Chapter 4

Integrability

4.1 Introduction

The problem of integrability in classical mechanics has been a seminal one. Motivated by celestial mechanics, it has stimulated a wealth of analytical methods and results. For example, as we have discussed in Chapter 2, the weaker requirement of only approximate integrability over finite times, or the existence of integrable regions in the phase space of a globally nonintegrable system, has led to the development of classical perturbation theory, with all its important achievements. However, deciding whether a given Hamiltonian system is globally integrable still remains a difficult task, for which a general constructive framework is lacking.

The topic of integrability is a vast one, and reviewing it is beyond the aim of the present monograph. For the sake of completeness, in this chapter we briefly discuss how the classical problem of integrability is rephrased in the Riemannian-geometric framework for the Hamiltonian dynamics introduced in Chapter 3.

In general, the existence of conservation laws, and of conserved quantities along the trajectories of a Hamiltonian system, is related to the existence of symmetries. The link is made by Noether's theorem [133]. A symmetry is seen as an invariance under the action of a group of transformations, and in the case of continuous symmetries, this can be related also to the existence of special vector fields: Killing vector fields on the mechanical manifold generating the transformations.

On a generic manifold M, a flow $\sigma : \mathbb{R} \times M \to M$ is generated by the ensemble of the integral curves of a vector field X on the manifold:

$$\frac{d}{dt}\sigma^\mu(t, x_0) = X^\mu(\sigma(t, x_0)) \ , \quad \mu \in \{1, \ldots, \dim(M)\} \ . \tag{4.1}$$

Given $t \in \mathbb{R}$, $\sigma(t, \cdot)$ is a diffeomorphism of M to itself. The summation in $\mathbb{R}$ endows σ with the structure of a commutative group

$$\sigma(t, \cdot) \circ \sigma(s, \cdot) = \sigma(t + s, \cdot); \tag{4.2}$$

such a group is called a one-parameter group of transformations $\sigma_t : M \to M$. Under the action of σ_ϵ, with ϵ infinitesimal, a point x of coordinates x^μ is transformed as

$$\sigma^\mu_\epsilon(x) = \sigma^\mu(\epsilon, x) = x^\mu + \epsilon X^\mu(x). \tag{4.3}$$

In this framework, the vector field X is called the infinitesimal generator of the transformation σ_ϵ. If two flows are given, $\sigma(t, x)$ and $\tau(t, x)$, generated by the vector fields X and Y respectively,

$$\frac{d\sigma^\mu(s, x)}{ds} = X^\mu(\sigma(s, x)) \ , \tag{4.4}$$

$$\frac{d\tau^\mu(t, x)}{dt} = Y^\mu(\tau(t, x)) \ , \tag{4.5}$$

the Lie derivative $\mathcal{L}_X Y$ of the vector field Y along the flow σ of X is defined by

$$(\mathcal{L}_X Y)(x) = \lim_{\epsilon \to 0} \frac{1}{\epsilon}[(\sigma_{-\epsilon})_*(Y(\sigma_\epsilon(x))) - Y(x)] \ , \tag{4.6}$$

where by $(\sigma_{-\epsilon})_* : T_{\sigma_\epsilon(x)}M \to T_x M$ we denote the derivative of $\sigma_{-\epsilon}$. This amounts to evaluating the variation of a vector field Y along a flow of σ, and this can also be extended to a tensor field A:

$$(\mathcal{L}_X A)(x) = \lim_{\epsilon \to 0} \frac{1}{\epsilon}[(\sigma_{-\epsilon})_*(A(\sigma_\epsilon(x))) - A(x)] \ . \tag{4.7}$$

On Riemannian manifolds (M, g), a special class of vector fields can be defined: Killing vector fields. A field X is such a vector field if

$$\mathcal{L}_X g = 0 \ . \tag{4.8}$$

It directly follows from (4.7) that a vector field is a Killing field iff the one-parameter group of transformations associated with it is an isometry.[1] This means that along the flow σ_t, geometry does not change, and therefore a Killing field represents an infinitesimal symmetry of the manifold. However, through Noetherian symmetries, and thus Killing vector fields, only a limited set of conservation laws can be accounted for. This is easily understood because only invariants that are linear functions of the momenta can be constructed by means of Killing vectors, while the energy, an invariant for any autonomous Hamiltonian system, is already a quadratic function of the momenta. The possibility of constructing invariants along a geodesic flow that are of higher order than linear in the momenta is related to the existence of Killing *tensor* fields on the mechanical manifolds [134–136].

In general, the components of any Killing tensor field on a mechanical manifold are solutions of a linear inhomogeneous system of first-order partial differential equations. Since the number of these equations always exceeds

[1] An isometry $f : M \to M$ is a diffeomorphism preserving distance: $f^* g = g$.

the number of the unknowns, the system is always *overdetermined*. The existence of Killing tensors thus requires *compatibility*. However, compatibility is generically very unusual, which suggests a possible explanation, at least of a qualitative kind, of the exceptionality of integrability with respect to nonintegrability.

4.2 Killing Vector Fields

On a Riemannian manifold, for any pair of vectors V and W, the following relation holds:

$$\frac{d}{ds}\langle V, W\rangle = \left\langle \frac{\nabla V}{ds}, W\right\rangle + \left\langle V, \frac{\nabla W}{ds}\right\rangle \ , \tag{4.9}$$

where $\langle V, W\rangle = g_{ij}V^iW^j$ and ∇/ds is the covariant derivative along a curve $\gamma(s)$. If the curve $\gamma(s)$ is a geodesic, for a generic vector X we have

$$\frac{d}{ds}\langle X, \dot\gamma\rangle = \left\langle \frac{\nabla X}{ds}, \dot\gamma\right\rangle + \left\langle X, \frac{\nabla \dot\gamma}{ds}\right\rangle = \left\langle \frac{\nabla X}{ds}, \dot\gamma\right\rangle \equiv \langle \nabla_{\dot\gamma} X, \dot\gamma\rangle \ , \tag{4.10}$$

where $(\nabla_{\dot\gamma}X)^i = \frac{dx^l}{ds}\frac{\partial X^i}{\partial x^l} + \Gamma^i_{jk}\frac{dx^j}{ds}X^k$, so that in components it reads

$$\frac{d}{ds}(X_i v^i) = v^i\nabla_i(X_j v^j) \ , \tag{4.11}$$

where $v^i = dx^i/ds$; with $X_j v^i \nabla_i v^j = X_j \nabla_{\dot\gamma}\dot\gamma^j = 0$, because geodesics are autoparallel, this can obviously be rewritten as

$$\frac{d}{ds}(X_i v^i) = \frac{1}{2}v^j v^i(\nabla_i X_j + \nabla_j X_i) \ , \tag{4.12}$$

telling that the vanishing of the left-hand side, i.e., the conservation of $X_i v^i$ along a geodesic, is guaranteed by the vanishing of the right-hand side, i.e.,

$$\nabla_{(i}X_{j)} \equiv \nabla_i X_j + \nabla_j X_i = 0 \ , \qquad i,j = 1,\ldots,\dim M_E. \tag{4.13}$$

If such a field exists on a manifold, it is called a Killing vector field (KVF). Equation (4.13) is equivalent to $\mathcal{L}_X g = 0$. On the mechanical manifolds (M_E, g_J), the unit vector $\frac{dq^k}{ds}$ is proportional to the canonical momentum $p_k = \frac{\partial L}{\partial \dot q^k} = \dot q^k$, $(a_{ij} = \delta_{ij})$, and is tangent to a geodesic. The existence of a KVF X implies that the quantity, linear in the momenta,

$$J(q,p) = X_k(q)\frac{dq^k}{ds} = \frac{1}{\sqrt{2}(E - V(q))}X_k(q)\frac{dq^k}{dt} = \frac{1}{\sqrt{2}W(q)}\sum_{k=1}^{N} X_k(q)p_k \tag{4.14}$$

is a constant of motion along the geodesic flow. Thus, for an N-degrees-of-freedom Hamiltonian system, a physical conservation law involving a conserved quantity linear in the canonical momenta can always be related to a symmetry on the manifold (M_E, g_J) due to the action of a KVF on the manifold. These are conservation laws of Noetherian kind. Equation (4.13) is equivalent to the vanishing of the Poisson brackets

$$\{H, J\} = \sum_{i=1}^{N} \left(\frac{\partial H}{\partial q^i} \frac{\partial J}{\partial p_i} - \frac{\partial H}{\partial p_i} \frac{\partial J}{\partial q^i} \right) = 0 \ , \tag{4.15}$$

the standard definition of a constant of motion $J(q, p)$. In fact, a linear function of the momenta

$$J(q, p) = \sum_i C_i(q) p_i \ , \tag{4.16}$$

if conserved, can be associated with the vector of components

$$X_k = [E - V(q)] C_k(q). \tag{4.17}$$

The explicit expression of the system of equations (4.13) is obtained by writing in components the covariant derivatives associated with the connection coefficients (3.17), and it finally reads

$$[E - V(q)] \left[\frac{\partial C_i(q)}{\partial q^j} + \frac{\partial C_j(q)}{\partial q^i} \right] - \delta_{ij} \sum_{k=1}^{N} \frac{\partial V}{\partial q^k} C_k(q) = 0, \tag{4.18}$$

or equivalently

$$\frac{1}{2} \sum_{k=1}^{N} p_k^2 \left[\frac{\partial C_i(q)}{\partial q^j} + \frac{\partial C_j(q)}{\partial q^i} \right] - \delta_{ij} \sum_{k=1}^{N} \frac{\partial V}{\partial q^k} C_k(q) = 0, \tag{4.19}$$

which, according to the principle of polynomial identity, yields the following conditions for the coefficients $C_i(q)$:

$$\begin{aligned} \frac{\partial C_i(q)}{\partial q^j} + \frac{\partial C_j(q)}{\partial q^i} &= 0 \ , \quad i \neq j \ , \quad i, j = 1, \dots, N \ , \\ \frac{\partial C_i(q)}{\partial q^i} &= 0 \ , \quad i = 1, \dots, N \ , \\ \sum_{k=1}^{N} \frac{\partial V}{\partial q^k} C_k(q) &= 0 \ . \end{aligned} \tag{4.20}$$

One can easily check that the same conditions stem from (4.15). As an elementary example, we can give the explicit expression of the components of the Killing vector field associated with the conservation of the total momentum $P(q, p) = \sum_{k=1}^{N} p_k$.

In this case the coefficients are $C_i(q) = 1$, so that the momentum conservation can be geometrically related to the action of the vector field of components $X_i = E - V(q),\ i = 1, \dots, N$, on the mechanical manifold. At least this class of invariants has a geometric counterpart in a symmetry of (M_E, g_J).

However, in order to achieve a fully geometric rephrasing of integrability, we need something similar for any constant of motion. If a one-to-one correspondence is to exist between conserved physical quantities along a Hamiltonian flow and suitable symmetries of the mechanical manifolds (M_E, g_J), then *integrability* will be equivalent to the existence of a number of symmetries at least equal to the number of degrees of freedom ($=$ dim M_E).

If a Lie group G acts on the phase space manifold through completely canonical transformations, and there exists an associated *momentum mapping*,[2] then every Hamiltonian having G as a symmetry group, with respect to its action, admits the momentum mapping as a constant of motion [137]. These symmetries are usually referred to as *hidden symmetries*, because even though their existence is ensured by integrability, they are not easily recognizable.[3]

4.3 Killing Tensor Fields

Let us now extend what has been just presented about KVFs in an attempt trying to generalize the form of the conserved quantity along a geodesic flow from $J = X_i v^i$ to $J = K_{j_1 j_2 \dots j_r} v^{j_1} v^{j_2} \cdots v^{j_r}$, with $K_{j_1 j_2 \dots j_r}$ a tensor of rank r. Thus, we look for the conditions that entail

$$\frac{d}{ds}(K_{j_1 j_2 \dots j_r} v^{j_1} v^{j_2} \cdots v^{j_r}) = v^j \nabla_j (K_{j_1 j_2 \dots j_r} v^{j_1} v^{j_2} \cdots v^{j_r}) = 0 \ . \qquad (4.21)$$

In order to work out from this equation a condition for the existence of a suitable tensor $K_{j_1 j_2 \dots j_r}$, which is called a *Killing tensor field* (KTF), let us first consider the rank-$2r$ tensor $K_{j_1 j_2 \dots j_r} v^{i_1} v^{i_2} \cdots v^{i_r}$ and its covariant derivative along a geodesic, i.e.,

$$\begin{aligned} & v^j \nabla_j (K_{j_1 j_2 \dots j_r} v^{i_1} v^{i_2} \dots v^{i_r}) \\ & \quad = v^j \left(\frac{\partial K_{j_1 \dots j_r}}{\partial x^j} - K_{l j_2 \dots j_r} \Gamma^l_{j_1 j} - \cdots - K_{j_1 \dots l} \Gamma^l_{j_r j} \right) v^{i_1} \cdots v^{i_r} \end{aligned}$$

[2] This happens whenever this action corresponds to the lifting to the phase space of the action of a Lie group on the configuration space.

[3] An interesting account of these hidden symmetries can be found in [138], where it is surmised that integrable motions of N-degrees-of-freedom systems are the "shadows" of free motions in symmetric spaces (for example, Euclidean spaces $\mathbb{R}^n$, hyperspheres $\mathbb{S}^n$, hyperbolic spaces $\mathbb{H}^n$) of sufficiently large dimension $n > N$.

$$+K_{j_1\dots j_r}\left(v^j\frac{\partial v^{i_1}}{\partial x^j}+\Gamma^{i_1}_{jl}v^lv^j\right)v^{i_2}\cdots v^{i_r}+\cdots$$

$$+K_{j_1\dots j_r}v^{i_1}\cdots v^{i_{r-1}}\left(v^j\frac{\partial v^{i_r}}{\partial x^j}+\Gamma^{i_r}_{jl}v^lv^j\right)$$

$$=v^{i_1}v^{i_2}\cdots v^{i_r}v^j\nabla_jK_{j_1j_2\dots j_r}\ , \tag{4.22}$$

where we have again used $v^j\nabla_jv^{i_k}=0$ along a geodesic, and a standard covariant differentiation formula B. Now, by contraction on the indices i_k and j_k, the rank-$2r$ tensor of the right-hand side of (4.22) provides a new expression for the right-hand side of (4.21), which reads

$$\frac{d}{ds}(K_{j_1j_2\dots j_r}v^{j_1}v^{j_2}\cdots v^{j_r})=v^{j_1}v^{j_2}\cdots v^{j_r}v^j\nabla_{(j}K_{j_1j_2\dots j_r)}\ \ , \tag{4.23}$$

where $\nabla_{(j}K_{j_1j_2\dots j_r)}=\nabla_jK_{j_1j_2\dots j_r}+\nabla_{j_1}K_{jj_2\dots j_r}+\cdots+\nabla_{j_r}K_{j_1j_2\dots j_{r-1}j}$, as can be easily understood by rearranging the indices of the summations in the contraction of the $2r$-rank tensor in the last part of (4.22) (a direct check for the case $N=r=2$ is immediate). The vanishing of (4.23), entailing the conservation of $K_{j_1j_2\dots j_r}v^{j_1}v^{j_2}\cdots v^{j_r}$ along a geodesic flow, is therefore guaranteed by the existence of a tensor field satisfying the conditions

$$\nabla_{(j}K_{j_1j_2\dots j_r)}=0\ . \tag{4.24}$$

These equations generalize (4.13) and give the definition of a KTF on a Riemannian manifold. These N^{r+1} equations in $(N+r-1)!/r!(N-1)!$ unknown independent components[4] of the Killing tensor constitute an *over-determined* system of equations. Thus, a priori, we can expect that the existence of KTFs has to be rather exceptional.

If a KTF exists on a Riemannian manifold, then the scalar

$$K_{j_1j_2\dots j_r}\frac{dq^{j_1}}{ds}\frac{dq^{j_2}}{ds}\cdots\frac{dq^{j_r}}{ds} \tag{4.25}$$

is a constant of motion for the geodesic flow on the same manifold.

Let us consider, as a generalization of the special case of rank one given by (4.16), the constant of motion

$$J(q,p)=\sum_{\{i_1,i_2,\dots,i_N\}}C_{i_1i_2\dots i_N}p_1^{i_1}p_2^{i_2}\cdots p_N^{i_N}\,, \tag{4.26}$$

which, with the constraint $i_1+i_2+\cdots+i_N=r$, is a homogeneous polynomial of degree r. The index i_j denotes the power with which the momentum p_j contributes. If $r<N$ then necessarily some indices i_j must vanish. By repeating the procedure developed in the case $r=1$, and by identifying

[4] This number of independent components, i.e., the binomial coefficient $\binom{N+r-1}{r}$, is due to the totally symmetric character of Killing tensors.

$$J(q,p) \equiv K_{j_1 j_2 \dots j_r} \frac{dq^{j_1}}{ds}\frac{dq^{j_2}}{ds}\cdots\frac{dq^{j_r}}{ds} \ , \tag{4.27}$$

we get the relationship between the components of the Killing tensor of rank r and the coefficients $C_{i_1 i_2 \dots i_N}$ of the invariant $J(q,p)$, that is,

$$K_{\underbrace{1\dots1}_{i_1},\underbrace{2\dots2}_{i_2},\dots,\underbrace{N\dots N}_{i_N}} = 2^{r/2}[E - V(q)]^r C_{i_1 i_2 \dots i_N} \ . \tag{4.28}$$

With the only difference of a more tedious combinatorics, also in this case it turns out that the equations (4.24) are equivalent to the vanishing of the Poisson brackets of $J(q,p)$, that is,

$$\{H, J\} = 0 \Longleftrightarrow \nabla_{(j} K_{j_1 j_2 \dots j_r)} = 0 \ . \tag{4.29}$$

Thus, the existence of Killing tensor fields satisfying (4.24) on a mechanical manifold (M, g_J) provides the rephrasing of integrability of Newtonian equations of motion, or equivalently, of standard Hamiltonian systems, within the Riemannian-geometric framework.

At first sight, it might appear too restrictive that prime integrals of motion have to be homogeneous functions of the components of p. However, as we shall discuss in the next section, the integrals of motion of the known integrable systems can actually be cast in this form. This is in particular the case of total energy, a quantity conserved by any autonomous Hamiltonian system.

4.4 Explicit KTFs of Known Integrable Systems

The first natural question to address concerns the existence of a KT field, on any mechanical manifold (M, g_J), to be associated with total energy conservation. Such a KT field actually exists and coincides with the metric tensor g_J. In fact, by definition it satisfies[5] (4.24).

One of the simplest case of integrable system is represented by a decoupled system described by a generic Hamiltonian

$$H = \sum_{i=1}^{N} \left[\frac{p_i^2}{2} + V_i(q_i)\right] = \sum_{i=1}^{N} H_i(q_i, p_i) \tag{4.30}$$

for which all the energies E_i of the subsystems H_i, $i = 1, \dots, N$, are conserved. On the associated mechanical manifold, N KT fields of rank 2 exist. They are given by

$$K_{jk}^{(i)} = \delta_{jk}\{V_i(q_i)[E - V(q)] + \delta_j^i [E - V(q)]^2\} \ . \tag{4.31}$$

[5] A property of the canonical Levi-Civita connection, on which the covariant derivative is based, is just the vanishing of ∇g.

In fact, these tensor fields satisfy (4.24), which explicitly reads

$$\nabla_k K^{(i)}_{lm} + \nabla_l K^{(i)}_{mk} + \nabla_m K^{(i)}_{kl} = \frac{\partial K^{(i)}_{lm}}{\partial q^k} + \frac{\partial K^{(i)}_{mk}}{\partial q^l} + \frac{\partial K^{(i)}_{kl}}{\partial q^m} - 2\Gamma^j_{kl} K^{(i)}_{jm} - 2\Gamma^j_{km} K^{(i)}_{jl} - 2\Gamma^j_{lm} K^{(i)}_{jk} = 0 \ , \tag{4.32}$$

$$k, l, m = 1, \dots, N \ .$$

The conserved quantities $J^{(i)}(q,p)$ are then obtained by saturation of the tensors $K^{(i)}$ with the velocities dq/ds:

$$J^{(i)}(q,p) = \sum_{jk=1}^{N} K^{(i)}_{jk} \frac{dq^j}{ds}\frac{dq^k}{ds} = V_i(q_i) \frac{1}{E - V(q)} \sum_{k=1}^{N} \frac{p_k^2}{2} + \frac{p_i^2}{2} = E_i \ . \tag{4.33}$$

This equation suggests that to require that the constants of motion be homogeneous polynomials of the momenta is not so restrictive as might appear. In fact, through the constant quantity

$$\frac{1}{E - V(q)} \sum_{k=1}^{N} \frac{p_k^2}{2} = 1 \ , \tag{4.34}$$

homogeneous of second degree in the momenta, any even-degree polynomial of the momenta can be made homogeneous. The possibility of inferring the existence of a conservation law from the existence of a KTF on (M, g_J) is thus extended to the constants of motion given by a sum of homogeneous polynomials whose degrees differ by an even integer,

$$J(p,q) = P^{(r)}(p) + P^{(r-2)}(p) + \cdots + P^{(r-2n)}(p) \in C^\infty(q)[p_1, \dots, p_N] \tag{4.35}$$

$$\text{homdeg } P^s = s \ , \quad s = r, r-2, \dots, r - 2\left[\frac{r}{2}\right] \ ,$$

so that it can be recast in the homogeneous form

$$J(p,q) = P^{(r)}(p) + P^{(r-2)}(p) \frac{1}{E - V(q)} \sum_{k=1}^{N} \frac{p_k^2}{2} + \cdots + P^{(r-2n)}(p) \left[\frac{1}{E - V(q)} \sum_{k=1}^{N} \frac{p_k^2}{2} \right]^n \ . \tag{4.36}$$

4.4.1 Nontrivial Integrable Models

It is worth noting that the geodesic flow on an ellipsoid immersed in Euclidean three-dimensional space provides one of the simplest nontrivial examples of integrability. Besides the constant of motion obtained through the metric tensor (which corresponds to the energy for physical geodesic flows), the second constant of the motion is given by [139]

$$J^2 = c\sum_{i=1}^{3}(a^i)^{-2}\left(\frac{dx^i}{ds}\right)^2 ,$$

where c is a constant, a^i are half the major semiaxes, and x^i are the coordinates in the immersion space. According to what has been discussed above, this extra constant of motion has to correspond to a rank-2 KT.[6]

Nontrivial examples of nonlinear integrable Hamiltonian systems are provided by the following Hamiltonians:

$$H = \sum_{i=1}^{N}\left\{\frac{p_i^2}{2} + \frac{a}{b}[e^{-b(q_{i+1}-q_i)} - 1]\right\} , \tag{4.37}$$

known as the Toda model, which is integrable for any given pair of the constants a and b; and

$$H = \sum_{i=1}^{N}\frac{p_i^2}{2} + \frac{1}{2}\left(\sum_{i=1}^{N} q_i^2\right)^2 - \sum_{i=1}^{N}\lambda_i q_i^2 , \tag{4.38}$$

which is completely integrable for any $\lambda_1, \dots, \lambda_N$ [140]. Recursive formulas are available for all the constants of motion of the Toda model at any N [141]; and also for the second Hamiltonian, the exact form of first integrals is known [140]. In both cases, the first integrals are polynomials of given parity of the momenta so that on the basis of what we have said above, each invariant $J^{(i)}$, $i = 1, \dots, N$ can be derived from a KTF on (M, g_J). Thus, integrability of these systems admits a Riemannian-geometric interpretation.

Let us mention here another remarkable example of integrability that seems to demand a generalization of this Riemannian approach. It concerns a one-parameter family of Hamiltonian deformations of the Kepler problem leading to nonsymplectomorphic systems. Such deformations represent the motion of a charged particle in the field of a magnetic monopole with a special choice of the potential [142]. The components of a Runge–Lenz vector Poisson commute with the Hamiltonian and are quadratically dependent on the velocity. In order to associate with a geodesic flow the trajectories of a system subject to velocity-dependent forces, as is the case of the deformed Kepler models, the use of Finsler manifolds is necessary [118, 122], and thus

[6] Notice that the set of variables x^i is here redundant because of the algebraic equation defining the ellipsoid. In this case one has to consider the metric on the surface induced from $\mathbb{R}^3$, which, in contrast to the Jacobi metric on the mechanical manifolds, is not conformally flat.

a rephrasing of integrability through KT fields on Finsler manifolds could be necessary. To the best of our knowledge this is still an open problem.[7]

4.4.2 The Special Case of the $N = 2$ Toda Model

Let us consider the special case of a two-degrees-of-freedom Toda model described by the integrable Hamiltonian[8]

$$H = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{24}\left[e^{2y+2\sqrt{3}x} + e^{2y-2\sqrt{3}x} + e^{-4y}\right] - \frac{1}{8} \ . \tag{4.39}$$

From what is already reported in the literature [141], we know that a third-order polynomial of the momenta has to be found eventually. Therefore, we look for a rank-3 KT satisfying

$$\nabla_i K_{jkl} + \nabla_j K_{ikl} + \nabla_k K_{ijl} + \nabla_l K_{ijk} = 0 \ , \qquad i,j,k,l = 1,2 \ , \tag{4.40}$$

where, associating the label 1 to x and the label 2 to y,

$$\{(i,j,k,l)\} = \{(1,1,1,1); (1,1,1,2); (1,1,2,2); (1,2,2,2); (2,2,2,2)\} \ .$$

The computation of the Christoffel coefficients according to (3.17) yields

$$\begin{aligned}
\Gamma^1_{11} &= \frac{-\partial_x V}{2[E - V(x,y)]} \ , \quad \Gamma^1_{22} = \frac{\partial_x V}{2[E - V(x,y)]} \ , \quad \Gamma^2_{11} = \frac{\partial_y V}{2[E - V(x,y)]} \ , \\
\Gamma^2_{22} &= \frac{-\partial_y V}{2[E - V(x,y)]} \ , \quad \Gamma^1_{12} = \frac{-\partial_y V}{2[E - V(x,y)]} \ , \quad \Gamma^2_{12} = \frac{-\partial_x V}{2[E - V(x,y)]} \ .
\end{aligned} \tag{4.41}$$

From (4.40) we get the system

$$\begin{aligned}
\nabla_1 K_{111} &= 0 \ , \\
\nabla_1 K_{122} + \nabla_2 K_{112} &= 0 \ , \\
\nabla_2 K_{111} + 3\nabla_1 K_{211} &= 0 \ , \\
\nabla_1 K_{222} + 3\nabla_2 K_{122} &= 0 \ , \\
\nabla_2 K_{222} &= 0 \ ,
\end{aligned} \tag{4.42}$$

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[7] The Killing–vector equations in Finsler spaces can be found in [118]. More recently these equations are studied in [143], where it is argumented that Killing vectors in Finsler spaces can yield invariants of higher order than linear in the momenta.

[8] This is derived from an $N = 3$ Hamiltonian (4.37) by means of two canonical transformations of variables removing translational invariance; see, for example, [49]; the third-order expansion of this new Hamiltonian yields the Hénon–Heiles model of (4.46) with $C = D = 1$.

whence

$$\begin{aligned}
&\partial_x K_{111} - 3\Gamma^1_{11} K_{111} - 3\Gamma^2_{11} K_{211} = 0 \ , \\
&\partial_x K_{122} + \partial_y K_{211} - \Gamma^1_{11} K_{122} - \Gamma^2_{11} K_{222} - 4\Gamma^1_{12} K_{112} \\
&\quad -4\Gamma^2_{11} K_{212} - \Gamma^1_{22} K_{111} - \Gamma^2_{22} K_{211} = 0 \ , \\
&\partial_y K_{111} + 3\partial_x K_{211} - 6\Gamma^1_{12} K_{111} - 6\Gamma^2_{12} K_{112} \\
&\quad -6\Gamma^1_{11} K_{211} - 6\Gamma^2_{11} K_{212} = 0 \ , \\
&\partial_x K_{222} + 3\partial_y K_{122} - 6\Gamma^1_{21} K_{122} - 6\Gamma^2_{21} K_{222} \\
&\quad -6\Gamma^1_{22} K_{112} - 6\Gamma^2_{22} K_{212} = 0 \ , \\
&\partial_y K_{222} - 3\Gamma^1_{22} K_{122} - 3\Gamma^2_{22} K_{222} = 0 \ ,
\end{aligned} \tag{4.43}$$

with the Christoffel coefficients given by (4.41), where one has to replace $V(x, y)$ with the potential part of the Hamiltonian (4.39) and $\partial_x V$, $\partial_y V$ with its derivatives. The general method of solving a linear inhomogeneous system of first-order partial differential equations in more than one dependent variables can be found in [144]. However, finding the explicit solution to the system of equations (4.43) is much facilitated because we already know a priori that this system is compatible and thus admits a solution, and we also have strong hints about the solution itself because the general form of the integrals of the Toda model is known [141]. The KTF, besides the metric tensor, for the model (4.39) is eventually found to have the components [145, 146]

$$\begin{aligned}
K_{111} &= 2(E-V)^2[3\partial_y V + 4(E-V)] \ , \\
&= 8(E-V)^3 + \frac{1}{2}(E-V)^2[e^{2y-2\sqrt{3}x} + e^{2y+2\sqrt{3}x} - 2e^{-4y}] \ , \\
K_{122} &= 2(E-V)^2[\partial_y V - 4(E-V)] \ , \\
&= -24(E-V)^3 + \frac{1}{2}(E-V)^2[e^{2y-2\sqrt{3}x} + e^{2y+2\sqrt{3}x} - 2e^{-4y}] \ , \\
K_{112} &= -2(E-V)^2\partial_x V = \frac{\sqrt{3}}{6}(E-V)^2(e^{2y+2\sqrt{3}x} - e^{2y-2\sqrt{3}x}) \ , \\
K_{222} &= -6(E-V)^2\partial_x V = \frac{\sqrt{3}}{2}(E-V)^2(e^{2y+2\sqrt{3}x} - e^{2y-2\sqrt{3}x}) \ ,
\end{aligned} \tag{4.44}$$

as can be easily checked by substituting them into (4.43). Hence, the second constant of motion, besides energy, is given by

$$\begin{aligned}
&J(x, y, p_x, p_y) \\
&= K_{ijk}\frac{dq^i}{ds}\frac{dq^j}{ds}\frac{dq^k}{ds} = K_{ijk}\frac{dq^i}{dt}\frac{dq^j}{dt}\frac{dq^k}{dt}\frac{1}{2\sqrt{2}[E-V(x,y)]^3} \\
&= \frac{1}{2\sqrt{2}[E-V(x,y)]^3}(K_{111}p_x^3 + 3K_{122}p_x p_y^2 + 3K_{112}p_x^2 p_y + K_{222}p_y^3) \\
&= 8p_x(p_x^2 - 3p_y^2) + (p_x + \sqrt{3}p_y)e^{2y-2\sqrt{3}x} - 2p_x e^{-4y} + (p_x - \sqrt{3}p_y)e^{2y+2\sqrt{3}x} ,
\end{aligned} \tag{4.45}$$

which coincides with the expression already reported in the literature [49] for the Hamiltonian (4.39).

4.4.3 The Generalized Hénon–Heiles Model

Let us now consider the two-degrees-of-freedom system described by the Hamiltonian

$$H = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}(x^2 + y^2) + Dx^2 y - \frac{1}{3}Cy^3 \ . \tag{4.46}$$

This model, originally derived to describe the motion of a test star in an axisymmetric galactic mean gravitational field, provided some of the first numerical evidence of the chaotic transition in nonlinear Hamiltonian systems [48]. Hénon and Heiles considered the case $C = D = 1$. The existence of a chaotic layer in the phase space of this model means lack of *global* integrability. However, by means of the Painlevé method, it has been shown [147] that for special choices of the parameters C and D this system is globally integrable. Let us now tackle integrability of this model from the viewpoint of the existence of KT fields on the manifold (M, g_J). We first begin with the equations for a Killing vector field. By means of (4.20) we look for possible coefficients $C_1(x,y)$, $C_2(x,y)$, thus obtaining

$$\begin{gathered} C_1 = C_1(y), \quad C_2 = C_2(x) \ , \\ \frac{dC_1(y)}{dy} + \frac{dC_2(x)}{dx} = 0 \ , \\ x(1+2Dy)C_1(y) + (y + Dx^2 - Cy^2)C_2(x) = 0 \ . \end{gathered} \tag{4.47}$$

From the second equation of (4.47) it follows that

$$\frac{dC_1(y)}{dy} = -\frac{dC_2(x)}{dx} = \text{cost.} \ , \tag{4.48}$$

whence, denoting the constant by α, the possible expressions for $C_1(y)$ and $C_2(x)$ are only of the form $C_1(y) = -\alpha y + \beta$, $C_2(x) = \alpha x + \gamma$, which, after substitution into the last equation of (4.47), implies

$$(x + 2Dxy)(-\alpha y + \beta) + (y + Dx^2 - Cy^2)(\alpha x + \gamma) = 0, \tag{4.49}$$

which has a non-trivial solution only for $C = D = 0$. On the other hand, for these values of the parameters the potential simplifies to $V(x,y) = \frac{1}{2}x^2 + \frac{1}{2}y^2$, whence the existence of the Killing vector field X of components $X_1 = y$ and $X_2 = -x$, which is due to the invariance under rotations in the xy plane.

Let us now consider the case of a rank-2 KTF. Equations (4.40) become

$$\nabla_i K_{jk} + \nabla_j K_{ik} + \nabla_k K_{ij} = 0 \ , \qquad i,j,k = 1,2 \ , \tag{4.50}$$

where, associating again the label 1 to x and the label 2 to y, $\{(i,j,k)\} = \{(1,1,1);(1,1,2);(1,2,2);(2,2,2)\}$. The Christoffel coefficients are still given by (4.41), where we have to use the potential part of Hamiltonian (4.46) so that $\partial_x V(x,y) = x + 2Dxy$ and $\partial_y V(x,y) = y + Dx^2 - Cy^2$. The KTF equations are then

$$\begin{aligned} \nabla_1 K_{11} &= 0 \ , \\ 2\nabla_1 K_{12} + \nabla_2 K_{11} &= 0 \ , \\ \nabla_1 K_{22} + 2\nabla_2 K_{12} &= 0 \ , \\ \nabla_2 K_{22} &= 0 \ , \end{aligned} \tag{4.51}$$

whence

$$\begin{aligned} \partial_x K_{11} - 2\Gamma^1_{11} K_{11} - 2\Gamma^2_{11} K_{21} &= 0 \ , \\ 2\partial_x K_{12} + \partial_y K_{11} - 4\Gamma^1_{12} K_{11} - (4\Gamma^2_{12} + 2\Gamma^1_{11}) K_{12} - 2\Gamma^2_{11} K_{22} &= 0 \ , \\ \partial_x K_{22} + 2\partial_y K_{12} - 2\Gamma^1_{22} K_{11} - (4\Gamma^1_{12} + 2\Gamma^2_{22}) K_{12} - 4\Gamma^2_{12} K_{22} &= 0 \ , \\ \partial_y K_{22} - 2\Gamma^1_{22} K_{12} - 2\Gamma^2_{22} K_{22} &= 0 \ . \end{aligned} \tag{4.52}$$

Since the Hamiltonian (4.46) is not integrable for a generic choice of the parameters C and D, we can reasonably expect that the generic property of the above *overdetermined* system of equations is *incompatibility*, i.e., only the trivial solution $K_{ij} = 0$ exists for the overwhelming majority of the pairs (C, D). However, the existence of special choices of C and D for which the Hamiltonian is integrable suggests that this overdetermined system can be *compatible* in special cases. For example, when $D = 0$ the variables x and y in (4.46) are decoupled, and thus two KT fields of rank 2 exist according to (4.31).

A non trivial solution for the system (4.52) must exist at least for the pair $(C = -6, D = 1)$. In fact, in this case the modified Hénon–Heiles model is known to be integrable [147]. An explicit solution for the system (4.52) is eventually found to be given by [145, 146]

$$\begin{aligned} K_{11} &= (3 - 4y)(E - V(x,y))^2 + x^2(x^2 + 4y^2 + 4y + 3)(E - V(x,y)) \ , \\ K_{12} &= 2x(E - V(x,y)) \ , \\ K_{22} &= \frac{1}{2}(x^2 + 4y^2 + 4y + 3)(E - V(x,y)) \ . \end{aligned} \tag{4.53}$$

The associated constant of motion is therefore

$$\begin{aligned} J(x,y,p_x,p_y) &= \frac{1}{(E - V(x,y))^2}(K_{11}p_x^2 + 2K_{12}p_xp_y + K_{22}p_y^2) \\ &= x^4 + 4x^2y^2 - p_x^2 y + 4p_xp_y x + 4x^2 y + 3p_x^2 + 3x^2. \end{aligned} \tag{4.54}$$

This expression is identical to that reported in [147], worked out for the same values of C and D with a completely different method based on the Painlevé property.[9]

4.5 Open Problems

Let us now summarize the meaning of the results presented above and point out some open problems.

- Besides qualitative and quantitative descriptions of chaos, within the framework of Riemannian geometrization of Newtonian mechanics, also *integrability* has its own place. The idea of associating KTFs with integrability has been essentially developed in the context of classical general relativity; see, for example, [148–150] and references quoted therein. That Killing tensors generate "hidden" symmetries associated with constants of the motion in classical Newtonian mechanics has been considered in [135, 136], and, more recently, in [151]. In particular, the integrability conditions for quadratic invariants were obtained in [151].
- The reduction of the problem of integrability of a given Hamiltonian system to the existence of suitable KTFs on (M_E, g_J) offers several points of interest; in particular, we have seen that the system of equations in the unknown components of a KTF of a preassigned rank is overdetermined. Thus at a qualitative level, integrability seems a rather exceptional property, and the larger N, the "more exceptional" it seems to be, because of the rapidly growing mismatch between the number of unknowns and the number of equations. In principle, the existence of compatibility conditions for systems of linear first-order partial differential equations could allow one to decide about integrability prior to any explicit attempt at solving the equations for the components of a KTF. Even better, there are geometric constraints to the existence of KTFs. Early results in this sense are reported in [152], so that it seems possible, at least in some cases, to devise purely geometric criteria of *nonintegrability*. For example, hyperbolicity of compact manifolds excludes [152] the existence of KTFs, and this is consistent with the property of geodesic flows on compact hyperbolic manifolds of being strongly chaotic (Anosov flows).
- In general, we lack a criterion to restrict the search for KTFs to a small interval of ranks, and this constitutes a practical difficulty. Nevertheless, since the involution of two invariants translates into the vanishing of special brackets—the Schouten brackets [150]—between the corresponding Killing tensors, a shortcut to proving integrability, for a large class of systems satisfying the conditions of the Poincaré–Fermi theorem (see Chapter 2), might be to find *only one* KTF of vanishing Schouten brackets with

[9] This result, worked out in Chapter 2 of [145], was independently found also in [143] following a different computational strategy.

the metric tensor. In fact, for quasi-integrable systems with $N \geq 3$, the Poincaré–Fermi theorem states that generically only energy is conserved. Thus if another constant of motion is known to exist (apart from Noetherian ones such as angular momentum), then the system must be integrable and in fact there must be N constants of motion.

- Unlike Killing vectors, which are associated with Noetherian symmetries and conservation laws, Killing tensors no longer have a simple geometric interpretation [149, 153]. Therefore the associated symmetries are non-Noetherian and hidden.

The Riemannian-geometric approach to integrability deserves further attention and investigation. In fact, among the other reasons of interest, by considering, for example, the standard Hénon–Heiles model ($C = D = 1$), we might wonder whether the regular regions of phase space correspond to a *local* satisfaction of the compatibility conditions of the system (4.52), which would lead to a better understanding of the relationship between geometry and stability of Newtonian mechanics. Moreover, we could imagine that by suitably defining *weak* and *strong violations* of these compatibility conditions, we could better understand the geometric origin of *weak* and *strong chaos* in Hamiltonian dynamics (see Chapter 2), and perhaps this might even suggest a starting point to developing a "geometric perturbation theory" complementary to the more standard canonical perturbation theory.