Z}_5, \mathrm{Z}_6$, $\mathrm{Z}_7 = \frac{\partial}{\partial p} + \frac{q}{p}\frac{\partial}{\partial q}$, $\mathrm{Z}_8 = s\frac{\partial}{\partial v} - y\frac{\partial}{\partial p}$ $-\frac{yq}{p}\frac{\partial}{\partial q}$.
$pF(pe^q)$	$\mathrm{X}_1, \mathrm{X}_2, \mathrm{X}_3, \mathrm{X}_4$, Nonlocal	$\mathrm{Y}_1, \mathrm{Y}_2, \mathrm{Y}_3, \mathrm{Y}_4, \mathrm{Y}_5$, $\mathrm{Y}_8 = t\frac{\partial}{\partial t} + 2\alpha^1\frac{\partial}{\partial x} - v\frac{\partial}{\partial v}$ $- 2p\frac{\partial}{\partial p} + 2\rho^2\frac{\partial}{\partial \rho}$.	$\mathrm{Z}_1, \mathrm{Z}_2, \mathrm{Z}_3, \mathrm{Z}_4$, $\mathrm{Z}_9 = s\frac{\partial}{\partial s} - v\frac{\partial}{\partial v}$ $- 2p\frac{\partial}{\partial p} + 2\frac{\partial}{\partial q}$.
$F(q)$	$\mathrm{X}_1, \mathrm{X}_2, \mathrm{X}_3, \mathrm{X}_4$, $\mathrm{X}_8 = \frac{\partial}{\partial p}$. Nonlocal	$\mathrm{Y}_1, \mathrm{Y}_2, \mathrm{Y}_3, \mathrm{Y}_4, \mathrm{Y}_5$, $\mathrm{Y}_9 = \frac{\partial}{\partial p}$, $\mathrm{Y}_{10} = \frac{t^2}{2}\frac{\partial}{\partial x} + t\frac{\partial}{\partial v} - \alpha^1\frac{\partial}{\partial p}$.	$\mathrm{Z}_1, \mathrm{Z}_2, \mathrm{Z}_3, \mathrm{Z}_4$, $\mathrm{Z}_{10} = \frac{\partial}{\partial p}$, $\mathrm{Z}_{11} = s\frac{\partial}{\partial v} - y\frac{\partial}{\partial p}$.
$F(p + nq)$ $n \neq 0$	$\mathrm{X}_1, \mathrm{X}_2, \mathrm{X}_3, \mathrm{X}_4$. Nonlocal Nonlocal	$\mathrm{Y}_1, \mathrm{Y}_2, \mathrm{Y}_3, \mathrm{Y}_4, \mathrm{Y}_5$, $\mathrm{Y}_{11} = n\alpha^1\frac{\partial}{\partial x} - \frac{\partial}{\partial p} - \rho^2\frac{\partial}{\partial \rho}$, $\mathrm{Y}_{12} = \frac{nt^2 + (\alpha^1)^2}{2}\frac{\partial}{\partial x} + nt\frac{\partial}{\partial v}$ $- n\alpha^1\frac{\partial}{\partial p} - \rho^2\alpha^1\frac{\partial}{\partial \rho}$.	$\mathrm{Z}_1, \mathrm{Z}_2, \mathrm{Z}_3, \mathrm{Z}_4$, $\mathrm{Z}_{12} = \frac{\partial}{\partial q} - n\frac{\partial}{\partial p}$, $\mathrm{Z}_{13} = ns\frac{\partial}{\partial v}$ $-ny\frac{\partial}{\partial p} + y\frac{\partial}{\partial q}$.
$F(p)$	$\mathrm{X}_1, \mathrm{X}_2, \mathrm{X}_3, \mathrm{X}_4$. Nonlocal Nonlocal	$\mathrm{Y}_1, \mathrm{Y}_2, \mathrm{Y}_3, \mathrm{Y}_4, \mathrm{Y}_5$, $\mathrm{Y}_\Psi = \Psi(\alpha^1)\frac{\partial}{\partial x} - \rho^2\Psi'(\alpha^1)\frac{\partial}{\partial \rho}$. Nonlocal $\Psi(\alpha^1)$ arbitrary.	$\mathrm{Z}_1, \mathrm{Z}_2, \mathrm{Z}_3, \mathrm{Z}_4$, Nonlocal $\mathrm{Z}_\Theta = \Theta\Big(y, q$ $+ \int \frac{dp}{F(p)}\Big)\frac{\partial}{\partial q}$, $\Theta(y, z)$ arbitrary.

tem $\mathbf{E}\{x,t\,;v,p,\rho\}$; the infinite number of symmetries Y_Ψ are local for the system $\mathbf{EA^1}\{x,t;v,p,\rho,\ \alpha^1\}$ but yield nonlocal symmetries of the systems $\mathbf{E}\{x,t\,;v,p,\rho\}$ and $\mathbf{L}\{y,s\,;v,p,q\}$.

(2) Nonlocal symmetries of polytropic PGD equations
Now consider symmetries of the nonlocally related PDE systems of planar gas dynamics equations in the tree $\mathcal{T}_b$ for the polytropic case $B(p,q) = \gamma p/q$, $\gamma \neq 0$. Comparisons are made for the complete point symmetry classifications of several such PDE systems: systems $\mathbf{E}\{x,t\,;v,p,\rho\}$ (3.39), $\mathbf{L}\{y,s\,;v,p,q\}$ (3.42) and $\underline{\mathbf{L}}\{y,s\,;p,q\}$ (3.46) [Table 4.17], as well as for the potential systems $\mathbf{LW}^1\{y,s\,;v,p,q,w^1\}$ (3.97), $\mathbf{LW}^4\{y,s\,;v,p,q,w^4\}$ (3.120) and $\underline{\mathbf{LW}}^4\{y,s\,;p,q,w^4\}$ (3.123) [Table 4.18].

Table 4.17 Point symmetries of the PGD systems $\mathbf{E}\{x,t\,;v,p,\rho\}$, $\mathbf{L}\{y,s\,;v,p,q\}$ and $\underline{\mathbf{L}}\{y,s\,;p,q\}$ in the polytropic case

γ	Point Symmetries		
	$\mathbf{E}\{x,t\,;v,p,\rho\}$	$\mathbf{L}\{y,s\,;v,p,q\}$	$\underline{\mathbf{L}}\{y,s\,;p,q\}$
Arbitrary	$\mathrm{X}_1 = \frac{\partial}{\partial x}$, $\mathrm{X}_2 = \frac{\partial}{\partial t}$, $\mathrm{X}_3 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}$, $\mathrm{X}_4 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial v}$, $\mathrm{X}_5 = x\frac{\partial}{\partial x} + v\frac{\partial}{\partial v} + p\frac{\partial}{\partial p} - \rho\frac{\partial}{\partial \rho}$, $\mathrm{X}_6 = p\frac{\partial}{\partial p} + \rho\frac{\partial}{\partial \rho}$.	$\mathrm{Z}_1 = \frac{\partial}{\partial s}$, $\mathrm{Z}_2 = s\frac{\partial}{\partial s} + y\frac{\partial}{\partial y}$, $\mathrm{Z}_3 = \frac{\partial}{\partial v}$, $\mathrm{Z}_4 = v\frac{\partial}{\partial v} + p\frac{\partial}{\partial p} + q\frac{\partial}{\partial q}$, $\mathrm{Z}_5 = y\frac{\partial}{\partial y} + p\frac{\partial}{\partial p} - q\frac{\partial}{\partial q}$, $\mathrm{Z}_6 = \frac{\partial}{\partial y}$.	$\hat{\mathrm{Z}}_1 = \mathrm{Z}_1$, $\hat{\mathrm{Z}}_2 = \mathrm{Z}_2$, $\hat{\mathrm{Z}}_3 = p\frac{\partial}{\partial p} + q\frac{\partial}{\partial q}$, $\hat{\mathrm{Z}}_4 = \mathrm{Z}_5$, $\hat{\mathrm{Z}}_5 = \mathrm{Z}_6$, $\hat{\mathrm{Z}}_6 = y^2\frac{\partial}{\partial y} + yp\frac{\partial}{\partial p} - 3yq\frac{\partial}{\partial q}$.
3	$\mathrm{X}_1, \mathrm{X}_2, \mathrm{X}_3, \mathrm{X}_4, \mathrm{X}_5, \mathrm{X}_6$, $\mathrm{X}_7 = xt\frac{\partial}{\partial x} + t^2\frac{\partial}{\partial t} + (x - vt)\frac{\partial}{\partial v} - 3tp\frac{\partial}{\partial p} - t\rho\frac{\partial}{\partial \rho}$.	$\mathrm{Z}_1, \mathrm{Z}_2, \mathrm{Z}_3, \mathrm{Z}_4, \mathrm{Z}_5, \mathrm{Z}_6$.	$\hat{\mathrm{Z}}_1, \hat{\mathrm{Z}}_2, \hat{\mathrm{Z}}_3, \hat{\mathrm{Z}}_4, \hat{\mathrm{Z}}_5, \hat{\mathrm{Z}}_6$, $\hat{\mathrm{Z}}_7 = s^2\frac{\partial}{\partial s} - 3sp\frac{\partial}{\partial p} + sq\frac{\partial}{\partial q}$.
-1	$\mathrm{X}_1, \mathrm{X}_2, \mathrm{X}_3, \mathrm{X}_4, \mathrm{X}_5, \mathrm{X}_6$.	$\mathrm{Z}_1, \mathrm{Z}_2, \mathrm{Z}_3, \mathrm{Z}_4, \mathrm{Z}_5, \mathrm{Z}_6$, $\mathrm{Z}_7 = \frac{\partial}{\partial p} + \frac{q}{p}\frac{\partial}{\partial q}$, $\mathrm{Z}_8 = -s\frac{\partial}{\partial v} + y\frac{\partial}{\partial p} + \frac{yq}{p}\frac{\partial}{\partial q}$.	$\hat{\mathrm{Z}}_1, \hat{\mathrm{Z}}_2, \hat{\mathrm{Z}}_3, \hat{\mathrm{Z}}_4, \hat{\mathrm{Z}}_5, \hat{\mathrm{Z}}_6$, $\hat{\mathrm{Z}}_8 = \mathrm{Z}_7$, $\hat{\mathrm{Z}}_9 = y\frac{\partial}{\partial p} + \frac{yq}{p}\frac{\partial}{\partial q}$, $\hat{\mathrm{Z}}_{10} = s\frac{\partial}{\partial p} + \frac{sq}{p}\frac{\partial}{\partial q}$, $\hat{\mathrm{Z}}_{11} = sy\frac{\partial}{\partial p} + \frac{syq}{p}\frac{\partial}{\partial q}$.

Observe that the symmetry $\widehat{\mathrm{Z}}_7$ yields nonlocal symmetries of each of the systems $\mathbf{L}\{y,s\,;v,p,q\}$ and $\mathbf{LW}^4\{y,s\,;v,p,q,w^4\}$ but yields local sym-

Table 4.18 Point symmetries of the PGD systems $\underline{\mathbf{LW}}^4\{y,s\,;p,q,w^4\}$, $\mathbf{LW}^4\{y,s;v,p,q,w^4\}$ and $\mathbf{LW}^1\{y,s\,;v,p,q,w^1\}$ in the polytropic case

γ	Point Symmetries		
	$\underline{\mathbf{LW}}^4\{y,s\,;p,q,w^4\}$	$\mathbf{LW}^4\{y,s\,;v,p,q,w^4\}$	$\mathbf{LW}^1\{y,s\,;v,p,q,w^1\}$
Arbitrary	$\hat{\mathrm{J}}_1=\frac{\partial}{\partial w^4}$, $\hat{\mathrm{J}}_2=\frac{\partial}{\partial s}$, $\hat{\mathrm{J}}_3=y\frac{\partial}{\partial y}+s\frac{\partial}{\partial s}+w^4\frac{\partial}{\partial w^4}$, $\hat{\mathrm{J}}_4=p\frac{\partial}{\partial p}+q\frac{\partial}{\partial q}+(\gamma+1)w^4\frac{\partial}{\partial w^4}$ $\hat{\mathrm{J}}_5=y\frac{\partial}{\partial y}+p\frac{\partial}{\partial p}-q\frac{\partial}{\partial q},+(2-\gamma)w^4\frac{\partial}{\partial w^4}$, $\hat{\mathrm{J}}_6=\frac{\partial}{\partial y}$.	$\mathrm{J}_1=\hat{\mathrm{J}}_1$, $\mathrm{J}_2=\hat{\mathrm{J}}_2$, $\mathrm{J}_3=\hat{\mathrm{J}}_3$, $\mathrm{J}_4=\frac{\partial}{\partial v}$, $\mathrm{J}_5=v\frac{\partial}{\partial v}+\hat{\mathrm{J}}_4$, $\mathrm{J}_6=\hat{\mathrm{J}}_5$, $\mathrm{J}_7=\hat{\mathrm{J}}_6$.	$\mathrm{Y}_1=\frac{\partial}{\partial w^1}$, $\mathrm{Y}_2=\hat{\mathrm{J}}_2$, $\mathrm{Y}_3=y\frac{\partial}{\partial y}+s\frac{\partial}{\partial s}+w^1\frac{\partial}{\partial w^1}$ $\mathrm{Y}_4=\frac{\partial}{\partial v}+s\frac{\partial}{\partial w^1}$, $\mathrm{Y}_5=v\frac{\partial}{\partial v}+p\frac{\partial}{\partial p}+q\frac{\partial}{\partial q}+w^1\frac{\partial}{\partial w^1}$, $\mathrm{Y}_6=y\frac{\partial}{\partial y}+p\frac{\partial}{\partial p}-q\frac{\partial}{\partial q}$, $\mathrm{Y}_7=\hat{\mathrm{J}}_6$.
3	$\hat{\mathrm{J}}_1,\hat{\mathrm{J}}_2,\hat{\mathrm{J}}_3,\hat{\mathrm{J}}_4$, $\hat{\mathrm{J}}_5,\hat{\mathrm{J}}_6$, $\hat{\mathrm{J}}_7=s^2\frac{\partial}{\partial s}-3sp\frac{\partial}{\partial p}+sq\frac{\partial}{\partial q}$.	$\mathrm{J}_1,\mathrm{J}_2,\mathrm{J}_3,\mathrm{J}_4$, $\mathrm{J}_5,\mathrm{J}_6,\mathrm{J}_7$.	$\mathrm{Y}_1,\mathrm{Y}_2,\mathrm{Y}_3,\mathrm{Y}_4$, $\mathrm{Y}_5,\mathrm{Y}_6,\mathrm{Y}_7$, $\mathrm{Y}_8=\hat{\mathrm{J}}_7+(w^1-sv)\frac{\partial}{\partial v}+sw^1\frac{\partial}{\partial w^1}$.
-1	$\hat{\mathrm{J}}_1,\hat{\mathrm{J}}_2,\hat{\mathrm{J}}_3,\hat{\mathrm{J}}_4$, $\hat{\mathrm{J}}_5,\hat{\mathrm{J}}_6$, $\hat{\mathrm{J}}_7=\mathrm{Z}_7$, $\hat{\mathrm{J}}_8=\mathrm{Z}_8$, $\hat{\mathrm{J}}_9=\hat{\mathrm{Z}}_{10}$, $\hat{\mathrm{J}}_{10}=\hat{\mathrm{Z}}_{11}$.	$\mathrm{J}_1,\mathrm{J}_2,\mathrm{J}_3,\mathrm{J}_4$, $\mathrm{J}_5,\mathrm{J}_6,\mathrm{J}_7$, $\mathrm{J}_8=\mathrm{Z}_7$, $\mathrm{J}_9=\mathrm{Z}_8$.	$\mathrm{Y}_1,\mathrm{Y}_2,\mathrm{Y}_3,\mathrm{Y}_4$, $\mathrm{Y}_5,\mathrm{Y}_6,\mathrm{Y}_7$.
1	$\hat{\mathrm{J}}_1,\hat{\mathrm{J}}_2,\hat{\mathrm{J}}_3,\hat{\mathrm{J}}_4$, $\hat{\mathrm{J}}_5,\hat{\mathrm{J}}_6$, $\hat{\mathrm{J}}_{11}=\hat{\mathrm{Z}}_6$.	$\mathrm{J}_1,\mathrm{J}_2,\mathrm{J}_3,\mathrm{J}_4$, $\mathrm{J}_5,\mathrm{J}_6,\mathrm{J}_7$.	$\mathrm{Y}_1,\mathrm{Y}_2,\mathrm{Y}_3,\mathrm{Y}_4$, $\mathrm{Y}_5,\mathrm{Y}_6,\mathrm{Y}_7$.

metries of the other four considered systems $\mathbf{E}\{x,t\,;v,p,\rho\}$, $\underline{\mathbf{L}}\{y,s\,;p,q\}$, $\mathbf{LW}^1\{y,s\,;v,p,q,w^1\}$ and $\underline{\mathbf{LW}}^4\{y,s\,;p,q,w^4\}$; the symmetries Z_7 and Z_8 yield nonlocal symmetries of the systems $\mathbf{E}\{x,t\,;v,p,\rho\}$ and $\mathbf{LW}^1\{y,s;v,p,q,w^1\}$ but local symmetries of the other four considered systems $\mathbf{L}\{y,s;v,p,q\}$, $\underline{\mathbf{L}}\{y,s\,;p,q\}$, $\mathbf{LW}^4\{y,s\,;v,p,q,w^4\}$ and $\underline{\mathbf{LW}}^4\{y,s\,;p,q,w^4\}$; the symmetries $\hat{\mathrm{Z}}_{10}$ and $\hat{\mathrm{Z}}_{11}$ are local symmetries of the Lagrange subsystem $\underline{\mathbf{L}}\{y,s;p,q\}$ and the subsystem $\underline{\mathbf{LW}}^4\{y,s\,;p,q,w^4\}$ but yield nonlocal symmetries of the other four considered systems $\mathbf{E}\{x,t\,;v,p,\rho\}$, $\mathbf{L}\{y,s\,;v,p,q\}$, $\mathbf{LW}^1\{y,s;\ v,p,\ q,w^1\}$ and $\mathbf{LW}^4\{y,s\,;v,p,q,w^4\}$. Interestingly, the symmetry $\hat{\mathrm{Z}}_6$, a local symmetry of the Lagrange subsystem $\underline{\mathbf{L}}\{y,s\,;p,q\}$ for *any* value of the polytropic constant γ, yields a local symmetry of the subsystem $\underline{\mathbf{LW}}^4\{y, s\,; p, q, w^4\}$ *only in the case* $\gamma = 1$ (and yields a nonlocal symmetry otherwise), and is a nonlocal symmetry of the other four considered PGD systems $\mathbf{E}\{x, t\,; v, p, \rho\}$, $\mathbf{L}\{y, s\,; v, p, q\}$, $\mathbf{LW}^1\{y, s\,; v, p, q, w^1\}$ and $\mathbf{LW}^4\{y, s\,; v, p, q, w^4\}$ for all values of γ.

(3) Nonlocal symmetries of generalized polytropic PGD equations
As another example, consider a nonlocal symmetry classification problem for PGD equations with a generalized polytropic equation of state

$$B(p, q) = \frac{M(p)}{q}, \quad M''(p) \neq 0, \tag{4.34}$$

which excludes the polytropic case considered in the previous example.

For the sake of brevity, consideration is only given for the extended tree $\mathcal{T}_a$ (3.105) of PDE systems of planar gas dynamics equations. [These follow from local conservation laws of the Lagrange PGD system $\mathbf{L}\{y, s\,; v, p, q\}$ that arise from zeroth-order multipliers $\Lambda_i = \Lambda_i(y, s, v, p, q)$; see Section 3.5.4, Figure 3.5.]

The extended tree $\mathcal{T}'_a$ includes ten nonlocally related PDE systems.

- The Euler system $\mathbf{E}\{x, t\,; v, p, \rho\}$ (3.39).
- The Lagrange system $\mathbf{L}\{y, s\,; v, p, q\}$ (3.42).
- Three singlet potential systems $\mathbf{LW^1}\{y, s\,; v, p, q, w^1\}$ (3.97), $\mathbf{LW}^2\{y, s;$ $v, p, q, w^2\}$(3.98), and $\mathbf{LW^3}\{y, s\,; v, p, q, w^3\}$ (3.99).
- Three couplets $\mathbf{LW^1W^2}\{y, s\,; v, p, q, w^1, w^2\}$ (3.100), $\mathbf{LW}^1\mathbf{W}^3\{y, s; v, p, q,$ $w^1, w^3\}$ (3.101), and $\mathbf{LW^2W^3}\{y, s\,; v, p, q, w^2, w^3\}$ (3.102).
- One triplet $\mathbf{LW^1W^2W^3}\{y, s\,; v, p, q, w^1, w^2, w^3\}$ (3.103).
- The nonlocally related subsystem $\underline{\mathbf{L}}\{y, s\,; p, q\}$ (3.46).

The point symmetry classification of each of the above seven potential systems (modulo the equivalence transformations (3.96)), i.e., the three singlets, three couplets and one triplet, yields Table 4.19 that lists point symmetries and nonlocal symmetries for the Lagrange PGD system $\mathbf{L}\{y, s\,; v, p, q\}$ with the equation of state (4.34).

In Table 4.19, the symmetries of each PDE system arise as projections of infinitesimal generators presented in the right-hand column on the space of variables of that system.

From Table 4.19, observe that the Euler system $\mathbf{E}\{x, t\,; v, p, \rho\}$ has the same symmetries for any $M(p)$. The infinitesimal generators $\mathrm{Z}_9, \ldots, \mathrm{Z}_{12}$ yield point symmetries of the systems $\underline{\mathbf{L}}\{y, s\,; p, q\}$, $\mathbf{L}\{y, s\,; v, p, q\}$ and $\mathbf{LW}^2\{y, s;$ $v, p, q, w^2\}$, and nonlocal symmetries of all other systems; the infinitesimal generators $\mathrm{Z}_{13}, \mathrm{Z}_{14}$ yield point symmetries of the systems $\underline{\mathbf{L}}\{y, s\,; p, q\}$ and

Table 4.19 Symmetries of the generalized polytropic planar gas dynamics equations

System	$M(p)$	Point Symmetries
$\mathbf{E}$	Arbitrary	$\mathrm{X}_1 = \frac{\partial}{\partial t}$, $\mathrm{X}_2 = \frac{\partial}{\partial x}$, $\mathrm{X}_3 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial v}$, $\mathrm{X}_4 = x\frac{\partial}{\partial x} + v\frac{\partial}{\partial v} - 2\rho\frac{\partial}{\partial \rho}$, $\mathrm{X}_5 = x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t}$.
$\underline{\mathbf{L}}$, $\mathbf{L}$, $\mathbf{LW}^1$, $\mathbf{LW}^2$, $\mathbf{LW}^3$, $\mathbf{LW}^1\mathbf{W}^2$, $\mathbf{LW}^1\mathbf{W}^3$, $\mathbf{LW}^2\mathbf{W}^3$, $\mathbf{LW}^1\mathbf{W}^2\mathbf{W}^3$	(i) Arbitrary	$\mathrm{Z}_1 = \frac{\partial}{\partial s} + w^2\frac{\partial}{\partial w^3}$, $\mathrm{Z}_2 = \frac{\partial}{\partial y} + w^1\frac{\partial}{\partial w^3}$, $\mathrm{Z}_3 = \frac{\partial}{\partial v} + s\frac{\partial}{\partial w^1} + y\frac{\partial}{\partial w^2} + sy\frac{\partial}{\partial w^3}$, $\mathrm{Z}_4 = -y\frac{\partial}{\partial y} + 2q\frac{\partial}{\partial q} + v\frac{\partial}{\partial v} + w^1\frac{\partial}{\partial w^1}$, $\mathrm{Z}_5 = s\frac{\partial}{\partial s} + y\frac{\partial}{\partial y} + w^1\frac{\partial}{\partial w^1} + w^2\frac{\partial}{\partial w^2}$ $+2w^3\frac{\partial}{\partial w^3}$, $\mathrm{Z}_6 = \frac{\partial}{\partial w^1}$, $\mathrm{Z}_7 = \frac{\partial}{\partial w^2}$, $\mathrm{Z}_8 = \frac{\partial}{\partial w^3}$.
$\underline{\mathbf{L}}$, $\mathbf{L}$, $\mathbf{LW}^2$	(ii) $-p\ln p$	$\mathrm{Z}_9 = y\frac{\partial}{\partial y} + 2p\frac{\partial}{\partial p} + \frac{2q}{\ln p}\frac{\partial}{\partial q} + v\frac{\partial}{\partial v} + 2w^2\frac{\partial}{\partial w^2}$.
	(iii) $\gamma p + \delta p^{\frac{\gamma+1}{\gamma}}$ $\gamma \neq 0, -1$	$\mathrm{Z}_{10} = \frac{(\gamma+1)y}{2\gamma}\frac{\partial}{\partial y} + p\frac{\partial}{\partial p} - \frac{q}{\delta p^{1/\gamma}+\gamma}\frac{\partial}{\partial q}$ $+\frac{(\gamma-1)v}{2\gamma}\frac{\partial}{\partial v} + w^2\frac{\partial}{\partial w^2}$.
	(iv) $1 + \alpha e^p$, $\alpha = \pm 1$	$\mathrm{Z}_{11} = \frac{\partial}{\partial p} + \frac{\alpha e^p}{1+\alpha e^p}q\frac{\partial}{\partial q} - s\frac{\partial}{\partial w^2}$, $\mathrm{Z}_{12} = y\frac{\partial}{\partial p} + \frac{\alpha e^p}{1+\alpha e^p}yq\frac{\partial}{\partial q} - s\frac{\partial}{\partial v} - sy\frac{\partial}{\partial w^2}$.
$\underline{\mathbf{L}}$, $\mathbf{LW}^2$	(ii) $-p\ln p$	$\mathrm{Z}_{13} = y^2\frac{\partial}{\partial y} + yp\frac{\partial}{\partial p} - \left(3 - \frac{1}{\ln p}\right)yq\frac{\partial}{\partial q}$ $-(yv - w^2)\frac{\partial}{\partial v} + yw^2\frac{\partial}{\partial w^2}$.
	(iii) $\gamma p + \delta p^{\frac{\gamma+1}{\gamma}}$ $\gamma \neq 0, -1$	$\mathrm{Z}_{14} = y^2\frac{\partial}{\partial y} + yp\frac{\partial}{\partial p} - \left(3 - \frac{\delta}{\gamma}\frac{p^{1/\gamma}}{\delta p^{1/\gamma}+\gamma}\right)yq\frac{\partial}{\partial q}$ $-(yv - w^2)\frac{\partial}{\partial v} + yw^2\frac{\partial}{\partial w^2}$.
$\underline{\mathbf{L}}$	(iii) with $\gamma = 3$: $3p + \delta p^{\frac{4}{3}}$	$\hat{\mathrm{Z}}_{15} = \frac{1}{3}s^2\frac{\partial}{\partial s} - sp\frac{\partial}{\partial p} + \frac{1}{\delta p^{4/3}+3}spq\frac{\partial}{\partial q}$.

$\mathbf{LW^2}\{y, s\,; v, p, q, w^2\}$, and nonlocal symmetries of all other systems, including the Euler system $\mathbf{E}\{x, t\,; v, p, \rho\}$ and the Lagrange system $\mathbf{L}\{y, s\,; v, p, q\}$.

The point symmetries of the Lagrange subsystem $\underline{\mathbf{L}}\{y, s\,; p, q\}$ include all corresponding point symmetries of the system $\mathbf{LW^2}\{y, s\,; v, p, q, w^2\}$; for $M(p) = 3p + \delta p^{4/3}$, one additional symmetry $\hat{\mathrm{Z}}_{15}$ is obtained that is a nonlocal symmetry of the Euler system $\mathbf{E}\{x, t\,; v, p, \rho\}$, the Lagrange system $\mathbf{L}\{y, s\,; v, p, q\}$ and all its seven potential systems considered in this example.

All other infinitesimal generators in Table 4.19 project onto point symmetries for both the Euler system $\mathbf{E}\{x, t\,; v, p, \rho\}$ and the Lagrange system $\mathbf{L}\{y, s\,; v, p, q\}$.

Exercises 4.2

4.2.1. Find equivalence transformations of the nonlocally related systems in the extended tree $\mathcal{T}_4$ for the nonlinear diffusion equation (3.18). Determine the point symmetry classifications of each of the nonlocally related subsystems $\mathbf{A}\{x,u\,;\alpha\}$, $\mathbf{V}\{u,t\,;v\}$ and $\mathbf{X}\{u,v\,;x\}$ within the extended tree $\mathcal{T}_4$ for the nonlinear diffusion equation (3.18).

4.2.2. Find the infinite set of point symmetries of the potential system $\mathbf{UW}\{x,t\,;u,w\}$ (3.82) of the nonlinear diffusion equation (3.18). Find a point transformation that maps $\mathbf{UW}\{x,t\,;u,w\}$ into a linear PDE system.

4.2.3. Show that the potential system $\mathbf{UB}\{x,t\,;u,\beta\}$ (3.84) of the nonlinear wave equation $u_{tt}=(c^2(u)u_x)_x$ (3.76), in the case $c(u)=u^{-2/3}$, has an infinite number of point symmetries. For this case, find an explicit form of the linearizing transformation. [Hint: In this case, instead of computing an infinite number of point symmetries and applying Theorem 2.4.2, one may start by introducing new independent variables $s=x^{-1}$, $\beta=x^3u$. The resulting PDE system is linearizable by a hodograph transformation.]

4.2.4.

(a) Find the point symmetries M_7 and M_8 of the potential system $\mathbf{TQ}\{u,v;\,t,q\}$ (3.93) of the wave equation $u_{tt}=(c^2(u)u_x)_x$ (3.76) [Table 4.7].
(b) Find the point symmetries N_5 and N_6 of the potential system $\mathbf{TR}\{u,v;\,t,r\}$ (3.94) of the wave equation (3.76) [Table 4.8].

4.2.5. Find the point symmetries of the linear wave equation $\mathbf{X}\{u,v\,;x\}$ (3.86). Deduce whether this linear wave equation can be mapped by a point transformation into a constant coefficient linear PDE.

4.2.6. Calculate the components of the nontrivial infinite-parameter set of point symmetries of the linear wave equation

$$q_{tt}=x^2q_{xx} \tag{4.35}$$

(equivalent to the equation (3.87) after a suitable renaming of the variables). Show that the scalar PDE (4.35) can be mapped into the constant coefficient linear wave equation $Q_{XT}=0$ by the point transformation

$$X=1/x+t,\quad T=1/x-t,\quad Q=q/x+t$$

[Bluman (1983); Bluman & Kumei (1987)].

4.2.7. Show that the symmetry $\hat{\mathrm{Z}}_6$ [Table 4.17], which yields a nonlocal symmetry of both the polytropic Euler and Lagrange PGD systems and a local symmetry of the Lagrange subsystem $\underline{\mathbf{L}}\{y, s\,; p, q\}$, also yields a local symmetry of both the potential Lagrange system $\mathbf{LW^2}\{y, s\,; v, p, q, w^2\}$ (3.98) and the triplet potential Lagrange system $\mathbf{LW^1W^2W^3}\{y, s\,; v, p, q, w^1, w^2, w^3\}$ (3.103). Find the components of the infinitesimal symmetry generator corresponding to each of v, w^1, w^2, and w^3.

4.3 Construction of Non-invertible Mappings Relating PDEs

In this section, nonlocally related systems are used to extend the work presented in Sections 2.4–2.6 on the invertible mapping of a given PDE system to one of a simpler type that can draw on an arsenal of well-known solution techniques. In particular, it is shown how to find useful nonlocal mappings relating PDEs through the use of nonlocally related potential systems.

Firstly, the invertible mapping algorithm presented in Section 2.4 is extended to include nonlocal mappings of nonlinear PDEs to linear PDEs. Here, if a nonlocally related potential system has a point symmetry that satisfies the criteria of Theorems 2.4.1 and 2.4.2 and yields a nonlocal (potential) symmetry of a given PDE system, then one can construct an invertible mapping of the potential system to a linear system that in turn yields a nonlocal mapping of the given nonlinear PDE system to a linear PDE system. A similar extension occurs when such a nonlocal mapping exists of a nonlinear PDE system to a linear PDE system when the nonlocally related potential system of the nonlinear PDE system has a set of local conservation law multipliers that satisfies the criteria of Theorems 2.6.1 and 2.6.2.

Secondly, it is shown how to extend the invertible mapping algorithm presented in Section 2.4 to include nonlocal mappings of linear PDEs with variable coefficients to linear PDEs with constant coefficients. Here one starts from the observation that each solution set of the adjoint PDE system of a given linear PDE system is a set of conservation law multipliers of the given PDE system and correspondingly yields a nonlocally related linear potential system of the given PDE system. The aim is to find a particular solution set of the adjoint PDE system that yields an invertible mapping of its corresponding nonlocally related linear potential system to a constant coefficient linear system. In turn this yields a non-invertible (nonlocal) mapping of the given linear PDE with variable coefficients to a linear PDE with constant coefficients. As examples, we consider nonlocal transformations of Kolmogorov equations to the backward heat equation [Bluman & Shtelen (2004)]. There also exists related work on nonlocal transformations of Schrödinger equations to the free particle equation [Bluman & Shtelen (1996a)].

4.3.1 Non-invertible mappings of nonlinear PDE systems to linear PDE systems

Suppose a given nonlinear PDE system does not have local (point or contact) symmetries (or, equivalently, does not have local conservation law multipliers) that yield an invertible mapping to a linear PDE system. In particular, this means that its local symmetries do not satisfy the criteria of Theorems 2.4.1, 2.4.2 (or, equivalently, that its local conservation law multipliers do not satisfy the criteria of Theorems 2.6.1, 2.6.2) so that there does not exist an invertible mapping of the nonlinear PDE system to any linear PDE system. However, it could happen that a nonlocally related system has an infinite set of local symmetries (an infinite set of local conservation law multipliers) that yields an invertible mapping of the nonlocally related system to some linear PDE system. Consequently, through the invertible mapping of the nonlocally related system to a linear system, one obtains a nonlocal (non-invertible) mapping of the given nonlinear PDE system to a linear PDE system. Of course, the local symmetries (local conservation law multipliers) yielding such a linearization of the nonlocally related system, must have an essential dependence on nonlocal variables.

For illustration, the following examples are considered.

(1) Linearization of Burgers' equation
As a first example, consider Burgers' equation

$$u_t + uu_x - u_{xx} = 0. \tag{4.36}$$

One can show that equation (4.36) has at most a finite number of contact symmetries. Hence there exists no point or contact transformation that linearizes Burgers' equation. As written (for convenience, after multiplication by the factor 2), the PDE (4.36) can be expressed as the conservation law $\mathrm{D}_t(2u)+\mathrm{D}_x(u^2-2u_x) = 0$. Correspondingly, one obtains the potential system

$$\begin{aligned} v_x &= 2u, \\ v_t &= 2u_x - u^2. \end{aligned} \tag{4.37}$$

The potential system (4.37) has an infinite number of point symmetries given by the infinitesimal generator

$$\mathrm{X} = e^{v/4}\left\{[2h(x,t) + g(x,t)u]\frac{\partial}{\partial u} + 4g(x,t)\frac{\partial}{\partial v}\right\}, \tag{4.38}$$

where $(g(x,t), h(x,t))$ is an arbitrary solution of the linear PDE system

$$\begin{aligned} h &= g_x, \\ h_x &= g_t \end{aligned} \tag{4.39}$$

[Vinogradov & Krasil'shchik (1984); Kersten (1987)]. Consequently, one can apply Theorems 2.4.1, 2.4.2 to obtain the well-known nonlocal Hopf–Cole transformation that linearizes Burgers' equation (4.36) [Exercise 2.4.4].

Note that from the form of the infinitesimal generator (4.38), one can immediately see that the locally related subsystem of (4.37), known as the integrated form of Burgers' equation, given by

$$v_t = v_{xx} - \tfrac{1}{4}(v_x)^2, \tag{4.40}$$

has the linearizing point symmetries

$$\mathrm{X} = e^{v/4} g(x,t) \frac{\partial}{\partial v},$$

where $g(x,t)$ is any solution of the linear heat equation

$$g_t - g_{xx} = 0.$$

(2) Linearization of a nonlinear heat conduction equation
The nonlinear heat conduction equation

$$u_t - (u^{-2} u_x)_x = 0, \tag{4.41}$$

which arises directly as a conservation law, does not have linearizing contact symmetries. However, one can show that the corresponding potential system given by

$$\begin{aligned} v_t &= u^{-2} u_x, \\ v_x &= u, \end{aligned} \tag{4.42}$$

has the infinite set of linearizing point symmetries

$$\mathrm{X} = g(t,v) \frac{\partial}{\partial x} - h(t,v) u^2 \frac{\partial}{\partial u}, \tag{4.43}$$

where $(g(t,v), h(t,v))$ is an arbitrary solution of the linear system

$$\begin{aligned} h &= g_v, \\ h_v &= g_t \end{aligned} \tag{4.44}$$

[Bluman, Kumei, & Reid (1988)]. See Exercise 2.4.3 for the corresponding transformation to a linear system.

Again, note that from the form of the infinitesimal generator (4.43), it follows that the locally related subsystem of (4.42), given by

$$v_t = (v_x)^{-2} v_{xx}, \tag{4.45}$$

has the infinite set of linearizing point symmetries

$$\mathrm{X} = g(t,v)\frac{\partial}{\partial x},$$

where $g(t,v)$ is an arbitrary solution of the linear heat equation

$$g_t = g_{vv}.$$

See Exercise 2.4.3 for the corresponding linearizing transformation.

(3) Linearization of the Thomas equations
As a third example, consider the nonlinear system of Thomas equations given by

$$\begin{aligned} v_t - u_x &= 0,\\ v_t - uv - u - v &= 0, \end{aligned} \tag{4.46}$$

that describes a fluid flow through a reacting medium [Thomas (1944); see also Whitham (1974)] and also can be related to the equations for two-wave interaction [Hasegawa (1974); Hashimoto (1974); Yoshikawa & Yamaguti (1974)]. Since the nonlinear PDE system (4.46) does not have an infinite number of point symmetries, it cannot be linearized by a point transformation. The first equation of (4.46) is written as a conservation law, which in turn leads directly to the corresponding potential system given by

$$\begin{aligned} w_x &= v,\\ w_t &= u,\\ v_t - uv - u - v &= 0. \end{aligned} \tag{4.47}$$

One can show [Bluman & Kumei (1990b)] that the potential system (4.47) has the infinite set of point symmetries

$$\begin{aligned} \mathrm{X} = e^w \Big\{ &[F(x,t)u + H(x,t)]\frac{\partial}{\partial u} + [F(x,t)v + G(x,t)]\frac{\partial}{\partial v}\\ &+F(x,t)\frac{\partial}{\partial w}\Big\}, \end{aligned} \tag{4.48}$$

where $(F(x,t), G(x,t), H(x,t))$ is an arbitrary solution of the linear PDE system

$$\begin{aligned} F_x &= G,\\ F_t &= H,\\ G_t &= G + H. \end{aligned} \tag{4.49}$$

Applying Theorems 2.4.1 and 2.4.2, one obtains the point transformation

$$\begin{aligned} z^1 &= x, \\ z^2 &= t, \\ w^1 &= e^{-w}, \\ w^2 &= e^{-w}v, \\ w^3 &= e^{-v}u, \end{aligned} \tag{4.50}$$

that invertibly maps the nonlinear system (4.47) to the linear system given by

$$\begin{aligned} \frac{\partial w^1}{\partial z^1} &= w^2, \\ \frac{\partial w^1}{\partial z^2} &= w^3, \\ \frac{\partial w^2}{\partial z^2} &= w^2 + w^3. \end{aligned} \tag{4.51}$$

Consequently, any solution $(w^1(z^1, z^2), w^2(z^1, z^2), w^3(z^1, z^2))$ of the linear system (4.51) yields the solution

$$(u(x,t), v(x,t)) = -\left(\frac{w^3(x,t)}{w^1(x,t)}, \frac{w^2(x,t)}{w^1(x,t)}\right),$$

of the Thomas equations (4.47).

Note that from the form of the infinitesimal generator (4.48), it follows that the locally related subsystem of (4.47), given by

$$w_{xt} - w_t w_x - w_t - w_x = 0, \tag{4.52}$$

has the linearizing infinite set of point symmetries

$$\mathrm{X} = F(x,t)e^{w}\frac{\partial}{\partial w} \tag{4.53}$$

where $F(x,t)$ is any solution of the linear PDE

$$F_{xt} - F_t - F_x = 0.$$

In particular, one obtains the point transformation $W = e^{-w}$ that maps the nonlinear PDE (4.52) to the linear PDE $W_{xt} - W_t - W_x = 0$.

(4) Linearization of a nonlinear reaction-diffusion equation
Consider the nonlinear reaction-diffusion equation given by

$$u_t - u^2 u_{xx} - 2u^2 = 0. \tag{4.54}$$

One can show that the PDE (4.54) has no linearizing set of contact symmetries and hence cannot be linearized by an invertible transformation. Multiplying the PDE (4.54) by u^{-2} yields the conservation law

$$\mathrm{D}_t(u^{-1}) + \mathrm{D}_x(u_x + 2x) = 0,$$

and the corresponding potential system ($u \neq 0$)

$$\begin{aligned} v_x &= u^{-1}, \\ v_t &= -(u_x + 2x) = -(u + x^2)_x. \end{aligned} \tag{4.55}$$

The nonlinear PDE system (4.55) also has no linearizing set of point symmetries. However, since the second PDE in (4.55) is written as a conservation law, one can accordingly introduce a second potential variable w to obtain the nonlocally related potential system

$$\begin{aligned} v_x &= u^{-1}, \\ w_x &= v, \\ w_t &= -(u + x^2). \end{aligned} \tag{4.56}$$

On can show [Exercise 2.4.8] that the potential system (4.56) has an infinite number of linearizing point symmetries given by the infinitesimal generator

$$\begin{aligned} \mathrm{X} = e^{(w-xv)}\Big\{&(F(t,v) - xH(t,v))\frac{\partial}{\partial x} \\ &+(G(t,v) - 2xF(t,v) + (x^2 - u)H(t,v))\frac{\partial}{\partial u} \\ &+(vF(t,v) - (1+xv)H(t,v))\frac{\partial}{\partial w}\Big\}, \end{aligned} \tag{4.57}$$

where $(F(t,v), G(t,v), H(t,v))$ is an arbitrary solution of the linear system

$$\frac{\partial H(t,v)}{\partial v} = F(t,v), \quad \frac{\partial H(t,v)}{\partial t} = G(t,v), \quad \frac{\partial F(t,v)}{\partial v} = G(t,v). \tag{4.58}$$

Consequently, one can show that the application of Theorems 2.4.1, 2.4.2 to the point symmetries (4.57) yields the point transformation

$$\begin{aligned} z^1 &= t, \\ z^2 &= v, \\ w^1 &= xe^{(xv-w)}, \\ w^2 &= (x^2 + u)e^{(xv-w)}, \\ w^3 &= e^{(xv-w)} - 1, \end{aligned}$$

that invertibly maps the nonlinear PDE system (4.56) to the linear system

$$\frac{\partial w^1}{\partial z^2} = w^2, \quad \frac{\partial w^3}{\partial z^2} = w^1, \quad \frac{\partial w^3}{\partial z^1} = w^2.$$

Correspondingly, one can show that any solution $(w^1, w^2, w^3) \neq (0, 0, -1)$ of this linear system yields the solution

$$u = \frac{w^2(w^3+1) - (w^1)^2}{(w^3+1)^2}$$

of the nonlinear reaction-diffusion equation (4.54).

(5) Linearization of a nonlinear telegraph equation

As a final example, consider the nonlinear telegraph equation [Varley & Seymour (1985)]

$$\phi_{tt} = (\phi_t)^2 \phi_{xx} + \phi_t(1 - \phi_t). \tag{4.59}$$

One can show that PDE (4.59) does not have contact symmetries yielding its linearization by an invertible point or contact transformation.

Let $u = \phi_t$, $v = \phi_x$. Then the corresponding PDE system

$$\begin{aligned} u &= \phi_t, \\ v &= \phi_x, \\ u_t &= u^2 v_x + u(1-u), \end{aligned} \tag{4.60}$$

is equivalent to and locally related to the scalar PDE (4.59), and hence (4.60) is also not linearizable by an invertible transformation.

Clearly, the nonlinear PDE system (4.60) has a nonlocally related subsystem given by

$$\begin{aligned} u_x &= v_t, \\ u_t &= u^2 v_x + u(1-u). \end{aligned} \tag{4.61}$$

As shown in Section 2.4.1, the nonlinear telegraph system (4.61) has an infinite set of point symmetries yielding its linearization by the point transformation (2.92) to the linear PDE system given by (2.93). In turn, this yields the linearization of the nonlinear telegraph equation (4.59) by a non-invertible (nonlocal) transformation.

Of course, one could consider the nonlinear PDE system (4.61) as the given PDE system with the nonlocally related potential system (4.60) arising from its first equation written as a conservation law. In turn, the scalar equation (4.59) is a locally related subsystem of the potential system (4.60).

4.3.2 Non-invertible mappings of linear PDEs with variable coefficients to linear PDEs with constant coefficients

In Section 2.5, there was consideration of the problem of determining whether a given linear PDE with variable coefficients can be mapped invertibly to a linear PDE with constant coefficients. The basis of the presented algorithm was the observation that a linear PDE with constant coefficients is completely characterized by its point symmetries connected with its linearity and invariance under the abelian group of translations of its independent variables. This led to a definitive answer to the posed problem and also to the construction of such an invertible mapping when one exists. Parabolic and hyperbolic equations were considered as specific examples.

Now suppose a given linear PDE with variable coefficients cannot be mapped invertibly to a linear PDE with constant coefficients. Using the linear parabolic PDE as a canonical example, it is shown how to construct non-invertible mappings to extend the class of linear PDEs with variable coefficients that can be mapped to linear PDEs with constant coefficients. This is accomplished through consideration of an appropriate potential system. In particular, for any given linear PDE, *any* solution of its adjoint equation is a multiplier for a conservation law that yields an equivalent nonlocally related potential system. The aim is to find such a multiplier so that the corresponding potential system can be mapped invertibly into a linear PDE system with constant coefficients. As a consequence, the given linear PDE could be mapped, non-invertibly, into an equivalent constant coefficient linear PDE. When the given PDE is a linear parabolic equation (without loss of generality, PDE (2.176)), then the constant coefficient PDE can be taken to be the backward heat equation.

The explicit relationship between the solutions of any given linear PDE system and its local conservation law multipliers (which satisfy the adjoint system of the given system) is exhibited by equations (2.219) and (2.220) in Section 2.6.

Now suppose the given PDE is the linear parabolic PDE in the standard form (see Section 2.5.1 and the discussion following equation (2.176)) given by

$$\mathrm{L}u = u_{xx} + u_y + V(x,y)u = 0. \tag{4.62}$$

The results presented in Section 2.5.1 can be summarized in terms of the following theorem [Bluman & Shtelen (2004)] which can be proven by direct calculation.

Theorem 4.3.1. *A linear parabolic PDE* (4.62) *can be mapped invertibly by a point transformation to the backward heat equation*

$$w_{z^1z^1} + w_{z^2} = 0 \tag{4.63}$$

if and only if $V(x,y)$ *is of the form*

$$V(x,y) = a(y)x^2 + b(y)x + c(y) \tag{4.64}$$

for some functions $a(y), b(y), c(y)$. *The point transformation that yields the mapping is given by*

$$\begin{aligned} z^1 &= \sigma(y)x + \rho(y), \\ z^2 &= \int^y \sigma^2(\hat{y})d\hat{y}, \\ w &= u \exp \tfrac{1}{4}[\sigma^{-1}\sigma'(y)x^2 + 2\sigma^{-1}\rho'(y)x + \lambda(y)], \end{aligned} \tag{4.65}$$

where $(\sigma(y), \rho(y), \lambda(y))$ *is a solution of the nonlinear system of ODEs*

$$\begin{aligned} \sigma^{-2}(\sigma\sigma'' - 2\sigma'^2) &= 4a(y), \\ (\sigma\rho'' - 2\sigma'\rho') &= 2\sigma^2 b(y), \\ \lambda' &= \sigma^{-2}(\rho'^2 - 2\sigma\sigma') + c(y). \end{aligned} \tag{4.66}$$

The solution of ODE system (4.66) appears in Bluman & Shtelen (2004).

Now the result of Theorem 4.3.1 is extended to include nonlocal (non-invertible) transformations of linear parabolic equations of the form (4.62) to the backward heat equation (4.63), i.e., through nonlocal transformations arising from related potential systems, one can widen the class of functions $V(x,y)$ for which a linear PDE (4.62) can be mapped into the backward heat equation (4.63). The work presented here appears in Bluman & Shtelen (2004).

A multiplier $\phi(x,y)$ that yields a local conservation law of the linear parabolic PDE (4.62) is any solution $\phi(x,y)$ of its adjoint PDE

$$\mathrm{L}^*\phi = \phi_{xx} - \phi_y + V(x,y)\phi = 0. \tag{4.67}$$

In particular, for arbitrary functions $(U(x,y), \Phi(x,y))$, one has the relationship

$$\begin{aligned} &\Phi \mathrm{L}U - U\mathrm{L}^*\Phi \\ &\quad = \Phi[U_{xx} + U_y + V(x,y)U] - U[\Phi_{xx} - \Phi_y + V(x,y)\Phi] \\ &\quad = \mathrm{D}_x(\Phi U_x - \Phi_x U) + \mathrm{D}_y(\Phi U). \end{aligned} \tag{4.68}$$

Consequently, for any solution $\phi(x,y)$ of the adjoint equation (4.67), the given linear parabolic scalar PDE (4.62) is nonlocally equivalent to the corresponding linear potential system

$$\begin{aligned} v_x &= \phi u, \\ v_y &= \phi_x u - \phi u_x. \end{aligned} \tag{4.69}$$

By direct calculation, one can prove the following extended theorem.

Theorem 4.3.2. *Let $\psi(x,y)$ be any solution of the linear PDE*

$$\psi_{xx} + \psi_y + [a(y)x^2 + b(y)x + c(y)]\psi = 0, \tag{4.70}$$

for some specific coefficients $a(y), b(y), c(y)$. Let $\phi(x,y) = \psi^{-1}$. For the same coefficients $a(y), b(y), c(y)$, consider the linear parabolic PDE (4.62) *with*

$$V(x,y) = -2\frac{\partial^2}{\partial x^2}\log|\phi(x,y)| + a(y)x^2 + b(y)x + c(y). \tag{4.71}$$

The corresponding potential system (4.69) *can be mapped invertibly by a point transformation to the backward heat potential system*

$$\begin{aligned} \frac{\partial w^2}{\partial z^1} &= w^1, \\ \frac{\partial w^2}{\partial z^2} &= -\frac{\partial w^1}{\partial z_1}, \end{aligned} \tag{4.72}$$

for which each component satisfies the backward heat equation, i.e., $w^i_{z^1z^1} + w^i_{z^2} = 0,\ i = 1,\ 2$. In particular, such a mapping is given by

$$\begin{aligned} z^1 &= \sigma(y)x + \rho(y), \\ z^2 &= \int^y \sigma^2(\hat{y})d\hat{y}, \\ w^1 &= \sigma^{-1}e^{g(x,y)}\left\{u + \left(\tfrac{1}{2}\sigma^{-1}(\sigma'(y)x + \rho'(y)) - \psi^{-1}\psi_x\right)\psi v\right\}, \\ w^2 &= e^{g(x,y)}\psi v, \end{aligned} \tag{4.73}$$

where $(\sigma(y), \rho(y), \lambda(y))$ is a solution of the corresponding nonlinear ODE system (4.66) *and*

$$g(x,y) = \tfrac{1}{4}[\sigma^{-1}\sigma'(y)x^2 + 2\sigma^{-1}\rho'(y)x + \lambda(y)].$$

The mapping (4.73) defines a point transformation acting on (x,t,u,v)-space that projects onto a nonlocal transformation acting on (x,t,u)-space if the coefficient of v is nonzero in the third equation of the mapping.

It is easy to see that the mapping (4.73) yields a nonlocal transformation of the linear PDE (4.62) to the backward heat equation if and only if $V(x,y)$ is of the form (4.71), $V(x,y)$ is not quadratic in x, and $\phi(x,y)$ satisfies the condition

$$\frac{\partial^5}{\partial x^5}\log|\phi(x,y)| \not\equiv 0.$$

Let $\hat{\psi}(z^1,z^2)$ be *any* solution of the backward heat equation $\hat{\psi}_{z^1z^1}+\hat{\psi}_{z^2}=0$. Then from the mapping equations (4.65) it follows that

$$\psi(x,y)=\hat{\psi}(z^1,z^2)\exp\{-\tfrac{1}{4}[\sigma^{-1}\sigma'(y)x^2+2\sigma^{-1}\rho'(y)x+\lambda(y)]\}$$

is a solution of the linear parabolic PDE (4.70), and accordingly, $V(x,y)$ given by the equation (4.71) becomes

$$V(x,y)=a(y)x^2+b(y)x+c(y)-2\sigma^2\left[\frac{\hat{\psi}_{z^2}}{\hat{\psi}}+\left(\frac{\hat{\psi}_{z^1}}{\hat{\psi}}\right)^2\right]-\frac{\sigma'(y)}{\sigma(y)}, \quad (4.74)$$

where $z^1=\sigma(y)x+\rho(y)$, $z^2=\int^y\sigma^2(\hat{y})d\hat{y}$, with $\sigma(y),\rho(y)$ related to $a(y),b(y)$ through the first two ODEs of the system (4.66). Hence *every* solution of the backward heat equation yields a coefficient $V(x,y)$ given by (4.74) for which the corresponding linear parabolic PDE (4.62) can be mapped to the backward heat equation. Moreover, one can prove the following theorem [Exercise 4.3.3].

Theorem 4.3.3. *Let* $w=\hat{\psi}(z^1,z^2)$ *be a solution of the backward heat equation* $w_{z^1z^1}+w_{z^2}=0$. *Such a solution yields a coefficient* $V(x,y)$ *given by* (4.74). *The corresponding linear parabolic PDE* (4.62) *can be mapped to the backward heat equation* only *through a nonlocal transformation if and only if* $\hat{\psi}(z^1,z^2)$ *is* not *one of the forms*

(I) $\hat{\psi}(z^1,z^2)=e^{(Pz^1-P^2z^2)}$,

(II) $\hat{\psi}(z^1,z^2)=\dfrac{1}{\sqrt{(z^2-\hat{z}^2)}}\exp\left\{\dfrac{(z^1-\hat{z}^1)^2}{4(z^2-\hat{z}^2)}\right\}$,

where $P,\hat{z}^1,\hat{z}^2$ *are arbitrary constants.*

In Bluman & Shtelen (2004), a recycling procedure [See also Bluman & Reid (1989).] is described that can further extend the class of linear parabolic equations that can be mapped into the heat equation by explicit nonlocal transformations. Interesting special cases include d-Bessel processes of the form

$$\frac{\partial u}{\partial t}+\frac{\partial^2 u}{\partial R^2}+\frac{(d-1)}{R}\frac{\partial u}{\partial R}=0, \quad d=2k+1, \quad k=1,2,\dots\,.$$

For related work on classes of Schrödinger equations that can be mapped into the free particle equation by nonlocal transformations, see Bluman & Shtelen (1996a).

Exercises 4.3

4.3.1. Consider the potential system $\mathbf{UW}\{x, t\,; u, w\}$ (3.82) of the nonlinear wave equation (4.8) in the case $c(u) = (u + B)^{-2}$.

(a) Find an infinite set of point symmetries of the potential system $\mathbf{UW}\{x, t\,; u, w\}$ (3.82).
(b) Find a point transformation that maps $\mathbf{UW}\{x, t\,; u, w\}$ into a linear PDE system.

4.3.2. Show that the potential system $\mathbf{UB}\{x, t\,; u, \beta\}$ (3.84) of the nonlinear wave equation $u_{tt} = (u^{-4/3}u_x)_x$ has an infinite number of point symmetries. Find the explicit form of a linearizing transformation.

4.3.3. Prove Theorem 4.3.3.

4.4 Discussion

Pucci & Saccomandi (1993) give some necessary conditions for the existence of potential symmetries that arise from the potential system for a given scalar PDE written as a conservation law.

For diffusion-convection equations of the form

$$u_t - [f(u)u_x + k(u)]_x = 0,$$

Sophocleous (1996) classifies all functions $f(u)$ and $k(u)$ for which there exist potential symmetries through analyzing the potential system that arises from the equation as written. He also finds the corresponding potential symmetries.

Chou & Qu (1999) consider the potential system and potential equation, respectively given by

$$\begin{aligned} v_x &= u, \\ v_t &= D(u)(u_x)^n + E(u) \end{aligned} \tag{4.75}$$

and

$$v_t = D(v_x)(v_{xx})^n + E(v_x) \tag{4.76}$$

for the class of diffusion-convection equations of the form

$$u_t - [D(u)(u_x)^n + E(u)]_x = 0. \tag{4.77}$$

They classify the cases when the potential system (4.75) yields a potential symmetry of (4.77) and classify the point symmetries of the potential equation (4.76). It is not noted in this paper that (1) each point symmetry of (4.76) yields a local symmetry of (4.75); and (2) each potential symmetry of

(4.77) that results from a point symmetry of (4.75), must yield a local symmetry (not necessarily a point symmetry) of (4.76). [It is easy to see that the potential system (4.75) and the potential equation (4.76) are locally related.]

Sophocleous (2005) finds potential symmetries of the class of nonlinear diffusion equations with variable coefficients of the form

$$u_t = [g(x)u^n u_x]_x \tag{4.78}$$

by considering the potential system that arises from the equation as written. In particular, he shows that such potential symmetries arise in two cases: (i) $n = -2,\ g(x) = x^2$; (ii) $n = -2,\ g(x) = x^{-2}$. For the first case, he obtains potential symmetries that yield the linearization of PDE (4.78) and also exhibits invariant solutions of (4.78), arising from potential symmetries.

Ivanova, Popovych & Sophocleous (2008a,b) classify potential systems and resulting nonlocal conservation laws and potential symmetries for variable coefficient diffusion-convection equations of the form

$$f(x)u_t - [g(x)f(u)u_x]_x - H(x)G(u)u_x = 0.$$

Ivanova & Sophocleous (2008) classify potential systems and find resulting potential symmetries of systems of diffusion equations of the form

$$\begin{aligned} u_t &= [f(u,v)u_x]_x, \\ v_t &= [g(u,v)u_x]_x. \end{aligned}$$

Senthilvelan & Torrisi (2000) find potential symmetries and resulting invariant solutions for a nonlinear PDE system representing a simplified model for reacting mixtures. Potential symmetries are exhibited that yield the linearization of the given PDE system by a nonlocal transformation.

Bluman, Cheviakov, & Ganghoffer (2008) consider the complete set of equations of nonlinear elasticity in a dynamical context. A tree of nonlocally related systems is constructed that includes both the Lagrange and Euler PDE systems. As a consequence, nonlocal symmetries are found for both systems. Invariant solutions are constructed from such a nonlocal symmetry of the Euler system.

Formally, nonlocal symmetries have been found for PDEs through infinitesimals depending on nonlocal variables that are integrals of the given dependent variables of a given PDE system [Konopelchenko & Mokhnachev [(1979), (1980)]; Kumei (1981); Kapcov (1982); Pukhnachev (1987)]. In these works nonlocal symmetries are not realized as local symmetries of potential systems.

A particular way of obtaining nonlocal symmetries of PDEs is to seek recursion operators, depending on inverse differentiation (integral) operators, that generate sequences of nonlocal symmetries from local symmetries. For further details, see Kapcov (1982), Bluman & Kumei (1989), and Guthrie (1994).

In Krasil'shchik & Vinogradov (1984) [see also Vinogradov & Krasil'shchik (1984); Kersten (1987); Vinogradov (1989); Krasil'shchik & Kersten (2000)], nonlocal symmetries are defined as local symmetries of an associated auxiliary PDE system whose integrability conditions yield the given PDE system. A rather general form is assumed for the auxiliary system which involves unspecified functions. In principle, these unspecified functions are determined by requiring that the integrability conditions of the auxiliary PDE system yield the given PDE system. In order to apply their method (related to an idea introduced by Wahlquist and Estabrook (1975)) it seems that one has to impose very strong assumptions on the form of the unspecified functions.

In the final chapter, the complexity in finding nonlocal symmetries and nonlocal conservation laws of a given PDE system in the case of three or more independent variables is considered. It is seen that in order that such nonlocal symmetries and/or nonlocal conservation laws can arise from local symmetries and/or local conservation laws, respectively, of a potential system, it is necessary to append the potential system with gauge constraints that relate the potential variables. On the other hand, it is shown that local symmetries of nonlocally related systems arising as subsystems of a given PDE system can yield nonlocal symmetries of the given PDE system as in the situation for two independent variables. Moreover, unlike potential systems arising from divergence-type conservation laws, potential systems arising from *lower-degree* (e.g., curl-type) conservation laws may require fewer or no gauge constraints in order to yield nonlocal symmetries and/or nonlocal conservation laws.