CHAPTER 6

THE BINOMIAL COEFFICIENTS $\binom{\nu}{m}$

Binomial coefficients occur widely throughout mathematics; for example in the expansions discussed in Section 6:14 and in the Leibniz theorem, equation 0:10:6.

Binomial coefficients are so named because they are the numerical multipliers that arise when a two-term sum such as $(a + b)$, a so-called *binomial*, is raised to a power. The corresponding numbers that arise in expansions of powers of such extended sums as $(a + b + c + d)$ are termed multinomial coefficients and are discussed briefly in Section 6:12.

6:1 NOTATION

We refer to ν and m, respectively, as the upper index and the lower index of the binomial coefficient. The lower index of the binomial coefficient $\binom{\nu}{m}$ is invariably a nonnegative integer, whereas the upper index may take any real value. Positive integer values of the upper index are frequent, however, and in that circumstance the notation $\binom{n}{m}$ is used. Alternative to $\binom{n}{m}$ are the notations ${}_nC_m$ and $C_m^{(n)}$ which have their origins in the role played by binomial coefficients in expressing the number of combinations of m objects selected from a group of n different objects [Section 2.14].

6:2 BEHAVIOR

When it is not zero, the binomial coefficient $\binom{n}{m}$ is invariably a positive integer. The values of such integers are shown in Figure 6-1 for small values of the indices. For a given n, the maximal value of $\binom{n}{m}$ occurs when $m = n/2$ if n is even, or jointly at $m = (n \pm 1)/2$ when n is odd.

When ν is not restricted to a nonnegative integer, the behavior of the binomial coefficient $\binom{\nu}{m}$ is illustrated in Figure 6-2, and (except when $m = 0$) includes positive, negative, and zero values. The magnitude of these values is modest for $-1 < \nu < m+1$ but increases towards $\pm\infty$ outside this range.

K.B. Oldham et al., *An Atlas of Functions, Second Edition*,
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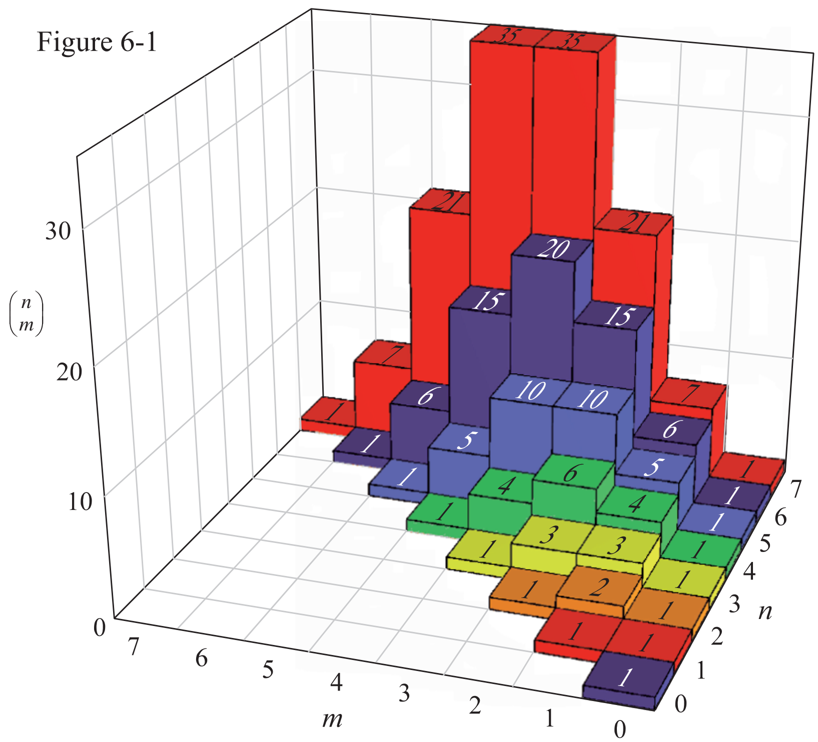
Figure 6-1



6:3 DEFINITIONS

The binomial coefficient is defined as the m-fold product

6:3:1
$$\binom{\nu}{m} = \left(\frac{\nu-m+1}{1}\right)\left(\frac{\nu-m+2}{2}\right)\left(\frac{\nu-m+3}{3}\right)\cdots\left(\frac{\nu}{m}\right) = \prod_{j=0}^{m-1}\frac{\nu-j}{m-j}$$

As is standard for empty products, this definition includes unity as the definition of $\binom{\nu}{0}$. The generating function [Section 0:3]

6:3:2
$$(1+t)^{\nu} = \sum_{m=0}^{\infty}\binom{\nu}{m}t^m \qquad -1<t<1$$

also defines the binomial coefficient.

A definition in terms of Pochhammer polynomials [Chapter 18]

6:3:3
$$\binom{\nu}{m} = \frac{(\nu-m+1)_m}{m!} = \frac{(-)^m(-\nu)_m}{m!}$$

is possible generally, whereas a definition exclusively in terms of the factorial function [Chapter 2]

6:3:4
$$\binom{n}{m} = \frac{n!}{m!(n-m)!}$$

requires that the upper index be a nonnegative integer.

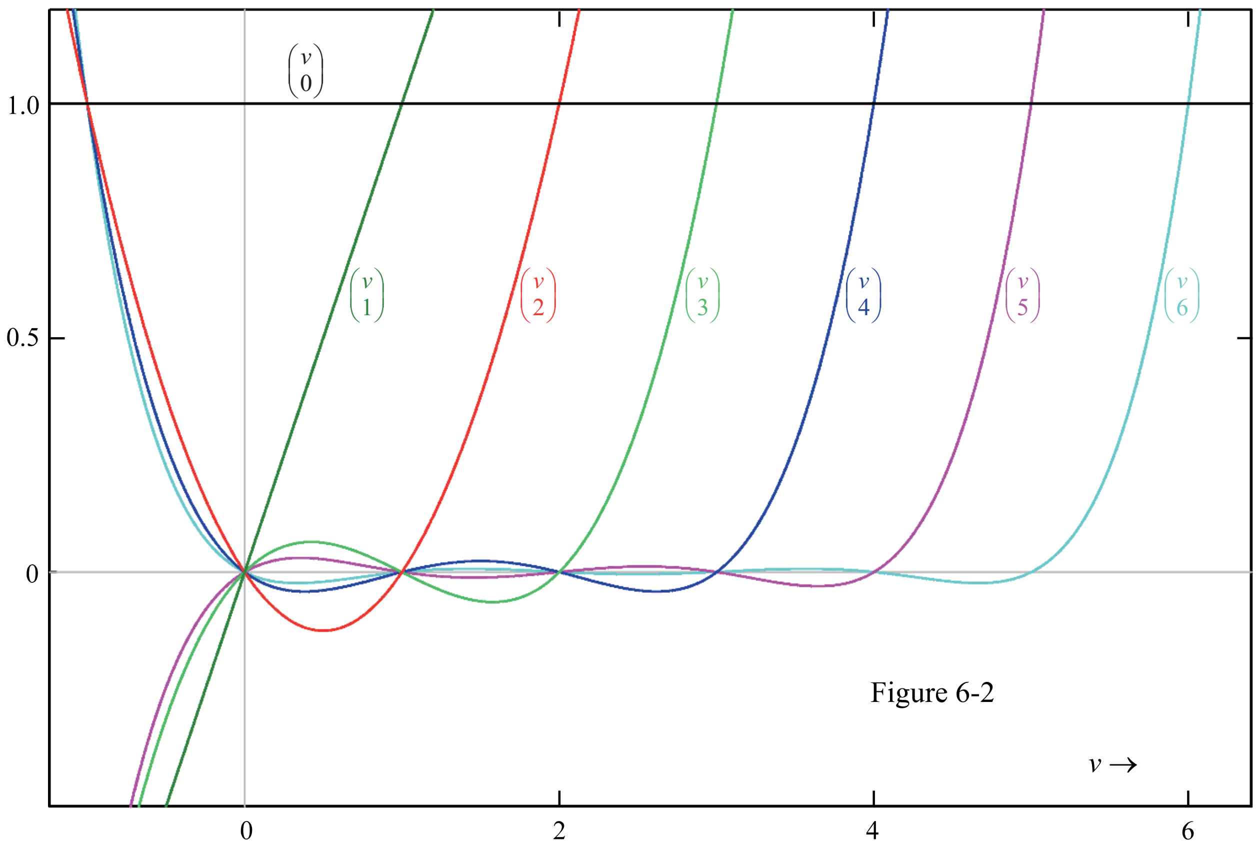


Figure 6-2

6:4 SPECIAL CASES

Reduction to an expression involving the double factorial [Section 2:13] occurs when the upper index is equal to twice the lower index

6:4:1 $$\binom{2m}{m} = \frac{4^m(2m-1)!!}{(2m)!!} = \frac{(2m)!}{(m!)^2} \qquad m = 0,1,2,\cdots$$

or differs by unity from twice the lower index

6:4:2 $$\binom{2m-1}{m-1} = \binom{2m-1}{m} = \frac{2^{2m-1}(2m-1)!!}{(2m)!!} = \frac{(2m)!}{2(m!)^2} \qquad m = 1,2,3,\cdots$$

6:5 INTRARELATIONSHIPS

There exist reflection formulas for the upper

6:5:1 $$\binom{-\nu}{m} = (-)^m \binom{\nu+m-1}{m}$$

and lower

6:5:2 $$\binom{n}{n-m} = \binom{n}{m}$$

indices, as well as recursion formulas

6:5:3 $$\binom{\nu+1}{m} = \binom{\nu}{m} + \binom{\nu}{m-1}$$

6:5:4 $$\binom{\nu}{m+1} = \frac{\nu-m}{m+1}\binom{\nu}{m}$$

for each index. The addition formula

6:5:5 $$\binom{\nu+\mu}{m} = \sum_{j=0}^{m}\binom{\nu}{j}\binom{\mu}{m-j}$$

known as *Vandermonde's convolution*, applies to the upper index.

There are a number of intrarelationships involving sums of binomial coefficients, including

6:5:6 $$\sum_{j=0}^{n-m}\binom{j+m}{m} = \binom{n+1}{m+1} \qquad n > m$$

6:5:7 $$\sum_{j=0}^{m}(-)^j\binom{\nu}{j} = (-)^m\binom{\nu-1}{m}$$

and

6:5:8 $$\sum_{j=0}^{J}\binom{n}{j}\binom{n}{n-J+j} = \binom{2n}{J} \qquad n \ge J$$

Formula 6:5:5 provides an expression for a sum of products of binomial coefficients; a second such formula is

6:5:9 $$\sum_{j=0}^{J}\binom{j}{m}\binom{j+\nu}{m} = \binom{m+\nu}{m}\binom{J+1+\nu}{J-m} \qquad J \ge m$$

When the upper index is an integer, finite, or infinite series of binomial coefficients frequently have simple expressions; examples include

6:5:10 $$\sum_{j=0}^{J}\binom{n}{j} = 2^n \qquad J \ge n$$

6:5:11 $$\sum_{j=0}^{J}(-)^j\binom{n}{j} = 0 \qquad J \ge n$$

6:5:12 $$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots + \binom{n}{J} = 2^{n-1} \qquad n \le J = 2, 4, 6, \cdots$$

6:5:13 $$\binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots + \binom{n}{J} = 2^{n-1} \qquad n \le J = 1, 3, 5, \cdots$$

and

6:5:14 $$\sum_{j=0}^{J} j\binom{n}{j} = 2^{n-1}n \qquad J \ge n$$

Similarly, the sum of squares of binomial coefficients gives

6:5:15
$$\sum_{j=0}^{J}\binom{n}{j}^2=\binom{2n}{n} \qquad J\geq n$$

6:6 EXPANSIONS

A binomial coefficient may be expanded as a power series in its upper index by the formula

6:6:1
$$\binom{v}{m}=\frac{1}{m!}\sum_{j=0}^{m}\mathrm{S}_m^{(j)}v^j$$

in which $\mathrm{S}_m^{(j)}$ is a Stirling number of the first kind [Section 18:6]. Additionally, definition 6:3:1 constitutes the expansion of $\binom{v}{m}$ as a finite product.

6:7 PARTICULAR VALUES

	$\binom{-1}{m}$	$\binom{-1/2}{m}$	$\binom{0}{m}$	$\binom{1/2}{m}$	$\binom{1}{m}$	$\binom{2m-1}{m}$	$\binom{2m}{m}$	$\binom{2m+1}{m}$
$m=0$	1	1	1	1	1	1	1	1
$m=1$	−1	$-1/2$	0	$1/2$	1	1	2	3
$m=2$	1	$3/8$	0	$-1/8$	0	3	6	10
$m=2,3,\cdots$	$(-1)^m$	$\frac{(-)^m(2m-1)!!}{(2m)!!}$	0	$\frac{(-)^{m+1}(2m-3)!!}{(2m)!!}$	0	$\frac{4^m(2m-1)!!}{2(2m)!!}$	$\frac{4^m(2m-1)!!}{(2m)!!}$	$\frac{4^{m+1}(2m-1)!!}{2(2m+2)!!}$

In addition to the cases tabulated above, there are the particular values

6:7:1
$$\binom{v}{0}=1 \qquad \text{and} \qquad \binom{v}{1}=v$$

which apply for all values of the upper index.

6:8 NUMERICAL VALUES

Binomial coefficients of integer indices may be arranged in the triangular arrangement shown on the right, in which each entry is the sum of the two above. This is commonly called *Pascal's triangle* (Blaise Pascal, French physicist and philosopher, 1623 – 1662), though it was described much earlier by the twelfth century mathematician Yanghui.

Equator's binomial coefficient routine (keyword **bincoef**) for calculating accurate values of $\binom{v}{m}$ is based on equation 6:3:1. The largest value of v for which all $\binom{v}{m}$ are calculable is 1029.3.

6:9 LIMITS AND APPROXIMATIONS

The limits

6:9:1 $$\binom{v}{m} \to \frac{v^m}{m!}\left[1 - \frac{m(m-1)}{2v} + \frac{m(m-1)(m-2)(3m-1)}{24v^2} - \cdots\right] \qquad v \to \pm\infty$$

and

6:9:2 $$\binom{v}{m} \to \frac{(-1)^m}{m^{v+1}\Gamma(-v)}\left[1 + \frac{v(v-1)}{2m} + \frac{v(v-1)(v-2)(3v-1)}{24m^2} + \cdots\right] \qquad m \to \infty$$

govern the behavior of the binomial coefficient when one index, but not both, is large. Of course limit 6:9:2, though universally valid, is useful only when v is not a nonnegative integer; otherwise $\binom{v}{m}$ is zero for all $m > v$.

If both indices are large and positive, and especially when m lies in the vicinity of $v/2$, the binomial coefficient is well approximated by the *Laplace-de Moivre formula*

6:9:3 $$\binom{v}{m} \approx 2^v \sqrt{\frac{2}{v\pi}} \exp\left\{\frac{-2}{v}\left(m - \frac{v}{2}\right)^2\right\} \qquad v \text{ large}$$

which finds statistical applications [Sokolnikoff and Redheffer pages 623−626]. Even for v as small as 10, the approximation leads to small absolute errors, as shown by Figure 6-3, in which the points are $\binom{10}{m}$ with $m = 0, 1, 2, \cdots, 11$ and the line is based on approximation 6:9:3 with $v = 10$ and m treated as a continuous variable.

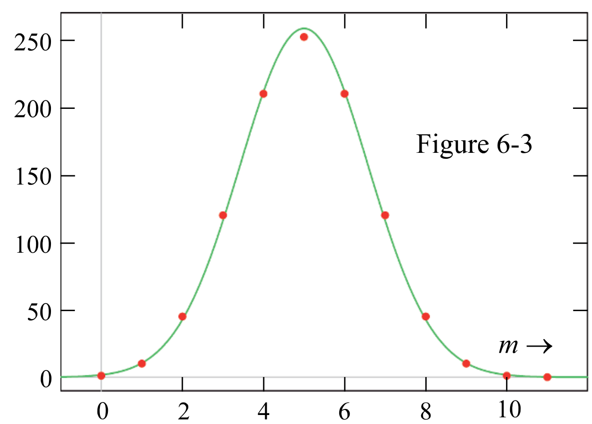


Figure 6-3

6:10 OPERATIONS OF THE CALCULUS

Differentiation with respect to the upper index gives a derivative involving a difference of two digamma functions [Chapter 44]:

6:10:1 $$\frac{\partial}{\partial v}\binom{v}{m} = \binom{v}{m}\left[\frac{1}{v} + \frac{1}{v-1} + \frac{1}{v-2} + \cdots + \frac{1}{v-m+1}\right] = \binom{v}{m}\left[\psi(-v) - \psi(m-v)\right]$$

6:11 COMPLEX ARGUMENT

The equivalence presented in equation 6:13:1 permits the formulas of Section 43:11 to be used to evaluate a binomial coefficient when one or both of the indices are imaginary or complex.

6:12 GENERALIZATIONS

The restriction that the lower index of a binomial coefficient be an integer may be relaxed by making use of the identity

6:12:1 $$\binom{\nu}{m} = \frac{1}{m\,\mathrm{B}(m,\nu-m+1)} \qquad m \neq 0$$

involving the complete beta function B [Section 43:13]. Thus the quantity 1/[μB(μ,ν−μ+1)] can be regarded as a generalized binomial coefficient, where μ, replacing m, is not necessarily an integer.

A generalization in a different direction is provided by *multinomial coefficients*. These are the coefficients that arise in such expansions as

6:12:2 $$\begin{aligned}(a+b+c)^5 &= \left[a^5+b^5+c^5\right]+5\left[a^4b+a^4c+ab^4+ac^4+b^4c+bc^4\right] \\ &+10\left[a^3b^2+a^3c^2+a^2b^3+a^2c^3+b^3c^2+b^2c^3\right]+20\left[a^3bc+ab^3c+abc^3\right]+30\left[a^2b^2c+a^2bc^2+ab^2c^2\right]\end{aligned}$$

When only integer powers are considered, all multinomial coefficients are integers. The general expression is

6:12:3 $$(a_1+a_2+a_3+\cdots+a_n)^N = \sum \mathrm{M}(N,m_1,m_2,m_3,\cdots,m_n)\left[a_1^{\,m_1}a_2^{\,m_2}\cdots a_n^{\,m_n}+a_1^{\,m_2}a_2^{\,m_1}\cdots a_n^{\,m_n}+\cdots\right]$$

where the summation embraces all combinations of the nonnegative integer m parameters that satisfy

6:12:4 $$m_1+m_2+m_3+\cdots+m_n = N$$

These parameters are not necessarily all distinct. In the 6:12:2 example, where n = 3 and N = 5, there are five summands because of the five ways (5+0+0 , 4+1+0 , 3+2+0 , 3+1+1 , 2+2+1) in which the number 5 can be composed by addition of three nonnegative integers. The multinomial coefficients themselves are given by

6:12:5 $$\mathrm{M}(N,m_1,m_2,m_3,\cdots,m_n) = \frac{N!}{(m_1)!(m_2)!(m_3)!\cdots(m_n)!}$$

For example, the second right-hand term in 6:12:2 has the multinomial (specifically a trinomial) coefficient of 5, equal to 5!/(4!1!0!). See Abramowitz and Stegun (pages 831−832) for a lengthy listing of multinomial coefficients. N need not be positive; see equation 15:6:1 for a counterexample.

6:13 COGNATE FUNCTIONS

The Pochhammer polynomial [Chapter 18], the factorial function [Chapter 2], and the (complete) beta function [Section 43:13] are all allied to the binomial coefficient, to which they are related through equations 6:3:3, 6:3:4, and 6:12:1. Because Γ(ν) generalizes (ν−1)!, the gamma function [Chapter 43] too is related through

6:13:1 $$\frac{\Gamma(\nu+1)}{\Gamma(m+1)\Gamma(\nu-m+1)} = \binom{\nu}{m}$$

to the binomial coefficient.

6:14 RELATED TOPIC: binomial expansions

The *binomial theorem*, also known as *Newton's formula*, permits a binomial to be raised to any power:

6:14:1 $$(a+b)^\nu = \begin{cases} a^\nu\left[1+\binom{\nu}{1}\frac{a}{b}+\binom{\nu}{2}\frac{a^2}{b^2}+\binom{\nu}{3}\frac{a^3}{b^3}+\cdots\right] = \sum_{m=0}^{\infty}\binom{\nu}{m}a^{\nu-m}b^m & -b<a<b \\ b^\nu\left[1+\binom{\nu}{1}\frac{b}{a}+\binom{\nu}{2}\frac{b^2}{a^2}+\binom{\nu}{3}\frac{b^3}{a^3}+\cdots\right] = \sum_{m=0}^{\infty}\binom{\nu}{m}a^m b^{\nu-m} & -a<b<a \end{cases}$$

This is a special case of the Taylor expansion, equation 0:5:1. The resulting series, known as a *binomial series*, terminates if v is a positive integer (in which case there is no restriction on the a/b ratio), but is infinite otherwise.

This chapter concludes with a catalog of some important binomial series. Frequently, binomials consist of a variable paired with a constant; the simplest of such pairings, $1\pm x$, serve in this role in the following examples.

6:14:2 $$(1\pm x)^n = 1\pm nx + \frac{n(n-1)}{2}x^2 \pm \frac{n(n-1)(n-2)}{6}x^3 + \cdots + n(\pm x)^{n-1} + (\pm x)^n = \sum_{m=0}^{n}\frac{n!}{m!(m-n)!}(\pm x)^n$$

6:14:3 $$(1\pm x)^4 = 1\pm 4x + 6x^2 \pm 4x^3 + x^4$$

6:14:4 $$(1\pm x)^3 = 1\pm 3x + 3x^2 \pm x^3$$

6:14:5 $$(1\pm x)^2 = 1\pm 2x + x^2$$

6:14:6 $$(1\pm x)^{3/2} = 1\pm \frac{3}{2}x + \frac{3}{8}x^2 \mp \frac{1}{16}x^3 + \frac{3}{128}x^4 \mp \frac{3}{256}x^5 + \frac{7}{1024}x^6 \mp \frac{9}{2048}x^7 + \cdots = 3\sum_{m=0}^{\infty}\frac{(2m-5)!!}{m!}\left(\frac{\mp x}{2}\right)^m$$

6:14:7 $$(1\pm x)^{1/2} = 1\pm \frac{1}{2}x - \frac{1}{8}x^2 \pm \frac{1}{16}x^3 - \frac{5}{128}x^4 \pm \frac{7}{256}x^5 - \frac{21}{1024}x^6 \pm \frac{33}{2048}x^7 - \cdots = -\sum_{m=0}^{\infty}\frac{(2m-3)!!}{m!}\left(\frac{\mp x}{2}\right)^m$$

6:14:8 $$(1\pm x)^{-1/2} = 1\mp \frac{1}{2}x + \frac{3}{8}x^2 \mp \frac{5}{16}x^3 + \frac{35}{128}x^4 \mp \frac{63}{256}x^5 + \frac{231}{1024}x^6 \mp \frac{429}{2048}x^7 + \cdots = \sum_{m=0}^{\infty}\frac{(2m)!}{(m!)^2}\left(\frac{\mp x}{4}\right)^m$$

6:14:9 $$(1\pm x)^{-1} = 1\mp x + x^2 \mp x^3 + x^4 \mp x^5 + x^6 \mp x^7 + x^8 \mp x^9 + x^{10} \mp x^{11} + x^{12} \mp \cdots = \sum_{m=0}^{\infty}(\mp x)^m$$

6:14:10 $$(1\pm x)^{-3/2} = 1\mp \frac{3}{2}x + \frac{15}{8}x^2 \mp \frac{35}{16}x^3 + \frac{315}{128}x^4 \mp \frac{693}{256}x^5 + \frac{2003}{1024}x^6 \mp \frac{6435}{2048}x^7 + \cdots = \sum_{m=0}^{\infty}\frac{(2m+1)!}{(m!)^2}\left(\frac{\mp x}{4}\right)^m$$

6:14:11 $$(1\pm x)^{-2} = 1\mp 2x + 3x^2 \mp 4x^3 + 5x^4 \mp 6x^5 + 7x^6 \mp 8x^7 + 9x^8 \mp 10x^9 + 11x^{10} \mp \cdots = \sum_{m=0}^{\infty}(m+1)(\mp x)^m$$

6:14:12 $$(1\pm x)^{-3} = 1\mp 3x + 6x^2 \mp 10x^3 + 15x^4 \mp 21x^5 + 28x^6 \mp 36x^7 + 45x^8 \mp \cdots = \sum_{m=0}^{\infty}\frac{(m+1)(m+2)}{2}(\mp x)^m$$

6:14:13 $$(1\pm x)^{-4} = 1\mp 4x + 10x^2 \mp 20x^3 + 35x^4 \mp 56x^5 + 84x^6 \mp 120x^7 + \cdots = \sum_{m=0}^{\infty}\frac{(m+1)(m+2)(m+3)}{6}(\mp x)^m$$

6:14:14 $$(1\pm x)^{-n} = 1\mp nx + \frac{n(n+1)}{2!}x^2 \mp \frac{n(n+1)(n+2)}{3!}x^3 + \cdots = \sum_{m=0}^{\infty}\frac{(m+1)(m+2)\ldots(m+n-1)}{(n-1)!}(\mp x)^m$$

See Section 2:13 for the double factorial function ( )!!. Series 6:14:6–14 converge if $-1 < x < 1$; sometimes they are also convergent if $x = 1$ or if $x = -1$. The general expression, valid for any value of v may be written in terms of the Pochhammer polynomial as

6:14:15 $$(1\pm x)^{-v} = \sum_{m=0}^{\infty}\frac{(v)_m}{m!}(\mp x)^m \qquad -1 < x < 1$$

Sometimes useful is the finite series plus remainder

6:14:16 $$(1\pm x)^{v} = \sum_{j=0}^{J-1}\frac{(-v)_j}{j!}(\mp x)^j + \frac{(-v)_J}{(J-1)!}(\mp x)^J\int_0^1 (1-t)^{J-1}(1+xt)^{v-J}\,\mathrm{d}t \qquad -1 < x < 1$$