CHAPTER 63

THE JACOBIAN ELLIPTIC FUNCTIONS

The bivariate functions of this chapter have several interesting properties. One is their ability to bridge the gap between circular functions and hyperbolic functions. Another, discussed in Section 63:11, is their double periodicity. Three of these twelve functions – $\mathrm{cn}(k,x)$, $\mathrm{sn}(k,x)$, and $\mathrm{dn}(k,x)$ – were described by the prolific Prussian mathematician Karl Gustav Jacob Jacobi (1804–1851), and these receive emphasis here. The other nine, introduced by the Englishman James Whitbread Lee Glaisher (1848–1928), are often regarded as subordinate, because they can be constructed so easily from Jacobi's trio.

Unlike most other functions, the symbols of the Jacobian elliptic function have, in themselves, mathematical significance. These symbols consist of two letters. The first is drawn from the set (c,s,d,n); the second is a different letter from the same set, for a total of 4×3 = 12. It is useful to think of an elliptic function as a quotient of two functions, for example

63:0:1 $$\mathrm{cs}(k,x) = \frac{a\text{ “c” }\mathit{function}}{an\text{ “s” }\mathit{function}}$$

As described in Section 63:8, these single-letter "c", "s", "n" and "d" functions are, in fact, Neville theta functions, but this is unimportant here. The value of representation 63:0:1 is that the important rules for multiplying or dividing two or more Jacobian elliptic functions, exemplified by the following:

63:0:2 $$\mathrm{sc}(k,x)\,\mathrm{cs}(k,x) = 1$$

63:0:3 $$\mathrm{cn}(k,x)\,\mathrm{nd}(k,x) = \mathrm{cd}(k,x)$$

63:0:4 $$\frac{\mathrm{dn}(k,x)}{\mathrm{sn}(k,x)} = \mathrm{ds}(k,x)$$

63:0:5 $$\mathrm{ns}(k,x)\,\mathrm{dc}(k,x) = \mathrm{ds}(k,x)\,\mathrm{nc}(k,x)$$

become self-evident on the basis of such partitioning. Other interrelations among elliptic functions, not evident from the symbolism, are:

63:0:6 $$1 = \mathrm{cn}^2(k,x) + \mathrm{sn}^2(k,x) = \mathrm{dn}^2(k,x) + k^2\,\mathrm{sn}^2(k,x)$$

63:0:7 $$\mathrm{nd}^2(k,x) = \mathrm{cd}^2(k,x) + \mathrm{sd}^2(k,x) = 1 + k^2\,\mathrm{sd}^2(k,x)$$

63:0:8 $$\mathrm{ns}^2(k,x) = \mathrm{cs}^2(k,x) + 1 = \mathrm{ds}^2(k,x) - k^2$$

63:0:9 $$\mathrm{nc}^2(k,x) = 1 + \mathrm{sc}^2(k,x) = \mathrm{dc}^2(k,x) - k^2\,\mathrm{sc}^2(k,x)$$

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Having a pythagorean flavor, these latter formulas, and others, follow easily from their geometric interpretation, as explored in Section 63:3.

63:1 NOTATION

The "Jacobian" (sometimes "Jacobi") adjective is not always attached to these elliptic functions. Alternatively, only the sn, cn, and dn functions may be associated with Jacobi, the others being called *Glaisher functions*. The name *cosine-amplitude* is given to cn, *sine-amplitude* to sn, and *delta-amplitude* to dn; the others have not been individually named.

The Jacobian elliptic functions are bivariate, with *modulus* k and *argument* x as the standard variables. You may encounter notations such as cn(x), suggesting a single variable only; the second variable is then implied, being treated as a constant unworthy of mention. The symbol p has been used for the modulus and u commonly replaces x. As well, the order of citation of the variables may be reversed, as in sn(u,k). Rarely, tn is used for sc, because of its tangent-like properties. In common with the functions of Chapters 61 and 62, Jacobian elliptic functions are often symbolized with "modulus substitutes"; their use may be signaled by replacement of the comma by some other separator, as in dn($x|m$) or cs($x\backslash\alpha$) where $m = k^2$ and $\alpha =$ arcsin(k).

The twelve Jacobian elliptic functions form four groups, according to the second letter of the function's name. Thus sc, dc, and nc are said to be *copolar*: they all possess poles of type c.

Several supplementary univariate and bivariate functions arise in discussions of Jacobian elliptic functions. These are the *complementary modulus* $k' = \sqrt{1-k^2}$, the *complete elliptic integrals* [Chapter 61] K(k) or K, and E(k) or E, the incomplete elliptic integrals [Chapter 62] F(k,φ) and E(k,φ), and the *amplitude*. The symbol φ is appropriate for the last, but am(k,x) often replaces it, to emphasize that it shares variables with the elliptic functions, to which it is related through the identities

63:1:1 $$\mathrm{am}(k,x) = \arcsin\{\mathrm{sn}(k,x)\} = \arccos\{\mathrm{cn}(k,x)\} = \arcsin\left\{\sqrt{1-\mathrm{dn}^2(k,x)}/k\right\} = \varphi \qquad x \le K$$

The function that is denoted dn(k,x) in the *Atlas* may be symbolized Δ(k,φ) elsewhere.

Capitalizing the initial letter of the symbol for a Jacobian elliptic function has been used to indicate the indefinite integral of the square of the function [Section 63:10]

63:1:2 $$\mathrm{Ef}(k,x) = \int_0^x \mathrm{ef}^2(k,t)\,\mathrm{d}t \qquad \mathrm{ef} = \mathrm{cn, sn, dn, cd, sd, nd, cs, ds, ns, sc, dc, nc}$$

but this convention is not adopted here. Note our usage, in 63:1:2 and elsewhere, of "ef" as a stand-in for certain – or, as here, all – elliptic functions.

63:2 BEHAVIOR

The Jacobian elliptic functions display interesting properties when the modulus and/or the argument are imaginary or complex. However, except in Sections 63:11, this chapter treats k and x as real. Moreover, we generally assume $0 \le k \le 1$, which covers the most important values of the modulus, though equations 63:5:1 and 63:5:16−19 show how this domain may be extended to all real values.

Figure 63-1, is a three dimensional representation of the cn(k,x), sn(k,x) and dn(k,x) functions; that is, the three Jabobian elliptic functions that belong to the copolar group n. Likewise, Figures 63-2, 63-3 and 63-4 each depict a trio of functions belonging to the other copolar groups. Many of the properties of the twelve functions are evident

Figure 63-1

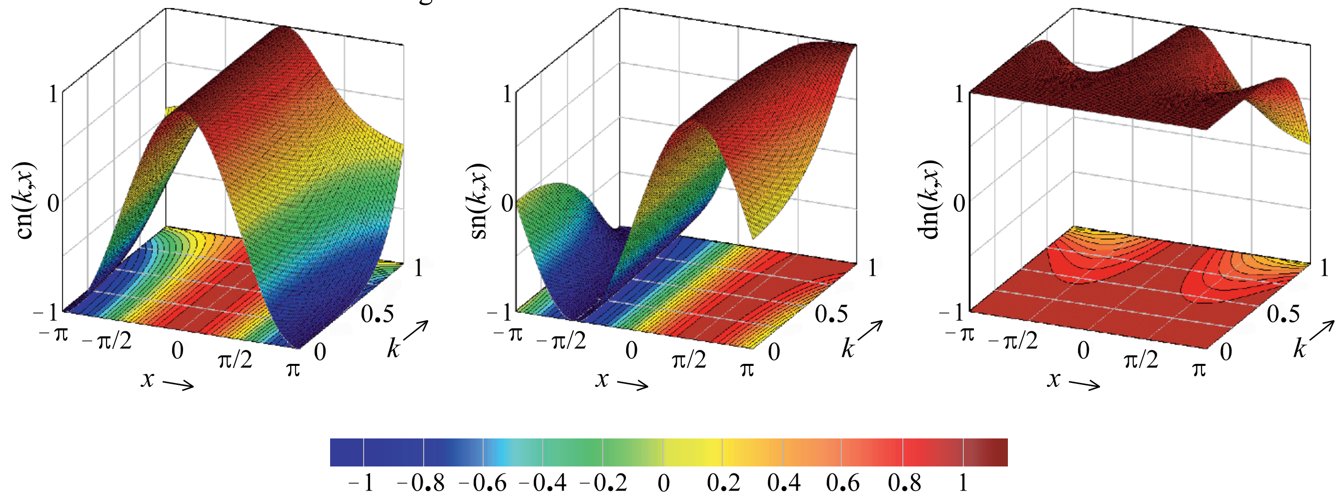


Figure 63-2

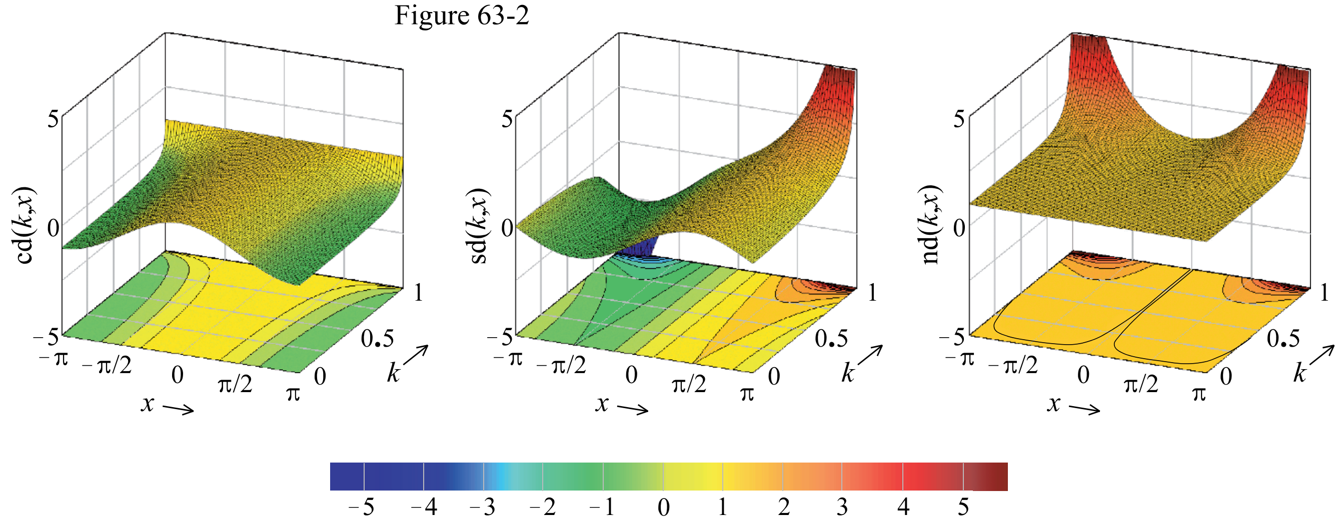


Figure 63-3

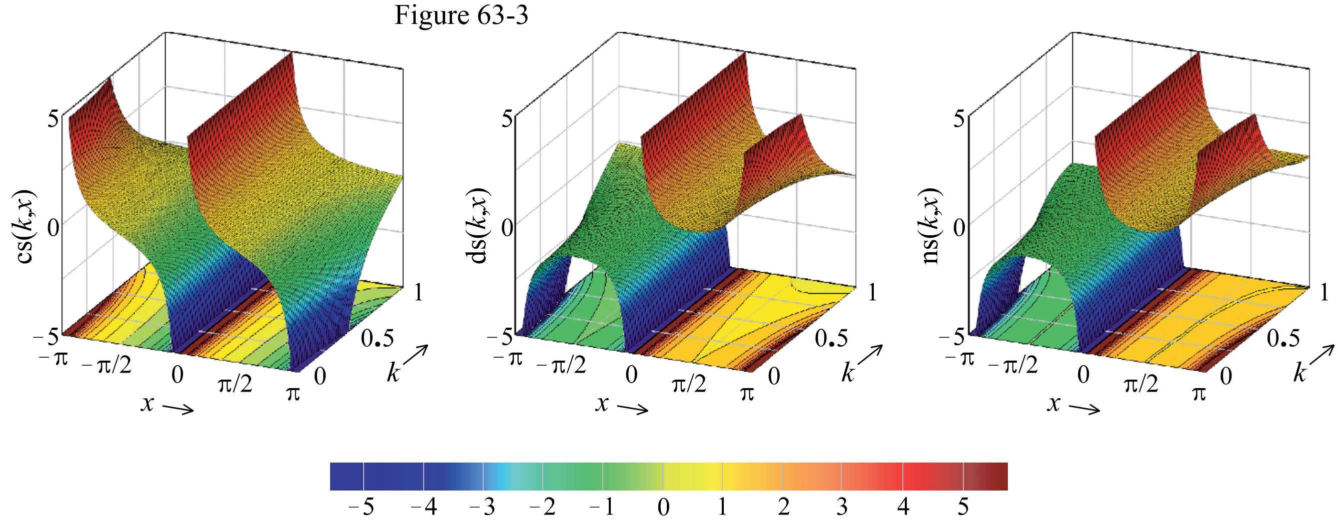
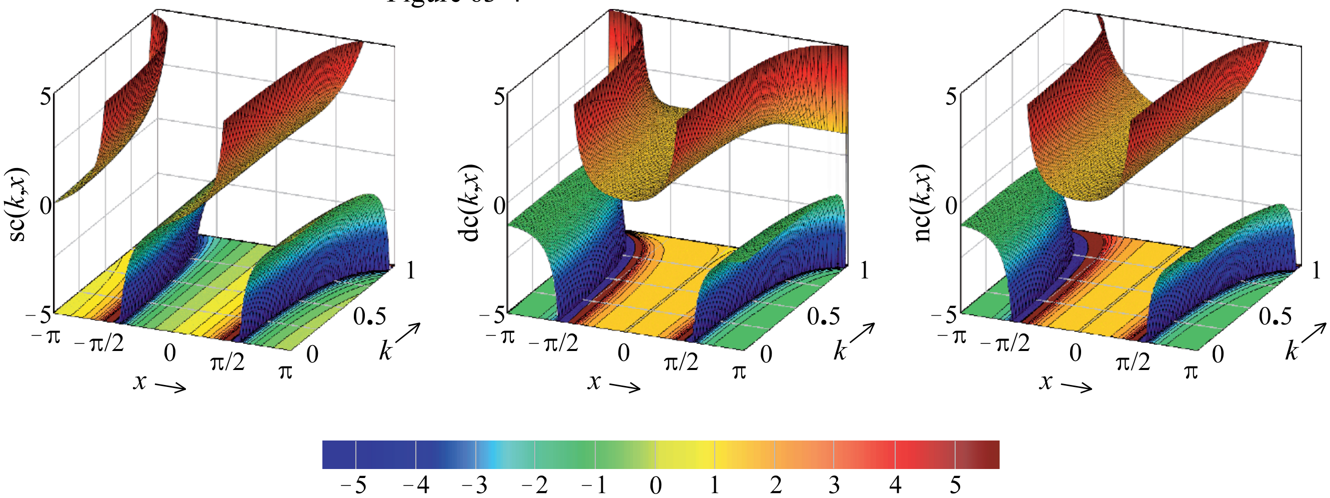


Figure 63-4



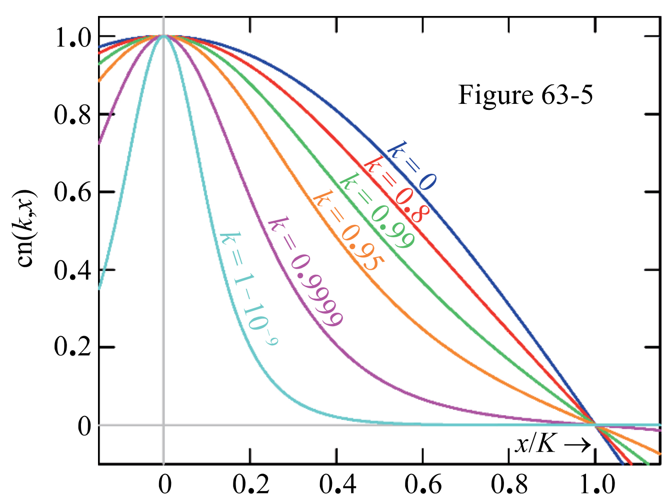
from these graphs. For the most part, the functions coincide with circular functions [those of Chapter 32, 33, and 34] when $k = 0$ but evolve as k increases to become hyperbolic functions [from Chapters 28−30] when $k = 1$. For example, $\mathrm{sc}(0,x) = \tan(x)$, whereas $\mathrm{sc}(1,x) = \sinh(x)$. See Section 63:4 for a complete listing.

Except when $k = 1$, all elliptic functions $\mathrm{ef}(k,x)$ are periodic in their argument x, with a period of either $4K$ or $2K$; specifically:

63:2:1
$$\left.\begin{array}{ll} \mathrm{ef}(k,x+4nK)=\mathrm{ef}(k,x) & \mathrm{ef} = \mathrm{sn, cn, ds, ns, dc, nc, sd, cd} \\ \mathrm{ef}(k,x+2nK)=\mathrm{ef}(k,x) & \mathrm{ef} = \mathrm{dn, cs, sc, nd} \end{array}\right\} n=\pm1,\pm2,\cdots$$

In this behavior, the elliptic functions are analogous to the circular functions, which have periods of 2π or π. This reflects the fact that, in elliptic algebra, the complete elliptic integral K plays the role that the right-angle, $\pi/2$, fills in circular trigonometry. Likewise, the concept of quadrants, familiar in the context of circular functions, can usefully be extended to elliptic functions. The table opposite reports the ranges of values adopted by the twelve elliptic functions in each of the four "quadrants". Information about the ranges, zeros, extrema and discontinuities of the functions can also be gleaned from a careful inspection of this tabulation.

The lengthening period as k increases is brought out particularly clearly in Figure 63-4. These three diagrams illustrate that when k reaches unity, the period becomes infinite, as appropriate for the hyperbolic functions that most elliptic functions become in that limit. The argument x is used as one of the variables in Figures 63-1 through 63-4, but it is the ratio x/K, where K denotes the complete elliptic integral $K(k)$, that is more significant in many respects and that was chosen in drawing Figures 63-5, 63-6 and 63-7. These diagrams show how the behavior of Jacobi's three functions – $\mathrm{cn}(k,x)$, $\mathrm{sn}(k,x)$, and $\mathrm{dn}(k,x)$ – is affected by the value of the modulus. Notice that the dependence on k is weak when k is small, but becomes dramatic as k approaches unity.



	$0 \le x \le K$	$K \le x \le 2K$	$2K \le x \le 3K$	$3K \le x \le 4K$
cn	$1 \ge \mathrm{cn}(k,x) \ge 0$	$0 \ge \mathrm{cn}(k,x) \ge -1$	$-1 \le \mathrm{cn}(k,x) \le 0$	$0 \le \mathrm{cn}(k,x) \le 1$
sn	$0 \le \mathrm{sn}(k,x) \le 1$	$1 \ge \mathrm{sn}(k,x) \ge 0$	$0 \ge \mathrm{sn}(k,x) \ge -1$	$-1 \le \mathrm{sn}(k,x) \le 0$
dn	$1 \ge \mathrm{dn}(k,x) \ge k'$	$k' \le \mathrm{dn}(k,x) \le 1$	$1 \ge \mathrm{dn}(k,x) \ge k'$	$k' \le \mathrm{dn}(k,x) \le 1$
cd	$1 \ge \mathrm{cd}(k,x) \ge 0$	$0 \ge \mathrm{cd}(k,x) \ge -1$	$-1 \le \mathrm{cd}(k,x) \le 0$	$0 \le \mathrm{cd}(k,x) \le 1$
sd	$0 \le \mathrm{sd}(k,x) \le 1/k'$	$1/k' \ge \mathrm{sd}(k,x) \ge 0$	$0 \ge \mathrm{sd}(k,x) \ge -1/k'$	$-1/k' \le \mathrm{sn}(k,x) \le 0$
nd	$1 \le \mathrm{nd}(k,x) \le 1/k'$	$1/k' \ge \mathrm{nd}(k,x) \ge 1$	$1 \le \mathrm{nd}(k,x) \le 1/k'$	$1/k' \ge \mathrm{nd}(k,x) \ge 1$
cs	$+\infty \ge \mathrm{cs}(k,x) \ge 0$	$0 \ge \mathrm{cs}(k,x) \ge -\infty$	$+\infty \ge \mathrm{cs}(k,x) \ge 0$	$0 \ge \mathrm{cs}(k,x) \ge -\infty$
ds	$+\infty \ge \mathrm{ds}(k,x) \ge k'$	$k' \le \mathrm{ds}(k,x) \le +\infty$	$-\infty \le \mathrm{ds}(k,x) \le -k'$	$-k' \ge \mathrm{ds}(k,x) \ge -\infty$
ns	$+\infty \ge \mathrm{ns}(k,x) \ge 1$	$1 \le \mathrm{ns}(k,x) \le +\infty$	$-\infty \le \mathrm{ns}(k,x) \le -1$	$-1 \ge \mathrm{ns}(k,x) \ge -\infty$
sc	$0 \le \mathrm{sc}(k,x) \le +\infty$	$-\infty \le \mathrm{sc}(k,x) \le 0$	$0 \le \mathrm{sc}(k,x) \le +\infty$	$-\infty \le \mathrm{sc}(k,x) \le 0$
dc	$1 \le \mathrm{dc}(k,x) \le +\infty$	$-\infty \le \mathrm{dc}(k,x) \le -1$	$-1 \ge \mathrm{dc}(k,x) \ge -\infty$	$+\infty \ge \mathrm{dc}(k,x) \ge 1$
nc	$1 \le \mathrm{nc}(k,x) \le +\infty$	$-\infty \le \mathrm{nc}(k,x) \le -1$	$-1 \ge \mathrm{nc}(k,x) \ge -\infty$	$+\infty \ge \mathrm{nc}(k,x) \ge 1$
am	$0 \le \mathrm{am}(k,x) \le \pi/2$	$\pi/2 \le \mathrm{am}(k,x) \le \pi$	$\pi \le \mathrm{am}(k,x) \le 3\pi/2$	$3\pi/2 \le \mathrm{am}(k,x) \le 2\pi$

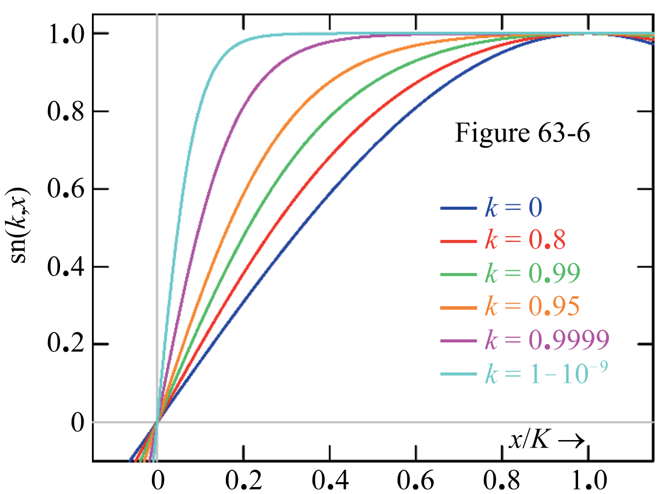


Figure 63-6

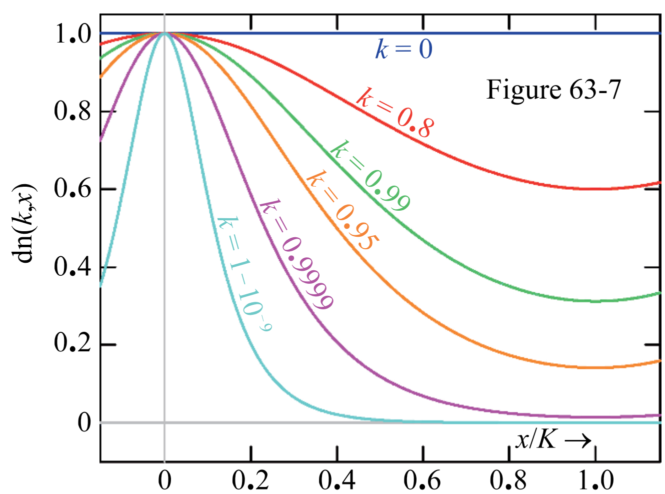


Figure 63-7

63:3 DEFINITIONS

Let the symbol x be assigned to the incomplete elliptic integral of the first kind, of modulus k and amplitude φ. Then the expression

63:3:1
$$x = \mathrm{F}(k,\varphi) = \int_0^{\varphi} \frac{1}{\sqrt{1-k^2\sin^2(\theta)}}\,d\theta$$

gives x as a function of k and φ. Equally well, one may regard the amplitude φ as a function of k and x. The six circular functions of φ then serve to define six of the Jacobian elliptic functions, as follows:

cos(φ) =	sin(φ) =	sec(φ) =	csc(φ) =	tan(φ) =	cot(φ) =
cn(k,x)	sn(k,x)	nc(k,x)	ns(k,x)	sc(k,x)	cs(k,x)

These definitions raise the pertinent question of how φ is to be found from known values of k and x. There is no straightforward way of doing this, but that does not undermine the validity of the definition. One may think of elliptic functions as being inverse functions [Section 0:3] of incomplete elliptic integrals; for example, treating k as a constant, cn(k,x) is the inverse function of F(k,arccos(x)), as detailed in Section 63:13.

The delta-amplitude is defined by

63:3:2 $$\mathrm{dn}(k,x)=\frac{\partial\varphi}{\partial x}=\sqrt{1-k^2\sin^2(\varphi)}$$

and the remaining five Jacobian elliptic functions may be defined from this, either by the definitions shown in the panel below

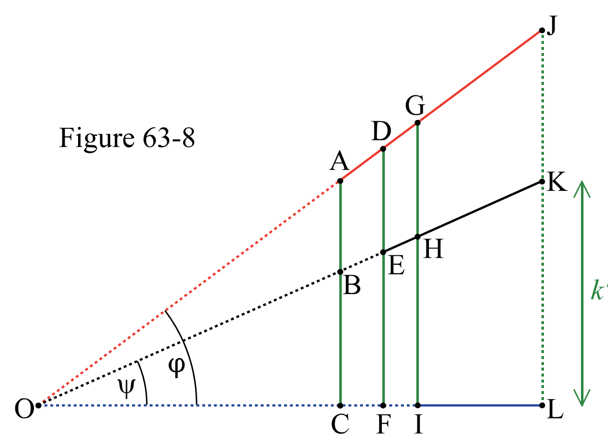
nd(k,x) =	cd(k,x) =	sd(k,x) =	dc(k,x) =	ds(k,x) =
1/dn(k,x)	cn(k,x)/dn(k,x)	sn(k,x)/dn(k,x)	dn(k,x)nc(k,x)	dn(k,x)ns(k,x)

or in numerous other ways allowed by the partitioning rules exemplified in equations 63:0:1−4.

In Section 33:3, a set of three similar triangles is described by means of which the six circular functions may be defined. A similar exercise is undertaken in Section 29:3 of the *Atlas* for the six hyperbolic functions. In much the same way, there exists a trigonometric construct that permits the defining of the twelve elliptic functions in an appealing way.

First construct the triangle OAC, right-angled at C, with the hypotenuse OA of unit length and with the angle AOC equal to the elliptic amplitude φ. Then cn(k,x) and sn(k,x) are defined as the lengths OC and CA. The OAC triangle, and the ensuing construction, are illustrated in Figure 63-8. In this diagram, dotted lines are all of unity length. Now extend the line OC to point I, such that OI is of unity length, and erect a perpendicular at that point to meet an extension of line OA at point G. Then OG and GI have lengths equal to nc(k,x) and sc(k,x) respectively. The next construction is to further extend lines OI and OG, to points L and J respectively, until the perpendicular distance JL between them becomes equal to unity. Then the lengths OL and OJ equal cs(k,x) and ns(k,x). Six of the elliptic functions have now been defined; the other six require further construction. Construct a line at an angle

Figure 63-8



63:3:3 $$\psi=\arctan\{k'\tan(\varphi)\}\qquad k'=\sqrt{1-k^2}$$

to line OL, as shown in the figure. This line will cut lines AC, GI and JL at points B, H and K. The length KL is thereby equal to the complementary modulus k'. The lengths of lines OB, OH and OK now define the elliptic functions dn(k,x), dc(k,x), and ds(k,x). The final construction is to measure unity length along line OK to a point E and erect the vertical line FED through that point. Lengths OD, OF and DF then define the remaining elliptic functions nd(k,x), cd(k,x), and sd(k,x).

Figure 63-9 is an exploded view of the Figure 63-8, marked with the length elements assigned to the various functions. Note that each of the four diagrams in this figure corresponds to a copolar group, and that the first

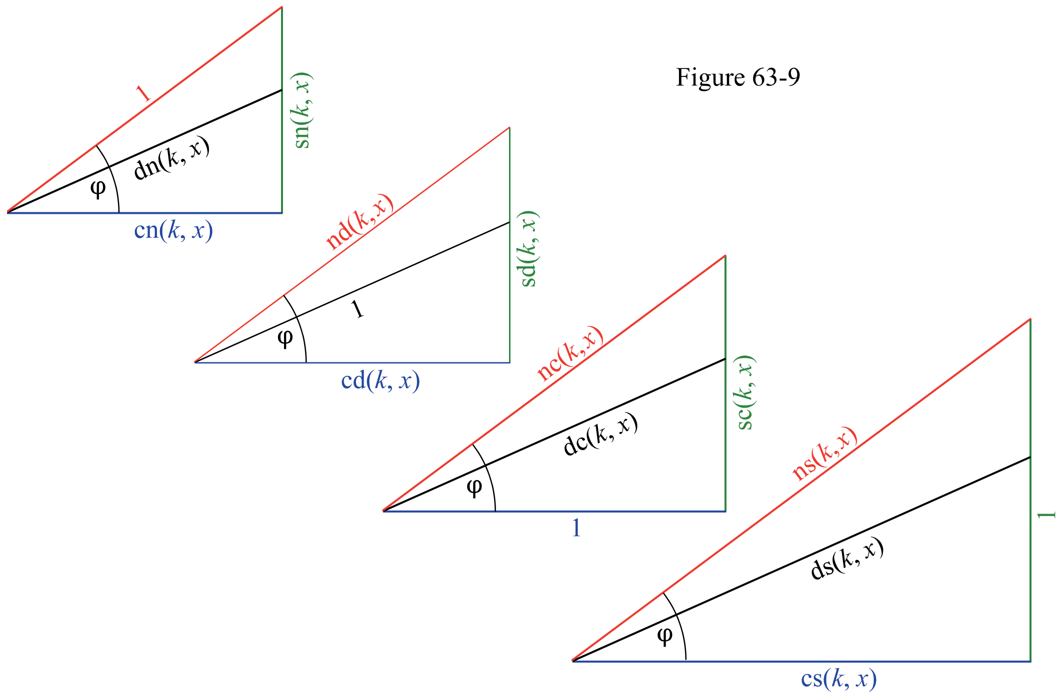


Figure 63-9

vindicates the early equalities in equation 63:1:1. Pythagorean arguments applied to various triangles lead directly to formulas 63:0:4−7. The four diagrams are "similar" in the geometric sense; that is, they differ in size but not in shape. This similarity enables one to assert, for example, that the ratio of the lengths of two red lines must equal the ratio of the lengths of two blue lines and therefore

63:3:4
$$\frac{\mathrm{nd}(k,x)}{1}=\frac{\mathrm{cd}(k,x)}{\mathrm{cn}(k,x)}$$

This relation leads directly to 63:0:3. In fact, all the "partitioning" rules discussed in Section 63:0 are validated by similarity arguments arising from Figure 63-9.

63:4 SPECIAL CASES

As reported in Section 63:2, each Jacobian elliptic function ef(k,x) reduces, when $k=0$, to a circular function of x, or to unity, whereas it reduces to a hyperbolic function or unity, when $k=1$.

	cn	sn	dn	cd	sd	nd	cs	ds	ns	sc	dc	nc	am
$k=0$	cos	sin	1	cos	sin	1	cot	csc	csc	tan	sec	sec	x
$k=1$	sech	tanh	sech	1	sinh	cosh	csch	csch	coth	sinh	1	cosh	gd

The table identifies the particular circular or hyperbolic function for each ef(0,x) and ef(1,x). The tabulation includes the elliptic amplitude am(k,x), which is seen to equal its argument when $k=0$ and the gudermannian function [Section 33:14] of x when $k=1$.

63:5 INTRARELATIONSHIPS

Every elliptic function is even with respect to its modulus

63:5:1 $$\mathrm{ef}(-k,x)=\mathrm{ef}(k,x) \qquad \text{all ef}$$

and either even or odd with respect to its argument

63:5:2 $$\mathrm{ef}(k,-x)=\begin{cases}\mathrm{ef}(k,x) & \mathrm{ef}=\mathrm{dn,\ cn,\ nc,\ dc,\ nd,\ cd}\\ -\mathrm{ef}(k,x) & \mathrm{ef}=\mathrm{sn,\ ns,\ cs,\ ds,\ sc,\ sd}\end{cases}$$

Note that the presence of an "s" in the symbol of the elliptic function ensures its oddness with respect to its argument.

Jacobi's trio of elliptic functions satisfies the following addition formulas:

63:5:3 $$\mathrm{cn}(k,x\pm y)=\frac{\mathrm{cn}(k,x)\mathrm{cn}(k,y)\mp\mathrm{sn}(k,x)\mathrm{sn}(k,y)\mathrm{dn}(k,x)\mathrm{dn}(k,y)}{1-k^2\mathrm{sn}^2(k,x)\mathrm{sn}^2(k,y)}$$

63:5:4 $$\mathrm{sn}(k,x\pm y)=\frac{\mathrm{sn}(k,x)\mathrm{cn}(k,y)\mathrm{dn}(k,y)\pm\mathrm{cn}(k,x)\mathrm{dn}(k,x)\mathrm{sn}(k,y)}{1-k^2\mathrm{sn}^2(k,x)\mathrm{sn}^2(k,y)}$$

63:5:5 $$\mathrm{dn}(k,x\pm y)=\frac{\mathrm{dn}(k,x)\mathrm{dn}(k,y)\mp k^2\mathrm{sn}(k,x)\mathrm{cn}(k,x)\mathrm{sn}(k,y)\mathrm{cn}(k,y)}{1-k^2\mathrm{sn}^2(k,x)\mathrm{sn}^2(k,y)}$$

These equations are easily converted to argument-duplication formulas that give values of ef(k,2x). Other important special cases are listed in the following table, which also includes expressions for the three prime elliptic functions of half-argument.

	ef = cn	ef = sn	ef = dn
ef(k,$x\pm K$)	$\mp k'\mathrm{sd}(k,x)$	$\pm\mathrm{cd}(k,x)$	$k'\mathrm{nd}(k,x)$
ef(k,$x\pm 2K$)	$-\mathrm{cn}(k,x)$	$-\mathrm{sn}(k,x)$	$\mathrm{dn}(k,x)$
$\mathrm{ef}\left(k,\frac{1}{2}x\right)$	$\pm\sqrt{\dfrac{\mathrm{cn}(k,x)+\mathrm{dn}(k,x)}{1+\mathrm{dn}(k,x)}}$	$\pm\sqrt{\dfrac{1-\mathrm{cn}(k,x)}{1+\mathrm{dn}(k,x)}}$	$\sqrt{\dfrac{\mathrm{cn}(k,x)+\mathrm{dn}(k,x)}{1+\mathrm{cn}(k,x)}}$

The principle of *Landen transformation* is explained in Section 62:5. When employed to transform the delta-amplitude in the ascending mode, the procedure is

63:5:6 $$\frac{2\sqrt{k}}{1+k}=k \qquad \frac{(1+k)}{2}x=x \qquad \frac{\sqrt{\mathrm{dn}^2(k,x)-1+k^2}+\mathrm{dn}(k,x)}{1+k}=\mathrm{dn}(k,x)$$

and serves to increase the modulus at the expense of a decrease in the argument. The corresponding formulation for the descending Landen transformation is

63:5:7 $$\frac{1-k'}{1+k'}=k \qquad (1+k')x=x \qquad \frac{\mathrm{dn}^2(k,x)+k'}{(1+k')\mathrm{dn}(k,x)}=\mathrm{dn}(k,x)$$

and similar – though generally more complicated – formulas apply to other elliptic functions.

What is called the *Jacobi real transformation* establishes a relationship between a Jacobian elliptic functions with modulus in the domain $1\le k\le\infty$ and one in the standard domain $0\le k\le 1$. All these transformations are of the form

63:5:8 $$\mathrm{ef}\left(\frac{1}{k},x\right)=w\,\mathrm{ef}'\left(k,\frac{x}{k}\right)$$

where the multiplier w is either 1, k, or $1/k$, and ef′ is another – or sometimes the same – elliptic function. The panel below lists the correspondences between ef and $w$ef′

ef =	cn	sn	dn	cd	sd	nd	cs	ds	ns	sc	dc	nc
$w$ef′ =	dn	$k$sn	cn	dc	$k$sc	nc	$\frac{1}{k}$ds	$\frac{1}{k}$cs	$\frac{1}{k}$ns	$k$sd	cd	nd

63:6 EXPANSIONS

The first five terms in the power series for the sine-amplitude, cosine-amplitude and delta-amplitude functions, as well as for the elliptic amplitude itself, are

63:6:1 $$\mathrm{cn}(k,x)=1-\frac{1}{2!}x^2+\frac{1+4k^2}{4!}x^4-\frac{1+44k^2+16k^4}{6!}x^6+\frac{1+408k^2+912k^4+64k^6}{8!}x^8-\cdots$$

63:6:2 $$\mathrm{sn}(k,x)=x-\frac{1+k^2}{3!}x^3+\frac{1+14k^2+k^4}{5!}x^5-\frac{1+135(k^2+k^4)+k^6}{7!}x^7+\frac{1+1228(k^2+k^6)+5478k^4+k^8}{9!}x^9-\cdots$$

63:6:3 $$\mathrm{dn}(k,x)=1-\frac{k^2}{2!}x^2+\frac{4k^2+k^4}{4!}x^4-\frac{16k^2+44k^4+k^6}{6!}x^6+\frac{64k^2+912k^4+408k^6+k^8}{8!}x^8-\cdots$$

63:6:4 $$\mathrm{am}(k,x)=x-\frac{k^2}{3!}x^3+\frac{4k^2+k^4}{5!}x^5-\frac{16k^2+44k^4+k^6}{7!}x^7+\frac{64k^2+912k^4+408k^6+k^8}{9!}x^9-\cdots$$

General formulas for the coefficients in these series are unknown. These four expansions are computationally useful whenever x is close to zero.

Gradshteyn and Ryzhik [Section 8.146] give a comprehensive listing of expansions of the Jacobian elliptic functions, as well as some of their logarithms and squares, in terms of the nome q [Section 61:15]. The most important are:

63:6:5 $$\mathrm{sn}(k,x)=\frac{2\pi}{kK}\left[\frac{q^{1/2}}{1-q}\sin\left(\frac{\pi x}{2K}\right)+\frac{q^{3/2}}{1-q^3}\sin\left(\frac{3\pi x}{2K}\right)+\frac{q^{5/2}}{1-q^5}\sin\left(\frac{5\pi x}{2K}\right)+\cdots\right]$$

63:6:6 $$\mathrm{cn}(k,x)=\frac{2\pi}{kK}\left[\frac{q^{1/2}}{1+q}\cos\left(\frac{\pi x}{2K}\right)+\frac{q^{3/2}}{1+q^3}\cos\left(\frac{3\pi x}{2K}\right)+\frac{q^{5/2}}{1+q^5}\cos\left(\frac{5\pi x}{2K}\right)+\cdots\right]$$

63:6:7 $$\mathrm{dn}(k,x)=\frac{2\pi}{K}\left[\frac{1}{4}+\frac{q}{1+q^2}\cos\left(\frac{\pi x}{K}\right)+\frac{q^2}{1+q^4}\cos\left(\frac{2\pi x}{K}\right)+\frac{q^3}{1+q^6}\cos\left(\frac{3\pi x}{K}\right)+\cdots\right]$$

63:6:8 $$\mathrm{am}(k,x)=\frac{\pi x}{2K}+2\left[\frac{q}{1+q^2}\sin\left(\frac{\pi x}{K}\right)+\frac{q^2/2}{1+q^4}\sin\left(\frac{2\pi x}{K}\right)+\frac{q^3/3}{1+q^6}\sin\left(\frac{3\pi x}{K}\right)+\cdots\right]$$

In these equations we are using K to represent K(k).

63:7 PARTICULAR VALUES

Simple expressions arise for each Jacobian elliptic function when the argument is $K/2$, corresponding to an argument that bisects the first "quadrant". Moreover, each elliptic function often coincides in value there with another of its eleven congeners. The pairings, and the values acquired, are listed below. Note that $k' = \sqrt{1-k^2}$.

$\mathrm{ds}\left(k,\tfrac{1}{2}K\right)$	$\mathrm{cn}\left(k,\tfrac{1}{2}K\right)$	$\mathrm{sn}\left(k,\tfrac{1}{2}K\right)$ $\mathrm{cd}\left(k,\tfrac{1}{2}K\right)$	$\mathrm{dn}\left(k,\tfrac{1}{2}K\right)$ $\mathrm{cs}\left(k,\tfrac{1}{2}K\right)$	$\mathrm{nc}\left(k,\tfrac{1}{2}K\right)$	$\mathrm{ns}\left(k,\tfrac{1}{2}K\right)$ $\mathrm{dc}\left(k,\tfrac{1}{2}K\right)$	$\mathrm{sc}\left(k,\tfrac{1}{2}K\right)$ $\mathrm{nd}\left(k,\tfrac{1}{2}K\right)$	$\mathrm{sd}\left(k,\tfrac{1}{2}K\right)$
$\sqrt{k'(1+k')}$	$\sqrt{\dfrac{k'}{1+k'}}$	$\dfrac{1}{\sqrt{1+k'}}$	$\sqrt{k'}$	$\sqrt{\dfrac{1+k'}{k'}}$	$\sqrt{1+k'}$	$\dfrac{1}{\sqrt{k'}}$	$\dfrac{1}{\sqrt{k'(1+k')}}$

The same values reoccur, possibly with a change of sign, at the midpoints of *all* quadrants.

The particular values when x is a multiple of K (that is, where adjacent quadrants meet) is evident from the table in Section 63:2. All such values are drawn from the nine-member set 0, $\pm k'$, ± 1, $\pm 1/k'$, and $\pm\infty|\mp\infty$.

63:8 NUMERICAL VALUES

Using $\mathrm{dn}(k_0,x_0)$ as illustrative, one popular technique for evaluating Jacobian elliptic functions is to use the Landen transformation in either its descending mode [equation 63:5:7] or ascending mode [equation 63:5:6], to progressively decrease or increase the modulus of the delta-amplitude until it has reached (after, say, n transformations) a value so close to unity or zero that the approximation

63:8:1 $$\mathrm{dn}(k,x) \approx 1 - \tfrac{1}{2}k^2 \sin^2(x) \qquad k \text{ small}$$

or

63:8:2 $$\mathrm{dn}(k,x) \approx \mathrm{sech}(x)\left[1 + \tfrac{1}{4}\left(1-k^2\right)\left(\sinh^2(x) + x\tanh(x)\right)\right] \qquad (1-k) \text{ small}$$

may be applied validly. These approximations arise from limits 63:9:3.

Another route to calculating Jacobian elliptic functions exploits the partitioning principle described in Section 63:0. The single-letter functions are, in fact, the Neville's theta functions and accordingly any of the twelve elliptic can be calculated as

63:8:3 $$\mathrm{ef}(k,x) = \frac{\vartheta_{\mathrm{e}}(k,x)}{\vartheta_{\mathrm{f}}(k,x)} \qquad \text{all ef functions}$$

Applying the routines described in Section 61:15, this is the procedure used by *Equator*'s Jacobian elliptic cn function routine and eleven other similarly named functions (keywords **cn**, **sn**, **dn**, **sd**, **cd**, **nd**, **sc**, **dc**, **nc**, **cs**, **ds**, and **ns**). Values of all twelve elliptic functions are available for variables in the domains $0 \le k \le 1$ and $-8\mathrm{K}(k) \le x \le 8\mathrm{K}(k)$.

Equator also has a routine (keyword **am**) which calculates the elliptic amplitude by the algorithm

63:8:4 $$\mathrm{am}(k,x) = \pi\,\mathrm{Int}\left(\frac{x}{2K}\right) + \begin{cases} \arcsin\{\mathrm{sn}(k,2yK)\} & 0 \le y \le 0.1 \\ \arccos\{\mathrm{cn}(k,2yK)\} & 0.1 < y < 0.9 \\ \pi - \arcsin\{\mathrm{sn}(k,2yK)\} & 0.9 \le y < 1 \end{cases} \qquad \begin{array}{l} y = \mathrm{frac}(x/2K) \\ K = \mathrm{K}(k) \end{array}$$

63:9 LIMITS AND APPROXIMATIONS

The following limiting approximations apply as the modulus of certain elliptic functions approaches the value zero or unity:

63:9:1 $$\mathrm{sn}(k,x) \approx \begin{cases} \sin(x) - \frac{1}{4}k^2\left[x - \sin(x)\cos(x)\right]\cos(x) & k \to 0 \\ \tanh(x) + \frac{1}{4}(k')^2\left[\sinh(x) - x\,\mathrm{sech}(x)\right]\mathrm{sech}(x) & k \to 1 \end{cases}$$

63:9:2 $$\mathrm{cn}(k,x) \approx \begin{cases} \cos(x) + \frac{1}{4}k^2\left[x - \sin(x)\cos(x)\right]\sin(x) & k \to 0 \\ \mathrm{sech}(x) - \frac{1}{4}(k')^2\left[\sinh(x) - x\,\mathrm{sech}(x)\right]\tanh(x) & k \to 1 \end{cases}$$

63:9:3 $$\mathrm{dn}(k,x) \approx \begin{cases} 1 - \frac{1}{2}k^2\sin^2(x) & k \to 0 \\ \mathrm{sech}(x) + \frac{1}{4}(k')^2\left[\sinh(x) + x\,\mathrm{sech}(x)\right]\tanh(x) & k \to 1 \end{cases}$$

63:9:4 $$\mathrm{am}(k,x) \approx \begin{cases} x - \frac{1}{4}k^2\left[x - \sin(x)\cos(x)\right] & k \to 0 \\ \mathrm{gd}(x) + \frac{1}{4}(k')^2\left[\sinh(x) - x\,\mathrm{sech}(x)\right] & k \to 1 \end{cases}$$

Limiting expressions as $x \to 0$ are available by curtailing expansions 63:6:1–4.

63:10 OPERATIONS OF THE CALCULUS

The derivative of an arbitrary elliptic function ef(k,x) with respect to its argument is proportional to the product of the two other elliptic functions (e′f and e″f) that, with ef, constitute a copolar group. The constant α of proportionality may, or may not, depend on k, as follows

63:10:1 $$\frac{\partial}{\partial x}\mathrm{ef}(k,x) = \alpha\,\mathrm{e'f}(k,x)\,\mathrm{e''f}(k,x) \qquad \begin{cases} \alpha=1 \text{ for ef} = \text{sc, sn, sd, nc} \\ \alpha=-1 \text{ for ef} = \text{cn, cs, ds, ns} \\ \alpha = k^2 \text{ for ef} = \text{nd or } -k^2 \text{ for ef} = \text{dn} \\ \alpha = k'^2 \text{ for ef} = \text{dc or} -k'^2 \text{ for ef} = \text{cd} \end{cases}$$

The derivatives with respect to the modulus again involve the proportionality constant α but another term β, reflecting the individuality of the ef elliptic function, is also involved

63:10:2 $$\frac{\partial}{\partial k}\mathrm{ef}(k,x) = \alpha\,\mathrm{e'f}(k,x)\,\mathrm{e''f}(k,x)\left[\frac{x}{k} + \beta\frac{\mathrm{sn}(k,x)}{k'^2} - \frac{E(k,\varphi)}{k\,k'^2}\right] \qquad \begin{cases} \beta = 0 \text{ for ef} = \text{cd,dc} \\ \beta = k\,\mathrm{cd}(k,x) \text{ for ef} = \text{cn,nc,sc,sn,cs,ns} \\ \beta = k\,\mathrm{dc}(k,x) \text{ for ef} = \text{sd,ds} \\ \beta = [\mathrm{dc}(k,x)]/k \text{ for ef} = \text{dn,nd} \end{cases}$$

Expressions for indefinite integrals of the forms

63:10:3 $$I_1 = \int_0^x \mathrm{ef}(k,t)\,\mathrm{d}t \qquad \text{and} \qquad I_2 = \int_0^x \mathrm{ef}^2(k,t)\,\mathrm{d}t \qquad \mathrm{f} \neq \mathrm{s}$$

exist for nine of the elliptic functions as listed below. Although those of pole type s diverge, their complements

63:10:4 $$I_1^* = \int_x^K \mathrm{es}(k,t)\,\mathrm{d}t \qquad \text{and} \qquad I_2^* = \int_x^K \mathrm{es}^2(k,t)\,\mathrm{d}t \qquad \mathrm{e} = \mathrm{n, c, d}$$

remain finite and are tabulated instead. For brevity in this table, each elliptic function ef(k,x) is denoted simply by the symbol ef.

	I_1 or I_1^*	I_2 or I_2^*
sn	$[\ln\{(\mathrm{dn}-k\mathrm{cn})/(1-k)\}]/k$	$[x-\mathrm{E}(k,\varphi)]/k^2$
cn	$[\arccos(\mathrm{dn})]/k$	$[\mathrm{E}(k,\varphi)-(k')^2x]/k^2$
dn	φ	$\mathrm{E}(k,\varphi)$
nc	$[\ln(\mathrm{dc}+k'\mathrm{sc})]/k'$	$x+[\mathrm{sn}\,\mathrm{dc}-\mathrm{E}(k,\varphi)]/k'^2$
sc	$[\ln\{(\mathrm{dc}+k'\mathrm{nc})/(1+k')\}]/k'$	$[\mathrm{sn}\,\mathrm{dc}-\mathrm{E}(k,\varphi)]/k'^2$
dc	$\ln(\mathrm{nc}+\mathrm{sc})$	$x+\mathrm{sn}\,\mathrm{dc}-\mathrm{E}(k,\varphi)$
nd	$[\arccos(\mathrm{cd})]/k'$	$[\mathrm{E}(k,\varphi)-k^2\mathrm{sn}\,\mathrm{cd}]/k'^2$
sd	$[\arcsin\{kk'(\mathrm{nd}-\mathrm{cd})\}]kk'$	$[\mathrm{E}(k,\varphi)-(k')^2x-k^2\mathrm{sn}\,\mathrm{cd}]/(k\,k')^2$
cd	$[\ln(\mathrm{nd}+k\mathrm{sd})]/k$	$[x+k^2\mathrm{sn}\,\mathrm{cd}-\mathrm{E}(k,\varphi)]/k^2$
ns	$\ln\{k'/(\mathrm{ds}-\mathrm{cs})\}$	$\mathrm{E}(k,\varphi)+\mathrm{cn}\,\mathrm{ds}-\mathrm{E}(k)+K-x$
cs	$\ln\{(1-k')/(\mathrm{ns}-\mathrm{ds})\}$	$\mathrm{E}(k,\varphi)+\mathrm{cn}\,\mathrm{ds}-\mathrm{E}(k)$
ds	$-\ln(\mathrm{ns}-\mathrm{cs})$	$\mathrm{E}(k,\varphi)+\mathrm{cn}\,\mathrm{ds}-\mathrm{E}(k)+k'^2(K-x)$

Equation 63:10:1 is helpful in evaluating the integrals of many products and quotients of elliptic functions; some of these are listed by Gradshteyn and Ryzhik [Sections 5.131−139].

63:11 COMPLEX ARGUMENT

The real and imaginary parts of Jacobi's three functions are:

63:11:1 $$\mathrm{cn}(k,x+iy) = \frac{\mathrm{cs}(k,x)\mathrm{ns}(k',y) - i\mathrm{dn}(k,x)\mathrm{dc}(k',y)}{\mathrm{ns}(k,x)\mathrm{cs}(k',y) + k^2\mathrm{sn}(k,x)\mathrm{sc}(k',y)}$$

63:11:2 $$\mathrm{sn}(k,x+iy) = \frac{\mathrm{ds}(k',y)\mathrm{nc}(k',y) + i\mathrm{ds}(k,x)\mathrm{cn}(k,x)}{\mathrm{ns}(k,x)\mathrm{cs}(k',y) + k^2\mathrm{sn}(k,x)\mathrm{sc}(k',y)}$$

63:11:3 $$\mathrm{dn}(k,x+iy) = \frac{\mathrm{ds}(k,x)\mathrm{ds}(k',y) - ik^2\mathrm{cn}(k,x)\mathrm{nc}(k',y)}{\mathrm{ns}(k,x)\mathrm{cs}(k',y) + k^2\mathrm{sn}(k,x)\mathrm{sc}(k',y)}$$

When the argument is imaginary, these formulas, and their nine other cohorts, reduce to Jacobi imaginary transformations, each of which establishes a relationship between a Jacobian elliptic functions of imaginary argument and one with real argument. All these transformations are of the form

63:11:4 $$\mathrm{ef}\left(k,iy\right)=w\,\overline{\mathrm{ef}}\;\left(k',y\right) \qquad k'=\sqrt{1-k^2}$$

where the multiplier w is either 1, i, or $-i$, and $\overline{\mathrm{ef}}$ is another – or sometimes the same – elliptic function. The panel below lists the correspondences between ef and $w\overline{\mathrm{ef}}$

ef =	cn	sn	dn	cd	sd	nd	cs	ds	ns	sc	dc	nc
$w\overline{\mathrm{ef}}$ =	nc	$i$sc	dc	nd	$i$sd	cd	$-i$ns	$-i$ds	$-i$cs	$i$sn	dn	cn

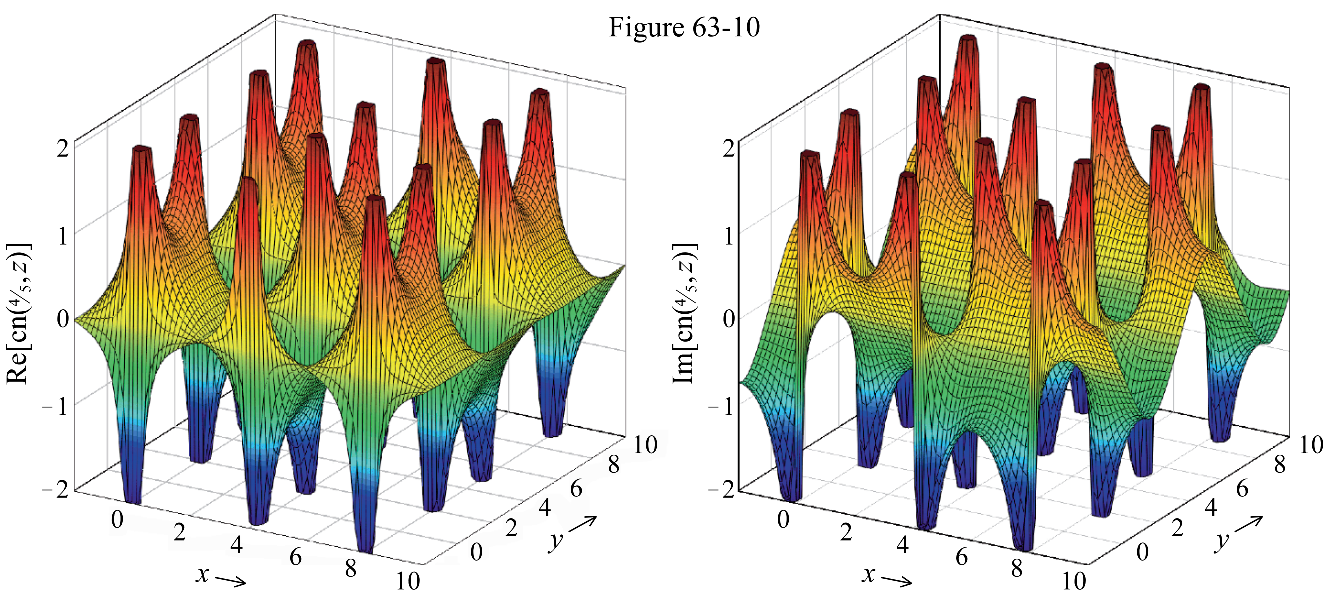
Notice that whether this transformation produces an imaginary or a real result depends on whether or not an "s" appears in the symbol for the elliptic function.

As elaborated in Section 63:2, an elliptic function of real argument is periodic in x with a period that is either 4K(k) or 2K(k). This periodicity is retained when the argument becomes complex but, as the previous paragraph demonstrates, an elliptic function of complex argument is periodic along the imaginary axis, too, with a period that is either 4iK(k') or 2iK(k'). The assignment of real and imaginary periods is made in the table to the right.

	Period	
	real	imaginary
cn,ds,nc,sd	4K(k)	4iK(k')
dn,cs,sc,nd	2K(k)	4iK(k')
sn,ns,dc,cd	4K(k)	2iK(k')

This double periodicity is clearly exemplified in Figure 63-10, which depicts the real and imaginary parts of cn(k,z) = cn(k,x+iy) with $k=4/5$. Note that, because there are two poles per period, the spacing of poles is $\frac{1}{2}[4\mathrm{K}(4/5)] = 3.9906$ along the real dimension and $\frac{1}{2}[4\mathrm{K}(3/5)] = 3.5015$ along the imaginary axis.

Figure 63-10



63:12 GENERALIZATIONS

We are aware of no direct generalizations of the Jacobian elliptic functions having been made.

63:13 COGNATE FUNCTIONS: inverse elliptic functions

Among a number of functions that are related to Jacobian elliptic functions, we here mention only their inverses. As with other inverses of periodic functions, ambiguities arise in assigning the principal values of the inverse elliptic functions. Beware of differing conventions.

With k invariant, the inverse elliptic functions are various incomplete elliptic integrals of the first kind [Chapter 62]. From the prototype

63:13:1 $$\mathrm{invam}(k,y)=\int_0^y \frac{1}{\sqrt{1-k^2\sin^2(\theta)}}\,\mathrm{d}\theta=\mathrm{F}(k,y)$$

it follows that

63:13:2 $$\mathrm{invcn}(k,y)=\int_y^1 \frac{1}{\sqrt{1-t^2}\sqrt{k^2t^2+1-k^2}}\,\mathrm{d}t=\mathrm{F}(k,\arccos(y))$$

63:13:3 $$\mathrm{invsn}(k,y)=\int_0^y \frac{1}{\sqrt{1-t^2}\sqrt{1-k^2t^2}}\,\mathrm{d}t=\mathrm{F}(k,\arcsin(y))$$

63:13:4 $$\mathrm{invdn}(k,y)=\int_y^1 \frac{1}{\sqrt{1-t^2}\sqrt{t^2-1+k^2}}\,\mathrm{d}t=\mathrm{F}\left(k,\arcsin\left\{\frac{\sqrt{1-y^2}}{k}\right\}\right) \qquad k'<y<1$$

with similar results for the other nine inverse Jacobian elliptic functions. Of course, when k equals zero or unity, they generally reduce to the functions of Chapters 35 or 31. Jeffrey [Chapter 12] lists a few other properties of these inverse functions.

63:14 RELATED TOPIC: Weierstrassian elliptic functions

The elliptic family of functions addressed in Chapters 61−63 are largely the creation of Legendre and Jacobi, but there is a parallel formalism due to the German Karl Theodor Wilhelm Weierstrass (teacher and mathematical innovator, 1815 −1897). Of course, the two rival systems are related. Here we shall point out some of those relationships, but stop well short of a comprehensive description of the Weierstrass system.

There are three interrelated parameters in the Weierstrass system that play a role equivalent to that played by Legendre's modulus k and complementary modulus k'. The equivalences are

63:14:1 $$k=\sqrt{\frac{e_2-e_1}{e_1-e_3}} \qquad k'=\sqrt{\frac{e_1-e_2}{e_1-e_3}} \qquad \text{where} \qquad e_1+e_2+e_3=0$$

Likewise, the role of determining the real and imaginary periods of the Weierstrass's elliptic functions, played in the Legendre system by K(k) and K(k'), is taken by two new variables

63:14:2 $$\omega_1=\frac{2\mathrm{K}(k)}{\sqrt{e_1-e_2}} \qquad \text{and} \qquad \omega_2=\frac{2i\mathrm{K}(k')}{\sqrt{e_1-e_2}}$$

The principal Weierstrassian elliptic function, usually symbolized P(z) with some fancy typographic rendering of the "P", is expressible in terms of particular Jacobian elliptic functions as the alternatives

63:14:3 $$e_1+(e_1-e_3)\mathrm{cs}^2\left(k,z\sqrt{e_1-e_3}\right)=e_2+(e_1-e_3)\mathrm{ds}^2\left(k,z\sqrt{e_1-e_3}\right)=e_3+(e_1-e_3)\mathrm{sn}^2\left(k,z\sqrt{e_1-e_3}\right)$$