Chapter 5
Equations Reducible to Pell's Type Equations

5.1 The Equations $x^2 - kxy^2 + y^4 = 1$ and $x^2 - kxy^2 + y^4 = 4$

An interesting problem concerning the Pell's equation $u^2 - Dv^2 = 1$ is to study when the second component of a solution (u, v) is a perfect square. This question is equivalent to solving the equation

$$X^2 - DY^4 = 1. \tag{5.1.1}$$

The equation (5.1.1) was intensively studied in a series of papers (see [116–118]). We begin this section by mentioning the main result about the above equation.

Theorem 5.1.1. *For a positive nonsquare integer D there are at most two solutions to the equation (5.1.1). If two solutions exist, and ε_D denotes the fundamental unit in the quadratic field $\mathbb{Q}(\sqrt{D})$, then they are given by (x_1, y_1), (x_2, y_2), $x_1 < x_2$, where $x_1 + y_1^2\sqrt{D} = \varepsilon_D$ and $x_2 + y_2^2\sqrt{D}$ is either ε_D^2 or ε_D^4, with the latter case occurring for only finitely many D.*

Following the recent paper [221] we first prove a generalization of Theorem 5.1.1. We then use this result to completely solve the equations

$$x^2 - kxy^2 + y^4 = 1 \tag{5.1.2}$$

and

$$x^2 - kxy^2 + y^4 = 4. \tag{5.1.3}$$

Let $D = e^2d$, with e an integer and d a positive squarefree integer. Then $\varepsilon_D = \dfrac{a + b\sqrt{d}}{2}$, where a and b are positive integers with the same parity, and satisfy

T. Andreescu, D. Andrica, *Quadratic Diophantine Equations*,
Developments in Mathematics 40, DOI 10.1007/978-0-387-54109-9_5

$a^2 - db^2 = (-1)^\alpha 4$, where $\alpha \in \{0,1\}$. Define $\lambda_D = \lambda_d$ to be the fundamental solution $u + v\sqrt{d}$ to $X^2 - dY^2 = 1$, with u and v positive integers. Then $\lambda_D = (\varepsilon_D)^c$, where

$$c = \begin{cases} 1 \text{ if } a \text{ and } b \text{ are even and } \alpha = 0 \\ 2 \text{ if } a \text{ and } b \text{ are even and } \alpha = 1 \\ 3 \text{ if } a \text{ and } b \text{ are odd and } \alpha = 0 \\ 6 \text{ if } a \text{ and } b \text{ are odd and } \alpha = 1. \end{cases} \tag{5.1.4}$$

Lemma 5.1.2 ([55]). *Let D be a nonsquare positive integer. If the equation $X^4 - DY^2 = 1$ is solvable in positive integers X, Y, then either $X^2 + Y\sqrt{D} = \lambda_D$ or λ_D^2. Solutions to $X^4 - DY^2 = 1$ arise from both λ_D and λ_D^2 only for $D \in \{1785, 7140, 28560\}$.*

Lemma 5.1.3. *If there are two solutions to equation (5.1.1), then they are given by $X + Y^2\sqrt{D} = \varepsilon_D$, ε_D^4 for $D \in \{1785, 28560\}$, and $X + Y^2\sqrt{D} = \varepsilon_D$, ε_D^2 otherwise.*

Proof. Let $T + U\sqrt{D}$ denote the fundamental solution in positive integers to the Pell's equation $x^2 - Dy^2 = 1$, and for $k \geq 1$ let $T_k + U_k\sqrt{D} = (T + U\sqrt{D})^k$.

If there exist two indices k_1 and k_2 for which U_{k_1} and U_{k_2} are squares, then by Theorem 5.1.1, $(k_1, k_2) = (1, 4)$ or $(k_1, k_2) = (1, 2)$. If there are integers x and y such that $U_1 = x^2$ and $U_4 = y^2$, then since $U_4 = 2T_2U_2$, there exist integers w and z such either $(T_2, U_2) = (w^2, 2z^2)$, or $(T_2, U_2) = (2w^2, z^2)$. The latter case is not possible, since it would imply the existence of three solutions to $X^2 - DY^4 = 1$, contradicting Theorem 5.1.1. In the former case, since $2z^2 = U_2 = 2T_1U_1$, there are integers u and $v > 1$ such that $T_1 = v^2$ and $U_1 = u^2$. We thus have solutions to $X^4 - DY^2 = 1$ arising from both ε_D and ε_D^2. By Lemma 5.1.2, we deduce that $D \in \{1785, 7140, 28560\}$, and since $U_1 = 2$ and $D = 7140$, we have finally that $D \in \{1785, 28560\}$. ☐

Lemma 5.1.4 ([117]). *The only positive integer solutions to the equation $X^2 - 2Y^4 = -1$ are $(X,Y) = (1,1)$, $(239, 13)$.*

Lemma 5.1.5 ([45]). *The only positive integer solutions to the equation $3X^4 - 2Y^2 = 1$ are $(X,Y) = (1,1)$, $(3,11)$.*

Lemma 5.1.6 ([38]). *With the notations in the proof to Lemma 5.1.3, if $T_k = 2x^2$ for some integer x, then $k = 1$.*

Theorem 5.1.7. *Let D be a nonsquare positive integer with $D \notin \{1785, 7140, 28560\}$. Then there are at most two positive indices k for which $U_k = 2^\delta y^2$ with y an integer and $\delta = 0$ or 1. If two solutions $k_1 < k_2$ exist, then $k_1 = 1$ and $k_2 = 2$, and provided that $D \neq 5$, $T + U\sqrt{D}$ is the fundamental unit in $\mathbb{Q}(\sqrt{D})$, or its square. For $D \in \{1785, 7140, 28560\}$, the only solutions to $U_k = 2^\delta y^2$ are $k = 1$, $k = 2$, and $k = 4$.*

Proof. If one of the equation $x^2 - Dy^4 = 1$, $x^2 - 4Dy^4 = 1$ is not solvable, then the result follows from Lemma 5.1.3 applied to $4D$ and D respectively.

Therefore we may assume that both of these equations are solvable. Let k and l be indices for which $U_k = z^2$ and $U_l = 2w^2$. It follows from the binomial theorem that not both of k and l are odd.

Assume first that k and l are both even. We will show that this leads to $D \in \{1785, 7140, 28560\}$. Letting $l = 2m$, then there are integers $u > 1$ and v such that $T_m = u^2$ and $U_m = v^2$. Then by Lemma 5.1.2, either $m = 1$ or $m = 2$. Also, by Lemma 5.1.3, and the fact that k is even, either $(k,m) = (2,1)$, $(k,m) = (4,1)$ and $D \in \{1785, 28560\}$, or else $k = m$. The first case is not possible since it would imply $k = l = 2$, and this contradicts the assumed forms of U_k and U_l. Thus, for $D \notin \{1785, 28560\}$, we have that $k = m$, and furthermore, the only possibility is $k = m = 2$. Since $U_2 = 2T_1U_1$, there are positive integers a, b for which either $(T_1, U_1) = (a^2, 2b^2)$ or $(T_1, U_1) = (2a^2, b^2)$. From the identity $T_2 = 2T_1^2 - 1$, these two possibilities yield the respective equations $u^2 = 2a^4 - 1$ or $u^2 = 8a^4 - 1$. The equation $u^2 = 8a^4 - 1$ is not solvable modulo 4. By Lemma 5.1.4, the only positive integer solution to the equation $u^2 = 2a^4 - 1$, with $u > 1$, is $u = 239$ and $a = 13$. Therefore, $T_1 = 169$, and $U_1 = 2b^2$ for some integer b. The only choice for b is $b = 1$, which results in $D = 7140$.

We can assume that k and l are of opposite parity. First assume that l is even, $l = 2m$, and that k is odd. Thus, we have that $U_{2m} = 2w^2$. From the identity $U_{2m} = 2T_mU_m$, and the fact that $(T_m, U_m) = 1$, it follows that there are integers u and v such that $T_m = u^2$ and $U_m = v^2$. By Lemma 5.1.2, either $m = 1$ or $m = 2$, and $T_1 + U_1\sqrt{D} = \lambda_D$. Furthermore, by Lemma 5.1.3, either $k = m$ or $k = 1$, $m = 2$. If $k = m$, then since k is odd and $m = 1$ or 2, we have that $k = 1$ and $l = 2$, which is our desired result. On the other hand, if $k = 1$ and $m = 2$, then $l = 4$, and we have that $U_4 = 2w^2$, $U_2 = v^2$, and $T_2 = u^2$. As in the previous paragraph, this leads to $D = 7140$.

Now assume that l is odd and k is even, $k = 2m$. Therefore, $U_{2m} = 2T_mU_m = z^2$, and it follows that there are integers u and v such that either $(T_m, U_m) = (2u^2, v^2)$ or $(T_m, U_m) = (u^2, 2v^2)$. In the first case, Lemma 5.1.3 implies that $(m,k) = (1,2)$, since U_m and U_k are both squares. Therefore U_1 is a square, and $2^{2\alpha}$ properly divides U_1 for some integer $\alpha \geqq 0$, Since $U_l = 2w^2$, $2^{2\beta+1}$ properly divides U_l for some integer $\beta \geqq 0$. From the fact that l is odd, the binomial theorem exhibits that the same power of 2 divides U_1 and U_l, thus leading to a contradiction. In the case that $(T_m, U_m) = (u^2, 2v^2)$, Lemma 5.1.2 shows that $m = 1$ or $m = 2$, and that $T_1 + U_1\sqrt{D} = \lambda_D$. Also, by Lemma 5.1.3 applied to $4D$, either $m = l$ or $(l,m) = (1,2)$. The former possibility leads to $l = 1$ and $k = 2$, which is the desired result. The latter possibility implies that $k = 4$, and that $T_2 = u^2$, $U_2 = 2v^2$, Since $U_2 = 2T_1U_1$, there are integers a and b such that $T_1 = a^2$, and $U_1 = b^2$. Therefore, $u^2 = T_2 = 2T_1^2 - 1 = 2a^4 - 1$, and by Lemma 5.1.4, it follows that $T_1 = 169$, and hence that $D = 1785$ or $D = 28560$.

It remains to prove that for $D \neq 5$, $T + U\sqrt{D} = T_1 + U_1\sqrt{D}$ is the fundamental unit ε_D in $\mathbb{Q}(\sqrt{D})$, or its square. Letting $T + U\sqrt{D} = \varepsilon_d^c$, then we need to prove that $c = 1$ or $c = 2$, where c is defined in (5.1.4).

Let $D = l^2d$ with d squarefree. Let $\lambda_d = t + u\sqrt{d}$, and for $k \geqq 1$, define $\lambda_d^k = t_k + u_k\sqrt{d}$. Then $T + U\sqrt{D} = \lambda_d^r = t_r + u_r\sqrt{d}$ for some integer r, and

$u_{ir} = lU_i$ for each $i \geqq 1$. We assume now that $U_1 = 2^{\delta_1}x^2$ and $U_2 = 2^{\delta_2}y^2$ for some integers x and y. Then $u_r = 2^{\delta_1}lx^2$ and $u_{2r} = 2^{\delta_2}ly^2$. Since $u_{2r} = 2t_r u_r$, it follows that $t_r = z^2$ or $2z^2$ for some integer z. By Lemma 5.1.3 and Lemma 5.1.6, either $r = 1$ or $r = 2$. This implies that c divides 12. We wish to show that 4 does not divide c. If 4 divides c, then $r = 2$ and $N(\varepsilon_d) = -1$, and so there are the integers $V > 1$ and W such that $V^2 - W^2 d = -1$, with $t_2 + u_2\sqrt{d} = \lambda_d^2 = (V + W\sqrt{d})^4$. Since $r = 2$, Lemma 5.1.6 shows that $t_2 = z^2$. Therefore, $t_2 = z^2 = 8V^4 + 8V^2 + 1$, and as it was shown in [117] that this equation implies $V = 0$, we have a contradiction. Therefore c divides 6, and to complete the proof of the theorem, we need to show that 3 does not divide c.

Assume that 3 divides c. Then $T + U\sqrt{D}$ is the cube of a unit in $\mathbb{Q}(\sqrt{D})$ of the form $\dfrac{a + b\sqrt{D}}{2}$, where a and b are odd, and $a^2 - b^2 D = 4$. Moreover, $T = a\left(\dfrac{a^2 - 3}{2}\right)$ is odd, and so either $T + U\sqrt{D} = X^2 + Y^2\sqrt{D}$ or $T + U\sqrt{D} = X^2 + 2Y^2\sqrt{D}$, i.e., T is not of the form $2X^2$. It follows that $a(a^2 - 3) = 2X^2$. If $(a, a^2 - 3) = 1$, then since a is odd, $a = A^2$ and $a^2 - 3 = 2B^2$ for some integers A, B, which is not possible by considering this last equation modulo 8. Therefore $(a, a^2 - 3) = 3$, and there are integers A, B for which $a = 3A^2$ and $a^2 - 3 = 6B^2$, which results in the equation $3A^4 - 2B^2 = 1$. By Lemma 5.1.5 the only positive integer solutions to this equation are $(A, B) = (1, 1)$ and $(A, B) = (3, 11)$. This shows that either $a = 3$ or $a = 27$. The case $a = 3$ yields $D = 5$, which we have excluded. The case $a = 27$ yields that either $D = 29$ or $D = 725$. It is easily checked that the hypotheses are not satisfied for both of these values of D. □

Corollary 5.1.8. *For $k = 169$, the only positive integer solutions to $x^2 - (k^2 - 1)$ $y^4 = 1$ are $(x, y) = (169, 1)$, $(6525617281, 6214)$.*

For $k > 1$ and $k \neq 169$, the only positive integer solution (x, y) to $x^2 - (k^2 - 1)$ $y^4 = 1$ is $(x, y) = (k, 1)$, unless $k = 2v^2$ for some integer v, in which case $(x, y) = (8v^4 - 1, 2v)$ is the only other solution.

For $k > 1$ there is no positive integer solutions (x, y) to $x^2 - (k^2 - 1)y^4 = 4$, unless $k = v^2$ for some integer v, in which case $(x, y) = (4v^4 - 2, 2v)$ is the only solution.

Proof. The particular case $k = 169$ is easily verified for both equations, and so we assume that $k > 1$ and $k \neq 169$. The fundamental solution to $x^2 - (k^2 - 1)y^2 = 1$ is $(k, 1)$. For $i \geq 1$ define $T_i + U_i\sqrt{k^2 - 1} = (k + \sqrt{k^2 - 1})^i$. There is always the solution $(x, y) = (k, 1)$ to $x^2 - (k^2 - 1)y^4 = 1$, and so by Theorem 5.1.7, if there is another solution, it must come from $T_2 + U_2\sqrt{k^2 - 1} = 2k^2 - 1 + 2k\sqrt{k^2 - 1}$, i.e., $(x, y) = (2k^2 - 1, \sqrt{2k})$. This entails that $2k$ is a perfect square, and hence that $k = 2v^2$ for some integer v, This gives $(x, y) = (8v^4 - 1, 2v)$.

We note that if k is odd, then the minimal solution to $x^2 - \left(\dfrac{k^2 - 1}{4}\right)y^2 = 1$ is $(x, y) = (k, 2)$, from which it follows that for k even or odd, any solution to $x^2 - (k^2 - 1)y^2 = 4$ has both x and y even. Now let (x, y) be a positive integer solution to $x^2 - (k^2-1)y^4 = 4$, then x and y are even, and $(u, v) = (x/2, y/2)$ is a positive integer solution to $u^2 - 4(k^2-1)v^4 = 1$, and hence there is a positive integer i for which $U_i = 2v^2$. By Theorem 5.1.7, since $U_1 = 1$ is already a square, $i = 2$. Therefore $u + 2v^2\sqrt{k^2-1} = T_2 + U_2\sqrt{k^2-1} = 2k^2 - 1 + 2k\sqrt{k^2-1}$, and hence $k = v^2$. This leads to the solution $(x, y) = (4v^4 - 2, 2v)$ to the equation $x^2 - (k^2-1)y^4 = 4$. This completes the proof. □

Theorem 5.1.9. *Let k be an even positive integer.*

1) *The only solutions to equation (5.1.2) in nonnegative integers* (x, y) *are* $(k, 1)$, $(1, 0)$, $(0, 1)$, *unless either k is a perfect square, in which case there are also the solutions* $(1, \sqrt{k})$, $(k^2-1, \sqrt{k})$, *or* $k = 338$ *in which case there are the solutions* $(x, y) = (114243, 6214)$, $(13051348805, 6214)$.
2) *The only solution in nonnegative integers* x, y *to the equation (5.1.3) is* $(x, y) = (2, 0)$, *unless* $k = 2v^2$ *for some integer* v, *in which case there are also the solutions* $(2, \sqrt{2k})$, $(2k^2-2, \sqrt{2k})$.

Proof. Letting $k = 2s$, then we can rewrite the equation $x^2 - kxy^2 + y^4 = 1$ as

$$(x - sy^2)^2 - (s^2-1)y^4 = 1.$$

Aside from the trivial solution $(x, y) = (1, 0)$, Corollary 5.1.8 implies that the only solutions are $y = 1$, $x - sy^2 = \pm s$, unless $s = 2v^2$ for some integer v, in which case there is also the solutions $y = 2v$ and $x - sy^2 = \pm(8v^4 - 1)$, or $k = 338$. In either case, the solutions listed in Corollary 5.1.8 lead to the solutions given in Theorem 5.1.9.

The equation $x^2 - kxy^2 + y^4 = 4$ can be rewritten as

$$(x - sy^2)^2 - (s^2-1)y^4 = 4.$$

Corollary 5.1.8 shows that, aside from the trivial solution $(x, y) = (2, 0)$, there is no solution in positive integers unless $s = v^2$ for some integer v, in which case $y = 2v$ and $x - sy^2 = \pm 4v^4 - 2$. It follows that $k = 2v^2$, $y = \sqrt{2k}$, and either $x = 2$ or $x = 2k^2 - 2$. □

5.2 The Equation $x^{2n} - Dy^2 = 1$

In this section we will discuss the solvability of the equation

$$x^{2n} - Dy^2 = 1, \tag{5.2.1}$$

where D is a nonsquare positive integer and n is an integer greater than 1. When $n = 2$ its solvability was discussed in the papers [51, 231, 232] and in the section above.

In what follows we also employ the equations

$$x^p - 2y^2 = -1, \tag{5.2.2}$$

and

$$x^p - 2y^2 = 1, \tag{5.2.3}$$

where p is a prime ≥ 5.

They were studied by elementary methods in the paper [51].

We first present two useful results.

Lemma 5.2.1. *If the equation (5.2.2) has positive integer solution* $(x, y) \neq (1, 1)$*, then* $2p|y$.

Proof. Suppose (x, y) is a positive integer solution of (5.2.2). Then

$$(x+1) \cdot \frac{x^p+1}{x+1} = 2y^2.$$

Since $\left(x+1, \dfrac{x^p+1}{x+1}\right) = 1$ or p, we have

$$x+1 = 2y_1^2, \quad \frac{x^p+1}{x+1} = y_2^2, \quad y = y_1 y_2, \tag{5.2.4}$$

or

$$x+1 = 2py_1^2, \quad \frac{x^p+1}{x+1} = py_2^2, \quad y = py_1 y_2. \tag{5.2.5}$$

By the result of [119], $\dfrac{x^p+1}{x+1} = y_2^2$, therefore $x = 1$. Thus (5.2.4) gives $x = y = 1$.

For (5.2.5) clearly $p|y$. We will prove $2|y$ with the elementary method given in [51].

If $2 \nmid y$, from (5.2.2), we have $x \equiv 1 \pmod 8$. Put

$$A(t) = \frac{x^p+1}{x+1}, \quad t \geq 1 \text{ and } 2 \nmid t,$$

and so $A(t) \equiv 1 \pmod 8$. Let $1 < l < p$ be a positive odd integer. Then there exist an integer r, odd, $0 < r < l$, and $2k$ such that $p = 2kl + r$ or $p = 2kl - r$.

If $p = 2kl + r$, then

$$A(p) = \frac{((x+1)A(l) - 1)^{2k} x^r + 1}{x+1} \equiv \frac{x^r + 1}{x+1} \equiv A(r) \pmod{A(l)}, \tag{5.2.6}$$

since $x^l = (x+1)A(l) - 1$. Now $(A(p), A(l)) = A((p,l)) = A(1) = 1$. Thus (5.2.6) gives

$$\frac{A(p)}{A(l)} = \frac{A(r)}{A(l)}.$$

If $p = 2kl - r$, then $l - r$ is even. Thus

$$\frac{A(p)}{A(l)} = \left(-x^{l-r}\frac{A(r)}{A(l)}\right) = \left(\frac{A(r)}{A(l)}\right),$$

since $A(l) \equiv 1 \pmod 8$ and

$$A(p) = x^{l-r}A(l(2k-1)) + A(l) - x^{l-r}A(r).$$

For l, r, we have

$$\begin{array}{ll}
l = 2k_1 r + \varepsilon_1 r_1, & 0 < r_1 < r, \\
r = 2k_2 r_1 + \varepsilon_2 r_2, & 0 < r_2 < r_1, \\
\cdots & \\
r_{s-1} = 2k_{s+1} r_s + \varepsilon_{s+1} r_{s+1}, & 0 < r_{s+1} < r_s, \\
r_s = k_{s+2} r_{s+1}, &
\end{array}$$

where $\varepsilon_i = \pm 1$ $(i = 1, \ldots, s+1)$ and r_i $(i = 1, \ldots, s+1)$ are odd integers. Since $(l, p) = 1$, we have $r_{s+1} = 1$. Hence

$$\begin{aligned}
\left(\frac{A(p)}{A(l)}\right) &= \left(\frac{A(r)}{A(l)}\right) = \left(\frac{A(l)}{A(r)}\right) = \left(\frac{A(r_1)}{A(r)}\right) = \left(\frac{A(r)}{A(r_1)}\right) = \left(\frac{A(r_2)}{A(r_1)}\right) \\
&= \cdots = \left(\frac{A(r_{s+1})}{A(r_s)}\right) = \left(\frac{A(l)}{A(r_s)}\right) = \left(\frac{1}{A(r_s)}\right) = 1.
\end{aligned}$$

Now, from $\dfrac{x^p+1}{x+1} = py_2^2$, we have

$$(py_2)^2 \equiv pA(p) \pmod{A(l)}.$$

Thus

$$\left(\frac{pA(p)}{A(l)}\right) = \left(\frac{p}{A(l)}\right) = \left(\frac{A(l)}{p}\right) = \left(\frac{l}{p}\right) = 1,$$

since $x \equiv -1 \pmod p$ and so $A(l) \equiv l \pmod p$, We have a contradiction if l is taken as an odd quadratic nonresidue of p. This proves the result. □

Lemma 5.2.2. *The equation (5.2.3) has only positive integer solution $x = 3$, $y = 11$ (when $p = 5$).*

Proof. From (5.2.3), we have

$$\frac{x^p - 1}{x - 1} = a^2 \tag{5.2.7}$$

if $p \nmid y$. By the result of [119], the solution of (5.2.7) is $x = 3$ (when $p = 5$). Thus (5.2.3) has positive integer solution $x = 3$, $y = 11$ (when $p = 5$).

If $p|y$, then $2|y$ by Remark 1. From (5.2.3), $(1 + \sqrt{-2y})(1 - \sqrt{-2y}) = x^p$. With the assumption $(1 + \sqrt{-2y}, 1 - \sqrt{-2y}) = 1$, we have

$$1 + \sqrt{-2y} = (a + b\sqrt{-2})^p, \quad x = a^2 + 2b^2, \tag{5.2.8}$$

where a, b are integers. Since $2|y$, from (5.2.3), it follows that

$$x \equiv 1 \pmod 8. \tag{5.2.9}$$

From (5.2.8) and (5.2.9), we have $2|b$ and $b \neq 0$. Now, (5.2.8) gives

$$1 = a^p + \binom{p}{2} a^{p-2}(b\sqrt{-2})^2 + \cdots + \binom{p}{p-1} a(b\sqrt{-2})^{p-1}. \tag{5.2.10}$$

Thus $a|1$ and so $a = \pm 1$.

If $a = -1$, then (5.2.10) gives

$$-2 = \binom{p}{2}(b\sqrt{-2})^2 + \cdots + \binom{p}{p-1}(b\sqrt{-2})^{p-1},$$

and so $p|2$ which is impossible.

If $a = 1$, then we have

$$0 = \binom{p}{2}(b\sqrt{-2})^2 + \cdots + \binom{p}{p-1}(b\sqrt{-2})^{p-1}. \tag{5.2.11}$$

Since $2|b$ and $b \neq 0$, let $2^{s_k} || \binom{p}{2k}(b\sqrt{-2})^{2k}$ $\left(1 \leq k \leq \frac{p-1}{2}\right)$, clearly $s_k > s_j$ $(k > j)$. Thus (5.2.11) is impossible. □

Theorem 5.2.3. *If $n > 2$ and the negative Pell's equation $u^2 - Dv^2 = -1$ is solvable, then the equation (5.2.1) has only one solution in positive integers: $x = 3$, $y = 22$ (when $n = 5$, $D = 122$).*

Proof. Let $\Omega = u_0 + v_0\sqrt{D}$ be the smallest solution to the equation $u^2 - Dv^2 = -1$, $\overline{\Omega} = u_0 - v_0\sqrt{D}$, and let $\eta = U_0 + V_0\sqrt{D}$ be the fundamental solution of the equation $U^2 - DV^2 = 1$, $\overline{\eta} = U_0 - V_0\sqrt{D}$. Then, we have $\eta = \Omega^2$.

Suppose (x, y) is any positive integer solution of (5.2.1). Then

$$x^n = \frac{\eta^m + \overline{\eta}_m}{2} = \frac{\Omega^{2m} + \overline{\Omega}^{2m}}{2}, \quad m > 0. \tag{5.2.12}$$

Clearly, without loss of generality, we may assume that $n = 4$ or $n = p$ (p is odd prime).

(a) If $n = 4$, then (5.2.12) gives

$$x^4 = 2\left(\frac{\Omega^m + \overline{\Omega}^m}{2}\right)^2 - (-1)^m,$$

and so $x = 1$, $m = 0$, which is impossible since $m > 0$.

(b) If $n = p$ (p is odd prime), then (5.2.12) gives

$$x^p = 2\left(\frac{\Omega^m + \overline{\Omega}^m}{2}\right)^2 - (-1)^m. \tag{5.2.13}$$

(b.1) When $2|m$, let $m = 2s$, $s > 0$; then (5.2.13) gives

$$x^p + 1 = 2\left(\frac{\Omega^{2s} + \overline{\Omega}^{2s}}{2}\right)^2. \tag{5.2.14}$$

Suppose $p = 3$. Then by (5.2.14), we have (see [218])

$$x = \frac{\Omega^{2s} + \overline{\Omega}^{2s}}{2} = 1, \tag{5.2.15}$$

and

$$x = 23, \quad \frac{\Omega^{2s} + \overline{\Omega}^{2s}}{2} = 78. \tag{5.2.16}$$

Clearly (5.2.15) is impossible, since $s > 0$, and (5.2.16) is also impossible since

$$\frac{\Omega^{2s} + \overline{\Omega}^{2s}}{2} = 2\left(\frac{\Omega^s + \overline{\Omega}^s}{2}\right)^2 - (-1)^s$$

is odd.

Thus $p > 3$. For (5.2.14) we have $2p|\dfrac{\Omega^{2s}+\overline{\Omega}^{2s}}{2}$ by Lemma 5.2.1. However, $2|\frac{\Omega^{2s}+\overline{\Omega}^{2s}}{2}$ is impossible.

(b.2) When $2 \nmid m$, we have

$$x^p - 1 = 2\left(\frac{\Omega^m+\overline{\Omega}^m}{2}\right)^2, \tag{5.2.17}$$

and so $x = 1$, $(\Omega^m+\overline{\Omega}^m)/2 = 0$ when $p = 3$ (see [218]). If $p > 3$, then (5.2.17) gives $x = 3$, $(\Omega^m+\overline{\Omega}^m)/2 = 11$ (when $p = 5$) by Lemma 5.2.2. Thus (5.2.1) has only positive integer solution $x = 3$, $y = 22$ (when $n = 5, D = 122$). ☐

Theorem 5.2.4. *If $\eta = U_1 + V_1\sqrt{D}$ is the fundamental solution to Pell's equation $U^2 - DV^2 = 1$, then the positive integer solutions to equation (5.2.1) do not satisfy*

$$x^n + y\sqrt{D} = \eta^{4m}, \quad n > 2,\ m > 0.$$

Proof. If

$$x^n + y\sqrt{D} = \eta^{4m}, \quad n > 2, m > 0,$$

then we have

$$x^n = \frac{\eta^{4m}+\overline{\eta}^{4m}}{2} = 2\left(\frac{\eta^{2m}+\overline{\eta}^{2m}}{2}\right)^2 - 1. \tag{5.2.18}$$

By Lemma 5.2.1, the equality (5.2.18) is impossible since $2 \nmid (\eta^{2m}+\overline{\eta}^{2m})/2$ and $m > 0$. ☐

As applications of the above results we will discuss now some interesting problems in number theory.

In 1939 (see [70]) it was conjectured that the equation

$$\binom{n}{m} = y^k, \quad n > m \geq 2,\ k \geq 3 \tag{5.2.19}$$

has no integer solution. In [70] it is proved that the conjecture is right when $m > 4$, leaving the cases $m = 2$ and $m = 3$ unsolved. Now, we can deduce the following result:

Corollary 5.2.5. *The equation*

$$\binom{n}{2} = y^{2k}$$

has no positive integer solution (n, y) with $n \geq 3$ and $k \geq 2$.

Proof. From $\binom{n}{2} = \frac{n(n-1)}{2} = y^{2k}$, we have

$$n-1 = 2y_1^{2k}, \quad n = y_2^{2k}, \quad y = y_1y_2,$$

or

$$n-1 = y_2^{2k}, \quad n = 2y_1^{2k}, \quad y = y_1y_2.$$

Hence

$$y_2^{2k} \mp 1 = 2y_1^{2k}. \tag{5.2.20}$$

If $2|k$, then (5.2.20) clearly gives $|y_1y_2| \leq 1$; on the other hand, $n \geq 3$ and $\binom{n}{2} = y^{2k}$ imply $|y| = |y_1y_2| > 1$. Here we have a contradiction. If $2 \nmid k$, $k \geq 2$, we may conclude from Theorem 5.2.3 and Lemma 5.2.1 that (5.2.20) is impossible. $\square$

Define the generalized Pell sequence by

$$x_0 = 1, \quad x_1 = a, \quad x_{n+2} = 2ax_{n+1} - x_n, \tag{5.2.21}$$

where a is an integer greater than 1.

Corollary 5.2.6. *The equation*

$$x_{4n} = y^m$$

has no positive integer solution (n, y)*, when* $m \geq 3$.

Proof. From (5.2.21) we have $x_n = \frac{\alpha^n + \overline{\alpha}^n}{2}$, $n \geq 0$, where $\alpha = \alpha + \sqrt{a^2-1}$ and $\overline{\alpha} = \alpha - \sqrt{\alpha^2 - 1}$ are roots of the trinomial $z^2 - 2az + 1$. Let $a^2 - 1 = Db^2$, where $D > 0$ is squarefree and b is positive integer. Then

$$\alpha = a + b\sqrt{D}, \quad \overline{\alpha} = a - b\sqrt{D} \quad \text{and} \quad \alpha\overline{\alpha} = 1.$$

Thus $y_n = \frac{\alpha^n - \overline{\alpha}^n}{2\sqrt{D}}$ satisfies

$$x_n^2 - Dy_n^2 = 1. \tag{5.2.22}$$

By Theorem 5.2.4, the relation (5.2.22) is impossible when $4|n$, $x_n = y^m$ and $m \geq 3$. $\square$

Clearly, if $a = 2u^2 + 1$ $(u > 0)$, then $Db^2 = a^2 - 1 = 4u^2(u^2 + 1)$. Thus $2u|b$. Letting $b = 2uv$, we have $u^2 + 1 = Dv^2$. Hence, using Theorem 5.2.3, we obtain

Corollary 5.2.7. *For the generalized Pell sequence*

$$x_0 = 1, \quad x_1 = 2u^2 + 1, \quad x_{n+2} = 2(2u^2 + 1)x_{n+1} - x_n,$$

where u is positive integer, x_n is never an m^{th} power if $m \geq 3$, except for $x_1 = 2 \cdot 11^2 + 1 = 3^5$.

Remarks. 1) In the paper [129] it is studied the equation (5.2.1), where $n = p$ is a prime. The main two results given there are:

1. If $p = 2$ and $D > \exp(64)$, then (5.2.1) has at most one positive integer solution (x, y).
2. If $p > 2$ and $D > \exp(\exp(\exp(\exp(10))))$, then $2 \nmid m$, where (x, y) is a solution to (5.2.1) expressed as

$$x^p + y\sqrt{D} = \varepsilon_1^m$$

and $\varepsilon_1 = u_1 + v_1\sqrt{D}$ is the fundamental solution to the Pell's equation $u^2 - Dv^2 = 1$.

2) In the paper [189] it is studied the equation $m^4 - n^4 = py^2$, where $p \geq 3$ is a prime, and then the equations $x^4 + 6px^2y^2 + p^2y^4 = z^2$, $c_k(x^4 + 6px^2y^2 + p^2y^4) + 4pd_k(x^3y + pxy^3) = z^2$, for $p \in \{3, 7, 11, 19\}$ and (c_k, d_k) is a solution to the Pell's equation $c^2 - pd^2 = 1$ or to the negative Pell's equation $c^2 - pd^2 = -1$.

3) In the paper [190] is considered the equation $x^4 - q^4 = py^3$, with the following conditions: p and q are distinct primes, x is not divisible by p, $p \equiv 11 \pmod{12}$, $q \equiv 1 \pmod 3$, x is not divisible by p, $p \equiv 11 \pmod{12}$, $q \equiv 1 \pmod 3$, $\overline{p}$ is a generator of the group $(\mathbb{Z}_q^*, \cdot)$, and 2 is a cubic residue mod q. This equation has been solved in the general case in the paper [121].

5.3 The Equation $x^2 + (x+1)^2 + \cdots + (x+n-1)^2 = y^2 + (y+1)^2 + \cdots + (y+n+k-1)^2$

In the paper [2] the relation $5^2 = 3^2 + 4^2$ was considered as the simplest solution in positive integers to various Diophantine equations, in particular, as the simplest solution for the case $n = 1$ to

$$\begin{aligned} &x^2 + (x+1)^2 + (x+2)^2 + \cdots + (x+n-1)^2 \\ &= y^2 + (y+1)^2 + (y+2)^2 + \cdots + (y+n)^2, \end{aligned} \tag{5.3.1}$$

i.e., the case where the sum of n consecutive squares equals the sum of $n+1$ consecutive squares. The complete set of solutions of (5.3.1) for all positive integers n, for which n and $n+1$ are squarefree, was given in [2] and [4].

The relation $5^2 = 3^2 + 4^2$ may also be considered as the simplest solution in positive integers for the case $k = 2$ of the sum of k consecutive squares is a perfect square. This problem is treated in [5].

In this section, we consider the equation (5.3.1) as the special case for $k = 1$ of

$$x^2 + (x+1)^2 + (x+2)^2 + \cdots + (x+n-1)^2$$
$$= y^2 + (y+1)^2 + (y+2)^2 + \cdots + (y+n+k-1)^2, \tag{5.3.2}$$

i.e., the case where the sum of n consecutive squares equals the sum of $n+k$ consecutive squares, and present results for $k \geq 2$. We will use the approach in [3].

Theorem 5.3.1. *The equation (5.3.2) is not solvable for* $k \equiv 3, 4,$ *or* $5 \pmod 8$.

Proof. The sum S of squares of n consecutive integers, modulo 4, is listed in the table:

n	1	2	3	4	5	6	7	8
$S \pmod 4$	0 or 1	1	1 or 2	2	2 or 3	3	0 or 3	0

Clearly, beginning with $n = 9$, the row for $S \pmod 4$ must repeat itself and continue to do so with the length of the period equal to 8. Now, if the sum of n consecutive squares is to equal the sum of $n+3$ consecutive squares, there must be, for some n, a number in the S-row which also appears in the S-row for $n+3$. This, however, is not the case for any value of n. Since the column of entries in the S-row repeats with period 8, the same is true for any value of $k \equiv 3 \pmod 8$. The same argument can be used to prove the nonexistence of solutions for $k \equiv 4$ or 5 $\pmod 8$. □

Theorem 5.3.2. *The equation (5.3.2) is not solvable for* $k \equiv 7, 11, 16,$ *or* $20 \pmod{27}$.

Proof. Using the formula for the sum of the first n squares, (5.3.2) can be rewritten as

$$nx^2 + n(n-1)x + n(n-1)(2n-1)/6$$
$$= (n+k)y^2 + (n+k)(n+k-1)y + (n+k)(n+k-1)(2n+2k-1)/6$$

or

$$n(2x+n-1)^2 = (n+k)(2y+n+k-1)^2 + kn^2 + k^2n + k(k^2-1)/3. \tag{5.3.3}$$

Letting

$$z = 2x + n - 1 \quad \text{and} \quad w = 2y + n + k - 1,$$

we can rewrite (5.3.3) as

$$nz^2 = (n + k)w^2 + kn^2 + k^2n + k(k^2 - 1)/3. \tag{5.3.4}$$

Considering first the case where $k \equiv \pmod{27}$, we substitute into (5.3.4) $k = 27\lambda + 7$ and obtain

$$\begin{aligned} nz^2 = (n + 27\lambda + 7)w^2 + (27\lambda + 7)n^2 + (27\lambda + 7)^2 n \\ + (27\lambda + 7)(243\lambda^2 + 126\lambda + 16). \end{aligned} \tag{5.3.5}$$

If $n \equiv 0 \pmod 3$, the left-hand side is congruent to 0, modulo 3, while the right-hand side is congruent to $w^2 + 1$, modulo 3, so that $w^2 \equiv 2$, which is a contradiction. If $n \equiv 2 \pmod 3$, a contradiction is similarly obtained as the left-hand side is congruent to $2z^2$, modulo 3, and the right-hand side congruent to 1, modulo 3.

If in (5.3.5), $n \equiv 1 \pmod 3$, we obtain

$$z^2 \equiv 2w^2 \pmod 3,$$

which is satisfies only if $z \equiv w \equiv 0 \pmod 3$, so that we can set

$$z = 3z' \quad \text{and} \quad w = 3w', \quad n = 3m + 1,$$

which, when substituted into (5.3.5) yields

$$\begin{aligned} (3m + 1)9z'^2 = (3m{+}27\lambda{+}8)9w'^2{+}(27\lambda{+}7)(3m{+}1)^2{+}(27\lambda{+}7)^2(3m{+}1) \\ +(27\lambda + 7)(243\lambda^2 + 126\lambda + 16), \end{aligned}$$

which immediately leads to a contradiction, since the left-hand side is congruent to 0, modulo 9, while the right-hand side is congruent to 6, modulo 9.

By substituting into (5.3.4) $k = 27\lambda + 11$, $k = 27\lambda + 16$, and $k = 27\lambda + 20$, and using the procedure shown above for $k = 27\lambda + 7$, we can similarly show that there are no solutions for (5.3.2) if $k \equiv 11, 16, \text{ or } 20 \pmod{27}$. □

We now turn to the question of finding values of k for which solutions to (5.3.2) exist. Such solutions can be obtained either by an analysis of (5.3.4) which is equivalent to (5.3.2) or by programming a computer to find solutions directly from (5.3.2). Using both methods, all values of $k \leq 100$, not excluded by Theorems 5.3.1 and 5.3.2, were considered and solutions found for the values of k indicated in the following table. In each case, we also list a solution for the indicated value of n and give x and y, as defined by (5.3.2).

k	n	x	y	k	n	x	y	k	n	x	y
6	5	28	15	39	2	25169	5539	64	16	740	294
8	3	137	67	40	5	378	104	71	2	378	23
9	3	23	6	41	11	1551	690	72	10	163	13
10	5	25	8	42	25	77	18	73	73	217	102
15	2	2743	933	46	1	3854	539	78	5	754	143
17	17	33	11	48	2	2603	496	79	79	312	166
18	3	127	38	49	2	210	14	80	3	2196	376
22	11	38	7	50	3	243	30	81	3	1257	195
23	2	8453	2379	54	39	160	67	86	43	188	51
24	2	24346	6740	55	55	128	51	87	2	510565	76493
25	25	123	70	56	14	151	33	89	89	227	97
26	3	1417	442	57	19	183	56	90	3	3521	586
31	4	196	49	58	11	36927	14712	94	33	608	253
32	3	239723	70167	62	25	5316	2813	95	2	716	51
33	11	313	137	63	2	236	5	96	1	679	15

Thus the first entry in the table means that

$$28^2+29^2+30^2+31^2+32^2=15^2+16^2+\cdots+25^2.$$

In the table above, no attempt was made to list for each given value of k the smallest value of n for which there exists a solution, since, as is evident from an inspection of the table, small values of n are frequently associated with very large values of x and y.

For each pair of values (k,n) for which a solution is given in the above table, infinitely many additional solutions can be obtained as follows.

Letting

$$kn^2+k^2n+\frac{k(k^2-1)}{3}=A$$

equation (5.3.4) can be rewritten as $nz^2-(n+k)w^2=A$ or, multiplying both sides by n,

$$u^2-Dw^2=N, \tag{5.3.6}$$

where $u=nz$, $D=n(n+k)$, $N=nA$.

Now if (h_m,k_m) is the general solution of Pell's equation

$$h^2-Dk^2=1 \tag{5.3.7}$$

and (u_1, w_1) is any solution to the general Pell's equation (5.3.6), then

$$u_m = u_1 h_m \pm D w_1 k_m, \quad w_m = u_1 k_m \pm w_1 h_m, \quad m \geq 0 \tag{5.3.8}$$

are solutions to the equation (5.3.6). For details we refer to Section 4.3.

On the other hand, equations (5.3.8) do not necessarily give all solutions for a given pair of values (k, n). Indeed, any attempt to find all solutions for given n and k is bound to lead to presently unsurmountable difficulties as complete solutions of (5.3.6) are available only for $n < \sqrt{D}$, and this condition will generally not be satisfied for $k > 1$ (see Section 4.3.4).

While in the case $k = 1$ (see [4] and [5]) it was shown that solutions exist for all values of n, for which n and $n+1$ are squarefree, it can easily be shown that for $k > 1$, even if there exists a solution for some n, there may be none for others. Thus, for example, it can be shown that for $k = 6$, solutions can exist only if $n \equiv 1$ or $5 \pmod 6$. Such facts can be established by arguments similar to those used in the proof of Theorem 5.3.1, making use of the facts that the sequence of values of $S \pmod 4$ in the table of that proof has period 8 and that, if a similar table were constructed for the sequence of values of $S \pmod 3$, it would have period 9.

It is of interest to note that Theorems 5.3.1 and 5.3.2, together with the table of solutions of equation (5.3.2) for $k \leq 100$, presented above, answers for all but 6 values of k the question as to whether or not a solution of (5.3.2) exists for values of $k \leqq 100$. These 6 values are $k = 2, 14, 30, 66, 82, 98$. No general method for proving the existence or nonexistence of solutions in individual cases seems to suggest itself. To illustrate typical proofs, we show below the ones for the case $k = 2$, where the knowledge of the Jacobi symbol is involved leads to a solution, and $k = 14$, where an analysis of the highest power of 2 dividing the constant term of (5.3.4) solves the problem.

Theorem 5.3.3. *The equation (5.3.2) is not solvable for $k = 2$.*

Proof. The sum S of the squares of n consecutive integers, modulo 12, is listed in the following table.

n	$S \pmod{12}$
1	$0, 1, 4,$ or 9
2	1 or 5
3	2 or 5
4	2 or 6
5	$3, 6, 7,$ or 10
6	7
7	$4, 7, 8,$ or 11
8	0 or 8
9	0 or 9
10	1 or 9
11	$1, 2, 5,$ or 10
12	2

Obviously, for $13 \leqq n \leqq 24$, the values of $S \pmod{12}$ are those of the above table increased by 2, while those for $25 \leqq n \leqq 36$ are those of the above table increased by 4, etc. From this, it is immediately seen that all values of n except $n \equiv 5$ or $11 \pmod{12}$ cannot give solutions.

Now substituting $n = 12m + 5$ into (5.3.4) yields

$$(12m+5)z^2 = (12m+7)w^2 + 72(2m+1)^2,$$

so that

$$-2z^2 \equiv 72(2m+1)^2 \pmod{12m+7}$$

or

$$z^2 \equiv -36(2m+1)^2 \pmod{12m+7},$$

which means that the Jacobi symbol $(-1/12m+7)$ must have the value +1, which is a contradiction.

An entirely similar analysis for the case $n = 12m + 11$ leads to another contradiction. □

Theorem 5.3.4. *The equation (5.3.2) is not solvable for* $k = 14$.

Proof. By simple congruence analysis we find that all values of n except $n \equiv 1 \pmod{4}$ can be excluded. Now substituting $n = 4m + 1$ into (5.3.4) yields

$$(4m+1)z^2 = (4m+15)w^2 + 14(4m+1)^2 + 196(4m+1) + 910. \tag{5.3.9}$$

Considering the above equation, modulo 4, we obtain $z^2 \equiv 3w^2 \pmod{4}$, which shows that z and w must both be even.

Now, if m is even, then (5.3.9) can be rewritten by letting $m = 2m'$ as

$$(8m'+1)z^2 = (8m'+15)w^2 + 224(4m'^2 + 8m' + 5).$$

Since letting $z = 2z'$, $w = 2w'$ leads exactly as shown above to the conclusion that z' and w' must be even, we let $z = 4z'$ and $w = 4w'$ and divide by 16 to obtain

$$(8m'+1)z'^2 = (8m'+15)w'^2 + 14(4m'^2 + 8m' + 5).$$

Considering this equation modulo 8, we obtain

$$z'^2 = -w'^2 + 6 \pmod{8},$$

which is a contradiction, since the left-hand side is congruent to 0,1, or $4 \pmod{8}$, while the right-hand side is congruent to 2,5, or $6 \pmod{8}$.

Now m is odd, then (5.3.9) can be rewritten by letting $m = 2m' + 1$ as

$$(8m' + 5){z'}^2 = (8m' + 19){w'}^2 + 448(2{m'}^2 + 6m' + 5)$$

or, diving both sides by 64 and letting $w = 8w'$, $z = 8z'$, as

$$(8m' + 5){z'}^2 = (8m' + 19){w'}^2 + 7(2{m'}^2 + 6m' + 5).$$

Considering this equation modulo 4, we obtain

$${z'}^2 \equiv 3{w'}^2 + 3 \pmod 4,$$

which again is a contradiction. □

5.4 The Equation $x^2 + 2(x+1)^2 + \cdots + n(x+n-1)^2 = y^2$

In this section, following [229], we will discuss the equation

$$x^2 + 2(x+1)^2 + \cdots + n(x+n-1)^2 = y^2 \tag{5.4.1}$$

determining the values of n for which it has finitely or infinitely many positive integer solutions (x, y).

Theorem 5.4.1. *For each $n \geq 2$ the equation (5.4.1) is solvable and it has infinitely many solutions unless $\dfrac{n(n+1)}{2}$ is a perfect square.*

Proof. The equation (5.4.1) can be written immediately into the form

$$\frac{n(n+1)}{2}x^2 + \frac{2(n-1)n(n+1)}{3}x + \frac{(n-1)n(n+1)(3n-2)}{12} = y^2. \tag{5.4.2}$$

The substitutions

$$k = \frac{n(n+1)}{2}, \quad u = 3x + 2(n-1), \quad v = \frac{3y}{k}$$

along with the observation

$$\frac{(n-1)n(n+1)(n+2)}{4} = k(k-1)$$

reduce (5.4.2) to the general Pell's equation

$$u^2 - kv^2 = 1 - k. \tag{5.4.3}$$

For all positive integral values of k, the equation (5.4.3) admits the solution $u_0 = 2n+1$, $v_0 = 3$, corresponding to the solution $x = 1$, $y = k$ of (5.4.1) which is the familiar formula for the sum of the first n cubes. Thus (5.4.1) has always at least one solution.

Now, let $k = \dfrac{n(n+1)}{2}$ be a nonsquare. In this case the Pell's equation

$$U^2 - kV^2 = 1 \tag{5.4.4}$$

has the solutions $(U_m, V_m)_{m\geq 0}$ given in Sections 3.2, 3.3, 3.4.

By using the theory of general Pell's equation developed in Chapter 4, it follows that if (u_0, v_0) is a solution of (5.4.3), then

$$u_m = u_0 U_m + k v_0 V_m, \quad v_m = v_0 U_m + u_0 V_m, \quad m = 0, 1, \ldots \tag{5.4.5}$$

are solutions to (5.4.3).

These will give solutions to (5.4.1) in all cases where

$$x = \frac{u_m + 2 - 2n}{3} \quad \text{and} \quad y = \frac{k v_m}{3} \tag{5.4.6}$$

are integers. We proceed to examine these.

If $n \equiv 1 \pmod 3$, then $k \equiv 1$, $u_0 \equiv 0$, $v_0 \equiv 0 \pmod 3$, and each u_m, v_m given in (5.4.5) will satisfy $u_m \equiv 0$, $v_m \equiv 0 \pmod 3$, which imply that x and y in (5.4.6) are integers.

If $n \equiv 2 \pmod 3$, then $k \equiv 0$, $u_0 \equiv 2$, $v_0 \equiv 0$, $U_0^2 \equiv 1 \pmod 3$ hence $U_0 \equiv 1$ or $2 \pmod 3$. For $U_0 \equiv 1 \pmod 3$ the relations (5.4.5) show that $u_m \equiv 2$, $u_m + 2 - 2n \equiv 0$, $kv_m \equiv 0 \pmod 3$, hence x and y in (5.4.6) are integers. For $U_0 \equiv 2 \pmod 3$ we have $u_m \equiv 1 \pmod 3$, and x is not an integer. However, in this case, from (5.4.5), $u_{m+1} \equiv 2 \pmod 3$ so that the corresponding x and y are integers.

Analogous study of the case $n \equiv 0 \pmod 3$ gives a similar result. Hence, in all cases, at least alternate members of the infinite sequence of solutions to (5.4.3) give integral values of x, y which satisfy the equation (5.4.1). $\square$

Remark. One may determine explicitly (see [15]) the integers n for which $k = \dfrac{n(n+1)}{2}$ is a perfect square. This reduces to finding the solutions to the equations $n(n+1) = 2s^2$ or, equivalently, $(2n+1)^2 - 8s^2 = 1$. The last Pell's equation has solutions $(2n_l + 1, s_l)_{l\geq 1}$, where

$$2n_l + 1 = \frac{1}{2}\left[\left(3+\sqrt{8}\right)^l + \left(3-\sqrt{8}\right)^l\right]. \tag{5.4.7}$$

From (5.4.7) it follows that all positive integers n with the above property are given by

$$n_l = \begin{cases} \left[\dfrac{\left(\sqrt{2}+1\right)^l - \left(\sqrt{2}-1\right)^l}{2}\right]^2 & \text{if } l \text{ is odd} \\ 2\left[\dfrac{\left(1+\sqrt{2}\right)^l - \left(1-\sqrt{2}\right)^l}{2\sqrt{2}}\right]^2 & \text{if } l \text{ is even} \end{cases} \tag{5.4.8}$$

5.5 The Equation $(x^2 + a)(y^2 + b) = F^2(x, y, z)$

In this section we study the general class of Diophantine equations

$$(x^2 + a)(y^2 + b) = F^2(x, y, z) \tag{5.5.1}$$

where $F : \mathbb{Z}_+ \times \mathbb{Z}_+ \times \mathbb{Z} \to \mathbb{Z}^*$ is a given function and a, b are nonzero integers satisfying $|a|, |b| \leq 4$.

It is clear that if only one of $x^2 + a$ or $y^2 + b$ is a perfect square, then the equation (5.5.1) is not solvable. In the given hypothesis, $x^2 + a$ and $y^2 + b$ are simultaneously nonzero perfect squares only if $|a| = 3$ and $|b| = 3$ in which situation (x, y) is one of the pairs (1,1), (1,2), (2,1), (2,2). For these pairs we must have

$$F(1,1,z) = \pm 4, \quad F(1,2,z) = \pm 2, \quad F(2,1,z) = \pm 2, \quad F(2,2,z) = \pm 1. \tag{5.5.2}$$

It remains to find z from the corresponding equations in (5.5.2), a problem that is strictly dependent upon the function F.

In order to have a unitary presentation of our general method, we may assume that $x \geq 3$ and $y \geq 3$.

From the above considerations we may assume that none of $x^2 + a$ and $y^2 + b$ is a perfect square. From (5.5.1) it follows that $x^2 + a = du^2$ and $y^2 + b = dv^2$ for some positive integers d, u, v. The last two equations can be written as

$$x^2 - du^2 = -a \quad \text{and} \quad y^2 - dv^2 = -b \tag{5.5.3}$$

which are general Pell's equations of the form $X^2 - dY^2 = N$, where $|N| \leq 4$.

Define the set

$$\mathcal{P}(a, b) = \{d \in \mathbb{Z} : \ d \text{ is nonsquare } \geq 2 \text{ and (5.5.3) are solvable}\} \tag{5.5.4}$$

and for any d in $\mathcal{P}(a, b)$ consider the general solutions $(x(d), u(d))$ and $(y(d), v(d))$ to the equations (5.5.3) (see Chapter 4 for details).

We have $x^2(d)+a = du^2(d)$ and $y^2(d)+b = dv^2(d)$ hence

$$F(x(d),y(d),z) = \pm du(d)v(d). \tag{5.5.5}$$

Denote by $\mathcal{Z}_d$ the set of all integers z satisfying the equation (5.5.5).

The solutions to the equation (5.5.1) are $(x(d),y(d),z)$, where $d \in \mathcal{P}(a,b)$ and $z \in \mathcal{Z}_d$.

To illustrate this method let us consider the following concrete examples.

5.5.1 *The Equation* $x^2+y^2+z^2+2xyz = 1$

In the book [25] the above equation is solved in integers. Indeed, it is equivalent to

$$(x^2-1)(y^2-1) = (xy+z)^2, \tag{5.5.6}$$

an equation of the form (5.5.1), where $a = b = -1$ and $F(x,y,z) = xy+z$.

In this case $\mathcal{P}(-1,-1) = \{d > 0: \ d \text{ nonsquare}\}$, as we have seen in Chapter 3.

Let $(s_l(d),t_l(d))_{l\geq 0}$ be the general solution to Pell's equation $s^2-dt^2 = 1$. From the general method, it follows that the integral solutions to the given equation are

$$(\pm s_m(d), \pm s_n(d), -s_m(d)s_n(d) \pm dt_m(d)t_n(d)), \tag{5.5.7}$$

for all $m,n \geq 0$ and $d \in \mathcal{P}(-1,-1)$.

Using either of relations (3.2.2), (3.2.5), or (3.2.6), one can prove the following equalities

$$s_m(d)s_n(d)+dt_m(d)t_n(d) = s_{m+n}(d), \quad m,n \geq 0$$
$$s_m(d)s_n(d)-dt_m(d)t_n(d) = s_{m-n}(d), \quad m \geq n \geq 0.$$

The triples (5.5.7) become

$$\begin{array}{l}(\pm s_m(d), \pm s_n(d), -s_{m+n}(d)), \ m,n \geq 0 \\ (\pm s_m(d), \pm s_n(d), -s_{m-n}(d)), \ m \geq n \geq 0,\end{array} \tag{5.5.8}$$

where the signs + and – correspond.

Given the symmetry of the equation in x,y,z, in order to obtain all of its solutions, we need to also consider the triples obtained from (5.5.8) by cyclic permutations. We mention that the solutions found in [216] are not complete.

Remark. The equation

$$x^2+y^2+z^2+2xyz = 1 \tag{5.5.9}$$

has an interesting history. Its geometric interpretation has been pointed out in [30], where it is shown that it reduces to finding all triangles whose angles have rational cosines.

The general solution in rational numbers of this equation is given [33]:

$$x = \frac{b^2 + c^2 - a^2}{2bc}, \quad y = \frac{a^2 + c^2 - b^2}{2ac}, \quad z = \frac{a^2 + b^2 - c^2}{2ab}.$$

In the paper [167] it is noted that apart from the trivial solutions $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$, $(0, 0, \pm 1)$, all integral solutions to the equation (5.5.9) are given by the following rule: if p, q, r are any integers with greatest common divisor 1 such that one of them is equal to the sum of the other two and if $u \geq 1$ is any integer, then

$$x = \pm \mathrm{ch}(p\theta), \quad y = \pm \mathrm{ch}(q\theta), \quad z = \pm \mathrm{ch}(r\theta)$$

where $\theta = \ln\left(u + \sqrt{u^2 - 1}\right)$ and $u \geq 1$ is an arbitrary integer.

In the papers [148, 149] it is studied the more general Diophantine equation

$$x^2 + y^2 + z^2 + 2xyz = n. \tag{5.5.10}$$

It is proved that this equation has no solutions in integers if $n \equiv 3 \pmod 4$, $n \equiv 6 \pmod 8$, $n \equiv \pm 3 \pmod 9$, $n = 1 - 4k^2$ with $k \not\equiv 0 \pmod 4$ and k has no prime factors of the form $4j+3$, or $n = 1 - 3k^2$ with $(k, 4) = 2$, $(k, 3) = 1$ and k has no prime factors of the form $3j + 2$. On the other hand, one solution to the equation (5.5.10) implies infinitely many such solutions, except possibly when n is a perfect square having no prime factors of the form $4j + 1$. Also, there are infinitely many solutions if $n = 2^r$ and r is odd, but only the solution $x = y = 0$, $z = 2^{\frac{r}{2}}$ and its cyclic permutations when r is even.

5.5.2 The Equation $x^2 + y^2 + z^2 - xyz = 4$

The problem of finding all triples of positive integers (x, y, z) with the property mentioned above appears in [7]. These triples were found by using our general method described at the beginning of this section. Indeed, writing the equation in the equivalent form

$$(x^2 - 4)(y^2 - 4) = (xy - 2z)^2$$

we note that in (5.5.1) we have $a = b = -4$ and $F(x, y, z) = xy - 2z$. Both of the equations (5.5.3) reduce to the special Pell's equation $s^2 - dt^2 = 4$, which was extensively discussed in Section 4.3.2. Let $(s_l(d), t_l(d))_{l \geq 0}$ be the general solution to the equation $s^2 - dt^2 = 4$ given in (4.4.2) or (4.4.5). From (5.5.5) we obtain

$$z_1 = \frac{1}{2}(s_m(d)s_n(d) + dt_m(d)t_n(d)) = s_{m+n}(d)$$

and

$$z_2 = \frac{1}{2}(s_m(d)s_n(d) - dt_m(d)t_n(d)) = s_{|m-n|}(d).$$

The general positive integral solutions to the equation $x^2+y^2+z^2-xyz=4$ are

$$(s_m(d), s_n(d), s_{m+n}(d)),\ m,n \geq 0 \text{ and } (s_m(d), s_n(d), s_{m-n}(d)),\ m \geq n \geq 0$$

along with the corresponding permutations.

5.5.3 The Equation $(x^2+1)(y^2+1)=z^2$

In order to solve this equation in positive integers x, y, z note that $a = b = 1$ and $F(x,y,z) = z$. The equations (5.5.3) become the negative Pell's equation $s^2 - dt^2 = -1$. As we have seen in Section 3.6 the set $\mathcal{P}(1,1)$ is far from easy to describe. The general solution of this equation is

$$(s_m, s_n, dt_m t_n),$$

where $m, n \geq 0$ and $d \in \mathcal{P}(1,1)$.

In a similar way one can solve the equations in $(k+1)$ variables:

$$(x_1^2 \pm 1)(x_2^2 \pm 1)\dots(x_k^2 \pm 1) = y^2 \tag{5.5.11}$$

for any choice of the signs $+$ and $-$.

5.5.4 The Equation $(x^2-1)(y^2-1)=(z^2-1)^2$

The problem of finding all solutions in positive integers to the equation

$$(x^2-1)(y^2-1) = (z^2-1)^2 \tag{5.5.12}$$

is still open [82]. Partial results concerning this equation were published in [120, 222, 227], and [228].

In what follows we will describe the set of solutions to the equation (5.5.12). Our description will show the complexity of the problem of finding all of its solutions. The equation (5.5.12) is of the type (5.5.1), where $a = b = -1$ and $F(x,y,z) = z^2 - 1$. By using the general method presented at the beginning of this section, we can describe the set of solutions to (5.5.12) in the following way: Fix a nonsquare $d \geq 2$ and consider the Pell's equation $s^2 - dt^2 = 1$. It is clear that the solutions to (5.5.12) are of the form $(s_m, s_n, z_{m,n})_{m,n \geq 0}$, where $(s_k, t_k)_{k \geq 0}$ is the general solution to the above Pell's equation, $z_{m,n} = \sqrt{1 + dt_m t_n}$, $m, n \geq 0$ and $1 + dt_m t_n$ is a perfect square.

Let

$$\mathcal{C}_d = \{(s_m, s_n, z_{m,n}) : \ 1 + dt_m t_n \text{ is a square}, \ m, n \geq 0\}.$$

The set of all solutions to (5.5.12) is

$$\mathcal{C} = \bigcup_{\substack{d \geq 2 \\ \sqrt{d} \notin \mathbb{Q}}} \mathcal{C}_d.$$

Note that for all nonsquare $d \geq 2$, $\mathcal{C}_d$ contains the infinite family of solutions (s_m, s_m, s_m), $m \geq 0$, but this is far from determining all elements in $\mathcal{C}_d$.

5.6 Other Equations with Infinitely Many Solutions

5.6.1 The Equation $x^2 + axy + y^2 = 1$

In the book [26] we determine all integers a for which the equation

$$x^2 + axy + y^2 = 1 \tag{5.6.1}$$

has infinitely many integer solutions (x, y). In case of solvability, we find all such solutions. Clearly, (5.6.1) is a special case of the general equation (4.8.1).

Theorem 5.6.1. *The equation (5.6.1) has infinitely many integer solutions if and only if* $|a| \geq 2$.

If $a = -2$, *the solutions are* $(m, m+1)$, $(m+1, m)$, $(-m, -m-1)$, $(-m-1, -m)$, $m \in \mathbb{Z}$.

If $a = 2$, *the solutions are* $(-m, m+1)$, $(m+1, -m)$, $(m, -m-1)$, $(-m-1, m)$, $m \in \mathbb{Z}$.

If $|a| > 2$, *the solutions are*

$$(-v_n, v_{n+1}), (v_n, -v_{n+1}), (-v_{n+1}, v_n), (v_{n+1}, -v_n), \tag{5.6.2}$$

where

$$v_n = \frac{1}{\sqrt{a^2 - 4}} \left[\left(\frac{a + \sqrt{a^2 - 4}}{2} \right)^n - \left(\frac{a - \sqrt{a^2 - 4}}{2} \right)^n \right], \quad n \geq 0. \tag{5.6.3}$$

Proof. Rewrite the given equation in the form

$$(2x+ay)^2-(a^2-4)y^2=4. \tag{5.6.4}$$

If $|a|<2$, then the curve described by (5.6.4) is an ellipse, and so only finitely many integer solutions occur.

If $|a|=2$ then the equation (5.6.1) has infinitely many solutions, since it can be written as $(x\pm y)^2=1$.

If $|a|>2$, then a^2-4 is not perfect square. In this case we have a special Pell's equation of the form

$$u^2-(a^2-4)v^2=4. \tag{5.6.5}$$

This type of equations was extensively studied in Section 4.3.2.

Note that a nontrivial solution to (5.6.5) in $(a,1)$. Using the formula (4.4.2) we obtain the general solution $(u_n,v_n)_{n\geq 1}$ to (5.6.5), where

$$u_n=\left(\frac{a+\sqrt{a^2-4}}{2}\right)^n+\left(\frac{a-\sqrt{a^2-4}}{2}\right)^n,$$

$$v_n=\frac{1}{\sqrt{a^2-4}}\left[\left(\frac{a+\sqrt{a^2-4}}{2}\right)^n-\left(\frac{a-\sqrt{a^2-4}}{2}\right)^n\right],\quad n\geq 1.$$

From formulas (4.4.3) the sequences $(u_n)_{n\geq 1}$, $(v_n)_{n\geq 1}$ satisfy the recursive system

$$\begin{cases} u_{n+1}=\dfrac{1}{2}[au_n+(a^2-4)v_n] \\[2ex] v_{n+1}=\dfrac{1}{2}(u_n+av_n),\ n\geq 1. \end{cases} \tag{5.6.6}$$

From (5.6.4) it follows that the nontrivial integer solutions (x,y) to the equation (5.6.1) satisfy

$$2x+ay=\pm u_n \quad \text{and} \quad y=\pm v_n,\ n\geq 1 \tag{5.6.7}$$

where the signs $+$ and $-$ correspond.

If $2x+ay=u_n$ and $y=v_n$, then from (5.6.6) it follows that

$$x=\frac{1}{2}(u_n-av_n)=\frac{1}{4}[au_{n-1}+(a^2-4)v_{n-1}-au_{n-1}-a^2v_{n-1}]=-v_{n-1}.$$

We obtain the solution $(-v_n,v_{n+1})_{n\geq 1}$. The choice $2x+ay=-u_n$ and $y=-v_n$ yield the solution $(v_n,-v_{n+1})_{n\geq 1}$ which in fact reflects the symmetry $(x,y)\to(-x,-y)$ of (5.6.1).

The last two solutions in (5.6.2) follow from the symmetry $(x,y) \to (y,x)$ of the equation (5.5.1). □

Remark. In the case $a = -m$, where m is a positive integer, the equation (5.6.1) was solved in positive integers by using a complicated method involving planar transformations in [31, pp. 70–73].

5.6.2 The Equation $\dfrac{x^2+1}{y^2+1} = a^2+1$

We will prove that if a is any fixed positive integer, then there exist infinitely many pairs of positive integers (x,y) such that

$$\frac{x^2+1}{y^2+1} = a^2+1. \tag{5.6.8}$$

This means that the set $\{J_m = m^2+1 : \ m = 1,2,\dots\}$ contains infinitely many pairs (J_x, J_y) such that $J_x = J_a J_y$.

The equation (5.6.8) is equivalent to the general Pell's equation

$$x^2 - (a^2+1)y^2 = a^2. \tag{5.6.9}$$

We notice that equation (5.6.9) has particular solutions $(a^2-a+1, a-1)$ and $(a^2+a+1, a+1)$. Let $(u_n, v_n)_{n\ge 0}$ be the general solution of its Pell's resolvent $u^2-(a^2+1)v^2 = 1$. The fundamental solution to the Pell's resolvent equation is $(u_1, v_1) = (2a^2+1, 2a)$.

Following (4.1.3), we construct the sequences of solutions $(x_n, y_n)_{n\ge 0}$ and $(x'_n, y'_n)_{n\ge 0}$ to the equation (5.6.8):

$$\begin{cases} x_n = (a^2-a+1)u_n + (a-1)(a^2+1)v_n \\ y_n = (a-1)u_n + (a^2-a+1)v_n \end{cases} \tag{5.6.10}$$

and

$$\begin{cases} x'_n = (a^2+a+1)u_n + (a+1)(a^2+1)v_n \\ y'_n = (a+1)u_n + (a^2+a+1)v_n \end{cases} \tag{5.6.11}$$

We will show now that for all $a \ge 3$ the solutions $(x_n, y_n)_{n\ge 0}$, $(x'_n, y'_n)_{n\ge 0}$ are all distinct. In this respect, following the criterion given in Section 4.1 it suffices to see that at least one of the numbers

$$\frac{xx'-yy'd}{N} \quad \text{and} \quad \frac{yx'-xy'}{N}$$

is not an integer. Here (x, y) and (x', y') are solutions to the general Pell's equation $X^2 - dY^2 = N$. Indeed,

$$\frac{xx' - yy'd}{N} = \frac{(a^2 - a + 1)(a^2 + a + 1) - (a - 1)(a + 1)(a^2 + 1)}{a^2} =$$
$$= \frac{a^4 + a^2 + 1 - a^4 + 1}{a^2} = 1 + \frac{2}{a^2} \notin \mathbb{Z}.$$

Remarks. 1) The equation

$$(x^2 + 1)(y^2 + 1) = z^2 + 1 \tag{5.6.12}$$

was known even to Diophantus. It was him who pointed out the solutions $(k, 0, k)$, $(k, k \pm 1, k^2 \pm k + 1)$, where k is a positive integer and the signs + and – correspond.

The problem of finding all solutions to (5.6.12) in positive integers appears in [173]. Unfortunately, the solution presented there was incorrect.

2) It is clear that if $y = x + 1$, then $(x^2 + 1)(y^2 + 1) = (x^2 + 1)(x^2 + 2x + 2) = (x^2 + x + 1)^2 + 1$, hence the equation (5.6.12) has infinitely solutions (x, y, z), where x and y are consecutive positive integers.

In the case when x is fixed, the problem of finding infinitely many y and z satisfying (5.6.12) also appears in [205] and it is solved by using a suitable negative Pell's equation.

3) A weaker version of the same problem appears in [89] as follows: the sequence of numbers $J_l = l^2 + 1$, $l = 1, 2, \dots$ contains an infinity of composite numbers $J_N = J_m \cdot J_n$. In fact, in the mentioned reference, for an arbitrary fixed m, only three pairs of corresponding n, N are found:

$$J_{m^2-m+1} = J_m \cdot J_{m-1}, \quad J_{m^2+m+1} = J_m \cdot J_{m+1} \quad \text{and} \quad J_{2m^2+m} = J_m \cdot J_{2m^2}.$$

4) The equation (5.6.8) is completely solved in rational numbers in [50]. Its general solution is given by

$$x = f(a)$$
$$y = \pm a - 1 + \frac{2}{r^2(a) + 1} - \frac{2r(a)}{r^2(a) + 1} f(a),$$

where

$$f(a) = -a \pm \frac{(a^2 + 1)(r(a) \pm 1)^2}{r^2(a) + 2ar(a) - 1}$$

and $r(a)$ is any rational function of a.

5) A slightly modified equation is given by

$$\frac{x^2-1}{y^2-1} = z^2. \tag{5.6.13}$$

This equation can be solved completely [139]. Indeed, it is equivalent to

$$x^2 - (y^2-1)z^2 = 1.$$

It is not difficult to see that $(y, 1)$ is the fundamental solution to this equation and that all solutions are given by $(x_n, y, z_n)_{n\geq 0}$, where y is any integer greater than 1 and

$$\begin{cases} x_n = \dfrac{1}{2}\left[\left(y+\sqrt{y^2-1}\right)^n + \left(y-\sqrt{y^2-1}\right)^n\right] \\ z_n = \dfrac{1}{2\sqrt{y^2-1}}\left[\left(y+\sqrt{y^2-1}\right)^n - \left(y-\sqrt{y^2-1}\right)^n\right] \end{cases}$$

6) Another equation related to (5.6.12) is

$$\frac{x^2+1}{y^2+1} = z^2. \tag{5.6.14}$$

This equation can be solved completely as well. We write it under the form

$$x^2 - (y^2+1)z^2 = -1,$$

a negative Pell's equation with minimal solution $(y, 1)$. Using formulas (3.6.3) we obtain the general solution $(x_n, z_n)_{n\geq 0}$,

$$\begin{cases} x_n = \dfrac{1}{2}\left[\left(y+\sqrt{y^2+1}\right)^{2n+1} + \left(y-\sqrt{y^2+1}\right)^{2n+1}\right] \\ y_n = \dfrac{1}{2\sqrt{y^2+1}}\left[\left(y+\sqrt{y^2+1}\right)^{2n+1} - \left(y-\sqrt{y^2+1}\right)^{2n+1}\right] \end{cases}$$

All solutions to (5.6.14) are given by $(x_n, y, z_n)_{n\geq 0}$, where y is any positive integer.

5.6.3 The Equation $(x+y+z)^2 = xyz$

Generally, integer solutions to equations of three or more variables are given in various parametric forms. In this section we will construct different families of infinite nonzero integer solutions to the equation:

$$(x+y+z)^2 = xyz. \tag{5.6.15}$$

Following our paper [8] we will indicate a general method of generating such families of solutions. We start by performing the transformations

$$x = \frac{u+v}{2} + a, \quad y = \frac{u-v}{2} + a, \quad z = b \tag{5.6.16}$$

where a and b are nonzero integer parameters that will be determined in a convenient manner. The equation becomes

$$(u+2a+b)^2 = \frac{b}{4}(u^2-v^2) + abu + a^2b.$$

Imposing the conditions $2(2a+b) = ab$ and $b(b-4) > 0$ yields the general Pell's equation

$$(b-4)u^2 - bv^2 = 4[(2a+b)^2 - a^2b]. \tag{5.6.17}$$

The imposed conditions are equivalent to $(a-2)(b-4) = 8$, and $b < 0$ or $b > 4$. A simple case analysis shows that the only pairs of integers (a,b) satisfying them are: $(1,-4)$, $(3,12)$, $(4,8)$, $(6,6)$, $(10,5)$.

The following table contains the general Pell's equations (5.6.17) corresponding to the above pairs (a,b), their Pell's resolvents, both equations with their fundamental solutions.

(a,b)	General Pell's equation (5.6.17) and its fundamental solution	Pell's resolvent and its fundamental solution
$(1,-4)$	$2u^2 - v^2 = -8,\ (2,4)$	$r^2 - 2s^2 = 1,\ (3,2)$
$(3,12)$	$2u^2 - 3v^2 = 216,\ (18,12)$	$r^2 - 6s^2 = 1,\ (5,2)$
$(4,8)$	$u^2 - 2v^2 = 128,\ (16,8)$	$r^2 - 2s^2 = 1,\ (3,2)$
$(6,6)$	$u^2 - 3v^2 = 216,\ (18,6)$	$r^2 - 3s^2 = 1,\ (2,1)$
$(10,5)$	$u^2 - 5v^2 = 500,\ (25,5)$	$r^2 - 5s^2 = 1,\ (9,4)$

By using the formula (4.5.2) we obtain the following sequences of solutions to the equations (5.6.17):

$$u_m^{(1)} = 2r_m^{(1)} + 4s_m^{(1)}, \quad v_m^{(1)} = 4r_m^{(1)} + 4s_m^{(1)}$$

where $r_m^{(1)} + s_m^{(1)}\sqrt{2} = (3+2\sqrt{2})^m$, $m \geq 0$;

$$u_m^{(2)} = 18r_m^{(2)} + 36s_m^{(2)}, \quad v_m^{(2)} = 12r_m^{(2)} + 36s_m^{(2)}$$

where $r_m^{(2)} + s_m^{(2)}\sqrt{6} = (5+2\sqrt{6})^m$, $m \geq 0$;

$$u_m^{(3)} = 16r_m^{(3)} + 16s_m^{(3)}, \quad v_m^{(3)} = 8r_m^{(3)} + 16s_m^{(3)}$$

where $r_m^{(3)} + s_m^{(3)}\sqrt{2} = (3+2\sqrt{2})^m$, $m \geq 0$;

$$u_m^{(4)} = 18r_m^{(4)} + 18s_m^{(4)}, \quad v_m^{(4)} = 6r_m^{(4)} + 18s_m^{(4)}$$

where $r_m^{(4)} + s_m^{(4)}\sqrt{3} = (2+\sqrt{3})^m$, $m \geq 0$;

$$u_m^{(5)} = 25r_m^{(5)} + 25s_m^{(5)}, \quad v_m^{(5)} = 5r_m^{(5)} + 25s_m^{(5)}$$

where $r_m^{(5)} + s_m^{(5)}\sqrt{5} = (9+4\sqrt{5})^m$, $m \geq 0$.

Formulas (5.6.16) yield the following five families of nonzero integer solutions to the equation (5.6.15):

$$\begin{aligned}
x_m^{(1)} &= 3r_m^{(1)} + 4s_m^{(1)} + 1, \quad y_m^{(1)} = -r_m^{(1)} + 1, \quad z_m^{(1)} = -4, \quad m \geq 0\\
x_m^{(2)} &= 15r_m^{(2)} + 36s_m^{(2)} + 3, \quad y_m^{(2)} = 3r_m^{(2)} + 3, \quad z_m^{(2)} = 12, \quad m \geq 0\\
x_m^{(3)} &= 12r_m^{(3)} + 16s_m^{(3)} + 4, \quad y_m^{(3)} = 4r_m^{(3)} + 4, \quad z_m^{(3)} = 8, \quad m \geq 0\\
x_m^{(4)} &= 12r_m^{(4)} + 18s_m^{(4)} + 6, \quad y_m^{(4)} = 6r_m^{(4)} + 6, \quad z_m^{(4)} = 6, \quad m \geq 0\\
x_m^{(5)} &= 15r_m^{(5)} + 25s_m^{(5)} + 10, \quad y_m^{(5)} = 10r_m^{(5)} + 10, \quad z_m^{(5)} = 5, \quad m \geq 0.
\end{aligned}$$

Remark. In the recent paper [78] the following approach to generates solutions to the equation (5.6.15) is indicated. Taking $z = kx - y$, for some integer k, our equation is equivalent to $y^2 - kxy + x(k+1)^2 = 0$, which is a quadratic equation in y, hence

$$y = \frac{1}{2}\left(kx \pm \sqrt{k^2x^2 - 4(k+1)^2x}\right).$$

Now, let $k^2x^2 - 4(k+1)^2x = a^2$, for some integer a. Treat this relation as a quadratic equation in x, we have

$$x = 2(k+1)^2 \pm \sqrt{4(k+1)^4 + k^2a^2}.$$

Again, consider $4(k+1)^4 + k^2a^2 = b^2$, for some integer b. Considering the last equation as $(2(k+1)^2)^2 + (ka)^2 = b^2$, which is a Pythagorean, we get the following two possible situations

$$\begin{cases} b = u^2 + v^2 \\ ka = u^2 - v^2 \\ (k+1)^2 = uv \end{cases} \quad \text{and} \quad \begin{cases} b = u^2 + v^2 \\ ka = 2uv \\ 2(k+1)^2 = u^2 - v^2 \end{cases}$$

where $u, v \in \mathbb{Z}$. Therefore, in order to generate solutions to equation (5.6.15), we start with two integers u, v such that uv or $\dfrac{u^2 - v^2}{2}$ is a perfect square $(k+1)^2$. Then, we find $a = \dfrac{u^2 - v^2}{k}$ or $a = \dfrac{2uv}{k}$, and $b = u^2 + v^2$. Finally, we obtain

$$x = 2(k+1)^2 \pm b, \quad y = \frac{1}{2}(kx \pm a), \quad z = kx - y.$$

Clearly, every pair (u, v) generates at most two values of k for each system considered above. Let us illustrate the method by the following special situation.

Example. Let $ka = \dfrac{3}{2}(k+1)^2$, $b = \dfrac{5}{2}(k+1)^2$, be the special solutions to the equation $(2(k+1)^2)^2 + (ka)^2 = b^2$. Then we obtain from families (x, y) the solutions:

$$\left(\frac{9(k+1)^2}{2k^2}, \frac{6(k+1)^2}{k}\right), \quad \left(\frac{9(k+1)^2}{2k^2}, \frac{3(k+1)^2}{k}\right),$$
$$\left(-\frac{(k+1)^2}{2k^2}, \frac{(k+1)^2}{k}\right), \quad \left(-\frac{(k+1)^2}{2k^2}, -\frac{2(k+1)^2}{k}\right).$$

In order to get integer solutions, the only possibilities are $k = -3, -1, 1, 3$, giving the solutions $(0, 0, 0)$, $(18, 12, 6)$, $(8, 16, 8)$, $(-2, 2-4)$, $(-2, -8, 6)$.

5.6.4 The Equation $(x+y+z+t)^2 = xyzt$

Using the method described in Section 5.6.3 we will generate nine infinite families of positive integer solutions to the equation

$$(x+y+z+t)^2 = xyzt. \tag{5.6.18}$$

We will follow the paper [9].

The transformations

$$x = \frac{u+v}{2} + a, \quad y = \frac{u-v}{2} + a, \quad z = b, \quad t = c \tag{5.6.19}$$

where a, b, c are positive integers, bring the equation (5.6.18) to the form

$$(u + 2a + b + c)^2 = \frac{bc}{4}(u^2 - v^2) + abcu + a^2bc.$$

Setting the conditions $2(2a+b+c)=abc$ and $bc>4$, we obtain the following general Pell's equation

$$(bc-4)u^2-bcv^2=4[(2a+b+c)^2-a^2bc]. \tag{5.6.20}$$

There are nine triples (a,b,c) up to permutations satisfying the above conditions: (1,6,4), (1,10,3), (2,2,6), (3,4,2), (3,14,1), (5,2,3), (4,1,9), (7,1,6), (12,1,5).

The following table contains the general Pell's equations (5.6.20) corresponding to the above triples (a,b,c), their Pell's resolvent, both equations with their fundamental solutions.

(a,b,c)	General Pell's equation (5.6.20) and its fundamental solution	Pell's resolvent and its fundamental solution
$(1,6,4)$	$5u^2-6v^2=120,\ (12,10)$	$r^2-30s^2=1,\ (11,2)$
$(1,10,3)$	$13u^2-15v^2=390,\ (15,13)$	$r^2-195s^2=1,\ (14,1)$
$(2,2,6)$	$2u^2-3v^2=96,\ (12,8)$	$r^2-6s^2=1,\ (5,2)$
$(3,4,2)$	$u^2-2v^2=72,\ (12,6)$	$r^2-2s^2=1,\ (3,2)$
$(3,14,1)$	$5u^2-7v^2=630,\ (21,15)$	$r^2-35s^2=1,\ (6,1)$
$(4,1,9)$	$5u^2-9v^2=720,\ (42,30)$	$r^2-45s^2=1,\ (161,24)$
$(5,2,3)$	$u^2-3v^2=150,\ (15,5)$	$r^2-3s^2=1,\ (2,1)$
$(7,1,6)$	$u^2-3v^2=294,\ (21,7)$	$r^2-3s^2=1,\ (2,1)$
$(12,1,5)$	$u^2-5v^2=720,\ (30,6)$	$r^2-5s^2=1,\ (9,4)$

By using the formula (4.4.2) we obtain the following sequences of solutions to equations (5.6.20):

$$u_m^{(1)}=12r_m^{(1)}+60s_m^{(1)},\quad v_m^{(1)}=10r_m^{(1)}+60s_m^{(1)},$$

where $r_m^{(1)}+s_m^{(1)}\sqrt{30}=(11+2\sqrt{30})^m,\ m\geq 0$;

$$u_m^{(2)}=15r_m^{(2)}+195s_m^{(2)},\quad v_m^{(2)}=13r_m^{(2)}+195s_m^{(2)},$$

where $r_m^{(2)}+s_m^{(2)}\sqrt{195}=(14+\sqrt{195})^m,\ m\geq 0$;

$$u_m^{(3)}=12r_m^{(3)}+24s_m^{(3)},\quad v_m^{(3)}=8r_m^{(3)}+24s_m^{(3)},$$

where $r_m^{(3)}+s_m^{(3)}\sqrt{6}=(5+2\sqrt{6})^m,\ m\geq 0$;

$$u_m^{(4)}=12r_m^{(4)}+12s_m^{(4)},\quad v_m^{(4)}=6r_m^{(4)}+12s_m^{(4)},$$

where $r_m^{(4)} + s_m^{(4)}\sqrt{2} = (3+2\sqrt{2})^m,\ m \ge 0$;

$$u_m^{(5)} = 21r_m^{(5)} + 105s_m^{(5)}, \quad v_m^{(5)} = 15r_m^{(5)} + 105s_m^{(5)},$$

where $r_m^{(5)} + s_m^{(5)}\sqrt{35} = (6+\sqrt{35})^m,\ m \ge 0$;

$$u_m^{(6)} = 42r_m^{(6)} + 270s_m^{(6)}, \quad v_m^{(6)} = 30r_m^{(6)} + 210s_m^{(6)},$$

where $r_m^{(6)} + s_m^{(6)}\sqrt{45} = (161+24\sqrt{45})^m,\ m \ge 0$;

$$u_m^{(7)} = 15r_m^{(7)} + 15s_m^{(7)}, \quad v_m^{(7)} = 5r_m^{(7)} + 15s_m^{(7)},$$

where $r_m^{(7)} + s_m^{(7)}\sqrt{3} = (2+\sqrt{3})^m,\ m \ge 0$;

$$u_m^{(8)} = 21r_m^{(8)} + 21s_m^{(8)}, \quad v_m^{(8)} = 7r_m^{(8)} + 21s_m^{(8)},$$

where $r_m^{(8)} + s_m^{(8)}\sqrt{3} = (2+\sqrt{3})^m,\ m \ge 0$;

$$u_m^{(9)} = 30r_m^{(9)} + 30s_m^{(9)}, \quad v_m^{(9)} = 6r_m^{(9)} + 30s_m^{(9)},$$

where $r_m^{(9)} + s_m^{(9)}\sqrt{5} = (9+4\sqrt{5})^m,\ m \ge 0$.

Formulas (5.6.19) yield the following nine families of positive integers solutions to the equation (5.6.18):

$$x_m^{(1)} = 11r_m^{(1)} + 60s_m^{(1)} + 1, \quad y_m^{(1)} = r_m^{(1)} + 1, \quad z_m^{(1)} = 6, \quad t_m^{(1)} = 4$$
$$x_m^{(2)} = 14r_m^{(2)} + 195s_m^{(2)} + 1, \quad y_m^{(2)} = r_m^{(2)} + 1, \quad z_m^{(2)} = 10, \quad t_m^{(2)} = 3$$
$$x_m^{(3)} = 10r_m^{(3)} + 24s_m^{(3)} + 2, \quad y_m^{(3)} = 2r_m^{(3)} + 2, \quad z_m^{(3)} = 2, \quad t_m^{(3)} = 6$$
$$x_m^{(4)} = 12r_m^{(4)} + 12s_m^{(4)} + 3, \quad y_m^{(4)} = 3r_m^{(4)} + 3, \quad z_m^{(4)} = 4, \quad t_m^{(4)} = 2$$
$$x_m^{(5)} = 18r_m^{(5)} + 105s_m^{(5)} + 3, \quad y_m^{(5)} = 3r_m^{(5)} + 3, \quad z_m^{(5)} = 14, \quad t_m^{(5)} = 1$$
$$x_m^{(6)} = 36r_m^{(6)} + 240s_m^{(6)} + 4, \quad y_m^{(6)} = 6r_m^{(6)} + 30s_m^{(6)} + 4, \quad z_m^{(6)} = 1, \quad t_m^{(6)} = 9$$
$$x_m^{(7)} = 10r_m^{(7)} + 15s_m^{(7)} + 5, \quad y_m^{(7)} = 5r_m^{(7)} + 5, \quad z_m^{(7)} = 2, \quad t_m^{(7)} = 3$$
$$x_m^{(8)} = 14r_m^{(8)} + 21s_m^{(8)} + 7, \quad y_m^{(8)} = 7r_m^{(8)} + 7, \quad z_m^{(8)} = 1, \quad t_m^{(8)} = 6$$
$$x_m^{(9)} = 18r_m^{(9)} + 30s_m^{(9)} + 12, \quad y_m^{(9)} = 12r_m^{(9)} + 12, \quad z_m^{(9)} = 1, \quad t_m^{(9)} = 5.$$

Remarks. 1) In [194] only solution $(x_m^{(7)}, y_m^{(7)}, z_m^{(7)}, t_m^{(7)})$ is found.
2) Note the atypical form of solution $(x_m^{(6)}, y_m^{(6)}, z_m^{(6)}, t_m^{(6)})_{m\ge 0}$.

5.6.5 The Equation $(x+y+z+t)^2 = xyzt+1$

The equation

$$(x+y+z+t)^2 = xyzt+1 \tag{5.6.21}$$

is considered in the paper [79], where the method to generate families of solutions is similar to the one described in the previous section. Introduction of the linear transformations

$$x = u+v+a,\ y = u-v+a,\ z = b,\ t = c, \tag{5.6.22}$$

where a, b, c are positive integers, leads (5.6.21) to the form

$$(bc-4)u^2 - bcv^2 = (2a+b+c)^2 - a^2bc - 1, \tag{5.6.23}$$

in which $bc > 4$ and $2(2a+b+c) = abc$. There are six triples (a, b, c) satisfying the above conditions, namely (5,2,3), (7,1,6), (12,1,5), (1,10,3), (3,14,1), (4,1,9). The following table contains the general Pell's equations (5.6.23) corresponding to the above triples (a, b, c), their Pell's resolvent, both equations with their fundamental solutions.

(a,b,c)	General Pell's equation (5.6.23) and its fundamental solution	Pell's resolvent and its fundamental solution
$(5,2,3)$	$u^2-3v^2=37,\ (7,2)$	$r^2-3s^2=1,\ (2,1)$
$(7,1,6)$	$u^2-3v^2=73,\ (10,3)$	$r^2-3s^2=1,\ (2,1)$
$(12,1,5)$	$u^2-5v^2=179,\ (28,11)$	$r^2-5s^2=1,\ (9,4)$
$(1,10,3)$	$13u^2-15v^2=97,\ (7,6)$	$r^2-195s^2=1,\ (14,1)$
$(3,14,1)$	$5u^2-7v^2=157,\ (10,7)$	$r^2-35s^2=1,\ (6,1)$
$(4,1,9)$	$5u^2-9v^2=179,\ (50,37)$	$r^2-45s^2=1,\ (161,24)$

In view of the formula (4.4.2), the following sequences are six families of positive integer solutions to the corresponding general Pell's equations (5.6.22):

$$u_m^{(1)} = 7r_m^{(1)} + 6s_m^{(1)},\ v_m^{(1)} = 2r_m^{(1)} + 7s_m^{(1)},$$

where $r_m^{(1)} + s_m^{(1)}\sqrt{3} = (2+\sqrt{3})^m,\ m \geq 0.$

$$u_m^{(2)} = 10r_m^{(2)} + 9s_m^{(2)},\ v_m^{(2)} = 3r_m^{(2)} + 10s_m^{(2)},$$

where $r_m^{(2)} + s_m^{(2)}\sqrt{3} = (2+\sqrt{3})^m,\ m \geq 0.$

$$u_m^{(3)} = 28r_m^{(3)} + 55s_m^{(3)},\ v_m^{(3)} = 11r_m^{(3)} + 28s_m^{(3)},$$

where $r_m^{(3)} + s_m^{(3)}\sqrt{5} = (9 + 4\sqrt{5})^m,\ m \geq 0.$

$$u_m^{(4)} = 7r_m^{(4)} + 90s_m^{(4)},\ v_m^{(4)} = 6r_m^{(4)} + 91s_m^{(4)},$$

where $r_m^{(4)} + s_m^{(4)}\sqrt{195} = (14 + \sqrt{195})^m,\ m \geq 0.$

$$u_m^{(5)} = 10r_m^{(5)} + 49s_m^{(5)},\ v_m^{(5)} = 7r_m^{(5)} + 50s_m^{(5)},$$

where $r_m^{(5)} + s_m^{(5)}\sqrt{35} = (6 + \sqrt{35})^m,\ m \geq 0.$

$$u_m^{(6)} = 50r_m^{(6)} + 333_m^{(6)},\ v_m^{(6)} = 37r_m^{(6)} + 250s_m^{(6)},$$

where $r_m^{(6)} + 3s_m^{(6)}\sqrt{5} = (161 + 72\sqrt{5})^m,\ m \geq 0.$

Formulas (5.6.22) yield the following six families of positive integers solutions to the equation (5.6.21):

$$x_m^{(1)} = 9r_m^{(1)} + 13s_m^{(1)} + 5,\ y_m^{(1)} = 5r_m^{(1)} - s_m^{(1)} + 5,\ z_m^{(1)} = 2,\ t_m^{(1)} = 3$$
$$x_m^{(2)} = 13r_m^{(2)} + 19s_m^{(2)} + 7,\ y_m^{(2)} = 7r_m^{(2)} - s_m^{(2)} + 7,\ z_m^{(2)} = 1,\ t_m^{(2)} = 6$$
$$x_m^{(3)} = 39r_m^{(3)} + 83s_m^{(3)} + 12,\ y_m^{(3)} = 17r_m^{(3)} + 27s_m^{(3)} + 12,\ z_m^{(3)} = 1,\ t_m^{(3)} = 5$$
$$x_m^{(4)} = 13r_m^{(4)} + 181s_m^{(4)} + 1,\ y_m^{(4)} = r_m^{(4)} - s_m^{(4)} + 1,\ z_m^{(4)} = 10,\ t_m^{(4)} = 3$$
$$x_m^{(5)} = 17r_m^{(5)} + 99s + 3,\ y_m^{(5)} = 3r_m^{(5)} - s_m^{(5)} + 3,\ z_m^{(5)} = 14,\ t_m^{(5)} = 1$$
$$x_m^{(6)} = 87r_m^{(6)} + 583s_m^{(6)} + 4,\ y_m^{(6)} = 13r_m^{(6)} + 83s_m^{(6)} + 4,\ z_m^{(6)} = 1,\ t_m^{(6)} = 9.$$

5.6.6 The Equation $x^3 + y^3 + z^3 + t^3 = n$

We will prove that if the equation

$$x^3 + y^3 + z^3 + t^3 = n \tag{5.6.24}$$

has an integral solution (a, b, c, d) such that $a \neq b$ or $c \neq d$ and $-(a+b)(c+d) > 0$ is not a perfect square, then it has infinitely many integral solutions.

For this, let us perform the transformations:

$$x = X + a, \quad y = -X + b, \quad z = Y + c, \quad t = -Y + d.$$

Then $(a + b)X^2 + (a^2 - b^2)X + (c + d)Y^2 + (c^2 - d^2)Y = 0$. The last equation is equivalent to

$$(a+b)\left(X+\frac{a-b}{2}\right)^2+(c+d)\left(Y+\frac{c-d}{2}\right)^2$$
$$=\frac{(a+b)(a-b)^2}{4}+\frac{(c+d)(c-d)^2}{4}. \tag{5.6.25}$$

From the hypothesis, (5.6.25) is a general Pell's equation:

$$AU^2-BV^2=C \tag{5.6.26}$$

where $A=a+b$, $B=-(c+d)$, $C=\frac{1}{4}[(a+b)(a-b)^2+(c+d)(c-d)^2]$ and $U=X+\frac{a-b}{2}$, $V=Y+\frac{c-d}{2}$.

We note that $(U_0,V_0)=\left(\frac{a-b}{2},\frac{c-d}{2}\right)$ satisfies the equation (5.6.26) and consider the Pell's resolvent $r^2-Ds^2=1$, where $D=-(a+b)(c+d)$, with the general solution $(r_m,s_m)_{m\geq 0}$. From the formula (4.5.2), we obtain the solutions $(U_m,V_m)_{m\geq 0}$ where

$$U_m=\frac{a-b}{2}r_m-(c+d)\frac{c-d}{2}s_m$$
$$V_m=\frac{c-d}{2}r_m+(a+b)\frac{a-b}{2}s_m.$$

It follows that

$$X_m=\frac{a-b}{2}r_m-\frac{c^2-d^2}{2}s_m-\frac{a-b}{2}$$
$$Y_m=\frac{c-d}{2}r_m+\frac{a^2-b^2}{2}s_m-\frac{c-d}{2}.$$

From these formulas we generate an infinite family of solutions $(x_m,y_m,z_m,t_m)_{m\geq 0}$ to the equation (5.6.24):

$$\begin{cases} x_m=\frac{a-b}{2}r_m-\frac{c^2-d^2}{2}s_m+\frac{a+b}{2} \\ y_m=-\frac{a-b}{2}r_m+\frac{c^2-d^2}{2}s_m+\frac{a+b}{2} \\ z_m=\frac{c-d}{2}r_m+\frac{a^2-b^2}{2}s_m+\frac{c+d}{2} \\ t_m=-\frac{c-d}{2}r_m-\frac{a^2-b^2}{2}s_m+\frac{c+d}{2}. \end{cases} \tag{5.6.27}$$

Remarks. 1) The main idea of the approach described above comes from [150] and all computations are given in [10].

2) A special case of equation (5.6.24) appears in the book [24]: Prove that the equation

$$x^3 + y^3 + z^3 + t^3 = 1999$$

has infinitely many integral solutions (1999 Bulgarian Mathematical Olympiad). In this case, one simple solution to the given equation is $(a, b, c, d) = (10, 10, -1, 0)$. By using formulas (5.6.27), we obtain the following infinite family of solutions:

$$(x_m, y_m, z_m, t_m) = \left(-\frac{1}{2}s_m + 10, \frac{1}{2}s_m + 10, -\frac{1}{2}(r_m + 1), \frac{1}{2}(r_m - 1)\right),$$

where $r_m + s_m\sqrt{20} = (9 + 2\sqrt{20})^m$, $m \geq 0$. It is not difficult to see that the integers r_m are all odd and that s_m are all even.