7
Old Babylonian Hand Tablets with Practical Mathematics

7.1. MS 2317. Division of a Funny Number by a Non-Regular Factor

7.1 a. Interpretation of the Three Numbers in the Text

MS 2317 (Fig. 7.1.1) is a quite small square hand tablet, inscribed on the obverse with three lines of numbers. At first sight, the text looks unpromising. The meaning of the four wedges in line 1 is not immediately clear, and the numbers 13 and 4 41 37 in lines 2 and 3 are, quite obviously, *non-regular* sexagesimal numbers.

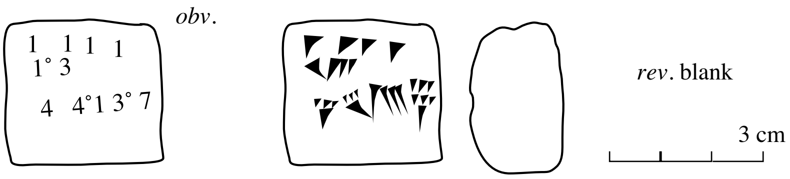


Fig. 7.1.1. MS 2317. A division exercise for a funny number.

A renewed look at the text reveals that it is unexpectedly interesting. The key to understanding what is going on here is to *work in sexagesimal arithmetic*. Indeed, a moment's reflection leads to the insight that the 3-place sexagesimal number in line 3 can be factorized as

$$4\ 41\ 37 = 4\ 37 \cdot 1\ 01.$$

For verification, note that $4\ 37 \cdot 1\ 01 = 4\ 37\ 00 + 4\ 37 = 4\ 41\ 37$. Hence, 4 41 37 is the product of two non-regular prime numbers, 4 37 (= 277) and 1 01 (= 61).

A connection between the number 13 in line 2 and the product $4\ 41\ 37 = 4\ 37 \cdot 1\ 01$ in line 3 is that *4 37 is an approximate reciprocal to 13*, since

$$13 \cdot 4\ 37 = 52\ 00 + 6\ 30 + 1\ 31 = 1\ 00\ 01.$$

(In absolute values, $13 \cdot ;04\ 37 = 1;00\ 01$.) Combining the observations that $4\ 41\ 37 = 4\ 37 \cdot 1\ 01$ and that $13 \cdot 4\ 37 = 1\ 00\ 01$, one finds that

$$13 \cdot 4\ 41\ 37 = 13 \cdot 4\ 37 \cdot 1\ 01 = 1\ 00\ 01 \cdot 1\ 01 = 1\ 00\ 01\ 00 + 1\ 00\ 01 = 1\ 01\ 01\ 01.$$

Consequently, the four wedges in line 1 of MS 2317 must be understood as the sexagesimal number 1 01 01 01, written as 1 1 1 1 in Babylonian place value notation without zeros. The whole text can then be interpreted as *a rather curious division exercise*:

What is 1 01 01 01 divided by 13? Answer: 4 41 37.

The division exercise is curious, because 1 01 01 01, written with just four ones, is what may be called a "funny number", and because it seems rather strange that anyone would have bothered to find out that this large and non-regular sexagesimal number is not a prime number but a product of two (or more) smaller numbers.

7.1 b. A Proposed Solution Algorithm for the Division Problem

In Old Babylonian mathematics, division problems could be solved in two ways. If the set task was to divide a given number *a* by a *regular* sexagesimal number *b*, then the reciprocal igi *b* of *b* was first computed, and then *a* was multiplied by this reciprocal. Thus, the rule was that

a divided by *b* = igi *b* · *a* if *b* is a regular sexagesimal number.

(Cf. the discussion of the division exercise MS 3871 in Sec. 1.3 above. In that example, all the numbers involved are regular sexagesimal numbers.)

On the other hand, if *b* was a *non-regular* sexagesimal number, then a question like one of the following ones was asked:

mi-nam a.na *b lu-uš-ku-un ša a i-na-di-nam* (as in YBC 4608, *MCT* D)
mi-nam a.na *b* ḫé.gar *ša a in.sì* (as in Str. 363, *MKT 1*)

In both cases, the translation would be something like

What shall I put as much as *b* that will give me *a*?

In ordinary language: What times *b* is equal to *a*? Thus, also in the case of a non-regular divisor, the division problem was transformed into an equivalent multiplication problem. No details are known about how an Old Babylonian mathematician would attack a problem of this kind. However, there are reasons to believe that a systematic approach like the "recursive division algorithm" described below may have been used, at least in more complicated cases. (See the discussion in Secs. A6 e-g in App. 6 of division problems in mathematical cuneiform texts from the third millennium BC.)

Take the concrete example when *b* = 13 and *a* = 1 01 01 01. The idea would be to start by first finding approximate solutions to the successive equations

13 · ? = 1 00 (1 géš), 13 · ? = 10 00 (10 géš), 13 · ? = 1 00 00 (1 šár),
13 · ? = 10 00 00 (10 šár), 13 · ? = 1 00 00 00 (1 šár.gal),

and then combine the results in the proper way. This could be done in the following way:

13 ·	**5**	=	1 05	=**1** 00	(+ 5)
13 ·	46	=	9 58	=10 00	(– 2)
13 ·	**4 37**	=	1 00 01	=**1** 00 00	(+ 1)
13 ·	46 09	=	9 59 57	=10 00 00	(– 3)
13 ·	**4 36 55**	=	59 59 55	=**1** 00 00 00	(– 5).

The addition of the approximate equations in lines 5, 3, and 1, gives the intermediate result that

13 · (4 36 56 + 4 37 + 5) = 1 00 00 00 (– 5) + 1 00 00 (+ 1) + 1 00 (+ 5).

Hence, the final result of the division algorithm is that

13 · 4 41 38 = 1 00 00 + 1 00 00 + 1 00 (+ 1) = 1 01 01 01, exactly!

Each step of the algorithm in the example above can be based on the result of the preceding step. When it has been shown that, for instance, 13 · 5 = 1 00 + 5, then a multiplication of both sides of this equation by 10 will show that 13 · 50 = 10 00 + 50. More precisely, 13 · 46 = 10 00 + 50 – 52 = 10 00 – 2. In the next step, multiplying both sides of the equation 13 · 46 = 10 00 – 2 by 6 will show that 13 · 4 36 = 1 00 00 – 12, or, more precisely 13 · 4 37 = 1 00 00 – 12 + 13 = 1 00 00 + 1. And so on.

From a modern, anachronistic point of view, the method is equivalent to finding successively improved approximations to 1/13 and 10/13 in terms of *sexagesimal fractions*. Indeed, it follows from the indicated computations, lines 1, 3, and 5, that successively better approximations to 1/13 are ;05, ;04 37, and ;04 36 56. Similarly, the intermediate lines 2 and 4 in the same series of computations show that successively better approximations to 10/13 are ;47 and ;46 10. It is tempting to conjecture that similar considerations prompted some anonymous Old Babylonian mathematician to compose the text of the cuneiform tablet M 10 (Free Library of Philadelphia), a curious table of approximate values, in sexagesimal fractions, for 1/7, 1/11, and 1/13, followed by approximate values for 10/14 and 10/17. See Sachs, *JCS* 6 (1952).

7.1 c. UET 5, 121 § 2. A Parallel Text from Early Old Babylonian Ur

In the extensive corpus of known Old Babylonian mathematical texts, there are only three examples of division exercises. Two of those are MS 2317 immediately above and MS 3817 in Sec. 1.3 above.

The third example is ***UET* 5, 121**, a small clay tablet from Ur with a series of inheritance problems (parallel to MS 1844 in Fig. 7.4.2 below) on the obverse, and three division problems on the reverse, all pretending to be examples of practical mathematics. (See Friberg, *RA* 94 (2000), 138-139.)

One of the three division problems is *UET 5*, 121 § 2 b, a *dressed up version* of the division problem in MS 2317. In *UET 5*, 121, just as in MS 2317, the answers to the division problems are given, but not the details of the solution procedures. In the dressed up version of § 2b, the prescribed data are that 1 01 01 01 (= 1 01 · 1 00 01) sheep are allotted to 13 shepherds, and the explicitly given answer is that each shepherd is allotted precisely 4 41 37 (= 1 01 · 4 37) sheep. (Note that 1 00 01 = 3,601 = 4 37 · 13.)

In § 2a of the same text, 1 01 01 01 (= 1 01 · 1 00 01) goats are divided among 13 13 (= 1 01 · 13) shepherd boys, and in § 2c, 1 01 01 sheep are divided among 7 shepherds.

***UET* 5, 121 §§ 2 a-c.**

Sheep, shepherds, and a funny number. A dressed up division exercise.

1	1(šár'u).gal 1(šár) 1(géš) 1 ud$_5$.ḫá / 1(géš'u) 3 13 kab.ra	$1^{!}$ 01 01 01 goats, 13 13 shepherd boys.
2	kab.ra.1.e / en.nam íb.ši.ti	1 shepherd boy, what does he approach to (take)?
3	4$_v$ 37 íb.ši.ti	4 37 he approaches to.
1	1(šár'u).gal 1(šár) 1(géš) 1 udu.ḫá 13 sipa /	$1^{!}$ 01 01 01 sheep, 13 shepherds.
2	sipa.1.e en.nam íb.ši.ti /	1 shepherd, what does he approach to?
3	4 41 37 íb.ši.ti	4 41 37 he approaches to.
1	1(šár) 1(géš) 1 udu.ḫá 7$_v$ sipa /	1 01 01 sheep, 7 shepherds.
2	sipa.1.e en.nam íb.<ši.ti> /	1 shepherd, what does he <approach to>?
3	8 43 íb.ši.ti	8 43he approaches to.

UET 5, 121 is interesting in several ways. It is one of a group of four small mathematical clay tablets found in what remains of a rich man's house at "1 Broad Street" in Ur. This group of clay tablets can be dated rather exactly to a relatively early part of the Old Babylonian period, among other things because the city of Ur was abandoned in 1763 BC (in the middle chronology), after Hammurabi had defeated Rim-Sîn of Larsa. The clay tablets are written almost exclusively in Sumerian, and they all use variant number signs. What is particularly interesting is that in the questions (but not in the answers) in *UET 5*, 121, large sexagesimal numbers are expressed in the Sumerian *non-positional* system, with special notations for 60, 10 · 60, and 60 · 60.

The implications of all this are vague but exciting. Did the division problems §§ 2a-c on the tablet *UET 5*, 121, form part of a Sumerian corpus of mathematics, of which virtually nothing else is known? Is the clay tablet MS 2317 with its undressed version of one of the division problems also from Ur, or is it from someplace else and from a later part of the Old Babylonian period? If it is from Ur, why is it not a round tablet, as all other known mathematical texts from Ur inscribed exclusively with numbers? If it is not from Ur, is it an example of how a problem type invented in one of the Mesopotamian cities could spread to other parts of Mesopotamia?

Note: Also *decimal* funny numbers can have interesting factorizations. Thus, for instance, the decimal funny number 1,001 (as in *Thousand and One Nights*), equal to the sexagesimal number 16 41, is the product of 7, 11, and 13. (Cf. the discussion in Sec. 7.2 c below of MS 2297, YBC 7353, YBC 11125, and VAT 7530 § 3.)

7.2. Combined Market Rate Exercises

7.2 a. Combined Market Rate Exercises with Regular Sexagesimal Market Rates

MS 2830 (Fig. 7.2.1, top) is a small rectangular hand tablet, similar in format to hand tablets with single multiplication tables. Two tabular arrays are inscribed on **MS 2830, *rev.*** The way in which these tabular arrays are constructed is fairly obvious. They are reproduced schematically below, in a somewhat more readable transliteration:

MS 2830 § 2a

1 shekel of silver

1	1	28 48	28 48
2	30	14 24	28 48
3	20	9 36	28 48
4	15	7 12	28 48

MS 2830 § 2b

1 shekel of silver

2	30	28 07 30	56 15
3	20	18 45	56 15
15	4	3 45	56 15
6	10	9 22 30	56 15

In the case of the first array (§ 2a), for instance, one may assume that the numbers 1, 2, 3, 4 in col. *i* are *arbitrarily prescribed data*. The numbers 1, 30, 20, 15 in col. *ii* are *the reciprocals* of the numbers in col. *i*, presumably with the values 1, ;30, ;20, and ;15, the "inverted values" of the four given numbers. *The sum of these inverted values* is

1 + ;30 + ;20 + ;15 = 2;05, a *regular* sexagesimal number.

The number 28 48 appearing in all four lines of col. *iv* is the reciprocal of 2 05, since

2 05 · 28 48 = 5 · 5 · 5 · 28 48 = 5 · 5 · 2 24 = 5 · 12 = 1 (in relative numbers).

Hence, it is reasonable to interpret the number 28 48 recorded in all four lines of col. *iv* as *the inverted value ;28 48 of the sum 2;05 of the inverted values in col. ii.*

The numbers in the four lines of col. *iii* can now be seen to be equal to *the values in the four lines of col. ii, multiplied by the constant factor ;28 48*. Indeed,

;28 48 · 1 = ;28 48,
;28 48 · ;30 = ;14 24,
;28 48 · ;20 = ;09 36,
;28 48 · ;15 = ;07 12.

The computation described above is the solution algorithm for what may be called a "combined market rate exercise". (See Friberg, *RlA 7* Sec. 5.2 h.) Let the "market rate" *r* of a given commodity be the number of "units" of that particular commodity that *can be purchased for 1 shekel of silver*. The nature of a unit depends, of course, on the commodity considered. For *barley*, *etc*., it could be the gur (a capacity unit of 5 00 sìla, each equal to about 1 liter), for *metals* it could be the mina (a weight unit equal to about 500 grams), for *fish* it could be a basket of sixty fishes, and so on.

Note that in a market economy *before the invention of money*, it was more convenient to operate with market rates (Sum.: gán.ba or ki.lam, Akk.: *maḫīrum*) than with prices!

In col. *i* of MS 2830 § 2a, four market rates are listed,

r = 1, 2, 3, and 4 units, respectively, for 1 shekel of silver.

The inverted value of the market rate of a given commodity may be understood as its "unit price" *p* in silver, because

if *r* units can be bought for 1 shekel of silver, then *p* = 1/*r* shekels of silver is the price of 1 unit.

In col. *ii* of § 2a are inscribed the unit prices of the four given commodities:

p = 1, 1/2 (;30), 1/3 (;20), and 1/4 (;15) shekels of silver per unit.

The sum of the four different unit prices can be called the "combined unit price" *P*:

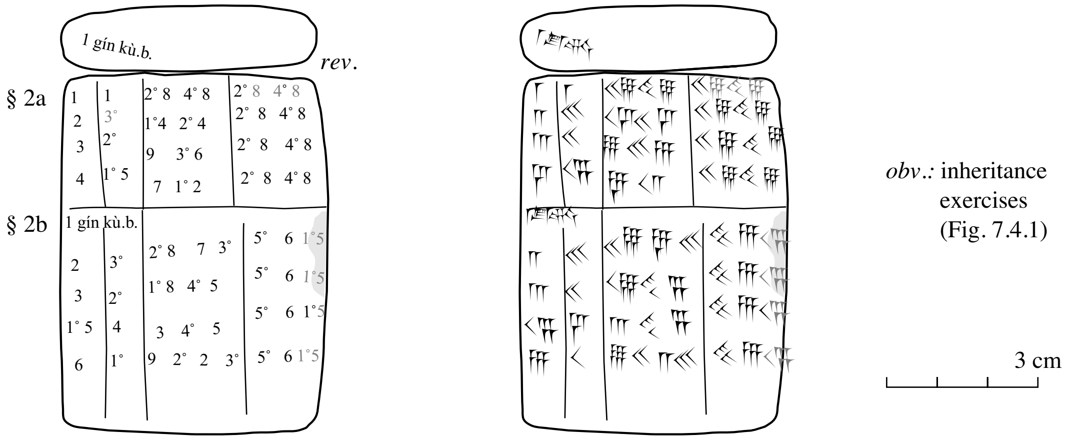
P = 2;05 shekels of silver for a combination of four units, one of each kind of commodity.

The inverse value of the combined unit price is the "combined market rate" *R*:

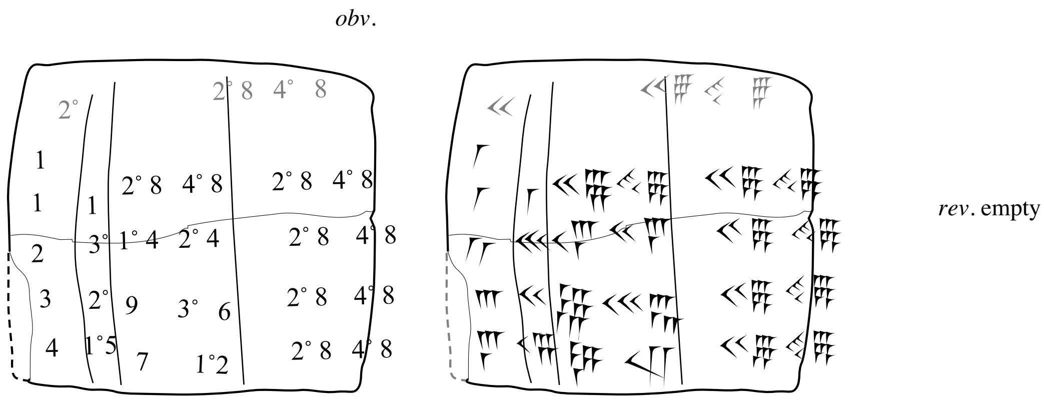
R = ;28 48 combinations of 1 unit of each kind of commodity per shekel of silver.

Another, equivalent, way of characterizing the combined market rate *R* is to say that

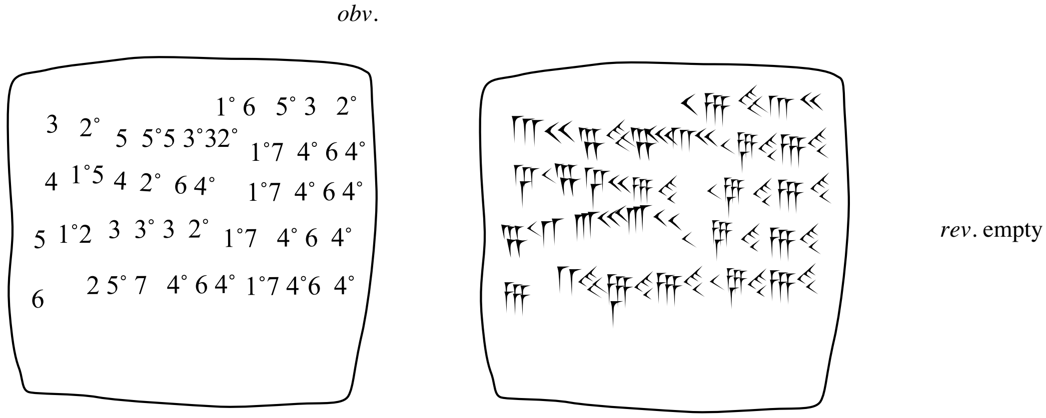
R is the combined market rate if the total price of *R* units of each kind of commodity is 1 shekel of silver.



MS 2830, *rev*. Market rates: a) 1, 2, 3, 4 b) 2, 3, 15, 6
Combined unit price: a) 2 05 b) 1 04 (both regular)
Total price: a) 1 shekel b) 1 shekel



MS 2832. Market rates: 1, 2, 3, 4. Combined unit price: 2 05 (regular). Total price: 1.



MS 2299. Market rates: 3, 4, 5, 6. Combined unit price: 57 (non-regular). Total price: 16 53 20.

Fig. 7.2.1. Three Old Babylonian combined market rate exercises with regular market rates.

In col. *iii* of § 2a on MS 2830 are inscribed the prices of R units of each kind of commodity, when R = ;28 48:

;28 48 · 1 = ;28 48, ;28 48 · 1/2 = ;14 24, ;28 48 · 1/3 = ;09 36, and ;28 48 · 1/4 = ;07 12 (shekels of silver).

It is easy to check that, indeed, the total of these four individual prices is precisely 1 shekel of silver:

;28 48 + ;14 24 + ;09 36 + ;07 12 = 1 (shekel).

Col. *iv* seems to be the final result of the computation, namely that in order to get a total price equal to precisely 1 shekel of silver, one has to purchase R = 28 48 units of *each* kind of commodity.

Briefly, it seems to be clear that the purpose of the first tabular array (§ 2a) on MS 2830, *rev.* is *to compute a number R such that 1 shekel of silver is the total price of R units of each of four commodities with the individual market rates 1, 2, 3, and 4 units per shekel of silver*. Note that the prescribed total price, 1 shekel of silver, is inscribed above the array, on the edge of the clay tablet.

The second tabular array on MS 2830, *rev.* (§ 2b), is similar. Here the four given market rates are 2, 3, 15, and 6 units per shekel, the corresponding unit prices ;30, ;20, ;04, and ;10 shekels per unit. Consequently,

the combined unit price in § 2b is ;30 + ;20 + ;04 + ;10 shekels = 1;04 shekels.

Since 1 04 is a *regular* sexagesimal number with the reciprocal 56 15 (see Sec. 2.5), it follows that

the combined market rate R in § 2b is equal to ;56 15 units per shekel (the value inscribed in col. *iv*).

The individual prices of R units of each kind of commodity are recorded in col. *iii*:

;56 15 · ;30 = ;28 07 30, ;56 15 · ;20 = ;18 45, ;56 15 · ;04 = ;03 45, and ;56 15 · ;10 = ;09 22 30 shekels.

Again, it is easy to check that the total of these four individual prices is precisely 1 shekel of silver:

;28 07 30 + ;18 45 + ;03 45 + ;09 22 30 = 1 (shekel).

This prescribed total price, 1 shekel of silver, is recorded above the array.

MS 2832 (Fig. 7.2.1, middle) is a square clay tablet inscribed with an array of rather large number signs. In this respect, the text is reminding of the simple multiplication, squaring, and division exercises discussed in Secs. 1.1.1-1.1.3 above. As for its content, MS 2832 is an almost exact parallel to MS 2830 § 2a. The only significant difference is that the prescribed total price is written above the array as '1 shekel silver' in MS 2830 § 2a, but simply as '1' in MS 2832. There is also a half erased number, 20 above the array on MS 2832, probably having little to do with the ensuing calculation. The similarly half erased number 28 48 in the upper right corner of the obverse was probably erased because it was misplaced. The four copies of the computed combined market rate 28 48 are correctly placed in lower positions on the obverse.

MS 2299 (Fig 7.2.1, bottom) is, just like MS 2832, a "combined market rate text" on a square clay tablet inscribed with relatively large number signs. The text looks rather messy, as the cuneiform signs are badly formed and weakly imprinted. There are no ruled lines between the columns of the array, and the numbers in different columns are not adequately separated from each other. Anyway, the intended arrangement is shown in the corrected transliteration to the right below:

MS 2832

20			28 48
1			
1	1	28 48	28 48
2	30	14 24	28 48
3	20	9 36	28 48
4	15	7 12	28 48

MS 2299

		16 53 20	
3	20	5 55 33 20	17 46 40
4	15	4 26 40	17 46 40
5	12	3 33 20	17 46 40
6	(10)	2 57 46 40	17 46 40

Just like the numbers 1, 2, 3, and 4 in col. *i* of MS 2830 § 2a, and the numbers 2, 3, 15, 6 in col. *i* of MS 2830 § 2b, the given numbers 3, 4, 5, and 6 in col. *i* of MS 2299 are all *regular* sexagesimal numbers. The numbers in col. *ii* are their reciprocals, with the values ;20, ;15, ;12, and ;10. (By mistake, the author or copyist of the text forgot to write down the number 10 in col. *ii,* row 4.)

An added complication in MS 2299, in comparison with the examples in MS 2830 Secs. 2a and 2b, is that the sum of the reciprocals of the given market rates, that is the combined unit price *P*, is a *non-regular* sexagesimal number in MS 2299:

$$P = ;20 + ;15 + ;12 + ;10 = ;57.$$

The combined market rate is the reciprocal of the combined unit price. Therefore, *when the combined unit price is a non-regular sexagesimal number, as in MS 2299, the value of the combined market rate cannot be computed exactly as a sexagesimal fraction*. For this reason, the problem stated and solved in MS 2299 is of a slightly different type compared with the corresponding problems in MS 2830 Secs. 2a and 2b. In particular, in MS 2299 the given total price is no longer 1 shekel of silver, as in the preceding problems. Instead, the given amount of silver is '16 52 30', the number recorded above cols. *iii* and *iv*.

The purpose of a tabular array such as the one on MS 2299 seems to have been *to compute a number N such that a given amount S of silver is the total price of N units of each of several commodities with given individual market rates*. The given market rates are inscribed in col. *i* of the array, the given amount *S* of silver is recorded above the array, and the computed number *N* is repeated several times in col. *iv*.

The first step of the solution algorithm, to compute the combined unit price *P*, gave the result in the case of MS 2299 that *P* has the non-regular value ;57 (shekels for 1 unit of each commodity). The second step of the solution algorithm is to find *N* as a solution to the linear equation

$$N \cdot P = S.$$

When *P* is a non-regular sexagesimal number, as in the case of MS 2299, the solution *N* to this linear equation is an *exact* sexagesimal number only if *S* is a multiple of *P*. It is not difficult to check that this requirement is satisfied here, since 16 53 20 = 57 · 17 46 40. Indeed, in sexagesimal arithmetic,

$$57 \cdot 17\ 46\ 40 = 16\ 09\ (00\ 00) + 43\ 42\ (00) + 38\ (00) = 16\ 53\ 20\ (00).$$

It remains to find out how an Old Babylonian student could find an answer to the question

What times 57 is 16 53 20?

The simplest way would have been to start by finding a factorization of 16 53 20, since the author of the problem was kind enough to let the given value 16 53 20 be the combined unit price 57 multiplied by a *regular* sexagesimal number. The factorization could be achieved in a series of simple steps:

$$16\ 53\ 20 = 20 \cdot 50\ 40, \quad 50\ 40 = 40 \cdot 1\ 16, \quad 1\ 16 = 4 \cdot 19, \quad \text{hence} \quad 16\ 53\ 20 = 20 \cdot 40 \cdot 4 \cdot 19.$$

Since 57 = 3 · 19, it was then easy to see that 16 52 30/57 = 20 · 40 · 4/ 3 = 17 46 40.

Remark: In the absence of any specific indications in the text of MS 2299 how the recorded numbers should be interpreted, there is more than one conceivable explanation of the problem and its solution. Instead of commodities purchased, one may think of wares produced, or work finished, *etc*., in a given period of time. (Cf. Friberg, *RlA 7* Sec. 5.6 h.) Thus, instead of a combined market rate problem of the kind described above, the problem behind the numerical array on MS 2299 may have been, for instance, a "combined work norm problem" of the following kind:

Given four kinds of wares produced or work finished *in equal quantities* at four different work rates, namely 3, 4, 5, and 6 units per man-day, and given a total of 16;53 20 man-days (understood as 16 and 2/3 and 1/3 of 2/3 man-days). Then the combined cost in labor is ;57 man-days for 1 unit of each kind, and 17;46 40 units of each kind can be produced or finished in the given 16;53 20 man-days. The cost in labor for 17;46 40 units of the first kind is 17;46 40 · ;20 man-days = 5;55 33 20 man-days, *etc*.

7.2 b. YBC 7234, 7235, 7354, 7355, 7358, and 11127, Six Parallel Texts in MCT

A number of cuneiform texts with tabular arrays were published by Neugebauer and Sachs in *MCT* (1945), 17. Although the general structure of the arrays was correctly analyzed, no explanation of the meaning of the arrays was offered. Photos of YBC 7358 and 11127, were published by Nemet-Nejat in *JNES* 54 (1995), and of YBC 7234, 7235, 7354, 7355, 7358, and 11127, again by Nemet-Nejat, in *UOS* (2002), in both cases with no explanation offered. Actually, the characterization of such hand tablets with tabular arrays as "help tables for combined market rate exercises”, and the explanation of “combined market rate exercises” as parallels to the more generally understood “combined work norm exercises”, appeared for the first time in Friberg, *RlA 7* (1990) Sec. 5.2 h and §§ 5.6 h-i.

Two of the tabular arrays published in *MCT* are the following:

YBC 7358

1 42 45

1	1	45	45
2	30	22 30	45
3	20	15	45
4	15	11 15	45
[5]	12	9	45

YBC 7235

1 03 20

1 40	1	1	40	1 06 40
5	3	20	13 20	1 06 40
6 40	4	15	10	1 06 40
				40

The first of these arrays, **YBC 7358**, can immediately be interpreted as the data for a combined market rate problem similar to the one in MS 2299. According to this interpretation, five given market rates are 1, 2, 3, 4, and 5 (units per shekel). The corresponding unit prices are 1, ;30, ;20, ;15, and ;12 (shekels per unit). The combined unit price (as usual, not explicitly indicated in the text) is 2;17 (shekels for 1 unit of each kind). Clearly, 2 17 is a *non-regular* sexagesimal number. As in MS 2299, the given total price in this case (recorded above the array in YBC 7358), is equal to the combined unit price 2 17 multiplied by a *regular* sexagesimal number. It is easy to see that 1 42 45 = 15 · 6 51 = 15 · 3 · 2 17 = 45 · 2 17. Consequently the number of units purchased of each commodity is 45, the number recorded five times in col. *iv*.

In **YBC 7235**, the numerical array has *five* columns, one more than the usual number. The most likely interpretation in this case is that three given market rates are 1 40 (= 100), 5 00 (= 300), and 6 40 (= 400) units per shekel. (col. *i*), and that the given total price is 1 03;20 shekels (recorded above the array). Expressed differently, the given market rates are 1, 3, and 4 *hundreds per shekel* (col. *ii*). The corresponding unit prices are 1, ;20, and ;15 *shekels per hundred* (col. *iii*). Consequently, if 1 hundred of each kind were purchased, the combined price would become1;35 shekels. The given total price, presumably to be understood as 1 03;20 shekels, is 40 times larger. Therefore, the number 40 is recorded under the other numbers in col. *v*. The final result is that *40 hundreds* of each kind must be purchased. The number 40 hundreds = 1 06 40 (in decimal numbers 4,000) is recorded three times in col. *v*, and the corresponding individual prices 40, 13;20, and 10 (shekels) are recorded in col. *iv*. It is easy to check that, as required, 40 + 13;20 + 10 = 1 03;20 (shekels).

7.2 c. Combined Market Rate Exercises with One Non-Regular Factor in the Data

In all the market rate tables considered in Sec. 7.2 a above, the given market rates in col. *i* were *regular* sexagesimal numbers. Otherwise it would not have been possible to compute exactly the unit prices in col. *ii* as the inverted values of the market rates.

In **MS 2268/19, *obv.*** (Fig. 7.2.2, top), on the other hand, a somewhat changed approach allows the presence of a *non-regular* factor, the same in one or more of the given market rates:

MS 2268/19, *obv.*

1 00 05 20

3 30	1	21 20	1 14 40
5 50	36	12 48	1 14 40
7	30	10 40	1 14 40
7 30	28	9 57 20	1 14 40
14	15	5 20	1 14 40

The given numbers in col. *i* of this text are 3 30, 5 50, 7, 7 30, and 14, presumably with the intended values 3;30 = 7 · ;30, 5;50 = 7 · ;50, 7, 7;30, and 14 = 7 · 2. Thus, all of these, except 7;30, contain the *non-regular* sexagesimal number 7 as a factor. Clearly, the sexagesimal reciprocals of the numbers with 7 as a factor do not exist. Therefore, assuming that the numbers in col. *i* are market rates, the numbers in col. *ii* cannot be the corresponding unit prices. Instead, they are the prices of 3;30 = 3 1/2 units of each kind of commodity. The number 3;30 can be thought of as the "false" (in the sense of "posited") size of the equal purchases made of each commodity, in an application of the method of false value. (See Friberg, *RlA 7* (1990) Sec. 5.7 d.) It is easy to check that

The price of 3;30 units at a market rate of	3;30	units per shekel	is	1	shekel,	
the price of 3;30 units at a market rate of	5;50	units per shekel	is	;36	shekel	(3;30/5;50 = 3/5 = ;36),
the price of 3;30 units at a market rate of	7	units per shekel	is	;30	shekel	(3;30/7 = 1/2 = ;30),
the price of 3;30 units at a market rate of	7;30	units per shekel	is	;28	shekel	(3;30/7;30 = 7/15 = ;28),
the price of 3;30 units at a market rate of	14	units per shekel	is	;15	shekel	(3;30/14 = 1/4 = ;15).

Hence, the combined price for 3 1/2 units of each commodity is

1 + ;36 + ;30 + ;28 + ;15 = 2;49 (shekels).

Here 2 49 is a *non-regular* sexagesimal number. As could be expected, 1 00 05 20, the given total price, is a multiple of this combined price. Assuming that the intended value of 1 00 05 20 is, for instance, 1;00 05 20 shekels, one finds that the value of the total price is precisely ;21 20 times the false price 2;49 shekels, as shown by the factorization

1 00 05 20 = 20 · 3 00 16 = 20 · 4 · 45 04 = 20 · 4 · 4 · 11 16 = 20 · 4 · 4 · 4 · 2 49 = 21 20 · 2 49.

Therefore, the "true prices" in col. *iii* are ;21 20 times the "false prices" in col. *ii*. Indeed,

;21 20 · 1			=	;21 20,
;21 20 · ;36	=	;12 + ;00 36 + ;00 12	=	;12 48,
;21 20 · ;30	=	;21 20/2	=	;10 40,
;21 20 · ;28	=	;09 20 + ;00 28 + ;00 09 20	=	;09 57 20,
;21 20 · ;15	=	;21 20/4	=	;05 20.

The sum of these true prices is, of course, equal to the given total price:

;21 20 + ;12 48 + ;10 40 + ;09 57 20 + ;05 20 = 1;00 05 20.

Finally, the true number of units that can be purchased of each commodity for this total price must be ;21 20 times the initially chosen false size of the equal purchases. In other words, it is

;21 20 · 3;30 = 1;10 + ;03 30 + ;01 10 = 1;14 40 (units).

This is, then, the value of the number 1 14 40 recorded in all 5 rows of col. *iv* of MS 2268/19.

The strange form of the given total price, 1;00 05 20 shekels = 1 shekel + 1/4 and 1/60 barley-corn, can be explained as follows. It was shown above that the combined price for 3 1/2 units of each kind is 2;49 shekels. Therefore, if the reciprocal of 2;49 had existed, the combined market price could have been computed as

3;30/2;49 = 3;30 · rec. 2;49.

Since 2;49 is non-regular, the reciprocal does not exist, but ;21 20 is a good *approximative reciprocal*. Indeed,

;21 20 · 2;49 = 1;00 05 20.

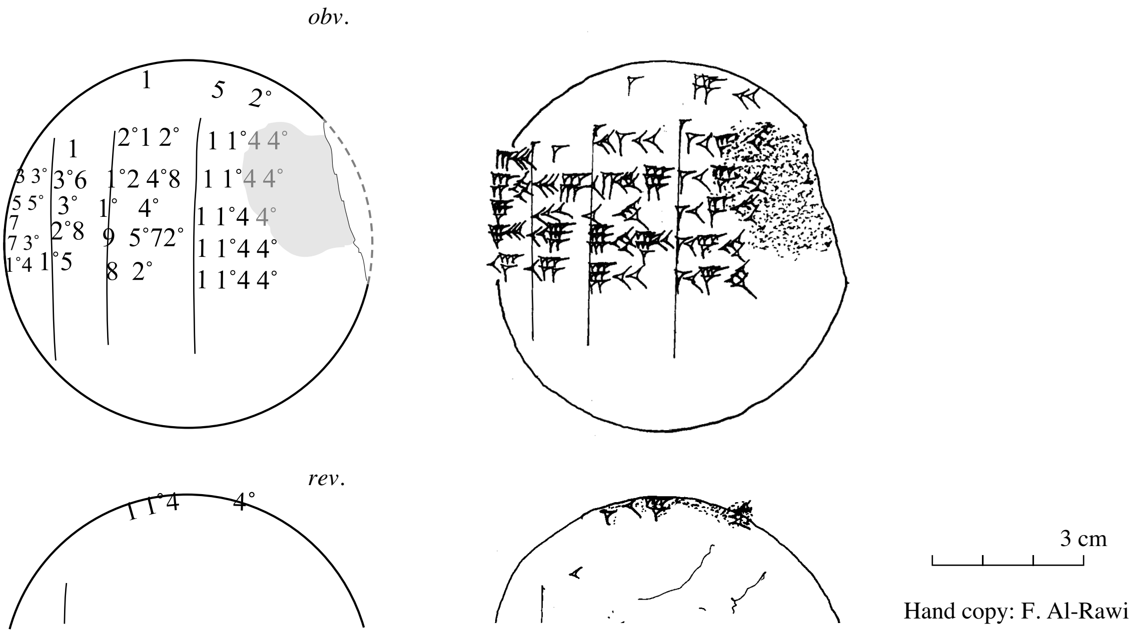
Hence, the number 1 14 40 recorded in the fourth column of MS 2268/19 can be interpreted as the *approximate combined market rate*

3;30 / 2;49 = appr. ;21 20 · 3;30 = 1;14 40 (units of each kind per shekel).

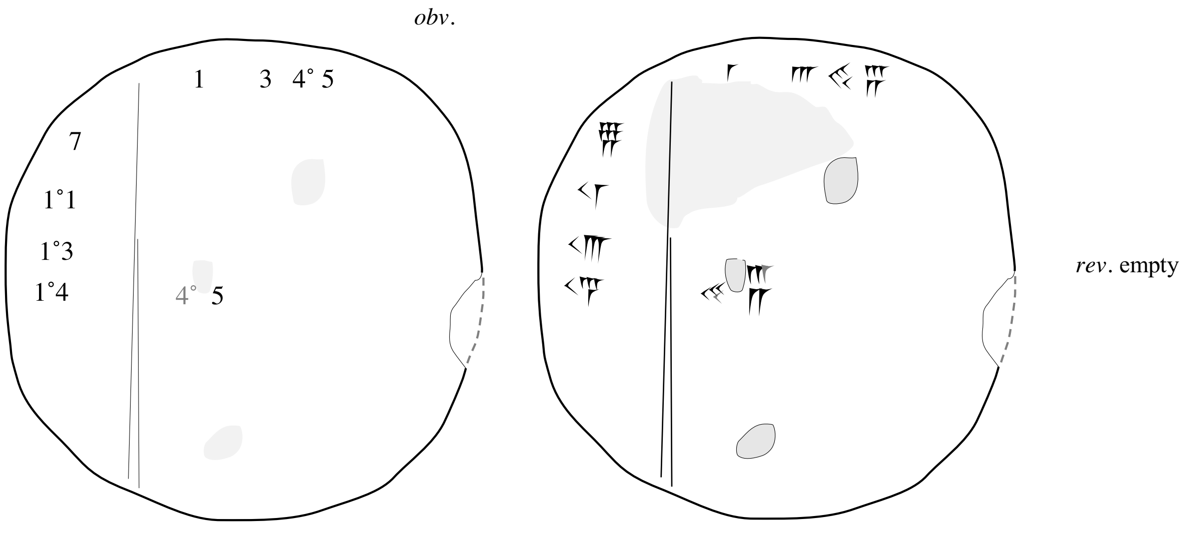
(Approximate solutions of this kind are extremely rare in Old Babylonian mathematics!)

7.2 d. N 3914, a Parallel Text with 10 Given Numbers

Another text of a similar type is **N 3914**, an Old Babylonian clay tablet from Nippur, of the same size and shape as the YBC clay tablets mentioned above with numerical data for combined market rate problems. (All those clay tablets are unprovenanced). N 3914 was published by Robson in *SCIAMVS* 1 (2000), 28, as a hand copy with a transliteration, but without any valid interpretation.



MS 2268/19, *obv*. Market rates: 3 30, 5 50, 7, 7 30, 14 (non-regular). Combined price: 2 49 for 3 30.
Total price: 1 00 05 20. Number of purchases: 1;14 40 units of each kind.
rev. The number recorded on the upper edge looks like 1 14 00 40 but may be a badly written 1 14 40.



MS 2297. Market rates: 7, 11, 13, 14 (non-regular).
[Combined price: 6 22 30 for 16 41 units. 10 · 6 22 30 = 1 03 45.]

Fig. 7.2.2. Two Old Babylonian combined market rate tables with non-regular market rates.

In N 3914, the given market rates range from 1 to 10, and the given total price is 3 25 01 40, possibly to be understood as 3 25;01 40 shekels = 3 1/3 minas 5 shekels 5 barley-corns. One of the given market rates, 7, is *non-regular*. For that reason, the initially chosen false size of the equal purchases is 7 (units). The numbers recorded in col. *ii* are the ten false prices, the individual unit prices multiplied by 7. The combined price for 7 units of each kind is

7 + 3;30 + 2;20 + 1;45 + 1;24 + 1;10 + 1 + ;52 30 + ;46 40 + 42 = 20;30 10.

This combined price is one tenth of the given total price 3 25;01 40. Consequently, the correct size of the equal purchases is ten times as large as the false size. Therefore, it is 10 · 7 = 1 10 (units), the number repeatedly recorded in col. *iv*. The total of all the ten purchases, 11 40 = 10 · 1 10 (units), is recorded in the right margin.

Here follows a slightly amended transliteration of the text, with damaged parts reconstructed:

N 3914

			3 25 01 40	
1	7	1 10	1 10	11 40
2	3 30	35	1 10	
3	2 20	23 20	1 10	
4	1 45	17 30	1 10	
5	1 24	14	1 10	
6	1 10	11 40	*etc.*	
7	1	10		
8	52 30	8 45		
9	46 40	7 46 40		
10	42	7		

7.2 e. Combined Market Rate Exercises with Several Non-Regular Factors in the Data

MS 2297 (Fig. 7.2.2, bottom) is a round clay tablet inscribed with what looks like only the first column of a tabular array, plus traces of additional numbers. The sexagesimal number 1 03 45 can be seen near the upper edge, at the place where the given total price is usually recorded in combined market rate exercises. Conceivably, a student first watched a teacher's solution to a combined market rate problem, then entered on this tablet the given market rates and the given total price, intending to try later to find on his own the solution to the problem. Which he never got around to do. Here follows a transliteration of the unfinished text:

MS 2297

		1 03 45
7		
11		
13		
14	45?	

A further comment to this text will have to wait until some possibly parallel texts have been considered. (See the note inserted below, after the discussion of the text VAT 7530 § 3.)

Two possibly parallel texts are **YBC 7353** and **YBC 11125**, published in transliteration, but without any attempted explanation, by Neugebauer and Sachs in *MCT*, 17:

YBC 7353

			3 11 15
7	2 23	1 11 30	8 20 30
11	1 31	45 30	8 20 30
13	1 17	38 30	8 20 30
14	1 11 30	35 45	8 20 30

YBC 11125

			4 15
7	2 23	1 35 20	11 07 20
11	1 31	1 00 40!	11 07 20
13	1 17	51 20	11 07 20
14	1 11 30	47 40	11 07 20

In both texts, four given market rates are 7, 11, 13, and 14 (units per shekel). Since the given market rates are *non-regular* sexagesimal numbers (with three different non-regular factors, 7, 11, and 13), the corresponding unit prices cannot be expressed exactly as finite sexagesimal numbers. The difficulty is sidestepped by assuming *false equal purchases of a sufficiently large quantity*, namely

$$7 \cdot 11 \cdot 13 = 16\ 41 \text{ (in decimal notation 1,001).}$$

The corresponding false prices are then

$$16\ 41/7 = 11 \cdot 13 = 2\ 23,\ 16\ 41/11 = 7 \cdot 13 = 1\ 31,\ 16\ 41/13 = 7 \cdot 11 = 1\ 17, \text{ and } 16\ 41/14 = 1\ 11;30.$$

The resulting combined price (not indicated in the texts) is

$$2\ 23 + 1\ 31 + 1\ 17 + 1\ 11;30 = 6\ 22;30.$$

In YBC 7353, the first of the two texts, the given total price can be interpreted as, for instance, 3 11;15 (shekels), half the combined price. Then, the correct size of the equal purchases is ;30 · 16 41 = 8 20;30 (col. *iv*), and the true prices in col. *iii* are precisely half the false prices in col. *ii*.

In YBC 11125, the second of the two texts, the given total price can be interpreted as 4 15, two-thirds of the combined price. Consequently, the true size of the equal purchases is ;40 · 16 41 = 11 07;20 (col. *iv*), and the true prices in col. *iii* are precisely two-thirds of the false prices in col. *ii*.

7.2 f. VAT 7530. A Theme Text with Combined Market Rate Problems

In Secs. 7.2 a-c above, five MS texts, 4 parallel YBC texts, and a single text from Nippur were discussed as examples of texts with tabular arrays for combined market rate problems. All those texts, with the exception of MS 2830 *rev*., are in the form of small round or squarish clay tablets on which are recorded numerical data without any explaining text. They are, in other words hand tablets of the kind discussed in Robson, *MMTC* (1999), App. 5, and Friberg, "Mathematics at Ur", *RA* 94 (2000). The text on the reverse of MS 2830 may have been copied from two such hand tablets.

In addition to these brief and practically wordless texts, there is also one Old Babylonian problem text dealing with combined market rate problems. This is the "theme text" **VAT 7530** published by Neugebauer in *MKT 1*, 287-289, with photo and hand copy in *MKT 2,* and with a renewed, partly successful attempt to analyze its meaning in *MCT*, 18. Six combined market rate problems are formulated in VAT 7530 §§ 1-6. No answers or detailed solution algorithms are provided.

The problems formulated in §§ 1-2 of VAT 7530 are unlike previously known examples of combined market rate problems, and so damaged that it is difficult to find valid interpretations of them. The problems in §§ 5-6 are similar to the one on N 3914 (one non-regular given market rate), while the problems in §§ 3-4 are similar to YBC 7353 (several non-regular market rates). Here is, for instance, the text of VAT 7530 § 5 in transliteration and translation:

VAT 7530 § 5 (*obv*. 17-21).

1	1 ma.na.ta.àm 2 ma.na [3 ma.n]a 4 ma.n[a]	1 mina each, 2 minas, *3 min*as, 4 minas,
2	[5 ma.na] / 6 ma.na 7 ma.na 8 ma.na	*5 minas*, 6 minas, 7 minas, 8 minas,
	9 ma.na 10 m[a.n]a /	9 minas, 10 m*in*as.
3	10 gín igi.4_{v}.gál *ù*	10 shekels a 4th-part (of a shekel) and
	igi.4_{v}-*a-at* še kù.babbar /	a 4th part of a barley-corn.
4	kù.babbar *li-li li-ri-da-ma* /	The silver may go up (and) go down, but
5	[gán.]ba *li-im-ta-ḫi-ra*	the market rates may be equal.

In this problem, the given market rates are the same as in N 3914. Hence, just as in N 3914, if the initially chosen false size of the equal purchases is 7 (units), then the combined price is 20;30 10 (shekels). Here, however, the given total price is 10 1/4 shekel 1/4 barley-corn = 10;15 05 shekels, not 3 25; 01 40 (shekels) as in N 3914. This means that the given total price is precisely 1/2 of the combined price. Accordingly, the true size of the equal purchases is 1/2 · 7 = 3;30 (units), and so on. However, no solution algorithm is provided, and no answer to the stated problem is given.

Slightly more complicated is the problem formulated in VAT 7530 § 6:

VAT 7530 § 6 (*rev.* 1-7).

1	1 ma.na.ta.àm 1 ma.na *ù* 10 gín.ta.à[m] /	1 mina each, 1 mina and 10 shekels each,
2	2 ma.na.ta.àm 2 3' ma.na /	2 minas each, 2 1/3 minas,
3	3 ma.na 3 2' ma.na 4_v ma.na 4_v 3" ma.na /	3 minas, 3 1/2 minas, 4 minas, 4 2/3 minas,
4	5 ma.na 5 6" ma.na 2 3' 25 še /	5 minas, 5 5/6 minas, 2 1/3 (shekels) 25 barley-corns
5	*ù* igi.4_v-*a-at* še /	and a 4th part of a barley-corn.
6	[kù.babbar] *li-li li-ri-da-ma* /	*The silver* may go up (and) go down, but
7	[gán.]ba *li-im-ta-ḫa-ar*	*the mar*ket rates may be equal.

The total price given in lines 4-5 is 2 1/3 shekels 25 1/4 barley-corns = 2;28 25 shekels. Therefore, *the problem* stated *in words* in the text of VAT 7530 can be reformulated, *together with its solution* (not given in the text), *in a tabular array* of the following form:

			2 28 25
1	7	35	35
1 10	6	30	35
2	3 30	17 30	35
2 20	3	15	35
3	2 20	11 40	35
3 30	2	10	35
4	1 45	8 45	35
4 40	1 30	7 30	35
5	1 24	7	35
5 50	1 12	6	35

Note that 7 is the only non-regular a factor in 1 10, 2 20, *etc*.
(actually, 1;10 = 7 · ;10, 2;20 = 7 · ;20, *etc*.)

The problem stated in VAT 7530 § 3 is the following:

VAT 7530 § 3 (*obv.* 7-10).

1	7 ma.na.ta.àm *ù* 11 ma.na.[ta.àm] /	7 minas each and 11 minas *each*,
2	13 ma.na.ta.àm *ù* 14 ma.na.[ta.àm] /	13 minas each and 14 minas *each*.
3	1 gín 11 še igi.4_v-*a-at* še kù.babbar /	1 shekel 11 barley-corns, a 4th part of a barley-corn.
4	kù.babbar *li-li ù li-ri-da*	The silver may go up and go down, but
	ma-ḫi-[*rum*] *li-im-ta-ḫar*	the market r*ates* may be equal.

The given total price is 1 shekel 11 1/4 barley-corns = 1;03 45 shekel. Hence, the solution to the stated problem, if given in the form of a tabular array, would have taken the following form:

		1 03 45	
7	2 23	23 50	2 46 50
11	1 31	15 10	2 46 50
13	1 17	12 50	2 46 50
14	1 11 30	11 55	2 46 50

Cf. the discussion of the arrays in YBC 7353 and 11125. With false equal purchases of 7 · 11 · 13 = 16 41 minas, the combined price is 6 22;30 shekels, 6 00 times the given total price 1;03 45 shekel. Therefore, the true equal purchases are 16 41 / 6 00 = 2;46 50 minas = 2 2/3 minas 6 5/6 shekels.

Note: The text of VAT 7530 § 3 contains only a question, but no answer and no solution procedure. Stated in the form of a tabular array, the question alone would take the form of only the first column and the superscript 1 03 45 in the array above. This is precisely the form of the array on MS 2297 above! Thus, it appears that MS 2297 was an assignment. The student was expected to complete the array with the missing columns and/or write a text along the lines of VAT 7530 § 3.

The problem stated in VAT 7530 § 4 is particularly interesting:

VAT 7530 § 4 (*obv*. 11-16).

1	7 ma.na.ta.àm 11 ma.na.ta.àm 13 ma.na.ta.àm /	7 minas each, 11 minas each, 13 minas each,
2	14 ma.na.ta.àm 19 ma.na.ta.àm *ma-ḫi-rum* /	14 minas each, 19 minas each is the market rate.
3	[1 ma].na 8 6" gín 11 2' še	*1 mi*na 8 5/6 shekels 11 1/2 barley-corns,
	igi.4_v-*a-at* še [kù.babbar] /	a 4th part barley-corn of *silver*.
4	*ma-ḫi-ir* 7 ma.na *ù* 19 ma.na [··· ···] /	The market rate of 7 minas and 19 minas ··· ··· ,
5	*ma-ḫi-ir* 11 ma.na *ù* 14 ma.na [··· ···] /	the market rate of 11 minas and 14 minas ··· ··· ,
6	*ma-ḫi-ir* 13 ma.na *li-*[*te*]*-er*	the market rate of 13 minas may go beyond.

Expressed as a tabular array, the solution to the problem stated in lines 1-3 would look like this:

			1 08 54 15
7	45 17	22 38 30	2 38 29 30
11	28 49	14 24 30	2 38 29 30
13	24 23	12 11 30	2 38 29 30
14	22 38 30	11 19 15	2 38 29 30
19	16 41	8 20 30	2 38 29 30

The meaning of lines 4-6 in VAT 7530 § 4 is far from obvious, in particular in view of the damage to the ends of lines 4 and 5 and to the middle part of the crucial inflected verb form *li-*[*te*]*-er* in line 6. Nevertheless, an interpretation will be attempted here: In more explicit form, the displayed solution to the stated problem may be rephrased as follows:

> If five different commodities have the five different market rates 7, 11, 13, 14, and 19 units per shekel,
> and if the total amount of silver available is 1 08;54 15 shekels, then 2 38;29 30 units can bought of each kind
> of commodity, that is altogether 5 · 2 38;29 30 units = 13 12;27 30 units.
> Therefore, a mixed bag of the five different commodities, with an equal number of each kind, may be bought
> at an average market rate of 13 12;27 30 units for 1 08;54 15 shekels = (approximately) 11;30 units per shekel.

This "average market rate" is, actually, the *harmonic mean* of the five given market rates. In modern mathematical terms:

> If $r_1, \ldots, r_5$ are the five given market rates then the combined market rate is $5/(1/r_1 + \ldots + 1/r_5)$.

As a kind of average of the given market rates, the harmonic mean is smaller than the more familiar *arithmetical mean*, which in the case considered here is (7 + 11 + 13 + 14 + 19)/5 = 64/5 = 12;48. A sharp-eyed Babylonian mathematician may have observed that this arithmetical mean of the five market rates is close to 13, the value of the market rate in the middle, as well as to the arithmetical mean (11 + 14)/2 = 12;30 of the two nearest market rates on each side of 13, and even to the arithmetical mean (7 + 19)/2 = 13 of the two extreme market rates on each side of 13. He may then have suspected that something similar may be shown in the case of the average market rate (that is, the harmonic mean). Actually, the average market rate of 7 and 19 is 2 · 7 · 19/(19 + 7) = 7 · 19/13 = 10 3/13 = appr. 10;14, which is not far away from, but *less than* 11;30 (line 4). The average market rate of 11 and 14 is 2 · 11 · 14/(14 + 11) = 12 8/25 = appr. 12;19, which again is not far away from, yet *greater than* 11;30 (line 5). The market rate in the middle, 13, is also *greater than* 11;30 (line 6).

Of all the mentioned Old Babylonian texts concerned with combined market rate problems, MS 2830 *rev*. is almost certainly the *earliest* one, being (probably) from early Old Babylonian Ur. VAT 7530 may also be an early Old Babylonian text, from one of the southern cities in Mesopotamia. This is indicated by the use of the variant number sign 4_v in certain expressions, such as 4_v ma.na, igi.4_v, and igi.4_v-*a-at*.

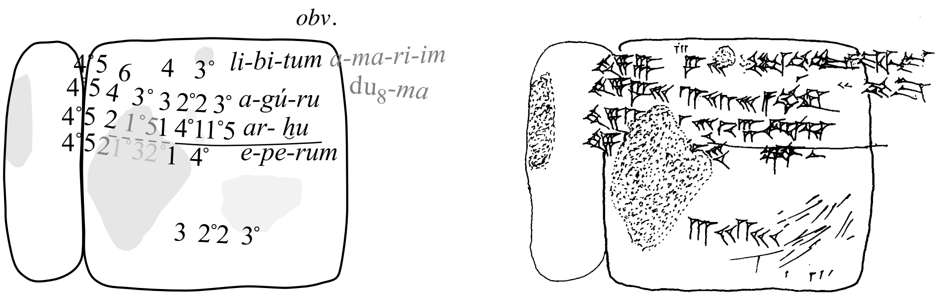
It is interesting to note that MS 2830 *rev*. is also the *simplest* example of all the known texts of this kind, with only regular sexagesimal numbers used for the given market rates, and with the given total price equal to precisely 1 shekel, written explicitly as 1 gín kù.babbar, not just as '1'.

7.3. Old Babylonian Brick Types and Brick Constants

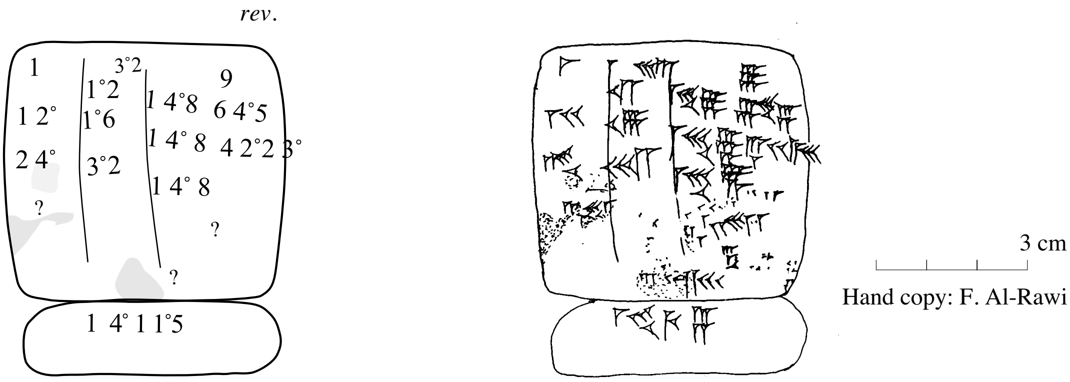
For obvious reasons, counting with bricks was a popular topic in Old Babylonian practical mathematics, exemplified by many entries in tables of constants and several different kinds of problem texts. Detailed surveys of the subject have been published by Friberg in *ChV* (2001), and by Robson in *MMTC* (1999) §§ 4-6. An effort to combine and refine the results of these two parallel but independent investigations was made in a review of Robson's *MMTC* by Friberg in *AfO* 46/47 (1999/2000). The mentioned surveys can now be completed through the addition to the corpus of a text from the Schøyen Collection with important implications for the subject.

7.3 a. MS 2221, obv. Walking Numbers, Loading Numbers, and Carrying Numbers for Three Kinds of Bricks, and for Mud

MS 2221 (Fig. 7.3.1) is a small square hand tablet, with a table of constants on the left edge and the obverse, and a numerical tabular array on the reverse. The text is an exercise in the use of constants for bricks and mud.



MS 2221 *obv*.: Computations of carrying numbers for three kinds of bricks and for mud.



MS 2221 *rev*.: Computation of a combined work norm for the carrying of three kinds of bricks

Fig. 7.3.1. An Old Babylonian hand tablet with brick constants and a brick problem.

Here follows a transliteration of the inscription on the left edge and on the obverse of MS 2221. (Note that the inscription on the right edge following after the first line of text on the obverse is badly readable. It seems to be a half erased leftover from an earlier inscription on the clay tablet. It will not be translated below, even if it seems to mention the molding of bricks and making a 'brick pile'.)

MS 2221, *obv.*

45	6	4 [30]	*li-bi-tum a*?*-ma*?*-ri*?*-im*? / du$_8$?*-ma*
45	4 30	3 22 30	*a-gú-ru*
45	2 [15]	[1] 41 15	*ar-ḫu*
45	2 [13 20]	[1] 40 3 22 30	*e-pe-rum*
		1 41 15	

It is impossible to understand the meaning of this inscription without a preceding discussion of Old Babylonian "brick metrology", like the one inserted below for the readers' convenience.

Listed in the last column of text on the obverse are the following Akkadian words:

libittu	(Sum. sig$_4$)	a word for *rectangular* bricks, with the width equal to 2/3 of the length,
agurru	(Sum. sig$_4$.al.ur$_5$.ra)	a word for *square* bricks, with the width equal to the length,
arḫu	(Sum. sig$_4$.áb)	a word for *half square* bricks, with the width equal to 1/2 of the length,
eperu	(Sum. saḫar)	a word for mud or earth, used for the fabrication of bricks.

For lack of more precise translations, the words '(regular) bricks', 'tile-bricks', and 'cow-bricks' will be used in the following as free translations of *libittu, agurru,* and *arḫu* (*arḫu* is Akkadian both for half square bricks and for 'cow'). In the cases attested in Old Babylonian mathematical texts and tables of constants, the length of the side of a square brick may be either 1/3, 2/3, or 1 cubit (= 10, 20, and 30 fingers), or 3 or 4 sixtieths of a ninda (= 18 and 24 fingers). The length of a rectangular brick may be either 1/2 cubit or 3 sixtieths of a ninda, and the length of a half brick may be either 2/3 cubit or 4 sixtieths of a ninda. The thickness of all these diverse types of bricks is usually 5 fingers (= 1/6 of a cubit), but it may also be 6 fingers (= 1/5 of a cubit = 1 sixtieth of a ninda).

Ever since the first discussion of brick types in Old Babylonian mathematical texts, in the comment by Neugebauer and Sachs to the metro-mathematical theme text *MCT*, 91 = YBC 4607, it has been customary to refer to brick types by number (type 1, type 2, *etc*.). A more informative kind of notation is to let the names of the various brick types refer to the *format*, with S for square bricks, H for half bricks, R for rectangular bricks, as well as to the *size*, with the tags 1/3c, 2/3c, and 1c for bricks of length 1/3, 2/3, or 1 cubit, and the tags 3n and 4n for bricks of length 3 or 4 sixtieths of a ninda. Thus, the attested brick types may be referred to as

S1/3c, S2/3c, S1c, S3n, and S4n	in the case of *square* bricks of regular thickness (5 fingers),
H2/3c and H4n	in the case of *half bricks* of regular thickness,
R1/2c and R4n	in the case of *rectangular* bricks of regular thickness.

For bricks of extra thickness (6 fingers), these notations may be augmented by a tag v for "variant", so that one writes S1/3cv, *etc*.

The dimensions of the bricks figuring in *MCT* O = YBC 4607, for example, are mentioned explicitly, which makes it easy to see that those bricks are of the four types R1/2c, R3n, S2/3c, S1c. The dimensions of the bricks figuring in Haddad 104 §§ 9-10 (Al-Rawi and Roaf, *Sumer* 43 (1984)) are mentioned explicitly, too, so that it is clear that the bricks in this latter case are of type S2/3cv.

In most other cases, the dimensions of bricks figuring in Old Babylonian mathematical texts or tables of constants are not mentioned explicitly.The brick type is specified only indirectly, often by reference to its "molding number" (Akk. *nalbanum*), which may be defined as follows:

Bricks of a given type have the molding number L if the volume of L brick šar of such bricks is 1 volume šar.
Here 1 brick šar = 12 00 bricks, and 1 volume šar = 1 square ninda · 1 cubit (= 1 n. · 1 n. · 1 c.).

Take, for instance, the most common type of bricks in Old Babylonian mathematical texts, rectangular bricks of type R1/2c. For such "standard bricks", one can compute the following parameters:

Base area A = 1/2 c. · 1/3 c. = ;02 30 n. · ;01 40 n. = ;00 04 10 sq. n.,
Volume $V = A$ · 1/6 c. = ;00 04 10 n. · n. · ;10 c. = ;00 00 41 40 volume šar,
Volume of 1 brick šar = 12 00 · ;00 00 41 40 volume šar = ;08 20 volume šar,
Molding number L = 7;12 (brick šar per volume šar), since 7;12 · ;08 20 = 1.

For “variant unit bricks”, bricks of type S1cv, the corresponding computations are simpler:

Base area A = 1 c. · 1 c. = ;05 n. · ;05 n. = ;00 25 sq. n.,
Volume $V = A$ · 1/5 c. = ;00 25 n. · n. · ;12 c. = ;00 05 volume šar,
Volume of 1 brick šar = 12 00 · ;00 05 volume šar = 1 volume šar,
Molding number L = 1 (brick šar per volume šar).

In other words,

The volume of 1 brick šar of variant unit bricks is 1 volume šar.

“Unitary relations” of this type are relatively frequent in Sumerian and Babylonian metrology. It is likely that this particular unitary relation is the explanation for the seemingly strange introduction of the brick šar = 12 00 bricks as a counting unit for bricks.

Another, even more striking, unitary relation involving variant unit bricks is the following:

The weight of 1 baked variant unit brick is 1 talent.

Now, the volume of L brick šar of bricks of molding number L is 1 volume šar, which is the volume of 1 brick šar of variant unit bricks. This means that a variant unit brick is L times more massive than a brick of molding number L. Consequently,

The weight of L baked bricks of molding number L is 1 talent (= 1 00 minas).
Consequently, the weight of 1 baked standard brick (type R1/2c) is 8 1/3 minas (= 1 talent / 7;12).

The weight of a (baked) standard brick is mentioned explicitly in the text *MCT* Oa = **YBC 7284**, a round hand tablet with the following inscriptions (plus an unrelated number) on obverse and reverse:

obv.	41 40	;00 00 41 40 (volume šar)	the volume of 1 brick
	8 20	;08 20 (volume šar)	the volume of 1 brick šar
	igi.gub.ba.bi 12	its constant 12(00)	bricks in 1 brick šar
rev.	1 sig_4	1 brick,	
	ki.lá.bi en.nam	its weight is what?	
	ki.lá.bi 8 3' ma.na	Its weight is 8 1/3 minas.	

The weights of *baked* bricks of types R1/2c, H2/3c, S2/3c, and R3n are listed in § 4' of the Old Babylonian table of constants RAFb = **BM 36776** (Fig. 7.3.2; Robson, *MMTC* (1999), 206; Friberg, *ChV* (2001), 67):

[8] 20	ki.lá	sig_4	*ṣa-rip-ti*	*8*;20	the weight of a	brick,	baked	RAFb 13
[11 06 40]	ki.lá	[sig_4].áb	*ṣa-rip-ti*	*11;06 40*	the weight of a	cow-*brick*,	baked	RAFb 14
[22 13 20]	[ki.lá]	[sig_4].al.ùr.ra	ki.2	*22 ;3 20*	the *weight* of a	tile-*brick*,	"	RAFb 15
[12]	[ki.lá]	[sig_4.3"]-*ti*	ki.2	*12*	the *weight* of a	*2/3-brick*,	"	RAFb 16

The weights of *sun-dried* bricks (also known as *mud bricks*) of the same four types are mentioned in § 3' of the same table of constants:

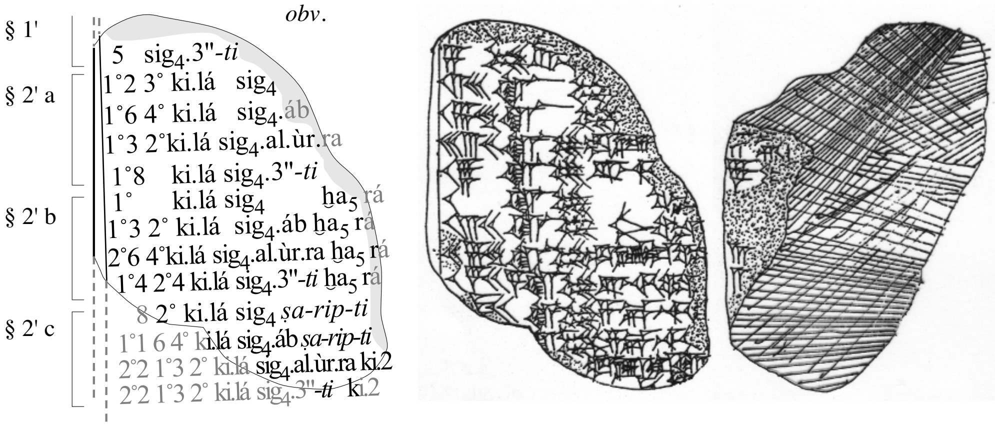
10	ki.lá	sig_4	ḫa$_5$.rá	10	the weight of a	brick,	dried	RAFb 9
13 20	ki.lá	sig_4.[áb]	ḫa$_5$.rá	13;20	the weight of a	*cow*-brick,	dried	RAFb 10
26 40	ki.lá	sig_4.al.ùr.ra	ḫa$_5$.rá	26;40	the weight of a	tile-brick,	dried	RAFb 11
14 24	ki.lá	sig_4.3"-*ti*	ḫa$_5$.rá	14;24	the weight of a	2/3-brick,	dried	RAFb 12

A comparison of the values listed in § 3' and § 4' shows that *the weight of a baked brick was assumed to be 5/6 of the weight of a dried brick*. Similarly, in § 2' of the same table of constants (RAFb 5-8), the weights of [freshly made, *wet*] bricks of the same four types are mentioned. Apparently, *the weight of a baked brick was assumed to be 2/3 of the weight of a wet brick*.

Much is lost of § 1' of BM 36776, but the paragraph can be partly reconstructed as follows:

[7 12]	[sig_4]	[...]	*7;12*	*brick,*	[...]	RAFb 1
[5 24]	[sig_4.áb]	[...]	*5;24*	*cow-brick,*	[...]	RAFb 2
2 42	[sig_4.al.ùr.ra]	[...]	2;42	*tile-brick,*	[...]	RAFb 3
5	sig_4.3"-*ti*	[...]	5	2/3-brick,	[...]	RAFb 4

The four sexagesimal numbers listed in § 1' stand for the *molding numbers* of bricks of the four types R1/2c, H2/3c, S2/3c, and R3n, with the values 7;12, 5;24, 2;42, and 5. They are the reciprocals of the weights of baked bricks of the four types, listed in § 4'.



Hand copy: F. Al-Rawi

Fig. 7.3.2. RAFb = BM 36776. A fragment of a table of constants for four kinds of bricks.

After this excursion into Old Babylonian brick metrology, it will be easier to understand what is going on in the inscription on MS 2221. In col. *ii* of MS 2221 *obv*., the numbers 6, 4 30, and 2 15 are mentioned as parameters for three kinds of bricks, rectangular (*libittu)*, square (*agurru*), and half square (*arḫu*). The same numbers appear also together in a small cluster of entries in the table of constants Kb = **CBS 10996**, which is a post-Old Babylonian, possibly Late Babylonian, copy of an older text (see Friberg, *ChV* (2001), 65):

[6]	^{giš}má.lá	sig$_4$	*6*	heavy boat	brick	Kb	B1
[4 30]	^{giš}má.lá	sig$_4$.áb	*4;30*	heavy boat	cow-brick	Kb	B2
[2] 15	^{giš}má.lá	sig$_4$.al.ùr.ra	2;15	heavy boat	tile-brick	Kb	B3
4 10?	^{giš}má.lá	sig$_4$.3"-*ti*	4;10?	heavy boat	2/3-brick	Kb	B4

Cf. the following entry in the Old Babylonian table of constants NSd = **YBC 5022**:

6	*ša*	^{giš}má.lá	6	of heavy boat	NSd 29

The four types of bricks mentioned in Kb B1-4, presumably R1/2c, H2/3c, S2/3c, and R3n, are the same as those mentioned repeatedly in §§ 1'-4' of BM 36776 above. The listed numbers 6, 4 30, 2 15, 4 10 are 5/6 of the numbers 7 12, 5 24, 2 42, 5 listed in BM 36776 § 1' (the molding numbers). At the same time, they are reciprocals of the numbers 10, 13 20, 26 40, 14 24 listed in BM 36776 § 3' (the weights of dried bricks). Hence, it is conceivable that a paragraph listing these numbers was once inscribed just before § 1' on RAFb = BM 36776, on the lost upper part of that clay tablet.

As for the meaning of this cluster of entries in a table of constants, the term ^{giš}má.lá(Akk. *malallū*) gives little information. The word normally stands for some kind of freight boat. (The translation here, 'heavy boat', is an ad hoc translation, meant to point out that lá is the Sumerian equivalent to Akk. *šaqālu* 'to weigh'.) The word ^{giš}má (Akk. *eleppu)* by itself normally means 'boat', but it appears also, out of context, in a few tables of constants (G = **IM 52916**, NSe = **YBC 7243**, and E = **IM 49949**), apparently with the same meaning there as *nalbanu* 'molding number':

7 12	*i-gi-gu-bu-ša*	*e-le-pu-um*	7;12	its constant	boat	G D 1
5	^{giš}má		5		boat	NSe 19
4 48	^{giš}má	*e-le-pí-im*	4;48	boat	boat	E 12

The numbers 7 12, 5, 4 48 in these entries presumably stand for the molding numbers associated with bricks of the types R1/2c, R3n, and S1/2c.

The fact that ^{giš}má.lá is a variant of ^{giš}má 'boat' with the meaning 'molding number' seems to suggest that ^{giš}má.lá 'heavy boat' refers to molding numbers, 6, 4 30, *etc*., of particularly heavy kinds of bricks, more precisely "variant" bricks that are 6/5 times more massive than bricks with the molding numbers 7 12, 5 24, *etc*.

This is the interpretation proposed in Friberg, *ChV* (2001), 85.

The numbers 6, 4;30, 2;15, 4;10 can also (or instead?) be interpreted as "loading numbers". Cf. the discussion in Robson, *MMTC* (1999) Sec. 5.4. Indeed, just as, incidentally,

1 talent is the weight of 7;12, 5;24, 2;42, and 5 *baked* bricks of the four types R1/2c, H2/3c, S2/3c, and R3n,

so, perhaps more importantly,

1 talent is the weight of 6, 4;30, 2;15, and 4;10 *sun-dried* bricks of the four types R1/2c, H2/3c, S2/3c, and R3n.

In this connection, it may be relevant that 'talent' is the modern translation (with a Greek origin) of the Sumerian word gú or gú.un (Akk. *biltum*) with the general meaning 'load'. Therefore, it is likely that 1 talent (about 30 kg) was deemed, by compassionless Mesopotamian administrators, to be a "man's-load", the weight a man could carry over long periods of time.

If this interpretation of 1 talent as a man's-load is correct, then the meaning of the alleged loading numbers may have been that

6, 4;30, 2;15, and 4;10 *sun-dried* bricks of the four types R1/2c, H2/3c, S2/3c, and R3n constitute a man's-load.

This interpretation is strongly supported by the explicit evidence of the table of constants FM = ***UET* 5, 881** (see the hand copy of the tablet in Friberg, *ChV* (2001), Fig. 3.1):

6	sig_4.si.sá	gú.un.lú.1.e	6	standard-bricks	load-of -1 -man	FM 1
1	sig_4.1.kùš.íb.si_8	gú.un.lú.1.e	1	brick-1-cubit-square	load-of -1 -man	FM 2
2 15	sig_4.3".kùš.ta	gú.un.lú.1.e	2;15	bricks-2/3-cubit-each	load-of -1 -man	FM 3

Line 2 can be interpreted as referring to the following *unitary relation*:

The weight of a *sun-dried* unit brick (type S1c) is 1 talent, that is, 1 man's-load.

The talent as a large unit of weight measure was possibly originally deliberately *defined*, at some time about the middle of the third millennium BC, so that this unitary relation would hold true!

The loading number 6 is mentioned in two entries in the table of constants NSd = YBC 5022:

6	*ma-aš-šu-ú-um* [*ša*] sig_4	6	carrying tray, of bricks	NSd 41

The number '45' appears 4 times in col. *i* of MS 2221, *obv*., on the left edge. This is the "walking number" *muttalliktum* or *tallaktum* (Sum. a.rá), which is known (cf. Robson, *MMTC* (1999), 79; Friberg, *ChV* (2001), 101) to have the value

45 00 ninda/man-day (around 16 km/man-day) of walking with 1 man's-load.

The significance of the numbers 4 30, 3 22 30, 1 41 15 (MS 2221 *obv*., col. *iii*, rows 1-3) is also well known (cf. Robson, *op. cit*., 83; Friberg, *op. cit*., 72). These are the "carrying numbers" (Akk. *nazbalum*) for bricks of types R1/2c, H2/3c, S2/3c, with the values

4 30 00, 3 22 30, and 1 41 15 brick-ninda per man-day.

Here the term 'brick-ninda' stands for '1 brick carried for a distance of 1 ninda').

Until now, it has not been quite clear how these carrying numbers were understood by Old Babylonian mathematicians. The tabular array on MS 2221 *obv*. (including the numbers inscribed on the left edge) makes the situation absolutely clear, at least after an obvious error has been corrected — the words *a-gú-ru* and *ar-ḫu* in lines 2-3 must change places.

The meaning of lines 1-3 is elucidated in the following expanded version of the array:

walking number	loading number	carrying number (work norm)	brick type
45 00 ninda/man-day	6 bricks	4 30 00 brick-ninda/man-day	bricks (type R1/2c)
45 00 ninda/man-day	4;30 bricks	3 22 30 brick-ninda/man-day	cow-bricks (type H2/3c)
45 00 ninda/man-day	2;15 bricks	1 41 15 brick-ninda/man-day	tile-bricks (type S2/3c)

With the 3 first lines of the array in this expanded form, it becomes clear that what takes place here is the *computation of the carrying numbers* (work norms for the carrying of bricks) for three common types of bricks.

Indeed, it is easy to check that

45 00 · 6 = 4 30 00, 45 00 · 4;30 = 3 22 30, and 45 00 · 2;15 = 1 41 15.

What, then, is the meaning of line 4 of the tabular array on MS 2221 *obv*., in which are mentioned the numbers 45, 2 13 20, and 1 40, followed by the word *e-pe-rum* 'mud'? The first number, 45, of course, is again the walking number. The second number, 2 13 20, appears in three different tables of constants as the constant for a *tupšikkum* (Sum. giš.dusu) 'mud basket'. (See Robson, *op. cit*., 78; Friberg, *op. cit*., 101). It can be shown that the value of this number is ;02 13 20 gín (= 1/27 volume shekel), and that it is simply the standardized volume of a basket used to carry mud in. Therefore, line 4 of the array on MS 2221 *obv*. can now be reformulated as:

walking number	loading number	carrying number (work norm)	kind
45 00 ninda/man-day	;02 13 20 gín (volume shekel)	1 40 gín-ninda/man-day	mud

Here the term 'gín-ninda' stands for '1 volume-shekel carried for a distance of 1 ninda'. Thus, this line is devoted to the computation of the carrying number for mud. It is, indeed, easy to check that 45 00 · ;02 13 20 = 1 40, for instance by counting in the following way:

45 00 · ;02 13 20 = 45 · 2;13 20 = 15 · 6;40 = 1 30 + 10 = 1 40.

The interpretation of 1 40 as the carrying number for mud is confirmed by entries in three different tables of constants mentioning 1 40 as the constant for *na-az-ba-al* saḫar 'carrying of mud' (Robson, *op. cit*., 78; Friberg, *op. cit*., 101.)

7.3 b. On Carrying Numbers in Old Babylonian Metro-Mathematical Problem Texts

In the large Old Babylonian metro-mathematical recombination text **Haddad 104** (Al-Rawi and Roaf, *Sumer* 43(1984)), the last problem (# 10) is a "combined work norm problem" concerned with the carrying of mud used for the fabrication of bricks. The part of the text that is solely concerned with the carrying of mud is reproduced below:

Hadddad 104 # 10

1	… … … *iš-tu* 5 *ṣú-up-pa-am za-bi-lam-ma* /	… … … From 5 (ninda), a *ṣuppān*, bring here (mud).
	… … …	… … …
	[*pa*]-*ni* 5 *me-ṣe-tim pu-ṭur-ma* 12 *i-lí*	The opposite of 5, the distance, resolve, 12 comes up.
4	12 *a-na* 45 *mu-ta-lik-tim* / *i-ši-ma* 9 *i-lí*	12 to 45, the walking (number), lift, then 9 comes up.
5	9 *a-na* 2 13 20 *tu-up-ši-ki i-ši-ma* / 20 saḫar	9 to 2 13 20, the basket, lift, then 20, the mud
6	*iš-tu* 5 *ṣú-up-pa-am ta-az-bi-lam i-lí*	(that) from 5, a *ṣuppān*, you carried here, comes up.
	… … …	… … …

In this text, mud is carried for a distance of 5 ninda from the place where it is dug up to the place where it is used for the making of bricks. The walking number 45 00 ninda is found to be 9 00 times this distance. This means that the man carrying the mud can make 9 00 trips with his mud basket in 1 day, the standardized volume of the mud basket being ;02 13 20 volume shekel. Consequently, the work norm of carrying (that is, the carrying number for mud) is evaluated in the given situation as

9 00 · 5 ninda · ;02 13 20 volume shekel = 20 volume shekels · 5 ninda.

Note that, indeed, 20 gín (volume shekels) · 5 ninda is equal to the established carrying number for mud, which is 1 40 gín-ninda. The constant 1 40 itself is not mentioned in the text.

In a similar way, carrying numbers for various types of bricks are never mentioned explicitly in Old Babylonian metro-mathematical problem texts. Four such texts concerned with the carrying of bricks are known at present; they are all discussed in Friberg, *ChV* (2001). The texts in question are **YBC 4673** §§ 2-3 (*ibid.,* 94), **YBC 4669** § B10 (*ibid.,* 113), **YBC 10722** (*ibid.,* 141), and **AO 8862** § 2.2-3 (*ibid.*, 141, 143). All these texts deal exclusively with standard bricks (type R1/2c) and mention the work norm for carrying such bricks only indirectly, as *9 sixties of bricks carried a distance of 30* ninda *(= 3 ropes) in a man-day*. Indeed,

9 00 bricks · 30 ninda = 4 30 00 brick-ninda = the carrying number for bricks of type R1/2c.

Here is a survey of how a carrying distance of 30 ninda is presented in the mentioned texts:

a.na 30 ninda uš / lú.1.e / 9 *šu-ši* sig_4 íl-*ma*	(For) as much as 30 ninda of length, 1 man 9 sixties of bricks he carried, then	YBC 4673 § 2
lú.šidim.e / *i-na* 30 ninda uš / 9 *šu-ši* sig_4 íl.íl-*ma*	A brick worker, from 30 ninda of length, 9 sixties of bricks he kept carrying	YBC 4673 § 3
lú.1.e / 30[!] ninda uš / ḫé.gul.gul. ḫé.íl.íl	1 man, 30 ninda of length, may he keep demolishing and carrying	YBC 4669 § B10
ag-ra-am a-gu-ur-ma / [*i-na* 30 ninda] uš / 9 *šu-ši* sig_4 *iz-bi-lam*	A paid worker I hired, from 30 ninda of length, 9 sixties of bricks he carried here	YBC 10722 (MCT P)
a.na *ša-la-<ša>-aš-li-i* / *iš-te-en a-wi-lu-um* / 9 *šu-ši* sig_4 / *iz-bi-la-am-ma*	As much as three ropes, one man, 9 sixties of bricks he carried here, then	AO 8862 § 2.2
šum-ma a.na *ša-la-ša-aš-li* [*ši*]-*du-um-mi*	If as much as three ropes the length, say,	AO 8862 § 2.3

7.3 c. MS 2221, rev. A Combined Carrying Number Problem for Three Types of Bricks

The numbers on the reverse of MS 2221 are not properly aligned, and a separating ruled line is missing between columns 3 and 4 of the tabular array. (It is likely that the student who wrote the text copied it from someone else's clay tablet, without having a good idea of what was going on.) There are also several half erased numbers from earlier computations on the tablet. It is clear, anyway, that the intended form of the array must have been something like this:

MS 2221 *rev.*

1	12	1 48	9	
1 20	16	1 48	6 45	
2 40	32	1 48	4 22 30	should be 3[!] 22 30

This looks very much like the tabular arrays for combined market rate problems discussed in Sec. 7.2. The similarity would have been even greater if what is now col. *iv* had been placed before the other columns. Anyway, it is not difficult to find a satisfying interpretation of this text.

A good starting point for an interpretation attempt is the assumption that the text on the reverse of MS 2221, just like the text on the obverse, is concerned with carrying numbers for three common types of bricks, rectangular (R1/2c), half square (H2/3c), and square (S2/3c). If that is so, then the number 9 in row 1 of col. *iv* may be interpreted as 9 sixties of bricks (carried over a distance of 30 ninda.) Now, recall that the computation on the obverse showed that the carrying numbers for the three mentioned types of bricks are

4 30 00, 3 22 30, and 1 41 15 brick-ninda.

In agreement with the apparent convention that the typical carrying distance is 30 ninda, these numbers can be expressed in the following, alternative form:

9, 6;45, and, 3;22 30 sixties of bricks, carried over a distance of 30 ninda.

Indeed, just as 4 30 00 = 9 00 · 30, so 3 22 30 = 6 45 · 30, and 1 41 15 = 3;22 30 · 30. Thus, if the carrying distance is supposed to be the typical 30 ninda, then the numbers in col. *iv* of the tabular array above stand for the correspondingly "reduced" carrying numbers for the three types of bricks. The number written '4 22 30' can be assumed to be an error. It should be '3 22 30'.

To find the "combined carrying number" is a "combined work norm problem" of the kind discussed in Friberg, *RlA 7* (1990) Sec. 5.6 h, and Friberg, *ChV* (2001) § 6. The question to be answered is the following:

If three types of bricks have the carrying numbers 9, 6;45, and 3;22 30 sixties of bricks per 30 ninda and man-day, find a number *N* so that *N sixties of bricks of each kind* can be carried 30 ninda in one man-day.

Here is the solution to this problem suggested by the tabular array above: Begin by letting 9 be the “false” value of *N*, that is, assume that a man carries *9 sixties of each kind of brick*. To carry 9 sixties of the first type of bricks, the one with the carrying number 9 (sixties per 30 ninda and man-day), will take him 1 day. To carry 9 sixties of the second type of bricks, the one with the carrying number 6;45 (sixties per 30 ninda and man-day) will take him 1;20 days, since

1; 20 · 6;45 = 6;45 + 2;15 = 9.

Similarly, to carry 9 sixties of the third type of bricks will take him 2;40 days, since

2;40 · 3;22 30 = 6;45 + 2;15 = 9.

Consequently, the numbers in col. *i* of the array are the multiples of 1 day that the man will need to carry 9 sixties of each type of bricks over a distance of 30 ninda. Together, the man will need

1 + 1;20 + 2;40 days = 5 days to carry 9 sixties of each type of bricks (over a distance of 30 ninda).

In 1 day, that is, in 1/5 of that time he will be able to carry

1/5 of 9 sixties = 1;48 sixties = 1 48 of each type of bricks (over a distance of 30 ninda).

The value 1 48 is inscribed in each row of col. *iii* of the array. Finally, in order to carry 1 48 of each type of bricks, the man will need

1/5 · 1	= ;12 days	for the first type of bricks,
1/5 · 1;20	= ;16 days	for the second type of bricks,
1/5 · 2;40	= ;32 days	for the third type of bricks.

Accordingly, the numbers 12, 16, and 32 are inscribed in col. *ii* of the array.

Here is an expanded form of the tabular array, with detailed information about what is counted in each case:

carrying 9 sixties over 30 ninda	1/5 of the time	1/5 of the bricks	carrying numbers	types of bricks
1 man-day	;12 man-day	1 48 bricks	9 sixties · 30 ninda	bricks, type R1/2c
1;20 man-day	;16 man-day	1 48 bricks	6;45 sixties · 30 ninda	cow-bricks, type H2/3c
2;40 man-day	;32 man-day	1 48 bricks	3;22 30 sixties · 30 ninda	tile-bricks, type S2/3c

7.3 d. Other Texts with Combined Work Norm Problems Involving Bricks and Mud

There are no known texts parallel to MS 2221 *obv.* with its calculation of carrying numbers for bricks and mud, or to MS 2221 *rev.* with its combined carrying number problem. On the other hand, several previously published texts with combined work norm problems for bricks and mud can now be better understood, in the light of the discussion above of both the obverse and the reverse of MS 2221. In particular, tabular arrays such as the one on MS 2221 *rev.* turn out to be very useful in presenting the solutions to such problems. Here follows a brief survey:

In **YBC 4669 § B10** (Friberg, *op. cit.*, 113), the vaguely stated problem appears to be the following: A man wrecks an old wall and carries away the bricks for a distance of 30 ninda. The answer given, without an explicit solution algorithm, is that he spends 1/5 of a day wrecking the wall and the remaining 4/5 of the day carrying away the bricks. The omitted solution algorithm could have been given in the form of a tabular array like the one below:

wrecking and carrying 25 volume shekels	1/5 of the time	1/5 of the volume	work norms	type of bricks
1 man-day wrecking	;12 man-day	5 gín	25 gín/man-day for wrecking	any type
4 man-days carrying	;48 man-day	5 gín	6;15 gín/man-day for carrying	the same type

The work norm for wrecking is not given in the text. It must have been well known. The work norm for carrying, *in terms of volume measure*, may have been computed as the product of the volume of, say, one standard brick (type R1/2c) and the carrying number for such bricks:

;00 41 40 gín/brick · 9 00 bricks · 30 ninda = 6;15 gín · 30 ninda.

The sum of the numbers in col. *i* is 5, so that a man will be able to wreck and carry 25 volume shekels in 5 days. Hence, he can wreck and carry 5 volume shekels in 1 day (cols. *ii-iii*).

YBC 4673 § 5 (Robson, *MMTC* (1999), 90-91) mentions 1 man (lú.1.e) repeatedly carrying mud (íl.íl) over a distance of 30 ninda and molding ($du_8.du_8$) bricks. The work norm for carrying mud (the carrying number) is, of course,

1 40 gín · ninda = 3;20 gín · 30 ninda.

Since part of the answer to the stated problem is given (2 40 bricks per man-day), it is possible to count backwards from this answer to find what the work norm for molding bricks from mud must be:

2 40 bricks = ;13 20 (2/9) · 12 00 bricks = ;13 20 brick šar,
the volume of ;13 20 brick šar of regular bricks (R1/2c) is ;13 20 / 7;12 volume šar = ;01 51 06 40 volume šar,
1;51 06 40 volume gín = ;33 20 (5/9) · 3;20 volume gín.

Now, if 5/9 of the day is spent carrying mud, 4/9 of the day is spent molding bricks. Consequently, it is tacitly assumed in this text that *the work norm for molding bricks* is

2;15 (9/4) · 1;51 06 40 volume gín/man-day = 4;10 volume gín/man-day.

In particular,

the work norm for molding regular bricks (R1/2c) is 2;15 (9/4) · 2 40 bricks/man-day =1/2 brick šar/man-day.

It follows that this is the solution algorithm for the stated problem, in the form of a tabular array:

processing 3;20 gín	5/9 of the time	5/9 of 3;20 gín	bricks	work norms	type
1 man-day (íl.íl)	;33 20 man-day	1;51 06 40 gín		3;20 gín · 30 ninda/man-day	mud
;48 man-day ($du_8.du_8$)	;26 40 man-day	1;51 06 40 gín	2 40	4;10 gín/man-day	R1/2c

The necessary computations are the following: The sum of the numbers in col. *i* is 1;48 (= 9/5). The reciprocal of 1;48 is ;33 20 (= 5/9), which is the first entry in col. *ii*. The second entry in col. *ii* is ;48 · ;33 20 = (4/5 · 5/9 = 4/9 =) ;26 40. The numbers in col. *iii* stand for 5/9 of 3;20 volume shekels = 1;51 06 40 volume shekels. The number of bricks in col. *iv* can be computed as the product of the volume (;01 51 06 40 volume šar) and the molding number for standard bricks:

;01 51 06 40 volume šar · 7;12 brick šar/volume šar = ;13 20 brick šar = ;13 20 · 12 00 bricks = 2 40 bricks.

In the problem text **Haddad 104 § 9** (Friberg, *op. cit.*, 110-111; Robson, *op. cit.*, 79), a combined work norm for brick making is computed, through combination of the three work norms for *alli ḫabātim* 'crushing', *alli labānim* 'molding', and *alli balālim* 'mixing'. Without going into details here, the given solution algorithm can be explained in terms of the following array:

work norms		inverted work norms	5 times the time	5 times the volume	bricks made	type
20 gín/man-day	crushing	;03 man-day/ gín	;15 man-day	5 gín	2 15	S2/3cv
20 gín/man-day	molding	;03 man-day/ gín	;15 man-day	5 gín	2 15	S2/3cv
10 gín/man-day	mixing	;06 man-day/ gín	;30 man-day	5 gín	2 15	S2/3cv

Here, the sum of the numbers in col. *ii* is ;12 man-day/gín = 1/5 man-day/gín. In col. *iii*, the times are 5 times bigger, and the sum is 1 man-day. In col. *iv*, the volume is 5 times bigger, that is 5 gín. The numbers in col. *v* are obtained as follows. First, the volume of 1 brick of type S2/3cv is

2/3 cubit · 2/3 cubit · 6 fingers = ;03 20 ninda · ;03 20 ninda · ;12 cubit = ;00 02 13 20 volume šar.

Since 2 13 20 is the reciprocal of 27, it follows that the volume of 1 brick of type S2/3cv is 1/27 of a volume shekel. Hence, the number of bricks of type S2/3cv contained in 5 volume shekels is

5 gín / 1/27 gín/brick = 5 · 27 bricks = 2 15 bricks.

Therefore, the result of the computation is that in this text

the combined wok norm for making bricks is 2 15 bricks (S2/3cv) /man-day, made out of 5 volume gin of mud.

In **Haddad 104 § 10**, a fourth work norm is added, that of carrying mud over a distance of 5 ninda. Since 1 40 gín-ninda/man-day = 20 gín/man-day · 5 ninda, the added work norm is 20 gín/man-day. Hence, the solution algorithm for the new problem may be presented in the form of the following correspondingly expanded tabular array:

work norms		inverted work norms	4 times the time	4 times the volume	bricks made	type
20 gín/man-day	crushing	;03 man-day/gín	;12 man-day	4 gín	1 48	S2/3cv
20 gín/man-day	molding	;03 man-day/gín	;12 man-day	4 gín	1 48	S2/3cv
10 gín/man-day	mixing	;06 man-day/gín	;24 man-day	4 gín	1 48	S2/3cv
20 gín/man-day	carrying	;03 man-day/gín	;12 man-day	4 gín	1 48	S2/3cv

Therefore, in this text,

the combined wok norm for carrying mud and making bricks is 1 48 bricks (S2/3cv) /man-day.

7.3 e. An Early Text with a Brick Problem and Sexagesimal Numbers in Place Value Notation

The small hand tablet ***RTC*, 413** (Fig. 7.3.3 below; Thureau-Dangin 1903; Friberg, *RlA* (1990), Fig. 3) has on its obverse the numerical parameters for a brick problem and on its reverse two sexagesimal numbers in place value notation, written with oversize digits.

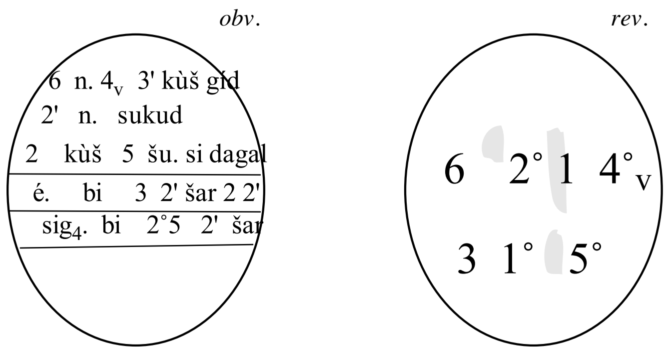


Fig. 7.3.3. *RTC* 413. A hand tablet with a brick problem and sexagesimal numbers in place value notation.

Here is a transliteration and translation of the text on the obverse:

***RTC*, 413**

1	6 ninda 4_v 3' kùš gíd /	6 ninda 4 1/3 cubits the length,
2	2' ninda sukud /	1/2 ninda the height,
3	2 kùš 5 šu.si dagal /	2 cubits 5 fingers the width.
4	é.bi 3 2' šar 2 2' /	Its house (volume) 3 1/2 šar 2 1/2 <gín>.
5	sig_4.bi 25 2' šar	Its bricks 25 1/2 šar.

In spite of its modest appearance, this is a very interesting text. It appears to be a very early Old Babylonian mathematical text. As such it may be the oldest known example of the use of sexagesimal numbers in place value notation in a mathematical cuneiform text.

Apparently, it is silently understood that the object of this exercise is a wall with a triangular cross section, built of standard mud bricks (type R1/2c). The text on the obverse of *RTC*, 413 specifies the linear dimensions of the wall, and then mentions the volume of the wall and the corresponding number of bricks.

The first step of the solution procedure is the following preliminary computation

1. u = 6 n. 4 1/3 c. = 6;21 40 n., s = 2;10 c., h = 1/2 n. = ;30 n., $u \cdot h$ = 3;10 50 sq. n.

This is clearly the explanation for the numbers recorded on the reverse. The second step is the computation of the volume of the wall, which rightly should have been carried out as follows:

2. $u \cdot h \cdot$ 1/2 $\cdot\, s$ = 1; 35 25 sq. n $\cdot$ 2;10 c. = 3;26 44 10 volume šar = 3 1/3 šar 6 2/3 gín 12 1/2 barley-corns.

Instead, however, the student who wrote the hand tablet made a stupid but interesting mistake and calculated as follows:

2*. 1;35 25 sq. n. · 2;10 c. = (1;35 25 · 2 + 2;10 (sic!)· ;10) volume šar
= (3;10 50 + ;21 40) volume šar = 3;32 30 volume šar = 3 1/2 šar 2 1/2 <gín>.

He then used this incorrect result in his final computation of the number of standard rectangular bricks in the wall, multiplying the volume by the molding number 7 12 for bricks of type R1/2 c:

3. 3;32 30 volume šar · 7;12 brick šar/volume šar = 25;30 brick šar = 25 1/2 brick šar.

Remember that an Old Babylonian school boy making this kind of computation could make use of the multiplication table with the head number 7 12, one of the standard head numbers in the Old Babylonian combined multiplication table!

7.4. Inheritance Problems with the Shares Forming a Geometric Progression

7.4 a. MS 2830, obv. A Theme Text with Five Inheritance Problems

MS 2830 (Fig. 7.4.1) is a small clay tablet, inscribed on both sides, but with totally unrelated texts on the two sides. Two tabular arrays on the *reverse* can be shown to be the numerical details of two combined market rate problems with regular market rates. (See Sec. 7.2 a.) The upper part of the *obverse* is badly damaged. Nevertheless, it seems to be clear that this side of the text originally was a small "theme text" with a series of five closely related "inheritance problems". The problems are written in Sumerian, and the number signs used for the numbers 4 and 7 (two wedges over two and four over three) are the kind of variant number signs that appear to be characteristic for early Old Babylonian mathematical texts from the southern part of Mesopotamia.

Actually, MS 2830 has several traits in common with a group of four important mathematical texts in the British Museum from early Old Babylonian Ur, such as the small rectangular format of the clay tablet, the almost exclusive use of Sumerian, and the use of variant number signs.(See Friberg, *R A* 94 (2000).) A plausible conjecture is that MS 2830 and the four texts from Ur have a common origin, although only MS 2830 found its way separately to the open market

The text of §§ 1d-1e is relatively well preserved, and it is obvious that at least these two problems are *variations on a common theme*. The little that remains of the text of §§ 1a-1c seems to make it clear that those three problems are variations on the same theme. It is also clear what that theme is, in spite of the damage to all lines of text on the obverse close to the right edge of the tablet, namely divisions of various amounts of silver between four brothers, with the shares always forming a geometric progression. This interpretation of the problems on the obverse of MS 2830 is supported by the existence of a parallel text, namely the first paragraph of ***UET 5*, 121**, one of the four early Old Babylonian mathematical texts from Ur mentioned above (reproduced in transliteration and translation in Sec. 7.4 b below). Incidentally, the second paragraph of *UET 5*, 121 is a parallel to MS 2317, the small clay tablet with a division problem for the funny number 1 01 01 01, discussed in Sec. 7.1 above.

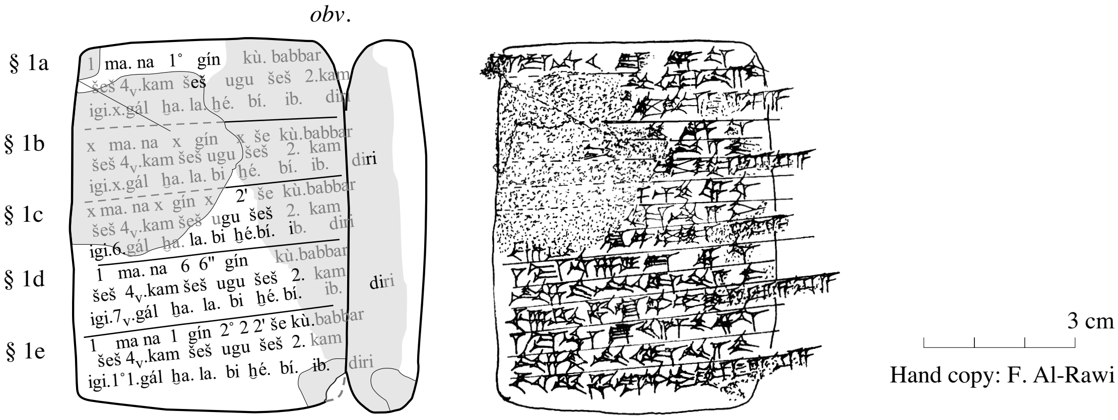


Fig. 7.4.1. MS 2830, *obv*. A theme text with five inheritance problems.

Here follows a transliteration and translation of the text of the five problems on MS 2830, *obv*. As usual, reconstructed parts of the text are marked by straight brackets in the translation, and by italic style in the transliteration.

MS 2830 §§ 1a-1e.

1a	1	[1] ma.na 10 gín [kù.babbar] /	*1* mina 10 shekels *of silver*,
	2	[šeš 4_v.kam š]eš [ugu šeš 2.kam] /	*4 brothers. Brother over 2nd brother*
	3	[igi.x.gál ḫa.la.bi ḫé.bí.ib$^?$.diri]	*(by) the … of its share may it go beyond.*
1b	1	[… ma.na … gín … še kù.babbar] /	*… mina … shekels … barley-corns of silver,*
	2	[šeš 4_v.kam šeš ugu šeš 2.kam] /	*4 brothers. Brother over 2nd brother*
	3	[igi.x.gál ḫa.la.bi ḫé.bí.ib$^?$.di]ri	*(by) the … of its share may it go bey*ond.
1c	1	[… ma.na… g]ín 2' [še kù.babbar] /	*… mina … she*kels 1/2 *barley-corn of silver,*
	2	[šeš 4_v.kam šeš u]gu šeš 2.[kam] /	*4 brothers. Brother o*ver 2*nd* brother
	3	igi.6.[gál ḫa].la ḫé.bí.[ib$^?$.diri]	(by) the 6th-*part of its sh*are may it *go beyond.*
1d	1	1 ma.na 6 6" gín [kù.babba]r /	1 mina 6 5/6 shekel *silver*,
	2	šeš 4_v.kam šeš ugu šeš 2.[kam] /	4 brothers. Brother over 2*nd* brother
	3	igi.7_v.gál ḫa.la.bi ḫé.bí.[ib]$^?$.di[ri]	(by) the 7th-part of its share may it go bey*ond.*
1e	1	1 ma.na 1 gín 22 2' še kù.[babbar] /	1 mina 1 shekel 22 1/2 barley-corn *silver*,
	2	šeš 4_v.kam šeš ugu šeš 2.[kam] /	4 brothers. Brother over 2*nd* brother
	3	igi.11.gál ḫa.la.bi ḫé.bí.ib$^?$.[diri]	(by) the 11th-part of its share may i*t go beyond.*

If MS 2830 § 1 and *UET 5,* 121 § 1 are parallel texts, then the well preserved problem briefly stated in **§ 1e** of MS 2830 can be restated in a more intelligible form as follows:

> Silver amounting to 1 mina 1 shekel 22 1/2 barley-corns (= 1 01;07 30 shekels) is divided among 4 brothers.
> Each brother's share, less the 11th part of that share, equals the next brother's share. Find the 4 shares.

The text of MS 2830 contains neither a solution algorithm nor an answer to this problem, so that it looks like an *assignment* given to a student by his teacher in an Old Babylonian scribe school. However, in his construction of the problem, the teacher must have made himself guilty of a miscalculation. As it is stated, the problem has no simple solution. Instead of 1 mina 1 shekel 22 1/2 barley-corns = 1 01;07 30 shekels, the given amount of silver should have been

1 mina 4 1/3 shekels 22 1/2 barley-corns = 1 04;27 30 shekels.

(An explanation of the error is given below.)

If the problem had been correctly stated, the student could have argued as follows in his solution algorithm:

Let the oldest brother's "false share" be	22 11 (= 11 · 11 · 11).	The 11th part of that is 2 01.
Then the 2nd brother's false share is	22 11 – 2 01 = 20 10.	The 11th part of that is 1 50.
Then the 3rd brother's false share is	20 10 – 1 50 = 18 20.	The 11th part of that is 1 40.
Then the 4th brother's false share is	18 20 – 1 40 = 16 40.	
The "false sum" of the four shares is	22 11 + 20 10 + 18 20 + 16 40 = 1 17 21.	

The given sum is the false sum times ;00 50 shekels (1 17 21 · ;00 50 = 1 04;27 30).
Therefore, the "true shares" must be the false shares times ;00 50 shekels, *etc*.

Note: Instead of starting with the oldest brother's share, going from there to the shares of the younger brothers, the student could just as well have started with the youngest bother's share, going from there to the shares of the older brothers. As the false share of the youngest brother, he could have chosen, for instance, 16 40 = 10 · 10 · 10. Then the next brother's share would have been 16 40 + 16 40 · 1/10 = 16 40 + 1 40 = 18 20, and so on. More about this alternative approach later, in the discussion of MS 1844 in Sec. 7.4.b below.

The problem in **§ 1d** is just as well preserved as the problem in § 1e. It can be reformulated as

Silver amounting to 1 mina 6 5/6 shekels (= 1 06;50 shekels) is divided among 4 brothers.
Each brother's share, less the 7th part of that share, equals the next brother's share. Find the 4 shares.

In this case, too, the problem as it is stated has no simple solution. The given amount of silver, 1 mina 6 5/6 shekel = 1 06;50 shekel, is apparently a mistake for

1/2 mina 6 5/6 shekel = 36;50 shekels.

Indeed, with the initial data corrected in this way, the problem can be solved as follows:

Let the oldest brother's "false share" be	5 43 (= 7 · 7 · 7).	The 7th part of that is 49.
Then the 2nd brother's false share is	5 43 – 49 = 4 54.	The 7th part of that is 42.
Then the 3rd broth's false share is	4 54 – 42 = 4 12.	The 7th part of that is 36.
Then the 4th brother's false share is	4 12 – 36 = 3 36.	
The "false sum" of the four shares is	5 43 + 4 54 + 4 12 + 3 36 = 18 25.	

The given sum is the false sum times ;02 shekels (18 25 · 2 = 36 50).
Therefore, the "true shares" must be the false shares times ;02 shekels, *etc*.

In **§ 1c**, the number specifying the given amount of silver is almost completely obliterated. The problem can be partly reconstructed as follows:

Silver amounting to [… …] 1/2 barley-corns is divided among [4] brothers.
Each brother's share, less the 6th part of that share, equals the next brother's share. Find the [4] shares.

The sign for 6 in *igi.6.gál* in this problem is damaged, but the reading is probably correct, and if it is, then the solution procedure in this case would begin as follows:

Let the oldest brother's "false share" be	3 36 (= 6 · 6 · 6).	The 6th part of that is 36.
Then the 2nd brother's false share is	3 36 – 36 = 3 (00).	The 6th part of that is 30.
Then the 3rd broth's false share is	3 (00) – 30 = 2 30.	The 6th part of that is 25.
Then the 4th brother's false share is	2 30 – 25 = 2 05.	
The "false sum" of the four shares is	3 36 + 3 (00) + 2 30 + 2 05 = 11 11.	

The given amount of silver in § 1c is probably [1 mina x shekels x] 1/2 barley-corn. It is possible that this given sum can be explained as the false sum 11 11 times ;06 30 shekel. Indeed,

11 11 · ;06 30 shekel = (1 07;06 + 5;35 30) shekels = 1 12;41 30 shekels = 1 mina 12 2/3 shekels 3 1/2 barley-corns.

Since the problems in §§ 1c and 1d are of the types 'by the 6th part beyond' and 'by the 7th part beyond', respectively, it seems to be a reasonable conjecture that the problems in §§ 1 a and 1b are of the corresponding types 'by the 4th part beyond' and 'by the 5th part beyond'. This conjecture can be tested. If, in **§ 1a**, a younger brother's share is equal to the next older brother's share, less 1/4 of that share, then the solution procedure for the problem in § 1a would start as follows:

Let the oldest brother's "false share" be	1 04 (= 4 · 4 · 4).	The 4th part of that is 16.
Then the 2nd brother's false share is	1 04 – 16 = 48.	The 4th part of that is 12.
Then the 3rd broth's false share is	48 – 12 = 36.	The 4th part of that is 9.
Then the 4th brother's false share is	36 – 9 = 27.	
The "false sum" of the four shares is	1 04 + 48 + 36 + 27 = 2 55.	

Now, the given amount of silver in § 1a is [x] minas 10 shekels = [xx] 10 shekels. This given sum can be explained as the false sum 2 55 times ;24 shekels. Indeed,

2 55 · ;24 shekels = 1 10 shekels = 1 mina 10 shekels.

Thus, the only problem on the obverse of MS 2830 for which no reconstruction can be suggested is the one in § **1b**, where there are no remaining traces of the data.

Remark: The numerical error in the given amount of silver in § 1d is easy to explain, since the sign for '1/2' in '1/2 mina' is written with a cuneiform sign consisting of a vertical wedge traversed by a horizontal wedge. The one who wrote or copied the text just failed to write the horizontal wedge, with the result that what should have been '1/2 mina' came to look as '1 mina' (a notational error).

The error in § 1e has a more interesting explanation. Apparently, in the construction of the data for the problem, the teacher wanted to compute the given amount of silver, in relative sexagesimal numbers, as 1 17 21 multiplied by 50. Below is shown both the correct computation, to the left, and the actual, incorrect computation, to the right:

1 17 21 · 50 =	50	1 17 21 · 50 =	50
	8 20		**5**
	5 50		5 50
	16 40		16 40
	50		50
	1 04 27 30		1 01 07 30

According to this reconstruction, the origin of the error was the incorrect multiplication 10 · 50 = 5 hundred = 5 (00), instead of 10 · 50 = 5 hundred = 8 20. In other words, what caused the error was that the one who performed the calculation was thinking in terms of decimal numbers when he was supposed to count with sexagesimal numbers. (It is well known that in the Akkadian language number words were decimal. In everyday life in Mesopotamia, decimal numbers were used for counting. Only well educated scribes knew how to count with sexagesimal numbers.)

7.4 b. MS 1844. A Lentil with the Solution Algorithm for an Inheritance Problem

MS 1844 (Fig. 7.4.2 below) is a round hand tablet. With its diameter of 11 cm and thickness of 3.5 cm it is easily the biggest and most massive of more than ten mathematical or metrological round hand tablets in the Schøyen Collection. The reverse of MS 1844 is blank, while the obverse of the tablet is inscribed with eight lines of sexagesimal numbers in place value notation, separated by ruled lines, and followed by one line with a brief subscript. The number of sexagesimal places (double digits) in the recorded numbers decreases from the first line to the eighth, a clear indication that the text is some kind of algorithm table. The first, and longest, number is written as four sexagesimal places followed, somewhat lower down, by four additional sexagesimal places. There is a numerical error in the number recorded in line 3. It is easy to check (as will be done below) that this error is *propagated upwards*, to the numbers recorded in lines 1 and 2. This means that the numbers in the algorithm table were computed *in reverse order*, beginning with the number '2' in line 8.

7.4 c. The Terms of a Geometric Progression

A transliteration of the text of MS 1844 is given below, to the left. Errors are indicated by bold script. Two simple *computational* or *notational* errors are 27 instead of 22 in line 3, and 15 instead of 18 in line 1. The remaining errors in lines 1 and 2 are caused by the error in line 3. To the right is a second copy of the algorithm table, with the errors corrected. The incorrect digits to the left are indicated by bold script. A key to the reconstruction of the table is the observation that the numbers in the successive lines of the table, counted *upwards* from line 8, were intended to form a geometric progression with the ratio between successive terms equal to 1 10 in relative place value notation.

Line	**The algorithm table on MS 1844.**	The same text, corrected.
1	23 1**5 20 36**	23 18 09 48
	1**2** 08 53 20	10 08 53 20
2	5 02 **41 3**2 35 33 20	5 02 35 42 35 33 20
3	4 19 2**7** 02 13 20	4 19 22 02 13 20
4	3 42 18 53 20	3 42 18 53 20
5	3 10 33 20	3 10 33 20
6	2 43 20	2 43 20
7	2 20	2 20
8	2	2
9	igi.7.gál.bi tur.šè	igi.7.gál.bi tur.šè

The subscript can probably be translated as 'for small by the 7th part' (more about this below), and may be interpreted as meaning that *each number in the table is equal to the number in the preceding line, diminished by 1/7 of its value*. (In modern notation, each number is 1 – 1/7 = 6/7 times the number in the preceding line.) Now, since 7 is not a regular sexagesimal number, Babylonian mathematicians would have found it difficult to *count* with a number like 1/7. On the other hand, it is known through explicit examples that they were well aware of a counting rule of the type

$$\text{if} \quad b = a - a \cdot 1/7, \quad \text{then} \quad a = b + b \cdot 1/6.$$

The correctness of this counting rule is obvious, at least in the case when a is a multiple of 7. Indeed,

$$\text{if} \quad a = n \cdot 7 \quad \text{and} \quad b = a - a \cdot 1/7, \quad \text{then} \quad b = n \cdot 7 - n = n \cdot 6, \quad \text{and} \quad b + b \cdot 1/6 = n \cdot 6 + n = a.$$

In view of this simple counting rule, the requirement that each number in the table shall be equal to the number in the preceding line, diminished by 1/7 of its value, can be replaced by the equivalent requirement that *each number in the table shall be equal to the number below it, increased by 1/6 of its value*. This reformulation of the requirement is a great simplification, since 6 is a regular sexagesimal number with the reciprocal 10. Therefore, increasing a given number by 1/6 of its value is equivalent to multiplying the given number by 1 10 in Babylonian *relative* place value notation, or by 1;10 in *absolute* place value notation.

This means that it was easier for the author of the text to count upwards in the algorithm table, multiplying with the factor '1 10' than to count downwards, subtracting seventh parts. That is also precisely what he did. Here follows a reconstruction of the way in which all the numbers in the algorithm table were computed, beginning with the number '2' in line 8. (An attempted explanation of the meaning of these successively computed numbers will be given below.)

2 · 1 10	=	2	+	20	=	2 20	(line 7)
2 20 · 1 10	=	2 20	+	23 20	=	2 43 20	(line 6)
2 43 20 · 1 10	=	2 43 20	+	27 13 20	=	3 10 33 20	(line 5)
3 10 33 20 · 1 10	=	3 10 33 20	+	31 45 33 20	=	3 42 18 53 20	(line 4)
3 42 18 53 20 · 1 10	=	3 42 18 53 20	+	37 03 08 53 20	=	4 19 22 02 13 20	(line 3)
4 19 22 02 13 20 · 1 10	=	4 19 22 02 13 20	+	43 13 40 22 13 20	=	5 02 35 42 35 33 20	(line 2)

Clearly, the seven numbers beginning with the number in line 8 and ending with the number in line 2 form a *geometric progression* with the *first term* 2 and the *factor* 1 10.

In the text, the mistake in line 3 (2**7** instead of 2**2**) is propagated to line 2 above it in the following way (incorrect digits in bold type):

4 19 2**7** 02 13 20 · 1 10 = 4 19 2**7** 02 13 20 + 43 1**4 3**0 22 13 20 = 5 02 **41 3**2 35 33 20.

The error in line 1, where the sum of the numbers in lines 2-8 is recorded, can be explained as the sum of the errors in lines 3 and 2 (more about this below).

The original error in line 3 is easy to explain. It is likely that the computation of 3 42 18 53 20 times 1 10 was set up on a counting board (or on another clay tablet) in the following way:

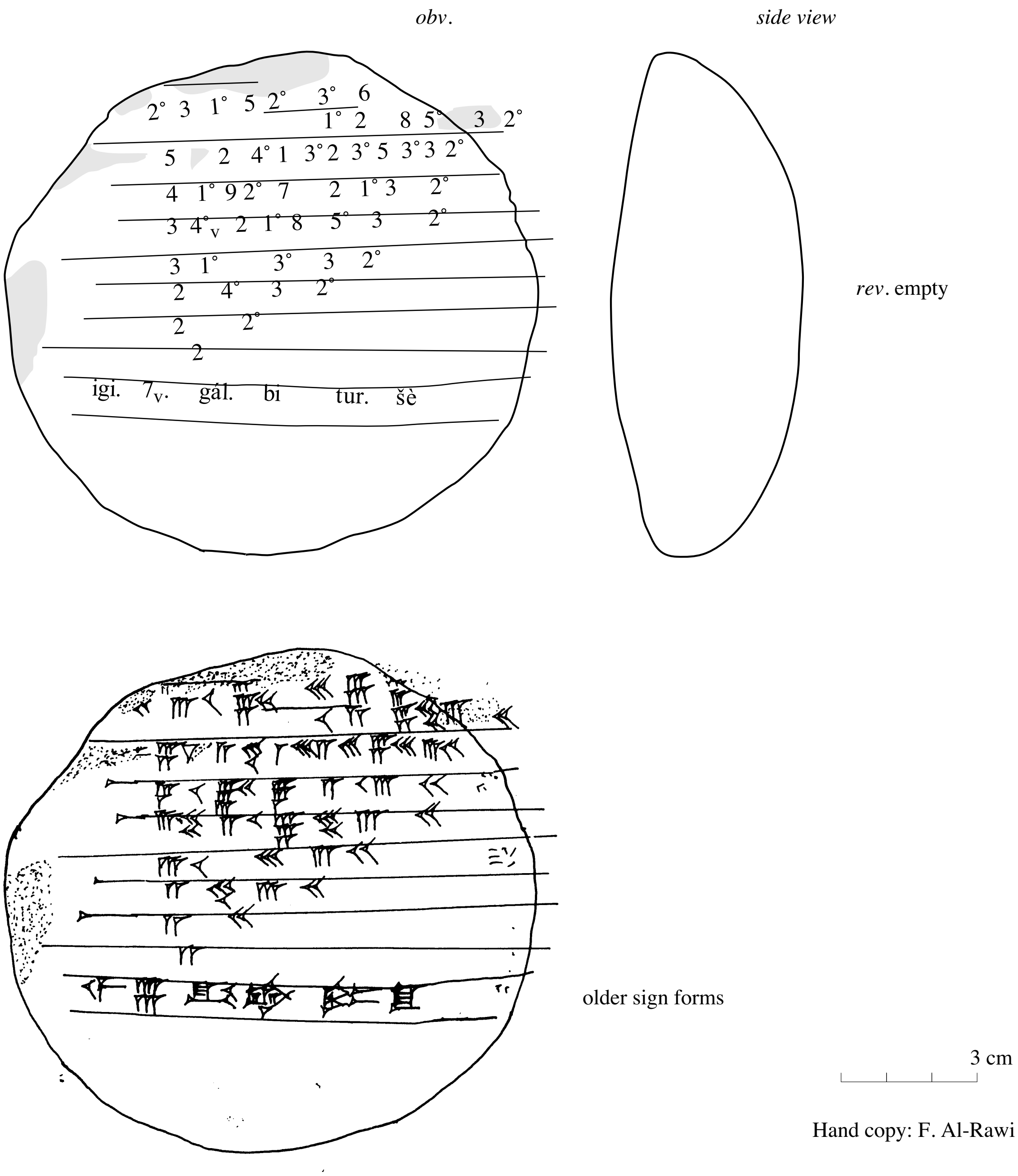


Fig. 7.4.2. MS 1844. A round hand tablet with the numerical solution algorithm for an inheritance problem.

1   1 1
3 42 18 53 20
+ 37 03 08 53 20
4 19 22 02 13 20

If this was done in a careless way, with poor vertical aligning of corresponding sexagesimal places, the correct addition 1+18 + 3 = 22 may have been replaced by the incorrect addition 1+ 18 + 8 = 27.

7.4 d. The Sum of the Geometric Progression

After the author of the text had computed the seven terms of the geometric progression, it would have been easy for him to compute the sum of the seven terms as follows (addition upwards):

23 18 **20 4**8 08 53 20
5 02 **41 3**2 35 33 20
4 19 2**7** 02 13 20
3 42 18 53 20
3 10 33 20
2 43 20
2 20
2

However, this cannot be the way he did it, since in the text the expected sum 23 18 20 48 08 53 20 is split in two parts, and is written as 23 1**5** 20 36 + 12 08 53 20. (Here, 1**5** instead of 18 is a simple notational error. The cuneiform signs for 5 and 8 can easily be mistaken for each other).

The question now is *why* the sum in line 1 should be split up in this way. The most likely explanation is that the author of MS 1844 had in mind the application of the algorithm to the solution of a specific problem (see below), and that it was not meaningful to use a more than four-place sexagesimal number in the final answer to that problem. Therefore, *on his counting board* (or whatever medium he used for his calculations) he may have prepared a *round-off* by writing each more than three-place term of the geometric progression as *the sum of a four-place number and a remainder*. Moreover, this was done in such a way that, in each case, *the last place of the four-place number would be divisible by the factor 6*. The reason for this peculiar kind of round-off is that the scribe made preparations for the division by 6 which was one of the recurrent steps in his computation of the successive terms of the progression. Thus, in line 5 the computed number 3 10 33 20 was rounded off to 3 10 33 18, with the remainder 2. Similarly, in line 4, the computed number 3 42 18 53 20 was rounded off to the four-place number 3 42 18 48, remainder 5 20. In line 3, with its small computational error, the corresponding round-off was 4 19 2**7** 02 13 20 = 4 19 2**7** + 2 13 20. And so on. Below is shown how the sum of the seven terms in the geometric progression may have been computed by separate addition of the four-place rounded numbers and of the remainders, respectively.

23 1**5 20 36**	+**12** 08 53 20	or, without the errors,	23 18 09 48	+ 10 08 53 20
5 02 **41 30**	+ **2** 35 33 20		5 02 35 42	+ 35 33 20
4 19 2**7**(00)	+ 2 13 20		4 19 22(00)	+ 2 13 20
3 42 18 48	+ 5 20		3 42 18 48	+ 5 20
3 10 33 18	+ 2		3 10 33 18	+ 2
2 43 20			2 43 20	
2 20			2 20	
2			2	

It is not clear why, after this computation of the sum, the four-place round-off and the corresponding remainder was rejoined in each of the seven terms of the progression.[1]

7.4 e. The Intended Application of the Algorithm

Inheritance problems, alternatively *division of property problems*, are usually formulated in the following way in Old Babylonian mathematical cuneiform texts: A given number of brothers, or partners, are required to share a given amount of silver, or a given piece of land, in unequal shares. The younger brothers, or the junior

1. A different interpretation of the data in the text is that there were no round-offs, and that the sexagesimal number in the first line was broken up in two parts only because there was not space enough on the obverse of the tablet to write the whole number in a single sequence of digits. As a matter of fact, the last two places of 12 08 53 20 are written around the edge of the tablet. However, this alternative interpretation is not very likely, because it does not explain why the fourth place of the number, which ought to be 48, was broken up in two parts and written as 36 with 12 immediately beneath.

partners, are supposed to get progressively smaller shares. More precisely, the shares are supposed to form a decreasing arithmetical or geometric progression, or some similar gradually decreasing set of weight or area numbers. An example is, of course, the theme text MS 2830, *obv.* (Fig. 7.4.1) with its series of five problems in which four brothers share given amounts of silver, with their shares forming geometric progressions.

In the case of the present text, MS 1844, the (corrected) data seem to constitute the numerical solution to an inheritance problem which (as we shall see) can be stated in the following way:

> Seven brothers, 23 minas 18 1/6 shekels of silver.
> Each brother's share, minus a 7th of that share, equals the next brother's share. Find the 7 shares.

In line with this interpretation, it is suggested here that the obscure sentence in the last line of the text refers to the circumstance that each share is 'a 7th less' than the preceding share.[2] The syntax of the sentence is peculiar, but maybe a clue to the correct interpretation is offered by the subscripts of three early Old Babylonian metrological tables for length measure, ***UET 7*, 114-115** and **BM 92698**, two from Ur and one from Larsa (mentioned already in Sec. 3.4 above, and shown in App.5, Figs. A5.3-4). In BM 92698, for instance, two sub-tables have the following subscripts in Sumerian:

nam.uš.sag aša$_5$.šè	(to be used) for lengths and fronts of fields (areas) in general
nam.sukud.bùr.saḫar.šè	(to be used) for heights and depths of mud (volumes) in general.

These subscripts indicate the *purpose* of the two sub-tables: one is to be used for *horizontal* length measures, the other for *vertical* length measures. In a similar way, it may be assumed that the subscript following the algorithm table on MS 1844 indicates that *the purpose* of the algorithm text was that it should be used for the construction or solution of inheritance problems where the share of one brother is less by a seventh than the share of the preceding brother. Thus, it is proposed here that the name of problems of this type was igi.7.gál.bi tur 'small by the 7th part', and that the subscript should be translated as

igi.7.gál.bi tur.šè	(to be used) for 'small by the 7th part'

The conjectured explanation of the intended application of the algorithm of MS 1844 will not be complete until an effort has been made to make clear what the share of the youngest brother actually was meant to be. In the text it is given just as '2', in *relative* place value notation and without any indication of the *chosen unit of measure*. Now, if the algorithm was to be used for the solution of an inheritance problem, the shares would be amounts of silver, expressed as multiples or fractions of 1 mina. In that case, the most plausible interpretation of the relative number '2' is that it stands for' 2 minas'. The computed shares of the six other brother would then be the following, with the errors corrected, and with the sexagesimal numbers conveniently rounded off:

5;02 35 40 minas	=	5 minas 2 1/2 shekels 17 barley-corns
4;19 22 00 minas	=	4 minas 19 1/3 shekels 6 barley-corns
3;42 18 50 minas	=	3 minas 42 1/6 shekels 26 1/2 barley-corns
3;10 33 20 minas	=	3 minas 10 1/2 shekels 10 barley-corns
2;43 20 minas	=	2 minas 43 1/3 shekels
2;20 minas	=	2 minas 20 shekels
2 minas		

A rounding off to multiples of ;00 00 10 mina makes sense, in view of the fact that the smallest weight measure occurring in Old Babylonian metrological tables for system *M* (Sec. 3.2) is 1/2 barley-corn = ;00 00 10 mina.

The sum of the shares computed in this way would be

23;18 09 50 minas=23 minas 18 shekels 29 1/2 barley-corns.

This result can be compared with the exact sum of the 7 shares, which is

23;18 09 58 08 53 20 minas.

Thus, the sum of the rounded shares and the exact sum of the shares are both very good approximations to the

2. Two more or less parallel situations are described in **BM 13901 # 10** (Neugebauer, *MKT 3*, 2), with the Akkadian phrase *mi-it-ḫar-tum a-na mi-it-ḫar-tim si-bi-a-tim im-ṭi* '(one) square side is less than (the other) square side by a seventh', and in **YBC 4714 § 6** (*MKT 1*, 487), with the Sumerian equivalent of that phrase: íb.si$_8$ íb.si$_8$.ra igi.7.gál ba.lá.

(fairly) round weight number

23;18 10 minas=23 minas 18 1/6 shekels.

For this reason, it is plausible that 23 minas 18 1/6 shekels was the given sum of the 7 shares. Apparently, the student who got as assignment to compute the solution to this inheritance problem of the type 'small by the 7th part' and the given sum of the 7 shares, seems to have started his computation by *cleverly guessing the correct size of the share of the youngest brother*. The way he reasoned may have been as follows: If the sum of the shares is about 23 minas, then the average share is about 3 minas. The smallest share must be less than the average share, so it is reasonable to assume that it is 2 minas.

Note: There is no known parallel in the corpus of Old Babylonian mathematics to this way of rounding off many-place sexagesimal numbers.

7.4 f. UET 5, 121. A Parallel Text from Early Old Babylonian Ur

The algorithm text MS 1844 was interpreted above as the numerical solution procedure for an inheritance problem of the type 'small by the 7th part' for *seven* brothers. Similarly, the five problems stated in the damaged theme text MS 2830, *obv*., were interpreted as inheritance problems for *four* brothers, all of the type 'by the *n*th part beyond', *n* equal to [4], [5], 6, 7, 11. These proposed interpretations are supported by the fact that there exists a known early Old Babylonian mathematical cuneiform text, written in Sumerian, in which the given problem is *explicitly* stated as an inheritance problem of the type "by the 5th part beyond" for *five* brothers. The text in question is Figulla and Martin, ***UET 5,* 121**, one of the mathematical texts from Ur, reproduced below in transliteration (cf. Friberg, *RA* 94 (2000) § 4a):

***UET 5,* 121 § 1. An inheritance problem of the type "by the 5th part beyond" for five brothers**.

1	26 ma.na 15 3" gín 15 [še] [kù.bab]bar /	26 minas 15 2/3 shekels 15 barley-corns of silver.
2	dumu.nita.bi 5	Its heirs are 5.
	šeš.gal šeš dumu.nita /	The big brother <over the next> brother-heir
3	igi.5.gál ḫa.la.kam ḫé.ib.diri /	<by> a 5th of the share may he be beyond.
4	šeš.gal.e	The big brother:
	7 3" ma.na 8 3" gín 15 še /	7 2/3 minas 8 2/3 shekels 15 barley-corns.
5	šeš.2.kam 6 ma.na 5 gín /	The 2nd brother: 6 minas 5 shekels.
6	šeš.3.kam 5 ma.na /	The 3rd brother: 5 minas.
7	šeš.4.kam 4 ma.na /	The 4th brother: 4 minas.
8	šeš.5.kam 3 ma.na 12 gín	The 5th brother: 3 minas 12 shekels.

It is interesting to see that the structure of this text is in a sense complementary to the structure of the algorithm text MS 1844. While the algorithm text consists exclusively of the *numerical solution procedure* for an inheritance problem of a certain kind, *UET 5* 121 consists exclusively of the *statement of the problem* and the *answer* to an inheritance problem of the same kind. The stated problem in the case of *UET 5* 121 can be formulated as follows:

Five heirs, 26 minas 15 2/3 shekels 15 barley-corns of silver.
The share of the older brother, less a 5th of that share, equals the share of the younger brother.
Find the 5 shares.

The solution procedure for this problem would proceed as follows, by use of the method of false value:

Let the oldest brother's false share be	(5 · 5 · 5 · 5 · 5 =) 10 25.	The 5th part of that is 2 05.
Then the 2nd brother's false share is	10 25 - 2 05 = 8 20.	The 5th part of that is 1 40.
Then the 3rd broth's false share is	8 20 - 1 40 = 6 40.	The 5th part of that is 1 20.
Then the 4th brother's false share is	6 40 - 1 20 = 5 20.	The 5th part of that is 1 04.
Then the 5th brother's false share is	5 20 - 1 04 = 4 16.	
The "false sum" of the five shares is	10 25 + 8 20 + 6 40 + 5 20 + 4 16 = 35 01.	

The given sum is ;26 minas 15 2/3 shekels 15 barley-corns = 26 15;45 shekels.
Thus, the given sum is the false sum times ;45 shekel (35 01 · ;45 = 26 15;45).
Therefore, the "true shares" must be the false shares times ;45 shekel, *etc*.