CHAPTER

7

THE LINEAR FUNCTION *bx* + *c* AND ITS RECIPROCAL

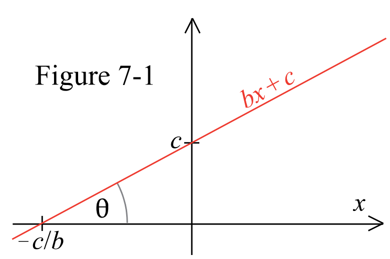
Many relationships in science and engineering take the form $f(x) = bx + c$, and many others can be cast into this form by a redefinition of the variables. The analysis of experimental results can thereby often be reduced to the determination of the coefficients b and c from paired x, $f(x)$ data. The *linear regression* or *least squares* method of performing this analysis is exposed in Section 7:14.

7:1 NOTATION

The linear function $bx + c$, or its special case $1+ x$, is sometimes referred to as a "binomial function" but confusingly it is also termed a "monomial function". Neither name is used in this *Atlas*.

The constant c is termed the *intercept*, whereas b is known as the *slope* or *gradient*. These names derive, of course, from the observation that, if the linear function $bx + c$ is plotted versus x, its graph is a straight line that intersects the vertical axis at altitude c and whose inclination from the horizontal is characterized by the number b, which measures the rate at which the function's value increases with x. The b coefficient is also the tangent [Chapter 34] of angle θ shown in Figure 7-1. The term "slope" is occasionally associated with the angle θ itself, but it more usually means $\tan(\theta)$, being negative if θ is an obtuse angle $(90^\circ < \theta < 180^\circ)$. The letter m often replaces b as a symbol for the slope. In the figures, but not elsewhere in the chapter, b and c are assumed positive.

Figure 7-1



The name "inverse linear function" is sometimes given to the function $1/(bx + c)$. Throughout this *Atlas*, however, we reserve the phrase "inverse function" for the relationship described in equation 0:3:3. The unambiguous name *reciprocal linear function* is used here.

The name *rectangular hyperbola* may also be associated with the function $1/(bx + c)$. However, this name is also applicable to other functions, described in Section 7:13, that share the same shape as, but possess a different orientation to, the reciprocal linear function illustrated in Figure 7-2.

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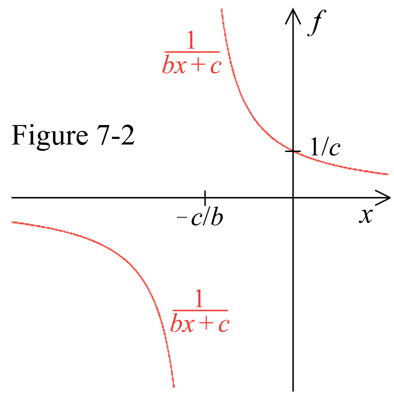
7:2 BEHAVIOR

The linear function $bx + c$ is defined for all values of the argument x and (unless $b = 0$) itself assumes all values. The same is true of the reciprocal linear function which, however, displays an infinite discontinuity at $x = -c/b$, as illustrated in Figure 7-2. A graph of the reciprocal linear function $\mathrm{f}(x) = 1/(bx + c)$ has a high degree of symmetry [Section 14:15], being inversion symmetric about the point $x = -c/b$, $f = 0$ and displaying mirror symmetry towards reflections in the lines $-x\mathrm{sgn}(b)-(c/b)$ and $x\mathrm{sgn}(b)+(c/b)$. Here sgn is the signum function [Chapter 8].

Figure 7-2



7:3 DEFINITIONS

The arithmetic operations of multiplication by b and addition of c fully define $\mathrm{f}(x) = bx + c$. The same operations, followed by division into unity, define the reciprocal linear function.

The linear function is completely characterized when its values, f_1 and f_2, are known at two (distinct) arguments, x_1 and x_2. The slope and intercept may be found from the formula

7:3:1 $$bx + c = \left(\frac{f_2 - f_1}{x_2 - x_1}\right)x + \frac{x_2 f_1 - x_1 f_2}{x_2 - x_1}$$

7:4 SPECIAL CASES

When $b = 0$, the linear function and its reciprocal reduce to a constant [Chapter 1]. When $c = 0$, the linear function is proportional to its argument x.

7:5 INTRARELATIONSHIPS

Both the linear function and its reciprocal obey the reflection formula

7:5:1 $$\mathrm{f}\left(-x - \frac{c}{b}\right) = -\mathrm{f}\left(x - \frac{c}{b}\right) \qquad \mathrm{f}(x) = bx + c \quad \text{or} \quad \frac{1}{bx + c}$$

Two linear functions that share the same b parameter represent straight lines that are *parallel*; the distance separating these lines is $|c_2 - c_1|/\sqrt{1 + b^2}$. If the slopes of two straight lines satisfy the relation $b_1 b_2 = -1$, the lines are mutually perpendicular. intersecting at the point $x = (c_2 - c_1)/(b_1 - b_2)$. The inverse of the linear function $\mathrm{f}(x) = bx + c$, defined by $\mathrm{F}\{\mathrm{f}(x)\} = x$, is another linear function $\mathrm{F}(x) = (x/b)-(c/b)$; graphically, these two straight lines usually cross at $x = c(1+b^2)/(1-b^2)$.

The sum or difference of two linear functions is a third linear function $(b_1 \pm b_2)x + c_1 \pm c_2$ and this property extends to multiple components. The product of two, three, or many linear functions is a quadratic function [Chapter 15], a cubic function [Chapter 16] or a higher polynomial function [Chapter 17]. Unless $b_2 c_1 = b_1 c_2$, the quotient of two linear functions is the infinite power series:

7:5:2
$$\frac{b_1x+c_1}{b_2x+c_2}=\begin{cases}\dfrac{c_1}{c_2}+\left(\dfrac{c_1}{c_2}-\dfrac{b_1}{b_2}\right)\displaystyle\sum_{j=1}^{\infty}\left(\dfrac{-b_2x}{c_2}\right)^j & |x|<\left|\dfrac{c_2}{b_2}\right| \\ \dfrac{b_1}{b_2}-\left(\dfrac{c_1}{c_2}-\dfrac{b_1}{b_2}\right)\displaystyle\sum_{j=1}^{\infty}\left(\dfrac{-c_2}{b_2x}\right)^j & |x|>\left|\dfrac{c_2}{b_2}\right|\end{cases}$$

The sum or difference of two reciprocal linear functions is a rational function [Section 17:13] of numeratorial and denominatorial degrees of 1 and 2 respectively. Finite series of certain reciprocal linear functions may be summed in terms of the digamma function [Chapter 44]

7:5:3
$$\frac{1}{c}+\frac{1}{x+c}+\frac{1}{2x+c}+\cdots+\frac{1}{Jx+c}=\sum_{j=0}^{J}\frac{1}{jx+c}=\frac{1}{x}\left[\psi\left(J+1+\frac{c}{x}\right)-\psi\left(\frac{c}{x}\right)\right]$$

or in terms of Bateman's G function [Section 44:13]

7:5:4
$$\frac{1}{c}-\frac{1}{x+c}+\frac{1}{2x+c}-\cdots\pm\frac{1}{Jx+c}=\sum_{j=0}^{J}\frac{(-1)^j}{jx+c}=\frac{1}{2x}\left[\mathrm{G}\left(\frac{c}{x}\right)\pm\mathrm{G}\left(J+1+\frac{c}{x}\right)\right]$$

In formula 7:5:4, the upper/lower signs are taken depending on whether J is even or odd. The corresponding infinite sum is

7:5:5
$$\frac{1}{c}-\frac{1}{x+c}+\frac{1}{2x+c}-\cdots=\sum_{j=0}^{\infty}\frac{(-1)^j}{jx+c}=\frac{1}{2x}\mathrm{G}\left(\frac{c}{x}\right)$$

See Section 44:14 for further information on this topic.

7:6 EXPANSIONS

The linear function may be expanded as a infinite series of Bessel functions [Chapter 52]

7:6:1
$$bx+c=c+2\mathrm{J}_1(bx)+6\mathrm{J}_3(bx)+10\mathrm{J}_5(bx)+\cdots=c+2\sum_{j=1}^{\infty}(2j-1)\mathrm{J}_{2j-1}(bx)$$

though this representation is seldom employed.

The reciprocal linear function is expansible as a *geometric series* in alternative forms

7:6:2
$$\frac{1}{bx+c}=\begin{cases}\dfrac{1}{c}-\dfrac{bx}{c^2}+\dfrac{b^2x^2}{c^3}-\dfrac{b^3x^3}{c^4}+\cdots=\dfrac{1}{c}\displaystyle\sum_{j=0}^{\infty}\left(\dfrac{-bx}{c}\right)^j & |x|<\left|\dfrac{c}{b}\right| \\ \dfrac{1}{bx}-\dfrac{c}{b^2x^2}+\dfrac{c^2}{b^3x^3}-\dfrac{c^3}{b^4x^4}+\cdots=\dfrac{1}{bx}\displaystyle\sum_{j=0}^{\infty}\left(\dfrac{-c}{bx}\right)^j & |x|>\left|\dfrac{c}{b}\right|\end{cases}$$

according to the magnitude of the argument x compared to that of the ratio c/b. Likewise there are two alternatives when the reciprocal linear function is expanded as an infinite product

7:6:3
$$\frac{1}{bx+c}=\begin{cases}\dfrac{c-bx}{c^2}\displaystyle\prod_{j=1}^{\infty}\left[1+\left(\dfrac{bx}{c}\right)^{2^j}\right] & -1<\dfrac{bx}{c}<1 \\ \dfrac{bx-c}{b^2x^2}\displaystyle\prod_{j=1}^{\infty}\left[1+\left(\dfrac{c}{bx}\right)^{2^j}\right] & \left|\dfrac{bx}{c}\right|>1\end{cases}$$

For example, if $|x| < 1$

7:6:4 $$\frac{1}{1 \pm x} = (1 \mp x)(1+x^2)(1+x^4)(1+x^8)(1+x^{16})\cdots$$

7:7 PARTICULAR VALUES

As Figure 7-1 shows, the linear function $bx + c$ equals c when $x = 0$ and equals zero when $x = -c/b$. The reciprocal linear function has neither an extremum nor a zero, but it incurs a discontinuity at $x = -c/b$ [Figure 7-2].

7:8 NUMERICAL VALUES

These are easily calculated by direct substitution. The construction feature of *Equator* enables a linear function to be used as the argument of another function.

7:9 LIMITS AND APPROXIMATIONS

The reciprocal linear function approaches zero asymptotically as $x \to \pm\infty$.

7:10 OPERATIONS OF THE CALCULUS

The rules for differentiation of the linear function and its reciprocal are

7:10:1 $$\frac{d}{dx}(bx+c) = b$$

and

7:10:2 $$\frac{d}{dx}\left(\frac{1}{bx+c}\right) = \frac{-b}{(bx+c)^2}$$

while those for indefinite integration are

7:10:3 $$\int_0^x (bt+c)\,dt = \frac{bx^2}{2} + cx$$

and

7:10:4 $$\int_0^x \frac{1}{bt+c}\,dt = \frac{1}{b}\ln\left(\left|\frac{bx+c}{c}\right|\right)$$

If $0 < -c/b < x$, the integrand in 7:10:4 encounters an infinity; in this event, the integral is to be interpreted as a *Cauchy limit*. This means that the ordinary definition of the integral is replaced by

7:10:5 $$\lim_{\varepsilon \to 0}\left\{\int_0^{-(c/b)-\varepsilon} \frac{1}{bt+c}\,dt + \int_{-(c/b)+\varepsilon}^{x} \frac{1}{bt+c}\,dt\right\}$$

Several other integrals in this section, including those that follow immediately, may also require interpretation as Cauchy limits, but mention of this will not always be made

7:10:6 $$\int_0^x \frac{1}{(Bt+C)(bt+c)}\mathrm{d}t = \frac{1}{bC-Bc}\ln\left(\left|\frac{C(bx+c)}{c(Bx+C)}\right|\right) \qquad Bc \neq bC$$

7:10:7 $$\int_0^x \frac{Bt+C}{bt+c}\mathrm{d}t = \frac{Bx}{b} + \frac{bC-Bc}{b^2}\ln\left(\left|\frac{bx+c}{c}\right|\right)$$

7:10:8 $$\int_0^x \frac{t^n}{bt+c}\mathrm{d}\,x = \frac{(-c)^n}{b^{n+1}}\left[\ln\left(\left|\frac{bx+c}{c}\right|\right) + \sum_{j=1}^{n}\frac{1}{j}\left(\frac{-bx}{c}\right)^j\right] \qquad n = 0,1,2,\cdots$$

Formulas for the semidifferentiation and semiintegration [Section 12:14] of the linear function are

7:10:9 $$\frac{\mathrm{d}^{1/2}}{\mathrm{d}x^{1/2}}(bx+c) = \frac{2bx+c}{\sqrt{\pi x}}$$

and

7:10:10 $$\frac{\mathrm{d}^{-1/2}}{\mathrm{d}x^{-1/2}}(bx+c) = \sqrt{\frac{x}{\pi}}\left[\frac{4bx}{3}+2c\right]$$

when the lower limit is zero. The table below shows the semiderivatives and semiintegrals of the reciprocal linear functions $1/(bx+c)$ and $1/(bx-c)$, when the lower limit is zero, and when it is $-\infty$. In this table, but not necessarily elsewhere in the chapter, b and c are positive.

	$\mathrm{f}(x) = \dfrac{1}{bx+c}$	$\mathrm{f}(x) = \dfrac{1}{bx-c}$
$\dfrac{\mathrm{d}^{1/2}\mathrm{f}}{\mathrm{d}x^{1/2}}$	$\dfrac{\sqrt{bx+c}-\sqrt{bx}\,\mathrm{arsinh}\left(\sqrt{bx/c}\right)}{\sqrt{\pi x(bx+c)^3}} \quad x>0$	$\dfrac{-\sqrt{c-bx}-\sqrt{bx}\,\arcsin\left(\sqrt{bx/c}\right)}{\sqrt{\pi x(c-bx)^3}} \quad 0<x<\dfrac{c}{b}$
$\dfrac{\mathrm{d}^{-1/2}\mathrm{f}}{\mathrm{d}x^{-1/2}}$	$\dfrac{2\mathrm{arsinh}\left(\sqrt{bx/c}\right)}{\sqrt{\pi b(bx+c)}} \quad x>0$	$\dfrac{-2\arcsin\left(\sqrt{bx/c}\right)}{\sqrt{\pi b(c-bx)}} \quad 0<x<\dfrac{c}{b}$
$\left.\dfrac{\mathrm{d}^{1/2}}{\mathrm{d}t^{1/2}}\mathrm{f}(t)\right\|_{-\infty}^{x}$	$\dfrac{1}{2}\sqrt{\dfrac{\pi b}{(-bx-c)^3}} \quad x<\dfrac{-c}{b}$	$\dfrac{-1}{2}\sqrt{\dfrac{\pi b}{(c-bx)^3}} \quad x<\dfrac{c}{b}$
$\left.\dfrac{\mathrm{d}^{-1/2}}{\mathrm{d}t^{-1/2}}\mathrm{f}(t)\right\|_{-\infty}^{x}$	$-\sqrt{\dfrac{\pi}{b(-bx-c)}} \quad x<\dfrac{-c}{b}$	$-\sqrt{\dfrac{\pi}{b(c-bx)}} \quad x<\dfrac{c}{b}$

See Sections 12:14 and 64:14 for the definitions and symbolism of semidifferintegrals with various lower limits.

The Laplace transforms of the linear and reciprocal linear functions are

7:10:11 $$\int_0^\infty (bt+c)\exp(-st)\,\mathrm{d}t = \mathfrak{L}\{bt+c\} = \frac{cs+b}{s^2}$$

and

7:10:12 $$\int_0^\infty \frac{1}{bt+c}\exp(-st)\mathrm{d}t = \mathfrak{L}\left\{\frac{1}{bt+c}\right\} = \frac{-1}{b}\exp\left(\frac{cs}{b}\right)\mathrm{Ei}\left(\frac{-cs}{b}\right)$$

the latter involving an exponential integral [Chapter 37]. A general rule for the Laplace transformation of the product of any transformable function f(t) and a linear function is

7:10:13 $$\int_0^\infty (bt+c)\mathrm{f}(t)\exp(-st)\mathrm{d}t = \mathfrak{L}\{(bt+c)\mathrm{f}(t)\} = c\mathfrak{L}\{\mathrm{f}(t)\} - b\frac{\mathrm{d}}{\mathrm{d}s}\mathfrak{L}\{\mathrm{f}(t)\}$$

Requiring a Cauchy-limit interpretation, the integral transform

7:10:14 $$\frac{1}{\pi}\int_{-\infty}^{\infty}\mathrm{f}(t)\frac{\mathrm{d}t}{t-y}$$

is called a *Hilbert transform* (David Hilbert, German mathematician, 1862 – 1943). The Hilbert transforms of many functions, mostly piecewise-defined functions [Section 8:4], are tabulated by Erdélyi, Magnus, Oberhettinger and Tricomi [*Tables of Integral Transforms*, Volume 2, Chapter 15]. For example, the Hilbert transform of the pulse function [Section 1:13] is

7:10:15 $$\frac{1}{\pi}\int_{-\infty}^{\infty} c\left[u\left(t-a+\frac{h}{2}\right)-u\left(t-a-\frac{h}{2}\right)\right]\frac{\mathrm{d}t}{t-y} = \frac{c}{\pi}\ln\left(\frac{2(a-y)+h}{2(a-y)-h}\right)$$

A valuable feature of Hilbert transformation is that the inverse transform is identical in form to the forward transformation, apart from a sign change.

7:11 COMPLEX ARGUMENT

The linear function of complex argument, and its reciprocal, split into the real and imaginary parts

7:11:1 $$bz+c = (bx+c)+iby$$

and

7:11:2 $$\frac{1}{bz+c} = \frac{bx+c}{(bx+c)^2+b^2y^2} - i\frac{by}{(bx+c)^2+b^2y^2}$$

if b and c are real.

The inverse Laplace transformation of the linear and reciprocal linear functions leads to functions from Chapters 10 and 26

7:11:3 $$\int_{\alpha-i\infty}^{\alpha+i\infty}(bs+c)\frac{\exp(ts)}{2\pi i}\,\mathrm{d}s = \mathfrak{I}\{bs+c\} = b\delta'(t)+c\delta(t)$$

and

7:11:4 $$\int_{\alpha-i\infty}^{\alpha+i\infty}\frac{1}{bs+c}\frac{\exp(ts)}{2\pi i}\,\mathrm{d}s = \mathfrak{I}\left\{\frac{1}{bs+c}\right\} = \frac{1}{b}\exp\left(\frac{-ct}{b}\right)$$

The Laplace inversion of a function of $bx+c$ is related to the inverse Laplace transform of the function itself through the general formula

7:11:5 $$\int_{\alpha-i\infty}^{\alpha+i\infty}\mathrm{f}(bs+c)\frac{\exp(ts)}{2\pi i}\,\mathrm{d}s = \mathfrak{I}\{\mathrm{f}(bs+c)\} = \frac{1}{b}\exp\left(\frac{-ct}{b}\right)\mathfrak{I}\{\overline{\mathrm{f}}(s)\}$$

7:12 GENERALIZATIONS

The linear and reciprocal linear functions are the $\nu = \pm 1$ cases of the more general function $(bx + c)^\nu$, two other instances of which are addressed in Chapter 11. More broadly, all the functions of Chapters 10−14 are particular examples of a wide class of algebraic functions generalized by the formula $(bx^n + c)^\nu$.

The linear function is an early member of a hierarchy in which the quadratic and cubic functions [Chapters 15 and 16] are higher members and which generalizes to the polynomial functions discussed Chapter 17. All the "named" polynomials [Chapters 18−24] have a linear function as one member of their families.

7:13 COGNATE FUNCTIONS

A frequent need in science and engineering is to approximate a function whose values, $f_0, f_1, f_2, \cdots, f_n$, are known only at a limited number of arguments x_0, x_1, x_2, $\cdots$, x_n, the so-called data points. The simplest way of constructing a function that fits all the known data, but one that is adequate in many applications, is by using a *piecewise-linear function*. In graphical terms, this implies simply "connecting the dots". For any argument lying between two adjacent data points, the interpolation

7:13:1
$$\mathrm{f}(x) = \frac{(x_{j+1} - x) f_j + (x - x_j) f_{j+1}}{x_{j+1} - x_j} \qquad x_j \le x \le x_{j+1}$$

applies. Usually a piecewise-linear function has discontinuities at all the interior data points. This defect is overcome by the "sliding cubic" and "cubic spline" interpolations exposed in Section 17:14.

The simplest reciprocal linear functions $1/(1 \pm x)$ serve as prototypes, or basis functions, for all $L = K$ hypergeometric functions [Section 18:14]. All the functions in Tables 18-1 and 18-2, as well as many others, may be "synthesized" [Section 43:14] from $1/(1+x)$ or $1/(1-x)$.

The reciprocal linear function is related to the function addressed in Section 15:4 because the shape of each is a *rectangular hyperbola*. Thus, clockwise rotation [Section 14:15] through an angle of $\pi/4$ of the curve $1/(bx + c)$ about the point $x = -c/b$ on the x-axis produces a new function,

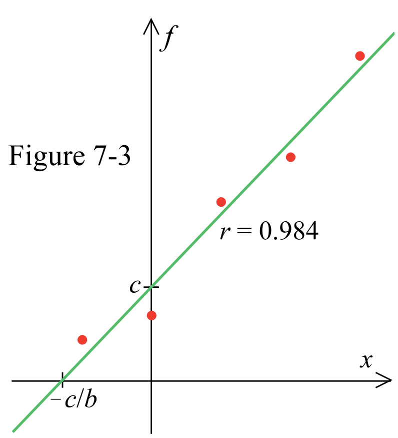
7:13:2
$$\pm\sqrt{\left(x + \frac{c}{b}\right)^2 + \frac{2}{b}}$$

that is a rectangular hyperbola of the class discussed in Section 15:4.

7:14 RELATED TOPIC: linear regression

Frequently experimenters collect data that are known, or believed, to obey the equation $f = \mathrm{f}(x) = bx + c$ but which incorporate errors. From the data, which consists of the n pairs of numbers (x_1,f_1), (x_2,f_2), (x_3,f_3), $\cdots$, (x_n,f_n), the scientist needs to find the b and c coefficients of the *best straight line* through the data, as in Figure 7-3. If the errors obey, or are assumed to obey, a Gaussian distribution [Section 27:14] and are entirely associated with the measurement of f (that is, the x values are exact), then the adjective "best" implies minimizing the sum of the squared deviations, $\sum (bx + c - f)^2$. The

Figure 7-3



procedure for finding the coefficients that achieve this minimization is known as *linear regression* or *least squares* and leads to the formulas

7:14:1 $$b = \frac{n\sum xf - \sum x \sum f}{n\sum x^2 - \left(\sum x\right)^2} = \frac{6\left[2\sum jf - (n+1)\sum f\right]}{n(n^2-1)h}$$

and

7:14:2 $$c = \frac{\sum x^2 \sum f - \sum x \sum xf}{n\sum x^2 - \left(\sum x\right)^2} = \frac{\sum f - b\sum x}{n} = \frac{\sum f}{n} - \frac{b(x_1+x_n)}{2}$$

An abbreviated notation, exemplified by

7:14:3 $$\sum xf = \sum_{j=1}^{n} x_j f_j$$

is used in the formulas of this section. Evaluation of these formulas simplifies considerably in the common circumstance in which data are gathered with equal spacing, this is when $x_2-x_1 = x_3-x_2 = \cdots = x_n-x_{n-1} = h$. The simplified formulas appear in red in equations 7:14:1, 7:14:2, 7:14:4 and 7:14:8.

A measure of how well the data obey the linear relationship is provided by the *correlation coefficient*, given by

7:14:4 $$r = \frac{n\sum xf - \sum x\sum f}{\sqrt{\left[n\sum x^2 - \left(\sum x\right)^2\right]\left[n\sum f^2 - \left(\sum f\right)^2\right]}} = b\sqrt{\frac{n\sum x^2 - \left(\sum x\right)^2}{n\sum f^2 - \left(\sum f\right)^2}} = \frac{nhb}{6}\sqrt{\frac{3(n^2-1)}{n\sum f^2 - \left(\sum f\right)^2}}$$

Values close to ± 1 imply a good fit of the data to the linear function, whereas r will be close to zero if there is little or no correlation between f and x. Sometimes r^2 is cited instead of r.

Commonly there is a need to know not only what the best values are of the slope b and the intercept c but also what uncertainties attach to these best values. Quoting their *standard errors* [Section 40:14] in the format

7:14:5 $$\text{slope} = b \pm \Delta b \qquad \text{where} \quad \Delta b = \text{standard error in } b$$

and

7:14:6 $$\text{intercept} = c \pm \Delta c \qquad \text{where} \quad \Delta c = \text{standard error in } c$$

is a succinct way of reporting the uncertainties associated with least squares determinations. the significance to be attached to these statements is that the probability is approximately 68% that the true slope will lie between $b-\Delta b$ and $b+\Delta b$. Similarly, there is a 68% probability that the true intercept lies in the range $c \pm \Delta c$. The formulas giving these standard errors are

7:14:7 $$\Delta b = \sqrt{\frac{n\sum f^2 - \left(\sum f\right)^2}{n\sum x^2 - \left(\sum x\right)^2} - \left[\frac{n\sum xf - \sum x\sum f}{n\sum x^2 - \left(\sum x\right)^2}\right]^2} = b\sqrt{\frac{1}{r^2} - 1}$$

and

7:14:8 $$\Delta c = \Delta b\sqrt{\frac{\sum x^2}{n}} = \frac{\Delta b}{2}\sqrt{\left(x_1+x_n\right)^2 + \frac{(n^2-1)h^2}{3}}$$

A related, but simpler, problem is the construction of the best straight line through the points (x_1,f_1), (x_2,f_2), (x_3,f_3), $\cdots$, (x_n,f_n), with the added constraint that the line *must* pass through the point (x_0,f_0). In practical problems this obligatory point is often the $x = 0$, $f = 0$ origin. Equations 7:14:1 and 7:14:2 should *not* be used in these circumstances, though they often are. The appropriate replacements are

7:14:9
$$b = \frac{\sum (x - x_0)(f - f_0)}{\sum (x - x_0)^2}$$

and

7:14:10
$$c = f_0 - bx_0$$

Equations 7:14:1−7:14:8 are based on the assumption that all data points are known with equal reliability, a condition that is not always valid. Variable reliability can be treated by assigning different *weights* to the points. If the value f_1 is more reliable than f_2, then a larger weight w_1 is assigned to the data pair (x_1, f_1) than the weight w_2 assigned to point (x_2, f_2). The weights then appear as multipliers in all summations, leading to the formulas

7:14:11
$$b = \frac{\sum w \sum wxf - \sum wx \sum wf}{\sum w \sum wx^2 - \left(\sum wx\right)^2}$$

and

7:14:12
$$c = \frac{\sum wf - b \sum wx}{\sum w}$$

for the slope and intercept. Only the *relative* weights are of import; the absolute values of w_1, w_2, w_3, $\cdots$, w_n have no significance beyond this. In practice, one attempts to assign a weight w_j to the jth point that is inversely proportional to the square of the uncertainty in f_j. Notice that formulas 7:14:1 and 7:14:2 are the special cases of 7:14:11 and 7:14:12 in which all w's are equal. Similarly, the formulas in 7:14:9 and 7:14:10 result from setting the weight of one point, (x_0, f_0), to be overwhelmingly greater than all the other weights, which are uniform.