CHAPTER 64

THE HURWITZ FUNCTION $\zeta(v, u)$

Together with the bivariate eta function and the Lerch function, both of which are also addressed in this chapter, the Hurwitz function provides a means of summing the interesting series listed in formulas 64:3:3, 64:13:1, and 64:12:3. These functions can all be characterized as "logarithm-like". The Hurwitz function plays an invaluable role in the Weyl differintegration of periodic functions, a topic discussed in Section 64:14.

64:1 NOTATION

The symbol $\zeta(\ ,\ )$ is standard for this function, but a variety of names − *generalized zeta function*, *Riemann's zeta function*, *Riemann's function*, *generalized Riemann zeta function*, *bivariate zeta function*, *Hurwitz zeta function*, and *Hurwitz function* − are commonly applied to it. We adopt the last of these names to avoid confusion with the function of Chapter 3 and to recognize the contributions of the German mathematician Adolf Hurwitz (1859 − 1919).

The variables v and u will be respectively termed the order and parameter of the Hurwitz function $\zeta(v,u)$. It is to achieve unity with the notation for the Lerch function [Section 64:12] that we resist the temptation to call u the argument of the function.

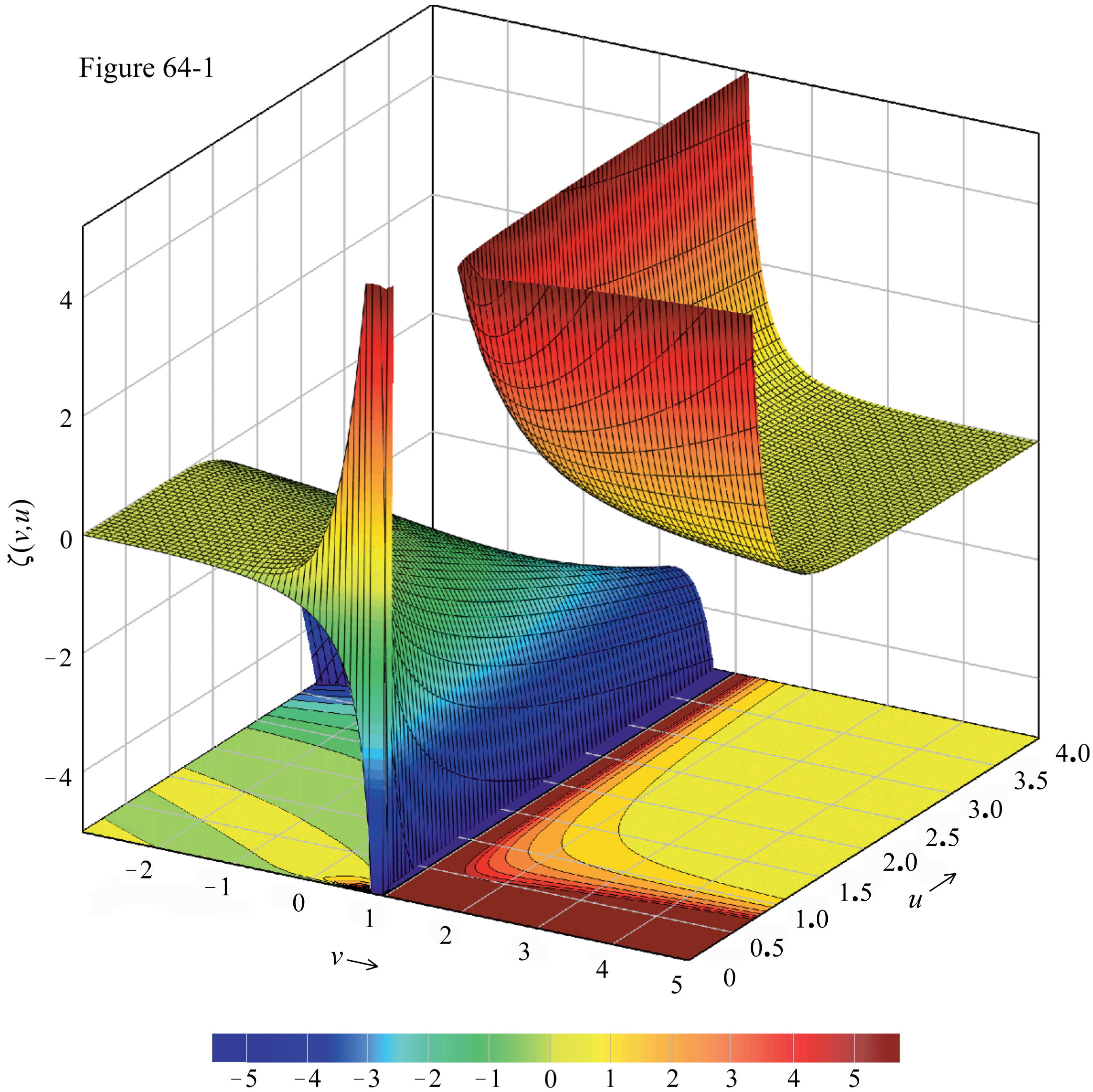
64:2 BEHAVIOR

We generally treat only real values of v and u, and exclude $v = 1$, where a $-\infty|+\infty$ discontinuity occurs. There is no unanimity in the definition of the Hurwitz function for negative parameter and accordingly the $u < 0$ domain is generally omitted from consideration in this *Atlas*, as it is in Figure 64-1. The status of $\zeta(v,0)$ is also questionable; here we regard it as infinite when v is greater then zero, but elsewhere it is considered to equal $\zeta(v,1)$.

The discontinuity in the Hurwitz function at $v = 1$ is the dominant feature in the landscape of the Hurwitz function shown in the figure. The function is invariably positive for $v > 1$, but it may have either sign for $v < 1$. In the latter domain, the Hurwitz function has a number of zeros, concentrated in the region of small u.

K.B. Oldham et al., *An Atlas of Functions, Second Edition*,
DOI 10.1007/978-0-387-48807-3_65,

Figure 64-1



64:3 DEFINITIONS

The difference between two digamma functions [Chapter 44] provides a generating function for Hurwitz functions of integer orders of 2 and greater

64:3:1 $$t\left[\psi(u)-\psi(u-t)\right]=t^2\zeta(2,u)+t^3\zeta(3,u)+t^4\zeta(4,u)+\cdots=\sum_{n=2}^{\infty}\zeta(n,u)t^n$$

The Hurwitz function may be defined through the integral

64:3:2 $$\Gamma(v)\zeta(v,u)=\int_0^\infty \frac{t^{v-1}\exp(-ut)}{1-\exp(-t)}\mathrm{d}t \qquad v>1,\ u>0$$

which may be regarded as a Laplace transform [Section 26:15]. More complicated is *Hermite's integral*

64:3:3 $$\zeta(v,u)=\frac{u^{-v}}{2}+\frac{u^{1-v}}{v-1}+2\int_0^\infty \frac{\sin\left(v\arctan(t/u)\right)}{\left(u^2+t^2\right)^{v/2}[\exp(2\pi t)-1]}\mathrm{d}t$$

and

64:3:4 $$\zeta(v,u)=\frac{1}{\Gamma(v)}\int_0^1\frac{t^{u-1}\left[-\ln(t)\right]^{v-1}}{(1-t)}\,\mathrm{d}t \qquad v>1,\ u>0$$

The most transparent definition is as the series

64:3:5 $$\zeta(v,u)=\frac{1}{u^v}+\frac{1}{(1+u)^v}+\frac{1}{(2+u)^v}+\frac{1}{(3+u)^v}+\cdots=\sum_{j=0}^{\infty}(j+u)^{-v} \qquad v>1$$

but this converges only when the order exceeds unity, and then often very slowly. To extend this definition to cover nonpositive integer u, some authorities, but not this *Atlas*, exclude from the defining series any term that generates an infinity. With a similar objective, other authors replace the summand in 64:3:3 by $[(j+u)^2]^{-v/2}$, but this modification is not adopted here either. A definition, valid only in the narrow domain of the parameter, is provided by *Hurwitz's formula*

64:3:6 $$\frac{\zeta(v,u)}{\Gamma(1-v)}=2\sum_{k=1}^{\infty}\frac{\sin(2k\pi u+\frac{1}{2}\pi v)}{(2k\pi)^{1-v}} \qquad v<0, \quad 0\le u\le 1$$

Ways to mitigate the restrictions imposed by these series definitions are discussed in Section 64:6.

The Hurwitz function may be defined also by Weyl differintegration [Section 64:14] of a simple algebraic function with respect to the logarithm of the differintegration variable:

64:3:7 $$\zeta(v,u)=\frac{\mathrm{d}^{-v}}{[\mathrm{d}\ln(t)]^{-v}}\frac{t^u}{1-t}\bigg|_{-\infty}^{0}$$

Some of the definitions in this section may be extended to noninteger negative parameters, $u<0$, but neither the *Atlas* nor *Equator* caters to this domain.

64:4 SPECIAL CASES

When the order is unity, the Hurwitz function suffers a discontinuity but, for any other positive integer order n, $\zeta(v,u)$ is equivalent to a polygamma function [Section 44:12]

64:4:1 $$\zeta(n,u)=\frac{(-)^n\psi^{(n-1)}(u)}{(n-1)!} \qquad n=2,3,4,\cdots$$

When the order is a nonpositive integer $-n$, the Hurwitz function can be expressed as a Bernoulli polynomial [Chapter 19]

64:4:2 $$\zeta(-n,u)=\frac{-\mathrm{B}_{n+1}(u)}{n+1} \qquad n=0,1,2,\cdots$$

The first few cases are

$\zeta(0,u)$	$\zeta(-1,u)$	$\zeta(-2,u)$	$\zeta(-3,u)$	$\zeta(-4,u)$
$\frac{1}{2}-u$	$\frac{-1}{12}+\frac{1}{2}u-\frac{1}{2}u^2$	$\frac{-1}{6}u+\frac{1}{2}u^2-\frac{1}{3}u^3$	$\frac{1}{120}-\frac{1}{4}u^2+\frac{1}{2}u^3-\frac{1}{4}u^4$	$\frac{1}{30}u-\frac{1}{3}u^3+\frac{1}{2}u^4-\frac{1}{5}u^5$

64:5 INTRARELATIONSHIPS

An obvious consequence of definition 64:3:5 is the recursion formula

64:5:1 $$\zeta(\nu,u+1)=\zeta(\nu,u)-u^{-\nu}$$

and this may be iterated to

64:5:2 $$\zeta(\nu,J+u)=\zeta(\nu,u)-\sum_{j=0}^{J-1}\frac{1}{(j+u)^{\nu}}$$

The duplication formula

64:5:3 $$\zeta(\nu,2u)=2^{-\nu}\left[\zeta(\nu,u)+\zeta(\nu,u+\tfrac{1}{2})\right]$$

may be generalized to

64:5:4 $$\zeta(\nu,mu)=m^{-\nu}\sum_{j=0}^{m-1}\zeta\left(\nu,u+\frac{j}{m}\right)\qquad m=2,3,4,\cdots$$

which leads to such relationships as

64:5:5 $$\zeta(\nu,u+\tfrac{1}{4})+\zeta(\nu,u+\tfrac{3}{4})=4^{\nu}\zeta(\nu,4u)-2^{\nu}\zeta(\nu,2u)$$

Certain series of Hurwitz functions of positive integer order, with monotone or alternating signs, may be summed:

64:5:6 $$\zeta(2,u)\pm\zeta(3,u)+\zeta(4,u)\pm\zeta(5,u)+\cdots=\begin{cases}1/(u-1) & u>1\\ 1/u & u>0\end{cases}$$

64:5:7 $$\frac{\zeta(2,u)}{2}\pm\frac{\zeta(3,u)}{3}+\frac{\zeta(4,u)}{4}\pm\frac{\zeta(5,u)}{5}+\cdots=\begin{cases}\psi(u)-\ln(u-1) & u>1\\ \ln(u)-\psi(u) & u>0\end{cases}$$

The latter sums involve the digamma function and logarithms [Chapters 44 and 25].

64:6 EXPANSIONS

The seminal expansion of the Hurwitz function is series 64:3:3. When this is incorporated into the Euler-Maclaurin formula 4:14:1, with h set to unity and the discrete j variable treated as a continuous variable t, the result

64:6:1 $$\zeta(\nu,u)=\sum_{j=0}^{\infty}(j+u)^{-\nu}\sim\int_0^{\infty}(t+u)^{-\nu}\mathrm{d}t+\sum_{n=0}\frac{\mathrm{B}_n}{n!}\frac{\mathrm{d}^{n+1}}{\mathrm{d}t^{n+1}}(t+u)^{-\nu}\bigg|_{t=0}^{t=\infty}$$

emerges. After carrying out the indicated integration and differentiations, one discovers the formula

64:6:2 $$\zeta(\nu,u)\sim\frac{1}{2u^{\nu}}+\frac{u^{1-\nu}}{\nu-1}\left[2-\sum_{j=0}\frac{(\nu-1)_{2j}\left|\mathrm{B}_{2j}\right|}{(2j)!}\left(\frac{-1}{u^2}\right)^{j}\right]$$

in which a Pochhammer polynomial [Chapter 18] occurs. Though technically asymptotic, this series converges well.

For computational purposes, it may sometimes be preferable to apply the Euler-Maclaurin transformation, not to the seminal series 64:3:3 itself, but to formula 64:5:2. This leads to

64:6:3 $$\zeta(\nu,u)\sim\sum_{j=0}^{J-1}\frac{1}{(j+u)^{\nu}}+\frac{(J+u)^{1-\nu}}{\nu-1}+\frac{1}{2(J+u)^{\nu}}+\frac{\nu}{12(J+u)^{1+\nu}}-\frac{\nu(\nu+1)(\nu+2)}{720(J+u)^{3+\nu}}+\cdots$$

where J is an arbitrary positive integer. The right-hand terms in this formula, other than the first, may be regarded as an expression for the remainder when the seminal series is truncated after the Jth term. Another expression for the same remainder is

64:6:4 $$\frac{1}{v-1}\left[\frac{J+u+v-1}{(J+u)^{v}}+\sum_{j=J}^{\infty}\frac{j+u+v}{(j+u+1)^{v}}-\frac{1}{(j+u)^{v-1}}\right]$$

Hurwitz's formula, equation 6:3:5, is valid over a narrow domain, but by invoking recursion 63:5:2 to derive

64:6:5 $$\zeta(v,u)=2\Gamma(1-v)\sum_{k=1}^{\infty}\frac{\sin(2k\pi u+\frac{1}{2}\pi v)}{(2k\pi)^{1-v}}-\sum_{j=0}^{\mathrm{Int}(u)-1}\left[j+\mathrm{frac}(u)\right]^{-v}\qquad v<1,\quad u\geq 0$$

its validity may be broadened.

64:7 PARTICULAR VALUES

Although the Hurwitz function is often known as the "generalized zeta function", there is an important mismatch between the definition $\sum(j+u)^{-v}$ of the Hurwitz function $\zeta(v,u)$ and the definition $\sum j^{-v}$ of the *zeta number* $\zeta(v)$ [Chapter 3] in that the former definition starts at $j=0$, whereas the latter starts at $j=1$. Hence when $u=0$, the Hurwitz function reduces to the zeta number only when the order is nonpositive. Particular values of the Hurwitz function for instances of positive integer parameters, and in the general case, are

$\zeta(v,0)$	$\zeta(v,1)$	$\zeta(v,2)$	$\zeta(v,3)$	$\zeta(v,4)$	$\zeta(v,m)$
$\begin{cases}\infty & v>0\\ \zeta(v) & v\leq 0\end{cases}$	$\zeta(v)$	$\zeta(v)-1$	$\zeta(v)-1-2^{-v}$	$\zeta(v)-1-2^{-v}-3^{-v}$	$\zeta(v)-\sum_{j=1}^{m-1}\frac{1}{j^{v}}$

For parameters that equal an odd multiple of ½, the Hurwitz function may be expressed in terms of the *lambda number* of Chapter 3

$\zeta(v,\frac{1}{2})$	$\zeta(v,\frac{3}{2})$	$\zeta(v,\frac{5}{2})$	$\zeta(v,m+\frac{1}{2})$
$2^{v}\lambda(v)$	$2^{v}\left[\lambda(v)-1\right]$	$2^{v}\left[\lambda(v)-1-3^{-v}\right]$	$2^{v}\left[\lambda(v)-\sum_{j=1}^{m}(2j-1)^{-v}\right]$

When the parameter is an odd multiple of ¼, the Hurwitz function involves also the *beta number* from the same chapter. The prototypes are

64:7:1 $$\zeta(v,\tfrac{1}{4})=4^{v}\left[\frac{\lambda(v)+\beta(v)}{2}\right]\qquad\text{and}\qquad\zeta(v,\tfrac{3}{4})=4^{v}\left[\frac{\lambda(v)-\beta(v)}{2}\right]$$

Notice that these formulas concur with the general rule 64:5:4, when 64:5:3 is taken into account.

64:8 NUMERICAL VALUES

Equator provides accurate values of $\zeta(v,u)$ for all $|v|\leq 100$ and $0\leq u\leq 100$. With keyword **Hurwitz**, the Hurwitz function routine uses formula 64:4:2 for negative integer orders and equation 64:6:5 for negative noninteger v not greater than -3.5. For all other orders, the expansion 64:6:2 is exploited via the ε-transformation [Section 10:14].

64:9 LIMITS AND APPROXIMATIONS

When the order is unity, the Hurwitz function suffers a discontinuity, but certain modifications of the Hurwitz function remain finite as $v = 1$ is approached. Thus

64:9:1 $$\lim_{v\to 1}\left\{\frac{\zeta(v,u)}{\Gamma(1-v)}\right\} = -1$$

and

64:9:2 $$\lim_{v\to 1}\left\{\zeta(v,u) - \frac{1}{v-1}\right\} = -\psi(u)$$

These results apply irrespective of whether unity is approached from smaller or larger values.

When u is small, the approximation

64:9:3 $$\zeta(v,u) \approx u^{-v} + \zeta(v) - vu\zeta(v+1) \qquad \text{small } u$$

involving the *digamma function* [Chapter 44], holds. Adding terms $\binom{-v}{j}u^j\,\zeta(v+j)$ with $j = 2,3,\cdots$ progressively improves the approximation.

64:10 OPERATIONS OF THE CALCULUS

Differentiation with respect to the *order* yields

64:10:1 $$\frac{\partial}{\partial v}\zeta(v.u) = -\sum_{j=0}^{\infty}\frac{\ln(j+u)}{(j+u)^v} \qquad v > 0$$

of which the special case

64:10:2 $$\frac{\partial\zeta}{\partial v}(0,u) = \ln\left(\frac{\Gamma(u)}{\sqrt{2\pi}}\right)$$

is noteworthy. Single and multiple differentiations with respect to the *parameter* yield

64:10:3 $$\frac{\partial}{\partial u}\zeta(v,u) = -v\zeta\left(v+1,u\right) \qquad \text{and} \qquad \frac{\partial^n}{\partial u^n}\zeta(v,u) = (-)^n\left(v\right)_n\zeta\left(v+n,u\right)$$

and these formulas may be generalized to the differintegration [Section 12:14] result

64:10:4 $$\frac{\partial^\mu}{\partial u^\mu}\zeta(v,u) = \frac{\Gamma(1-v)}{\Gamma(1-v-\mu)}\zeta\left(v+\mu,u\right)$$

where μ is not necessarily an integer. In these formulas $(v)_n$ denotes a Pochhammer polynomial [Chapter 18] and Γ symbolizes the gamma function [Chapter 43]. Note that both formulas in 64:10:3 accord with 64:10:4, as does 64:10:5.

Formulas for indefinite integration with respect to the parameter include:

64:10:5 $$\int_u^\infty \zeta(v,t)\,\mathrm{d}t = \frac{\zeta(v-1,u)}{v-1} \qquad v > 2$$

64:10:6 $$\int_1^u \zeta(v,t)\,\mathrm{d}t = \frac{\zeta(v-1,u) - \zeta(v-1)}{1-v} \qquad v < 2$$

64:10:7
$$\int_0^u \zeta(v,t)\mathrm{d}t = \frac{\zeta(v-1,u)-\zeta(v-1)}{1-v} \qquad v<1$$

and lead to the following interesting definite integrals

64:10:8
$$\int_0^1 \zeta(v,t)\mathrm{d}t = 0 \qquad v<1$$

64:10:9
$$\int_1^2 \zeta(v,t)\mathrm{d}t = \frac{1}{v-1}$$

The parts-integration procedure [Section 0:10] that produces the result

64:10:10
$$\int_1^u t\,\zeta(v,t)\mathrm{d}t = \frac{u\zeta(v-1,u)-\zeta(v-1)}{1-v} - \frac{\zeta(v-2,u)-\zeta(v-2)}{(1-v)(2-v)}$$

may be iterated to generate expressions for integrals of $t^n\zeta(v,t)$ where $n=2,3,4,\cdots$.

The *Böhmer integrals* that are the subject of Section 39:12 appear in the integration formulas

64:10:11
$$\int_x^{1+x} \genfrac{}{}{0pt}{}{\sin}{\cos}(2\pi t)\zeta(v,t)\mathrm{d}t = (2\pi)^{v-1}\genfrac{}{}{0pt}{}{\mathrm{S}}{\mathrm{C}}(2\pi x,1-v) \qquad v>1, \quad x\ge 0$$

There is a close connection between these results and the discussion in Section 64:14.

64:11 COMPLEX ARGUMENT

There is interest in the function $\zeta\left(\frac{1}{2}+iy,u\right)$ in the context of *Riemann's hypothesis* [Section 3:11] but this topic will not be pursued here.

Inverse Laplace transformation leads to hyperbolic functions [Chapters 29 and 30]:

64:11:1
$$\int_{\alpha-i\infty}^{\alpha+i\infty} \zeta(v,s)\frac{\exp(st)}{2\pi i}\mathrm{d}s = \mathfrak{I}\{\zeta(v,s)\} = \frac{t^{v-1}}{2\Gamma(v)}\left[\coth\left(\frac{t}{2}\right)+1\right]$$

64:11:2
$$\int_{\alpha-i\infty}^{\alpha+i\infty} \zeta(v,s+\tfrac{1}{2})\frac{\exp(st)}{2\pi i}\mathrm{d}s = \mathfrak{I}\{\zeta(v,s+\tfrac{1}{2})\} = \frac{t^{v-1}\operatorname{csch}(t/2)}{2\Gamma(v)}$$

64:12 GENERALIZATION: the Lerch function

Named for the Czech mathematician Mathias Lerch (1860 – 1922), the trivariate function $\Phi(x,v,u)$ generalizes the Hurwitz function because

64:12:1
$$\Phi(1,v,u) = \zeta(v,u)$$

However, inasmuch as, for appropriate of the nonunity variables,

64:12:2
$$\Phi(x,1,u) = \frac{1}{x^u}\ln_u\left(\frac{1}{1-x}\right) \qquad \text{and} \qquad \Phi(x,v,1) = \frac{-1}{x}\mathrm{polyln}_v(1-x)$$

it may equally well be regarded as a generalization of either of the functions mentioned in Section 25:12 – the generalized logarithmic function, or the polylogarithm.

Note that, though the *Atlas* prefers that the terminal character in the symbol of a multivariate function be the argument, we defer to the convention that the three variables of the Lerch function are cited in the order: argument x, order ν, and parameter u. Though extensions may be possible, we generally impose the restrictions $|x| \le 1$, $\nu \ne 1$, and $u > 0$ throughout this section. Analogous to equations 64:3:2 and 64:3:5, are the definitions of the Lerch function as an integral

64:12:3 $$\Phi(x,\nu,u) = \frac{1}{\Gamma(\nu)} \int_0^\infty \frac{t^{\nu-1}\exp(-ut)}{1-x\exp(-t)}\,\mathrm{d}t$$

or as an infinite series

64:12:4 $$\Phi(x,\nu,u) = \frac{1}{u^\nu} + \frac{x}{(1+u)^\nu} + \frac{x^2}{(2+u)^\nu} + \frac{x^3}{(3+u)^\nu} + \cdots = \sum_{j=0}^\infty \frac{x^j}{(j+u)^\nu}$$

Yet another definition of the Lerch function is as a Weyl differintegral [Section 64:14] with respect to the logarithm of the argument:

64:12:5 $$\Phi(x,\nu,u) = x^{-u} \frac{\mathrm{d}^{-\nu}}{[\mathrm{d}\ln(t)]^{-\nu}} \frac{t^u}{1-t}\bigg|_{-\infty}^{\ln(x)} \qquad x > 0$$

Many of the properties of the Lerch function, such as its recursion

64:12:6 $$\Phi(x,\nu,1+u) = \frac{1}{x}\left[\Phi(x,\nu,u) - \frac{1}{u^\nu}\right]$$

echo those of the Hurwitz function. The following limit governs the approach of the argument to unity

64:12:7 $$\lim_{x\to 1}\left\{\frac{\Phi(x,\nu,u)}{(1-x)^{\nu-1}}\right\} = \Gamma(1-\nu) \qquad \nu < 1$$

A surprisingly large number of familiar functions arise by specializing one or more of the variables of the Lerch function. Specializations of the order to integer values lead to the following special cases:

$\Phi(x,1,u)$	$\Phi(x,0,u)$	$\Phi(x,-1,u)$	$\Phi(x,-n,u)$	$\Phi(-x,1,\tfrac{1}{2})$
$\dfrac{\mathrm{B}(u,0,x)}{x^u}$	$\dfrac{1}{1-x}$	$\dfrac{u+x-xu}{(1-x)^2}$	$u\Phi(x,1-n,u) + x\dfrac{\partial}{\partial x}\Phi(x,1-n,u)$	$\dfrac{2}{\sqrt{x}}\arctan\left(\sqrt{x}\right)$

in which an *incomplete beta function* [Chapter 58] is found. When the parameter is specialized, *polylogarithms* [Section 25:12] often appear:

$\Phi(x,1,1)$	$\Phi(x,2,1)$	$\Phi(x,1,\tfrac{1}{2})$	$\Phi(x,\nu,0)$	$\Phi(x,\nu,\tfrac{1}{2})$
$\dfrac{-\ln(1-x)}{x}$	$\dfrac{-\mathrm{diln}(1-x)}{x}$	$\dfrac{2}{\sqrt{x}}\mathrm{arctanh}\left(\sqrt{x}\right)$	$-\mathrm{polyln}_\nu(1-x)$	$\dfrac{\mathrm{polyln}_\nu(1-x)}{\sqrt{x}} - \dfrac{\mathrm{polyln}_\nu\left(1-\sqrt{x}\right)}{2^{-\nu}\sqrt{x}}$

Other special cases include

$\Phi(1,\nu,u)$	$\Phi(0,\nu,u)$	$\Phi(-1,\nu,u)$	$\Phi(1,\nu,2u)$	$\Phi(-x,1,\tfrac{1}{2})$	$\Phi(-\tfrac{1}{2},1,\tfrac{1}{2})$
$\zeta(\nu,u)$	$u^{-\nu}$	$\eta(\nu,u)$	$\dfrac{\zeta(\nu,u)+\zeta(\nu,u+\tfrac{1}{2})}{2^\nu}$	$\dfrac{2}{\sqrt{x}}\arctan\left(\sqrt{x}\right)$	$\dfrac{\pi}{\sqrt{2}}$

Equator's Lerch function routine (keyword **Lerch**) relies on expansion 64:12: 4. Because, when x approaches unity, this series is slow to converge, a more convergent series representing the remainder is appended following curtailment of the original expansion. The formula used by *Equator* is

64:12:8 $$\Phi(x,u,v) \approx \sum_{j=0}^{J-1} \frac{x^j}{(j+u)^v} - x^J \sum_{m=0}^{M} \binom{-v}{m} \frac{\text{polyln}_{-m}(x)}{(J+u)^{m+v}}$$

This expression for the remainder has its origin in equation 25:12:5. J is chosen large enough that only a few terms of the m-series are needed.

64:13 COGNATE FUNCTION: the bivariate eta function

The definition of this function as a series

64:13:1 $$\eta(v,u) = \frac{1}{u^v} - \frac{1}{(1+u)^v} + \frac{1}{(2+u)^v} - \frac{1}{(3+u)^v} + \cdots = \sum_{j=0}^{\infty} \frac{(-1)^j}{(j+u)^v}$$

differs from the corresponding definition of the Hurwitz function only by the presence of alternating signs. Other similarities to $\zeta(v,u)$ are its recurrence and duplication formulas:

64:13:2 $$\eta(v,u+1) = u^{-v} - \eta(v,u)$$

64:13:3 $$\eta(v,2u) = 2^{-v}\left[\zeta(v,u) - \zeta(v,u+\tfrac{1}{2})\right]$$

which likewise differ from their Hurwitz analogues only by signs. Equation 64:13:3 provides a route to calculate the bivariate eta function from the Hurwitz function; another way is

64:13:4 $$\eta(v,u) = 2^{1-v}\zeta(v,\tfrac{1}{2}u) - \zeta(v,u)$$

This latter equation is the one used by *Equator*'s bivariate eta function routine, which uses the keyword **eta**.

Some special cases of the bivariate eta function are

$\eta(v,0)$	$\eta(v,1)$	$\eta(v,2)$	$\eta(v,\frac{1}{2})$	$\eta(1,u)$	$\eta(1,1)$	$\eta(1,\frac{1}{2})$
$\begin{cases} \infty & v>0 \\ -\eta(v) & v<0 \end{cases}$	$\eta(v)$	$1-\eta(v)$	$2^v\beta(v)$	$\frac{G(u)}{2}$	$\ln(2)$	$\frac{\pi}{2}$

Here $\eta(v)$ and $\beta(v)$ are *eta* and *beta numbers* from Chapter 3; $G(u)$ is *Bateman's G function* [Section 44:13].

64:14 RELATED TOPIC: Weyl differintegration

The Hurwitz and Lerch functions of this chapter have strong connections with the fractional calculus [Section 12:14], as does the bivariate eta function. For example, equations 64:3:6 and 64:12:4 show how the first two of these functions can be generated from simple algebraic expressions by the operations of the fractional calculus.

Differintegration is the operation that unifies differentiation and integration and extends the concept to fractional orders. Except when the order μ of differintegration is a nonnegative integer, a lower limit must be specified for the differintegral of a function $f(x)$ to be fully characterized. Any number will serve as this lower limit but the most common are 0 and $-\infty$.

Differintegration with a lower limit of $-\infty$ is called *Weyl differintegration* (Hermann Klaus Hugo Weyl, German mathematician, 1885–1955). Notations vary greatly, but our symbolism for a Weyl differintegral, and a definition applicable when $\mu < 0$, is through the *Riemann-Liouville integral transform*

64:14:1 $$\frac{d^\mu}{dt^\mu}f(t)\bigg|_{-\infty}^{x} = \frac{1}{\Gamma(-\mu)}\int_{-\infty}^{x}\frac{f(t)}{(x-t)^{1+\mu}}dt \qquad \mu<0$$

When μ is positive, the definition still relies on this transform but subsequently differentiates it a sufficient number of times

64:14:2 $$\frac{d^\mu}{dt^\mu}f(t)\bigg|_{-\infty}^{x} = \frac{d^n}{dx^n}\left\{\frac{1}{\Gamma(n-\mu)}\int_{-\infty}^{x}\frac{f(t)}{(x-t)^{1+\mu-n}}dt\right\} \qquad n>\mu\geq 0$$

Consider, for example, the function $f(x)=\exp(-bx)$ where b is a positive constant. Then a change of the integration variable in 64:14:1 to $b(x-t)$ easily establishes that

64:14:3 $$\frac{d^\mu}{dt^\mu}\exp(-bt)\bigg|_{-\infty}^{x} = b^\mu\exp(-bt) \qquad b>0$$

and the same result is also given by 64:14:2. As a second example, the *Randles-Sevčik function*, important in electrochemistry, may be defined, for negative x, as the Weyl semiderivative of the function $1/[1+\exp(-x)]$

64:14:4 $$\frac{d^{1/2}}{dt^{1/2}}\frac{1}{1+\exp(-t)}\bigg|_{-\infty}^{x} = \frac{d^{1/2}}{dt^{1/2}}\sum_{j=1}^{\infty}(-)^{j+1}\exp(jt)\bigg|_{-\infty}^{x} = \sum_{j=1}^{\infty}(-)^{j+1}\sqrt{j}\exp(jx) \qquad x<0$$

Other representations of this functions are

64:14:5 $$\exp(x)\Phi\left(-\exp(x),\tfrac{1}{2},1\right) \qquad \text{and} \qquad \sqrt{\frac{\pi}{2}}\sum_{j=1}^{\infty}\frac{\sqrt{X_j-x}\left(X_j+2x\right)}{X_j^3} \qquad X_j=\sqrt{(2j-1)^2\pi^2+x^2}$$

the latter not being restricted to negative x.

Of course, as with regular differentiation and integration, one may differintegrate with respect to a function, instead of with respect to a variable. For example, replacing t in equation 64:14:3 by a logarithm leads to

64:14:6 $$\frac{d^\mu}{[d\ln(t)]^\mu}t^{-b}\bigg|_{0}^{\ln(x)} = b^\mu x^{-b} \qquad b>0$$

This formula lies at the heart of definitions 64:3:6 and 64:12:4.

The Hurwitz and bivariate eta functions play vital roles in the *Weyl differintegration of periodic functions*. Let $\mathrm{per}(x)$ be such a function and its period be P. With definition 64:14:1 applied to this periodic function [Chapter 36],

64:14:7 $$\Gamma(-\mu)\frac{d^\mu}{dt^\mu}\mathrm{per}(t)\bigg|_{-\infty}^{x} = \int_{-\infty}^{x}\frac{\mathrm{per}(t)}{(x-t)^{1+\mu}}dt = \sum_{j=0}^{\infty}\int_{x-P-jP}^{x-jP}\frac{\mathrm{per}(t)}{(x-t)^{1+\mu}}dt = \frac{1}{P^{1+\mu}}\sum_{j=0}^{\infty}\int_{0}^{P}\frac{\mathrm{per}(x-\lambda-jP)}{[j+(\lambda/P)]^{1+\mu}}d\lambda$$

where, in the final step, the integration variable was changed to $\lambda = x-t-jP$. The final integrand is seen to involve the Hurwitz summands from equation 64:3:3, whence

64:14:8 $$\frac{d^\mu}{dt^\mu}\mathrm{per}(t)\bigg|_{-\infty}^{x} = \frac{P^{-1-\mu}}{\Gamma(-\mu)}\int_{0}^{P}\mathrm{per}(x-\lambda)\zeta\left(1+\mu,\frac{\lambda}{P}\right)d\lambda \qquad \mu<0$$

For $-1<\mu<0$, this result requires that the integrals in 64:14:7 converge which, in turn, requires that the mean value of the periodic function be zero over its period. This requirement can be discarded when μ is positive, in which case the formula for Weyl differintegration of a periodic function, derived from 64:14:2, is

64:14:9 $$\left.\frac{d^\mu}{dt^\mu}\text{per}(t)\right|_{-\infty}^{x} = \frac{P^{-1-\mu}}{\Gamma(-\mu)}\int_0^P [\text{per}(x-\lambda)-\text{per}(x)]\zeta\left(1+\mu,\frac{\lambda}{P}\right)d\lambda \qquad \mu>0$$

Because increasing x by P leaves the right-hand members of 64:14:8 and 64:14:9 unchanged, it is evident that the Weyl differintegral of a periodic function is itself periodic and of unchanged period.

Though the rather complicated formulas of the previous paragraph apply to *all* periodic functions, Weyl differintegration of the cosine function merely scales the function and shifts its phase

64:14:10 $$\left.\frac{d^\mu}{dt^\mu}\cos(\omega t)\right|_{-\infty}^{x} = \omega^\mu\cos\left(\omega x+\frac{\mu\pi}{2}\right)$$

and a similar result holds for the sine. As a final example, consider the *square-wave function* $(-1)^{\text{Int}(2x/P)}$ [Section 36:14]. The result of differintegrating this periodic function to order μ can be expressed succinctly in terms of the bivariate eta function:

64:14:11 $$\left.\frac{d^\mu}{dt^\mu}(-1)^{\text{Int}(2x/P)}\right|_{-\infty}^{x} = \frac{(2/P)^\mu}{\Gamma(1-\mu)}\eta\left(\mu,\frac{2x}{P}\right)$$

The waveforms produced for the cases of orders $\mu = 1, \frac{5}{6}, \frac{2}{3}, \frac{1}{2}, \frac{1}{3}, \frac{1}{6}, 0, -\frac{1}{6}, -\frac{1}{3}, -\frac{1}{2}, -\frac{2}{3}, \frac{5}{6}$, and -1 are illustrated in Figure 64-2. They demonstrate a transition from square-wave to triangular-wave behavior as the order moves from 0 to -1, corresponding to increasingly robust integration. Increasing the order of differentiation, as μ transitions from 0 to 1, leads ultimately to a set of Dirac functions [Chapter 9], spiking alternately in the positive and negative directions. In the figure, the curves have been normalized to accentuate the familial pattern.

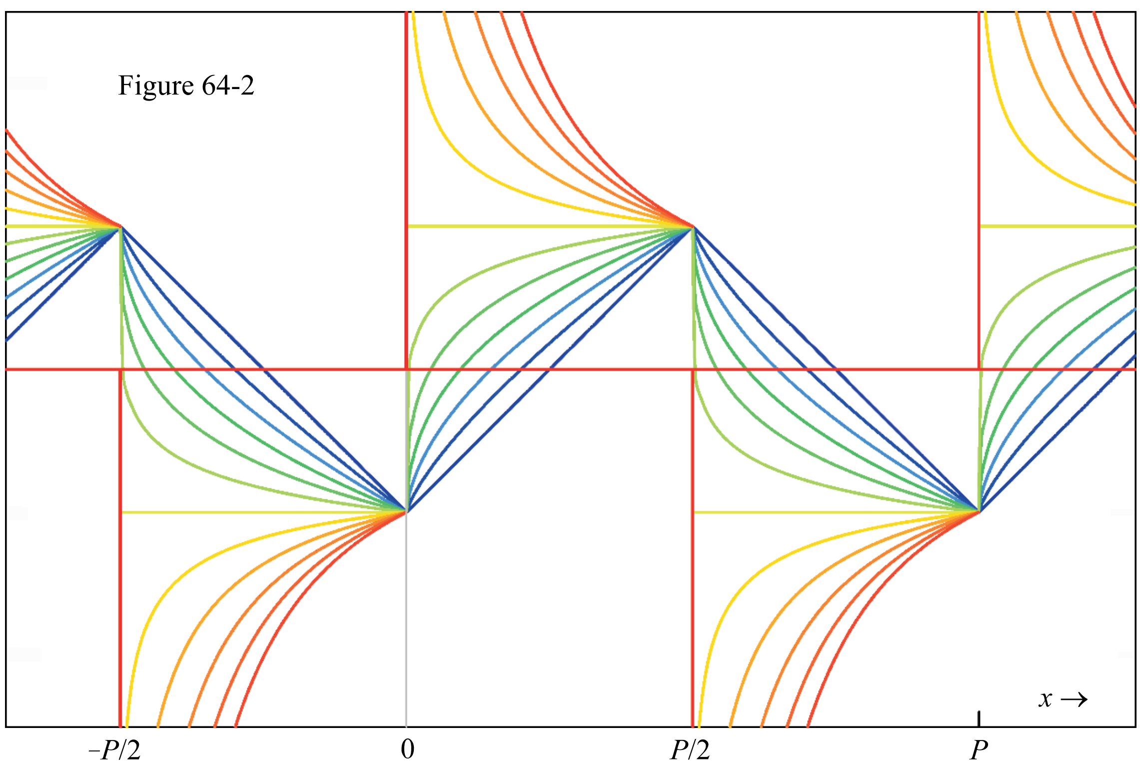


Figure 64-2