7

Highest Weight Theory

By studying the L^2 functions on a compact Lie group G, the Peter–Weyl Theorem gives a simultaneous construction of all irreducible representations of G. Two important problems remain. The first is to parametrize $\widehat{G}$ in a reasonable manner and the second is to individually construct each irreducible representation in a natural way. The solution to both of these problems is closely tied to the notion of *highest weights*.

7.1 Highest Weights

In this section, let G be a compact Lie group, T a maximal torus, and $\Delta^+(\mathfrak{g}_\mathbb{C})$ a system of positive roots with corresponding simple system $\Pi(\mathfrak{g}_\mathbb{C})$. Write

$$\mathfrak{n}^\pm = \bigoplus_{\alpha \in \Delta^\pm(\mathfrak{g}_\mathbb{C})} \mathfrak{g}_\alpha,$$

so that

$$\mathfrak{g}_\mathbb{C} = \mathfrak{n}^- \oplus \mathfrak{t}_\mathbb{C} \oplus \mathfrak{n}^+ \tag{7.1}$$

by the root space decomposition. Equation 7.1 is sometimes called a *triangular decomposition* of $\mathfrak{g}_\mathbb{C}$ since $\mathfrak{n}^\pm$ can be chosen to be the set of strictly upper, respectively lower, triangular matrices in the case where G is $GL(n, \mathbb{F})$. Notice that $[\mathfrak{t}_\mathbb{C} \oplus \mathfrak{n}^+, \mathfrak{n}^+] \subseteq \mathfrak{n}^+$ and $[\mathfrak{t}_\mathbb{C} \oplus \mathfrak{n}^-, \mathfrak{n}^-] \subseteq \mathfrak{n}^-$.

Definition 7.2. Let V be a representation of $\mathfrak{g}$ with weight space decomposition $V = \bigoplus_{\lambda \in \Delta(V)} V_\lambda$.
(a) A nonzero $v \in V_{\lambda_0}$ is called a *highest weight vector* of weight λ_0 with respect to $\Delta^+(\mathfrak{g}_\mathbb{C})$ if $\mathfrak{n}^+ v = 0$, i.e., if $Xv = 0$ for all $X \in \mathfrak{n}^+$. In this case, λ_0 is called a *highest weight* of V.
(b) A weight λ is said to be *dominant* if $B(\lambda, \alpha) \geq 0$ for all $\alpha \in \Pi(\mathfrak{g}_\mathbb{C})$, i.e., if λ lies in the closed Weyl chamber corresponding to $\Delta^+(\mathfrak{g}_\mathbb{C})$.

As an example, recall that the action of $\mathfrak{su}(2)_{\mathbb{C}} = \mathfrak{sl}(2, \mathbb{C})$ on $V_n(\mathbb{C}^2)$, $n \in \mathbb{Z}^{\geq 0}$, from Equation 6.7 is given by

$$E \cdot (z_1^k z_2^{n-k}) = -k\, z_1^{k-1} z_2^{n-k+1}$$
$$H \cdot (z_1^k z_2^{n-k}) = (n-2k)\; z_1^k z_2^{n-k}$$
$$F \cdot (z_1^k z_2^{n-k}) = (k-n)\, z_1^{k+1} z_2^{n-k-1},$$

and recall that $\{V_n(\mathbb{C}^2) \mid n \in \mathbb{Z}^{\geq 0}\}$ is a complete list of irreducible representations for $SU(2)$. Taking $i\mathfrak{t} = \operatorname{diag}(\theta, -\theta)$, $\theta \in \mathbb{R}$, there are two roots, $\pm\epsilon_{12}$, where $\epsilon_{12}(\operatorname{diag}(\theta, -\theta)) = 2\theta$. Choosing $\Delta^+(\mathfrak{sl}(2, \mathbb{C})) = \{\epsilon_{12}\}$, it follows that z_2^n is a highest weight vector of $V_n(\mathbb{C}^2)$ of weight $n\frac{\epsilon_{12}}{2}$. Notice that the set of dominant analytically integral weights is $\{n\frac{\epsilon_{12}}{2} \mid n \in \mathbb{Z}^{\geq 0}\}$. Thus there is a one-to-one correspondence between the set of highest weights of irreducible representations of $SU(2)$ and the set of dominant analytically integral weights. This correspondence will be established for all connected compact groups in Theorem 7.34.

Theorem 7.3. *Let G be a connected compact Lie group and V an irreducible representation of G.*
(a) *V has a unique highest weight, λ_0.*
(b) *The highest weight λ_0 is dominant and analytically integral, i.e., $\lambda_0 \in A(T)$.*
(c) *Up to nonzero scalar multiplication, there is a unique highest weight vector.*
(d) *Any weight $\lambda \in \Delta(V)$ is of the form*

$$\lambda = \lambda_0 - \sum_{\alpha_i \in \Pi(\mathfrak{g}_{\mathbb{C}})} n_i \alpha_i$$

for $n_i \in \mathbb{Z}^{\geq 0}$.
(e) *For $w \in W$, $wV_\lambda = V_{w\lambda}$, so that* $\dim V_\lambda = \dim V_{w\lambda}$. *Here $W(G)$ is identified with $W(\Delta(\mathfrak{g}_{\mathbb{C}}))$, as in Theorem 6.43 via the* Ad*-action from Equation 6.35.*
(f) *Using the norm induced by the Killing form, $\|\lambda\| \leq \|\lambda_0\|$ with equality if and only if $\lambda = w\lambda_0$ for $w \in W(\mathfrak{g}_{\mathbb{C}})$.*
(g) *Up to isomorphism, V is uniquely determined by λ_0.*

Proof. Existence of a highest weight λ_0 follows from the finite dimensionality of V and Theorem 6.11. Let v_0 be a highest weight vector for λ_0 and inductively define $V_n = V_{n-1} + \mathfrak{n}^- V_{n-1}$ where $V_0 = \mathbb{C}v_0$. This defines a filtration on the $(\mathfrak{n}^- \oplus \mathfrak{t}_{\mathbb{C}})$-invariant subspace $V_\infty = \cup_n V_n$ of V. If $\alpha \in \Pi(\mathfrak{g}_{\mathbb{C}})$, then $[\mathfrak{g}_\alpha, \mathfrak{n}^-] \subseteq \mathfrak{n}^- \oplus \mathfrak{t}_{\mathbb{C}}$. Since $\mathfrak{g}_\alpha V_0 = 0$, a simple inductive argument shows that $\mathfrak{g}_\alpha V_n \subseteq V_n$. In particular, this suffices to demonstrate that V_∞ is $\mathfrak{g}_{\mathbb{C}}$-invariant. Irreducibility implies $V = V_\infty$ and part (d) follows.

If λ_1 is also a highest weight, then $\lambda_1 = \lambda_0 - \sum n_i\alpha_i$ and $\lambda_0 = \lambda_1 - \sum m_i\alpha_i$ for $n_i, m_i \in \mathbb{Z}^{\geq 0}$. Eliminating λ_1 and λ_0 shows that $-\sum n_i\alpha_i = \sum m_i\alpha_i$. Thus $-n_i = m_i$, so that $n_i = m_i = 0$ and $\lambda_1 = \lambda_0$. Furthermore, the weight decomposition shows that $V_\infty \cap V_{\lambda_0} = V_0 = \mathbb{C}v_0$, so that parts (a) and (c) are complete.

The proof of part (e) is done in the same way as the proof of Theorem 6.36. For part (b), notice that $r_{\alpha_i}\lambda_0$ is a weight by part (e). Thus

$$\lambda_0 - 2\frac{B(\lambda_0, \alpha_i)}{B(\alpha_i, \alpha_i)}\alpha_i = \lambda_0 - \sum_{\alpha_j \in \Pi(\mathfrak{g}_\mathbb{C})} n_j \alpha_j$$

for $n_j \in \mathbb{Z}^{\geq 0}$. Hence $2\frac{B(\lambda_0,\alpha_i)}{B(\alpha_i,\alpha_i)} = n_i$, so that $B(\lambda_0, \alpha_i) \geq 0$ and λ_0 is dominant. Theorem 6.27 shows that λ_0 (in fact, any weight of V) is analytically integral.

For part (f), Theorem 6.43 shows that it suffices to take λ dominant by using the Weyl group action. Write $\lambda = \lambda_0 - \sum n_i \alpha_i$. Solving for λ_0 and using dominance in the second line,

$$\begin{aligned} \|\lambda_0\|^2 &= \|\lambda\|^2 + 2 \sum_{\alpha_i \in \Pi(\mathfrak{g}_\mathbb{C})} n_i B(\lambda, \alpha_i) + \Bigg\| \sum_{\alpha_i \in \Pi(\mathfrak{g}_\mathbb{C})} n_i \alpha_i \Bigg\|^2 \\ &\geq \|\lambda\|^2 + \Bigg\| \sum_{\alpha_i \in \Pi(\mathfrak{g}_\mathbb{C})} n_i \alpha_i \Bigg\|^2 \geq \|\lambda\|^2 . \end{aligned}$$

In the case of equality, it follows that $\sum_{\alpha_i \in \Pi(\mathfrak{g}_\mathbb{C})} n_i \alpha_i = 0$, so that $n_i = 0$ and $\lambda = \lambda_0$.

For part (g), suppose V' is an irreducible representation of G with highest weight λ_0 and corresponding highest weight vector v_0'. Let $W = V \oplus V'$ and define $W_n = W_{n-1} + \mathfrak{n}^- W_{n-1}$, where $W_0 = \mathbb{C}(v_0, v_0')$. As above, $W_\infty = \cup_n W_n$ is a subrepresentation of $V \oplus V'$. If U is a nonzero subrepresentation of W_∞, then U has a highest weight vector, (u_0, u_0'). In turn, this means that u_0 and u_0' are highest weight vectors of V and V', respectively. Part (a) then shows that $\mathbb{C}(u_0, u_0') = W_0$. Thus $U = W_\infty$ and W_∞ is irreducible. Projection onto each coordinate establishes the G-intertwining map $V \cong V'$. □

The above theorem shows that highest weights completely classify irreducible representations. It only remains to parametrize all possible highest weights of irreducible representations. This will be done in §7.3.5 where we will see there is a bijection between the set of dominant analytically integral weights and irreducible representations of G.

Definition 7.4. Let G be connected and let V be an irreducible representation of G with highest weight λ. As V is uniquely determined by λ, write $V(\lambda)$ for V and write χ_λ for its character.

Lemma 7.5. *Let G be connected. If $V(\lambda)$ is an irreducible representation of G, then $V(\lambda)^* \cong V(-w_0\lambda)$, where $w_0 \in W(\Delta(\mathfrak{g}_\mathbb{C}))$ is the unique element mapping the positive Weyl chamber to the negative Weyl chamber (c.f. Exercise 6.40).*

Proof. Since $V(\lambda)$ is irreducible, the character theory of Theorems 3.5 and 3.7 show that $V(\lambda)^*$ is irreducible. It therefore suffices to show that the highest weight of $V(\lambda)^*$ is $-w_0\lambda$.

Fix a G-invariant inner product, $(\cdot, \cdot)$, on $V(\lambda)$, so that $V(\lambda)^* = \{\mu_v \mid v \in V(\lambda)\}$, where $\mu_v(v') = (v', v)$ for $v' \in V(\lambda)$. By the invariance of the form, $g\mu_v = \mu_{gv}$ for $g \in G$, so that $X\mu_v = \mu_{Xv}$ for $X \in \mathfrak{g}$. Since $(\cdot, \cdot)$ is Hermitian, it follows that $Z\mu_v = \mu_{\theta(Z)v}$ for $Z \in \mathfrak{g}_\mathbb{C}$.

Let v_λ be a highest weight vector for $V(\lambda)$. Identifying $W(G)$ with $W(\Delta(\mathfrak{g}_\mathbb{C})^\vee)$ and $W(\Delta(\mathfrak{g}_\mathbb{C}))$ as in Theorem 6.43 via the Ad-action of Equation 6.35, it follows from Theorem 7.3 that $w_0 v_\lambda$ is a weight vector of weight $w_0\lambda$ (called the *lowest weight vector*). As $\theta(Y) = -Y$ for $Y \in i\mathfrak{t}$ and since weights are real valued on $i\mathfrak{t}$, it follows that $\mu_{w_0 v_\lambda}$ is a weight vector of weight $-w_0\lambda$.

It remains to see that $\mathfrak{n}^- w_0 v_\lambda = 0$ since Lemma 6.14 shows $\theta\mathfrak{n}^+ = \mathfrak{n}^-$. By construction, $w_0\Delta^+(\mathfrak{g}_\mathbb{C}) = \Delta^-(\mathfrak{g}_\mathbb{C})$ and $w_0^2 = I$, so that $\mathrm{Ad}(w_0)\mathfrak{n}^- = \mathfrak{n}^+$. Thus

$$\mathfrak{n}^- w_0 v_\lambda = w_0 \left(\mathrm{Ad}(w_0^{-1})\mathfrak{n}^-\right) v_\lambda = w_0 \mathfrak{n}^+ v_\lambda = 0$$

and the proof is complete. □

7.1.1 Exercises

Exercise 7.1 Consider the representation of $SU(n)$ on $\bigwedge^p \mathbb{C}^n$. For T equal to the usual set of diagonal elements, show that a basis of weight vectors is given by vectors of the form $e_{l_1} \wedge \cdots \wedge e_{l_p}$ with weight $\sum_{i=1}^p \epsilon_{l_i}$. Verify that only $e_1 \wedge \cdots \wedge e_p$ is a highest weight to conclude that $\bigwedge^p \mathbb{C}^n$ is an irreducible representation of $SU(n)$ with highest weight $\sum_{i=1}^p \epsilon_i$.

Exercise 7.2 Recall that $V_p(\mathbb{R}^n)$, the space of complex-valued polynomials on $\mathbb{R}^n$ homogeneous of degree p, and $\mathcal{H}_p(\mathbb{R}^n)$, the harmonic polynomials, are representations of $SO(n)$. Let T be the standard maximal torus given in §5.1.2.3 and §5.1.2.4, let $h_j = E_{2j-1,2j} - E_{2j,2j-1} \in \mathfrak{t}$, $1 \le k \le m \equiv \lfloor \frac{n}{2} \rfloor$, i.e., h_j is an embedding of $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and let $\epsilon_j \in \mathfrak{t}^*$ be defined by $\epsilon_j(h_{j'}) = -i\delta_{j,j'}$ (c.f. Exercise 6.14).
(1) Show that h_j acts on $V_p(\mathbb{R}^n)$ by the operator $-x_{2j}\partial_{x_{2j-1}} + x_{2j-1}\partial_{x_{2j}}$.
(2) For $n = 2m+1$, conclude that a basis of weight vectors is given by polynomials of the form

$$(x_1 + ix_2)^{j_1} \cdots (x_{2m-1} + ix_{2m})^{j_m} (x_1 - ix_2)^{k_1} \cdots (x_{2m-1} - ix_{2m})^{k_m} x_{2m+1}^{l_0},$$

$l_0 + \sum_i j_i + \sum_i k_i = p$, each with weight $\sum_i (k_i - j_i)\epsilon_i$.
(3) For $n = 2m$, conclude that a basis of weight vectors is given by polynomials of the form

$$(x_1 + ix_2)^{j_1} \cdots (x_{n-1} + ix_n)^{j_m} (x_1 - ix_2)^{k_1} \cdots (x_{n-1} - ix_n)^{k_m},$$

$\sum_i j_i + \sum_i k_i = p$, each with weight $\sum_i (k_i - j_i)\epsilon_i$.
(4) Using the root system of $\mathfrak{so}(n,\mathbb{C})$ and Theorem 2.33, conclude that the weight vector $(x_1 - ix_2)^p$ of weight $p\epsilon_1$ must be the highest weight vector of $\mathcal{H}_p(\mathbb{R}^n)$ for $n \ge 3$.
(5) Using Lemma 2.27, show that a basis of highest weight vectors for $V_p(\mathbb{R}^n)$ is given by the vectors $(x_1 - ix_2)^{p-2j} \|x\|^{2j}$ of weight $(p-2j)\epsilon_1$, $1 \le j \le m$.

Exercise 7.3 Consider the representation of $SO(n)$ on $\bigwedge^p \mathbb{C}^n$ and continue the notation from Exercise 7.2.

(1) For $n = 2m+1$, examine the wedge product of elements of the form $e_{2j-1} \pm ie_{2j}$ as well as e_{2m+1} to find a basis of weight vectors (the weights will be of the form $\pm\epsilon_{j_1} \cdots \pm \epsilon_{j_r}$ with $1 \leq j_1 < \ldots < j_r \leq p$). For $p \leq m$, show that only one is a highest weight vector and conclude that $\bigwedge^p \mathbb{C}^n$ is irreducible with highest weight $\sum_{i=1}^p \epsilon_i$.
(2) For $n = 2m$, examine the wedge product of elements of the form $e_{2j-1} \pm ie_{2j}$ to find a basis of weight vectors. For $p < m$, show that only one is a highest weight vector and conclude that $\bigwedge^p \mathbb{C}^n$ is irreducible with highest weight $\sum_{i=1}^p \epsilon_i$. For $p = m$, show that there are exactly two highest weights and that they are $\sum_{i=1}^{m-1} \epsilon_i \pm \epsilon_m$. In this case, conclude that $\bigwedge^m \mathbb{C}^n$ is the direct sum of two irreducible representations.

Exercise 7.4 Let G be a compact Lie group, T a maximal torus, and $\Delta^+(\mathfrak{g}_\mathbb{C})$ a system of positive roots with respect to $\mathfrak{t}_\mathbb{C}$ with corresponding simple system $\Pi(\mathfrak{g}_\mathbb{C})$.
(1) If $V(\lambda)$ and $V(\lambda')$ are irreducible representations of G, show that the weights of $V(\lambda) \otimes V(\lambda')$ are of the form $\mu + \mu'$, where μ is a weight of $V(\lambda)$ and μ' is a weight of $V(\lambda')$.
(2) By looking at highest weight vectors, show $V(\lambda + \lambda')$ appears exactly once as a summand in $V(\lambda) \otimes V(\lambda')$.
(3) Suppose $V(\nu)$ is an irreducible summand of $V(\lambda) \otimes V(\lambda')$ and write the highest weight vector of $V(\nu)$ in terms of the weights of $V(\lambda) \otimes V(\lambda')$. By considering a term in which the contribution from $V(\lambda)$ is as large as possible, show that $\nu = \lambda + \mu'$ for a weight μ' of $V(\lambda')$.

Exercise 7.5 Recall that $V_{p,q}(\mathbb{C}^n)$ from Exercise 2.35 is a representations of $SU(n)$ on the set of complex polynomials homogeneous of degree p in $z_1, \ldots, z_n$ and homogeneous of degree q in $\overline{z_1}, \ldots, \overline{z_n}$ and that $\mathcal{H}_{p,q}(\mathbb{C}^n)$ is an irreducible subrepresentation.
(1) If $H = \operatorname{diag}(t_1, \ldots, t_n)$ with $\sum_j t_j = 0$, show that H acts on $V_{p,q}(\mathbb{C}^n)$ as $\sum_j t_j(-z_j \partial_{z_j} + \overline{z_j}\partial_{\overline{z_j}})$.
(2) Conclude that $z_1^{k_1} \cdots z_n^{k_n} \overline{z_1}^{l_1} \cdots \overline{z_n}^{l_n}$, $\sum_j k_j = p$ and $\sum_j l_j = q$, is a weight vector of weight $\sum_j (l_j - k_j)\epsilon_j$.
(3) Show that $-p\epsilon_n$ is a highest weight of $V_{p,0}(\mathbb{C}^n)$.
(4) Show that $q\epsilon_1$ is a highest weight of $V_{0,q}(\mathbb{C}^n)$.
(5) Show that $q\epsilon_1 - p\epsilon_n$ is the highest weight of $\mathcal{H}_{p,q}(\mathbb{C}^n)$.

Exercise 7.6 Since $\mathrm{Spin}_n(\mathbb{R})$ is the simply connected cover of $SO(n)$, $n \geq 3$, the Lie algebra of $\mathrm{Spin}_n(\mathbb{R})$ can be identified with $\mathfrak{so}(n)$ (a maximal torus for $\mathrm{Spin}_n(\mathbb{R})$ is given in Exercise 5.5).
(1) For $n = 2m+1$, show that the weights of the spin representation S are all weights of the form $\frac{1}{2}(\pm\epsilon_1 \cdots \pm \epsilon_m)$ and that the highest weight is $\frac{1}{2}(\epsilon_1 + \cdots + \epsilon_m)$.
(2) For $n = 2m$, show that the weights of the half-spin representation S^+ are all weights of the form $\frac{1}{2}(\pm\epsilon_1 \cdots \pm \epsilon_m)$ with an even number of minus signs and that the highest weight is $\frac{1}{2}(\epsilon_1 + \cdots + \epsilon_{m-1} + \epsilon_m)$.
(3) For $n = 2m$, show that the weights of the half-spin representation S^- are all weights of the form $\frac{1}{2}(\pm\epsilon_1 \cdots \pm \epsilon_m)$ with an odd number of minus signs and that the highest weight is $\frac{1}{2}(\epsilon_1 + \cdots + \epsilon_{m-1} - \epsilon_m)$.

7.2 Weyl Integration Formula

Let G be a compact connected Lie group, T a maximal torus, and $f \in C(G)$. We will prove the famous Weyl Integration Formula (Theorem 7.16) which says that

$$\int_G f(g)\,dg = \frac{1}{|W(G)|}\int_T d(t)\int_{G/T} f(gtg^{-1})\,dgT\,dt,$$

where $d(t) = \prod_{\alpha\in\Delta^+(\mathfrak{g}_\mathbb{C})} |1-\xi_{-\alpha}(t)|^2$ for $t \in T$. Using Equation 1.42, the proof will be based on a change of variables map $\psi : G/T \times T \to G$ given by $\psi(gT,t) = gtg^{-1}$. In order to ensure all required hypothesis are met, it is necessary to first restrict our attention to a distinguished dense open subset of G called the set of *regular* elements.

7.2.1 Regular Elements

Let G be a compact Lie group with maximal torus T and $X \in \mathfrak{g}$. Recall from Definition 5.8 that X is called a *regular* element of $\mathfrak{g}$ if $\mathfrak{z}_\mathfrak{g}(X)$ is a Cartan subalgebra. Also recall from Theorem 6.27 the bijection between the set of analytically integral weights, $A(T)$, and the character group, $\chi(T)$, that maps $\lambda \in A(T)$ to $\xi_\lambda \in \chi(T)$ and satisfies

$$\xi_\lambda(\exp H) = e^{\lambda(H)}$$

for $H \in \mathfrak{t}$.

Definition 7.6. Let G be a compact connected Lie group with maximal torus T.
(a) An element $g \in G$ is said to be *regular* if $Z_G(g)^0$ is a maximal torus.
(b) Write $\mathfrak{g}^{\text{reg}}$ for the set of regular element in $\mathfrak{g}$ and write G^{reg} for the set of regular elements in G.
(c) For $t \in T$, let

$$d(t) = \prod_{\alpha\in\Delta(\mathfrak{g}_\mathbb{C})} (1-\xi_{-\alpha}(t)).$$

Theorem 7.7. *Let G be a compact connected Lie group.*
(a) $\mathfrak{g}^{\text{reg}}$ *is open dense in* $\mathfrak{g}$,
(b) G^{reg} *is open dense in G,*
(c) *If T is a maximal torus and* $t \in T$, $t \in T^{\text{reg}}$ *if and only if* $d(t) \neq 0$,
(d) *For* $H \in \mathfrak{t}$, e^H *is regular if and only if* $H \in \Xi = \{H \in \mathfrak{t} \mid \alpha(H) \notin 2\pi i\mathbb{Z},\ \alpha \in \Delta(\mathfrak{g}_\mathbb{C})\}$,
(e) $G^{\text{reg}} = \cup_{g\in G}\left(gT^{\text{reg}}g^{-1}\right)$.

Proof. Let l be the dimension of a Cartan subalgebra and $n = \dim\mathfrak{g}$. Any element $X \in \mathfrak{g}$ lies in at least one Cartan subalgebra, so that $\dim(\ker(\text{ad}(X))) \geq l$. Thus

$$\det(\text{ad}(X) - \lambda I) = \sum_{k=l}^{n} c_k(X)\lambda^k,$$

where $c_k(X)$ is a polynomial in X. Since $\mathrm{ad}(X)$ is diagonalizable, X is regular if and only if $\dim(\ker(\mathrm{ad}(X))) = l$. In particular, X is regular if and only if $c_l(X) \neq 0$. Thus $\mathfrak{g}^{\mathrm{reg}}$ is open in $\mathfrak{g}$. It also follows that $\mathfrak{g}^{\mathrm{reg}}$ is dense since a polynomial vanishes on a neighborhood if and only if it is zero.

For part (b), similarly observe that each $g \in G$ lies in a maximal torus so that $\dim(\ker(\mathrm{Ad}(g) - I)) \geq l$. Thus

$$\det(\mathrm{Ad}(g) - \lambda I) = \sum_{k=l}^{n} \widetilde{c}_k(g)(\lambda - 1)^k,$$

where $\widetilde{c}_k(g)$ is a smooth function of g. From Exercise 4.22, recall that the Lie algebra of $Z_G(g)$ is $\mathfrak{z}_\mathfrak{g}(g) = \{X \in \mathfrak{g} \mid \mathrm{Ad}(g)X = X\}$. Since $Z_G(g)^0$ is a maximal torus if and only if $\mathfrak{z}_\mathfrak{g}(g)$ is a Cartan subalgebra, diagonalizability implies g is regular if and only if $\widetilde{c}_l(g) \neq 0$. Thus G^{reg} is open in G.

To establish the density of G^{reg}, fix a maximal torus T of G. Since the eigenvalues of $\mathrm{Ad}(e^H)$ are of the form $e^{\alpha(H)}$ for $\alpha \in \Delta(\mathfrak{g}_\mathbb{C}) \cup \{0\}$, it follows that e^H is regular if and only if $H \in \Xi$. Since Ξ differs from $\mathfrak{t}$ only by a countable number of hyperplanes, Ξ is dense in $\mathfrak{t}$ by the Baire Category Theorem. Because exp is onto and continuous, T^{reg} is therefore dense in T. Since the Maximal Torus Theorem shows that $G = \cup_{g \in G} \left(gTg^{-1}\right)$, counting eigenvalues of $\mathrm{Ad}(g)$ shows $G^{\mathrm{reg}} = \cup_{g \in G} \left(gT^{\mathrm{reg}}g^{-1}\right)$. Density of G^{reg} in G now follows easily from the density of T^{reg} in T. □

Definition 7.8. Let G be a compact connected Lie group and T a maximal torus. Define the smooth, surjective map $\psi : G/T \times T \to G$ by

$$\psi(gT, t) = gtg^{-1}.$$

Abusing notation, we also denote by ψ the smooth, surjective map $\psi : G/T \times T^{\mathrm{reg}} \to G^{\mathrm{reg}}$ defined by restriction of domain.

It will soon be necessary to understand the invertibility of the differential $d\psi : T_{gT}(G/T) \times T_t(T) \to T_{gtg^{-1}}(G)$ for $g \in G$ and $t \in T$. Calculations will be simplified by locally pulling $G/T \times T$ back to G with an appropriate cross section for G/T. Write $\pi : G \to G/T$ for the natural projection map.

Lemma 7.9. *Let G be a compact connected Lie group and T a maximal torus. Then $\mathfrak{g} = \mathfrak{t} \oplus \left(\mathfrak{g} \cap \bigoplus_{\alpha \in \Delta(\mathfrak{g}_\mathbb{C})} \mathfrak{g}_\alpha\right)$ and there exists an open neighborhood $U_\mathfrak{g}$ of 0 in $\left(\mathfrak{g} \cap \bigoplus_{\alpha \in \Delta(\mathfrak{g}_\mathbb{C})} \mathfrak{g}_\alpha\right)$ so that, if $U_G = \exp U_\mathfrak{g}$ and $U_{G/T} = \pi U_G$, then:*

***(a)** the map $U_\mathfrak{g} \overset{\exp}{\to} U_G \overset{\pi}{\to} U_{G/T}$ is a diffeomorphism,*

***(b)** $U_{G/T}$ is an open neighborhood of eT in G/T,*

***(c)** $U_G T = \{gt \mid g \in U_G, t \in T\}$ is an open neighborhood of e in G*

***(d)** The map $\xi : U_G T \to G/T \times T$ given by $\xi(gt) = (gT, t)$ is a smooth, well-defined diffeomorphism onto $U_{G/T} \times T$.*

Proof. The decomposition $\mathfrak{g} = \mathfrak{t} \oplus \left(\mathfrak{g} \cap \bigoplus_{\alpha \in \Delta(\mathfrak{g}_\mathbb{C})} \mathfrak{g}_\alpha\right)$ follows from Theorem 6.20. In fact, $\left(\mathfrak{g} \cap \bigoplus_{\alpha \in \Delta(\mathfrak{g}_\mathbb{C})} \mathfrak{g}_\alpha\right)$ is spanned by the elements $\mathcal{J}_\alpha$ and $\mathcal{K}_\alpha$ for $\alpha \in \Delta(\mathfrak{g}_\mathbb{C})$.

Since the map $(H, X) \to e^H e^X$, $H \in \mathfrak{t}$ and $X \in \left(\mathfrak{g} \cap \bigoplus_{\alpha\in\Delta(\mathfrak{g}_{\mathbb{C}})} \mathfrak{g}_\alpha\right)$, is therefore a local diffeomorphism at 0, it follows that there is an open neighborhood $U_\mathfrak{g}$ of 0 in $\left(\mathfrak{g} \cap \bigoplus_{\alpha\in\Delta(\mathfrak{g}_{\mathbb{C}})} \mathfrak{g}_\alpha\right)$ on which exp is a diffeomorphism onto U_G.

Recall that $T_{eT}(G/T)$ may be identified with $\mathfrak{g}/\mathfrak{t}$. Thus by construction, the differential of π restricted to $T_e(U_G)$ at e is clearly invertible, so that π is a local diffeomorphism from U_G at e. Thus, perhaps shrinking $U_\mathfrak{g}$ and U_G, we may assume that $U_{G/T}$ is an open neighborhood of eT in G/T and that the maps $U_\mathfrak{g} \overset{\exp}{\to} U_G \overset{\pi}{\to} U_{G/T}$ are diffeomorphisms. This finishes parts (a) and (b).

For part (c), $U_G T$ is a neighborhood of e since the map $(H, X) \to e^H e^X$ is a local diffeomorphism at 0. In fact, there is a subset V of T so that $U_G V$ is open. Taking the union of right translates by elements of T, it follows that $U_G T$ is open.

For part (d), suppose $gt = g't'$ with $g, g' \in U_G$ and $t, t' \in T$. Then $\pi g = \pi g'$, so that $g = g'$ and $t = t'$. Thus the map is well defined and the rest of the statement is clear. □

Using Lemma 7.9, it is now possible to study the differential $d\psi : T_{gT}(G/T) \times T_t(T) \to T_{gtg^{-1}}(G)$. This will be done with the map ξ and appropriate translations to pull everything back to neighborhoods of e in G.

Lemma 7.10. *Let G be a compact connected Lie group and T a maximal torus. Choose $U_G \subseteq G$ as in Lemma 7.9. For $g \in G$ and $t \in T$, let $\phi : U_G T \to G$ be given by*

$$\phi = l_{gt^{-1}g^{-1}} \circ \psi \circ (l_{gT} \times l_t) \circ \xi,$$

where ξ is defined as in Lemma 7.9. Then the differential $d\phi : \mathfrak{g} \to \mathfrak{g}$ is given by

$$d\phi(H + X) = \mathrm{Ad}(g) \left[\left(\mathrm{Ad}(t^{-1}) - I\right) X + H\right]$$

for $H \in \mathfrak{t}$ and $X \in \left(\mathfrak{g} \cap \bigoplus_{\alpha\in\Delta(\mathfrak{g}_{\mathbb{C}})} \mathfrak{g}_\alpha\right)$ and

$$\det(d\phi) = d(t).$$

Proof. Calculate

$$d\phi(H) = \frac{d}{ds}\phi(e^{sH})|_{s=0} = \frac{d}{ds} g e^{sH} g^{-1}|_{s=0} = \mathrm{Ad}(g)H$$

$$d\phi(X) = \frac{d}{ds}\phi(e^{sX})|_{s=0} = \frac{d}{ds} g t^{-1} e^{sX} t e^{-sX} g^{-1}|_{s=0} = \mathrm{Ad}(gt^{-1})X - \mathrm{Ad}(g)X,$$

so that the formula for $d\phi$ is established by linearity. For the calculation of the determinant, first note that $\det \mathrm{Ad}(g) = 1$. This follows from the three facts: (1) the determinant is not changed by complexifying, (2) each g lies in a maximal torus, and (3) the negative of a root is always a root. The problem therefore reduces to showing that the determinant of $\left(\mathrm{Ad}(t^{-1}) - I\right)$ on $\bigoplus_{\alpha\in\Delta(\mathfrak{g}_{\mathbb{C}})} \mathfrak{g}_\alpha$ is $\prod_{\alpha\in\Delta(\mathfrak{g}_{\mathbb{C}})}(1 - e^{-\alpha(H)})$. Since $\dim \mathfrak{g}_\alpha = 1$ and $\mathrm{Ad}(t^{-1})$ acts on $\mathfrak{g}_\alpha$ by $e^{-\alpha(\ln t)}$, where $e^{\ln t} = t$, the proof follows easily. The extra negative signs are taken care of by the even number of roots (since $\Delta(\mathfrak{g}_{\mathbb{C}}) = \Delta^+(\mathfrak{g}_{\mathbb{C}}) \amalg \Delta^-(\mathfrak{g}_{\mathbb{C}})$). □

Theorem 7.11. *Let G be a compact connected Lie group and T a maximal torus. The map*

$$\psi : G/T \times T^{\text{reg}} \to G^{\text{reg}} \text{ given by}$$
$$\psi(gT, t) = gtg^{-1}$$

is a surjective, $|W(G)|$-to-one local diffeomorphism.

Proof. For $g \in G$ and $t \in T^{\text{reg}}$, Lemma 7.10 and Theorem 7.7 show that ψ is a surjective local diffeomorphism at (gT, t). Moreover if $w \in N(T)$, then

$$\psi(gw^{-1}T, wtw^{-1}) = \psi(gT, t). \tag{7.12}$$

Since $gw^{-1}T = gT$ if and only if $w \in T$, it follows that $\left|\psi^{-1}(gtg^{-1})\right| \geq |W(G)|$.

To see that ψ is exactly $|W(G)|$-to-one, suppose $gtg^{-1} = hsh^{-1}$ for $h \in G$ and $s \in T^{\text{reg}}$. By Theorem 6.36, there is $w \in N(T)$, so that $s = wtw^{-1}$. Plugging this into $gtg^{-1} = hsh^{-1}$ quickly yields $w' = g^{-1}hw \in Z_G(t)$. Since t is regular, $Z_G(t)^0 = T$. Being the identity component of $Z_G(T)$, $c_{w'}$ preserves T, so that $w' \in N(T)$. Hence

$$(hT, s) = (gw'w^{-1}T, wtw^{-1}) = (gw'w^{-1}T, ww'^{-1}tw'w^{-1}).$$

Since this element was already known to be in $\psi^{-1}(gtg^{-1})$ by Equation 7.12, we seethat $\left|\psi^{-1}(gtg^{-1})\right| \leq |W(G)|$, as desired. □

7.2.2 Main Theorem

Let G be a compact connected Lie group and T a maximal torus. From Theorem 1.48 we know that

$$\int_G f(g)\, dg = \int_{G/T} \left(\int_T f(gt)\, dt \right) d(tT)$$

for $f \in C(G)$. Recall that the invariant measures above are given by integration against unique (up to ± 1) normalized left-invariant volume forms $\omega_G \in \bigwedge^*_{\text{top}}(G)$ and $\omega_{G/T} \in \bigwedge^*_{\text{top}}(G/T)$. In this section we make a change of variables based on the map ψ to obtain Weyl's Integration Formula. To this end write $n = \dim G$, $l = \dim T$ (also called the *rank* of G when $\mathfrak{g}$ is semisimple), and write $\iota : T \to G$ for the inclusion map. Recall that $\pi : G \to G/T$ is the natural projection map.

Lemma 7.13. *Possibly replacing ω_T by $-\omega_T$ (which does not change integration), there exists a G-invariant form $\widetilde{\omega_T} \in \bigwedge^*_l(G)$, so that*

$$\omega_T = \iota^* \widetilde{\omega_T}$$

and

$$\omega_G = \left(\pi^* \omega_{G/T}\right) \wedge \widetilde{\omega_T}$$

Proof. Clearly the restriction map $\iota^*|_e : \mathfrak{g}^* \to \mathfrak{t}^*$ is surjective. Choose any $(\widetilde{\omega_T})_e \in \bigwedge_l^*(G)_e$, so $\iota^*(\widetilde{\omega_T})_e = (\omega_T)_{eT}$. Using left translation, uniquely extend $(\widetilde{\omega_T})_e$ to a left-invariant form $\widetilde{\omega_T} \in \bigwedge_l^*(G)$. Since ι commutes with left multiplication by G, it follows that $\iota^*\widetilde{\omega_T} = \omega_T$. Since π also commutes with left multiplication by G, $\pi^*\omega_{G/T} \in \bigwedge_{n-l}^*(G)$ is left-invariant as well. Thus $\left(\pi^*\omega_{G/T}\right) \wedge \widetilde{\omega_T} \in \bigwedge_n^*(G)$ is left-invariant and therefore $\left(\pi^*\omega_{G/T}\right) \wedge \widetilde{\omega_T} = c\omega_G$ for some $c \in \mathbb{R}$ by uniqueness.

Write π_i for the two natural coordinate projections $\pi_1 : G/T \times T \to G/T$ and $\pi_2 : G/T \times T \to T$. Using the notation from Lemma 7.9, observe that $\pi|_{U_G T} = \pi_1 \circ \xi$, so that

$$\pi^*\omega_{G/T} = \xi^*\pi_1^*\omega_{G/T}$$

on $U_G T$. Similarly, observe that $I|_T = \pi_2 \circ \xi \circ \iota$, so that $\iota^*\left(\xi^*\pi_2^*\omega_T\right) = \omega_T$. Thus

$$\xi^*\pi_2^*\omega_T = \widetilde{\omega_T} + \omega$$

on $U_G T$ for some $\omega \in \bigwedge_l^*(U_G T)$ with $\iota^*\omega = 0$.

We claim that $\left(\pi^*\omega_{G/T}\right) \wedge \omega = 0$ on $U_G T$. Since ξ is a diffeomorphism, this is equivalent to showing $\left(\pi_1^*\omega_{G/T}\right) \wedge \omega' = 0$, where $\omega' = \left(\xi^{-1}\right)^* \omega \in \bigwedge_l^*(U_{G/T} \times T)$ satisfies $\iota^*\xi^*\omega' = 0$. Now ω' can be written as a sum $\omega' = \sum_{j=0}^l f_j \left(\pi_1^*\omega_j'\right) \wedge \left(\pi_2^*\omega_{l-j}''\right)$, where f_j is a smooth function on $G/T \times T$, $\omega_j' \in \bigwedge_j^*(U_{G/T})$, and $\omega_{l-j}'' \in \bigwedge_{l-j}^*(T)$. Without loss of generality, we may take $\pi_1^*\omega_0' = 1$. As $I|_T = \pi_2 \circ \xi \circ \iota$ and $(\pi_1 \circ \xi \circ \iota)(t) = eT$ for $t \in T$, it follows that $0 = \iota^*\xi^*\omega' = f_0\omega_l''$. Therefore $\omega' = \sum_{j=1}^l f_j \left(\pi_1^*\omega_j'\right) \wedge \left(\pi_2^*\omega_{l-j}''\right)$. Since $\omega_{G/T}$ is a top degree form, $\omega_{G/T} \wedge \omega_j' = 0$, $j \geq 1$, so that $\left(\pi_1^*\omega_{G/T}\right) \wedge \omega' = 0$, as desired.

It now follows that

$$\begin{aligned} c\omega_G &= \left(\pi^*\omega_{G/T}\right) \wedge \widetilde{\omega_T} = \left(\pi^*\omega_{G/T}\right) \wedge (\widetilde{\omega_T} + \omega) \\ &= \xi^*[\left(\pi_1^*\omega_{G/T}\right) \wedge \left(\pi_2^*\omega_T\right)] \end{aligned} \tag{7.14}$$

on $U_G T$. Looking at local coordinates, it is clear that $\left(\pi_1^*\omega_{G/T}\right) \wedge \left(\pi_2^*\omega_T\right) \neq 0$, so $c \neq 0$. Replacing ω_T by $-\omega_T$ if necessary, we may assume $c > 0$. Choose any continuous function f supported on $U_G T$ and use the change of variables formula to calculate

$$\begin{aligned} c\int_{G/T}\int_T f \circ \xi^{-1}(gT, t)\, dt\, dgT &= c\int_{G/T}\int_T f(gt)\, dt\, dgT = c\int_G f(g)\, dg \\ &= \int_{U_G T} f\, c\omega_G = \int_{U_G T} f\, \xi^*[\left(\pi_1^*\omega_{G/T}\right) \wedge \left(\pi_2^*\omega_T\right)] \\ &= \int_{U_{G/T}\times T} f \circ \xi^{-1} \left(\pi_1^*\omega_{G/T}\right) \wedge \left(\pi_2^*\omega_T\right). \end{aligned}$$

Since it follows immediately from the definitions (Exercise 7.7) that

$$(7.15)\quad \int_{U_{G/T}\times T} f\circ\xi^{-1}\left(\pi_1^*\omega_{G/T}\right)\wedge\left(\pi_2^*\omega_T\right)=\int_{G/T}\int_T f\circ\xi^{-1}(gT,t)\,dt\,dgT,$$

$c=1$, as desired. □

Theorem 7.16 (Weyl Integration Formula). *Let G be a compact connected Lie group, T a maximal torus, and $f\in C(G)$. Then*

$$\int_G f(g)\,dg=\frac{1}{|W(G)|}\int_T d(t)\int_{G/T} f(gtg^{-1})\,dgT\,dt,$$

where $d(t)=\prod_{\alpha\in\Delta^+(\mathfrak{g}_\mathbb{C})}|1-\xi_{-\alpha}(t)|^2$ for $t\in T$.

Proof. Since Theorem 7.7 shows that G^{reg} is open dense in G and T^{reg} is open dense in T, it suffices to prove that

$$\int_{G^{\text{reg}}} f(g)\,dg=\frac{1}{|W(G)|}\int_{T^{\text{reg}}} d(t)\int_{G/T} f(gtg^{-1})\,dgT\,dt.$$

To this end, recall that Theorem 7.11 shows that $\psi: G/T\times T^{\text{reg}}\to G^{\text{reg}}$ is a surjective, $|W(G)|$-to-one local diffeomorphism. We will prove that

$$(7.17)\quad \psi^*\omega_G=d(t)\left(\pi_1^*\omega_{G/T}\right)\wedge\left(\pi_2^*\omega_T\right),$$

where π_1 and π_2 are the projections from Lemma 7.13. Once this is done, the theorem follows immediately from Equation 1.42.

To verify Equation 7.17, first note that there is a smooth function $\delta: G/T\times T\to\mathbb{R}$, so that

$$\psi^*\omega_G|_{gtg^{-1}}=\left[\delta\,\left(\pi_1^*\omega_{G/T}\right)\wedge\left(\pi_2^*\omega_T\right)\right]|_{(gT,t)}$$

since the dimension of top degree form is 1 at each point. Since $U_{G/T}\times T$ is a neighborhood of (eT,e), Equation 7.14 shows $\left[\left(\pi_1^*\omega_{G/T}\right)\wedge\left(\pi_2^*\omega_T\right)\right]|_{(eT,e)}=\left(\xi^{-1}\right)^*\omega_G|_e$, so that

$$\begin{aligned}\psi^* l^*_{gt^{-1}g^{-1}}\omega_G|_e&=\psi^*\omega_G|_{gtg^{-1}}=\left(l_{g^{-1}}\times l_{t^{-1}}\right)^*\left[\delta\,\left(\pi_1^*\omega_{G/T}\right)\wedge\left(\pi_2^*\omega_T\right)\right]|_{(eT,e)}\\&=\left(l_{g^{-1}}\times l_{t^{-1}}\right)^*\left(\xi^{-1}\right)^*\left[\delta\circ\left(l_g\times l_t\right)\circ\xi\,\omega_G\right]|_e.\end{aligned}$$

Thus

$$\phi^*\omega_G|_e=\left(l_{gt^{-1}g^{-1}}\circ\psi\circ(l_g\times l_t)\circ\xi\right)^*\omega_G|_e=\left[\delta\circ\left(l_g\times l_t\right)\circ\xi\,\omega_G\right]|_e.$$

By looking at a basis of $\bigwedge_1^*(G)_e$, it follows that $\delta(gT,t)=\delta\circ\left(l_g\times l_t\right)\circ\xi|_e=\det(d\phi)$. This determinant was calculated in Lemma 7.10 and found to be

$$d(t)=\prod_{\alpha\in\Delta(\mathfrak{g}_\mathbb{C})}(1-\xi_{-\alpha}(t))=\prod_{\alpha\in\Delta^+(\mathfrak{g}_\mathbb{C})}|1-\xi_{-\alpha}(t)|^2.$$

□

7.2.3 Exercises

Exercise 7.7 Verify Equation 7.15.

Exercise 7.8 Let G be a compact connected Lie group and T a maximal torus. For $H \in \mathfrak{t}$, show that

$$d(e^H) = 2^{|\Delta(\mathfrak{g}_{\mathbb{C}})|} \prod_{\alpha \in \Delta^+(\mathfrak{g}_{\mathbb{C}})} \sin^2\left(\frac{\alpha(H)}{2i}\right).$$

Note that $\alpha(H) \in i\mathbb{R}$.

Exercise 7.9 Let f be a continuous class function on $SU(2)$. Use the Weyl Integration Formula to show that

$$\int_{SU(2)} f(g)\,dg = \frac{2}{\pi}\int_0^{\pi} f(\text{diag}(e^{i\theta}, e^{-i\theta}))\sin^2\theta\,d\theta,$$

c.f. Exercise 3.22.

Exercise 7.10 Let G be a compact connected Lie group and T a maximal torus (c.f. Exercise 6.29).
(1) If f is an L^1-class function on G, show that

$$\int_G f(g)\,dg = \frac{1}{|W(G)|}\int_T d(t) f(t)\,dt.$$

(2) Show that the map $f \to |W(G)|^{-1}\, df|_T$ defines a norm preserving isomorphism between the L^1-class functions on G and the W-invariant L^1-functions on T.
(3) Show that the map $f \to |W(G)|^{-\frac{1}{2}}\, Df|_T$ defines a unitary isomorphism between the L^2 class functions on G to the W-invariant L^2 functions on T, where $D(e^H) = \prod_{\alpha\in\Delta^+(\mathfrak{g}_{\mathbb{C}})}\left(1 - e^{-\alpha(H)}\right)$ for $H \in \mathfrak{t}$ (so $D\overline{D} = d$).

Exercise 7.11 For each group G below, verify $d(t)$ is correctly calculated.
(1) For $G = SU(n)$, $T = \{\text{diag}(e^{i\theta_k}) \mid \sum_k \theta_k = 0\}$, and $t = \text{diag}(e^{i\theta_k})$,

$$d(t) = 2^{n(n-1)} \prod_{1\le j<k\le n} \sin^2\left(\frac{\theta_j - \theta_k}{2}\right).$$

(2) For either $G = SO(2n+1)$, T as in §5.1.2.4, and

$$t = \text{blockdiag}\left(\begin{pmatrix} \cos\theta_k & \sin\theta_k \\ -\sin\theta_k & \cos\theta_k \end{pmatrix}, 1\right)$$

or $G = SO(E_{2n+1})$, T as in Lemma 6.12, and $t = \text{diag}(e^{i\theta_k}, e^{-i\theta_k}, 1)$,

$$d(t) = 2^{2n^2} \prod_{1\le j<k\le n} \sin^2\left(\frac{\theta_j - \theta_k}{2}\right)\sin^2\left(\frac{\theta_j + \theta_k}{2}\right) \prod_{1\le j\le n} \sin^2\left(\frac{\theta_j}{2}\right).$$

(3) For either $G = SO(2n)$, T as in §5.1.2.3, and

$$t = \text{blockdiag}\left(\begin{pmatrix} \cos\theta_k & \sin\theta_k \\ -\sin\theta_k & \cos\theta_k \end{pmatrix}\right)$$

or $G = SO(E_{2n})$, T as in Lemma 6.12, and $t = \text{diag}(e^{i\theta_k}, e^{-i\theta_k})$,

$$d(t) = 2^{2n(n-1)} \prod_{1\le j<k\le n} \sin^2\left(\frac{\theta_j - \theta_k}{2}\right)\sin^2\left(\frac{\theta_j + \theta_k}{2}\right).$$

(4) For $G = Sp(n)$ realized as $Sp(n) \cong U(2n) \cap Sp(n, \mathbb{C})$ and $T = \{t = \text{diag}(e^{i\theta_k}, e^{-i\theta_k})\}$,

$$d(t) = 2^{2n^2} \prod_{1\le j<k\le n} \sin^2\left(\frac{\theta_j - \theta_k}{2}\right)\sin^2\left(\frac{\theta_j + \theta_k}{2}\right) \prod_{1\le j\le n} \sin^2\left(\theta_j\right).$$

7.3 Weyl Character Formula

Let G be a compact Lie group with maximal torus T. Recall that Theorem 3.30 shows that the set of irreducible characters $\{\chi_\lambda\}$ is an orthonormal basis for the set of L^2 class functions on G.

Assume G is connected and, for the sake of motivation, momentarily assume G is simply connected as well. In §7.3.1 we will choose a skew-W-invariant function Δ defined on T, so that $|\Delta(t)|^2 = d(t)$. It easily follows from the Weyl Integration Formula that $\{\Delta\, \chi_\lambda|_T\}$ is therefore an orthonormal basis for the set of L^2 skew-W-invariant functions on T with respect to the measure $|W(G)|^{-1}\, dt$ (*c.f.* Exercise 7.10).

On the other hand, it is simple to write down another basis for the set of L^2 skew-W-invariant functions on T by looking at alternating sums over the Weyl group of certain characters on T. By decomposing $\chi_\lambda|_T$ into characters on T, it will follow rapidly that these two bases are the same. In turn, this yields an explicit formula for χ_λ called the Weyl Character Formula.

7.3.1 Machinery

Let G be a compact Lie group with maximal torus T. Recall that Theorem 6.27 shows there is a bijection between the set of analytically integral weights and the character group given by mapping $\lambda \in A(T)$ to $\xi_\lambda \in \chi(T)$. The next definition sets up similar notation for more general functions on $\mathfrak{t}$.

Definition 7.18. Let G be a compact Lie group with maximal torus T.
(a) Let $f : \mathfrak{t} \to \mathbb{C}$ be a function. We say f *descends* to T if $f(H + Z) = f(H)$ for $H, Z \in \mathfrak{t}$ with $Z \in \ker(\exp)$. In that case, write $f : T \to \mathbb{C}$ for the function given by

$$f(e^H) = f(H).$$

(b) If $f : \mathfrak{t} \to \mathbb{C}$ satisfies $f(wH) = f(H)$ for $w \in W(\Delta(\mathfrak{g}_{\mathbb{C}})^{\vee})$, f is called *W-invariant.*
(c) If $F : T \to \mathbb{C}$ satisfies $F(c_w t) = F(t)$ for $w \in N(T)$, F is called *W-invariant.*
(d) If $f : \mathfrak{t} \to \mathbb{C}$ satisfies $f(wH) = \det(w)\, f(H)$ for $w \in W(\Delta(\mathfrak{g}_{\mathbb{C}}))$, f is called *skew-W-invariant.*
(e) If $F : T \to \mathbb{C}$ satisfies $F(c_w t) = \det(\mathrm{Ad}(w)|_{\mathfrak{t}})\, F(t)$ for $w \in N(T)$, F is called *skew-W-invariant.*

In particular, for $\lambda \in A(T)$, the function $H \to e^{\lambda(H)}$ on $\mathfrak{t}$ descends to the function ξ_λ on T. Also note that $\det w \in \{\pm 1\}$ since w is a product of reflections.

Lemma 7.19. *Let G be a compact connected Lie group with maximal torus T.*
***(a)** If $f : \mathfrak{t} \to \mathbb{C}$ descends to T and is W-invariant, then $f : T \to \mathbb{C}$ is W-invariant.*
***(b)** Restriction of domain establishes a bijection between the continuous class functions on G and the continuous W-invariant functions on T.*

Proof. For part (a), recall that the identification of $W(G)$ with $W(\Delta(\mathfrak{g}_{\mathbb{C}})^{\vee})$ from Theorem 6.43 via the Ad-action of Equation 6.35. It follows that when f descends to T and is W-invariant, then $f(c_w t) = f(t)$ for $w \in N(T)$ and $t \in T$.

For part (b), suppose $F : T \to \mathbb{C}$ is W-invariant and fix $g_0 \in G$. By the Maximal Torus Theorem, there exists $h_0 \in G$, so $t_0 = c_{h_0} g_0 \in T$. Extend F to a class function on G by setting $F(g_0) = F(t_0)$. This is well defined by Theorem 6.36. It only remains to see that if F is continuous on T, then its extension to G is also continuous.

For this, suppose $g_n \in G$ with $g_n \to g_0$. Choose $h_n \in G$, so $t_n = c_{h_n} g_n \in T$. Since G is compact, passing to subsequences allows us to assume there is $h_0' \in G$ and $t_0' \in T$, so that $h_n \to h_0'$ and $t_n \to t_0'$. In particular, $t_0' = c_{h_0'} g_0$ so that, by Theorem 6.36, there exists $w \in N(T)$ with $wt_0 = t_0'$. Thus

$$F(g_n) = F(t_n) \to F(t_0') = F(t_0) = F(g_0).$$

Since we began with an arbitrary sequence $g_n \to g_0$, the proof is complete. □

Let G be a compact Lie group, T a maximal torus, and $\Delta^+(\mathfrak{g}_{\mathbb{C}})$ a system of positive roots with corresponding simple system $\Pi(\mathfrak{g}_{\mathbb{C}}) = \{\alpha_1, \ldots, \alpha_l\}$. Recall from Equation 6.39 the unique element $\rho \in (i\mathfrak{t})^*$ satisfying $\rho(h_{\alpha_i}) = 2\frac{B(\rho,\alpha_i)}{B(\alpha_i,\alpha_i)} = 1$, $1 \le j \le l$.

Lemma 7.20. *Let G be a compact Lie group with a maximal torus T.*
***(a)** $\rho = \frac{1}{2}\sum_{\alpha \in \Delta^+(\mathfrak{g}_{\mathbb{C}})} \alpha$.*
***(b)** For $w \in W(\Delta(\mathfrak{g}_{\mathbb{C}}))$, $w\rho - \rho \in R \subseteq A(T)$, and so the function $\xi_{w\rho-\rho}$ descends to T.*

Proof. For part (a), write $\Pi(\mathfrak{g}_{\mathbb{C}}) = \{\alpha_1, \ldots \alpha_l\}$ and let $\rho' = \frac{1}{2}\sum_{\alpha \in \Delta^+(\mathfrak{g}_{\mathbb{C}})} \alpha$ (c.f. Exercise 6.34). By the definitions, it suffices to show that $r_{\alpha_j}\rho' = \rho'$. For this, it suffices to show that r_{α_j} preserves the set $\Delta^+(\mathfrak{g}_{\mathbb{C}})\backslash\{\alpha_j\}$. If $\alpha \in \Delta^+(\mathfrak{g}_{\mathbb{C}})\backslash\{\alpha_j\}$ is written as $\alpha = \Sigma_k n_k \alpha_k$ with $n_{k_0} > 0$, $k_0 \ne j$, then the coefficient of α_{k_0} in $r_{\alpha_j}\alpha = \alpha - \alpha(h_{\alpha_j})\alpha_j$ is still n_{k_0}, so that $r_{\alpha_j}\alpha \in \Delta^+(\mathfrak{g}_{\mathbb{C}})\backslash\{\alpha_j\}$.

Part (b) is straightforward. In fact, it is immediate that

$$w\rho - \rho = \sum_{\alpha \in [w\Delta^+(\mathfrak{g}_{\mathbb{C}})] \cap \Delta^-(\mathfrak{g}_{\mathbb{C}})} \alpha.$$ □

Definition 7.21. For G a compact Lie group with a maximal torus T, let $\Delta : \mathfrak{t} \to \mathbb{C}$ be given by

$$\Delta(H) = \prod_{\alpha \in \Delta^+(\mathfrak{g}_\mathbb{C})} \left(e^{\alpha(H)/2} - e^{-\alpha(H)/2}\right)$$

for $H \in \mathfrak{t}$.

Lemma 7.22. *Let G be a compact Lie group with a maximal torus T.*
(a) *The function Δ is skew-symmetric on $\mathfrak{t}$.*
(b) *The function Δ descends to T if and only if the function $H \to e^{-\rho(H)}$ descends to T.*
(c) *The function $|\Delta|^2$ always descends to T and there $|\Delta(t)|^2 = d(t)$, $t \in T$.*

Proof. For part (a), it suffices to show that $\Delta \circ r_{h_\alpha} = -\Delta$ for $\alpha \in \Delta^+(\mathfrak{g}_\mathbb{C})$. This follows from three observations. The first is that composition with r_{h_α} maps $\left(e^{\alpha/2} - e^{-\alpha/2}\right)$ to $-\left(e^{\alpha/2} - e^{-\alpha/2}\right)$. The second is that if $\beta \in \Delta^+(\mathfrak{g}_\mathbb{C})$ satisfies $r_\alpha \beta = \beta$, then composition with r_{h_α} fixes $\left(e^{\beta/2} - e^{-\beta/2}\right)$. For the third, suppose $\beta \in \Delta^+(\mathfrak{g}_\mathbb{C}) \backslash \{\alpha\}$ satisfies $r_\alpha \beta \neq \beta$. Choose $\beta' \in \Delta^+(\mathfrak{g}_\mathbb{C})$, so that either $r_\alpha \beta = \beta'$ or $r_\alpha \beta = -\beta'$. Then composition with r_{h_α} fixes $\left(e^{\beta/2} - e^{-\beta/2}\right)\left(e^{\beta'/2} - e^{-\beta'/2}\right)$.

For part (b), write $\rho = \frac{1}{2}\sum_{\alpha \in \Delta^+(\mathfrak{g}_\mathbb{C})} \alpha$ to see that

$$e^{-\rho(H)}\Delta(H) = \prod_{\alpha \in \Delta^+(\mathfrak{g}_\mathbb{C})} \left(1 - e^{-\alpha(H)}\right) \tag{7.23}$$

for $H \in \mathfrak{t}$. Since the function $H \to \prod_{\alpha \in \Delta^+(\mathfrak{g}_\mathbb{C})} \left(1 - e^{-\alpha(H)}\right)$ clearly descends to T, part (b) is complete. For part (c), calculate

$$|\Delta(H)|^2 = e^{-\rho(H)}\Delta(H)\overline{e^{-\rho(H)}\Delta(H)} = \prod_{\alpha \in \Delta^+(\mathfrak{g}_\mathbb{C})} \left|1 - e^{-\alpha(H)}\right|^2$$

to complete the proof. □

Note that although $e^{-\rho}$ often descends to a function on T, it does not always descend (Exercise 7.12). Also note that the function $d(t)$ plays a prominent role in Weyl Integration Formula. In particular, we can now write the Weyl Integration Formula as

$$\int_G f(g)\, dg = \frac{1}{|W(G)|}\int_T |\Delta(t)|^2 \int_{G/T} f(gtg^{-1})\, dgT\, dt \tag{7.24}$$

for connected G and $f \in C(G)$.

For the next definition, recall from the proof of Theorem 7.7 that

$$\Xi = \{H \in \mathfrak{t} \mid \alpha(H) \notin 2\pi i\mathbb{Z} \text{ for all roots } \alpha\}$$

is open dense in $\mathfrak{t}$ and $\exp \Xi = T^{\text{reg}}$.

Definition 7.25. Let G be a compact Lie group with a maximal torus T. Fix an analytically integral weight $\lambda \in A(T)$. Let $\Theta_\lambda : \Xi \to \mathbb{C}$ be given by

$$\Theta_\lambda(H) = \frac{\sum_{w \in W(\Delta(\mathfrak{g}_\mathbb{C}))} \det(w)\, e^{[w(\lambda+\rho)](H)}}{\Delta(H)} = \frac{\sum_{w \in W(\Delta(\mathfrak{g}_\mathbb{C}))} \det(w)\, e^{[w(\lambda+\rho)-\rho](H)}}{\prod_{\alpha \in \Delta^+(\mathfrak{g}_\mathbb{C})} \left(1 - e^{-\alpha(H)}\right)}$$

for $H \in \Xi$.

Lemma 7.26. *Let G be a compact connected Lie group with a maximal torus T. Fix an analytically integral weight $\lambda \in A(T)$. The function Θ_λ descends to a smooth W-invariant function on T^{reg}. In turn, this function, still denoted by Θ_λ, uniquely extends to a smooth class function on G^{reg}.*

Proof. The first expression for Θ_λ shows that it is symmetric since the numerator and denominator are skew-symmetric. The second expression for Θ_λ shows it descends to a function on T^{reg} since the numerator and denominator both descend to T and the denominator is nonzero on Ξ. The final statement follows as in Lemma 7.19. □

7.3.2 Main Theorem

Let G be a compact connected Lie group with a maximal torus T. For $\lambda, \lambda' \in A(T)$, the function $\xi_\lambda : T \to \mathbb{C}$ can be viewed as a 1-dimensional irreducible representation of T. As a result, ξ_λ and $\xi_{\lambda'}$ are equivalent if and only if the are equal as functions. This happens if and only if $\lambda = \lambda'$. By the character theory of T, it follows that

$$\int_T \xi_\lambda(t)\xi_{-\lambda'}(t)\,dt = \begin{cases} 1 & \text{if } \lambda = \lambda' \\ 0 & \text{if } \lambda \neq \lambda'. \end{cases} \tag{7.27}$$

Theorem 7.28 (Weyl Character Formula). *Let G be a compact connected Lie group with a maximal torus T. If $V(\lambda)$ is an irreducible representation of G with highest weight λ, then the character of $V(\lambda)$, χ_λ, satisfies*

$$\chi_\lambda(g) = \Theta_\lambda(g)$$

for $g \in G^{\text{reg}}$.

Proof. First note it suffices to prove the theorem for $g = e^H$, $H \in \Xi$. Next for $\gamma \in A(T)$, let $D_\gamma : \mathfrak{t} \to \mathbb{C}$ be the skew-symmetric function defined by

$$D_\gamma(H) = \sum_{w \in W(\Delta(\mathfrak{g}_\mathbb{C}))} \det(w)\, e^{(w\gamma)(H)}.$$

The proof will be completed by showing that $\chi_\lambda(e^H)\Delta(H) = D_{\lambda+\rho}(H)$ for $H \in \mathfrak{t}$.

To this end, by considering the weight decomposition of $V(\lambda)$, write $\chi_\lambda = \sum_{\gamma_j \in A(T)} n_j \xi_{\gamma_j}$ as a finite sum on T for $n_j \in \mathbb{Z}^{\geq 0}$. Thus

$$\chi_\lambda(e^H)\Delta(H) = e^{\rho(H)} \prod_{\alpha\in\Delta^+(\mathfrak{g}_\mathbb{C})} \left(1 - e^{-\alpha(H)}\right) \sum_{\gamma_j\in A(T)} n_j e^{\gamma_j(H)}$$
$$= \sum_{\gamma_j\in A(T)} m_j e^{(\gamma_j+\rho)(H)}$$

for some $m_j \in \mathbb{Z}$. Since χ_λ is symmetric and Δ is skew-symmetric, $\chi_\lambda(e^H)\Delta(H)$ is skew-symmetric as well. Noting that the set of functions $\{e^{\gamma_j+\rho} \mid \gamma_j \in A(T)\}$ is independent, the action of r_α coupled with skew-symmetry shows that $m_j = 0$ if $\gamma_j + \rho$ is on a Weyl chamber wall. Recalling that the Weyl group acts simply transitively on the open Weyl chambers (Theorem 6.43), examination of the the Weyl group orbits of $A(T) + \rho$ and skew-symmetry imply that

$$\chi_\lambda(e^H)\Delta(H) = \sum_{\gamma_j\in A(T),\ \gamma_j+\rho \text{ strictly dominant}} m_j D_{\gamma_j+\rho}(H),$$

where *strictly dominant* means $B(\gamma_j + \rho, \alpha_i) > 0$ for $\alpha_i \in \Pi(\mathfrak{g}_\mathbb{C})$, i.e., $\gamma_j + \rho$ lies in the open positive Weyl chamber.

Next, character theory shows that $\int_G |\chi_\lambda|^2 \, dg = 1$. Thus the Weyl Integration Formula gives

$$\text{(7.29)} \qquad 1 = \frac{1}{|W(G)|} \int_T |\Delta|^2 \, |\chi_\lambda|^2 \, dt$$
$$= \frac{1}{|W(G)|} \int_T \left| \sum_{\gamma_j\in A(T),\ \gamma_j+\rho \text{ str. dom.}} m_j D_{\gamma_j+\rho} \right|^2 dt.$$

Here $\left|\sum_{\gamma_j\in A(T),\ \gamma_j+\rho \text{ str. dom.}} m_j D_{\gamma_j+\rho}\right|^2$ descends to T since $|\Delta|^2 \, |\chi_\lambda|^2$ descends to T. In fact, the function $H \to e^{-\rho(H)} D_{\gamma_j+\rho}(H)$ descends to T since $e^{w(\gamma_j+\rho)-\rho}$ does. Therefore $D_{\gamma_j+\rho}\overline{D_{\gamma_{j'}+\rho}} = \left(e^{-\rho} D_{\gamma_j+\rho}\right) \overline{\left(e^{-\rho} D_{\gamma_{j'}+\rho}\right)}$ descends to T and

$$\frac{1}{|W(G)|} \int_T D_{\gamma_j+\rho}\overline{D_{\gamma_{j'}+\rho}} \, dt = \frac{1}{|W(G)|} \sum_{w,w'\in W(\Delta(\mathfrak{g}_\mathbb{C}))} \det(ww') \int_T \xi_{w(\gamma_j+\rho)}\xi_{-w'(\gamma_{j'}+\rho)} \, dt.$$

Since $\gamma_j + \rho$ and $\gamma_{j'} + \rho$ are in the open Weyl chamber, $w(\gamma_j + \rho) = w'(\gamma_{j'} + \rho)$ if and only if $w = w'$ and $j = j'$. Thus

$$\frac{1}{|W(G)|} \int_T D_{\gamma_j+\rho}\overline{D_{\gamma_{j'}+\rho}} \, dt = \begin{cases} 1 & \text{if } j = j' \\ 0 & \text{if } j \neq j'. \end{cases}$$

In particular, this simplifies Equation 7.29 to

$$1 = \sum_{\gamma_j\in A(T),\ \gamma_j+\rho \text{ str. dom.}} m_j^2.$$

Finally, since $m_j \in \mathbb{Z}$, all but one are zero. Thus there is a $\gamma \in A(T)$ with $\gamma + \rho$ strictly dominant so that $\chi_\lambda(e^H)\Delta(H) = \pm D_{\gamma+\rho}(H)$. To determine γ and the $\pm$

sign, notice that the weight decomposition shows that $\chi_\lambda(e^H) = e^{\lambda(H)} + \ldots$ where the ellipses denote weights strictly lower than λ. Writing

$$\chi_\lambda(e^H)\Delta(H) = e^{\rho(H)}\chi_\lambda(e^H)e^{-\rho(H)}\Delta(H) = \left(e^{(\lambda+\rho)(H)} + \ldots\right) \prod_{\alpha\in\Delta^+(\mathfrak{g}_\mathbb{C})} \left(1 - e^{-\alpha(H)}\right),$$

we see $\chi_\lambda(e^H)\Delta(H) = e^{(\lambda+\rho)(H)} + \ldots$. In particular, expanding the function $H \to \chi_\lambda(e^H)\Delta(H)$ in terms of $\{e^{\gamma_j+\rho} \mid \gamma_j \in A(T)\}$, it follows that $e^{\lambda+\rho}$ appears with coefficient 1. On the other hand, similarly expanding $\pm D_{\gamma+\rho}$, we see that the only term of the form $e^{\gamma_j+\rho}$ appearing for which $\gamma_j + \rho$ is dominant is $\pm e^{\gamma+\rho}$. Therefore $\lambda = \gamma$, the undetermined $\pm$ sign is a $+$. □

7.3.3 Weyl Denominator Formula

Theorem 7.30 (Weyl Denominator Formula). *Let G be a compact connected Lie group with a maximal torus T. Then*

$$\Delta(H) = \sum_{w\in W(\Delta(\mathfrak{g}_\mathbb{C}))} \det(w)\, e^{(w\rho)(H)}$$

for $H \in \mathfrak{t}$.

Proof. Simply take the trivial representation $V(0) = \mathbb{C}$ with $\chi_0(g) = 1$ and apply the Weyl Character Formula to $g = e^H$ for $H \in \Xi$. The formula extends to all $\mathfrak{t}$ by continuity. □

Note the Weyl Denominator Formula allows the Weyl Character Formula to be rewritten in the form

$$\chi_\lambda(e^H) = \frac{\sum_{w\in W(\Delta(\mathfrak{g}_\mathbb{C}))} \det(w)\, e^{[w(\lambda+\rho)](H)}}{\sum_{w\in W(\Delta(\mathfrak{g}_\mathbb{C}))} \det(w)\, e^{(w\rho)(H)}} \tag{7.31}$$

for $H \in \mathfrak{t}$ with $e^H \in T^{\text{reg}}$, *i.e.*, $H \in \Xi$.

7.3.4 Weyl Dimension Formula

Theorem 7.32 (Weyl Dimension Formula). *Let G be a compact connected Lie group with a maximal torus T. If $V(\lambda)$ is the irreducible representation of G with highest weight λ, then*

$$\dim V(\lambda) = \prod_{\alpha\in\Delta^+(\mathfrak{g}_\mathbb{C})} \frac{B(\lambda+\rho,\alpha)}{B(\rho,\alpha)}.$$

Proof. Since $\dim V(\lambda) = \chi_\lambda(e)$, we ought to evaluate Equation 7.31 at $H = 0$. Unfortunately, Equation 7.31 is not defined at $H = 0$, so we take a limit. Let $u_\rho \in i\mathfrak{t}$, so that $\rho(H) = B(H, u_\rho)$ for $H \in \mathfrak{t}$. Then it is easy to see that $itu_\rho \in \Xi$ for small positive t (Exercise 7.13), so that

$$\dim V(\lambda) = \lim_{t\to 0} \Theta_\lambda(itu_\rho)$$

$$(7.33)\qquad = \lim_{t\to 0} \frac{\sum_{w\in W(\Delta(\mathfrak{g}_\mathbb{C}))} \det(w)\, e^{[w(\lambda+\rho)](itu_\rho)}}{\sum_{w\in W(\Delta(\mathfrak{g}_\mathbb{C}))} \det(w)\, e^{(w\rho)(itu_\rho)}}.$$

Now observe that

$$\begin{aligned}(w(\lambda+\rho))(itu_\rho) &= it(\lambda+\rho)(w^{-1}u_\rho) = itB(u_{\lambda+\rho}, w^{-1}u_\rho)\\ &= itB(wu_{\lambda+\rho}, u_\rho) = it\rho(wu_{\lambda+\rho}) = (w^{-1}\rho)(itu_{\lambda+\rho}).\end{aligned}$$

Since $\det w = \det(w^{-1})$, the Weyl Denominator Formula rewrites the numerator in Equation 7.33 as

$$\begin{aligned}\sum_{w\in W(\Delta(\mathfrak{g}_\mathbb{C}))} \det(w)\, e^{[w(\lambda+\rho)](itu_\rho)} &= \sum_{w\in W(\Delta(\mathfrak{g}_\mathbb{C}))} \det(w)\, e^{(w\rho)(itu_{\lambda+\rho})} = \Delta(itu_{\lambda+\rho})\\ &= \prod_{\alpha\in\Delta^+(\mathfrak{g}_\mathbb{C})} \left(e^{\alpha(itu_{\lambda+\rho})/2} - e^{-\alpha(itu_{\lambda+\rho})/2}\right)\\ &= \prod_{\alpha\in\Delta^+(\mathfrak{g}_\mathbb{C})} \left(it\alpha(u_{\lambda+\rho}) + \cdots\right)\\ &= (it)^{|\Delta^+(\mathfrak{g}_\mathbb{C})|} \prod_{\alpha\in\Delta^+(\mathfrak{g}_\mathbb{C})} B(\alpha, \lambda+\rho) + \cdots\end{aligned}$$

where the ellipses denote higher powers of t. Similarly, the Weyl Denominator Formula rewrites denominator in Equation 7.33 as

$$\sum_{w\in W(\Delta(\mathfrak{g}_\mathbb{C}))} \det(w)\, e^{(w\rho)(itu_\rho)} = (it)^{|\Delta^+(\mathfrak{g}_\mathbb{C})|} \prod_{\alpha\in\Delta^+(\mathfrak{g}_\mathbb{C})} B(\alpha, \rho) + \cdots$$

which finishes the proof. □

7.3.5 Highest Weight Classification

Theorem 7.34 (Highest Weight Classification). *For a connected compact Lie group G with maximal torus T, there is a one-to-one correspondence between irreducible representations and dominant analytically integral weights given by mapping $V(\lambda) \to \lambda$ for dominant $\lambda \in A(T)$.*

Proof. We saw in Theorem 7.3 that the map $V(\lambda) \to \lambda$ is well defined and injective. It remains to see it is surjective. For any $\lambda \in A(T)$, Lemma 7.26 shows the function Θ_λ descends to a smooth class function on G^{reg}. The Weyl Integral Formula to calculates

$$\begin{aligned}\int_G |\Theta_\lambda|^2\, dg &= \frac{1}{|W(G)|} \int_{T^{\text{reg}}} |\Delta(t)\Theta_\lambda|^2\, dt\\ &= \frac{1}{|W(G)|} \int_T \left| \sum_{w\in W(\Delta(\mathfrak{g}_\mathbb{C}))} \det(w)\, \xi_{w(\lambda+\rho)} \right|^2 dt\\ &= \frac{1}{|W(G)|} \sum_{w,w'\in W(\Delta(\mathfrak{g}_\mathbb{C}))} \det(ww') \int_T \xi_{w(\lambda+\rho)}\xi_{-w'(\lambda+\rho)}\, dt.\end{aligned}$$

When λ is also dominant, $\lambda + \rho$ is strictly dominant so that, as in the proof of the Weyl Character Formula, Equation 7.27 shows that

$$\int_T \xi_{w(\lambda+\rho)}\xi_{-w'(\lambda+\rho)}\,dt = \delta_{w,w'}.$$

As a result, $\int_G |\Theta_\lambda|^2\,dg = 1$ for any dominant $\lambda \in A(T)$. In particular, Θ_λ is a nonzero L^2 class function on G.

Now choose any irreducible representation $V(\mu)$ of G and note that the function Θ_μ extends to the character χ_μ. By the now typical calculation,

$$\begin{aligned}\int_G \chi_\mu\overline{\Theta_\lambda}\,dg &= \frac{1}{|W(G)|}\int_{T^{\text{reg}}} |\Delta(t)|^2\,\Theta_\mu\overline{\Theta_\lambda}\,dt \\ &= \frac{1}{|W(G)|}\sum_{w,w'\in W(\Delta(\mathfrak{g}_\mathbb{C}))} \det(ww')\int_{T^{\text{reg}}} \xi_{w(\mu+\rho)}\xi_{w'(\lambda+\rho)}\,dt \\ &= \begin{cases} 1 & \text{if } \mu=\lambda \\ 0 & \text{if } \mu\neq\lambda. \end{cases}\end{aligned}$$

Since Theorems 7.3 and 3.30 imply that $\{\chi_\mu \mid$ there exists an irreducible representation with highest weight $\mu\}$ is an orthonormal basis for the set of L^2 class functions on G, the value of $\int_G \chi_\mu\overline{\Theta_\lambda}\,dg$ cannot be zero for every such μ. In particular, this means that there is an irreducible representation with highest weight λ. □

7.3.6 Fundamental Group

Here we finish the proof of Theorem 6.30. This is especially important in light of the Highest Weight Classification. Of special note, it shows that when G is a simply connected compact Lie group with semisimple Lie algebra, then the irreducible representations are parametrized by the set of dominant algebraic weights, P. In turn, this also classifies the irreducible representations of $\mathfrak{g}$ (Theorem 4.16). At the opposite end of the spectrum, Theorem 6.30 shows that the irreducible representations of $\text{Ad}(G) \cong G/Z(G)$ (Lemma 5.11) are parametrized by the dominant elements of the root lattice, R. The most general group lies between these two extremes.

Lemma 7.35. *Let G be a compact connected Lie group with maximal torus T. Let $G^{\text{sing}} = G\backslash G^{\text{reg}}$. Then G^{sing} is a closed subset with* codim $G^{\text{sing}} \geq 3$ *in G.*

Proof. It follows from Theorem 7.7 that G^{sing} is closed and the map $\psi : G/T \times T^{\text{sing}} \to G^{\text{sing}}$ is surjective. Moreover $t \in T^{\text{sing}}$ if and only if there exists $\alpha \in \Delta^+(\mathfrak{g}_\mathbb{C})$, so $\xi_\alpha(t) = 1$ so that $T^{\text{sing}} = \cup_{\in\Delta^+(\mathfrak{g}_\mathbb{C})} \ker\xi_\alpha$. As a Lie subgroup of T, $\ker\xi_\alpha$ is a closed subgroup of codimension 1. Let $U_\alpha = \{gtg^{-1} \mid g \in G \text{ and } t \in \ker\xi_\alpha\}$, so that $G^{\text{sing}} = \cup_{\in\Delta^+(\mathfrak{g}_\mathbb{C})} U_\alpha$.

Recall that $\mathfrak{z}_\mathfrak{g}(t) = \{X \in \mathfrak{g} \mid \text{Ad}(t)X = X\}$ (Exercise 4.22). Since $\text{Ad}(t)$ acts on $\mathfrak{g}_\alpha$ as $\xi_\alpha(t)$, it follows that $\mathfrak{g} \cap (\mathfrak{g}_{-\alpha} \oplus \mathfrak{g}_\alpha) \subseteq \mathfrak{z}_\mathfrak{g}(t)$ when $t \in \ker\xi_\alpha$. Now choose a standard embedding $\varphi_\alpha : SU(2) \to G$ corresponding to α and let V_α be the compact manifold $V_\alpha = G/(\varphi_\alpha(SU(2))T) \times \ker \xi_\alpha$. Observe that $\dim V_\alpha = \dim G - 3$ and that ψ maps V_α onto U_α. Therefore the precise version of this lemma is that G^{sing} is a finite union of closed images of compact manifolds each of which has codimension 3 with respect to G. □

Thinking of a homotopy of loops as a two-dimensional surface, Lemma 7.35 coupled with standard transversality theorems ([42]), show that loops in G with a base point in G^{reg} can be homotoped to loops in G^{reg}. As a corollary, it is straightforward to see that

$$\pi_1(G) \cong \pi_1(G^{\text{reg}}).$$

Let G be a compact Lie group with maximal torus T. Recall from Theorem 7.7 that $e^H \in T^{\text{reg}}$ if and only if $H \in \{H \in \mathfrak{t} \mid \alpha(H) \notin 2\pi i\mathbb{Z}$ for all roots $\alpha\}$. The connected regions of $\{H \in \mathfrak{t} \mid \alpha(H) \notin 2\pi i\mathbb{Z}$ for all roots $\alpha\}$ are convex and are given a special name.

Definition 7.36. Let G be a compact Lie group with maximal torus T. The connected components of $\{H \in \mathfrak{t} \mid \alpha(H) \notin 2\pi i\mathbb{Z}$ for all roots $\alpha\}$ are called *alcoves*.

Lemma 7.37. *Let G be a compact connected Lie group with maximal torus T and fix a base $t_0 = e^{H_0} \in T^{\text{reg}}$ with $H_0 \in \mathfrak{t}$.*
(a) *Any continuous loop $\gamma : [0, 1] \to G^{\text{reg}}$ with $\gamma(0) = t_0$ can be written as*

$$\gamma(s) = c_{g_s} e^{H(s)}$$

with $g_0 = e$, $H(0) = H_0$, and the maps $s \to g_s T \in G/T$ and $s \to H(s) \in \mathfrak{t}^{\text{reg}}$ continuous. In that case, $g_1 \in N(T)$ and

$$H(1) = \text{Ad}(g_1)^{-1} H_0 + X_\gamma$$

for some $X_\gamma \in 2\pi i \ker \mathcal{E}$. The element X_γ is independent of the homotopy class of γ.
(b) *Write A_0 for the alcove containing H_0. Keeping the same base t_0, the map*

$$\pi_1(G^{\text{reg}}) \to A_0 \cap \{wH_0 + Z \mid w \in W(\Delta(\mathfrak{g}_\mathbb{C})^\vee) \text{ and } Z \in 2\pi i A(T)^*\}$$

induced by $\gamma \to X_\gamma$ is well defined and bijective.

Proof. Using the Maximal Torus Theorem, write $\gamma(s) = c_{g_s}\tau(s)$ with $\tau(s) \in T^{\text{reg}}$, $\tau(0) = t_0$, and $g_0 = e$. In fact, since $\psi : G/T \times T^{\text{reg}} \to G^{\text{reg}}$ is a covering, the lifts $s \to \tau(s) \in T^{\text{reg}}$ and $s \to g_s T \in G/T$ are uniquely determined by these conditions and continuity. Since $\exp : \mathfrak{t}^{\text{reg}} \to T^{\text{reg}}$ is also a local diffeomorphism (Theorem 5.14), there exists a unique continuous lift $s \to H(s) \in \mathfrak{t}^{\text{reg}}$ of τ, so $H(0) = H_0$ and $\gamma(s) = c_{g_s} e^{H(s)}$.

As γ is a loop, $\gamma(0) = \gamma(1)$, so $e^{H_0} = c_{g_1} e^{H(1)}$. Because e^{H_0} and $e^{H(1)}$ are regular, $T = Z_G(e^{H_0})^0 = c_{g_1} Z_G(e^{H(1)})^0 = c_{g_1} T$, so that $g_1 \in N(T)$ is a Weyl group element. Writing $w = \text{Ad}(g_1)$, it follows that $H_0 \equiv wH(1)$ modulo $2\pi i \ker \mathcal{E}$, the kernel of $\exp : \mathfrak{t} \to T$. Therefore write $H(1) = w^{-1}H_0 + X_\gamma$ for some $X_\gamma \in 2\pi i \ker \mathcal{E}$.

To see that X_γ is independent of the homotopy class of γ, suppose $\gamma' : [0, 1] \to G^{\text{reg}}$ with $\gamma'(0) = t_0$ is another loop and that $\gamma(s, t)$ is a homotopy between γ and γ'. Thus $\gamma(s, 0) = \gamma(s)$, $\gamma(s, 1) = \gamma'(s)$, and $\gamma(0, t) = \gamma(1, t) = t_0$. Using the same arguments as above and similar notational conventions, write $\gamma'(s) = c_{g'_s} e^{H'(s)}$ and $H'(1) = w'^{-1}H_0 + X'_\gamma$. Similarly, write $\gamma(s, t) = c_{g_{s,t}} e^{H(s,t)}$ and $H(1, s) = w_s^{-1}H_0 + X_\gamma(s)$. Notice that $w_0 = w$, $w_1 = w'$, $X_\gamma(0) = X_\gamma$, and $X_\gamma(1) = X'_\gamma$. Since w_s and $X_\gamma(s)$ vary continuously with s and since $W(T)$ and $2\pi i \ker \mathcal{E}$ are discrete, w_s and $X_\gamma(s)$ are constant. This finishes part (a).

For part (b), first note that continuity of $H(s)$ implies that $H(1)$ is still in A_0, so that the map is well defined. To see surjectivity, fix $H' \in A_0 \cap \{wH_0 + Z \mid w \in W(\Delta(\mathfrak{g}_\mathbb{C})^\vee) \text{ and } Z \in 2\pi i A(T)^*\}$ and write $H' = w'^{-1}H_0 + Z'$ for $w' \in W(\Delta(\mathfrak{g}_\mathbb{C})^\vee)$ and $Z' \in 2\pi i \ker \mathcal{E}$. Choose a continuous path $s \to g'_s \in G$, so that $g'_0 = e$ and $\text{Ad}(g'_1) = w'$. Let $H'(s) = H_0 + s(H' - H_0) \in A_0$ and consider the curve $\gamma'(s) = c_{g_s} e^{H'(s)}$. Since $\gamma'(0) = t_0$ and $\gamma'(1) = e^{w'H'} = e^{H_0 + Z'} = t_0$, γ' is a loop with base point t_0. By construction, $X_{\gamma'} = H'$, as desired. To see injectivity, observe that if $X_\gamma = X_{\gamma''}$ with $\gamma(s) = c_{g_s} e^{H(s)}$ and $\gamma''(s) = c_{g_s} e^{H''(s)}$, then $\gamma(s, t) = c_{g_s} e^{(1-t)H'(s) + tH''(s)}$ is a homotopy between the two. □

Lemma 7.38. *Let G be a compact connected Lie group with maximal torus T.*
(a) *Each homotopy class in G with base e can be represented by a loop of the form*

$$\gamma(s) = e^{sX_\gamma}$$

for some $X_\gamma \in 2\pi i \ker \mathcal{E}$, i.e., for some X_γ in the kernel of $\exp : \mathfrak{t} \to T$. *The surjective map from $2\pi i \ker \mathcal{E}$ to $\pi_1(G)$ induced by $X_\gamma \to \gamma$ is a homomorphism.*
(b) *Fix an alcove A_0 and $H_0 \in A_0$. The above map restricts to a bijection on $\{Z \in 2\pi i \ker \mathcal{E} \mid wH_0 + Z \in A_0 \text{ for some } w \in W(\Delta(\mathfrak{g}_\mathbb{C})^\vee)\}$.*

Proof. Lemma 7.37 shows that each homotopy class in G with base t_0 can be represented by a curve of the form $\gamma(s) = c_{g_s} e^{H(s)}$ with $H(1) = \text{Ad}(g_1)^{-1}H_0 + X_\gamma$ for some $X_\gamma \in 2\pi i \ker \mathcal{E}$. Using the homotopy $\gamma(s, t) = c_{g_s} e^{(1-t)H(s) + t[H_0 + s(H(1) - H_0)]}$, we may assume $H(s)$ is of the form $H(s) = H_0 + s\left(\text{Ad}(g_1)^{-1}H_0 + X_\gamma - H_0\right)$.

Translating back to the identity, it follows that each homotopy class in G with base e can be represented by a curve of the form

$$\gamma(s) = e^{-H_0} c_{g_s} e^{H_0 + s\left(\text{Ad}(g_1)^{-1}H_0 + X_\gamma - H_0\right)}.$$

Using the homotopy $\gamma(s, t) = e^{-tH_0} c_{g_s} e^{tH_0 + s\left(t\,\text{Ad}(g_1)^{-1}H_0 + X_\gamma - tH_0\right)}$, we may assume $\gamma(s) = c_{g_s} e^{sX_\gamma}$. Finally, using the homotopy $\gamma(s, t) = c_{g_{st}} e^{sX_\gamma}$, we may assume $\gamma(s) = e^{sX_\gamma}$. Verifying that the map $\gamma \to X_\gamma$ is a homomorphism is straightforward and left as an exercise (Exercise 7.24). Part (b) follows from Lemma 7.37. □

Note that a corollary of Lemma 7.38 shows that the inclusion map $T \to G$ induces a surjection $\pi_1(T) \to \pi_1(G)$.

Definition 7.39. Let G be a compact connected Lie group with maximal torus T. The *affine Weyl group* is the group generated by the transformations of $\mathfrak{t}$ of the form $H \to wH + Z$ for $w \in W(\Delta(\mathfrak{g}_{\mathbb{C}})^{\vee})$ and $Z \in 2\pi i R^{\vee}$.

Lemma 7.40. *Let G be a compact connected Lie group with maximal torus T.*
(a) *The affine Weyl group is generated by the reflections across the hyperplanes $\alpha^{-1}(2\pi i n)$ for $\alpha \in \Delta(\mathfrak{g}_{\mathbb{C}})$ and $n \in \mathbb{Z}$.*
(b) *The affine Weyl group acts simply transitively on the set of alcoves.*

Proof. Recall that $h_\alpha \in R^{\vee}$ and notice the reflection across the hyperplane $\alpha^{-1}(2\pi i n)$ is given by $r_{h_\alpha,n}(H) = r_{h_\alpha} H + 2\pi i h_\alpha$ (Exercise 7.25). Since the Weyl group is generated by the reflections r_{h_α}, part (a) is finished. The proof of part (b) is very similar to Theorem 6.43 and the details are left as an exercise (Exercise 7.26). □

Theorem 7.41. *Let G be a connected compact Lie group with semisimple Lie algebra and maximal torus T. Then $\pi_1(G) \cong \ker \mathcal{E}/R^{\vee} \cong P/A(T)$.*

Proof. By Lemma 7.38, it suffices to show that the loop $\gamma(s) = e^{sX_\gamma}$, $X_\gamma \in 2\pi i \ker \mathcal{E}$, is trivial if and only if $X_\gamma \in 2\pi i R^{\vee}$. For this, first consider the standard $\mathfrak{su}(2)$-triple corresponding to $\alpha \in \Delta(\mathfrak{g}_{\mathbb{C}})$ and let $\varphi_\alpha : SU(2) \to G$ be the corresponding embedding. The loop $\gamma_\alpha(s) = e^{2\pi i s h_\alpha}$ is the image under φ_α of the loop $s \to \operatorname{diag}(e^{2\pi i s}, e^{-2\pi i s})$ in $SU(2)$. As $SU(2)$ is simply connected, γ_α is trivial. Thus there is a well-defined surjective map $2\pi i \ker \mathcal{E}/2\pi i R^{\vee} \to \pi_1(G)$.

It remains to see that it is injective. Fix an alcove A_0 and $H_0 \in A_0$. Since $2\pi i \ker \mathcal{E} \subseteq 2\pi i P^{\vee}$, $A_0 - X_\gamma$ is another alcove. By Lemma 7.40, there is a $w \in W(\Delta(\mathfrak{g}_{\mathbb{C}})^{\vee})$ and $H \in 2\pi i R^{\vee}$, so that $wH_0 + H \in A_0 - X_\gamma$. Thus $wH_0 + (X_\gamma + H) \in A_0$. Because the loop $s \to e^{sH}$ is trivial, we may use a homotopy on γ and assume $H = 0$, so that $wH_0 + X_\gamma \in A_0$. But as $H_0 + 0 \in A_0$, Lemma 7.38 shows that γ must be homotopic to the trivial loop $s \to e^{s0}$. □

7.3.7 Exercises

Exercise 7.12 Show that the function e^{ρ} descends to the maximal torus for $SU(n)$, $SO(2n)$, and $Sp(2n)$, but not for $SO(2n+1)$.

Exercise 7.13 Let G be a compact Lie group with a maximal torus T. Let $u_\rho \in i\mathfrak{t}$, so that $\rho(H) = B(H, u_\rho)$ for $H \in \mathfrak{t}$. Show that $itu_\rho \in \Xi$ for small positive t.

Exercise 7.14 Show that the dominant analytically integral weights of $SU(3)$ are all expressions of the form $\lambda = n\pi_1 + m\pi_2$ for $n, m \in \mathbb{Z}^{\geq 0}$ where π_1, π_2 are the fundamental weights $\pi_1 = \frac{2}{3}\epsilon_{1,2} + \frac{1}{3}\epsilon_{2,3}$ and $\pi_2 = \frac{1}{3}\epsilon_{1,2} + \frac{2}{3}\epsilon_{2,3}$. Conclude that

$$\dim V(\lambda) = \frac{(n+1)(m+1)(n+m+2)}{2}.$$

Exercise 7.15 Let G be a compact Lie group with semisimple $\mathfrak{g}$ and a maximal torus T. The set of dominant weight vectors are of the form $\lambda = \sum_i n_i \pi_i$ where $\{\pi_i\}$ are the fundamental weights and $n_i \in \mathbb{Z}^{\geq 0}$. Verify the following calculations.
(1) For $G = SU(n)$,

$$\dim V(\lambda) = \prod_{1 \leq i < j \leq n} \left(1 + \frac{n_i + \cdots + n_{j-1}}{j - i}\right).$$

(2) For $G = Sp(n)$,

$$\dim V(\lambda) = \prod_{1 \leq i < j \leq m} \left(1 + \frac{n_i + \cdots + n_{j-1}}{j - i}\right) \cdot \prod_{1 \leq i < j \leq m} \left(1 + \frac{n_i + \cdots + n_{j-1} + 2\left(n_j + \cdots + n_{m-1}\right)}{2n + 2 - i - j}\right) \cdot \prod_{1 \leq i \leq m} \left(1 + \frac{n_i + \cdots + n_{m-1} + n_m}{n + 1 - i}\right).$$

(3) For $G = \mathrm{Spin}_{2m+1}(\mathbb{R})$,

$$\dim V(\lambda) = \prod_{1 \leq i < j \leq m} \left(1 + \frac{n_i + \cdots + n_{j-1}}{j - i}\right) \cdot \prod_{1 \leq i < j \leq m} \left(1 + \frac{n_i + \cdots + n_{j-1} + 2\left(n_j + \cdots n_{m-1}\right) + n_m}{2m + 1 - i - j}\right) \cdot \prod_{1 \leq i \leq m} \left(1 + \frac{2\left(n_i + \cdots + n_{m-1}\right) + n_m}{2n + 1 - 2i}\right).$$

(4) For $G = \mathrm{Spin}_{2m}(\mathbb{R})$,

$$\dim V(\lambda) = \prod_{1 \leq i < j \leq m} \left(1 + \frac{n_i + n_{j-1}}{j - i}\right) \cdot \prod_{1 \leq i < j \leq m} \left(1 + \frac{n_i + \cdots + n_{j-1} + 2\left(n_j + \cdots + n_{m-1}\right) + n_m}{2m - i - j}\right).$$

Exercise 7.16 For each group G below, show that the listed representation(s) V of G has minimal dimension among nontrivial irreducible representations.
(1) For $G = SU(n)$, V is the standard representation on $\mathbb{C}^n$ or its dual.
(2) For $G = Sp(n)$, V is the standard representation on $\mathbb{C}^{2n}$.
(3) For $G = \mathrm{Spin}_{2m+1}(\mathbb{R})$ with $m \geq 2$, $V = \mathbb{C}^{2m+1}$ and the action comes from the covering $\mathrm{Spin}_{2m+1}(\mathbb{R}) \to SO(2m+1)$.
(4) For $G = \mathrm{Spin}_{2m}(\mathbb{R})$ with $m > 4$, $V = \mathbb{C}^{2m}$ and the action comes from the covering $\mathrm{Spin}_{2m}(\mathbb{R}) \to SO(2m)$.

Exercise 7.17 Let G be a compact Lie group with a maximal torus T. Suppose V is a representation of G that possesses a highest weight of weight λ. If $\dim V = \dim V(\lambda)$, show that $V \cong V(\lambda)$ and, in particular, irreducible.

Exercise 7.18 Use Exercise 7.17 and the Weyl Dimension Formula to show that the following representation V of G is irreducible:
(1) $G = SU(n)$ with $V = \bigwedge^p \mathbb{C}^n$ (c.f. Exercise 7.1).
(2) $G = SO(n)$ with $V = \mathcal{H}_m(\mathbb{R}^n)$ (*c.f.* Exercise 7.2).
(3) $G = SO(2n+1)$ with $V = \bigwedge^p \mathbb{C}^{2n+1}$, $1 \leq p \leq n$ (c.f. Exercise 7.3).
(4) $G = SO(2n)$ with $V = \bigwedge^p \mathbb{C}^{2n}$, $1 \leq p < n$ (*c.f.* Exercise 7.3).
(5) $G = SU(n)$ with $V = V_{p,0}(\mathbb{C}^n)$ (c.f. Exercise 7.5).
(6) $G = SU(n)$ with $V = V_{0,q}(\mathbb{C}^n)$ (c.f. Exercise 7.5).
(7) $G = SU(n)$ with $V = \mathcal{H}_{p,q}(\mathbb{C}^n)$ (c.f. Exercise 7.5).
(8) $G = \mathrm{Spin}_{2m+1}(\mathbb{R})$ with $V = S$ (c.f. Exercise 7.6).
(9) $G = \mathrm{Spin}_{2m}(\mathbb{R})$ with $V = S^{\pm}$ (c.f. Exercise 7.6).

Exercise 7.19 Let λ be a dominant analytically integral weight of $U(n)$ and write $\lambda = \lambda_1\epsilon_1 + \cdots + \lambda_n\epsilon_n$, $\lambda_j \in \mathbb{Z}$ with $\lambda_1 \geq \cdots \geq \lambda_n$. For $H = \mathrm{diag}(H_1, \ldots, H_n) \in \mathfrak{t}$, show that the Weyl Character Formula can be written as

$$\chi_\lambda(e^H) = \frac{\det\left(e^{(\lambda_j + j - 1)H_k}\right)}{\det\left(e^{(j-1)H_k}\right)}.$$

Exercise 7.20 Let G be a compact connected Lie group with maximal torus T.
(1) If G is not Abelian, show that the dimensions of the irreducible representations of G are unbounded.
(2) If $\mathfrak{g}$ is semisimple, show that there are at most a finite number of irreducible representations of any given dimension.

Exercise 7.21 Let G be a compact connected Lie group with maximal torus T. For $\lambda \in (i\mathfrak{t})^*$, the *Kostant partition function* evaluated at λ, $\mathcal{P}(\lambda)$, is the number of ways of writing $\lambda = \sum_{\alpha\in\Delta^+(\mathfrak{g}_\mathbb{C})} m_\alpha \alpha$ with $m_\alpha \in \mathbb{Z}^{\geq 0}$.
(1) As a formal sum of functions on $\mathfrak{t}$, show that

$$\prod_{\alpha\in\Delta^+(\mathfrak{g}_\mathbb{C})} \left(1 + e^{-\alpha} + e^{-2\alpha} + \cdots\right) = \sum_\lambda \mathcal{P}(\lambda) e^{-\lambda}$$

to conclude that

$$1 = \left(\sum_\lambda \mathcal{P}(\lambda) e^{-\lambda}\right) \prod_{\alpha\in\Delta^+(\mathfrak{g}_\mathbb{C})} \left(1 - e^{-\alpha}\right).$$

For what values of $H \in \mathfrak{t}$ can this expression be evaluated?
(2) The *multiplicity*, m_μ, *of* μ *in* $V(\lambda)$ is the dimension of the μ-weight space in $V(\lambda)$. Thus $\chi_\lambda = \sum_\mu m_\mu \xi_\mu$. Use the Weyl Character Formula, part (1), and gather terms to show that m_μ is given by the expression

$$m_\mu = \sum_{w \in W(\Delta(\mathfrak{g}_\mathbb{C}))} \det(w)\, \mathcal{P}\left(w(\lambda+\rho) - (\mu+\rho)\right).$$

This formula is called the *Kostant Multiplicity Formula*.
(3) For $G = SU(3)$, calculate the weight multiplicities for $V(\epsilon_{1,2} + 3\epsilon_{2,3})$.

Exercise 7.22 Let G be a compact connected Lie group with maximal torus T. The *multiplicity*, m_μ, *of* $V(\mu)$ *in* $V(\lambda) \otimes V(\lambda')$ is the number of times $V(\mu)$ appears as a summand in $V(\lambda) \otimes V(\lambda')$. Thus $\chi_\lambda \chi_{\lambda'} = \sum_\mu m_\mu \chi_\mu$. Use part (1) of Exercise 7.21 and compare dominant terms to show m_μ is given by the expression

$$m_\mu = \sum_{w, w' \in W(\Delta(\mathfrak{g}_\mathbb{C}))} \det(ww')\, \mathcal{P}\left(w(\lambda+\rho) + w'(\lambda'+\rho) - (\mu+2\rho)\right).$$

This formula is called *Steinberg's Formula*.

Exercise 7.23 Let G be a compact connected Lie group with maximal torus T and $\alpha \in \Delta(\mathfrak{g}_\mathbb{C})$. Show that $\ker \xi_\alpha$ in T may be disconnected.

Exercise 7.24 Show that the map $\gamma \to X_\gamma$ from Lemma 7.38 is a homomorphism.

Exercise 7.25 Let G be a compact connected Lie group with maximal torus T. Show that the reflection across the hyperplane $\alpha^{-1}(2\pi i n)$ is given by the formula $r_{h_\alpha, n}(H) = r_{h_\alpha} H + 2\pi i n h_\alpha$ for $H \in \mathfrak{t}$.

Exercise 7.26 Let G be a compact connected Lie group with maximal torus T. Show that the affine Weyl group acts simply transitively on the set of alcoves.

7.4 Borel–Weil Theorem

The Highest Weight Classification gives a parametrization of the irreducible representations of a compact Lie group. Lacking is an explicit realization of these representations. The Borel–Weil Theorem repairs this gap.

7.4.1 Induced Representations

Definition 7.42. (a) A complex *vector bundle* $\mathcal{V}$ of rank n on a manifold M is a manifold $\mathcal{V}$ and a smooth surjective map $\pi : \mathcal{V} \to M$ called the *projection*, so that: **(i)** for each $x \in M$, the *fiber* over x, $\mathcal{V}_x = \pi^{-1}(x)$, is a vector space of dimension n and **(ii)** for each $x \in M$, there is a neighborhood U of x in M and a diffeomorphism $\varphi : \pi^{-1}(U) \to U \times \mathbb{C}^n$, so that $\varphi(\mathcal{V}_y) = (y, \mathbb{C}^n)$ for $y \in U$.
(b) The set of smooth (continuous) *sections* of $\mathcal{V}$ are denoted by $\Gamma(M, \mathcal{V})$ and consists of all smooth (continuous) maps $s : M \to \mathcal{V}$, so that $\pi \circ s = I$.
(c) An action of a Lie group G on $\mathcal{V}$ is said to *preserve fibers* if for each $g \in G$ and $x \in M$, there exists $x' \in M$, so that $g\mathcal{V}_x \subseteq \mathcal{V}_{x'}$. In this case, the action of G on $\mathcal{V}$ naturally descends to an action of G on M.

(d) $\mathcal{V}$ is a *homogeneous* vector bundle over M for the Lie group G if **(i)** the action of G on $\mathcal{V}$ preserves fibers; **(ii)** the resulting action of G on M is transitive; and **(iii)** each $g \in G$ maps $\mathcal{V}_x$ to $\mathcal{V}_{gx}$ linearly for $x \in M$.
(e) If $\mathcal{V}$ is a homogeneous vector bundle over M, the vector space $\Gamma(M, \mathcal{V})$ carries an action of G given by

$$(gs)(x) = g(s(g^{-1}x))$$

for $s \in \Gamma(M, \mathcal{V})$.
(f) Two homogeneous vector bundles $\mathcal{V}$ and $\mathcal{V}'$ over M for G are *equivalent* if there is a diffeomorphism $\varphi : \mathcal{V} \to \mathcal{V}'$, so that $\pi' \circ \varphi = \varphi \circ \pi$.

Note it suffices to study manifolds of the form $M = G/H$, H a closed subgroup of G, when studying homogenous vector bundles.

Definition 7.43. Let G be a Lie group and H a closed subgroup of G. Given a representation V of H, define the homogeneous vector bundle $G \times_H V$ over G/H by

$$G \times_H V = (G \times V)/^{\sim},$$

where $^{\sim}$ is the equivalence relation given by

$$(gh, v)^{\sim}(g, hv)$$

for $g \in G$, $h \in H$, and $v \in V$. The projection map $\pi : G \times_H V \to G/H$ is given by $\pi(g, v) = gH$ and the G-action is given by $g'(g, v) = (g'g, v)$ for $g' \in G$.

It is necessary to verify that $G \times_H V$ is indeed a homogeneous vector bundle over G/H. Since H is a regular submanifold, this is a straightforward argument and left as an exercise (Exercise 7.27).

Theorem 7.44. *Let G be a Lie group and H a closed subgroup of G. There is a bijection between equivalence classes of homogenous vector bundles $\mathcal{V}$ on G/H and representations of H.*

Proof. The correspondence maps $\mathcal{V}$ to $\mathcal{V}_{eH}$. By definition $\mathcal{V}_{eH}$ is a representation of H. Conversely, given a representation V of H, the vector bundle $G \times_H V$ inverts the correspondence. □

Definition 7.45. Let G be a Lie group and H a closed subgroup of G. Given a representation (π, V) of H, define the smooth (continuous) *induced representation* of G by

$$\operatorname{Ind}_H^G(V) = \operatorname{Ind}_H^G(\pi) = \{\text{smooth (continuous) } f : G \to V \mid f(gh) = h^{-1} f(g)\}$$

with action $(g_1 f)(g_2) = f(g_1^{-1} g_2)$ for $g_i \in G$.

Theorem 7.46. *Let G be a Lie group, H a closed subgroup of G, and V a representation of H. There is a linear G-intertwining bijection between $\Gamma(G/H, G \times_H V)$ and $\operatorname{Ind}_H^G(V)$.*

Proof. Identify $(G \times_H V)_{eH}$ with V by mapping $(h, v) \in (G \times_H V)_{eH}$ to $h^{-1}v \in V$. Given $s \in \Gamma(G/H, G \times_H V)$, let $f_s \in \operatorname{Ind}_H^G(V)$ be defined by $f_s(g) = g^{-1}(s(gH))$. Conversely, given $f \in \operatorname{Ind}_H^G(V)$, let $s_f \in \Gamma(G/H, G \times_H V)$ be defined by $s_f(gH) = (g, f(g))$. It is easy to use the definitions to see these maps are well defined, inverses, and G-intertwining (Exercise 7.28). □

Theorem 7.47 (Frobenius Reciprocity). *Let G be a Lie group and H a closed subgroup of G. If V is a representation of H and a W is a representation of G, then as vector spaces*

$$\operatorname{Hom}_G(W, \operatorname{Ind}_H^G(V)) \cong \operatorname{Hom}_H(W|_H, V).$$

Proof. Map $T \in \operatorname{Hom}_G(W, \operatorname{Ind}_H^G(V))$ to $S_T \in \operatorname{Hom}_H(W|_H, V)$ by $S_T(w) = (T(w))(e)$ for $w \in W$ and map $S \in \operatorname{Hom}_H(W|_H, V)$ to $T_S \in \operatorname{Hom}_G(W, \operatorname{Ind}_H^G(V))$ by $(T_S(w))(g) = S(g^{-1}w)$. Verifying these maps are well defined and inverses is straightforward (Exercise 7.28). □

In the special case of $H = \{e\}$ and $V = \mathbb{C}$, the continuous version gives $\Gamma(G/H, G \times_H V) \cong \operatorname{Ind}_H^G(V) = C(G)$. In this setting, Frobenius Reciprocity already appeared in Lemma 3.23.

7.4.2 Complex Structure on *G/T*

Definition 7.48. Let G be a compact connected Lie group with maximal torus T.
(a) Choosing a faithful representation, assume $G \subseteq U(n)$ for some n. By Theorem 4.14 there exists a unique connected Lie subgroup of $GL(n, \mathbb{C})$ with Lie algebra $\mathfrak{g}_{\mathbb{C}}$. Write $G_{\mathbb{C}}$ for this subgroup and call it the *complexification* of G.
(b) Fix $\Delta^+(\mathfrak{g}_{\mathbb{C}})$ a system of positive roots and recall $\mathfrak{n}^+ = \bigoplus_{\alpha \in \Delta^+(\mathfrak{g}_{\mathbb{C}})} \mathfrak{g}_\alpha$. The corresponding *Borel subalgebra* is $\mathfrak{b} = \mathfrak{t}_{\mathbb{C}} \oplus \mathfrak{n}^+$.
(c) Let N, B, and A be the unique connected Lie subgroups of $GL(n, \mathbb{C})$ with Lie algebras $\mathfrak{n}^+$, $\mathfrak{b}$, and $\mathfrak{a} = i\mathfrak{t}$, respectively.

For example, if $G = U(n)$ with the usual positive root system, $G_{\mathbb{C}} = GL(n, \mathbb{C})$, N is the subgroup of upper triangular matrices with 1's on the diagonal, B is the subgroup of all upper triangular matrices, and A is the subgroup of diagonal matrices with entries in $\mathbb{R}^{>0}$. Although not obvious from Definition 7.48, $G_{\mathbb{C}}$ is in fact unique up to isomorphism when G is compact. More generally for certain types of noncompact groups, complexifications may not be unique or even exist (e.g., [61], VII §1). In any case, what is important for the following theory is that $G_{\mathbb{C}}$ is a complex manifold.

Lemma 7.49. *Let G be a compact connected Lie group with maximal torus T.*
(a) *The map* $\exp : \mathfrak{n}^+ \to N$ *is a bijection.*
(b) *The map* $\exp : \mathfrak{a} \to A$ *is a bijection.*
(c) *N, B, A, and AN are closed subgroups.*
(d) *The map from $T \times \mathfrak{a} \times \mathfrak{n}^+$ to B sending $(t, X, H) \to te^X e^H$ is a diffeomorphism.*

Proof. Since T consists of commuting unitary matrices, we may assume T is contained in the set of diagonal matrices of $GL(n, \mathbb{C})$. By using the Weyl group of $GL(n, \mathbb{C})$, we may further assume $u_\rho = \text{diag}(c_1, \ldots, c_n)$ with $c_i \geq c_{i+1}$. Therefore if $X \in \mathfrak{g}_\alpha$, $\alpha \in \Delta^+(\mathfrak{g}_\mathbb{C})$, with $X = \sum_{i,j} k_{i,j} E_{i,j}$, then

$$\sum_{i,j} (c_i - c_j) k_{i,j} E_{i,j} = [u_\rho, X] = \alpha(u_\rho) X = \sum_{i,j} B(\alpha, \rho) k_{i,j} E_{i,j}.$$

Since $B(\alpha, \rho) > 0$, it follows that $k_{i,j} = 0$ whenever $c_i - c_j \leq 0$. In turn, this shows that X is strictly upper triangular.

It is well known and easy to see that the set of nilpotent matrices are in bijection with the set of unipotent matrices by the polynomial map $M \to e^M$ with polynomial inverse $M \to \ln(I + (M - I)) = \sum_k \frac{(-1)^{k+1}}{k} (M - I)^k$. In particular if $X, Y \in \mathfrak{n}^+$, there is a unique strictly upper triangular $Z \in \mathfrak{gl}(n, \mathbb{C})$, so that $e^X e^Y = e^Z$.

Dynkin's formula is usually only applicable to small X and Y. However, $\Delta^+(\mathfrak{g}_\mathbb{C})$ is finite, so $[X_n^{(i_n)}, \ldots, X_1^{(i_1)}]$ is 0 for sufficiently large i_j for $X_j \in \mathfrak{n}^+$. Thus all the sums in the proof of Dynkin's formula are finite and the formula for Z is a polynomial in X and Y. Coupled with the already mentioned polynomial formula for Z, Dynkin's Formula therefore actually holds for all $X, Y \in \mathfrak{n}^+$. As a consequence, $Z \in \mathfrak{n}^+$ and $\exp \mathfrak{n}^+$ is a subgroup. Since N is generated by $\exp \mathfrak{n}^+$, part (a) is finished. The group N is closed since $\exp : \mathfrak{n}^+ \to N$ is a bijection and the exponential map restricted to the strictly upper triangular matrices has a continuous inverse.

Part (b) and the fact that A is closed in $G_\mathbb{C}$ follows from the fact that $\mathfrak{a}$ is Abelian and real valued. Next note that AN is a subgroup. This follows from the two observations that $(an)(a'n') = (aa')((c_{a'^{-1}} n) n')$, $a, a' \in A$ and $n, n' \in N$, and that $c_{e^H} e^X = \exp\left(e^{\text{ad}(H)} X\right)$, $H \in \mathfrak{a}$ and $X \in \mathfrak{n}^+$. Since the map from $\mathfrak{b} = \mathfrak{t} \oplus \mathfrak{a} \oplus \mathfrak{n}^+ \to G_\mathbb{C}$ given by $(H_1, H_2, X) \to e^{H_1} e^{H_2} e^X$ is a local diffeomorphism near 0, products of the form tan, $t \in T$, $a \in A$, and $n \in N$, generate B. Just as with AN, TAN is a subgroup, so that $B = T e^{\mathfrak{a}} e^{\mathfrak{n}^+}$. It is an elementary fact from linear algebra that this decomposition is unique and the proof is complete. □

The point of the next theorem is that G/T has a G-invariant complex structure inherited from the fact that $G_\mathbb{C}/B$ is a complex manifold. This will allow us to study holomorphic sections on G/T.

Theorem 7.50. *Let G be a compact connected Lie group with maximal torus T. The inclusion $G \hookrightarrow G_\mathbb{C}$ induces a diffeomorphism*

$$G/T \cong G_\mathbb{C}/B.$$

Proof. Recall that $\mathfrak{g} = \{X + \theta X \mid X \in \mathfrak{g}_\mathbb{C}\}$, so that $\mathfrak{g}/\mathfrak{t}$ and $\mathfrak{g}_\mathbb{C}/\mathfrak{b}$ are both spanned by the projections of $\{X_\alpha + \theta X_\alpha \mid X_\alpha \in \mathfrak{g}_\alpha, \alpha \in \Delta^+(\mathfrak{g}_\mathbb{C})\}$. In particular, the differential of the map $G \to G_\mathbb{C}/B$ is surjective. Thus the image of G contains a neighborhood of eB in $G_\mathbb{C}/B$. As left multiplication by g and g^{-1}, $g \in G$, is continuous, the image of G is open in $G_\mathbb{C}/B$. Compactness of G shows that the image is closed so that connectedness shows the map $G \to G_\mathbb{C}/B$ is surjective.

It remains to see that $G \cap B = T$. Let $g \in G \cap B$. Clearly $\mathrm{Ad}(g)$ preserves $\mathfrak{g} \cap \mathfrak{b} = \mathfrak{t}$, so that $g \in N(T)$. Writing w for the corresponding element of $W(\Delta(\mathfrak{g}_{\mathbb{C}}))$, the fact that $g \in B$ implies that w preserves $\Delta^+(\mathfrak{g}_{\mathbb{C}})$. In turn, this means w preserves the corresponding Weyl chamber. Since Theorem 6.43 shows that $W(\Delta(\mathfrak{g}_{\mathbb{C}}))$ acts simply transitively on the Weyl chambers, $w = I$ and $g \in T$. □

7.4.3 Holomorphic Functions

Definition 7.51. Let G be a compact Lie group with maximal torus T. For $\lambda \in A(T)$, write $\mathbb{C}_\lambda$ for the one-dimensional representation of T given by ξ_λ and write L_λ for the *line bundle*

$$L_\lambda = G \times_T \mathbb{C}_\lambda.$$

By Frobenius Reciprocity, $\Gamma(G/T, L_\lambda)$ is a huge representation of G. However by restricting our attention to holomorphic sections, we will obtain a representation of manageable size.

Definition 7.52. Let G be a compact connected Lie group with maximal torus T and $\lambda \in A(T)$.
(a) Extend $\xi_\lambda : T \to \mathbb{C}$ to a homomorphism $\xi_\lambda^{\mathbb{C}} : B \to \mathbb{C}$ by

$$\xi_\lambda^{\mathbb{C}}(te^{iH}e^X) = \xi_\lambda(t)e^{i\lambda(H)}$$

for $t \in T$, $H \in \mathfrak{t}$, and $X \in \mathfrak{n}^+$.
(b) Let $L_\lambda^{\mathbb{C}} = G_{\mathbb{C}} \times_B \mathbb{C}_\lambda$ where $\mathbb{C}_\lambda$ is the one-dimensional representation of B given by $\xi_\lambda^{\mathbb{C}}$.

Lemma 7.53. *Let G be a compact connected Lie group with maximal torus T and $\lambda \in A(T)$. Then $\Gamma(G/T, L_\lambda) \cong \Gamma(G_{\mathbb{C}}/B, L_\lambda^{\mathbb{C}})$ and $\mathrm{Ind}_T^G(\xi_\lambda) \cong \mathrm{Ind}_B^{G_{\mathbb{C}}}(\xi_\lambda^{\mathbb{C}})$ as G-representations.*

Proof. Since the map $G \to G_{\mathbb{C}}/B$ induces an isomorphism $G/T \cong G_{\mathbb{C}}/B$, any $h \in G_{\mathbb{C}}$ can be written as $h = gb$ for $g \in G$ and $b \in B$. Moreover, if $h = g'b'$, $g' \in G$ and $b' \in B$, then there is $t \in T$ so $g' = gt$ and $b' = t^{-1}b$.

On the level of induced representations, map $f \in \mathrm{Ind}_T^G(\xi_\lambda)$ to $F_f \in \mathrm{Ind}_B^{G_{\mathbb{C}}}(\xi_\lambda^{\mathbb{C}})$ by $F_f(gb) = f(g)\xi_{-\lambda}^{\mathbb{C}}(b)$ for $g \in G$ and $b \in B$ and map $F \in \mathrm{Ind}_B^{G_{\mathbb{C}}}(\xi_\lambda^{\mathbb{C}})$ to $f_F \in \mathrm{Ind}_T^G(\xi_\lambda)$ by $f_F(g) = F(g)$. It is straightforward to verify that these maps are well defined, G-intertwining, and inverse to each other (Exercise 7.31). □

Definition 7.54. Let G be a compact connected Lie group with maximal torus T and $\lambda \in A(T)$.
(a) A section $s \in \Gamma(G/T, L_\lambda)$ is said to be *holomorphic* if the corresponding function $F \in \mathrm{Ind}_B^{G_{\mathbb{C}}}(\xi_\lambda^{\mathbb{C}})$, c.f. Theorem 7.46 and Lemma 7.53, is a holomorphic function on $G_{\mathbb{C}}$, i.e., if

$$dF(iX) = i\,dF(X)$$

at each $g \in G_{\mathbb{C}}$ and for all $X \in T_g(G_{\mathbb{C}})$ where $dF(X) = X(\mathrm{Re}\,F) + iX(\mathrm{Im}\,F)$.
(b) Write $\Gamma_{\mathrm{hol}}(G/T, L_\lambda)$ for the set of all holomorphic sections.

Since the differential dF is always $\mathbb{R}$-linear, the condition of being holomorphic is equivalent to saying that dF is $\mathbb{C}$-linear. Written in local coordinates, this condition gives rise to the standard Cauchy–Riemann equations (Exercise 7.32).

Definition 7.55. Let G be a connected (linear) Lie group with maximal torus T. Write $C^\infty(G_\mathbb{C})$ for the set of smooth functions on $G_\mathbb{C}$ and use similar notation for G.

(a) For $Z \in \mathfrak{g}_\mathbb{C}$ and $F \in C^\infty(G_\mathbb{C})$, let

$$[dr(Z)F]\,(h) = \frac{d}{dt} F(he^{tZ})|_{t=0}$$

for $h \in G_\mathbb{C}$. For $X \in \mathfrak{g}$ and $f \in C^\infty(G)$, let

$$[dr(X)f]\,(g) = \frac{d}{dt} f(ge^{tX})|_{t=0}$$

for $g \in G$.

(b) For $Z = X + iY$ with $X, Y \in \mathfrak{g}$, let

$$dr_\mathbb{C}(Z) = dr(X) + i dr(Y).$$

Note that $dr_\mathbb{C}(Z)$ is a well-defined operator on $C^\infty(G)$ but that $dr(Z)$ is not (except when $Z \in \mathfrak{g}$).

Lemma 7.56. *Let G be a compact connected Lie group with maximal torus T, $\lambda \in A(T)$, $F \in \operatorname{Ind}_B^{G_\mathbb{C}}(\xi_\lambda^\mathbb{C})$, and $f = F|_G$ the corresponding function in* $\operatorname{Ind}_T^G(\xi_\lambda)$.

(a) *Then F is holomorphic if and only if*

$$dr_\mathbb{C}(Z)F = 0$$

for $Z \in \mathfrak{n}^+$.

(b) *Equivalently, F is holomorphic if and only if*

$$dr_\mathbb{C}(Z)f = 0$$

for $Z \in \mathfrak{n}^+$.

Proof. Since $dl_g : T_e(G_\mathbb{C}) \to T_g(G_\mathbb{C})$ is an isomorphism, F is holomorphic if and only if

$$dF(dl_g\,(iZ)) = i dF(dl_g Z) \tag{7.57}$$

for all $g \in G_\mathbb{C}$ and $X \in \mathfrak{g}_\mathbb{C}$ where, by definition,

$$dF(dl_g Z) = \frac{d}{dt} F(ge^{tZ})|_{t=0} = [dr(Z)F]\,(g).$$

If $Z \in \mathfrak{n}^+$, then $e^{tZ} \in N$, so that $F(ge^{tZ}) = F(g)$. Thus for $Z \in \mathfrak{n}^+$, Equation 7.57 is automatic since both sides are 0. If $Z \in \mathfrak{t}_\mathbb{C}$, $F(ge^{tZ}) = F(g)e^{-t\lambda(Z)}$. Thus for $Z \in \mathfrak{t}_\mathbb{C}$, Equation 7.57 also holds since both sides are $-i\lambda(Z)F(g)$.

Since $\mathfrak{g}_{\mathbb{C}} = \mathfrak{n}^- \oplus \mathfrak{t}_{\mathbb{C}} \oplus \mathfrak{n}^+$, part (a) will be finished by showing Equation 7.57 holds for $Z \in \mathfrak{n}^-$. However, Equation 7.57 is equivalent to requiring $dr(iZ)F = idr(Z)F$ which in turn is equivalent to requiring $dr(Z)F = dr_{\mathbb{C}}(Z)F$. If $Z \in \mathfrak{n}^-$, then $\theta Z \in \mathfrak{n}^+$ and $Z + \theta Z \in \mathfrak{g}$. Thus

$$dr(Z)F = dr(Z+\theta Z)F - dr(\theta Z)F = dr_{\mathbb{C}}(Z+\theta Z)F = dr_{\mathbb{C}}(Z) + dr_{\mathbb{C}}(\theta Z),$$

so that $dr(Z)F = dr_{\mathbb{C}}(Z)F$ if and only if $dr_{\mathbb{C}}(\theta Z) = 0$, as desired.

For part (b), first, assume F is holomorphic. Since $f = F|_G$, it follows that $dr_{\mathbb{C}}(\mathfrak{n}^+)f = 0$. Conversely, suppose $dr_{\mathbb{C}}(\mathfrak{n}^+)f = 0$. Restricting the above arguments from $G_{\mathbb{C}}$ to G shows $dr(Z)F|_g = dr_{\mathbb{C}}(Z)F|_g$ for $g \in G$ and $Z \in \mathfrak{g}_{\mathbb{C}}$. Hence if $X \in \mathfrak{g}$,

$$\begin{aligned}(dr(X)F)(gb) &= \frac{d}{dt}F(gbe^{tX})|_{t=0} = \frac{d}{dt}F(ge^{t\,\mathrm{Ad}(b)X}b)|_{t=0}\\ &= \xi_{-\lambda}(b)\,\frac{d}{dt}F(ge^{t\,\mathrm{Ad}(b)X})|_{t=0}\\ &= \xi_{-\lambda}(b)\,(dr(\mathrm{Ad}(b)X)F)(g) = \xi_{-\lambda}(b)\,(dr_{\mathbb{C}}(\mathrm{Ad}(b)X)F)(g)\end{aligned}$$

for $g \in G$ and $b \in B$. Thus if $Z = X + iY \in \mathfrak{n}^+$ with $X, Y \in \mathfrak{g}$, note $\mathrm{Ad}(b)Z \in \mathfrak{n}^+$ and calculate

$$\begin{aligned}(dr_{\mathbb{C}}(Z)F)(gb) &= (dr(X)F)(gb) + i(dr(Y)F)(gb)\\ &= \xi_{-\lambda}(b)\,[(dr_{\mathbb{C}}(\mathrm{Ad}(b)X)F)(g) + (dr_{\mathbb{C}}(i\,\mathrm{Ad}(b)Y)F)(g)]\\ &= \xi_{-\lambda}(b)\,(dr_{\mathbb{C}}(\mathrm{Ad}(b)Z)F)(g) = 0,\end{aligned}$$

as desired. □

7.4.4 Main Theorem

The next theorem gives an explicit realization for each irreducible representation.

Theorem 7.58 (Borel–Weil). *Let G be a compact connected Lie group and $\lambda \in A(T)$.*

$$\Gamma_{\mathrm{hol}}(G/T, L_\lambda) \cong \begin{cases} V(w_0\lambda) & \textit{for } -\lambda \textit{ dominant} \\ \{0\} & \textit{else,} \end{cases}$$

where $w_0 \in W(\Delta(\mathfrak{g}_{\mathbb{C}}))$ is the unique Weyl group element mapping the positive Weyl chamber to the negative Weyl chamber (c.f. Exercise 6.40).

Proof. The elements of $\Gamma_{\mathrm{hol}}(G/T, L_\lambda)$ correspond to holomorphic functions in $\mathrm{Ind}_T^G(\xi_\lambda)$. It follows that the elements of $\Gamma_{\mathrm{hol}}(G/T, L_\lambda)$ correspond to the set of smooth functions f on G, satisfying

$$f(gt) = \xi_{-\lambda}(t)f(g) \tag{7.59}$$

for $g \in G$ and $t \in T$ and

$$dr_{\mathbb{C}}(Z)f = 0 \tag{7.60}$$

for $Z \in \mathfrak{n}^+$.

Using the C^∞-topology on $C^\infty(G)$, Corollary 3.47 shows that $C^\infty(G)_{G\text{-fin}} = C(G)_{G\text{-fin}}$ so that, by Theorem 3.24 and the Highest Weight Theorem,

$$C^\infty(G)_{G\text{-fin}} \cong \bigoplus_{\text{dom. } \gamma \in A(T)} V(\gamma)^* \otimes V(\gamma)$$

as a $G \times G$-module with respect to the $r \times l$-action. In this decomposition, tracing through the identifications (Exercise 7.33) **??** shows that the action of G on $\Gamma_{\text{hol}}(G/T, L_\lambda)$ intertwines with the trivial action on $V(\gamma)^*$ and the standard action on $V(\gamma)$. Recalling that Lemma 7.5, write φ for the intertwining operator

$$\varphi : \bigoplus_{\text{dom. } \gamma \in A(T)} V(-w_0\gamma) \otimes V(\gamma) \xrightarrow{\sim} C^\infty(G)_{G\text{-fin}}.$$

Given $f \in C^\infty(G)$, use Theorem 3.46 to write $f = \sum_{\text{dom. } \gamma \in A(T)} f_\gamma$ with respect to convergence in the C^∞-topology, where $f_\gamma = \varphi(x_\gamma)$ with $x_\gamma \in V(-w_0\gamma) \otimes V(\gamma)$.

Equation 7.60 is then satisfied by f if and only if it is satisfied by each f_γ. Tracing through the identifications, the action of $dr_{\mathbb{C}}(Z)$ corresponds to the standard (complexified) action of Z on $V(-w_0\gamma)$ and the trivial action on $V(\gamma)$. In particular, Theorem 7.3 shows that x_γ can be written as $x_\gamma = v_{-w_0\gamma} \otimes y_\gamma$ where $v_{-w_0\gamma}$ is a highest weight vector of $V(-w_0\gamma)$ and $y_\gamma \in V(\gamma)$.

Tracing through the identifications again, Equation 7.59 is then satisfied if and only if $tv_{-w_0\gamma} = \xi_{-\lambda}(t)v_{-w_0\gamma}$. But since $tv_{-w_0\gamma} = \xi_{-w_0\gamma}(t)v_{-w_0\gamma}$, it follows that Equation 7.59 is satisfied if and only if $w_0\gamma = \lambda$ and the proof is complete. □

As an example, consider $G = SU(2)$ with T the usual subgroup of diagonal matrices. Realizing $\Gamma_{\text{hol}}(G/T, \xi_{-n\frac{\epsilon_{12}}{2}})$ as the holomorphic functions in $\text{Ind}_B^{G_{\mathbb{C}}}(\xi^{\mathbb{C}}_{-n\frac{\epsilon_{12}}{2}})$,

$$\Gamma_{\text{hol}}(G/T, \xi_{-n\frac{\epsilon_{12}}{2}}) \cong$$
$$\{\text{holomorphic } f : SL(2,\mathbb{C}) \to \mathbb{C} \mid f(g \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}) = a^n f(g),\ g \in SL(2,\mathbb{C})\}.$$

Since $\begin{pmatrix} z_1 & z_3 \\ z_2 & z_4 \end{pmatrix}\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} z_1 & bz_1 + z_3 \\ z_2 & bz_2 + z_4 \end{pmatrix}$, the induced condition in the case of $a = 1$ shows $f \in \text{Ind}_B^{G_{\mathbb{C}}}(\xi^{\mathbb{C}}_{-n\frac{\epsilon_{12}}{2}})$ is determined by its restriction to the first column of $SL(2,\mathbb{C})$. Since $\begin{pmatrix} z_1 & z_3 \\ z_2 & z_4 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} = \begin{pmatrix} az_1 & a^{-1}z_3 \\ az_2 & a^{-1}z_4 \end{pmatrix}$, the induced condition for the case of $b = 0$ shows that f is homogeneous of degree n as a function on the first column of $SL(2,\mathbb{C})$. Finally, the holomorphic condition shows $\Gamma_{\text{hol}}(G/T, \xi_{-n\frac{\epsilon_{12}}{2}})$ can be identified with the set of homogeneous polynomials of degree n on the first column of $SL(2,\mathbb{C})$. In other words, $\Gamma_{\text{hol}}(G/T, \xi_{-n\frac{\epsilon_{12}}{2}}) \cong V_n(\mathbb{C}^2)$ as expected.

As a final remark, there is a (dualized) generalization of the Borel–Weil Theorem to the Dolbeault cohomology setting called the Bott–Borel–Weil Theorem. Although we only state the result here, it is fairly straightforward to reduce the calculation to a fact from Lie algebra cohomology ([97]). In turn this is computed by a theorem of Kostant ([64]), an efficient proof of which can be found in [86].

Given a complex manifold M, write $A^p(M) = \bigwedge_p^* T^{0,1}(M)$ for the smooth differential forms of type $(0, p)$ ([93]). The $\overline{\partial}_M$ operator maps $A^p(M)$ to $A^{p+1}(M)$ and is given by

$$\begin{aligned}\left(\overline{\partial}_M \omega\right)(X_0, \ldots, X_p) = &\sum_{k=0}^{p} (-1)^k X_k \omega(X_0, \ldots, \widehat{X_k}, \ldots, X_p) \\ &+ \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_0, \ldots, \widehat{X_i}, \ldots, \widehat{X_j}, \ldots, X_p)\end{aligned}$$

for antiholomorphic vector fields X_j. If $\mathcal{V}$ is a holomorphic vector bundle over M, the sections of $\mathcal{V} \bigotimes A^p(M)$ are the $\mathcal{V}$-valued differential forms of type $(0, p)$ and the set of such is denoted $A^p(M, \mathcal{V})$. The operator $\overline{\partial} : A^p(M, \mathcal{V}) \to A^{p+1}(M, \mathcal{V})$ is given by $\overline{\partial} = 1 \otimes \overline{\partial}_M$ and satisfies $\overline{\partial}^2 = 0$. The *Dolbeault cohomology* spaces are defined as

$$H^p(M, \mathcal{V}) = \ker \overline{\partial} / \operatorname{Im} \overline{\partial}.$$

Theorem 7.61 (Bott–Borel–Weil Theorem). *Let G be a compact connected Lie group and $\lambda \in A(T)$. If $\lambda + \rho$ lies on a Weyl chamber wall, then $H^p(G/T, L_\lambda) = \{0\}$ for all p. Otherwise,*

$$H^p(G/T, L_\lambda) \cong \begin{cases} V(w(\lambda+\rho)-\rho) & \text{for } p = \left|\{\alpha \in \Delta^+(\mathfrak{g}_\mathbb{C}) \mid B(\lambda+\rho, \alpha) < 0\}\right| \\ \{0\} & \text{else,} \end{cases}$$

where $w \in W(\Delta(\mathfrak{g}_\mathbb{C}))$ is the unique Weyl group element making $w(\lambda + \rho)$ dominant.

7.4.5 Exercises

Exercise 7.27 Let G be a Lie group and H a closed subgroup of G. Given a representation V of H, verify $G \times_H V$ is a homogeneous vector bundle over G/H.

Exercise 7.28 Verify the details of Theorems 7.46 and 7.47.

Exercise 7.29 Let G be a compact connected Lie group with maximal torus T and $\lambda \in A(T)$.
(1) Show that $\xi_\lambda^\mathbb{C}$ is a homomorphism.
(2) Show that $\xi_\lambda^\mathbb{C}$ is the unique extension of ξ_λ from T to B as a homomorphism of complex Lie groups.

Exercise 7.30 Let G be a compact connected Lie group with maximal torus T and $\lambda \in A(T)$. If V is an irreducible representation, show that $V \cong V(\lambda)$ if and only if there is a nonzero $v \in V$ satisfying $bv = \xi_\lambda^\mathbb{C}(b)v$ for $b \in B$. In this case, show that v is unique up to nonzero scalar multiplication and is a highest weight vector.

Exercise 7.31 Verify the details of Lemma 7.53.

Exercise 7.32 Let $G_{\mathbb{C}}$ be a *complex* (linear) connected Lie group with maximal torus T. Recall that a complex-valued function F on $G_{\mathbb{C}}$ is *holomorphic* if $dF(dl_g\,(iX)) = idF(dl_gX)$ for all $g \in G_{\mathbb{C}}$ and $X \in \mathfrak{g}_{\mathbb{C}}$, where $dF(dl_gX) = \frac{d}{dt}F(ge^{tX})|_{t=0}$. Note that dF is $\mathbb{R}$-linear.
(1) In the special case of $G_{\mathbb{C}} = \mathbb{C}\backslash\{0\} \cong GL(1,\mathbb{C})$, $z \in G_{\mathbb{C}}$, and $X = 1$, show that $dF(dl_z(iX)) = \frac{\partial}{\partial y}F|_z$ and $idF(dl_zX) = i\frac{\partial}{\partial x}F|_z$, where $z = x + iy$. Conclude that dF is not $\mathbb{C}$-linear for general F and that, in this case, F is holomorphic if and only if $u_x = v_y$ and $u_y = -v_x$, where $F = u + iv$.
(2) Let $\{X_j\}_{j=1}^n$ be a basis over $\mathbb{C}$ for $\mathfrak{g}_{\mathbb{C}}$. For $g \in G_{\mathbb{C}}$, show that the map $\varphi : \mathbb{R}^{2n} \to G_{\mathbb{C}}$ given by

$$\varphi(x_1, \ldots, x_n, y_1, \ldots, y_n) = ge^{x_1X_1}\cdots e^{x_nX_n}e^{iy_1X_1}\cdots e^{iy_nX_n}$$

is a local diffeomorphism near 0, c.f. Exercise 4.12.
(3) Identifying $\mathfrak{g}_{\mathbb{C}}$ with $T_e(G_{\mathbb{C}})$, show $d\varphi(\partial_{x_j}|_0) = dl_gX_j$ and $d\varphi(\partial_{y_j}|_0) = dl_g(iX_j)$.
(4) Given a smooth function F on $G_{\mathbb{C}}$, write F in local coordinates near g as $f = \varphi^*F$. Show that F is holomorphic if and only if for each $g \in G_{\mathbb{C}}$, $u_{x_j} = v_{y_j}$ and $u_{y_j} = -v_{x_j}$ where $f = u + iv$. In other words, F is holomorphic if and only if it satisfies the Cauchy–Riemann equations in local coordinates.

Exercise 7.33 In the proof of the Borel–Weil theorem, trace through the various identifications to verify that the claimed actions are correct.

Exercise 7.34 Let B be the subgroup of upper triangular matrices in $GL(n,\mathbb{C})$. Let $\lambda = \lambda_1\epsilon_1 + \cdots + \lambda_n\epsilon_n$ be a dominant integral weight of $U(n)$, i.e., $\lambda_k \in \mathbb{Z}$ and $\lambda_1 \geq \ldots \lambda_n$.
(1) Let $f : GL(n,\mathbb{C}) \to \mathbb{C}$ be smooth. For $i < j$, show that $dr(iE_{j,k})f|_g = idr(E_{j,k})f|_g$ if and only if

$$0 = \sum_{l=1}^n \overline{z_{l,j}}\frac{\partial f}{\partial \overline{z_{l,k}}}|_g,$$

where $g = (z_{j,k}) \in GL(n,\mathbb{C})$ and $\frac{\partial}{\partial \overline{z_{j,k}}} = \frac{1}{2}\left(\frac{\partial}{\partial x_{j,k}} + i\frac{\partial}{\partial y_{j,k}}\right)$ with $z_{j,k} = x_{j,k} + iy_{j,k}$. Conclude that $dr(iE_{j,k})f = idr(E_{j,k})f$ if and only if $\frac{\partial f}{\partial \overline{z_{l,k}}} = 0$.
(2) Show that the irreducible representation of $U(n)$ with highest weight λ is realized by

$$V_\lambda = \Big\{\text{holomorphic } F : GL(n,\mathbb{C}) \to \mathbb{C} \mid F(gb) = \xi^{\mathbb{C}}_{-\lambda_n\epsilon_1\cdots-\lambda_1\epsilon_n}(b)F(g),\; g \in GL(n,\mathbb{C}), b \in B\Big\}$$

with action given by left translation of functions, i.e., $(g_1F)(g_2) = F(g_1^{-1}g_2)$.
(3) Let $F_{w_0\lambda} : GL(n,\mathbb{C}) \to \mathbb{C}$ be given by

$$F_{w_0\lambda}(g) = (\det_1 g)^{\lambda_{n-1}-\lambda_n}\cdots(\det_{n-1} g)^{\lambda_1-\lambda_2}(\det_n g)^{-\lambda_1},$$

where $\det_k(g_{i,j}) = \det_{i,j \le k}(g_{i,j})$. Show that $F_{w_0\lambda}$ is holomorphic, invariant under right translation by N, and invariant under left translation by N^t.
(4) Show that $F_{w_0\lambda} \in V_\lambda$ and show $F_{w_0\lambda}$ has weight $\lambda_n\epsilon_1 + \cdots + \lambda_1\epsilon_n$. Conclude that $F_{w_0\lambda}$ is the lowest weight vector of V_λ, i.e., that $F_{w_0\lambda}$ is the highest weight vector for the positive system corresponding to the opposite Weyl chamber.
(5) Let $F_\lambda(g) = F_{w_0\lambda}(w_0 g)$, where $w_0 = E_{1,n} + E_{2,n-2} + \ldots, E_{n,1}$. Write F_λ in terms of determinants of submatrices and show F_λ is a highest weight for V_λ.

Exercise 7.35 Let G be a compact Lie group. Show G is *algebraic* by proving the following:
(1) Suppose G acts on a vector space V and $\mathcal{O}$ and $\mathcal{O}'$ are two distinct orbits. Show there is a continuous function f on V that is 1 on $\mathcal{O}$ and -1 on $\mathcal{O}'$.
(2) Show there is a polynomial p on V, so that $|p(x) - f(x)| < 1$ for $x \in \mathcal{O} \cup \mathcal{O}'$. Conclude that $p(x) > 0$ when $x \in \mathcal{O}$ and $p(x) < 0$ when $x \in \mathcal{O}'$.
(3) Let $\mathcal{P}$ be the convex set of all polynomials p on V satisfying $p(x) > 0$ when $x \in \mathcal{O}$ and $p(x) < 0$ when $x \in \mathcal{O}'$. With respect to the usual action, $(g \cdot p)(x) = p(g^{-1}x)$ for $g \in G$, use integration to show that there exists $p \in \mathcal{P}$ that is G-invariant.
(4) Show that G-invariant polynomials on V are constant on G-orbits.
(5) Let $\mathcal{I}$ be the ideal of all G-invariant polynomials on V that vanish on $\mathcal{O}$. Show that there is $p \in \mathcal{I}$, so that p is nonzero on $\mathcal{O}'$. Conclude that the set of zeros of $\mathcal{I}$ is exactly $\mathcal{O}$.
(6) By choosing a faithful representation, assume $G \subseteq GL(n, \mathbb{C})$ and consider the special case of $V = M_{n,n}(\mathbb{C})$ with G-action given by left multiplication of matrices. Show that G is itself an orbit in V and is therefore algebraic.