Chapter 6
Diophantine Representations of Some Sequences

In 1900, David Hilbert asked for an algorithm to decide whether a given Diophantine equation is solvable or not and put this problem tenth in his famous list of 23.

In 1970, it was proved that such an algorithm cannot exist, i.e., the problem is recursively undecidable. Proof was supplied by Yu. V. Matiyasevich [133], heavily leaning on results arrived at by M. Davis, J. Robinson, and H. Putnam [60]. This was accomplished by proving that any enumerable set $A \subseteq \mathbb{N} = \{0, 1, 2, \dots\}$ can be represented in the following form: There exists a polynomial $p(x, x_1, \dots, x_n)$ with $n \geq 0$ such that $a \in A$ if and only if $p(a, x_1, \dots, x_n) = 0$ is solvable for particular nonnegative integers $x_1, \dots, x_n$, i.e.,

$$a \in A \;\Leftrightarrow\; \exists\, x_1, \dots, x_n \geq 0 :\; p(a, x_1, \dots, x_n) = 0.$$

Therefore, the set A equals the set of parameters for which the equation $p = 0$ is solvable. Employing an idea of H. Putnam [178] this can be reformulated as follows. If $q(x, x_1, \dots, x_n) = x(1 - p(x, x_1, \dots, x_n)^2)$, then A equals the set of positive values of q, where its variables range over the nonnegative integers. Among the recursively enumerable sets there are many for which such representation is surprising. We will name some examples which are of importance in number theory.

(1) The primes and their recursively enumerable subsets, most outstanding Fermat-, Mersenne-, and Twin-primes.
(2) The set of partial denominators of the continued fraction expansion of numbers as e, π and $\sqrt[3]{2}$. (Whereas for e this is known to equal $\{1\} \cup \{2, 4, 6, \dots\}$, there is only computer-based research regarding the other numbers.

In this chapter we will introduce a Diophantine representation concept for sequences of integers that refines the idea of Diophantine set. This concept proves helpful in solving several types of Diophantine equations.

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6.1 Diophantine *r*-Representable Sequences

The sequence $(x_m)_{m\geq 1}$ is *Diophantine r-representable* if there exists a sequence $(P_n)_{n\geq 1}$ of polynomials of degree r, $P_n \in \mathbb{Z}[X_1,\dots,X_r]$, such that for any positive integer n the following equality holds:

$$P_n(x_{n-r+1},\dots,x_n)=0. \tag{6.1.1}$$

This means that the sequence $(x_m)_{m\geq 1}$ has the above property if and only if among the solutions to the Diophantine equation

$$P_n(y_1,y_2,\dots,y_r)=0$$

there are some for which $y_1^{(n)}=x_{n-r+1}$, $y_2^{(n)}=x_{n-r+2},\dots,$ $y_r^{(n)}=x_n$, for all positive integers n.

The main result of this section is that any sequence defined by a linear recurrence of order r is Diophantine r-representable. Our approach follows the method given in [27] and [47] (see also [28] in the case $r=2$).

Consider the sequence $(x_n)_{n\geq 1}$ defined recursively by

$$\begin{cases} x_i=\alpha_i,\ i=1,2,\dots,r \\ x_n=\displaystyle\sum_{k=1}^{r} a_k x_{n-r-1+k},\ n\geq r+1 \end{cases} \tag{6.1.2}$$

where $\alpha_1,\alpha_2,\dots,\alpha_r$ and $a_1,a_2,\dots,a_r$ are integers with $a_1\neq 0$.

For $n\geq r$, let

$$D_n=\det\begin{bmatrix} x_{n-r+1} & x_{n-r+2} & \cdots & x_{n-1} & x_n \\ x_{n-r+2} & x_{n-r+3} & \cdots & x_n & x_{n+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ x_n & x_{n+1} & \cdots & x_{n+r-2} & x_{n+r-1} \end{bmatrix} \tag{6.1.3}$$

Lemma 6.1.1. *For all integers $n\geq r$, the following equality holds:*

$$D_n=(-1)^{(r-1)(n-r)}a_1^{n-r}D_r. \tag{6.1.4}$$

Proof. Following the method of [104, 135] and [202] we introduce the matrix

$$A_n=\begin{bmatrix} x_{n-r+1} & x_{n-r+2} & \cdots & x_{n-1} & x_n \\ x_{n-r+2} & x_{n-r+3} & \cdots & x_n & x_{n+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ x_{n-1} & x_n & \cdots & x_{n+r-3} & x_{n+r-2} \\ x_n & x_{n+1} & \cdots & x_{n+r-2} & x_{n+r-1} \end{bmatrix}.$$

It is easy to see that

$$A_{n+1}=\begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 \\ a_1 & a_2 & a_3 & a_4 & \dots & a_{r-2} & a_{r-1} & a_r \end{bmatrix} \cdot A_n$$

and so that

$$A_n=\begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 \\ a_1 & a_2 & a_3 & a_4 & \dots & a_{r-2} & a_{r-1} & a_r \end{bmatrix}^{n-r} \cdot A_r. \tag{6.1.5}$$

Passing to determinants in (6.1.5), we obtain $((-1)^{r-1}a_1)^{n-r}D_r = D_n$ for $n \geq r$, that is the relation (6.1.4). □

Theorem 6.1.2. *Any sequence defined by a linear recurrence of order r is Diophantine r-representable.*

Proof. Consider the sequence $(x_n)_{n\geq 1}$ defined by (6.1.2) and let $P_n \in \mathbb{Z}[X_1,\dots,X_r]$ be the polynomial given by

$$P_n(y_1,\dots,y_r) = F_r(y_1,\dots,y_r) - (-1)^{(r-1)(n-r)}a_1^{n-r}F_r(\alpha_1,\dots,\alpha_r) \tag{6.1.6}$$

where $F_r \in \mathbb{Z}[X_1,\dots,X_r]$ is obtained from the determinant (6.1.3) and the recursive relation (6.1.2).

From the relation (6.1.4) it follows that for all $n \geq r$ the following equalities hold

$$\begin{aligned} P_n(x_{n-r+1},\dots,x_n) &= F_r(x_{n-r+1},\dots,x_n) - (-1)^{(r-1)(n-r)}a_1^{n-r}F_r(\alpha_1,\dots,\alpha_r) \\ &= D_n - (-1)^{(r-1)(n-r)}a_1^{n-r}D_r = 0, \end{aligned}$$

i.e., the sequence $(x_n)_{n\geq 1}$ is Diophantine r-representable. □

Remarks. 1) When $r=2$, the polynomial F_r in (6.1.6) is given by

$$F_2(x,y) = x^2 - a_2xy - a_1y^2 \tag{6.1.7}$$

and it follows that, for the sequence $(x_n)_{n\geq 1}$ defined by

$$\begin{cases} x_1 = \alpha_1,\ x_2 = \alpha_2 \\ x_n = a_1x_{n-2} + a_2x_{n-1},\ n \geq 3 \end{cases} \tag{6.1.8}$$

the relation $F_2(x_{n-1}, x_n) = (-1)^n a_1^{n-2} F_2(\alpha_1, \alpha_2)$ holds, i.e.,

$$x_n^2 - a_2 x_{n-1} x_n - a_1 x_{n-1}^2 = (-1)^n a_1^{n-2}(\alpha_2^2 - a_2 \alpha_1 \alpha_2 - a_1 \alpha_1^2). \tag{6.1.9}$$

The relation (6.1.9) is the first relation of [35] and [36].

2) In the particular case $r = 3$, after elementary calculation, we obtain

$$\begin{aligned} F_3(x, y, z) = -x^3 - (a_1 + a_2 a_3) y^3 - a_1^2 z^3 + 2a_3 x^2 y + a_2 x^2 z \\ -(a_2^2 + a_1 a_3) y^2 z - (a_3^2 - a_2) x y^2 \\ -a_1 a_3 x z^2 - 2a_1 a_2 y z^2 + (3a_1 - a_2 a_3) xyz, \end{aligned}$$

hence we get that, for the linear recurrence

$$\begin{cases} x_1 = \alpha_1,\ x_2 = \alpha_2,\ x_3 = \alpha_3 \\ x_n = a_1 x_{n-2} + a_2 x_{n-2} + a_3 x_{n-1},\ n \geq 4 \end{cases} \tag{6.1.10}$$

the relation

$$F_3(x_{n-2}, x_{n-1}, x_n) = a_1^{n-3} F_3(\alpha_1, \alpha_2, \alpha_3) \tag{6.1.11}$$

is true.

3) If in the proof of Theorem 6.1.2, the equation $P_n(x_{n-r+1}, \ldots, x_n) = 0$ can be solved with respect to x_n, then x_n can be written as a function in $r - 1$ variables $x_{n-r+1}, \ldots, x_{n-1}$.
4) Note that the polynomial $F_r \in \mathbb{Z}[X_1, \ldots, X_r]$ can be viewed as an "invariant" to the sequence $(x_n)_{n \geq 1}$ defined by (6.1.2).
5) For additional informations about the special case $r = 2$ we refer to [92].

6.2 A Property of Some Special Sequences

If $a_1 = a_2 = 1$ and $\alpha_1 = \alpha_2 = 1$, then (6.1.8) defines the Fibonacci sequence $(F_n)_{n \geq 1}$ (see [105, 114, 187]) and [217] for many interesting properties). From (6.1.9) we obtain

$$F_n^2 - F_n F_{n-1} - F_{n-1}^2 = (-1)^{n-1}. \tag{6.2.1}$$

If $a_1 = a_2 = 1$ and $\alpha_1 = 1$, $\alpha_2 = 3$, then (6.1.8) defines the Lucas sequence $(L_n)_{n \geq 1}$ (see [105]) and from (6.1.9) it follows that

$$L_n^2 - L_n L_{n-1} - L_{n-1}^2 = 5(-1)^n. \tag{6.2.2}$$

If $a_1 = 1$, $a_2 = 2$ and $\alpha_1 = 1$, $\alpha_2 = 3$, then (6.1.8) gives the Pell sequence $(P_n)_{n\geq 1}$ (see [106]) and the relation (6.1.9) becomes

$$P_n^2 - 2P_nP_{n-1} - P_{n-1}^2 = (-1)^{n-1}. \tag{6.2.3}$$

From the relations (6.2.1), (6.2.2), (6.2.3) we deduce

$$F_n = \frac{1}{2}\left(F_{n-1} + \sqrt{5F_{n-1}^2 + 4(-1)^{n-1}}\right) \tag{6.2.4}$$

$$L_n = \frac{1}{2}\left(L_{n-1} + \sqrt{5L_{n-1}^2 + 20(-1)^n}\right) \tag{6.2.5}$$

$$P_n = P_{n-1} + \sqrt{2P_{n-1}^2 + (-1)^{n-1}}. \tag{6.2.6}$$

These identities give the possibility for writing computer programs that facilitate the computation of the terms of each of the three sequences $(F_n)_{n\geq 1}$, $(L_n)_{n\geq 1}$, $(P_n)_{n\geq 1}$.

In [183] it is given a method for obtaining the relation (6.2.4) by using hyperbolic functions. Similar results are also presented in [90].

Proposition 6.2.1. *If the sequence $(x_n)_{n\geq 1}$ is given by (6.1.8), then for all integers $n \geq 3$, the integer*

$$(a_2^2 + 4a_1)x_{n-1}^2 + 4(-1)^{n-1}a_1^{n-2}(a_1\alpha_1^2 + a_2\alpha_1\alpha_2 - \alpha_2^2)$$

is a perfect square.

Proof. From (6.1.9) we obtain

$$(a_2^2 + 4a_1)x_{n-1}^2 + 4(-1)^{n-1}a_1^{n-2}(a_1\alpha_1^2 + a_2\alpha_1\alpha_2 - \alpha_2^2) = (2x_n - a_2x_{n-1})^2$$

which finishes the proof. □

Proposition 6.2.2. *Let α_1, α_2 and k be nonzero integers. The general Pell's equations*

$$x^2 - (k^2 + 4)y^2 = 4(\alpha_1^2 + k\alpha_1\alpha_2 - \alpha_2^2)$$

and

$$(k^2 + 4)u^2 - v^2 = 4(\alpha_1^2 + k\alpha_1\alpha_2 - \alpha_2^2)$$

are solvable.

Proof. In (6.1.8) consider $a_1 = 1$ and $a_2 = k$. From Proposition 6.2.1 it follows that $(x, y) = (2x_n - kx_{n-1}, x_{n-1})$ is a solution to the first equation whenever n is odd. If n is even, then $(u, v) = (x_{n-1}, 2x_n - kx_{n-1})$ is a solution to the second equation. □

Remark. Note that the first equation in Proposition 6.2.2 has solution $(2\alpha_1 + k\alpha_2, \alpha_2)$. From Theorem 4.5.1 it follows that it has infinitely many integral solutions.

Similarly, the second equation in Proposition 6.2.2 has solution $(k\alpha_1 + (k^2 + 2)\alpha_2, \alpha_1 + k\alpha_2)$, and by applying Theorem 4.5.1 we deduce that it has infinitely many integral solutions.

6.3 The Equations $x^2 + axy + y^2 = \pm 1$

The result in Theorem 5.6.1 shows that if $|a| > 2$ the pairs $(-v_n, v_{n+1})$, $(v_n, -v_{n+1})$, $(-v_{n+1}, v_n)$, $(v_{n+1}, -v_n)$ of consecutive terms in the sequence given by

$$v_n = \frac{1}{\sqrt{a^2-4}}\left[\left(\frac{a+\sqrt{a^2-4}}{2}\right)^n - \left(\frac{a-\sqrt{a^2-4}}{2}\right)^n\right]$$

can be characterized as solutions to the equation $x^2 + ax + y^2 = 1$.

On the other hand, the sequence $(v_n)_{n\geq 0}$ satisfies the linear recurrence of order 2

$$v_{n+1} = av_n - v_{n-1},\ n \geq 1,\ \text{where } v_0 = 0 \text{ and } v_1 = 1.$$

Therefore the solutions to the discussed equation consists of all pairs of consecutive terms in a sequence defined by a second order recursive linear relation.

In what follows, we will study the equation

$$x^2 + axy + y^2 = -1, \tag{6.3.1}$$

which is also a special case of (4.8.1).

Theorem 6.3.1. *The equation (6.3.1) is solvable in integers if and only if $a = \pm 3$.*
If $a = -3$, then the solutions are

$$(-F_{2n-1}, -F_{2n+1}),\ (-F_{2n+1}, -F_{2n-1}),$$
$$(F_{2n-1}, F_{2n+1}),\ (F_{2n+1}, F_{2n-1}),\ n \geq 1.$$

If $a = 3$, then the solutions are

$$(-F_{2n-1}, F_{2n+1}),\ (-F_{2n+1}, F_{2n-1}),$$
$$(F_{2n-1}, -F_{2n+1}),\ (F_{2n+1}, -F_{2n-1}),\ n \geq 1,$$

where $(F_m)_{m\geq 1}$ is the Fibonacci sequence.

Proof. First consider $a < 0$. If there is a solution (x, y), then $xy > 0$. Therefore, we may assume that $x > 0$, $y > 0$ and we may consider that x is minimal.

If $a \neq -3$, then $x \neq y$, for otherwise $(a+2)x^2 = -1$, which is impossible, because $a + 2 \neq -1$. We have

$$\begin{aligned} 0 = x^2 + axy + y^2 + 1 &= (x+ay)^2 - axy - a^2y^2 + y^2 + 1 \\ &= (-x-ay)^2 + a(-x-ay)y + y^2 + 1, \end{aligned}$$

hence $(-x - ay, y)$ is also a solution. It follows that $-x - ay > 0$.

If we prove that $-x - ay < x$, then we contradict the minimality of x. Indeed, from the symmetry of the equation, we may assume that $x > y$. Then $x^2 > y^2 + 1 = x(-x - ay)$, so $x > -x - ay$. It follows that in this case the equation (6.3.1) is not solvable.

Consider now $a > 0$ and let (x, y) be a solution. Then $xy < 0$ and we may assume for example that $x > 0$ and $y < 0$. Setting $z = -y$, we obtain the equivalent equation $x^2 + (-a)xz + z^2 = -1$, with $x > 0$, $z > 0$, which we examined above. It follows that the equation (6.3.1) is not solvable if $-a \neq -3$, i.e., $a \neq 3$.

It remains to solve the equation when $a = \pm 3$. First, consider the case $a = -3$ and write the equation $x^2 - 3xy + y^2 = -1$ in the following equivalent form $(2x - 3y)^2 - 5y^2 = -4$. This is a special Pell's equation:

$$u^2 - 5v^2 = -4. \tag{6.3.2}$$

Its minimal solution is $(1, 1)$. By the results in Section 4.3.2, it follows that the general solution (u_m, v_m) to (6.3.2) is given by

$$u_m + v_m\sqrt{5} = 2\left(\frac{1+\sqrt{5}}{2}\right)^m, \quad m = 1, 3, 5, \ldots$$

Since

$$u_m - v_m\sqrt{5} = 2\left(\frac{1-\sqrt{5}}{2}\right)^m, \quad m = 1, 3, 5, \ldots$$

we obtain

$$u_m = \left(\frac{1+\sqrt{5}}{2}\right)^m + \left(\frac{1-\sqrt{5}}{2}\right)^m$$

and

$$v_m = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^m - \left(\frac{1-\sqrt{5}}{2}\right)^m\right] = F_m$$

where $m = 1, 3, 5, \ldots$

It follows that $2x - 3F_m = u_m$, hence

$$x = \left(\frac{3}{2\sqrt{5}} + \frac{1}{2}\right)\left(\frac{1+\sqrt{5}}{2}\right)^m - \left(\frac{3}{2\sqrt{5}} - \frac{1}{2}\right)\left(\frac{1-\sqrt{5}}{2}\right)^m$$
$$= \frac{(1+\sqrt{5})^2}{4\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^m - \frac{(1-\sqrt{5})^2}{2\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^m$$
$$= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{m+2} - \left(\frac{1-\sqrt{5}}{2}\right)^{m+2}\right] = F_{m+2}.$$

We obtain the solutions (F_{2n+1}, F_{2n-1}), $n \geq 1$, and by the symmetries $(x, y) \to (y, x)$ and $(x, y) \to (-x, -y)$ we find the others.

If $a = 3$, the substitution $y = -z$ transforms the equation into

$$x^2 - 3xz + z^2 = -1.$$

From the above considerations we obtain the solutions

$$(x, z) = (F_{2n+1}, F_{2n-1})$$

and by using the same symmetries we get the four families of solutions given in the Theorem. □

Remarks. 1) The conclusion in Theorem 6.3.1 can be also obtained by considering the more general equation (see [200] or [25]):

$$x^2 + y^2 + 1 = xyz.$$

The integral solutions (x, y, z) to this equation are given by

$$(-F_{2n-1}, -F_{2n+1}, 3),\ (-F_{2n+1}, -F_{2n-1}, 3),\ (F_{2n-1}, F_{2n+1}, 3),$$
$$(F_{2n+1}, F_{2n-11}, 3),\ (-F_{2n-1}, F_{2n+1}, -3),\ (-F_{2n+1}, F_{2n-1}, -3),$$
$$(F_{2n-1}, -F_{2n+1}, -3),\ (F_{2n+1}, -F_{2n-1}, -3),\ n \geq 1.$$

2) In [150] it is considered the more general equation

$$f_1(x, y) = zf_2(x, y)$$

where $f_1(x, y) = ax^2 + bxy + cy^2 + dx + ey + f$, $f_2(x, y) = pxy + qx + ry + s$ are quadratic forms with integer coefficients and $ac \neq 0$, $a|\gcd(b, d, p, q)$, $c|\gcd(b, e, p, r)$.

3) In the paper [188] is considered the equation

$$x^4 - 6x^2y^2 + 5y^4 = 16F_{n-1}F_{n+1}, \tag{6.3.3}$$

when one of the Fibonacci numbers F_{n-1}, F_{n+1} is prime and another is prime or it is a product of two different prime numbers. There are such Fibonacci numbers, for example $F_5 = 5$ and $F_7 = 13$; $F_{11} = 89$ and $F_{13} = 233$; $F_{17} = 1597$ and $F_{19} = 165580141 = F_{29} = 514229$ and $F_{31} = 1346269 = 557 \cdot 2147$; $F_{41} = 165580141 = 2789 \cdot 59369$ and $F_{43} = 433494437$.

Using the equivalent form $(x^2 - 5y^2)(x^2 - y^2) = 16F_{n-1}F_{n+1}$ to the equation, and the result in Theorem 4.1.1, in the paper [188] is shown that all integral solutions (x, y, n) to (6.3.3) are $(x, y, n) = (\pm L_{6l}, \pm F_{6l}, 6l)$, $l \geq 1$, when $6l - 1$ are prime numbers, F_{6l+1} is a product of two different primes, and L_{6l} is the Lucas number.

6.4 Diophantine Representations of the Sequences Fibonacci, Lucas, and Pell

In this section we will consider some special cases of the Diophantine equation

$$x^2 + axy - y^2 = b \tag{6.4.1}$$

where a and b are integers and we will show that all nontrivial positive solutions to (6.4.1) are representable by pairs of consecutive terms in the sequences $(F_n)_{n\geq 1}$, $(L_n)_{n\geq 1}$, $(P_n)_{n\geq 1}$. These results are given in [47] but the method used there is different and more complicated. Note that this equation is a special case of (4.8.1).

Theorem 6.4.1. *(i) The nontrivial positive integer solutions to the equation*

$$x^2 + xy - y^2 = -1 \tag{6.4.2}$$

are given by (F_{2n}, F_{2n+1}), $n \geq 1$.

(ii) The nontrivial positive integer solutions to the equation

$$x^2 + xy - y^2 = 1 \tag{6.4.3}$$

are given by (F_{2n-1}, F_{2n}), $n \geq 1$.

Proof. (i) The equation is equivalent to

$$(2x + y)^2 - 5y^2 = -4.$$

This is a special Pell's equation of the form $u^2 - 5v^2 = -4$ and has solution

$$\frac{1}{2}(u_m + v_m\sqrt{5}) = \left(\frac{1+\sqrt{5}}{2}\right)^m, \quad m = 3, 5, \ldots$$

(see Theorem 4.4.1). It follows that

$$u_m = \left(\frac{1+\sqrt{5}}{2}\right)^m + \left(\frac{1-\sqrt{5}}{2}\right)^m$$

and

$$v_m = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^m - \left(\frac{1-\sqrt{5}}{2}\right)^m\right], \quad m = 3, 5, \ldots$$

Hence $y_m = v_m$ and

$$x_m = \frac{1}{2}(u_m - v_m) = \frac{1}{2}\left[\left(1 - \frac{1}{\sqrt{5}}\right)\left(\frac{1+\sqrt{5}}{2}\right)^m + \left(1 + \frac{1}{\sqrt{5}}\right)\left(\frac{1-\sqrt{5}}{2}\right)^m\right]$$
$$= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{m-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{m-1}\right], \; m = 3, 5, \ldots$$

Thus $(x_n, y_n) = (F_{2n}, F_{2n+1})$, $n \geq 1$.

(ii) Similarly, we obtain the equivalent equation

$$(2x + y)^2 - 5y^2 = 4$$

which is a special Pell's equation of the form $u^2 - 5v^2 = 4$ and has solution

$$u_n = \left(\frac{3+\sqrt{5}}{2}\right)^n + \left(\frac{3-\sqrt{5}}{2}\right)^n,$$
$$v_n = \frac{1}{\sqrt{5}}\left[\left(\frac{3+\sqrt{5}}{2}\right)^n - \left(\frac{3-\sqrt{5}}{2}\right)^n\right],$$

where $n \geq 1$ (see (4.3.2)).

It follows that

$$y_n = v_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{2n} - \left(\frac{1-\sqrt{5}}{2}\right)^{2n}\right] = F_{2n}$$

and

$$x_n = \frac{1}{2}(u_n - v_n) = \frac{1}{2}\left[\left(1 - \frac{1}{\sqrt{5}}\right)\left(\frac{1+\sqrt{5}}{2}\right)^{2n} + \left(1 + \frac{1}{\sqrt{5}}\right)\left(\frac{1-\sqrt{5}}{2}\right)^{2n}\right]$$
$$= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{2n-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{2n-1}\right] = F_{2n-1}.$$

□

Theorem 6.4.2. *(i) The nontrivial positive integer solutions to the equation*

$$x^2 + xy - y^2 = -5 \tag{6.4.4}$$

are given by (L_{2n-1}, L_{2n}), $n \geq 1$.
(ii) The nontrivial positive integer solutions to the equation

$$x^2 + xy - y^2 = 5 \tag{6.4.5}$$

are given by (L_{2n}, L_{2n+1}), $n \geq 1$.

Proof. Recall that the general term of the Lucas sequence is given by

$$L_m = \left(\frac{1+\sqrt{5}}{2}\right)^m + \left(\frac{1-\sqrt{5}}{2}\right)^m, \quad m \geq 1. \tag{6.4.6}$$

(i) Write the equation in the equivalent form

$$(2x+y)^2 - 5y^2 = -20$$

and let $2x + y = 5u$, $y = v$. We obtain the special Pell's equation $v^2 - 5u^2 = 4$, whose solutions are

$$v_n = \left(\frac{3+\sqrt{5}}{2}\right)^n + \left(\frac{3-\sqrt{5}}{2}\right)^n,$$
$$u_n = \frac{1}{\sqrt{5}}\left[\left(\frac{3+\sqrt{5}}{2}\right)^n - \left(\frac{3-\sqrt{5}}{2}\right)^n\right], \quad n \geq 1.$$

It follows that

$$y_n = v_n = \left(\frac{1+\sqrt{5}}{2}\right)^{2n} + \left(\frac{1-\sqrt{5}}{2}\right)^{2n} = L_{2n}$$

and

$$x_n = \frac{1}{2}(5u_n - v_n) = \left[(\sqrt{5}-1)\left(\frac{1+\sqrt{5}}{2}\right)^{2n} - (\sqrt{5}+1)\left(\frac{1-\sqrt{5}}{2}\right)^{2n}\right]$$
$$= \left(\frac{1+\sqrt{5}}{2}\right)^{2n-1} + \left(\frac{1-\sqrt{5}}{2}\right)^{2n-1} = L_{2n-1}.$$

(ii) Similarly, the equivalent equation $(2x+y)^2 - 5y^2 = 20$ reduces to $v^2 - 5u^2 = -4$, where $2x + y = 5u$ and $y = v$. We have

$$\frac{1}{2}(v_m + u_m\sqrt{5}) = \left(\frac{1+\sqrt{5}}{2}\right)^m, \quad m = 1, 3, 5, \ldots$$

(see Theorem 4.4.1), hence

$$y_m = v_m = \left(\frac{1+\sqrt{5}}{2}\right)^m + \left(\frac{1-\sqrt{5}}{2}\right)^m = L_m, \quad m = 1, 3, 5, \ldots$$

and

$$x_m = \frac{1}{2}(5u_m - v_m) = \frac{1}{2}\left[(\sqrt{5}-1)\left(\frac{1+\sqrt{5}}{2}\right)^m - (\sqrt{5}+1)\left(\frac{1-\sqrt{5}}{2}\right)^m\right]$$
$$= \left(\frac{1+\sqrt{5}}{2}\right)^{m-1} + \left(\frac{1-\sqrt{5}}{2}\right)^{m-1} = L_{m-1}, \quad m = 1, 3, 5, \ldots$$

□

Theorem 6.4.3. *(i) The nontrivial positive integer solutions to the equation*

$$x^2 + 2xy - y^2 = -1 \tag{6.4.7}$$

are given by (P_{2n}, P_{2n+1}), $n \geq 0$.

(ii) The nontrivial positive integer solutions to the equation

$$x^2 + 2xy - y^2 = 1$$

are given by (P_{2n-1}, P_{2n}), $n \geq 1$.

Proof. The general term of the Pell's sequence is given by

$$P_m = \frac{1}{2\sqrt{2}}[(1+\sqrt{2})^m - (1-\sqrt{2})^m], \quad m \geq 1. \tag{6.4.8}$$

(i) Write the equation in the equivalent form $(x+y)^2-2y^2=-1$. This is a negative Pell's equation of the form $u^2-2v^2=-1$, whose solutions are given by

$$u_n=\frac{1}{2}[(1+\sqrt{2})^{2n+1}+(1-\sqrt{2})^{2n+1}]$$

and

$$v_n=\frac{1}{2\sqrt{2}}[(1+\sqrt{2})^{2n+1}-(1-\sqrt{2})^{2n+1}],\quad n\geq 0.$$

It follows that

$$y_n=v_n=P_{2n+1}$$

and

$$\begin{aligned}x_n-u_n-v_n&=\frac{1}{2}\left[\left(1-\frac{1}{\sqrt{2}}\right)(1+\sqrt{2})^{2n+1}+\left(1+\frac{1}{\sqrt{2}}\right)(1-\sqrt{2})^{2n+1}\right]\\&=\frac{1}{2\sqrt{2}}[(1+\sqrt{2})^{2n}-(1-\sqrt{2})^{2n}]=P_{2n}.\end{aligned}$$

(ii) We obtain the Pell's equation $(x+y)^2-2y^2=1$, whose solutions are

$$x_n+y_n=\frac{1}{2}[(1+\sqrt{2})^{2n}+(1-\sqrt{2})^{2n}],$$

$$y_n=\frac{1}{2\sqrt{2}}[(1+\sqrt{2})^{2n}-(1-\sqrt{2})^{2n}],\quad n\geq 1.$$

It follows that $y_n=P_{2n}$ and

$$\begin{aligned}x_n&=\frac{1}{2}\left[\left(1-\frac{1}{\sqrt{2}}\right)(1+\sqrt{2})^{2n}+\left(1+\frac{1}{\sqrt{2}}\right)(1-\sqrt{2})^{2n}\right]\\&=\frac{1}{2\sqrt{2}}[(1+\sqrt{2})^{2n-1}-(1-\sqrt{2})^{2n-1}]=P_{2n-1}.\end{aligned}$$

□

The results in Theorems 6.4.1–6.4.3 can be summarized in the following Theorem proven by the infinite descent method in [47].

Theorem 6.4.4. *Let a be a positive integer and let $(\alpha_n)_{n\geq 1}$ be the sequence defined recursively by*

$$\begin{cases}\alpha_1=1,\ \alpha_2=a\\ \alpha_{n+1}=a\alpha_n+\alpha_{n-1},\ n\geq 2.\end{cases}\tag{6.4.9}$$

Then all positive integer solutions to the equation

$$|x^2 + axy - y^2| = 1 \tag{6.4.10}$$

are given by (α_n, α_{n+1}), $n \geq 1$.

Proof. The general term of the sequence $(\alpha_n)_{n\geq 1}$ in (6.4.9) is given by

$$\alpha_n = \frac{1}{\sqrt{a^2+4}}\left[\left(\frac{a+\sqrt{a^2+4}}{2}\right)^n - \left(\frac{a-\sqrt{a^2+4}}{2}\right)^n\right], \quad n \geq 1. \tag{6.4.11}$$

The equation $x^2+axy-y^2 = -1$ is equivalent to $(2x+ay)^2-(a^2+4)y^2 = -4$, which is a special Pell's equation of the form $u^2 - (a^2+4)v^2 = -4$. From Theorem 4.4.1 it follows that

$$\frac{1}{2}(u_m + v_m\sqrt{a^2+4}) = \left(\frac{a+\sqrt{a^2+4}}{2}\right)^m, \quad m = 1, 3, 5, \ldots$$

Hence

$$u_m = \left(\frac{a+\sqrt{a^2+4}}{2}\right)^m + \left(\frac{a-\sqrt{a^2+4}}{2}\right)^m$$

and

$$v_m = \frac{1}{\sqrt{a^2+4}}\left[\left(\frac{a+\sqrt{a^2+4}}{2}\right)^m - \left(\frac{a-\sqrt{a^2+4}}{2}\right)^m\right].$$

Therefore $y_m = v_m = \alpha_m$, $m = 1, 3, 5, \ldots$, and

$$x_m = \frac{1}{2}(u_m - av_m)$$

$$= \frac{1}{2}\left[\left(1-\frac{a}{\sqrt{a^2+4}}\right)\left(\frac{a+\sqrt{a^2+4}}{2}\right)^m + \left(1+\frac{a}{\sqrt{a^2+4}}\right)\left(\frac{a-\sqrt{a^2+4}}{2}\right)^m\right]$$

$$= \frac{1}{\sqrt{a^2+4}}\left[\left(\frac{a+\sqrt{a^2+4}}{2}\right)^{m-1} - \left(\frac{a-\sqrt{a^2+4}}{2}\right)^{m-1}\right] = \alpha_{m-1}.$$

The equation $x^2+axy-y^2 = 1$ is equivalent to $(2x+ay)^2-(a^2+4)y^2 = 4$. From Theorem 4.4.1 it follows that the general solution to the equation $u^2-(a^2+4)v^2 = 4$ is given by

$$\frac{1}{2}(u_n + v_n\sqrt{a^2+4}) = \left(\frac{a^2+2+a\sqrt{a^2+4}}{2}\right)^n = \left(\frac{a+\sqrt{a^2+4}}{2}\right)^{2n}.$$

We obtain

$$u_n = \left(\frac{a+\sqrt{a^2+4}}{2}\right)^{2n} + \left(\frac{a-\sqrt{a^2+4}}{2}\right)^{2n},$$

$$v_n = \frac{1}{\sqrt{a^2+4}}\left[\left(\frac{a+\sqrt{a^2+4}}{2}\right)^{2n} - \left(\frac{a-\sqrt{a^2+4}}{2}\right)^{2n}\right].$$

Hence $y_n = v_n = \alpha_{2n}$ and

$$x_n = \frac{1}{2}(u_n - av_n)$$

$$= \frac{1}{2}\left[\left(1-\frac{a}{\sqrt{a^2+4}}\right)\left(\frac{a+\sqrt{a^2+4}}{2}\right)^{2n} + \left(1+\frac{a}{\sqrt{a^2+4}}\right)\left(\frac{a-\sqrt{a^2+4}}{2}\right)^{2n}\right]$$

$$= \frac{1}{\sqrt{a^2+4}}\left[\left(\frac{a+\sqrt{a^2+4}}{2}\right)^{2n-1} - \left(\frac{a-\sqrt{a^2+4}}{2}\right)^{2n-1}\right] = \alpha_{2n-1}.$$

□

Remarks. 1) Theorem 6.4.4 characterizes the pairs of consecutive terms of the sequence $(\alpha_n)_{n\geq 1}$ defined by the linear recurrence (6.4.9).

2) The set consisting of α_{2k}, $k \geq 1$, is included in the set of positive values of the polynomial

$$P_1(x,y) = x[1-(x^2+axy-y^2-1)^2]$$

and the set consisting of α_{2k+1}, $k \geq 0$, is included in the set of positive values of the polynomial

$$P_2(x,y) = x[1-(x^2+axy-y^2+1)^2].$$

3) The result in Theorem 6.4.4 also appears in the paper [134].

6.5 Diophantine Representations of Generalized Lucas Sequences

We define the *generalized Lucas sequence* $(\gamma_n)_{n\geq 0}$, $\gamma_n = \gamma_n(a,b)$, with parameters $a,b \in \mathbb{Z}^*$ by

$$\gamma_{n+1} = f(a,b)\gamma_n - \gamma_{n-1}, \quad n \geq 1 \tag{6.5.1}$$

where $f : \mathbb{Z}^* \times \mathbb{Z}^* \to \mathbb{Z}$ is a given function and $\gamma_0 = \gamma_0(a,b)$, $\gamma_1 = \gamma_1(a,b)$ are given integers.

The following theorems generalize all results in Section 6.4.

Theorem 6.5.1. *Let a,b be nonzero integers such that $b \neq 1$ and $a^2 - 4b > 0$ is a nonsquare. All integral solutions to the equation*

$$x^2 + axy + by^2 = 1 \tag{6.5.2}$$

are given by $(\alpha_n, \beta_n)_{n\geq 1}$, $(-\alpha_n, -\beta_n)_{n\geq 1}$, where $(\alpha_n)_{n\geq 1}$, $(\beta_n)_{n\geq 1}$ are the generalized Lucas sequences defined by

$$\begin{aligned} &\alpha_{n+1} = u_0\alpha_n - \alpha_{n-1},\ \alpha_0 = 2,\ \alpha_1 = u_0 \textit{ and} \\ &\beta_{n+1} = u_0\beta_n - \beta_{n-1},\ \beta_0 = 1,\ \beta_1 = \frac{1}{2}(u_0 - av_0). \end{aligned} \tag{6.5.3}$$

Here $u_0 = u_0(a,b)$, $v_0 = v_0(a,b)$ are the minimal solutions to the special Pell's equation

$$u^2 - (a^2 - 4b)v^2 = 4. \tag{6.5.4}$$

Proof. The general terms of the sequences $(\alpha_n)_{n\geq 0}$ and $(\beta_n)_{n\geq 0}$ are given by

$$\alpha_n = \left(\frac{u_0 + v_0\sqrt{a^2-4b}}{2}\right)^n + \left(\frac{u_0 - v_0\sqrt{a^2-4b}}{2}\right)^n$$

and

$$\begin{aligned} \beta_n = \frac{1}{2}\Bigg[&\left(1 - \frac{a}{\sqrt{a^2-4b}}\right)\left(\frac{u_0 + v_0\sqrt{a^2-4b}}{2}\right)^n \\ &+ \left(1 + \frac{a}{\sqrt{a^2-4b}}\right)\left(\frac{u_0 - v_0\sqrt{a^2-4b}}{2}\right)^n\Bigg]. \end{aligned}$$

The equation (6.5.2) is equivalent to $(2x + ay)^2 - (a^2 - 4b)y^2 = 4$, i.e., to the general Pell's equation (6.5.3). From Theorem 4.4.1 its general solution is given by

$$\frac{1}{2}(u_n + v_n\sqrt{a^2 - 4b}) = \left(\frac{u_0 + v_0\sqrt{a^2 - 4b}}{2}\right)^n, \quad n \geq 1.$$

It follows that

$$u_n = \left(\frac{u_0 + v_0\sqrt{a^2 - 4b}}{2}\right)^n + \left(\frac{u_0 - v_0\sqrt{a^2 - 4b}}{2}\right)^n$$

and

$$v_n = \frac{1}{\sqrt{a^2 - 4b}}\left[\left(\frac{u_0 + v_0\sqrt{a^2 - 4b}}{2}\right)^n - \left(\frac{u_0 - v_0\sqrt{a^2 - 4b}}{2}\right)^n\right], \quad n \geq 1.$$

Thus $y_n = v_n = \alpha_n$ and $x_n = \frac{1}{2}(u_n - av_n) = \beta_n$. $\square$

Remark. Theorems 6.5.1 and 6.5.2 give an useful method for solving the Diophantine equations of degree three in four variables

$$x^2 + uxy + vy^2 = \pm 1.$$

Indeed, setting $u = a$, $v = b$, with $a, b \in \mathbb{Z}$, the above equations are equivalent to

$$(2x + ay)^2 - (a^2 - 4b)y^2 = \pm 4.$$

If $a^2 - 4b < 0$, there are at most finitely many solutions.

If $a^2 - 4b = 0$, the equations reduce to $(2x + ay)^2 = \pm 4$, and for a even we obtain solutions $\left(-\frac{ka}{2} \pm 1, k\right)$, $k \in \mathbb{Z}$.

If $a^2 - 4b > 0$ is a perfect square, there are at most finitely many solutions.

If $a^2 - 4b > 0$ is not a square, then all solutions to the equation $x^2 + uxy + vy^2 = 1$ are given by $(x, y, u, v) = (\pm\alpha_m, \pm\beta_m, a, b)$, $m \geq 1$, where (α_m), (β_m) are the generalized Lucas sequences defined in Theorem 6.5.1.

All solutions to the equation $x^2 + uxy + vy^2 = -1$ are given by

$$(x, y, u, v) = (\pm\alpha_{2n+1}, \pm\beta_{2n+1}, a, b), \quad n \geq 0,$$

where (α_m), (β_m) are defined in Theorem 6.5.2.

For some particular values of a and b the generalized Lucas sequences $(\alpha_n)_{n\geq 0}$ and $(\beta_n)_{n\geq 0}$ defined by (6.5.3) coincide with some classical sequences. In the following table we will give a few such situations (see also Section 6.4).

a	b	Equation (6.5.2)	Solutions
1	-1	$x^2+xy-y^2=1$	$(F_{2n-1},F_{2n}),(-F_{2n-1},-F_{2n})$
-1	-1	$x^2-xy-y^2=1$	$(F_{2n+1},F_{2n}),(-F_{2n+1},-F_{2n})$
2	-1	$x^2+2xy-y^2=1$	$(P_{2n-1},P_{2n}),(-P_{2n-1},-P_{2n})$
-2	-1	$x^2-2xy-y^2=1$	$(P_{2n+1},P_{2n}),(-P_{2n+1},-P_{2n})$

Theorem 6.5.2. *Let a,b be nonzero integers such that $b\neq 1$ and $a^2-4b>0$ is a nonsquare. Assume that the special Pell's equation*

$$s^2-(a^2-4b)t^2=-4 \tag{6.5.5}$$

is solvable and its minimal solution is (s_0,t_0), $s_0=s_0(a,b)$, $t_0=t_0(a,b)$. Then all integral solutions to the equation

$$x^2+axy+by^2=-1 \tag{6.5.6}$$

are given $(\alpha_{2n+1},\beta_{2n+1})$, $(-\alpha_{2n+1},-\beta_{2n+1})$, $n\geq 1$, where $(\alpha_m)_{m\geq 0}$, $(\beta_m)_{m\geq 0}$ are the generalized Lucas sequences defined by

$$\begin{aligned}&\alpha_{m+1}=s_0\alpha_m-\alpha_{m-1},\ \alpha_0=2,\ \alpha_1=s_0 \textit{ and}\\ &\beta_{m+1}=s_0\beta_m-\beta_{m-1},\ \beta_0=1,\ \beta_1=\frac{1}{2}(s_0-at_0).\end{aligned} \tag{6.5.7}$$

Proof. We proceed like in the previous theorem and take into account the results in Theorem 4.4.1 concerning the general solution to the equation (6.5.5). □

In some special cases, the generalized Lucas sequences defined by (6.5.7) yield to solutions involving well-known sequences. We will illustrate this by presenting the following table (see also Section 6.5).

a	b	Equation (6.5.6)	Solutions
1	-1	$x^2+xy-y^2=-1$	$(F_{2n},F_{2n+1}),(-F_{2n},-F_{2n+1})$
-1	-1	$x^2-xy-y^2=-1$	$(F_{2n},F_{2n-1}),(-F_{2n},-F_{2n-1})$
2	-1	$x^2+2xy-y^2=-1$	$(P_{2n},P_{2n+1}),(-P_{2n},-P_{2n+1})$
-2	-1	$x^2-2xy-y^2=-1$	$(P_{2n},P_{2n-1}),(-P_{2n},-P_{2n-1})$
5	5	$x^2+5xy+5y^2=-1$	$(L_{2n},-F_{2n+1}),(-L_{2n},F_{2n+1})$

Remarks. 1) The solvability condition for the general Pell's equation (6.5.5) in Theorem 6.5.2 is necessary. Indeed, for example, for $a = 5$, $b = 1$ the equation

$$s^2 - 21t^2 = -4$$

is not solvable (the left-hand side is congruent to 0 or 1 $\pmod 3$). The corresponding equation (6.5.6):

$$x^2 + 5xy - y^2 = -1$$

is also not solvable.

2) The special case $b = -1$ is studied in [134]. Particular Diophantine representations for the Fibonacci and Lucas sequences are given in [96] and [97]. We also mention the connection with the general Pell's equation given in [61]. A particular definition for generalized Lucas sequences appears in [91].

An interesting special case for the equation (6.5.2) is $a = b$. We obtain the Diophantine equation

$$x^2 + axy + ay^2 = 1. \tag{6.5.8}$$

The general Pell's equation (6.5.4) becomes $u^2 - (a^2 - 4a)v^2 = 4$, whose minimal solution is $(u_0, v_0) = (a - 2, 1)$.

With the notations in Theorem 6.5.1 the generalized Lucas sequences $(\alpha_n)_{n\geq 0}$, $(\beta_n)_{n\geq 0}$ are given by

$$\alpha_{n+1} = (a-2)\alpha_n - \alpha_{n-1}, \quad \alpha_0 = 2, \quad \alpha_1 = a - 2$$
$$\beta_{n+1} = (a-2)\beta_n - \beta_{n-1}, \quad \beta_0 = 1, \quad \beta_1 = -1.$$

From Theorem 6.5.1 we obtain the following Corollary:

Corollary 6.5.3. *The equation (6.5.8) is always solvable and all of its solutions are given by* $(\alpha_n, \beta_n)_{n\geq 0}$.

Next we study when the solutions to the equation (6.5.8) are linear combinations over $\mathbb{Q}$ of the classical Fibonacci and Lucas sequences. The results are obtained in the paper [20].

For other results we refer to the papers [61, 91, 96–98] and [134]. Also, the problem is connected to the Y.V. Matiasevich and J. Robertson way to solve the Hilbert's Tenth Problem, and it has applications to the problem of singlefold Diophantine representation of recursively enumerable sets. In the recent paper [102] the equations $x^2 - kxy + y^2 = 1$, $x^2 - kxy - y^2 = 1$ are solved in terms of generalized Fibonacci and Lucas numbers. Let us mention that in the paper [83] is defined the Hankel matrices involving the Pell, Pell-Lucas and modified Pell sequences, is computed their Frobenius norm, and it is investigated some spectral properties of them.

Recall the Binet's formulas for F_n and L_n:

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right],$$
$$L_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n.$$

These formulas can be extended to negative integers n in a natural way. We have $F_{-n} = (-1)^{n-1}F_n$ and $L_{-n} = (-1)^n L_n$, for all n.

Theorem 6.5.4. *The solutions to the positive equation (6.5.8) are linear combinations with rational coefficients of at most two Fibonacci and Lucas numbers if and only if $a = a_n = \pm L_{2n} + 2$, $n \geq 1$.*

For each n, all of its integer solutions (x_k, y_k) are given by

$$\begin{cases} x_k = \dfrac{\varepsilon_k}{2} L_{2kn} \mp \dfrac{a_n}{2F_{2n}} F_{2kn} \\ y_k = \pm \dfrac{1}{F_{2n}} F_{2kn}, \end{cases} \tag{6.5.9}$$

where $k \geq 1$, signs $+$ and $-$ depend on k and correspond, while $\varepsilon_k = \pm 1$.

Proof. The equation $x^2 + axy + ay^2 = 1$ is equivalent to the positive special Pell's equation

$$(2x + ay)^2 - (a^2 - 4a)y^2 = 4. \tag{6.5.10}$$

From formula (4.4.6) it follows that

$$2x_n + ay_m = \varepsilon_m \left[\left(\frac{u_1 + v_1\sqrt{D}}{2}\right)^m + \left(\frac{u_1 - v_1\sqrt{D}}{2}\right)^m\right]$$

and

$$y_m = \frac{\varepsilon_m}{\sqrt{D}}\left[\left(\frac{u_1 + v_1\sqrt{D}}{2}\right)^m - \left(\frac{u_1 - v_1\sqrt{D}}{2}\right)^m\right],$$

where $m \in \mathbb{Z}$, $\varepsilon_m = \pm 1$, $D = a^2 - 4a$, and (u_1, v_1) is the minimal positive solution to $u^2 - Dv^2 = 4$. we have $(u_1, v_1) = (a - 2, 1)$, and combining the above relations it follows

$$x_m = \frac{\varepsilon_m}{2}\left[\left(1 - \frac{a}{\sqrt{a^2-4a}}\right)\left(\frac{a-2+\sqrt{a^2-4a}}{2}\right)^m + \left(1 + \frac{a}{\sqrt{a^2-4a}}\right)\left(\frac{a-2-\sqrt{a^2-4a}}{2}\right)^m\right] \tag{6.5.11}$$

and

$$y_m = \frac{\varepsilon_m}{\sqrt{a^2-4a}}\left[\left(\frac{a-2+\sqrt{a^2-4a}}{2}\right)^m - \left(\frac{a-2-\sqrt{a^2-4a}}{2}\right)^m\right]. \tag{6.5.12}$$

Taking into account Binet's formulas, solution (x_m, y_m) is representable in terms of F_m and L_m only if $a^2 - 4a = 5s^2$, for some positive integer s. This is equivalent to the special Pell's equation

$$(a-2)^2 - 5s^2 = 4, \tag{6.5.13}$$

whose minimal solution is $(a_1 - 2, s_1) = (3, 1)$. The general integer solution to (6.5.13) is

$$a_n - 2 = \varepsilon_n\left[\left(\frac{3+\sqrt{5}}{2}\right)^n + \left(\frac{3-\sqrt{5}}{2}\right)^n\right] = \varepsilon_n L_{2n},$$

and

$$s_n = \frac{\varepsilon_n}{\sqrt{5}}\left[\left(\frac{3+\sqrt{5}}{2}\right)^n - \left(\frac{3-\sqrt{5}}{2}\right)^n\right] = \varepsilon_n F_{2n},$$

where n is an integer and $\varepsilon_n = \pm 1$.

From $(2x + ay)^2 - (a^2 - 4a)y^2 = 4$ we find $(2x + a_n y)^2 - 5(s_n y)^2 = 4$, with integer solution (x_m, y_m) given by

$$2x_m + a_n y_m = \varepsilon_{2m} L_{2m} \quad \text{and} \quad s_n y_m = \pm F_{2m}.$$

Hence

$$x_m = \frac{1}{2}\left[\varepsilon_{2m} L_{2m} \mp a_n \frac{F_{2m}}{F_{2n}}\right], \quad y_m = \pm \frac{F_{2m}}{F_{2n}}, \tag{6.5.14}$$

where signs + and $-$ correspond, and $\varepsilon_{2m} = \pm 1$.

Taking into account that F_{2n} divides F_{2m} if and only if n divides m (see [21, p. 180] and [90, p. 39]), it is necessary that $m = kn$, for some positive integer k. Formula (6.5.14) becomes (6.5.9).

A parity argument shows that in the equation

$$(x+ay)^2-(a^2-4a)y^2=4,$$

X is even, so x_k in (6.5.9) is always an integer. □

The following two tables give the integer solutions to equation (6.5.8) at level k, including the trivial solution obtained for $k=0$.

n	$a_n = L_{2n}+2$	Equation (6.5.8)	Solutions
1	5	$x^2+5xy+5y^2=1$	$x=\frac{\varepsilon_k}{2}L_{2k}\mp\frac{5}{2}F_{2k},\ y=\pm F_{2k}$
2	9	$x^2+9xy+9y^2=1$	$x=\frac{\varepsilon_k}{2}L_{4k}\mp\frac{5}{6}F_{4k},\ y=\pm\frac{1}{3}F_{4k}$
3	20	$x^2+20xy+20y^2=1$	$x=\frac{\varepsilon_k}{2}L_{6k}\mp\frac{5}{4}F_{6k},\ y=\pm\frac{1}{8}F_{6k}$
4	49	$x^2+49xy+49y^2=1$	$x=\frac{\varepsilon_k}{2}L_{8k}\mp\frac{7}{6}F_{8k},\ y=\pm\frac{1}{21}F_{8k}$
5	125	$x^2+125xy+125y^2=1$	$x=\frac{\varepsilon_k}{2}L_{10k}\mp\frac{25}{22}F_{10k},\ y=\pm\frac{1}{55}F_{10k}$
6	324	$x^2+324xy+324y^2=1$	$x=\frac{\varepsilon_k}{2}L_{12k}\mp\frac{9}{8}F_{12k},\ y=\pm\frac{1}{144}F_{12k}$

n	$a_n = -L_{2n}+2$	Equation (6.5.8)	Solutions
1	-1	$x^2-xy-y^2=1$	$x=\frac{\varepsilon_k}{2}L_{2k}\pm\frac{1}{2}F_{2k},\ y=\pm F_{2k}$
2	-5	$x^2-5xy-5y^2=1$	$x=\frac{\varepsilon_k}{2}L_{4k}\pm\frac{5}{6}F_{4k},\ y=\pm\frac{1}{3}F_{4k}$
3	-16	$x^2-16xy-16y^2=1$	$x=\frac{\varepsilon_k}{2}L_{6k}\pm F_{6k},\ y=\pm\frac{1}{8}F_{6k}$
4	-45	$x^2-45xy-45y^2=1$	$x=\frac{\varepsilon_k}{2}L_{8k}\pm\frac{15}{14}F_{8k},\ y=\pm\frac{1}{21}F_{8k}$
5	-121	$x^2-121xy-121y^2=1$	$x=\frac{\varepsilon_k}{2}L_{10k}\mp\frac{11}{10}F_{10k},\ y=\pm\frac{1}{55}F_{10k}$
6	-320	$x^2-320xy-320y^2=1$	$x=\frac{\varepsilon_k}{2}L_{12k}\mp\frac{10}{9}F_{12k},\ y=\pm\frac{1}{144}F_{12k}$

Next we will consider the "negative" equation of the type (6.5.8):

$$x^2+axy+ay^2=-1. \tag{6.5.15}$$

Unlike the result in Theorem 6.5.4, there are only two values of a for which the corresponding property holds.

Theorem 6.5.5. *The solutions to the negative equation (6.5.15) are linear combinations with rational coefficients of at most two Fibonacci and Lucas numbers if and only if $a=-1$ or $a=5$.*

If $a=-1$, all of its integer solutions (x_m, y_m) are given by

$$x_m=\frac{\varepsilon_m}{2}L_{2m+1}\pm\frac{1}{2}F_{2m+1},\quad y_m=\pm F_{2m+1},\quad m\ge 0. \tag{6.5.16}$$

If $a=5$, all integer solutions (x_m, y_m) are

$$x_m=\frac{\varepsilon_m}{2}L_{2m+1}\mp 5F_{2m+1},\quad y_m=\pm F_{2m+1},\quad m\ge 0. \tag{6.5.17}$$

The signs $+$ and $-$ depend on m and correspond, while $\varepsilon_m=\pm 1$.

Proof. As in the proof of Theorem 6.5.4 the equation is equivalent to

$$(2x + ay)^2 - (a^2 - 4a)y^2 = -4.$$

Suppose that this negative special Pell's equation is solvable. Its solution (x_m, y_m) is representable in terms of Fibonacci and Lucas numbers as a linear combination with rational coefficients only if $a^2 - 4a = 5s^2$. As in the proof of Theorem 6.5.4 we obtain $a_n = \pm L_{2n} + 2$ and $s_n = \pm F_{2n}$, $n \geq 1$.

The equation $(2x + ay)^2 - (a^2 - 4a)y^2 = -4$ becomes

$$(2x + ay)^2 - 5(s_n y)^2 = -4,$$

whose integer solutions are $2x_m + ay_m = \varepsilon_m L_{2m+1}$ and $s_n y_m = \pm F_{2m+1}$. It follows that

$$y_m = \pm \frac{F_{2m+1}}{F_{2n}}, \quad m \geq 1.$$

If $n \geq 2$, then $F_{2n} \geq 2$, and since $2n$ does not divide $2m + 1$, it follows that F_{2n} does not divide F_{2m+1} (see [21, pp. 180] and [90, pp. 39]), hence y_m is not an integer.

Thus $n = 1$ and so $a = \pm L_2 + 2$, i.e., $a = -1$ or $a = 5$.

For $a = -1$, it follows $y_m = \pm F_{2m+1}$ and $2x_m - y_m = \varepsilon_m L_{2m+1}$, and we obtain solutions (6.5.16).

If $a = 5$, then $y_m = \pm F_{2m+1}$ and $2x_m + 5y_m = \varepsilon_m L_{2m+1}$, yielding the solutions (6.5.17). □

Remark. On the other hand, it is more or less known Zeckendorf's theorem in [230], which states that every positive integer can be represented uniquely as the sum of one or more distinct Fibonacci numbers in such a way that the sum does not include two consecutive Fibonacci numbers. Such a sum is called *Zeckendorf representation* and it is related to the Fibonacci coding of a positive integer. Our results are completely different, because the number of terms is reduced to at most two, and the sum in the representation of solutions is a linear combination with rational coefficients.