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How to Get a Better Understanding of Mathematical Cuneiform Texts

0.1. On Avoiding Anachronisms in Translations of Mathematical Terms

The terminology used in modern elementary mathematics has a mixed origin, which may be characterized as Greek/Latin/early modern. The terminology used in Babylonian mathematical cuneiform texts, on the other hand, is pre-Greek. Therefore, the use of Greek and Latin words in a discussion of a mathematical cuneiform text is in itself an anachronism. If one wants to convey a proper understanding of the essence of Mesopotamian mathematics, one should try to avoid anachronisms by using *literal translations* of technical terms in cuneiform mathematical texts, whenever possible.

A clear example is the seemingly self-evident term *triangle*. It is derived from the Latin word *triangulum* which, like its Greek predecessor τρίγωνον, means 'three-cornered' or 'three-angled'. However, the idea of an angle between two sides in a rectilinear plane figure can be traced no further back than to classical Greek geometry. More specifically, in classical Greek geometry, a triangle was referred to as 'the triangle ABC' where A, B, C are the three corners of the triangle. In Old Babylonian geometry, on the other hand, a triangle was always specified in terms of the lengths of two or three of its sides, never in terms of its angles or the positions of its corners. The term used for 'triangle' was invariably the Sumerian word sag.kak, possibly with the literal meaning 'peg-head'. (According to the dictionaries, the corresponding Akkadian (i. e. Babylonian) word is *santakkum* 'wedge', but there is not necessarily an actual connection between the two words.) Other examples of modern mathematical terms, in the Greek tradition referring to angles or corners, are *rectangle* 'right-angled', *diagonal* 'across corners', and *pentagon* 'five-cornered', *hexagon* 'six-cornered', *heptagon* 'seven-cornered', for the latter of which the Old Babylonian names were sag.5 '5-front', sag.6 '6-front', sag.7 '7-front'.

Since the concept of angles was unknown (or, at least, never explicitly mentioned) in Old Babylonian mathematics, the concept *right-angled* was also unknown. It is likely that what we call a right-angled triangle was thought of as one of the two triangles into which a rectangle can be divided by an oblique transversal of maximal length. Perhaps a better word for such a triangle is simply a "right triangle". The term *ṣiliptum* used for such a transversal in a rectangle was derived from a verb meaning 'to cross over', while the rectangle itself was called simply uš sag, where uš 'length' (š = sh) is the term used for the long side of a rectangle, and sag 'head, front' the term used for the short side.

The modern terms 'square' and 'quadratic' are both derived from the Latin word *quadratum* 'fourish', probably derived from the Greek word τετράγωνον 'four-angled'. The terms normally used in Old Babylonian mathematical texts are the Sumerian íb.si$_8$ or the Akkadian *mitḫartum*, both derived from verbs with the meaning 'to be equal'. Another term, documented for the first time in MS 5112 (Sec. 11.2 below) is téš.a.sì, possibly with the literal meaning 'given together' or 'given equal'. A well known peculiarity of the Old Babylonian mathematical terminology is that all these words can denote both a square figure and a side of such a figure, depending on the context. As a reminder of this fact, and for lack of better alternatives, "equalside" is suggested here as a fairly literal translation of íb.si$_8$/*mitḫartum* and "sameside" as a translation of téš.a.sì.

It is clearly important that (more or less) *literal translations should be used for Babylonian or Sumerian mathematical terms*. One direct consequence of this principle is that different translations should be used for different words in the original texts, even when they correspond to the same word in modern mathematical terminology. It was this simple idea that led Jens Høyrup to the surprising observation that Old Babylonian mathematicians distinguished between different kinds of addition (in particular, *joining together* two equally important entities, as opposed to *adding* one entity to another, more important entity). Old Babylonian mathematicians also distinguished between different kinds of multiplication, *etc*. This observation, in its turn, led Høyrup to the important discovery that geometric models play a much greater role than earlier realized in Old Babylonian problems for squares and rectangles. The idea has been further developed by Høyrup in a long series of publications since 1982. See, for instance, Høyrup, *AoF* 17 (1990), *HSc* 34 (1996), and *LWS* (2002). See also the related discussion of "metric algebra" problems in the theme text MS 5112 in Sec. 11.2 below.

0.2. Conform Transliterations, Translations, and Interpretations

The idea of "conform" transliterations and translations of mathematical cuneiform texts was developed by the present author in the early 1980's as a reaction to the extremely reader-unfriendly way in which Babylonian mathematics was presented by Otto Neugebauer in the classical works *MKT = Mathematische Keilschrift-Texte* I-III (1935-37), by François Thureau-Dangin in *TMB = Textes mathématiques babyloniens* (1938), and by Otto Neugebauer together with Abraham Sachs in *MCT = Mathematical Cuneiform Texts* (1945). In *TMB,* for instance, all Sumerian words in the mathematical cuneiform texts were interpreted as "sumerograms" and replaced by the Akkadian (Babylonian) words that they were assumed to represent. For mathematically oriented readers trying to compare transliterations of mathematical cuneiform texts *sign by sign* with hand copies of the original texts, this made the task much more difficult than it needed to be. In all three of the classical works, transliterations of mathematical cuneiform texts were followed by translations into *standard* German, French, or English. For readers trying to compare the transliterations *word by word* with the translations, this, too, made the task much more difficult than it needed to be.

The following example of a *conform translations*, word by word (and sentence by sentence), is taken from MS 5112 § 2 b (Sec. 11.2 d below) :

1	a.šà 2 téš.a.sì gar.gar-*ma* 21 40	The fields of 2 samesides (I) heaped, then 21 40.
	téš.a.sì gar.gar-*ma* 50 /	The samesides (I) heaped, then 50.
2	téš.a.sì.meš en.nam	The samesides (are) what?

A translation of the same passage into standard English would be like this:

I added the areas of 2 squares: 21 40. I added the sides of the squares: 50. What are the sides of the squares?

Here is another example, taken from MS 5112 § 9 (Sec. 11.2 l):

1	uš sag gu$_7$.gu$_7$-*ma* 50 a.šà	The length (and) the front (I made) eat (each other), then 50, the field.
	i-na uš 30 ninda zi-*ma*	From the length 30 ninda (I) tore out, then
2	*a-na* sag / 5 ninda daḫ-*ma* 50 a.šà	to the front 5 ninda (I) added, then 50, the field.
	uš sag en.nam	The length (and) the front (are) what?

In this case, a translation to standard English would be like this:

I multiplied the length and the width: area 50. I subtracted 30 ninda from the length,
and I added 5 ninda to the width: area 50. What are the length and the width?

There are several differences between conform translations and translations into standard English. The most obvious difference is that in the conform translations the word order is (essentially) the same as in the transliterated texts. This should make it possible even for readers who know little Sumerian or Akkadian to connect most of the words in the transliterations with their translations.

Another difference is that in the conform translations an effort has been made to let the translations of individual words be as close as possible to the normal (non-mathematical) meanings of the words in the original texts. Note, in particular, that in the first of the examples above the term used for an addition (of the areas of two squares) is gar.gar '(make a) heap', while in the second example a different term is used for an addition (of an extra 5 ninda to the front), namely daḫ 'add'. Note also the peculiar use in the first example of the term téš.a.sì to denote both two squares (of which the areas are added together) and the sides of those squares (which are also added together).

The un-English word order and the strange-looking words in a conform translation should be no great obstacles to the average reader. After a little while the inverted word order will be familiar and the unusual words will be well known.[1] The trouble caused by the conform translation may even give the reader a pleasurable feeling of getting closer to the original language of the mathematical cuneiform text!

A *conform transliteration* is one where the cuneiform signs in the original text have been transliterated directly, without any attempt to replace sumerograms by the Akkadian words that they can be assumed to represent. In transliterations in this work, Sumerian words or sumerograms are written with plain letters with extra spacing, while Akkadian words are written with Italic letters. In order to emphasize the difference, dots are used the separate from each other the components of composite Sumerian words, while dashes are used to separate the syllables of Akkadian words. (The common practice of letting capital letters indicate uncertain readings of parts of Sumerian words is ignored here.)

Sometimes it is desirable to find a form for a conform transliteration which is even closer to the form of the original text. This is the case, in particular, when the text is damaged so that it becomes important to make clear where the damaged parts of the text are located and how great the chance is of being able to make a credible reconstruction of lost or damaged words. It is also often of interest to have a transliteration which clearly shows how the original text was structured, how the layout of the text on the clay tablet was organized, *etc*. In all such cases, what is needed is *a conform transliteration within an outline of the clay tablet*. See, for instance, the examples of such conform transliterations, side by side with hand copies of simple clay tablets with arithmetical exercises in Fig. 1.1.1 below, or the examples of conform transliterations within the outlines of large clay tablets with mathematical problem texts in Figs. 10.1.1, 10.2.2, 10.3.1, *etc*.

A final and crucial step in the complete presentation of a new mathematical cuneiform text is the interpretation of the text. In this work it will be attempted to give what may be called "conform interpretations", disclosing as clearly as possible the intentions of the writers of the texts, and explaining as faithfully as possible the methods they used to construct and solve the stated problems in the larger collections of exercises. Although normal language and normal mathematical terminology will be used to a great extent in these interpretations, also here an effort will be made to avoid possible anachronisms. In particular, no use will be made of modern abstract notations such as x, y, and z, or D_1, D_2, R_1, R_2, *etc*. Instead, easily remembered acrophonic abbreviations will be used, such as u and s for uš 'length' and sag 'front', or h, A, V for heights, areas, and volumes. Squares and square roots will never be expressed by use of exponents and square root signs. Instead the square of a number a will be written sq. a, and square roots, or rather *square sides*, will be written as sqs. b. Solutions to intricate mathematical problems will never be presented in the form of complicated mathematical formulas. Instead, *the successive steps of complicated solution procedures will be presented one by one* in "quasi-modern notations" that are only mildly anachronistic.

0.3. Babylonian Sexagesimal Numbers

0.3 a. Sexagesimal Numbers

1. E. Robson has less confidence in her own or her readers' ability to get past such minor obstacles. In her book *MMTC* (1999), 5, she writes: "Any of these translations have their own merits and demerits: it is almost impossible to find a satisfactory translation for any Old Babylonian mathematical word in modern English, as the concepts behind them are so different from ours. Friberg and Høyrup have each tried to overcome this difficulty in recent years by inventing a new mathematical vocabulary. The exercise, although well intentioned, has been to a great deal self-defeating, as one then has to translate a jargon-filled and visually unattractive 'conform translation' into standard English to understand it."

Every given *integer* (whole number) a can be written in the form

$$a = a_{n-1} \cdot 60^{n-1} + \cdots + a_2 \cdot 60^2 + a_1 \cdot 60^1 + a_0,$$

where n is a sufficiently large whole number, and where

all the n "double digits" or "sexagesimal places" $a_{n-1}, ..., a_2, a_1, a_0$ are whole numbers between 1 and 59.

In *sexagesimal place value notation*, an integer of this form is expressed more compactly as

$$a = a_{n-1} \ldots a_2\, a_1\, a_0.$$

Take, for instance, the large decimal number 1,000,000 (a million). It can be *converted into a sexagesimal number* (= a number in sexagesimal place value notation) in the following way:

1,000,000/60 = 16,666.66..., **16,666**.66.../60 = 277.77833..., **277**.77833.../60 = **4**.62963... .

Consequently, $n = 4$ and the first sexagesimal place is 4. Now, subtract 4 and multiply by 60:

0.62963... · 60 = **37**.7778333... .

Therefore, the second sexagesimal place is 37. Subtract 37 and multiply by 60:

0.7778333... · 60 = **46**.66... .

Thus, the third sexagesimal place is 46. Now, subtract 46 and multiply by 60:

0.66... · 60 = **40**.

This shows that the fourth and final sexagesimal place is 4. Therefore, the computation shows that

1,000,000 = 4 37 46 40.

Conversely, the sexagesimal number 4 37 46 40 can be *converted into a decimal number* (= a number in decimal place value notation) most easily by use of the following systematic procedure:

4 37 =	**4** · 60 + **37** = 240 + 37 = **277**,
4 37 46 =	**277** · 60 + **46** = 16,620 + 46 = **16,666**,
4 37 46 40 =	**16,666** · 60 + **40** = 999,960 + 40 = **1,000,000**.

The methods used in this couple of examples can be used generally to convert given decimal numbers into sexagesimal numbers or, conversely, given sexagesimal numbers into decimal numbers.

Sexagesimal place value notation can be used also for *fractions* or for numbers with both an integral and a fractional part. A "sexagesimal semi-colon" is then inserted to separate the integral part from the fractional part of the number. It is not true, however, that *any* fraction can be written as a (finite) sexagesimal number. (Neither can *any* fraction be written as a (finite) decimal number.) *The only numbers that can be written as sexagesimal numbers with a (finite) fractional part are numbers that are equal to a whole number divided by some power of 60*.

Important examples are

1/3 = 20/60 = ;20
1/2 = 30/60 = ;30
2/3 = 40/60 = ;40.

Other examples are

1,000,000/60 =	4 37 46;40
1,000,000/60^2 =	4 37;46 40
1,000,000/60^3 =	4;37 46 40
1,000,000/60^4 =	;04 37 46 40
1,000,000/60^5 =	;00 04 37 46 40.

Two useful *decimal-sexagesimal* and *sexagesimal-decimal* conversion tables are:

60	1 00	;00 01	1/3,600
70	1 10	;01	1/60

Continued

80	1 20	1 00	60
90	1 30	2 00	120
100	1 40	3 00	180
200	3 20	4 00	240
300	5 00	5 00	300
400	6 40	6 00	360
500	8 20	7 00	420
600	10 00	8 00	480
700	11 40	9 00	540
800	13 20	10 00	600
900	15 00	1 00 00	3,600
1,000	16 40	1 00 00 00	216,000

It is important to keep in mind that the predominantly Semitic population in Mesopotamia in the 2nd and 1st millennia BC normally counted with decimal numbers, using their own *Semitic decimal number words*. Only educated scribes had learned how to count with the originally Sumerian sexagesimal numbers in place value notation. Therefore, it is no wonder that there existed special "conversion tables" for the conversion of decimal numbers into sexagesimal numbers. An Old Babylonian conversion table is MS 3970, shown in Fig. 2.7.1 below. A Late Babylonian example is BM 36841, shown in Fig. 2.7.2. It is also no wonder that there existed multiplication tables for both 1 40 = 100 and 16 40 = 1,000. See, for instance, the reverse of the combined multiplication table MS 3974 (Fig. 2.6.13).

0.3 b. Sexagesimal Numbers in Relative Place Value Notation

The Old Babylonians (or, rather, their Sumerian predecessors in the Ur III period) invented place value notation for sexagesimal numbers, but they did not invent *final zeros* to distinguish, for instance, 1 from 1 00 = 60. Neither did they invent a *separator* like the sexagesimal semi-colon to separate the integral part of a sexagesimal number from its fractional part. They also did not invent *initial zeros* to distinguish, for instance ;01 and ;00 01 from 1. As a result, sexagesimal numbers in Old (and Late) Babylonian mathematical texts have only "relative" (or "floating") values. All Babylonian sexagesimal numbers are written as if they were integers, but the *intended* value of a sexagesimal number in a Babylonian cuneiform text can be its "nominal" value multiplied by any positive or negative power of 60. Thus, *only the context* can decide what the intended "absolute value" is of a "relative sexagesimal number" in such a text. This can seem to have been an awkward handicap for Old Babylonian calculators, but the truth is that it can be very convenient to be able to count with sexagesimal numbers without having to bother about the absolute value of the numbers or about the precise positions of the separating semicolons and final or initial zeros.

It is important to remember this peculiarity of Babylonian sexagesimal numbers. Therefore, it is advisable to follow the practice in this book, where in all *transliterations* and most *translations* of mathematical texts no use is made of separating semi-colons or of final and initial zeros in sexagesimal numbers. As a compromise, and only for the sake of clarity, *internal zeros* are inserted where needed, to indicate missing zeros or ones in the middle of a sexagesimal number. In the *commentaries*, on the other hand, full use is made of both separating semi-colons and final or initial zeros.

The Babylonians themselves overcame the ambiguity of their relative sexagesimal numbers in several ways. Thus, for instance, a missing internal double zero in a sexagesimal number could be indicated by a gap or by a special sign. An example is offered by the number 1 19 00 44 26 40 in MS 3049 § 5, line 9 (see Fig. 11.1.4 in Ch. 11 below), which is written as '1 19   44 26 40'. (Actually, the scribe at first wrote 44 where the gap should be, but then corrected himself by erasing 44 and writing it again a step to the right.)

Typically, in an Old Babylonian mathematical problem text, the *question* was stated in terms of "traditional" length numbers, area numbers, capacity numbers, *etc*., for which unambiguous special notations were used. In the ensuing *solution procedure*, the given traditional numbers were converted into sexagesimal numbers and all computations were carried out using those numbers. The *answer*, finally, was often again given in the form of traditional numbers.

A clear example of this practice is offered by the exercise MS 3052 § 1 c (Sec. 10.2.a below), where in the question the length of a wall is given as 5 uš (5 'lengths' = 5 · 60 ninda), the volume of the wall as 2_{iku} (2 'dykes' = 2 · 100 square ninda), and the length of a hole drilled through the wall as 1 kùš 7 1/2 šu.si ('1 cubit 7 1/2 fingers'). When these data reappear in the solution procedure, they are expressed as 5 (meaning 5 00 ninda), 3 20 (meaning 3 20 square ninda), and 6 15 (meaning ;06 15 ninda), respectively. (The ninda was the basic Old Babylonian unit of length measure, equal to 12 cubits of 30 fingers each.) The end result of the computation is the relative sexagesimal number 8, rephrased in the answer as 1/2 ninda 2 cubits (= 8 cubits).

In another exercise, MS 3049 § 5 (Sec. 11.1 d), the height of a gate is given in the question as '5 cubits, 25, and 10 fingers, 1 40', but reappears in the solution procedure as 26 40 (= ;25 + ;01 40).

A more confusing Old Babylonian way of specifying the intended size of a relative sexagesimal number is exemplified by BM 96957+ § 5 b (also in Sec. 11.1 d below), where the height of a gate is given in the question as 40 kùš '40, cubits' (meaning not 40 cubits but ;40 ninda = 8 cubits).

0.4. Counting with Sexagesimal Numbers in Relative Place Value Notation

0.4 a. Addition of Sexagesimal Numbers

It is not known how complicated additions of sexagesimal numbers were carried out by Old Babylonian school boys or their teachers. No exercise tablets with additions have been found, and when sexagesimal numbers are added as one of the steps of the solution procedure in an Old Babylonian mathematical problem text, the result is always given directly. There was simply not space enough on a clay tablet for non-essential text, such as the details of an addition algorithm. Thus, additions that could not be done in the head were carried out, presumably, either on some kind of counting board or on a separate clay tablet that was erased after each computation.

A simple example is the addition of the numbers of men in four troops of soldiers in the mathematical exercise MS 2792 # 2 (see Fig. 10.3.7). The numbers to be added are 4 30, 8 (00), 6 40, and 7 30, and the text states, quite laconically (in line 7): 'Add the soldiers, 26 40'. It is clear that the addition must have been done (essentially) in the following way:

1	
4 30	a) 30 + 40 + 30 = (100 =) 1 40, where
8	40 is used as the last sexagesimal place in the sum, while
6 40	1 is carried to the preceding column of sexagesimal places.
+ 7 30	b) 1 + 4 + 8 + 6 + 7 = 26, the preceding sexagesimal place in the sum.
26 40	

The example shows that the addition algorithm is similar to the familiar addition algorithm for *decimal* numbers. The only differences are that in sexagesimal addition sexagesimal *places* or *double digits* are used instead of decimal *single digits*, and that in sexagesimal addition *sixties* in the sum of a column of sexagesimal double-digits are carried to the preceding column of double-digits, while in decimal addition *tens* in the sum of a column of decimal digits are carried to the preceding column.

A more interesting example is the addition of three squares of sexagesimal numbers in MS 3049 § 5, an application of the Old Babylonian "diagonal rule" in three dimensions (see Fig. 11.1.5 below). The three squares are given in relative numbers as 26 40, 8 53 20, and 6 40, and their squares are, respectively, 11 51 06 40, 1 19 (00) 44 26 40, and 44 26 40. In the text, the addition of the three squares is announced simply as follows (in line 12): 'Add them, 13 54 34 14 26 40 you will see'.

The author of the text must have had some idea about the *absolute* values of the three numbers and their squares. Otherwise he would not have known how to add the squares correctly. As a matter of fact, the given numbers have to be interpreted as ;26 40 (= 12/27 = 4/9), ;08 53 20 (= 4/27), and ;06 40 (= 3/27 = 1/9). The corresponding interpretations of their squares as absolute sexagesimal numbers are ;11 51 06 40, ;01 19 00 44

26 40, and ;00 44 26 40.

Below is shown how the addition of the three squares can be set up in relative sexagesimal numbers (the display to the left), and in absolute sexagesimal numbers (the display to the right):

1 2	1 2
11 51 06 40	;11 51 06 40
1 19 00 44 26 40	;01 19 00 44 26 40
+ 44 26 40	+ ;00 44 26 40
13 54 34 04 26 40	;13 54 34 04 26 40

It is clear that it required considerable skill to get the addition right using only relative sexagesimal numbers!

0.4 b. Subtraction of Sexagesimal Numbers

In MS 2792 # 1, the object considered is a leaning ramp built by four troops of soldiers. The side of a ramp has the form of a trapezoid with three transversals parallel to the base and the top of the trapezoid (see Fig. 10.3.3). In the computation of the lengths of the three transversals, three subtractions have to be done (in lines 12, 15, 17). No details are given. Below is shown how the three subtractions can have been set up:

	60	~~60~~ 60
7 49 26 40	~~5~~ 36 06 40	3 ~~45~~
– 2 13 20	– 1 51 06 40	– 1 28 53 20
5 36 06 40	3 45	2 16 06 40

The first example is simple. The sexagesimal places or double-digits in the lower sexagesimal number are subtracted from the places in the upper number, one at a time, starting from the right.

The second example is more complicated. In the third place from the right, 51 cannot be subtracted from 36, so 1 is *borrowed* from the fourth place, being worth 60 in the third place.

In the third example, there are two such borrowings.

Thus, the indicated sexagesimal subtraction algorithm is similar to the usual decimal subtraction algorithm, with the exception that decimal digits are replaced by sexagesimal double-digits and that *sixties* are borrowed each time instead of *tens*.

0.4 c. Multiplication of Sexagesimal Numbers

Three Old Babylonian clay tablets with elementary multiplication exercises are known, **MS 2728**, **2729**, and **3944** (see Fig. 1.1.1). As could be expected, all the multiplication exercises consist of only a question and an answer, with a total lack of detailed computations.

There are several methods that an Old Babylonian school boy conceivably may have used to find the products of the numbers given in the mentioned multiplication exercises. Take, for instance, the first exercise in MS 2729, to find the product of 35 and 30. The most direct way of finding the product would have been through addition of "partial products":

$$35 \cdot 30 = (30 + 5) \cdot 30 = 30 \cdot 30 + 5 \cdot 30 = (900 + 150 =)\ 15\ (00) + 2\ 30 = 17\ 30.$$

A somewhat more ingenious way would be to think of 30 as the fraction 1/2 = ;30. Then, in *relative* place value notation,

$$35 \cdot 30 = 35/2 = 17\ 1/2 = 17\ 30.$$

In a similar way, the answers to the second multiplication exercise on MS 2729 may have been obtained as follows:

$$40 \cdot 35 = 2/3 \cdot 35 = 2/3 \cdot 30 + 2/3 \cdot 5 = 20 + 3\ 20 = 23\ 20.$$

The remaining multiplication exercises in MS 2728 and 2729 are equally simple. Somewhat more complicated are the multiplication exercises in MS 3944. Take, for instance, the first exercise in MS 3944, to find the product of 25 and 17 30. It can be used to illustrate three available techniques for finding the product of two sexagesimal numbers.

The method that may appear most natural to someone familiar only with *decimal* arithmetics is to convert the given sexagesimal numbers to decimal numbers, then multiply the decimal numbers in the usual way, and finally convert the result back to a sexagesimal number, as follows:

17 30 = 17 · 60 + 30 = 1,020 + 30 = 1,050,
25 · 17 30 = 25 · 1,050 = 26,250,
26,250 = 7 17 30 (see Sec. 0.3 a above).

The most obvious method for an Old Babylonian school boy probably was to rely on the multiplication table for 25, which he may have known by heart (see Appendix 2). Then:

25 · 17 30 = 25 ·17 + 25 · 30 = 7 05 + 12 30 = 7 17 30 (in *relative* place value notation).

Another method that might have appealed to an Old Babylonian school boy is an application of the following "factorization method", requiring only the multiplication table for 5, a small number:

25 · 17 30 = 5 · 5 · 17 30, 5 · 17 30 = 1 25 + 2 30 = 1 27 30, 5 · 1 27 30 = 5 + 2 15 + 2 30 = 7 17 30.

It has not been known before how Babylonian teachers and students of mathematics carried out multiplications of "many-place" sexagesimal numbers, too long to be multiplied in the head. There have been speculations about the use in Mesopotamia of some kind of abacus or counting board, but no archaeological remains of such devices have ever been found. Now, however, the present author has been able to show that some fragments of texts from the Late Babylonian/Seleucid period in Mesopotamia (in the second half of the first millennium BC) can be explained as what remains of explicit written multiplication algorithms for many-place sexagesimal numbers, organized in very much the same way as our own decimal multiplication algorithms. See Sec. A9 b in Appendix 9 below.

0.4 d. Division of Sexagesimal Numbers

In most cases, divisions in Old Babylonian mathematical texts were transformed into multiplications by use of the following simple device: A given sexagesimal number is "regular" if it is possible to find another sexagesimal number igi $a = a'$, the "reciprocal" of a, such that $a \cdot a'$ = '1' (= some power of 60). It is well known that this happens if, and only if, a is a product of powers of 2, 3, and 5 (2, 3, and 5 are the prime factors of the sexagesimal base 60). Now, if the divisor is a regular sexagesimal number a, then division by a can be replaced by a multiplication by igi a, the reciprocal of a.

Here are some examples of the application of this method in various exercises in MS 5112 (Sec. 11.2 below) and in MS 3052 (Sec. 10.2 below):

30/16	=	30 · igi 16	=	30 · 3 45	=	1 52 30	(MS 5112 § 12, lines 8-9)
20/1 20	=	20 · igi 1 20	=	20 · 45	=	15	(MS 5112 § 13, line 10)
2 30/3 20	=	2 30 · igi 3 20	=	2 30 · 18	=	45	(MS 3052 § 1 a, line 7)
26 15/22 30	=	26 15 · igi 22 30	=	26 15 · 2 40	=	1 10	(MS 3052 § 1 b, line 4)

The correctness of the last of the examples above may not be completely obvious, in particular since the reciprocal of 22 30 is not listed in the Old Babylonian standard table of reciprocals (Sec. 2.5 below). On the other hand, it is easy to find the reciprocal of 22 30, for instance as follows: Clearly 22 30 is equal to 1/ 2 · 45. Therefore, the reciprocal of 22 30 is equal to 2 times the reciprocal of 45, so that igi 22 30 = 2 · 1 20 = 2 40.

It is also possible to compute 26 15/22 30 by use of a detour over decimal arithmetic. Indeed, since 26 15 = 1,200 + 360 + 15 = 1,575 and 22 30 = 1,200 + 120 + 30 = 1,350, it follows that 26 15/22 30 = 1,575/1,350 = 1.166... = 1 1/6 = 1;10.

It is, of course, not always possible to transform a division problem into an equivalent multiplication problem. This happens when the divisor is either a "many-place" regular sexagesimal number, or a non-regular sexagesimal number. An example of the first kind can be found in MS 3871 (Fig. 1.3.1), the only known example of an Old Babylonian division exercise. No details of the division algorithm are given in MS 3871, but it is likely that the division there of 4 37 46 40 by 11 34 26 40 was achieved by use of a quite straightforward factorization method. (See the discussion in Sec. 1.3 below.)

Interesting examples of divisions by non-regular sexagesimal numbers can be found in some of the "combined market rate exercises" in Sec. 7.2 below, for instance MS 2299 (Fig. 7.2.1, bottom). There, four commodities have a "combined unit price" of ;57 shekel, while the silver available is given as 16 53 20, probably meaning 16;53 20 shekels. To find out how many units of each kind can be bought for this amount of silver, one has to divide 16 53 20 by 57. Since 57 = 3 · 19, it is a "semiregular" sexagesimal number, that is, the product of a regular and a non-regular number. Similarly,

16 53 20 = 20 · 50 40 = 20 · 40 · 1 16 = 20 · 40 · 4 · 19.

Therefore, 16 53 20/57 = 20 · 40 · 4 / 3 = 17 46 40.

Another example of a division of a semiregular sexagesimal number by a non-regular sexagesimal number can be found in the combined market rate exercise MS 2268/19 (Fig. 7.2.2, top). There, the combined price for 3 1/2 = 3;30 units of each of five different commodities is 2;49 shekels, while the silver available is 1;00 05 20 shekels. To find out how many units of each kind can be bought for this amount of silver, one has to divide 1 00 05 20 by 2 49, and then multiply by 3 30. Here 2 49 is non-regular, since it is equal to the square of 13. The answer given in the text is that the wanted number is 1 14 40, probably meaning 1;14 40 units of each kind. This number can be explained as follows:

(1 00 05 20/2 49) · 3 30 = 21 20 · 3 30 = 1 14 40 (in relative sexagesimal numbers).

Since 1 00 05 20 is semiregular, the division of 1 00 05 20 by 2 49 can have been achieved as follows:

1 00 05 20 = 20 · 3 00 16 = 20 · 16 · 11 16 = 20 · 16 · 4 · 2 49 = 21 20 · 2 49.

A more difficult task is to divide a sexagesimal number by another sexagesimal number, when neither number is semi-regular. How this could be done is shown in the discussion of MS 2317 in § 7.1. There, it is suggested that the division of the "funny number" 1 01 01 01 by the non-regular number 13 was achieved by use of a certain elegant division algorithm, known from division problems in mathematical cuneiform texts from the third millennium BC. (See Appendix 6, Secs. A6 e-g.) Essentially, that method is based on the computation of a succession of increasingly more accurate *approximate reciprocals* to the non-regular divisor.

0.4 e. Computing Square Sides (Square Roots) of Sexagesimal Numbers

Computations of square roots (or rather *square sides*) are common in Old Babylonian mathematical problem texts, in solution procedures involving quadratic equations. Here are, for instance, examples from MS 5112 (Sec. 11.2 below), a recombination text with the theme 'equations for squares or rectangles':

sqs. 2 01 = 11	(MS 5112 § 1)	sqs. 5 03 45 = 2 15	(MS 5112 § 9)
sqs. 10 25 = 25	(MS 5112 § 2 c)	sqs. 16 01 = 31	(MS 5112 § 12)
sqs. 42 15 = 6 30	(MS 5112 § 5)	sqs. 56 15 = 7 30	(MS 5112 § 13)
sqs. 1 46 40 = 1 20	(MS 5112 § 8)		

In MS 5112, the questions and answers are typically of the form

16 01, what is the square side? 31 each (way) is the square side (MS 5112 § 12, lines 7-8)

Note that the reference to 'each (way)', probably meaning 'in each direction', is a clear indication that the computation was interpreted as the computation of *all sides* of a square with the given area. However, no details of the computation method are given. In the first two and in the last but one of the examples listed above, the square sides can have been found by use of a *table of square sides* (see Sec. 2.2 below). In the remaining cases, a table of square sides would be of no help. Instead, it is likely that the square sides were computed by use of what may be called the "Old Babylonian square side rule", incorrectly known as "Heron's rule" (see Friberg, *BagM* 28 (1997) § 8). This is a rule for the computation of relatively good approximations to a square side. When the given area is an exact square, the computation will yield an exact answer, as in the following example:

sqs. 5 03 45 = sqs. (sq. 2 (00) + 1 03 45) = appr. 2 (00) + 1 03 45/(2 · 2 (00)) = appr. 2 (00) + 15 = 2 15
sq. 2 15 = sq. (2 (00) + 15) = 4 (00 00) + 1 (00 00) + 3 45 = 5 03 45.

The method works also in more complicated examples, as in MS 3049 § 5 (Sec. 11.1 d below). There the "inside diagonal" of a gate is computed by use of the three-dimensional diagonal rule. The last step of the solution procedure is the computation of the square side of 13 54 34 04 26 40. The answer is given directly as 28 53 20. No information is given about how this result was obtained. However, *if* the square side was obtained by repeated use of the square side rule, then the successive steps of the computation were (essentially) the following:

1. sq. 29 = 14 01, 13 54 = 14 01 – 7 (13 54 being the first couple of sexagesimal places in 13 54 | 34 04 | 26 40).
2. sqs. (13 54) = sqs. (sq. 29 – 7) = appr. 29 – 7 / (2 · 29) = appr. 29 – 7 / (2 · 30) = 28;53.
3. sq. 28 53 = 13 54 14 49, 13 54 | 34 04 = 13 54 14 49 + 19 15.
4. sqs. (13 54 | 34 04) = sqs. (sq. 28 53 + 19 15) = appr. 28 53 + 19 15 / (2 · 28 53) = appr. 28 53 + 20 / (2 · 30) = 28 53;20.
5. sq. 28 53 20 = 13 54 34 04 26 40, the given number.

Another possibility is that the square side of 13 54 34 04 26 40 was computed by use of a variant of the Old Babylonian "trailing part algorithm". (See the detailed discussion in Sec. 1.5 a below.)

Other interesting examples of computations of square sides of semiregular sexagesimal numbers can be found in Appendix 8 below, in connection with the discussion of the famous table text Plimpton 322. Thus, for instance, in Sec. A8 a it is shown that

sqs. 1 56 56 58 14 50 06 15 = 3 13 · 5^5 · 30 = 1 23 46 02 30, and that
sqs. 56 56 58 14 50 06 15 = 56 07 · 5^3 · 30 = 58 27 17 30.

0.4 f. Number Signs for Sexagesimal Numbers With and Without Place Value Notation

When writing sexagesimal numbers *in place value notation*, the only numbers signs needed are signs for the 'ones' from 1 to 9, and for the 'tens' from 10 to 50. The normal Babylonian forms of the cuneiform signs for the ones and tens are displayed in Fig. 0.4.1 below.

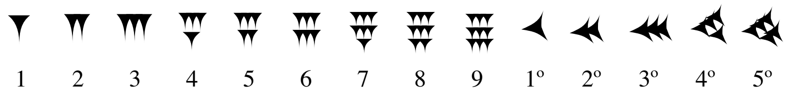

Figure 0.4.1. Babylonian cuneiform number signs in place value notation.

Note that since there are special cuneiform signs for the tens, it is appropriate to call those signs for instance 1°, 2°, 3°, 4°, and 5°, as here, rather than 10, 20, 30, 40, and 50.

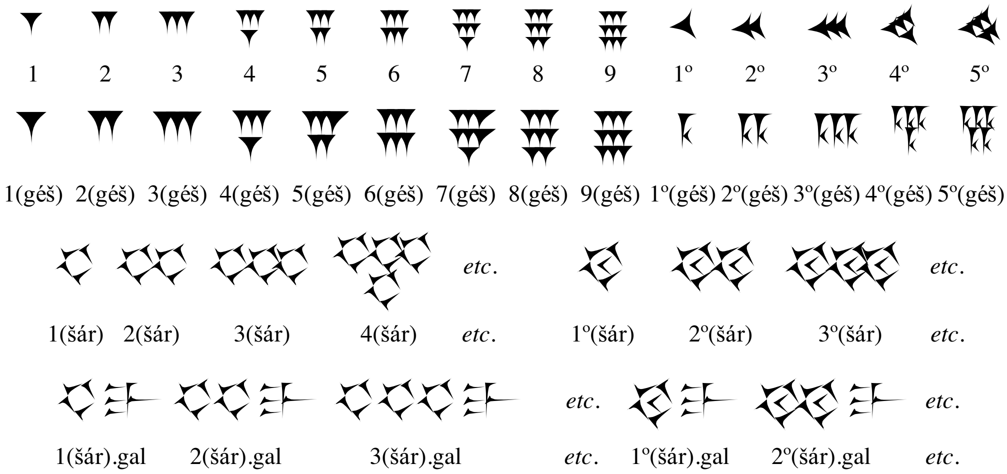

Figure 0.4.2. Babylonian cuneiform number signs in "sign value notation".

Occasionally sexagesimal numbers are written *without the use of place value notation* in Babylonian cuneiform texts, by use of notations inherited from the Sumerians. The Babylonian forms of the sexagesimal numbers signs in such "sign value notation" are displayed in Fig. 0.4.2 above. In the transliterations of the number signs, the following Sumerian words are used:

géš = 60, šár = $60 \cdot 60$, šár.gal = $60 \cdot 60 \cdot 60$ 'the great šár'.

See also the much more complete discussion of cuneiform systems of notations for numbers and measures in Appendix 4 below!

For completeness, also the forms of the Babylonian cuneiform signs for the frequently occurring "basic fractions" are shown in Fig. 0.4.3 below. (Cf. Fig. A4.2 in Appendix 4.)

3' (1/3) 2' (1/2) 3" (2/3) 6" (5/6)

Figure 0.4.3. Babylonian cuneiform signs for the "basic fractions".