Chapter 7
Other Applications

7.1 When Are $an + b$ and $cn + d$ Simultaneously Perfect Squares?

In [122] and [123] it is proven that there are infinitely many positive integers n such that $2n + 1$ and $3n + 1$ are both perfect squares. The proof relies on the theory of general Pell's equations.

In what follows we will present an extension of this result, based on our papers [13] and [14]. The main result is also cited in [41, Problem 1.13]. Recall that in Theorem 4.5.2 we proved that if a, c are relatively prime positive integers, not both perfect squares, and if b, d are integers, then the Diophantine equation

$$ax^2 - cy^2 = ad - bc$$

is solvable if and only if $an + b$ and $cn + d$ are simultaneously perfect squares for some positive integer n. In this case, the number of such n's is infinite. If $(x_m, y_m)_{m\geq 0}$ are solutions to $ax^2 - by^2 = ad - bc$ (see Theorem 4.5.1), then for all n_mm $m \geq 0$, where

$$n_m = \frac{y_m^2 - b}{a} = \frac{x_m^2 - d}{c},$$

$an_m + b$ and $cn_m + d$ are simultaneously perfect squares (see Theorem 4.5.2).

From the previous formulas, we see that the least positive n_0 for which $an_0 + b$ and $cn_0 + d$ are simultaneously perfect squares is

$$n_0 = \frac{x_0^2 - d}{c} = \frac{y_0^2 - b}{a},$$

where (x_0, y_0) is the minimal solution to the equation $ax^2 - cy^2 = ad - bc$.

The main result in this section is the following:

T. Andreescu, D. Andrica, *Quadratic Diophantine Equations*,
Developments in Mathematics 40, DOI 10.1007/978-0-387-54109-9_7

Theorem 7.1.1. *Let a, c be relatively prime positive integers, not simultaneously perfect squares and let b, d be integers. Each of the following conditions is sufficient for the numbers $an + b$ and $cn + d$ to be both perfect squares for infinitely many positive integers n:*

1) b and d are perfect squares;
2) $a + b$ and $c + d$ are perfect squares;
3) $\frac{a}{c} = \frac{b-1}{d-1}$.

Proof. Conditions 1) and 2) state that $an + b$ and $cn + d$ are simultaneously perfect squares for $n = 0$ and $n = 1$, respectively. From Theorem 4.5.2 it follows that they have this property for infinitely many positive integers n.

Condition 3) is equivalent to $a - c = ad - bc$, in which case the equation $ax^2 - by^2 = ad - bc$ has solution $(1, 1)$ and the conclusion follows from the same Theorem 4.5.2. □

Applications

1) The numbers $2n + 3$ and $5n + 6$ are both perfect squares for infinitely many positive integers n. Indeed, the equation $2x^2 - 5y^2 = -3$ has solution $(1, 1)$ and the result follows from Theorem 4.5.2.
2) If k is an arbitrary positive integer different from 3, then n and $(k^2 - 4)n - 1$ cannot be simultaneously perfect squares. Indeed, in Section 3.6 we saw that the negative Pell's equation $x^2 - (k^2 - 4)y^2 = -1$ is not solvable (see also [199]) and the conclusion follows from Theorem 4.5.2.
3) If p and q are relatively prime positive integers and pq is not a perfect square, then $pn + 1$ and $qn + 1$ are simultaneously perfect squares for infinitely many positive integers n. This property follows from Theorem 7.1.1.1). For $p = 2$ and $q = 3$ we obtain the result in [122]. For $p = 3$ and $q = 4$ we obtain Problem 8 in [25, p. 83].

   If $p = 1$ and $q = 3$ we obtain the first part of the result in [40]. The second part shows that if $n_1 < n_2 < \cdots < n_k < \dots$ are all positive integers satisfying the above property, then $n_k n_{k+1} + 1$ is also a perfect square, $k = 1, 2, \dots$. Indeed, the equation is $3x^2 - y^2 = 2$, which is equivalent to the Pell's equation $u^2 - 3v^2 = 1$, where $u = \frac{1}{2}(3x - y)$ and $v = \frac{1}{2}(y - x)$. The general solution is $(u_k, v_k)_{k \geq 1}$, where $u_k + v_k\sqrt{3} = (2 + \sqrt{3})^k$, $k \geq 1$, hence

$$n_k = x_k^2 - 1 = (u_k + v_k)^2 - 1 = \frac{1}{6}[(2 + \sqrt{3})^{2k+1} + (2 - \sqrt{3})^{2k+1} - 4].$$

   We have

$$n_k n_{k+1} + 1 = \left\{\frac{1}{6}[(2 + \sqrt{3})^{2k+2} + (2 - \sqrt{3})^{2k+2} - 8]\right\}^2, \quad k \geq 1.$$

4) For any nonzero integers k and l, the numbers $(k^2+1)n+2l$ and $2kn+l^2+1$ are simultaneously perfect squares for infinitely many positive integers n. This follows from Theorem 7.1.1.2).
5) The following application appeared in [11] (see also [25, p. 82]). The smallest positive integer m such that $19m+1$ and $95m+1$ are both perfect squares is 134232. Indeed, setting $19m=n$ we are looking for the smallest $n \equiv 0 \pmod{19}$ such that $n+1$ and $5n+1$ are simultaneously perfect square. In this case, the equation is $x^2-5y^2=-4$, whose general solution is given by

$$\frac{1}{2}(x_k+y_k\sqrt{5})=\left(\frac{1+\sqrt{5}}{2}\right)^k, \quad k=1,3,5,\ldots$$

(see also Section 4.3.2). It follows that $y_k=F_{2k-1}$, $k=1,2,\ldots$, and $n_k=F_{2k-1}^2-1$. The smallest k for which $n_k \equiv 0 \pmod{19}$ is $k=9$, hence the desired integers is $m=\frac{1}{19}n_9=134232$.

7.2 Triangular Numbers

Let $T_n=\frac{n(n+1)}{2}$ denote the n^{th} *triangular number*. In this section we will present several situations when some properties related to these numbers reduce to solving Pell-type equations.

7.2.1 Triangular Numbers with Special Properties

There are infinitely many positive integers n for which T_n is a perfect square. Indeed, if T_n is a perfect square, then so is $T_{4n(n+1)}$, because $\frac{n(n+1)}{2}=k^2$ implies

$$T_{4n(n+1)}=T_{8k^2}=4k^2(8k^2+1)=4k^2(4n^2+4n+1)=[2k(2n+1)]^2.$$

Taking into account that $T_1=1^2$, by the above procedure, we generate a sequence of perfect square triangular numbers. A formula for such integers n has already been given in (5.4.8).

It is natural to ask what are all triangular numbers that are perfect squares. We have [15]:

Theorem 7.2.1. *The triangular number T_x is a perfect square if and only if*

$$x = \begin{cases} 2P_m^2, & m \text{ even} \\ \left[\dfrac{(1+\sqrt{2})^m + (1-\sqrt{2})^m}{2}\right]^2, & m \text{ odd} \end{cases} \tag{7.2.1}$$

where $(P_m)_{m\geq 0}$ is the Pell's sequence.

Proof. The equation $T_x = y^2$ is equivalent to $(2x+1)^2 - 8y^2 = 1$. The Pell's equation $u^2 - 8v^2 = 1$ has solutions

$$u_m = \frac{1}{2}[(1+\sqrt{2})^{2m} + (1-\sqrt{2})^{2m}]$$

and

$$v_m = \frac{1}{2\sqrt{2}}[(1+\sqrt{2})^{2m} - (1-\sqrt{2})^{2m}] = P_{2m}$$

hence the conclusion. □

Remarks. 1) Every other x satisfying $T_x = y^2$ is a perfect square.
2) Every y satisfying $T_x = y^2$ is a Pell number.
3) The equation $T_x = (T_y)^2$ is more difficult. It has only solutions $(1,1)$ and $(8,3)$. A complicated proof was given by W. Ljunggren (see [150] for details).
4) Some extensions of the result in Theorem 7.2.1 are given in the paper [223].

Theorem 7.2.2. *If k is a positive integer that is not a perfect square, then the equation*

$$kT_x = T_y \tag{7.2.2}$$

has infinitely many solutions in positive integers.

Proof. Equation (7.2.2) is equivalent to $(2y+1)^2 - k(2x+1)^2 = 1-k$. Let (u_1, v_1) be the fundamental positive integral solution of Pell's equation $u^2 - kv^2 = 1$. If u_1 and v_1 are of opposite parity, we obtain infinitely many (but not necessarily all) positive integral solutions (x, y) by taking

$$2y+1+(2x+1)\sqrt{k} = (1+\sqrt{k})(u_1+v_1\sqrt{k})^j, \quad j = 1,2,3,\ldots$$

If u_1 and v_1 are both odd (which can occur only when $k \equiv 0 \pmod 8$), we set

$$u + v\sqrt{k} = (u_1 + v_1\sqrt{k})^2,$$

and since u_1 is odd and v_1 is even, we get infinitely many positive integral solutions x, y by taking

$$2y+1+(2x+1)\sqrt{k}=(1+\sqrt{k})(u+v\sqrt{k})^j, \quad j=1,2,3,\ldots$$

This completes the proof of the theorem. □

The equation (7.2.2) is also studied in [34].

It is interesting to see what is the asymptotic density of "composite" triangular numbers among all triangular numbers. In [214] it is shown that this density is zero. More specifically, if $F(n)$ denotes the number of triples a, b, c such that

$$T_aT_b = T_c, \quad 1 < a \le b < c \le n \tag{7.2.3}$$

we will show that

$$F(n) < 4n^{3/4}. \tag{7.2.4}$$

Denote $g(x)=A\sqrt{x^2-d^2}$ for $x \ge d$, where A and d are given positive numbers. Suppose h is a fixed positive number. Then

$$m_h(x)=\frac{1}{2x}[g(x+h)+g(x-h)]$$

is an increasing function of x for $x \ge d+h$.

Clearly,

$$m_h(x)=\frac{g(x+h)^2-g(x-h)^2}{2x\{g(x+h)-g(x-h)\}}=\frac{2A^2h}{g(x+h)-g(x-h)}.$$

Thus it suffices to show that $g(x+h)-g(x-h)$ is a decreasing function of x for $x \ge d+h$. But for $x > d+h$ we have

$$g'(x+h)-g'(x-h)=\frac{A(x+h)}{\sqrt{(x+h)^2-d^2}}-\frac{A(x-h)}{\sqrt{(x-h)^2-d^2}}<0,$$

since the derivative of $x(x^2-d^2)^{-1/2}$ is $-d^2(x^2-d^2)^{-3/2}$.

Because $F(n)=0$ for $n \le 7$ we may assume $n \ge 8$. For a given a with $1 < a < n$, let $s(a,n)$ denote the number of pairs (b,c) satisfying (7.2.3). If $b \ge a > 2^{1/4}n^{1/2}$, clearly $T_b \ge T_a > \frac{1}{2}\cdot 2^{1/4}n^{1/2}(2^{1/4}n^{1/2}+1)$ and, hence $T_aT_b > \frac{1}{2}(n+2^{3/4})$.

Thus $s(a,n)=0$ if $a > 2^{1/4}n^{1/2}$, and so

$$F(n)=\sum_{a=2}^{[2^{1/4}n^{1/2}]} s(a,n).$$

Suppose that a and n are fixed and $s(a,n) > 0$. Set $K = T_a$. Then the equation $T_aT_b = t_c$ is equivalent to $K(b^2+b) = c^2+c$ or

$$K\{(2b+1)^2-1\} = (2c+1)^2-1.$$

Set $u = 2b+1$, $v = 2c+1$. Because $v^2 = (2c+1)^2 \le (2n+1)^2$, we have

$$u^2 = \frac{v^2-1}{K}+1 \le \frac{4n^2+4n}{K}+1 = 8\frac{n(n+1)}{a(a+1)}+1 < 8\frac{n^2}{a^2}+1 < 9\frac{n^2}{a^2},$$

so

$$u < 3n/a. \tag{7.2.5}$$

On the other hand, $u = 2b+1 \ge 2a+1 = \sqrt{8K+1} > \sqrt{2K}$ and

$$v^2 = Ku^2 - K + 1 > K\{(2b+1)^2-1\} \ge K\{(2a+1)^2-1\} = 8K^2,$$

hence

$$0 < \sqrt{K}u - v = \frac{K-1}{\sqrt{K}u+v} < \frac{K}{2\sqrt{2}K+2\sqrt{2}K}$$

or

$$0 < \sqrt{K}u - v < 1/(4\sqrt{2}). \tag{7.2.6}$$

Now suppose (b_i, c_i), $i = 1,2,\ldots,s$ are the solutions to $KT_b = T_c$ with $a \le b_j < c_j \le n$ and $b_1 < b_2 < \cdots < b_s$, where $s = s(a,n)$. Set $u_i = 2b_i+1$ and $v_i = 2c_i+1$ for $i = 1,2,\ldots,s$. We claim that $u_{i+1}-u_i \ne u_{j+1}-u_j$ for $1 \le i < j \le s-1$.

Suppose to the contrary that $u_{i+1}-u_i = u_{j+1}-u_j$ for some pair (i,j) with $1 \le i < j \le s-1$. From (7.2.6) we have

$$-1/(4\sqrt{2}) < (\sqrt{K}u_{i+1}-v_{i+1}) - (\sqrt{K}u_i - v_i) < 1/(4\sqrt{2}),$$

so

$$\sqrt{K}(u_{i+1}-u_i) - (v_{i+1}-v_i) = \theta_i,$$

where $|\theta_i| < 1/(4\sqrt{2})$. Similarly,

$$\sqrt{K}(u_{j+1}-u_j) - (v_{j+1}-v_j) = \theta_j,$$

where $|\theta_j| < 1/(4\sqrt{2})$. Hence

$$v_{i+1}-v_i+\theta_i = \sqrt{K}(u_{i+1}-u_i) = \sqrt{K}(u_{j+1}-u_j) = v_{j+1}-v_j+\theta_j,$$

hence

$$[(v_{j+1}-v_j)-(v_{i+1}-v_i)] = |\theta_i-\theta_j| < 1/(2\sqrt{2}).$$

Because the left-hand side is an integer, we have

$$v_{i+1} - v_i = v_{j+1} - v_j. \qquad (7.2.7)$$

On the other hand (u_i, v_i), (u_{i+1}, v_{i+1}), (u_j, v_j), (u_{j+1}, v_{j+1}) are points with positive integral coordinates lying on the hyperbola $y^2 = Kx^2 - (K-1)$ and satisfying the conditions $u_{i+1} - u_i = u_{j+1} - u_j > 0$, $u_i < u_j$. Further,

$$\frac{v_{i+1} - v_i}{v_{j+1} - v_j} = \frac{K(u_{i+1}^2 - u_i^2)/(v_{i+1} + v_i)}{K(u_{j+1}^2 - u_j^2)/(v_{j+1} + v_j)} = \frac{(v_{j+1} + v_j)/(u_{j+1} + u_j)}{(v_{i+1} + v_i)/(u_{i+1} + u_i)}. \qquad (7.2.8)$$

Applying the monotonicity of function m_h with $2h = u_{i+1} - u_i = u_{j+1} - u_j$ and $g(x) = \sqrt{Kx^2 - (K-1)}$, we find that

$$(v_{j+1} + v_j)/(u_{j+1} + u_j) > (v_{i+1} + v_i)/(u_{i+1} + u_i),$$

and then (7.2.8) gives $v_{i+1} - v_j > v_{j+1} - v_j$. But this contradicts (7.2.7) and so our assumption that $u_{i+1} - u_i = u_{j+1} - u_j$ is untenable.

Thus we have shown that the gaps $u_2 - u_1, u_3 - u_2, \ldots, u_s - u_{s-1}$ are $s - 1$ different even positive integers. Hence,

$$\begin{aligned} u_s - u_1 &= (u_2 - u_1) + (u_3 - u_2) + \cdots + (u_s - u_{s-1}) \\ &\geq 2 + 4 + \cdots + 2(s-1) = s(s-1). \end{aligned}$$

Combining this with (7.2.5), we obtain

$$3n/a > u_s > u_s - u_1 \geq (s-1)^2,$$

so

$$s(a,n) < 1 + \sqrt{3n/a}.$$

Hence

$$\begin{aligned} F(n) = \sum_{a=2}^{[2^{1/4}n^{1/2}]} s(a,n) &< \sum_{a=2}^{[2^{1/4}n^{1/2}]} (1 + \sqrt{3n/a}) \\ &< 2^{1/4}n^{1/2} + \sqrt{3n}\int_1^{2^{1/4}n^{1/2}} t^{-1/2}dt \\ &< \sqrt{3n}\int_0^{2^{1/4}n^{1/2}} t^{-1/2}dt < 4n^{3/4}. \end{aligned}$$

Thus (7.2.4) is proved. □

The following result was proven in [170].

Theorem 7.2.3. *The equation*

$$T_m = T_n T_p \tag{7.2.9}$$

is solvable for infinitely many triples (m, n, p), $p \geq 2$, *and unsolvable for infinitely many triples* (m, n, p).

Proof. For the first part, we choose $p = 2$. The equation (7.2.9) becomes $T_m = 3T_n$. From Theorem 7.2.2 it follows that the last equation has infinitely many solutions.

For the second part, let m be an odd prime number. The equation (7.2.9) is equivalent to

$$2m(m+1) = n(n+1)p(p+1).$$

Without loss of generality, we may assume that $m|n$ or $m|n+1$, i.e., $n = km$ or $n+1 = km$.

Since $p(p+1) \geq 6$, in the first case we obtain

$$2(m+1) = k(km+1)p(p+1) \geq 6(m+1),$$

a contradiction. In the second case, when $n = km - 1$, we have

$$2(m+1) = (km-1)kp(p+1) \geq 6(m-1)$$

which is a contradiction, as well. It follows that equation (7.2.9) is not solvable. □

7.2.2 Rational Numbers Representable as $\frac{T_m}{T_n}$

The following results have been proven in [170]. The proof of the first result is based on some results contained in our papers [13] and [14].

Theorem 7.2.4. *If r is a positive rational number and $\sqrt{r}$ is irrational, then there exist positive integers m, n such that*

$$r = \frac{T_m}{T_n}. \tag{7.2.10}$$

Proof. Let $r = \frac{p}{q}$, where p, q are relatively prime positive integers. Then (7.2.10) is equivalent to $\frac{m(m+1)}{n(n+1)} = \frac{p}{q}$, i.e., $\frac{(2m+1)^2-1}{(2n+1)^2-1} = \frac{p}{q}$. Letting $2m+1 = x$ and $2n+1 = y$ yields

$$qx^2 - py^2 = q - p. \tag{7.2.11}$$

The irrationality of $\sqrt{r}$ implies that pq is not a perfect square.

Since (7.2.11) is solvable (it has solution $x = y = 1$), from Theorem 4.5.1 it follows that it has infinitely many solutions.

If $(u_k, v_k)_{k\geq 0}$ is the general solution to the Pell's equation $u^2 - pqv^2 = 1$, then $u_k + v_k\sqrt{pq} = (u_0 + v_0\sqrt{pq})^k$, $k \geq 1$. It follows that $u_1 = u_0^2 + pqv_0^2$, $v_1 = 2u_0v_0$.

From Theorem 4.5.1, $(X_k, Y_k)_{k\geq 0}$, where $X_k = u_k + pv_k$, $Y_k = u_k + qv_k$, are solutions to the equation (7.2.11). Since $u_1^2 - pqv_1^2 = 1$ and v_1 is even, it follows that u_1 is odd. Hence X_1 and Y_1 are both odd and we can choose $m = \frac{1}{2}(X_1 - 1)$ and $n = \frac{1}{2}(Y_1 - 1)$. □

Theorem 7.2.5. *Among the positive rational numbers r for which $\sqrt{r}$ is rational, infinitely many are representable in the form (7.2.10) and infinitely many are not.*

Proof. Let p be an odd integer and let $r = (2p)^2$. Choosing $m = p^2 - 1$ and $n = \frac{p-1}{2}$, we obtain $r = \frac{T_m}{T_n}$.

If p is an odd prime, we will prove that $r = \left(\frac{p+1}{p-1}\right)^2$ is not of the form (7.2.10). Indeed $r = \frac{T_m}{T_n}$ would imply $\frac{m(m+1)}{n(n+1)} = \frac{(p+1)^2}{(p-1)^2}$. Setting $2m + 1 = x$ and $2n+1 = y$, we have $\frac{x^2-1}{y^2-1} = \frac{(p+1)^2}{(p-1)^2}$, $x, y \geq 3$. The last equality is equivalent to

$$\left(\frac{p-1}{2}x - \frac{p+1}{2}y\right)\left(\frac{p-1}{2}x + \frac{p+1}{2}y\right) = -p$$

and so

$$\frac{p-1}{2}x - \frac{p+1}{2}y = -1 \text{ and } \frac{p-1}{2}x + \frac{p+1}{2}y = p,$$

which yields $x = 1$ and $y = 1$, a contradiction. □

7.2.3 When Is $\frac{T_m}{T_n}$ a Perfect Square?

In this subsection we are interested in finding all pairs (m, n) for which the ratio of triangular numbers T_m and T_n is the square of an integer.

In [140] it is shown that pairs $(4n(n+1), n)$, $n \geq 1$, satisfy the above property. In the recent paper [101] all pairs (m, n) are determined by using a suitable Pell's equation.

The relation $\frac{T_m}{T_n} = q^2$ is equivalent to

$$\frac{(2m+1)^2-1}{(2n+1)^2-1} = q^2.$$

Using now the result and notation in Remark 5), subsection 5.6.2, we obtain $2m_k + 1 = x_k$, $q_k = z_k$, $k \geq 0$. It follows that $m_k = \frac{x_k - 1}{2}$, $q_k = z_k$, where

$$\begin{cases} x_k = \frac{1}{2}\left[\left(2n+1+2\sqrt{n(n+1)}\right)^k + \left(2n+1-2\sqrt{n(n+1)}\right)^k\right] \\ z_k = \frac{1}{4\sqrt{n(n+1)}}\left[\left(2n+1+2\sqrt{n(n+1)}\right)^k - \left(2n+1-2\sqrt{n(n+1)}\right)^k\right] \end{cases}$$

$k \geq 0$.

It is clear that x_k is odd for all k, hence all such pairs (m, n) are given by $(m_k, n)_{k \geq 0}$, where n is an arbitrary positive integer.

7.3 Polygonal Numbers

The kth polygonal number of order n (or the kth n-gonal number) P_k^n is given by the equation

$$P_k^n = \frac{k}{2}[(n-2)(k-1)+2].$$

Diophantus (c. 250 A.D.) noted that if the arithmetic progression with first term 1 and common difference $n-2$ is considered, then the sum of the first k terms is P_k^n. The usual geometric realization, from which the name derives, is obtained by considering regular polygons with n sides sharing a common angle and having points at equal distances along each side with the total number of points being P_k^n.

The first forty pages of Dickson's *History of Number Theory*, Vol. II, are devoted to results on polygonal numbers.

In [201] it is shown that there are infinitely many triangular numbers which at the same time can be written as the sum, the difference, and the product of other triangular numbers. It is easy to show that $4(m^2+1)^2$ is the sum, difference, and product of squares. Since then, several authors have proved similar results for sums and differences of other polygonal numbers. In [85] are considered pentagonal numbers, in [162] and [163] are considered hexagonal and septagonal numbers, and in [6] it is proved that for any n infinitely many n-gonal numbers can be written as the sum and difference of other n-gonal numbers. Although [85] gives several examples of pentagonal numbers written as the product of two other pentagonal numbers, the existence of an infinite class was left in doubt.

In this section we show that for every n there are infinitely many n-gonal numbers that can be written as the product of two other n-gonal numbers, and in fact show how to generate infinitely many such products. We suspect that our method does not generate all of the solutions for every n, but we have not tried to prove this. Moreover, except for $n = 3$ and 4, it is still not known whether there are infinitely many n-gonal numbers which at the same time can be written as the sum, difference, and product of n-gonal numbers.

Our proof uses the theory of the Pell equation. We also use a result on the existence of infinitely many solutions of a Pell equation satisfying a congruence condition, given that one solution exists satisfying the congruence condition. Next we note some facts about the Pell equation and prove this latter result. Then we prove the theorem on products of polygonal numbers.

In what follows, $\mathbb{Z}_+$ denotes the set of positive integers and $(a,b) \equiv (c,d) \pmod{m}$ means that $a \equiv c$ and $b \equiv d \pmod{m}$.

Theorem 7.3.1. *If $D \in \mathbb{Z}_+$ is not a square, then for any $m \in \mathbb{Z}_+$ there are infinitely many integral solutions to the Pell's equation $u^2 - Dv^2 = 1$, with $(u,v) \equiv (1,0) \pmod{m}$.*

Proof. Suppose (u_1, v_1) is the fundamental solution to Pell's equation

$$u^2 - Dv^2 = 1$$

and that $(u_j, v_j)_{j\geq 1}$ is the general solution given by

$$u_j + v_j\sqrt{D} = (u_1 + v_1\sqrt{D})^j, \quad j \geq 1.$$

Since there are only m^2 distinct ordered pairs of integers modulo m, there must be $j, l \in \mathbb{Z}_+$ such that $(u_j, v_j) \equiv (u_l, v_l) \pmod{m}$. We notice that, for any $k \geq 2$,

$$u_k + v_k\sqrt{D} = (u_1 + v_1\sqrt{D})(u_{k-1} + v_{k-1}\sqrt{D})$$

so

$$u_k = u_1u_{k-1} + Dv_1v_{k-1} \text{ and } v_k = v_1u_{k-1} + u_1v_{k-1}$$

(see also Section 3.2).

Applying these equations to the above congruence, we deduce

$$\begin{aligned} &u_1u_{j-1} + Dv_1v_{j-1} \equiv u_1u_{l-1} + Dv_1v_{l-1} \pmod{m} \text{ and} \\ &v_1u_{j-1} + u_1v_{j-1} \equiv v_1u_{l-1} + u_1v_{l-1} \pmod{m}. \end{aligned} \tag{7.3.1}$$

From (7.3.1) it follows $(u_1^2 - Dv_1^2)v_{j-1} \equiv (u_1^2 - Dv_1^2)v_{l-1} \pmod{m}$. Since $u_1^2 - Dv_1^2 = 1$, we obtain $v_{j-1} \equiv v_{l-1} \pmod{m}$.

Similarly, from (7.3.1) we obtain $u_{j-1} \equiv v_{l-1} \pmod{m}$, so $(u_{j-1}, v_{j-1}) \equiv (u_{l-1}, v_{l-1}) \pmod{m}$.

We can conclude that for $i = |j - l|$, $(1, 0) = (u_0, v_0) \equiv (u_{si}, v_{si}) \pmod{m}$, for any $i \in \mathbb{Z}_+$. □

Theorem 7.3.2. *If $a, b, m, D \in \mathbb{Z}_+$, D is not a square, and the general Pell's equation $u^2 - Dv^2 = M$ has a solution (u^*, v^*) with $(u^*, v^*) \equiv (a, b) \pmod{m}$, then it has infinitely many solutions $(u_k^*, v_k^*)_{k \geq 1}$ such that $(u_k^*, v_k^*) \equiv (a, b) \pmod{m}$.*

Proof. Let $(u_k, v_k)_{k\geq 1}$ be the solutions to the Pell's equation $u^2 - Dv^2 = 1$, guaranteed by Theorem 7.3.1, i.e., $(u_k, v_k) \equiv (1, 0) \pmod{m}$. Then the solution $(u_k^*, v_k^*)_{k\geq 1}$ to the general Pell's equation obtained from these solutions are such that

$$u_k^* = u^* u_k + Dv^* v_k \equiv a \cdot 1 + Db \cdot 0 \equiv a \pmod{m}$$

and

$$v_k^* = v^* u_k + u^* v_k \equiv b \cdot 1 + a \cdot 0 \equiv b \pmod{m}, \quad k \geq 1$$

(see also Section 4.1). □

The following Corollary follows by taking m in Theorem 7.3.2 to be the least common multiple of m_1 and m_2.

Corollary 7.3.3. *If $a, b, m_1, m_2, D \in \mathbb{Z}_+$, D is not a square, and $a^2 - Db^2 = M$, then there are infinitely many solutions to the general Pell's equation $u^2 - Dv^2 = M$ with $u \equiv a \pmod{m_1}$ and $v \equiv b \pmod{m_2}$.*

Next we show that any nonsquare n-gonal number is infinitely often the quotient of two n-gonal numbers (see [68]). The theorem that n-gonal products are infinitely often n-gonal and a remark on the solvability of a related equation follow.

Theorem 7.3.4. *If the n-gonal number $P = P_s$ is not a square, then there exist infinitely many distinct pairs (P_x, P_y) of n-gonal numbers such that*

$$P_x = P_s P_y. \tag{7.3.2}$$

Proof. Recalling that $P_x = \dfrac{x}{2}[(n-2)(x-1)+1]$ and setting $n - 2 = r$, equation (7.3.2) becomes

$$rx^2 - (r-2)x = P[ry^2 - (r-2)y].$$

Multiplying by $4r$ to complete the square gives

$$(2rx-(r-2))^2-(r-2)^2=P[(2ry-(r-2))^2-(r-2)^2].$$

Setting

$$u=2rx-(r-2), \quad v=2ry-(r-2) \tag{7.3.3}$$

we get the general Pell's equation

$$u^2-Pv^2=M, \tag{7.3.4}$$

with $M=(r-2)^2-P(r-2)^2$.

Thus, in order to ensure infinitely many solution (x,y) to (7.3.2), it suffices to have infinitely many solutions (u,v) to (7.3.4) for which the pair (x,y) obtained from (7.3.3) is integral. Put another way, it suffices to show the existence of infinitely many solutions (u^*,v^*) to (7.3.4) for which the congruence

$$(u^*,v^*)\equiv(-(r-2),-(r-2))\equiv(r+2,r+2) \pmod{2r}$$

holds.

But notice that, since $P_1=1$, a particular solution of (7.3.2) is $x=s$, $y=1$, and these values of x and y give $u=(2s-1)r+2$, $v=r+2$, as a particular solution of (7.3.4). Thus, we have a solution (u^*,v^*) of (7.3.4) with $(u^*,v^*)\equiv(r+2,r+2) \pmod{2r}$. Theorem 7.3.2 guarantees the infinitely many solutions we are seeking. □

Our final result is now a straightforward corollary.

Theorem 7.3.5. *For any $n\geq 3$, there are infinitely many n-gonal numbers which can be written as a product of two other n-gonal numbers.*

Proof. The case $n=4$ is trivial. By Theorem 7.3.4, we need only show that P_s is not a square for some s. But for $n\neq 4$, at least one of $P_2=n$ and $P_9=9(4n-7)$ is not a square. □

Remarks. 1) Trying to prove that

$$P_k=\frac{k}{2}[(n-2)(k-1)+2]=P_xP_y$$

infinitely often by setting $P_x=k$ and

$$P_y=\frac{1}{2}[(n-2)(P_x-1)+2]$$

and solving the corresponding Pell's equation that results works if $n \neq 2t^2 + 2$, and thus, for these values of n, there are infinitely many solutions to the equation $P_p = P_x P_y$.

2) There are 51 solutions of $P_x^3 = P_s^3 P_y^3$ with $P_x^3 < 10^6$. There are 43 solutions of $P_x^n = P_s^n P_y^n$ with $5 \leq n \leq 36$ and $P_x^n < 10^6$. For $36 < n \leq 720$, there are no solutions with $P_x^n < 10^6$.
3) In [107] are considered the simultaneous equations $P_x^n = P_y^m = P_z^q$, where m, n, q, x, y, z are positive integers. By reducing these to systems of simultaneous Pell equations, one can show that if (m, n, q) is not a permutation of $(3, 6, k)$ (for $k > 3$), then all solutions of the above system of equations have $\max\{x, y, z\} < c$, where c is an effectively computable constant depending only on m, n and q. In fact, the remaining case may also be easily analyzed, upon noting the reduction to

$$Z^2 - jX^2 = (j - 1)(j - 4),$$

if we take $(m, n, q) = (3, 6, j + 2)$. If j is a square, this equation has at most finitely many solutions, while, if $j > 1$ is not a square, it has infinitely many, corresponding to classes of the given Pell's equation, upon noting that $(Z, X) = (j - 2, 1)$ gives one such solution.

7.4 Powerful Numbers

Define a positive integer r to be a *powerful number* if p^2 divides r whenever the prime p divides r. The following list contains all powerful numbers between 1 and 1000: 1, 4, 8, 9, 16, 25, 27, 32, 36, 49, 64, 72, 81, 100, 108, 121, 125, 128, 144, 169, 196, 200, 216, 225, 243, 256, 288, 289, 324, 343, 361, 392, 400, 432, 441, 484, 500, 512, 529, 576, 625, 648, 675, 676, 729, 784, 800, 841, 864, 900, 961, 968, 972, 1000. Let $k(x)$ denote the number of powerful numbers not exceeding x. Following [77] we show that

$$\lim_{x \to \infty} \frac{k(x)}{\sqrt{x}} = c,$$

with the constant $c = \dfrac{\zeta(3/2)}{\zeta(3)}$, where ζ is the well-known Riemann zeta function. We also prove that there are infinitely many pairs of consecutive powerful integers, such as 8, 9 and 288, 289. We conclude with some results and conjectures concerning the gaps between powerful numbers.

7.4.1 The Density of Powerful Numbers

Let

$$F(s) = \prod_p (1 + p^{-2s} + p^{-3s} + \dots) = \prod_p \left(1 + \frac{1}{p^s(p^s - 1)}\right) \tag{7.4.1}$$

where the products are extended over all primes p. It is evident that

$$F(s) = \sum_{r \in K} r^{-s}, \tag{7.4.2}$$

where K is the set of powerful numbers. Then, the sum of the reciprocals of the powerful numbers,

$$F(1) = \sum_{r \in K} \frac{1}{r} = \prod_p \left(1 + \frac{1}{p(p-1)}\right), \tag{7.4.3}$$

is seen to be convergent (see [136–138] for the theory of convergent series).

To estimate $k(x)$, the number of powerful numbers up to x, we observe first that $k(x) \geqq [\sqrt{x}]$, since every perfect square is powerful. Next, we observe that every powerful number r can be represented as a perfect square n^2 (including the case $n = 1$) times a perfect cube m^3 (including $m = 1$), and that this representation is unique if we require m to be square-free. That is, we set m equal to the product of those primes having odd exponents in the canonical factorization of r into powers of distinct primes, and the representation $r = n^2m^3$ is then unique.

Thus,

$$k(x) = \#(n^2m^2 \leqq x, \mu(m) \neq 0) = \sum_{m=1}^{\infty} \mu^2(m) \left[\left(\frac{x}{m^3}\right)^{1/2}\right] \sim cx^{1/2},\ x \to \infty, \tag{7.4.4}$$

where

$$\sum_{m=1}^{\infty} \mu^2(m) m^{-3/2} < \zeta(3/2) < \infty. \tag{7.4.5}$$

Explicitly,

$$c = \prod_p (1 + p^{-3/2}) = \prod_p (1 - p^{-3})/(1 - p^{-3/2}) = \zeta(3/2)\zeta(3), \tag{7.4.6}$$

where $\zeta(s)$ is the Riemann zeta function (see [213]). This evaluation of c comes from setting $s = 3/2$ in the identity

$$\begin{aligned}\sum_{m=1} \frac{\mu^2(m)}{m^s} &= \prod_p \left(1+\frac{1}{p^s}\right)\\ &= \prod_p \frac{1-p^{-2s}}{1-p^{-s}}\\ &= \prod_p (1-p^{-2s})\Big/\prod_p (1-p^{-s})\\ &= \zeta(s)/\zeta(2s) \qquad (7.4.7)\end{aligned}$$

for all $Re(s) > 1$, where $\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \prod_p (1-p^{-s})^{-1}$ for $Re(s) > 1$.

For purposes of estimation, we have the inequalities

$$cx^{1/2} \geqq k(x) \geqq cx^{1/2} - 3x^{1/3} \text{ for } x \geqq 1, \qquad (7.4.8)$$

because $cx^{1/2} = \sum_{m=1}^{\infty} \mu^2(m)(x/m^3)^{1/2} \geqq \sum_{m=1}^{\infty} \mu^2(m)[(x/m^3)^{1/2}] = k(x)$, and

$$\begin{aligned}cx^{1/2} - k(x) &= \sum_{m=1}^{\infty} |\mu(m)| \left\{ \left(\frac{x}{m^3}\right)^{1/2} - \left[\left(\frac{x}{m^3}\right)^{1/2}\right] \right\}\\ &\leqq \sum_{m=1}^{[x^{1/3}]-1} |\mu(m)| \cdot 1 + \sum_{m=[x^{1/3}]}^{\infty} |\mu(m)| \left(\frac{x}{m^3}\right)^{1/2}\\ &\leqq ([x^{1/3}]-1) + \left(1+\sqrt{x}\int_{[x^{1/3}]}^{\infty} u^{-3/2}du\right)\\ &\leqq x^{1/3} + 2x^{1/2}x^{-1/6} = 3x^{1/3}.\end{aligned}$$

Numerically, $c = 2 \cdot 173\ldots$

We have the further identities:

$$\begin{aligned}F(s) &= \sum_{r\in K} (1/r^s)\\ &= \sum_{n=1}^{\infty} n^{-2s} \sum_{m=1}^{\infty} \mu^2(m) m^{-2s}\\ &= \sum_{i=1}^{\infty} t^{-2s} \sum_{m|t} |\mu(m)|/m^s\end{aligned}$$

$$= \sum_{t=1}^{\infty} t^{-2s} \prod_{p|t} (1 + p^{-s}), \tag{7.4.9}$$

where we used the substitution $t = mn$;

$$F(s) = \sum_{n=1}^{\infty} n^{-2s} \sum_{m=1}^{\infty} \mu^2(m) m^{-2s} = \zeta(2s)\zeta(3s)/\zeta(6s), \tag{7.4.10}$$

and

$$F(1) = \zeta(2)\zeta(3)/\zeta(6) = \sum_{i=1}^{\infty} \Psi(t)/t^3, \tag{7.4.11}$$

where

$$\Psi(t) = t \prod_{p|t} \left(1 + \frac{1}{p}\right), \tag{7.4.12}$$

by setting $s = 1$ in the previous identities (7.4.9) and (7.4.10).

Since $\zeta(2) = \pi^2/6$ and $\zeta(6) = \pi^6/945$, we observe

$$F(1) = \frac{315}{2\pi^4}\zeta(3). \tag{7.4.13}$$

7.4.2 Consecutive Powerful Numbers

Four consecutive integers cannot all be powerful, since one of them is twice an odd number. No example of three consecutive powerful numbers is known, unless one is willing to accept -1, 0, 1. If such an example exists, it must be of the form

$$4k - 1, \quad 4k, \quad 4k + 1.$$

No case of $4k - 1$ and $4k + 1$ both being powerful is known. In fact, the only known example of consecutive odd numbers $2k - 1$ and $2k + 1$ both being powerful is $2k - 1 = 25$, $2k + 1 = 27$.

There are two infinite families of examples where two consecutive integers are powerful which correspond to the solutions of the Pell equations $x^2 - 2y^2 = 1$ and $x^2 - 2y^2 = -1$.

Let x_1, y_1 satisfy $x_1^2 - 2y_1^2 = \pm 1$. Then $8x_1^2 y_1^2 = A$ and $(x_1^2 + 2y_1^2)^2 = B$ are consecutive powerful numbers. The following table gives several examples of consecutive powerful numbers from solutions of the equations $x^2 - 2y^2 = \pm 1$.

x	y	A	B
1	1	$8 = 2^3$	$9 = 3^2$
3	2	$288 = 2^5 \cdot 3^2$	$289 = 17^2$
7	5	$9800 = 2^3 \cdot 5^2 \cdot 7^2$	$9801 = 3^4 \cdot 11^2$
17	12	$332928 = 2^5 \cdot 3^2 \cdot 17^2$	$332929 = 577^2$
$\sqrt{B_0}$	$\sqrt{A_0/2}$	$4A_0B_0$	$4A_0B_0 + 1$

If A and $B = A + 1$ are consecutive powerful numbers, and if B is a perfect square, $B = u^2$, then $A = (u - 1)(u + 1)$. If u is even, then $(u - 1, u + 1) = 1$, and both $u - 1$ and $u + 1$ are odd powerful numbers. As already remarked, the only known instance of this occurrence is $u - 1 = 25$, $u + 1 = 27$, leading to the isolated example $A = 675 = 3^3 \cdot 5^2$, $B = 676 = 2^2 \cdot 13^2$. If u is odd, then $(u - 1)/2$ and $(u + 1)/2$ are consecutive integers, with $((u - 1)/2, (u + 1)/2) = 1$. For $u^2 - 1$ to be powerful, $(u - 1)/2$ and $(u + 1)/2$ must be a powerful odd number and twice a powerful number, in either order. The two Pell equations produce examples in both orders. However, an example satisfying neither of these Pell equations is also known, with $(u - 1)/2 = 242 = 2 \cdot 11^2$ and $(u + 1)/2 = 243 = 3^5$. This leads to $A = 235.224 = 2^3 \cdot 3^5 \cdot 11^2$ and $B = 235.225 = 5^2 \cdot 97^2$.

Whenever A and B are consecutive powerful numbers, so too are $A' = 4AB$ and $B' = 4AB+1 = (2A+1)^2$. The solution $x_0 = 1, y_0 = 1$, of $x^2 - 2y^2 = -1$ generates all solutions of the Pell equations $x^2 - 2y^2 = \pm 1$, in the sense that $x_n + y_n\sqrt{2} = (x_0 + y_0\sqrt{2})^n$ yields the complete set of solutions (x_n, y_n) such that $x_n^2 - 2y_n^2 = \pm 1$.

Note that the consecutive powerful numbers $A = 675$, $B = 676$, come from the solution $x = 26$, $y = 15$, of the Pell equation $x^2 - 3y^2 = 1$, with $A = 3y^2$ and $B = x^2$. Similarly, the example $A = 235.224$, $B = 235.225$ of consecutive powerful numbers arises from the Pell equation $x^2 - 6y^2 = 1$ with $x = 485$, $y = 198$. More generally, any solution (x_1, y_1) of the Pell equation $x^2 - dy^2 = \pm 1$, with the extra condition that $d|y_1^2$, leads to an infinite family of consecutive powerful numbers, starting with $A_1 = x_1^2$, $B_1 = dy_1^2 = A_1 \pm 1$, and continuing with $A_n = x_n^2$, $B_n = dy_n^2$, where (x_n, y_n) are obtained from the computation $(x_1 + \sqrt{d}y_1)^n = x_n + \sqrt{d}y_n$.

Conversely, whenever we have two consecutive powerful integers, if one of them is a perfect square x^2, we can write the other in the form $n^2m^3 = my^2$, with m square-free, and we have a solution to the Pell equation $x^2 - my^2 = \pm 1$.

In all cases given thus far consecutive powerful numbers, the larger number is a perfect square. However, the Pell equation $x^2 - 5y^2 = -1$ with $5|y$ leads to infinitely many powerful numbers $x^2 + 1 = 5y^2$, such as $x^2 = (682)^2 = 465124$; $5y^2 = 5(305)^2 = 5^3 \cdot 61^2 = 465125$.

One example consisting of two consecutive powerful numbers where neither is a perfect square is given by $A = 23^3 = 12167$ and $B = 2^3 \cdot 3^2 \cdot 13^2 = 12168$. An interesting method based on the equation $ax^2 - by^2 = 1$ to generate such consecutive powerful numbers is presented in the paper [220]. For instance, in this paper is found $A = 7(2637362)^2 = 48689748233308$ and $B = 3(4028637)^2 = 48689748233307$.

7.4.3 Gaps Between Powerful Numbers

The set K of powerful numbers is closed under multiplication. Since there are infinitely many pairs of powerful numbers which differ by 1, there are infinitely many pairs of powerful numbers differing by r, for any $r \in K$.

Every positive integer not of the form $2(2b+1)$ is difference of two powerful numbers in at least one way (specifically, as a difference of two perfect squares). For numbers of the form $2(2b+1)$, $b \geqq 0$, such representations may also exist. Thus:

$2 = 3^3 - 5^2$	$30 = 83^2 - 19^3$
$6 = ?$	$34 = ?$
$10 = 13^3 - 3^7$	$38 = 37^2 - 11^3$
$14 = ?$	$42 = ?$
$18 = 19^2 - 7^3 = 3^2(3^3 - 5^2)$	$46 = 17^2 - 3^5$
$22 = 7^2 - 3^3 = 47^2 - 3^7$	$50 = 5^2(3^3 - 5^2)$
$26 = 3^3 - 1^2 = 7^2 \cdot 3^5 - 109^2$	$54 = 3^4 - 3^3 = 3^3(3^3 - 5^2) = 7^3 - 17^2.$

If u and v are both powerful numbers, $(u, v) = 1$, and $a = u - v$, we say that a has a proper representation as a difference of powerful numbers. We observe that

$$2b + 1 = (b+1)^2 - b^2$$

$$8c = (2c+1)^2 - (2c-1)^2$$

so that all odd numbers, as well as all multiples of 8, have proper representations. Among the numbers $2(2b+1)$, $b = 0, 1, \ldots, 13$ for which representations were found, there were proper representations included in every case except $2(2b+1) = 50$. Finally, for numbers $4(2b+1)$, $b = 0, 1, \ldots, 12$, we observe the following proper representations:

$4 = 5^3 - 11^2$	$60 = ?$
$12 = 47^2 - 13^3$	$68 = 3^3 \cdot 5^4 - 7^5$
$20 = ?$	$76 = 5^3 - 7^2$
$28 = ?$	$84 = ?$
$36 = ?$	$92 = ?$
$44 = 5^3 - 3^4 = 13^2 - 5^3$	$100 = 7^3 - 3^5$
$52 = ?$	

It is interesting that if u and $v = u + 4$ are both powerful, then so too are $u' = uv$ and $v' = u' + 4 = (u+2)^2$. Thus, from the example $4 = 5^3 - 11^2$, an infinite number of proper representations of 4 are obtainable. It would be interesting to determine whether or not any numbers other than 1 and 4 have infinitely many proper representations.

Among the powerful numbers which are not perfect squares, the smallest difference known to occur infinitely often is 4. Specifically, the equation $3x^2-2y^2=1$ has infinitely many solutions for which $3|x$, such as $x=9$, $y=11$. For any such solution, we have $12x^2-8y^2=4$, where $12x^2$ and $8y^2$ are both powerful, and neither is a square. The only known instances where the difference between nonsquare powerful numbers is less than 4 are: $2^7-5^3=3$, and (as previously mentioned) $2^3\cdot 3^2\cdot 13^2-23^3=1$.

It has been conjectured that 6 cannot be represented in any way as a difference between two powerful numbers. It is further conjectured that there are infinitely many numbers which cannot be so represented. Other interesting properties and open problems concerning powerful numbers are mentioned in [141].

7.5 The Diophantine Face of a Problem Involving Matrices in $M_2(\mathbb{Z})$

Let R be a ring with identity. An element $a\in R$ is called *unit-regular* if $a=bub$ with $b\in R$ and a unit u in R, *clean* if $a=e+u$ with an idempotent e and a unit u, and *nil-clean* if $a=e+n$ with an idempotent e and a nilpotent n. A ring is *unit-regular* (or *clean*, or *nil-clean*) if all its elements are so. In [48], it was proved that *every unit-regular ring is clean*. However, in [103], it was noticed that this implication, *for elements*, fails. In the paper, plenty of unit-regular elements which are not clean are found among 2×2 matrices of the type $\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$ with integer entries.

While it is easy to prove that *any nil-clean ring is also a clean ring*, the question whether *nil-clean elements are clean*, was left open (see [63] and restated in [64]) for some 7 years. In this section, following the paper [29], we answer in the negative this question.

7.5.1 Nil-Clean Matrices in $M_2(\mathbb{Z})$

As this was done (in a special case) in [103], we investigate elements in the 2×2 matrix ring $\mathcal{M}_2(\mathbb{Z})$. Since $\mathbb{Z}$ and direct sums of $\mathbb{Z}$ are not clean (not even exchange rings), it makes sense to look for elements which are not clean in this matrix ring.

We first recall some elementary facts.

Let R be an integral domain and $A\in\mathcal{M}_n(R)$. Then A is a zero divisor if and only if $\det A=0$. Therefore idempotents (excepting the identity matrix) and nilpotents have zero determinant.

For $A\in\mathcal{M}_n(R)$, $rk(A)<n$ if and only if $\det A$ is a zero divisor in R. A matrix A is a unit in $\mathcal{M}_n(R)$ if and only if $\det A\in U(R)$. Thus, *the units* in $\mathcal{M}_2(\mathbb{Z})$ are the 2×2 matrices of $\det=\pm 1$.

Lemma 7.5.1. *Nontrivial idempotents in $\mathcal{M}_2(\mathbb{Z})$ are matrices*

$$\begin{bmatrix} \alpha+1 & u \\ v & -\alpha \end{bmatrix}$$

with $\alpha^2+\alpha+uv=0$.

Proof. One way follows by calculation. Conversely, notice that excepting I_2, such matrices are singular. Any nontrivial idempotent matrix in $\mathcal{M}_2(\mathbb{Z})$ has rank 1. By Cayley–Hamilton Theorem, $E^2 - \mathrm{tr}(E)E + \det(E)I_2 = 0$. Since $\det(E) = 0$ and $E^2 = E$ we obtain $(1-\mathrm{tr}(E)).E = 0_2$ and so, since there are no zero divisors in $\mathbb{Z}$, $\mathrm{tr}(E) = 1$. □

Lemma 7.5.2. *Nilpotents in $\mathcal{M}_2(\mathbb{Z})$ are matrices*

$$\begin{bmatrix} \beta & x \\ y & -\beta \end{bmatrix}$$

with $\beta^2+xy=0$.

Proof. One way follows by calculation. Conversely, just notice that nilpotent matrices in $\mathcal{M}_2(\mathbb{Z})$ have the characteristic polynomial t^2 and so have trace and determinant equal to zero. □

Therefore *the set of all the nil-clean matrices* in $\mathcal{M}_2(\mathbb{Z})$, which use a nontrivial idempotent in their nil-clean decomposition, is

$$\left\{ \begin{bmatrix} \alpha+\beta+1 & u+x \\ v+y & -\alpha-\beta \end{bmatrix} |\alpha,\beta,u,v,x,y \in \mathbb{Z}, \alpha^2+\alpha+uv=0=\beta^2+xy \right\}.$$

Remarks. 1) Nil-clean matrices in $\mathcal{M}_2(\mathbb{Z})$ which use a nontrivial idempotent, have the trace equal to 1. Otherwise, this is 2 or 0.

2) Since only the absence of nonzero zero divisors is (essentially) used, the above characterizations hold in any integral domain.

It is easy to discard the triangular case.

Proposition 7.5.3. *Upper triangular nil-clean matrices, which are neither unipotent nor nilpotent, are idempotent, and so (strongly) clean.*

Proof. Such upper triangular idempotents are $\begin{bmatrix} \alpha+1 & u \\ 0 & -\alpha \end{bmatrix}$ with

$$-\det = \alpha^2+\alpha = 0,$$

so have $\alpha \in \{-1, 0\}$, that is, $\begin{bmatrix} 1 & u \\ 0 & 0 \end{bmatrix}$ or $\begin{bmatrix} 0 & u \\ 0 & 1 \end{bmatrix}$. Upper triangular nilpotents have the form $\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}$, and so upper triangular nil-clean matrices have the form $\begin{bmatrix} 1 & u \\ 0 & 0 \end{bmatrix}$ or $\begin{bmatrix} 0 & u \\ 0 & 1 \end{bmatrix}$. As noticed before, these are idempotent. □

In the sequel we shall use the quadratic equation (3.1.1)

$$ax^2 + bxy + cy^2 + dx + ey + f = 0,$$

where a, b, c, d, e, f and are integers.

Denote $D =: b^2 - 4ac$, $g =: \gcd(b^2 - 4ac; 2ae - bd)$ and

$$\Delta =: 4acf + bde - ae^2 - cd^2 - fb^2.$$

Then the equation reduces to

$$-\frac{D}{g}Y^2 + gX^2 + 4a\frac{\Delta}{g} = 0$$

which (if $D > 0$) is a general Pell equation. Here

$$Y = 2ax + by + d \text{ and } X = \frac{D}{g}y + \frac{2ae - bd}{g}.$$

Notice that this equation may be also written as $-DY^2 + X^2 + 4a\Delta = 0$ replacing X by gX (and so $X = Dy + 2ae - bd$).

7.5.2 *The General Case*

In order to find a nil-clean matrix in $\mathcal{M}_2(\mathbb{Z})$ which is not clean, we need integers $\alpha, \beta, u, v, x, y$ with $\alpha^2 + \alpha + uv = 0 = \beta^2 + xy$, such that for every $\gamma, s, t \in \mathbb{Z}$, with $\gamma^2 + \gamma + st = 0$, the determinant

$$\det\left[\begin{bmatrix} \alpha+\beta-\gamma & u+x-s \\ v+y-t & -\alpha-\beta+\gamma \end{bmatrix}\right] = -(\alpha+\beta-\gamma)^2 - (u+x-s)(v+y-t) \notin \{\pm 1\}.$$

That is, subtracting any idempotent $\begin{bmatrix} \gamma+1 & s \\ t & -\gamma \end{bmatrix}$ from

$$\begin{bmatrix} \alpha+1 & u \\ v & -\alpha \end{bmatrix} + \begin{bmatrix} \beta & x \\ y & -\beta \end{bmatrix},$$

the result should not be a unit in $\mathcal{M}_2(\mathbb{Z})$.

Remark. Notice that above we have excepted the trivial idempotents. However, this will not harm since, in finding a counterexample, we ask for the nil-clean example not to be idempotent, nilpotent nor unit (and so not unipotent).

In the sequel, to simplify the writing, the following notations will be used: firstly, $m := 2\alpha+2\beta+1$ (m is odd and so nonzero) and $n := (u+x)(v+y)+(\alpha+\beta)^2+1$, and secondly, $r := \alpha+\beta$ and $\delta := r^2+r+(v+y)(u+x)$. Then

$$m = 2r+1,\ n = (u+x)(v+y)+r^2+1 = \delta - r + 1.$$

This way an arbitrary nil-clean matrix which uses no trivial idempotents is now written

$$C = \begin{bmatrix} r+1 & u+x \\ v+y & -r \end{bmatrix}$$

and $\delta = -\det C$. To simplify the wording such nil-clean matrices will be called *nontrivial nil-clean*.

Theorem 7.5.4. *Let*

$$C = \begin{bmatrix} r+1 & u+x \\ v+y & -r \end{bmatrix}$$

be a nontrivial nil-clean matrix and let

$$E = \begin{bmatrix} \gamma+1 & s \\ t & -\gamma \end{bmatrix}$$

be a nontrivial idempotent matrix. With above notations, $C-E$ is invertible in $\mathcal{M}_2(\mathbb{Z})$ with $\det(C-E)=1$ if and only if

$$X^2 - (1+4\delta)Y^2 = 4(v+y)^2(2r+1)^2(\delta^2+2\delta+2)$$

with

$$X = (2r+1)[-(1+4\delta)t + (2\delta+3)(v+y)]$$

and

$$Y = 2(v+y)^2 s + (2r^2+2r+1+2\delta)t - (2\delta+3)(v+y).$$

Further, $C-E$ is invertible in $\mathcal{M}_2(\mathbb{Z})$ with $\det(C-E)=-1$ *if and only if*

$$X^2-(1+4\delta)Y^2=4(v+y)^2(2r+1)^2\delta(\delta-2)$$

with

$$X=(2r+1)[-(1+4\delta)t+(2\delta-1)(v+y)]$$

and

$$Y=2(v+y)^2s+(2r^2+2r+1+2\delta)t-(2\delta-1)(v+y).$$

Proof. For given α,β,u,v,x,y, $\det(C-E)=\pm 1$ amounts to a general inhomogeneous equation of the second degree with two unknowns, which we reduce to a canonical form, as mentioned in the previous section. Here are the details.

$$\begin{aligned}&-\gamma^2-st-(\alpha+\beta)^2+2(\alpha+\beta)\gamma+(v+y)s+(u+x)t-(u+x)(v+y)\\&=(2\alpha+2\beta+1)\gamma+(v+y)s+(u+x)t-(u+x)(v+y)-(\alpha+\beta)^2=\pm 1.\end{aligned}$$

The case $\det=1$. Since

$$-m\gamma=(v+y)s+(u+x)t-(u+x)(v+y)-(\alpha+\beta)^2-1=(v+y)s+(u+x)t-n,$$

we obtain from $(-m\gamma)^2-m(-m\gamma)+m^2st=0$, the equation

$$[(v+y)\mathbf{s}+(u+x)\mathbf{t}-n]^2-m[(v+y)\mathbf{s}+(u+x)\mathbf{t}-n]+m^2st=0,$$

or

$$\begin{aligned}&(v+y)^2\mathbf{s}^2+[2(v+y)(u+x)+m^2]\mathbf{st}+(u+x)^2\mathbf{t}^2\\&-(m+2n)(v+y)\mathbf{s}-(m+2n)(u+x)\mathbf{t}+(m+n)n=0.\end{aligned}$$

Thus, with the notations of the previous section

$$\mathbf{a}=(v+y)^2,\ \mathbf{b}=[2(v+y)(u+x)+m^2],\ \mathbf{c}=(u+x)^2$$

and

$$\mathbf{d}=-(m+2n)(v+y),\ \mathbf{e}=-(m+2n)(u+x),\ \mathbf{f}=(m+n)n.$$

Further

$$\mathbf{D}=[2(v+y)(u+x)+m^2]^2-4(v+y)^2(u+x)^2$$

$$= m^4 + 4m^2(v+y)(u+x) = m^2[m^2 + 4(v+y)(u+x)],$$
$$\mathbf{2ae - bd} = m^2(m+2n)(v+y) \text{ for } g = \gcd(D, 2ae - bd)$$

(notice that $m^2|g$) and

$$\boldsymbol{\Delta} = 4acf + bde - ae^2 - cd^2 - fb^2$$
$$= 4(v+y)^2(u+x)^2(m+n)n + [2(v+y)(u+x) + m^2](m+2n)^2(v+y)(u+x)$$
$$-(v+y)^2(m+2n)^2(u+x)^2 - (u+x)^2(m+2n)^2(v+y)^2$$
$$-(m+n)n[2(v+y)(u+x) + m^2]^2$$
$$= m^4[(v+y)(u+x) - (m+n)n].$$

The case $\det = -1$. Formally exactly the same calculation, but n is slightly modified: here

$$n' = (u+x)(v+y) + (\alpha+\beta)^2 - 1,$$

i.e., $n' := n - 2$.

These equations reduce to the canonical form

$$gX^2 - \frac{D}{g}Y^2 = -4a\frac{\Delta}{g}$$

with

$$D = m^2[m^2 + 4(v+y)(u+x)],$$
$$g = \gcd(D, m^2(m+2n)(v+y)), \; a = (v+y)^2$$

and

$$\Delta = m^4[(v+y)(u+x) - (m+n)n].$$

Since clearly $g = m^2 g'$, in the above equation we can replace D and Δ by $\frac{D}{m^2}$ and $\frac{\Delta}{m^2}$ (and $g = \gcd(m^2 + 4(v+y)(u+x); (m+2n)(v+y))$), that is $D = m^2 + 4(v+y)(u+x)$ and $\Delta = m^2[(v+y)(u+x) - (m+n)n]$.

Further, this amounts to $g^2X^2 - DY^2 = -4a\Delta$ and so we can eliminate g (by taking a new unknown: $X' = gX$). Hence we reduce to the equation

$$X^2 - [m^2 + 4(v+y)(u+x)]Y^2 = -4(v+y)^2 m^2[(v+y)(u+x) - (m+n)n].$$

which we can rewrite as

$$X^2 - (1+4\delta)Y^2 = 4(v+y)^2(2r+1)^2(\delta^2+2\delta+2).$$

Further, for $\det = -1$, we obtain a similar equation replacing n by $n-2$, i.e., $n = \delta - r - 1$:

$$X^2 - (1+4\delta)Y^2 = 4(v+y)^2(2r+1)^2\delta(\delta-2).$$

The linear systems in s and t corresponding to $\det = 1$ and $\det = -1$, are respectively:

$$\begin{cases} 2(v+y)^2s + (2r^2+2r+1+2\delta)t - (2\delta+3)(v+y) = Y \\ (2r+1)[-(1+4\delta)t + (2\delta+3)(v+y)] = X \end{cases}$$

for $\det = 1$
(here $-(2r+1)\gamma = (v+y)s + (u+x)t - n = (v+y)s + (u+x)t - \delta + r - 1$), and

$$\begin{cases} 2(v+y)^2s + (2r^2+2r+1+2\delta)t - (2\delta-1)(v+y) = Y \\ (2r+1)[-(1+4\delta)t + (2\delta-1)(v+y)] = X \end{cases}$$

for $\det = -1$
(here $-(2r+1)\gamma = (v+y)s + (u+x)t - n' = (v+y)s + (u+x)t - \delta + r + 1$). □

7.5.3 The Example

Since $1+4\delta \geq 1$ if $\delta \geq 0$, in this case, from the general theory of Pell equations, it is known that the equations emphasized in Theorem 7.5.4 have infinitely many solutions, and so we cannot decide whether all the linear systems corresponding to these equations have (or not) integer solutions. However, if $\delta \leq -1$, then $1+4\delta < 0$ and we have elliptic type of Pell equations, which clearly have only finitely many integer solutions.

Take $r = 2$, $\delta = -57$ and $v+y = -7$, $u+x = 9$, that is, the matrix we consider is

$$\begin{bmatrix} 3 & 9 \\ -7 & -2 \end{bmatrix};\ 1+4\delta = -227.$$

More precisely $\alpha = -1$, $\beta = 3$, $u = 0$, $v = -6$, $x = 9$, and $y = -1$, i.e., the nil-clean decomposition

$$\begin{bmatrix} 3 & 9 \\ -7 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -6 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 9 \\ -1 & -3 \end{bmatrix}.$$

The (elliptic) Pell equation which corresponds to a unit with $\det = 1$ is $X^2 + 227Y^2 = 15.371.300$ with $X = 3(227t + 777)$ (we shall not need Y).

Since $X = 227(3t + 10) + 61$ we deduce $X^2 = 227k + 89$ for a suitable integer k. However, since $15.371.300 = 67.715 \times 227 - 5$ from the Pell equation we obtain $X^2 = 227l - 5$ (for a suitable integer l) and so there are no integer solutions.

As for the equation which corresponds to $\det = -1$, $X^2 + 227Y^2 = 16.478.700$ with $X = 3(227t+805)$. Analogously, $X = 227(3t+10)+145$ and $X^2 = 227p+141$ (for some integer p). Since from the Pell equation $(16.478.700 = 72.593\times 227+89)$ we obtain $X^2 = 227q+89$ (for an integer q), and again we have no integer solutions.

7.5.4 How the Example Was Found

A deceptive good news is that both equations (in Theorem 7.5.4) are solvable (over $\mathbb{Z}$): the first equation admits the solutions

$$X = \pm(v + y)(2r + 1)(2\delta + 3) \text{ and } Y = \pm(v + y)(2r + 1),$$

and the second equation admits the solutions:

$$X = \pm(v + y)(2r + 1)(2\delta - 1) \text{ and } Y = \pm(v + y)(2r + 1).$$

Therefore, the main problem which remains with respect to the solvability of the initial equations in s and t (γ is determined by s and t) is whether the linear systems above (in s and t) also have solutions (over $\mathbb{Z}$). Here is an analysis of this problem, just for the solutions given above.

For a unit with $\det = 1$ we have four solutions:
for $+X = +(v + y)(2r + 1)(2\delta + 3)$ we obtain $t = 0$.
Then for $+Y = +(v + y)(2r + 1)$ we obtain

$$s = u + x + \frac{r^2 + 2r + 2}{v + y} \text{ and } \gamma = -1$$

and for $-Y = -(v + y)(2r + 1)$ we obtain

$$s = u + x + \frac{r^2 + 1}{v + y} \text{ and } \gamma = 0.$$

The corresponding clean decompositions are

$$\begin{bmatrix} r+1 & u+x \\ v+y & -r \end{bmatrix} = \begin{bmatrix} 0 & u+x+\frac{r^2+2r+2}{v+y} \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} r+1 & -\frac{r^2+2r+2}{v+y} \\ v+y & -r-1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & u+x+\frac{r^2+1}{v+y} \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} r & -\frac{r^2+1}{v+y} \\ v+y & -r \end{bmatrix}.$$

Notice that r^2+1 and $r^2+2r+2=(r+1)^2+1$ are nonzero.

For $-X=-(v+y)(2r+1)(2\delta+3)$ we obtain

$$t=(v+y)(1+\frac{5}{1+4\delta})$$

which is an integer if and only if $1+4\delta$ divides $5(v+y)$. However, this has to be continued with conditions on s.

For a unit with $\det=-1$ we also have four solutions:
for $+X=(v+y)(2r+1)(2\delta-1)$ we obtain $t=0$. Then for $+Y=(v+y)(2r+1)$ we obtain

$$s=u+x+\frac{r^2+2r}{v+y} \text{ and } \gamma=-1$$

and for $-Y=-(v+y)(2r+1)$ we obtain

$$s=u+x+\frac{r^2-1}{v+y} \text{ and } \gamma=0.$$

The corresponding clean decompositions are

$$\begin{bmatrix} r+1 & u+x \\ v+y & -r \end{bmatrix} = \begin{bmatrix} 0 & u+x+\frac{r^2+2r}{v+y} \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} r+1 & -\frac{r^2+2r}{v+y} \\ v+y & -r-1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & u+x+\frac{r^2-1}{v+y} \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} r & -\frac{r^2-1}{v+y} \\ v+y & -r \end{bmatrix}.$$

Notice that $r^2-1=0$ if and only if $r\in\{\pm1\}$ and $r^2+2r=0$ if and only if $r\in\{0,2\}$.

For $-X=-(v+y)(2r+1)(2\delta-1)$ we obtain $t=(v+y)\left(1+\frac{1}{1+4\delta}\right)$ which is an integer if and only if $1+4\delta$ divides $v+y$. Again, this has to be continued with conditions on s.

Generally the relations $\alpha^2+\alpha+uv=0$ and $\beta^2+xy=0$, do not imply that $v+y$ divides any of r^2+1, r^2-1, $r^2+2r=(r+1)^2-1$ or $r^2+2r+2=(r+1)^2+1$ (recall that $r=\alpha+\beta$), nor that $1+4\delta$ divides $5(v+y)$ (and so does not divide $v+y$).

Searching for a counterexample, we need integers $\alpha, \beta, u, v, x, y$ such that $\alpha^2 + \alpha + uv = 0 = \beta^2 + xy$, and $v + y$ does not divide any of the numbers: $r^2 + 1$, $r^2 - 1$, $(r+1)^2 - 1$ or $(r+1)^2 + 1$.

Further, $1 + 4\delta$ should not divide $5(v + y)$ and, moreover, to cover the trivial idempotents, we add two other conditions.

Since idempotents and units are clean in any ring, we must add:

$$\det \begin{bmatrix} r+1 & u+x \\ v+y & -r \end{bmatrix} \neq 0$$

(this way the nil-clean matrix is not idempotent, nor nilpotent) and

$$\det \begin{bmatrix} r+1 & u+x \\ v+y & -r \end{bmatrix} \neq \pm 1,$$

(it is not a unit, and so nor unipotent), that is $\delta \notin \{0, \pm 1\}$.

Notice that if $r \in \{-2, -1, 0, 1\}$, then 0 appears among our two numbers ($r^2 - 1$, $(r+1)^2 - 1$) and the fraction is zero (i.e., an integer).

Since a matrix is nil-clean if and only if its transpose is nil-clean, we should have symmetric conditions on the corners $v + y$ and $u + x$, respectively. That is why, $u + x$ should not divide any of the numbers: $r^2 + 1$, $r^2 - 1$, $(r+1)^2 - 1$, or $(r+1)^2 + 1$, and further, $1 + 4\delta$ should not divide $5(u + x)$.

Further, we exclude clean decompositions which use an idempotent of type

$$\begin{bmatrix} 0 & 0 \\ k & 1 \end{bmatrix}.$$

In this case the unit (supposed with $\det = -1$) should be

$$\begin{bmatrix} r+1 & u+x \\ (v+y)-k & -r-1 \end{bmatrix}$$

and if its determinant equals -1 then $u + x$ divides $r^2 + r$. Since idempotent, nilpotent, unit and so nil-clean matrices have the same property when transposed, to the conditions above we add $u + x$ and $v + y$ do not divide $r^2 + r$.

By inspection, one can see that there are no selections of $u + x$ and $v + y$ less than ± 7 and ± 9, at least for $r \in \{2, 3, \ldots 10\}$, which satisfy all the above nondivisibilities.

Therefore $v + y = -7$, $u + x = 9$ is some kind of minimal selection. In order to keep numbers in the Pell equation as low as possible we choose $r = 2$ and so $\delta = -57$.

Indeed, our matrix verifies all *these exclusion conditions*: -7 and 9 do not divide any of $r^2 \pm 1 = 3, 5$, $(r+1)^2 \pm 1 = 8, 10$ nor $r^2 + r = 6$; $1 + 4\delta = -227$ (prime number) does not divide $5 \times (-7) = -35$ nor $5 \times 9 = 45$, and $\delta \notin \{0, \pm 1\}$.

Remark. We found this example in terms of r, δ, $u + x$, and $v + y$. It was not obvious how to come back to the nil-clean decomposition, that is, to α, β, u, v, x, and y (indeed, this reduces to another elliptic Pell equation!). However, the following elementary argument showed more: there is only one solution, given by $(u, v) = (0, -6)$.

The system $\alpha + \beta = 2$, $u + x = 9$, $v + y = -7$, $\alpha^2 + \alpha + uv = 0 = \beta^2 + xy$ is equivalent to $(7u - 9v - 59)(7u - 9v - 54) + 25uv = 0$. Denote $t = 7u - 9v - 59$, hence $u = \frac{1}{7}(9v + t + 59)$. We obtain the equation

$$t(t + 5) + 25uv = 0.$$

Looking mod 5, it follows $t = 5k$, for some integer k. The equation simplifies to $k(k + 1) + uv = 0$. That is

$$k(k + 1) + \frac{1}{7}(9v + 5k + 59)v = 0.$$

Considering the last equation as a quadratic equation in k, we have

$$7k^2 + (5v + 7)k + 9v^2 + 59v = 0.$$

The discriminant of the last equation is

$$\Delta = (5v + 7)^2 - 28(9v^2 + 59v) = -227v^2 - 1582v + 49.$$

In order to have integer solutions for our last equation it is necessary $\Delta \geq 0$ and Δ to be a perfect square. The quadratic function

$$f(v) = -227v^2 - 1582v + 49$$

has the symmetry axis of the equation $v_{max} = -\frac{1582}{2 \cdot 227} < 0$, and $f(1) < 0$, hence there are no integers $v \geq 1$ such that $f(v) \geq 0$.

On the other hand, we have $f(-7) = 0$, giving $k = 2$, hence $t = 10$. Replacing in the equation (1) we obtain $6 - 7u = 0$, equation with no integer solution. Moreover, we have $f(v) < 0$ for all $v < -7$.

From the above remark, it follows that all possible integer solutions for v are -6, $-5, -4, -3, -2, -1, 0$. Checking all these possibilities we obtain $f(-6) = 37^2$ and then $k = -1$. We get $t = -5$, and equation (1) becomes $-6u = 0$, hence $u = 0$.

7.6 A Related Question

Since both unit-regular and nil-clean rings are clean, a natural question is whether these two classes are somehow related. First $\mathbb{Z}_3$ (more generally, any domain with at least 3 elements) is a unit-regular ring which is not nil-clean, and, $\mathbb{Z}_4$ (more generally, any nil clean ring with nontrivial Jacobson radical) is nil-clean but not unit-regular.

Finally, we give examples of nil-clean matrices in $\mathcal{M}_2(\mathbb{Z})$ which are not unit-regular, and unit-regular matrices which are not nil-clean.

Recall that the set of all the nontrivial nil-clean matrices in $\mathcal{M}_2(\mathbb{Z})$ is

$$\left\{ \begin{bmatrix} \alpha+\beta+1 & u+x \\ v+y & -\alpha-\beta \end{bmatrix} | \alpha, \beta, u, v, x, y \in \mathbb{Z}, \alpha^2+\alpha+uv=0=\beta^2+xy \right\},$$

and that the only nonzero unit-regular matrices with a zero second row are

$$\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix},$$

with (a,b) unimodular (i.e., a row whose entries generate the unit ideal) [see [103]).

Hence $\begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}$ is *unit-regular but not nil-clean* (nil-clean matrices have trace equal to 2,1 or 0; in the first case $\begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix} - I_2$ is not nilpotent). Conversely, first notice that the nil-clean matrices with a zero second row are exactly the matrices $\begin{bmatrix} 1 & b \\ 0 & 0 \end{bmatrix}, b \in \mathbb{Z}$. Being idempotent, these are also unit-regular (so not suitable).

However, consider the nil-clean matrix (with our notations $\alpha=\beta=v=x=0$, $u=1, y=2$)

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}.$$

Suppose A is unit-regular. Then, using an equivalent definition, $A=EU$ with $E=E^2$ and $U \in GL_2(\mathbb{Z})$. Since $\det A=-2 \neq \pm 1$, A is not a unit and so $E \neq I_2$. Hence $\det E=0$ and from $\det A=\det E \cdot \det U$, we obtain a contradiction.