CHAPTER 59

THE LEGENDRE FUNCTIONS $P_\nu(x)$ AND $Q_\nu(x)$

The two bivariate functions of this chapter arise in several physical contexts, most notably in describing phenomena within, or on the surface of, a sphere. Applications of these functions concentrate on real arguments of magnitudes less than unity and accordingly the domain $-1 < x < 1$ receives emphasis in this chapter. *Associated Legendre functions*, which are trivariate extensions of $P_\nu(x)$ and $Q_\nu(x)$, are discussed briefly in Section 59:12. The final section of this chapter illustrates how these functions arise in physical applications.

59:1 NOTATION

$P_\nu(x)$ and $Q_\nu(x)$ are known respectively as the *Legendre function of the first kind* and the *Legendre function of the second kind*, of degree ν. Not infrequently the argument x is replaced by a hyperbolic or circular cosine; the *Atlas* uses notations such as $P_\nu(\cosh(\alpha))$ and $Q_\nu(\cos(\theta))$ when these replacements are made.

The term *spherical harmonic* is used collectively to embrace both the Legendre functions and the associated Legendre functions [Section 59:12].

Regrettably, the definitions of $P_\nu(x)$ and $Q_\nu(x)$ are by no means uniform from one author to the next, and varying symbolism are in use. Abramowitz and Stegun [Page 332] list several of these alternatives. Sometimes different symbols are adopted according as the argument is real or complex. As elsewhere in this *Atlas*, our emphasis here is on real variables. Accordingly, an equation in this chapter will sometimes imply only that the real parts of each side are equal.

59:2 BEHAVIOR

Apart from the isolated instances addressed in Sections 59:4 and 59:5, both Legendre functions adopt complex values when $x \le -1$. This domain receives no further attention here. The Q Legendre function is mostly complex also when its argument exceeds -1, but the definition discussed in Section 59:11 accords it the real values discussed in most of this chapter. $Q_\nu(x)$ is unequivocally real only for $-1 < x < 1$.

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Each Legendre function exhibits such different properties on either side of $x = 1$ that effectively it behaves as a distinct function in each region. Accordingly, we display their landscapes as the four graphs shown opposite.

The values acquired by the two Legendre functions at arguments of -1, 0, and $+1$ are reported in detail in Section 59:7 and their general behavior within $-1 < x < 1$ is evident from Figures 59-1 and 59-2. Because of the degree-reflection formulas 59:5:3 and 59:5:4, it suffices to consider only degrees of $\nu \geq -\tfrac{1}{2}$, and the two figures are so restricted. The landscape is rippled for both functions, with multiple zeros and local extrema.

Figures 59-3 and 59-4 are three-dimensional graphs of the Legendre functions in the $x > 1$ domain. $P_\nu(x)$ exhibits a rather bland landscape; notice the reflection symmetry that exists across the plane $\nu = -\tfrac{1}{2}$, in consequence of formula 59:5:3. The repetitive $-\infty|+\infty$ discontinuities that occur in Figure 59-4 whenever the degree of $Q_\nu(x)$ encounters a negative integer value may be attributed to the cotangent term in formula 59:5:4.

59:3 DEFINITIONS

Several definite integrals, including the following, represent the Legendre functions:

59:3:1 $$P_\nu(x) = \frac{1}{\pi}\int_0^\pi \left[x + \sqrt{x^2-1}\cos(\phi)\right]^\nu d\phi \qquad x>1 \qquad \text{all } \nu$$

59:3:2 $$Q_\nu(x) = \int_0^\infty \left[x + \sqrt{x^2-1}\cosh(t)\right]^{-\nu-1} dt \qquad x>1 \qquad \nu>-1$$

There are also several definitions of the Legendre functions as indefinite integrals:

59:3:3 $$P_\nu(\cos(\theta)) = \frac{\sqrt{2}}{\pi}\int_0^\theta \frac{\cos\left((\nu+\tfrac{1}{2})\phi\right)}{\sqrt{\cos(\phi)-\cos(\theta)}} d\phi \qquad 0<\theta<\pi$$

59:3:4 $$P_\nu(\cosh(\alpha)) = \frac{\sqrt{2}}{\pi}\cot\left((\nu+\tfrac{1}{2})\pi\right)\int_\alpha^\infty \frac{\sinh\left((\nu+\tfrac{1}{2})t\right)}{\sqrt{\cosh(t)-\cosh(\alpha)}} dt \qquad \alpha>0 \qquad -1<\nu<0$$

59:3:5 $$Q_\nu(\cosh(\alpha)) = \frac{1}{\sqrt{2}}\int_\alpha^\infty \frac{\exp\left(-(\nu+\tfrac{1}{2})t\right)}{\sqrt{\cosh(t)-\cosh(\alpha)}} dt \qquad \alpha>0 \qquad \nu>-1$$

Despite appearances to the contrary, the integrands in the definitions

59:3:6 $$P_\nu(x) = \frac{1}{\pi}\int_1^\infty \left\{ i\left[x+it\sqrt{1-x^2}\right]^{-1-\nu} - \left[x-it\sqrt{1-x^2}\right]^{-1-\nu}\right\}\frac{dt}{\sqrt{t^2-1}} \qquad -1<x<1 \qquad \nu>-1$$

and

59:3:7 $$Q_\nu(x) = \frac{1}{2}\int_1^\infty \left\{\left[x+it\sqrt{1-x^2}\right]^{-1-\nu} + \left[x-it\sqrt{1-x^2}\right]^{-1-\nu}\right\}\frac{dt}{\sqrt{t^2-1}} \qquad -1<x<1 \qquad \nu>-1$$

are wholly real because all imaginary terms cancel after binomial expansions.

On account of formula 59:5:3, the degree ν in any definition of $P_\nu(x)$ may be replaced by $-\nu-1$. A similar replacement for $Q_\nu(x)$ is valid only if ν is an odd multiple of ½.

The traditional definition of the Legendre functions is as solutions of *Legendre's differential equation*

59:3:8 $$(1-x^2)\frac{d^2 f}{dx^2} - 2x\frac{df}{dx} + \nu(\nu+1)f = 0 \qquad f = w_1 P_\nu(x) + w_2 Q_\nu(x)$$

the w's being arbitrary constants.

Figure 59-1

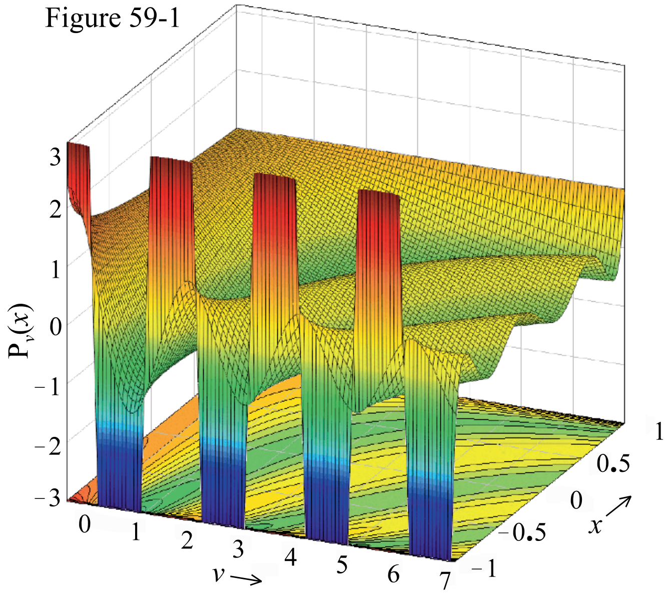


Figure 59-2

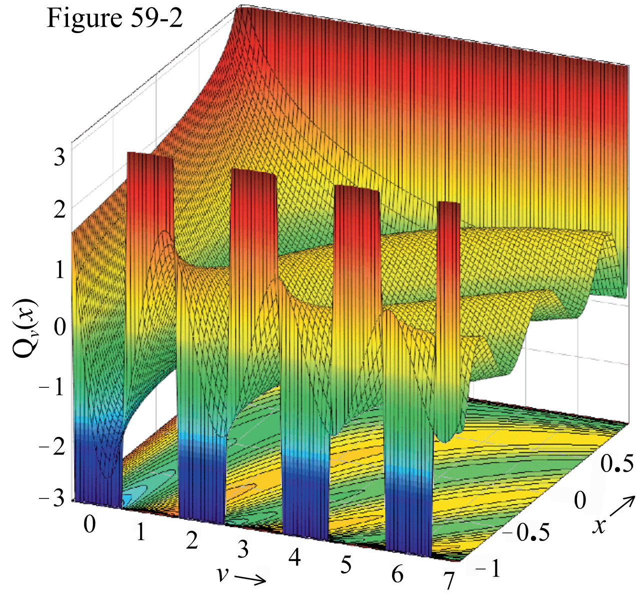


Figure 59-3

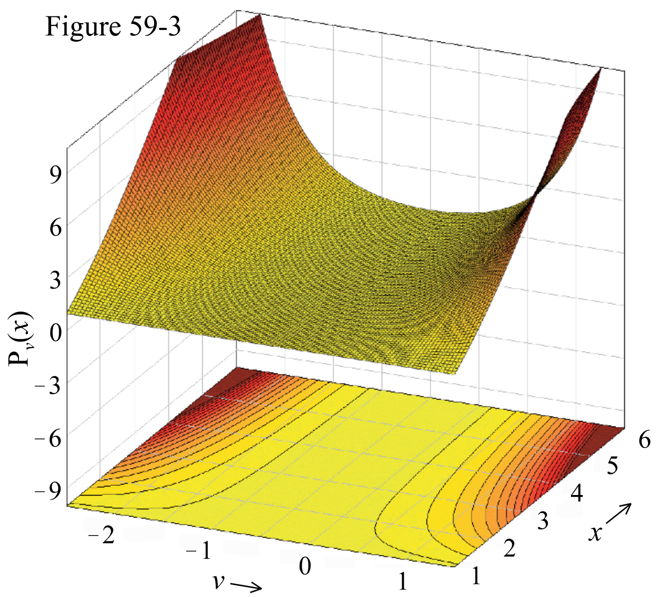
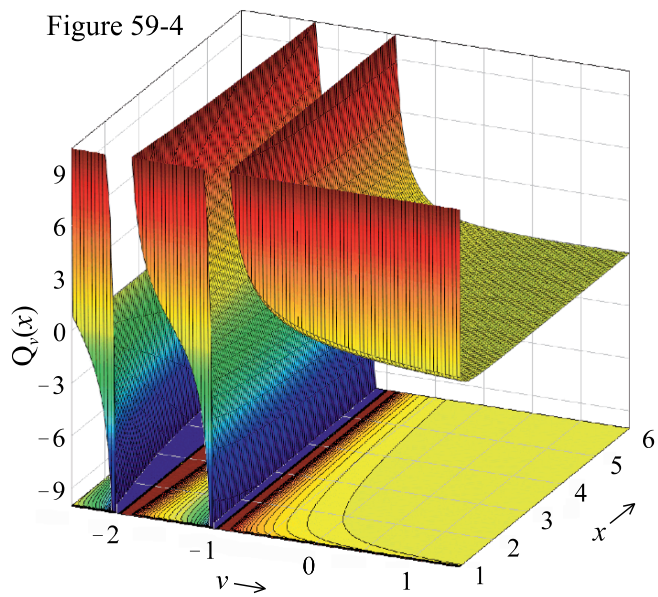


Figure 59-4



The hypergeometric formulation, equation 59:6:1, of the Legendre function of the first kind opens the way to its synthesis [Section 43:14] from a power function:

59:3:9
$$(1-x)^\nu \xrightarrow[1]{1+\nu} \mathrm{P}_\nu(1-2x)$$

Synthesis of the second Legendre function is more elaborate:

59:3:10
$$(1-x)^{-\nu/2} \xrightarrow[\nu+\frac{1}{2}]{(\nu+1)/2} \frac{2^\nu\Gamma(\nu+\frac{1}{2})}{\sqrt{\pi}\,\Gamma(\nu)x^{\nu/2}} Q_{\nu\text{-}1}\left(\frac{1}{\sqrt{x}}\right)$$

The Legendre function of the first kind can be expressed in many ways as special cases of the Gauss hypergeometric function [Section 59:13] and occasionally $P_\nu(x)$ is defined by this route.

59:4 SPECIAL CASES

When ν is an integer of either sign, $P_\nu(x)$ reduces to one of the Legendre polynomials discussed in Chapter 21

59:4:1
$$\left.\begin{matrix} P_\nu(x) \\ P_{-\nu-1}(x) \end{matrix}\right\} = P_n(x) = \text{polynomial} \qquad \begin{cases} \nu = n = 0,1,2,\cdots \\ \nu = -n-1 = -1,-2,-3,\cdots \end{cases}$$

Legendre functions of the second kind are not defined for negative integer degree, but when ν is a *non*negative integer, $Q_\nu(x)$ reduces to one of the functions $Q_n(x)$ described in Section 21:13, many of which are graphed in Figure 21-3. Some examples are

	$\nu=-2$	$\nu=-1$	$\nu=0$	$\nu=1$	$\nu=2$	$\nu=3$
$P_\nu(x)$	x	1	1	x	$\frac{3}{2}x^2-\frac{1}{2}$	$\frac{5}{2}x^3-\frac{3}{2}x$
$Q_\nu(x)$	undef	undef	$Q_0(x)$	$x\,Q_0(x)-1$	$P_2(x)Q_0(x)-\frac{3}{2}x$	$P_3(x)Q_0(x)-\frac{5}{2}x^2+\frac{2}{3}$

where the zero-degree instance of the Legendre function of the second kind is an inverse hyperbolic function [Chapter 31]

59:4:2
$$Q_0(x) = \begin{cases} \text{artanh}(x) & -1<x<1 \\ \text{arcoth}(x) & |x|>1 \end{cases}$$

When the degree ν is an odd multiple of ½, each of the Legendre functions reduces to an expression involving complete elliptic integrals [Chapter 61] with square-root moduli. The prime examples are for $\nu=-\frac{1}{2}$:

59:4:3
$$P_{-1/2}(x) = \frac{2}{\pi}K\left(\sqrt{\frac{1-x}{2}}\right)$$

59:4:4
$$Q_{-1/2}(x) = K\left(\sqrt{\frac{1+x}{2}}\right)$$

The modulus of the elliptic integral in these formulas is imaginary for $x>1$ in the case of P, or $x<-1$ for Q. Nevertheless, these formulas define real Legendre functions in these x domains because [see 61:11:4] complete elliptic integrals are real when their moduli are imaginary. However, for $x<-1$ in the case of P, or $x>1$ for Q, the modulus of the elliptic integral exceeds unity whereupon its value, and hence that of the Legendre function, becomes complex. Because the reflection formula $f_{-\nu-1}(x)=f_\nu(x)$ applies to *both* Legendre functions in these special cases, each expression for a Legendre function of a degree that is an odd multiple of ½ applies to two degrees that average $-\frac{1}{2}$. For example

59:4:5
$$P_{-3/2}(x) = P_{1/2}(x) = \frac{4}{\pi}E\left(\sqrt{\frac{1-x}{2}}\right) - \frac{2}{\pi}K\left(\sqrt{\frac{1-x}{2}}\right)$$

and

59:4:6 $$Q_{-3/2}(x) = Q_{1/2}(x) = K\left(\sqrt{\frac{1+x}{2}}\right) - 2E\left(\sqrt{\frac{1+x}{2}}\right)$$

By employing two equations from 59:4:3-6 as the starting point, recursion formula 59:5:5 may be used to develop formulas expressing the Legendre function of any other degree that is an odd multiple of ½.

59:5 INTRARELATIONSHIPS

The argument-reflection formulas

59:5:1 $$P_n(-x) = (-)^{|n+1/2|-1/2}\, P_n(x) \qquad n = 0, \pm1, \pm2, \cdots$$

59:5:2 $$Q_n(-x) = -(-)^n\, Q_n(x) \qquad n = 0, 1, 2, \cdots$$

apply only when the degree is an integer, whereas the degree-reflection formulas

59:5:3 $$P_{-\nu-1}(x) = P_\nu(x)$$

59:5:4 $$Q_{-\nu-1}(x) = Q_\nu(x) - \pi\cot(\nu\pi)\, P_\nu(x)$$

are of general applicability.

The same degree-recursion formula

59:5:5 $$(\nu+1)\,f_{\nu+1}(x) = (2\nu+1)x\,f_\nu(x) - \nu\,f_{\nu-1}(x) \qquad f = P \text{ or } Q$$

applies to both kinds of Legendre function.

The following equations link the two kinds of Legendre function but apply only when the degree is not an integer and when $-1 < x < 1$

59:5:6 $$Q_\nu(\pm x) = \frac{\pi}{2}\left[\cot(\nu\pi)\, P_\nu(\pm x) - \csc(\nu\pi)\, P_\nu(\mp x)\right]$$

59:5:7 $$P_\nu(\pm x) = -\frac{2}{\pi}\left[\cot(\nu\pi)\, Q_\nu(\pm x) + \csc(\nu\pi)\, Q_\nu(\mp x)\right]$$

59:6 EXPANSIONS

As detailed in Section 59:13, Legendre functions may be expressed as Gauss hypergeometric functions [Chapter 60] in a multitude of ways. Because each of these latter functions may be expanded as the (usually infinite) power series 60:6:1, the number of expansions of $P_\nu(x)$ and $Q_\nu(x)$ is huge. The listing presented here is not exhaustive. In all cases the strategy is to express the Legendre function as a Gauss hypergeometric function, or as a weighted sum of two such functions, and then expand the latter function(s) as power series.

For reasons explained in Section 59:13, the expansion

59:6:1 $$P_\nu(x) = F\left(-\nu, 1+\nu, 1, \frac{1-x}{2}\right) = \sum_{j=0}^{\infty} \frac{(-\nu)_j (1+\nu)_j}{(1)_j (1)_j} \left(\frac{1-x}{2}\right)^j = 1 - \frac{\nu(1+\nu)}{2}(1-x) + \cdots$$

valid in the range $-1 < x < 3$ range of argument, and the more complicated expansion

59:6:2 $$Q_\nu(x) = \frac{\sqrt{\pi}\,\Gamma(1+\nu)}{\Gamma\left(\frac{3}{2}+\nu\right)[2x]^{1+\nu}} F\left(\tfrac{1}{2}+\tfrac{1}{2}\nu, 1+\tfrac{1}{2}\nu, \tfrac{3}{2}+\nu, \frac{1}{x^2}\right) = \frac{\sqrt{\pi}\,\Gamma(1+\nu)}{\Gamma\left(\frac{3}{2}+\nu\right)[2x]^{1+\nu}} \left[1 + \frac{\nu^2+3\nu+2}{(8\nu+12)x^2} + \cdots\right]$$

which has validity for $x>1$, are seen as fundamental. Should v equal one of the values $-3/2, -5/2, -7/2, \cdots$, the first $(-v-½)$ terms in 59:6:2 are zero. If you wish to avoid this, replace each v on the right-hand side by $-v-1$.

To illustrate the twin-Gauss approach to expanding Legendre functions, first consider the two expansions

59:6:3 $$F\left(-\tfrac{1}{2}v,\tfrac{1}{2}+\tfrac{1}{2}v,\tfrac{1}{2},x^2\right)=\sum_{j=0}^{\infty}\frac{\left(\frac{-1}{2}v\right)_j\left(\frac{1}{2}+\frac{1}{2}v\right)_j}{\left(\frac{1}{2}\right)_j(1)_j}x^{2j}=1-\frac{v^2+v}{2}x^2+\frac{v^4+2v^3-5v^2-6v}{24}x^4-\cdots$$

and

59:6:4 $$F\left(\tfrac{1}{2}-\tfrac{1}{2}v,1+\tfrac{1}{2}v,\tfrac{3}{2},x^2\right)=\sum_{j=0}^{\infty}\frac{\left(\frac{1}{2}-\frac{1}{2}v\right)_j\left(1+\frac{1}{2}v\right)_j}{\left(\frac{3}{2}\right)_j(1)_j}x^{2j}=1-\frac{v^2+v-1}{6}x^2+\frac{v^4+2v^3-12v^2-13v+12}{120}x^4-\cdots$$

both applicable for $-1<x<1$. The Legendre functions are weighted sums of these two Gauss hypergeometric functions of argument x^2:

59:6:5 $$P_v(x)=\frac{\sqrt{\pi}}{\Gamma\left(\frac{1}{2}-\frac{1}{2}v\right)\Gamma\left(1+\frac{1}{2}v\right)}F\left(-\tfrac{1}{2}v,\tfrac{1}{2}+\tfrac{1}{2}v,\tfrac{1}{2},x^2\right)\pm\frac{2\sqrt{\pi}}{\Gamma\left(\frac{v}{2}\right)\Gamma\left(\frac{1}{2}+\frac{1}{2}v\right)}xF\left(\tfrac{1}{2}-\tfrac{1}{2}v,1+\tfrac{1}{2}v,\tfrac{3}{2},x^2\right)$$

59:6:6 $$Q_v(x)=\frac{\sqrt{\pi^3}\,\Gamma\left(1+\frac{1}{2}v\right)}{\Gamma^2\left(\frac{1}{2}+\frac{1}{2}v\right)\Gamma\left(\frac{1}{2}-\frac{1}{2}v\right)}xF\left(\tfrac{1}{2}-\tfrac{1}{2}v,1+\tfrac{1}{2}v,\tfrac{3}{2},x^2\right)+\frac{\sqrt{\pi^3}\,\Gamma\left(\frac{1}{2}+\frac{1}{2}v\right)}{2\Gamma^2\left(1+\frac{1}{2}v\right)\Gamma\left(\frac{-1}{2}v\right)}F\left(-\tfrac{1}{2}v,\tfrac{1}{2}+\tfrac{1}{2}v,\tfrac{1}{2},x^2\right)$$

Notice that, in each of these formulas, there is an x multiplier in one or other of the weights, ensuring that all nonnegative integer powers of x appear in the ultimate expansions.

There are also expansions as trigonometric functions:

59:6:7 $$P_v\left(\cos(\theta)\right)=\frac{2\Gamma(1+v)}{\sqrt{\pi}\,\Gamma\left(\frac{3}{2}+v\right)}\sum_{j=0}^{\infty}\frac{(1+v)_j\left(\frac{1}{2}\right)_j}{\left(\frac{3}{2}+v\right)_j(1)_j}\sin\left((1+v+2j)\theta\right)\qquad 0<\theta<\pi$$

and

59:6:8 $$Q_v\left(\cos(\theta)\right)=\frac{\sqrt{\pi}\,\Gamma(1+v)}{\Gamma\left(\frac{3}{2}+v\right)}\sum_{j=0}^{\infty}\frac{(1+v)_j\left(\frac{1}{2}\right)_j}{\left(\frac{3}{2}+v\right)_j(1)_j}\cos\left((1+v+2j)\theta\right)\qquad 0<\theta<\pi$$

though these sums are often slow to converge.

59:7 PARTICULAR VALUES

In this section we look into the values acquired by $P_v(x)$ and $Q_v(x)$ at the arguments $x=-1$, 0, 1 and ∞. The first and second tables below apply to Legendre function of the first and second kinds respectively. Observe that a plethora of formulas is needed to cover particular values at zero argument; mostly, these are special cases of the formulas

59:7:1 $$P_v(0)=\frac{\Gamma\left(\frac{1}{2}+\frac{1}{2}v\right)\cos\left(\frac{1}{2}v\pi\right)}{\sqrt{\pi}\Gamma\left(1+\frac{1}{2}v\right)}\qquad\text{and}\qquad Q_v(0)=\frac{-\sqrt{\pi}\,\Gamma\left(\frac{1}{2}+\frac{1}{2}v\right)\sin\left(\frac{1}{2}v\pi\right)}{2\Gamma\left(1+\frac{1}{2}v\right)}$$

Gauss's constant g [Section 1:7] occurs frequently in these tables, as do the double and quadruple factorial functions that are addressed in Section 2:3. Note the example $(15)!!!!=15\times11\times7\times3$, and that quadruple factorials of $1,-1$ and -3 all equal unity.

ν	$P_\nu(-1)$	$P_\nu(0)$	$P_\nu(1)$	$P_\nu(\infty)$
$-3,-5,-7,-9,\cdots$	1	$\dfrac{(-)^{(\nu+1)/2}(-\nu-2)!!}{(-\nu-1)!!}$	1	$+\infty$
$\frac{-5}{2},\frac{-9}{2},\frac{-13}{2},\frac{-17}{2},\cdots$	$+\infty$	$(-)^{(1+2\nu)/4}\dfrac{\sqrt{2}(-2\nu-4)!!!!g}{(-2\nu-2)!!!!}$	1	$+\infty$
$1,-2,3,-4,\cdots$	-1	0	1	$+\infty$
$\frac{-3}{2},\frac{-7}{2},\frac{-11}{2},\frac{-15}{2},\cdots$	$-\infty$	$(-)^{(3+2\nu)/4}\dfrac{\sqrt{2}(-2\nu-4)!!!!}{(-2\nu-2)!!!!\pi g}$	1	$+\infty$
$-1,0$	1	1	1	1
$\frac{-1}{2},\frac{3}{2},\frac{7}{2},\frac{11}{2},\cdots$	$+\infty$	$(-)^{(2\nu+1)/4}\dfrac{\sqrt{2}(2\nu-2)!!!!g}{(2\nu)!!!!}$	1	$+\infty$
$\frac{1}{2},\frac{5}{2},\frac{9}{2},\frac{13}{2},\cdots$	$-\infty$	$(-)^{(2\nu-1)/4}\dfrac{\sqrt{2}(2\nu-2)!!!!}{(2\nu)!!!!\pi g}$	1	$+\infty$
$2,4,6,8,\cdots$	1	$\dfrac{(-)^{\nu/2}(\nu-1)!!}{\nu!!}$	1	$+\infty$

ν	$Q_\nu(-1)$	$Q_\nu(0)$	$Q_\nu(1)$	$Q_\nu(\infty)$
$\frac{-5}{2},\frac{-9}{2},\frac{-13}{2},\frac{-17}{2},\cdots$	$\dfrac{\pi}{2}$	$(-)^{(1+2\nu)/4}\dfrac{(-2\nu-4)!!!!\pi g}{\sqrt{2}(-2\nu-2)!!!!}$	$+\infty\|+\infty$	0
$\frac{-3}{2},\frac{-7}{2},\frac{-11}{2},\frac{-15}{2},\cdots$	$\dfrac{-\pi}{2}$	$(-)^{(1-2\nu)/4}\dfrac{(-2\nu-4)!!!!}{\sqrt{2}(-2\nu-2)!!!!g}$	$+\infty\|+\infty$	0
$-1,-2,-3,-4,\cdots$	undef	undef	undef	undef
$\frac{-1}{2},\frac{3}{2},\frac{7}{2},\frac{11}{2},\cdots$	$\dfrac{\pi}{2}$	$(-)^{(2\nu+1)/4}\dfrac{(2\nu-2)!!!!\pi g}{\sqrt{2}(2\nu)!!!!}$	$+\infty\|+\infty$	0
$0,2,4,6,\cdots$	$-\infty$	0	$+\infty\|+\infty$	0
$\frac{1}{2},\frac{5}{2},\frac{9}{2},\frac{13}{2},\cdots$	$\dfrac{-\pi}{2}$	$(-)^{(2\nu+3)/4}\dfrac{(2\nu-2)!!!!}{\sqrt{2}(2\nu)!!!!g}$	$+\infty\|+\infty$	0
$1,3,5,7,\cdots$	$+\infty$	$(-)^{(\nu-1)/2}\dfrac{(\nu-1)!!}{\nu!!}$	$+\infty\|+\infty$	0

59:8 NUMERICAL VALUES

Equator's routines for the Legendre function of the first kind and Legendre function of the second kind (keywords **P** and **Q**) employ equations 59:6:1 or 59:6:5 and 59:6:2 or 59:6:6 to evaluate $P_\nu(x)$ and $Q_\nu(x)$ in the domains $|\nu| \le 100$ and $x \ge -1$. Where functions are complex, only the real part is returned. For integer ν values of the Legendre function of the first kind are also provided for the $x < -1$ region.

59:9 LIMITS AND APPROXIMATIONS

Close to $x = 1$, the Legendre function of the first kind becomes linear

59:9:1 $$P_\nu(x) \approx 1 + \frac{\nu+\nu^2}{2}(x-1) \qquad |1-x| \text{ small}$$

whereas that of the second kind approaches infinity in accord with the relationship

59:9:2 $$Q_\nu(x) \approx \ln\left(\sqrt{\frac{2}{|1-x|}}\right) - \gamma - \psi(1+\nu) \qquad |1-x| \text{ small}$$

The corresponding limiting behaviors as $-1 \leftarrow x$ are described by

59:9:3 $$P_\nu(x) \approx \cos(\nu\pi) + \frac{\sin(\nu\pi)}{\pi}\left[\gamma + 2\psi(1+\nu) + \ln\left(\frac{1+x}{2}\right)\right] \qquad (1+x) \text{ small and positive}$$

and

59:9:4 $$Q_\nu(x) \approx \frac{-\pi\sin(\nu\pi)}{2} + \frac{\cos(\nu\pi)}{2}\left[\gamma + 2\psi(1+\nu) + \ln\left(\frac{1+x}{2}\right)\right] \qquad (1+x) \text{ small and positive}$$

The γ and $\psi(\ )$ terms in these formulas are Euler's constant [Section 1:7] and the digamma function [Chapter 44].

When the argument is large and positive, the limiting expressions are

	$\nu < -\frac{1}{2}$	$\nu = -\frac{1}{2}$	$\nu \ge 0$
$P_\nu(x\to\infty)$	$\dfrac{\Gamma\left(-\nu-\frac{1}{2}\right)}{\sqrt{\pi}\,\Gamma(-\nu)}(2x)^{-\nu-1}$	$\dfrac{\sqrt{2}}{\pi}\dfrac{\ln(8x)}{\sqrt{x}}$	$\dfrac{\Gamma\left(\nu+\frac{1}{2}\right)}{\sqrt{\pi}\,\Gamma(\nu+1)}(2x)^{\nu}$

	$\nu < -\frac{1}{2}$ but $\neq -\frac{3}{2}, -\frac{5}{2}, -\frac{7}{2}, \cdots$	$\nu = -\frac{3}{2}, -\frac{5}{2}, -\frac{7}{2}, \cdots$	$\nu \le 3,\ \nu \neq -1, -2, -3, \cdots$
$Q_\nu(x\to\infty)$	$\dfrac{\sqrt{\pi}\,\Gamma\left(-\nu-\frac{1}{2}\right)\cot(\nu\pi)}{\Gamma(-\nu)}(2x)^{-\nu-1}$	$\dfrac{(-2\nu-2)!!\pi\sqrt{2}}{\left(-\frac{1}{2}-\nu\right)!}(4x)^{\nu}$	$\dfrac{\sqrt{\pi}\,\Gamma(1+\nu)}{\Gamma\left(\frac{3}{2}+\nu\right)[2x]^{\nu+1}}$

59:10 OPERATIONS OF THE CALCULUS

Either differentiation or indefinite integration of a Legendre function yields an associated Legendre function [Section 59:12]

59:10:1 $$\frac{d}{dx}f_\nu(x) = \begin{cases} -f_\nu^{(1)}(x)/\sqrt{1-x^2} & -1 < x < 1 \\ f_\nu^{(1)}(x)/\sqrt{x^2-1} & x > 1 \end{cases} \qquad f = P \text{ or } Q$$

59:10:2 $$\int_x^1 \mathrm{P}_\nu(t)\,\mathrm{d}t = \sqrt{1-x^2}\,\mathrm{P}_\nu^{(-1)}(x) \qquad -1<x<1$$

59:10:3 $$\int_1^x \mathrm{P}_\nu(t)\,\mathrm{d}t = \sqrt{x^2-1}\,\mathrm{P}_\nu^{(-1)}(x) \qquad x>1$$

As an alternative to 59:10:1, the derivatives of Legendre functions may be expressed, without recourse to associated functions, as

59:10:4 $$\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{f}_\nu(x) = \frac{\nu+1}{1-x^2}\left[x\,\mathrm{f}_\nu(x) - \mathrm{f}_{\nu+1}(x)\right] = \frac{\nu}{1-x^2}\left[\mathrm{f}_{\nu-1}(x) - x\,\mathrm{f}_\nu(x)\right] \qquad x>-1 \qquad \mathrm{f}=\mathrm{P}\text{ or }\mathrm{Q}$$

or by many other formulas discovered by incorporating the recursion 59:5:5.

Noteworthy integrals and Laplace transforms of the first kind of Legendre function include

59:10:5 $$\int_x^1 t\,\mathrm{P}_\nu(t)\,\mathrm{d}t = \frac{\left(1-x^2\right)\mathrm{P}_\nu(x) + x\sqrt{1-x^2}\,\mathrm{P}_\nu^{(1)}(x)}{(1-\nu)(2+\nu)} \qquad -1<x<1$$

59:10:6 $$\int_0^1 t^\lambda\,\mathrm{P}_\nu(t)\,\mathrm{d}t = \frac{\sqrt{\pi}\,\Gamma(1+\lambda)}{2^{1+\lambda}\,\Gamma\left(1+\tfrac{1}{2}\lambda-\tfrac{1}{2}\nu\right)\Gamma\left(\tfrac{3}{2}+\tfrac{1}{2}\lambda+\tfrac{1}{2}\nu\right)} \qquad \lambda>-1$$

59:10:7 $$\int_{-1}^1 (1+t)^\lambda\,\mathrm{P}_\nu(t)\,\mathrm{d}t = \frac{2^{1+\lambda}\,\Gamma^2(1+\lambda)}{\Gamma\left(2+\lambda+\nu\right)\Gamma\left(1+\lambda-\nu\right)} \qquad \lambda>-1$$

59:10:8 $$\int_0^\infty \mathrm{P}_\nu(1+bt)\exp(-st)\,\mathrm{d}t = \mathfrak{L}\left\{\mathrm{P}_\nu(1+bt)\right\} = \sqrt{\frac{2}{\pi bs}}\exp\left(\frac{s}{b}\right)\mathrm{K}_{\nu+1/2}\left(\frac{s}{b}\right)$$

and many others are given by Gradshteyn and Ryzhik [Section 7.1].

Formulas for integrals of products of Legendre functions include

59:10:9 $$\int_{-1}^1 \mathrm{P}_\nu(t)\,\mathrm{P}_\omega(t)\,\mathrm{d}t = \frac{2\pi\sin\left\{(\nu-\omega)\pi\right\} - 4\sin(\nu\pi)\sin(\omega\pi)[\psi(1+\nu)-\psi(1+\omega)]}{\pi^2(\nu-\omega)(1+\nu+\omega)} \qquad \nu+\omega\ne-1$$

59:10:10 $$\int_0^1 \mathrm{P}_\nu(t)\,\mathrm{P}_\omega(t)\,\mathrm{d}t = \frac{2\Gamma\left(1+\tfrac{1}{2}\nu\right)\Gamma\left(\tfrac{1}{2}+\tfrac{1}{2}\omega\right)\sin\left(\tfrac{1}{2}\nu\pi\right)\cos\left(\tfrac{1}{2}\omega\pi\right)}{\pi\Gamma\left(\tfrac{1}{2}+\tfrac{1}{2}\nu\right)\Gamma\left(1+\tfrac{1}{2}\omega\right)(\nu-\omega)(1+\nu+\omega)} + \frac{2\Gamma\left(1+\tfrac{1}{2}\omega\right)\Gamma\left(\tfrac{1}{2}+\tfrac{1}{2}\nu\right)\sin\left(\tfrac{1}{2}\omega\pi\right)\cos\left(\tfrac{1}{2}\nu\pi\right)}{\pi\Gamma\left(\tfrac{1}{2}+\tfrac{1}{2}\omega\right)\Gamma(1+\tfrac{1}{2}\nu)(\omega-\nu)(1+\omega+\nu)}$$

59:10:11 $$\int_{-1}^1 \mathrm{P}_\nu(t)\,\mathrm{Q}_\omega(t)\,\mathrm{d}t = \frac{\pi[1-\cos\left\{(\nu-\omega)\pi\right\}] - 2\sin(\nu\pi)\cos(\omega\pi)[\psi(1+\nu)-\psi(1+\omega)]}{\pi(\nu-\omega)(1+\nu+\omega)} \qquad \nu>0<\omega$$

and

59:10:12 $$\int_{-1}^1 \mathrm{Q}_\nu(t)\,\mathrm{Q}_\omega(t)\,\mathrm{d}t = \frac{\left[\psi(\nu+1)-\psi(\omega+1)\right]\left[1+\cos(\nu\pi)\cos(\omega\pi)\right] - \tfrac{1}{2}\pi\sin(\nu\pi-\omega\pi)}{(\omega-\nu)(1+\nu+\omega)}$$

$$\omega\ne-\nu-1,-1,-2,-3,\cdots$$

where $\Gamma(\ )$ and $\psi(\ )$ are the gamma and digamma functions [Chapters 43 and 44]. Other integrals of this sort are listed in Section 3.12 of Erdélyi et al. [*Higher Transcendental Functions*, Volume 1], though not all of them appear to be correct. One cannot set $\omega=\nu$ in any of these formulas without carefully investigating the behavior as $\omega\to\nu$. Some consequences are as follows; they mostly involve the trigamma function [Section 44:12],

$\int_{-1}^{1}P_\nu^2(t)\,dt =$	$\int_{-1}^{1}P_\nu(t)Q_\nu(t)\,dt =$	$\int_{-1}^{1}Q_\nu^2(t)\,dt =$	$\int_{1}^{\infty}P_\nu(t)Q_\nu(t)\,dt =$
$\dfrac{2\pi^2-4\sin^2(\nu\pi)\psi^{(1)}(1+\nu)}{\pi^2(1+2\nu)}$	$\dfrac{-\sin(2\nu\pi)\psi^{(1)}(1+\nu)}{\pi(1+2\nu)}$	$\dfrac{\frac{1}{2}\pi^2-[1+\cos^2(\nu\pi)]\psi^{(1)}(1+\nu)}{1+2\nu}$	∞

The formulas 59:10:9−12 are useful, not only in their own right, but also as the source of other definite integrals by taking advantage of such identities as $P_0(t)=1$, $P_1(t)=t$, and $\frac{2}{3}P_2(t)-\frac{1}{3}P_0(t)=t^2$. The orthogonality properties of Legendre functions of the first kind are revealed by setting $m=0$ in equation 59:12:11.

59:11 COMPLEX ARGUMENT

To avoid the Legendre functions of complex argument being multivalued, the complex plane must be cut. For the $P_\nu(z)$ function, it is conventional to make the cut along the line $-\infty \le x \le -1$, $y=0$, whereas a longer cut $-\infty \le x \le +1$, $y=0$ is needed for $Q_\nu(z)$. Resulting from the cut are ambiguities in the values of Legendre functions when the argument is real. To resolve the ambiguity, it is conventional, but not universal, to assign the average of the values on either side of the cut. That is

59:11:1
$$f_\nu(x)=\lim_{\delta\to 0}\left\{\frac{f_\nu(x+i\delta)+f_\nu(x-i\delta)}{2}\right\} \qquad f = P \text{ or } Q$$

When their arguments or degrees are complex, the Legendre functions are generally complex, too, but the *Atlas* does not pursue this topic except for some comments on *conical functions*. These are Legendre functions of complex degree, of which the real part is −½. They arise in solving differential equations such as those listed in Section 46:15 for a space shaped as a cone or possessing certain other geometries [Lebedev, Sections 8.5, 8.9 and 8.12]. Notwithstanding its complex degree, the conical function of the first kind is real

59:11:2
$$P_{\frac{-1}{2}+i\lambda}(x)=F\left(\tfrac{1}{2}+i\lambda,\tfrac{1}{2}-i\lambda,1,\frac{1-x}{2}\right)=\sum_{j=0}^{\infty}\frac{[(1-x)/2]^j}{(j!)^2}\prod_{k=0}^{j-1}\lambda^2+\left(k+\tfrac{1}{2}\right)^2$$

Inverse Laplace transformation of Legendre functions generates a spherical instance of the Macdonald [Section 26:13] or the modified Bessel [Section 28:13] functions

59:11:3
$$\int_{\alpha-i\infty}^{\alpha+i\infty}P_\nu(bs)\frac{\exp(ts)}{2\pi i}\,ds=\mathfrak{I}\{P_\nu(bs)\}=\frac{-2}{\pi^2 b}\sin(\nu\pi)k_\nu\left(\frac{t}{b}\right)$$

59:11:4
$$\int_{\alpha-i\infty}^{\alpha+i\infty}Q_\nu(bs)\frac{\exp(ts)}{2\pi i}\,ds=\mathfrak{I}\{Q_\nu(bs)\}=\frac{1}{b}i_\nu\left(\frac{t}{b}\right)$$

59:12 GENERALIZATIONS: the associated Legendre functions

The trivariate *associated Legendre functions* $P_\nu^{(\mu)}(x)$ and $Q_\nu^{(\mu)}(x)$ are generalizations of the functions that are the main subject of this chapter, inasmuch as

59:12:1
$$P_\nu^{(0)}(x)=P_\nu(x) \qquad \text{and} \qquad Q_\nu^{(0)}(x)=Q_\nu(x)$$

The third variable μ is the *order* of the function. Some authors describe $P_\nu^{(\mu)}(x)$ and $Q_\nu^{(\mu)}(x)$ as “Legendre functions" and consider $P_\nu(x)$ and $Q_\nu(x)$ as merely the zero-order instances. Associated Legendre functions having a degree that is an odd multiple of ½ are sometimes known as *toroidal functions*.

Linear combinations of $P_\nu^{(\mu)}(x)$ and $Q_\nu^{(\mu)}(x)$ satisfy the *associated Legendre differential equation*

59:12:2
$$(1-x^2)\frac{d^2 f}{dx^2}-2x\frac{df}{dx}+\left[\nu(1+\nu)-\frac{\mu^2}{1-x^2}\right]f=0 \qquad f=w_1 P_\nu^{(\mu)}(x)+w_2 Q_\nu^{(\mu)}(x)$$

This commonly arises as the trigonometric equivalent

59:12:3
$$\frac{d^2 f}{d\theta^2}+\cot(\theta)\frac{df}{d\theta}+\left[\nu(1+\nu)-\mu^2\csc^2(\theta)\right]f=0 \qquad f=w_1 P_\nu^{(\mu)}\left(\cos(\theta)\right)+w_2 Q_\nu^{(\mu)}\left(\cos(\theta)\right)$$

The latter is valid only for $|\theta| < \pi$, which is the primary domain of interest for associated Legendre functions, corresponding to $-1<x<1$. For reasons made evident in Section 59:14, applications almost invariably require that the order be a nonnegative integer and henceforth we mostly impose this restriction and replace μ by m.

The differential equation 59:12:2 might suggest that associated Legendre functions of orders m and $-m$ would be defined identically, but this is not so. Instead one has the order-reflection formulas

59:12:4
$$f_\nu^{(-m)}(x)=\frac{f_\nu^{(m)}(x)}{(-\nu)_m(1+\nu)_m} \qquad f = P \text{ or } Q \qquad m=0,1,2,\cdots$$

For example $f_\nu^{(-1)}(x)=-f_\nu^{(1)}(x)/(\nu^2+\nu)$. The degree-reflection formulas are

59:12:5
$$P_{-\nu}^{(m)}(x)=P_{\nu-1}^{(m)}(x) \qquad \text{and} \qquad Q_{-\nu}^{(m)}(x)=Q_{\nu-1}^{(m)}(x)-\pi\cot(\nu\pi)P_{\nu-1}^{(m)}(x) \qquad m=0,1,2,\cdots$$

Integral representations of the associated Legendre functions that are extensions of those listed in Section 59:3 exist, but are not reported here. The multiple differentiation formulas

59:12:6
$$P_\nu^{(m)}(x)=(-)^m\left(1-x^2\right)^{m/2}\frac{d^m}{dx^m}P_\nu(x)$$

and

59:12:7
$$Q_\nu^{(m)}(x)=(-)^m\left(1-x^2\right)^{m/2}\frac{d^m}{dx^m}Q_\nu(x) \qquad \nu\neq-1,-2,-3,\cdots$$

provide the simplest definitions for integer orders and lead to most of the formulas reported here. Be aware that the $(-)^m$ multiplier in these formulas is omitted by some authorities, so that odd-ordered associated Legendre functions may be encountered with signs that differ from those here. The full story around associated Legendre functions is quite complicated, especially when the argument is complex; see Gradshteyn and Ryzhik [Section 8.70] for enlightenment.

The associated Legendre function $P_\nu^{(\mu)}(x)$ may be expressed as a weighted sum of any two of its *contiguous functions*, that is, of any two of $P_{\nu-1}^{(\mu)}(x)$, $P_{\nu+1}^{(\mu)}(x)$, $P_\nu^{(\mu-1)}(x)$, and $P_\nu^{(\mu+1)}(x)$. The appropriate weights are contained in the table overleaf, which applies equally to associated Legendre functions of either kind. To illustrate the use of this table, note that the final row implies

59:12:8
$$f_\nu^{(\mu)}(x)=\frac{-\sqrt{1-x^2}}{2\mu x}\left[f_\nu^{(\mu+1)}+(\nu-\mu+1)(\nu+\mu)f_\nu^{(\mu-1)}\right] \qquad f=P \text{ or } Q$$

a result that may be reorganized into an order-recursion relationship. Likewise the top row of the table provides a degree-recursion formula, generalizing equation 59:5:5.

There are many formulas that connect associated Legendre functions with Gauss hypergeometric functions and thus facilitate expansion. In particular,

59:12:9
$$P_\nu^{(m)}(x)=\frac{(\nu-m+1)_{2m}}{(-2)^m m!}\left(1-x^2\right)^{m/2}F\left(m-\nu,\nu+m+1,m+1,\frac{1-x}{2}\right)$$

$f_{\nu-1}^{(\mu)}(x)$ f = P or Q	$f_{\nu+1}^{(\mu)}(x)$ f = P or Q	$f_{\nu}^{(\mu-1)}(x)$ f = P or Q	$f_{\nu}^{(\mu+1)}(x)$ f = P or Q
$\dfrac{\nu+\mu}{(2\nu+1)x}$	$\dfrac{\nu-\mu+1}{(2\nu+1)x}$		
$\dfrac{1}{x}$		$\dfrac{-(\nu-\mu+1)\sqrt{1-x^2}}{x}$	
$\dfrac{\nu+\mu}{(\nu-\mu)x}$			$\dfrac{\sqrt{1-x^2}}{(\nu-\mu)x}$
	$\dfrac{1}{x}$	$\dfrac{-(\nu+\mu)\sqrt{1-x^2}}{x}$	
	$\dfrac{\nu-\mu+1}{(\nu+\mu+1)x}$		$\dfrac{-\sqrt{1-x^2}}{(\nu+\mu+1)x}$
		$\dfrac{-(\nu-\mu+1)(\nu+\mu)\sqrt{1-x^2}}{2\mu x}$	$\dfrac{-\sqrt{1-x^2}}{2\mu x}$

and

59:12:10 $$Q_\nu^{(m)}(x)=\frac{(-)^m\sqrt{\pi}\,\Gamma(1+\nu+m)}{2^{\nu+1}\Gamma\left(\frac{3}{2}+\nu\right)}\frac{\left(x^2-1\right)^{m/2}}{x^{1+\nu+m}}F\left(\tfrac{1}{2}+\tfrac{1}{2}\nu+\tfrac{1}{2}m,1+\tfrac{1}{2}\nu+\tfrac{1}{2}m,\tfrac{3}{2}-\tfrac{1}{2}m,\frac{1}{x^2}\right)$$

are the analogues of equations 59:6:1 and 59:6:2, to which they reduce when $m=0$.

Some examples of associated Legendre functions in which both the degree and the order are positive integers are

$P_1^{(1)}(x)=P_{-2}^{(1)}(x)=-\sqrt{1-x^2}$	$Q_1^{(1)}(x)=\dfrac{-x}{\sqrt{1-x^2}}-\sqrt{1-x^2}\,\mathrm{artanh}(x)$
$P_2^{(1)}(x)=P_{-3}^{(1)}(x)=-3x\sqrt{1-x^2}$	$Q_2^{(1)}(x)=\dfrac{2-3x^2}{\sqrt{1-x^2}}-3x\sqrt{1-x^2}\,\mathrm{artanh}(x)$
$P_2^{(2)}(x)=P_{-3}^{(2)}(x)=3\left(1-x^2\right)$	$Q_2^{(2)}(x)=\dfrac{5x-3x^3}{1-x^2}+3\left(1-x^2\right)\mathrm{artanh}(x)$

Note that these are rarely polynomials, though they sometimes go by the misnomer "associated Legendre polynomials". $P_n^{(m)}(x)=0$ whenever $|m|$ exceeds the larger of n and $-n-1$, a result that has consequence in Section 59:14.

The associated Legendre function of the first kind satisfies the orthogonality relationship [Section 21:14]

59:12:11 $$\int_{-1}^{1}P_n^{(m)}(t)\,P_N^{(m)}(t)\,dt=\begin{cases}0 & N\neq n\\ \dfrac{2}{2n+1}\dfrac{(n+m)!}{(n-m)!} & N=n\end{cases}$$

Operative for arguments between −1 and 1, for $|\nu| \le 150$, and for integer m not exceeding 150 in magnitude, *Equator* provides an associated Legendre function of the first kind routine and an associated Legendre function of the second kind routine (keywords **assocP** and **assocQ**). For ν not less than −½ and positive m, these algorithms generally utilize the following double series

59:12:12 $$\mathrm{P}_\nu^{(m)}(x)=\frac{2^m}{(1-x^2)^{m/2}\sqrt{\pi}}\left[\cos\left(\frac{\nu+m}{2}\pi\right)\frac{\Gamma(\frac{1+\nu+m}{2})}{\Gamma\left(\frac{2+\nu-m}{2}\right)}\mathrm{F}\left(\frac{-\nu-m}{2},\frac{1+\nu-m}{2},\frac{1}{2},x^2\right)\right.$$
$$\left.+2x\sin\left(\frac{\nu+m}{2}\pi\right)\frac{\Gamma(\frac{2+\nu+m}{2})}{\Gamma\left(\frac{1+\nu-m}{2}\right)}\mathrm{F}\left(\frac{1-\nu-m}{2},\frac{2+\nu-m}{2},\frac{3}{2},x^2\right)\right]$$

and

59:12:13 $$\mathrm{Q}_\nu^{(m)}(x)=\frac{2^{m-1}\sqrt{\pi}}{(1-x^2)^{m/2}}\left[-\sin\left(\frac{\nu+m}{2}\pi\right)\frac{\Gamma(\frac{1+\nu+m}{2})}{\Gamma\left(\frac{2+\nu-m}{2}\right)}\mathrm{F}\left(\frac{-\nu-m}{2},\frac{1+\nu-m}{2},\frac{1}{2},x^2\right)\right.$$
$$\left.+2x\cos\left(\frac{\nu+m}{2}\pi\right)\frac{\Gamma(\frac{2+\nu+m}{2})}{\Gamma\left(\frac{1+\nu-m}{2}\right)}\mathrm{F}\left(\frac{1-\nu-m}{2},\frac{2+\nu-m}{2},\frac{3}{2},x^2\right)\right]$$

but, whenever equation 59:12:9 proves to provide answers with more precision, it is substituted. Formula 59:12:4 or 59:12:5 is employed, where necessary, if the order or degree is negative.

59:13 COGNATE FUNCTIONS: certain Gauss hypergeometric functions

When the variable x in Legendre's differential equation 59:3:18 is replaced by $X = (1-x)/2$, the result may be written as

59:13:1 $$X(1-X)\frac{\mathrm{d}^2\mathrm{f}}{\mathrm{d}X^2}+[1-\{(-\nu)+(1+\nu)+1\}X]\frac{\mathrm{d}\mathrm{f}}{\mathrm{d}X}-(-\nu)(1+\nu)\mathrm{f}=0$$

Compare this with Gauss's differential equation 60:3:7: the two are identical if $a = -\nu$, $b = 1+\nu$, and $c = 1$. It follows that a solution to Legendre's differential equation is

59:13:2 $$\mathrm{f}=\mathrm{F}\left(-\nu,1+\nu,1,\frac{1-x}{2}\right)$$

This is the result that is identified with $\mathrm{P}_\nu(x)$ in 59:6:1. Evidently the first kind of Legendre function is a particular Gauss hypergeometric function. This latter function is addressed in some detail in the next chapter; here the focus is on how the Legendre functions are expressible as Gauss functions.

Now set $\mathrm{g} = x^{1+\nu}\mathrm{f}$, where f is the dependent variable in Legendre's differential equation 59:3:8. Thereby that equation becomes

59:13:3 $$\left(x^4-x^2\right)\frac{\mathrm{d}^2\mathrm{g}}{\mathrm{d}x^2}+2x\left(1+\nu-\nu x^2\right)\frac{\mathrm{d}\mathrm{g}}{\mathrm{d}x}+(1+\nu)(2+\nu)\mathrm{g}=0$$

and if the independent variable is now replaced by $\chi = x^{-2}$, the differential equation adopts a form that can be rewritten

59:13:4 $$\chi(1-\chi)\frac{\mathrm{d}^2\mathrm{g}}{\mathrm{d}\chi^2}+\left[\left(\tfrac{3}{2}+\nu\right)-\left\{\left(\tfrac{1}{2}+\tfrac{1}{2}\nu\right)+\left(1+\tfrac{1}{2}\nu\right)+1\right\}\chi\right]\frac{\mathrm{d}\mathrm{g}}{\mathrm{d}\chi}-\left(\tfrac{1}{2}+\tfrac{1}{2}\nu\right)\left(1+\tfrac{1}{2}\nu\right)\mathrm{g}=0$$

This equation again identifies with the Gauss hypergeometric differential equation 60:3:7, this time if the choices

$a = (1+\nu)/2$, $b = (2+\nu)/2$, and $c = (3+2\nu)/2$ are made. We therefore have

59:13:5 $$\mathrm{f} = x^{-1-\nu}\mathrm{g} = x^{-1-\nu}\,\mathrm{F}\left(\tfrac{1}{2}+\tfrac{1}{2}\nu, 1+\tfrac{1}{2}\nu, \tfrac{3}{2}+\nu, x^{-2}\right)$$

as a second solution to the Legendre differential equation. This is the expression ascribed to $Q_\nu(x)$ in equation 59:6:2, apart from a constant factor of $\sqrt{\pi}\,\Gamma(1+\nu)/[2^{1+\nu}\Gamma(\tfrac{3}{2}+\nu)]$.

In summary, particular Gauss hypergeometric functions solve Legendre's differential equation:

59:13:6 $$\left(1-x^2\right)\frac{\mathrm{d}^2\mathrm{f}}{\mathrm{d}x^2} - 2x\frac{\mathrm{d}\mathrm{f}}{\mathrm{d}x} + \nu(\nu+1)\mathrm{f} = 0 \qquad \mathrm{f} = w_1\mathrm{F}\left(-\nu, 1+\nu, 1, \tfrac{1}{2}-\tfrac{1}{2}x\right) + \frac{w_2}{x^{1+\nu}}\mathrm{F}\left(1+\tfrac{1}{2}\nu, \tfrac{1}{2}+\tfrac{1}{2}\nu, \tfrac{3}{2}+\nu, x^{-2}\right)$$

Legendre functions are nothing but these Gauss functions "in disguise", with suitable weighting factors, as prescribed in equations 59:6:1 and 59:6:2. Gauss hypergeometric functions are unusually flexible, with the great number of intrarelationships described in Section 60:5. This flexibility passes on to Legendre functions, as the tabulation below amply demonstrates. This lists examples of hypergeometric representations of each kind of Legendre function.

$P_\nu(x)$	$Q_\nu(x)$
$\mathrm{F}\left(-\nu, 1+\nu, 1, \tfrac{1}{2}-\tfrac{1}{2}x\right)$	$\dfrac{\sqrt{\pi}\,\Gamma(1+\nu)}{\Gamma\left(\tfrac{3}{2}+\nu\right)[2x]^{1+\nu}}\mathrm{F}\left(\tfrac{1}{2}+\tfrac{1}{2}\nu, 1+\tfrac{1}{2}\nu, \tfrac{3}{2}+\nu, x^{-2}\right)$
$\left[\dfrac{x+1}{2}\right]^{\nu}\mathrm{F}\left(-\nu, -\nu, 1, \dfrac{x-1}{x+1}\right)$	$\dfrac{2^{-1-\nu}\sqrt{\pi}\,\Gamma(1+\nu)}{\Gamma\left(\tfrac{3}{2}+\nu\right)[x+1]^{1+\nu}}\mathrm{F}\left(1+\nu, 1+\nu, 2+2\nu, \dfrac{2}{1+x}\right)$
$\mathrm{F}\left(\tfrac{-1}{2}\nu, \tfrac{1}{2}+\tfrac{1}{2}\nu, 1, 1-x^2\right)$	$\dfrac{-\Gamma(1+\nu)}{\Gamma\left(\tfrac{3}{2}+\nu\right)}\sqrt{\dfrac{\pi\left(x-\sqrt{x^2-1}\right)^{1+2\nu}}{2\sqrt{x^2-1}}}\,\mathrm{F}\left(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{3}{2}+\nu, \dfrac{\sqrt{x^2-1}-x}{2\sqrt{x^2-1}}\right)$
$\left[x+\sqrt{x^2-1}\right]^{\nu}\mathrm{F}\left(-\nu, \tfrac{1}{2}, 1, \dfrac{2\sqrt{x^2-1}}{x+\sqrt{x^2-1}}\right)$	$\dfrac{\sqrt{\pi}\,\Gamma(1+\nu)}{\Gamma\left(\tfrac{3}{2}+\nu\right)\left[x+\sqrt{x^2-1}\right]^{1+\nu}}\mathrm{F}\left(\tfrac{1}{2}, 1+\nu, \tfrac{3}{2}+\nu, \dfrac{x-\sqrt{x^2-1}}{x+\sqrt{x^2-1}}\right)$
$x^{\nu}\,\mathrm{F}\left(\tfrac{-1}{2}\nu, \tfrac{1}{2}-\tfrac{1}{2}\nu, 1, \dfrac{x^2-1}{x^2}\right)$	$\dfrac{2^{-1-\nu}\sqrt{\pi}\,\Gamma(1+\nu)}{\Gamma\left(\tfrac{3}{2}+\nu\right)\left[x^2-1\right]^{(1+\nu)/2}}\mathrm{F}\left(1+\nu, 1+\nu, \tfrac{3}{2}+\nu, \dfrac{\sqrt{x^2-1}-x}{2\sqrt{x^2-1}}\right)$

In consequence of the reflection formula 59:5:5, additional entries may be added to the first (only!) column by replacing each ν by $-\nu-1$. The domains of the tabulated Gauss hypergeometric functions are restricted, because the argument of an $\mathrm{F}(a,b,c,x)$ function is limited to $-\infty < x < 1$ (sometimes $x = 1$ is admissible). Note, however, that though the Gauss hypergeometric function is defined for $-\infty < x < 1$, the corresponding series, equation 60:3:1, requires $-1 < x < 1$ for convergence.

The table above shows ways in which a Legendre function may be expressed as a *single* Gauss hypergeometric function. There are very many other representations as sums of two or more such functions; Section 60:5 provides routes for creating these.

59:14 RELATED TOPIC: solving Laplace's equation in spherical coordinates

Functions that satisfy Laplace's equation [Section 46:15] are known as *harmonic functions*. Legendre functions satisfy this equation when spherical coordinates [Section 46:14] are employed, explaining why "spherical harmonic" is often considered synonymous with "Legendre function".

In Section 46:14, the Laplacian operator was presented in several coordinate systems. Of course, the spherical coordinates are the preferred means of indexing three-dimensional space for studies involving the interior or exterior of spheres, or portions thereof. Laplace's equation in these coordinates is

59:14:1 $$\frac{\partial^2 F}{\partial r^2}+\frac{2}{r}\frac{\partial F}{\partial r}+\frac{1}{r^2}\frac{\partial^2 F}{\partial \phi^2}+\frac{\cot(\phi)}{r^2}\frac{\partial F}{\partial \phi}+\frac{\csc^2(\phi)}{r^2}\frac{\partial^2 F}{\partial \theta^2}=0$$

where F is the pertinent scalar quantity (temperature, potential, or the like). If coordinate separability is asserted, so that $F(r,\phi,\theta)=R(r)\Phi(\phi)\Theta(\theta)$, then we may trisect equation 59:14:1 in a manner strictly analogous to that used in Section 46:15, arriving at three separated equations. The first of these, addressing the dependence of F on the radial coordinate, is

59:14:2 $$\frac{d^2 R}{dr^2}+\frac{2}{r}\frac{dR}{dr}-\frac{v(v+1)}{r^2}R=0$$

where $v(v+1)$ is a conveniently formulated separation constant. The solution of this equation is mentioned in Section 32:13 but will not concern us here. The second separated equation, expressing how F is affected by the longitude is

59:14:3 $$\frac{d^2\Theta}{d\theta^2}=-\mu^2\Theta$$

with μ^2 being the new separation constant. The third separated equation, which incorporates both separation constants, conveys the dependence of F on the latitudinal angle ϕ and is

59:14:4 $$\frac{d^2\Phi}{d\phi^2}+\cot(\phi)\frac{d\Phi}{d\phi}+\left[v(v+1)-\mu^2\csc^2(\phi)\right]\Phi=0$$

Each of the three separated equations is a typical second-order ordinary differential equation, solutions of which are available through the procedures described in Section 24:14. Our prime interest here is in the solution of 59:14:4. Because this equation is identical, apart from notation, with 59:12:3, its solution is a weighted sum of associated Legendre functions:

59:14:5 $$\Phi(\phi)=w_1 P_v^{(\mu)}\left(\cos(\phi)\right)+w_2 Q_v^{(\mu)}\left(\cos(\phi)\right)$$

When the region of interest is the *surface* of a sphere, equation 59:14:2 is not pertinent and the overall solution is a composite of solutions of the latitudinal and longitudinal separated equations, 59:14:4 and 59:14:3, with arbitrary weighting factors

59:14:6 $$F(\phi,\theta)=\Phi(\phi)\Theta(\theta)=\left[w_5 P_v^{(\mu)}\left(\cos(\phi)\right)+w_6 Q_v^{(\mu)}\left(\cos(\phi)\right)\right]\left[w_3\sin(\mu\theta)+w_4\cos(\mu\theta)\right]$$

One can argue from the geometry of a sphere that μ must be an integer (as otherwise $\sin\{\mu(\theta+2\pi)\}$ would not equal $\sin\{\mu\theta\}$, as it must). In most physical situations, w_6 must be zero (as otherwise, F would be infinite at $\phi=0$, the "north pole"). Because $\theta=\pi$ corresponds to another point on the sphere's surface (the "south pole", in fact) and $P_v^{(m)}(-1)$ is infinite for all noninteger values of v, we are forced to conclude that v must also be an integer (which we can treat as nonnegative because P_{-n} merely duplicates P_{n-1}). As noted in the preceding section, the associated Legendre functions $P_n^{(m)}$ vanish for $|m|>n$. These various considerations permit us to simplify 59:14:6 to

59:14:7 $$F(\phi,\theta)=P_n^{(m)}\left(\cos(\phi)\right)\left[w_1\sin(m\theta)+w_2\cos(m\theta)\right] \qquad n=0,1,2,\cdots \qquad m=-n,-n+1,\cdots,n$$

Depending on the application, the n and m numbers may be referred to as *eigenvalues* or *quantum numbers*. The remaining weighting factors, w_1 and w_2, are independent of ϕ and θ but will generally depend on n and m, so that $w_1(n,m)$ and $w_2(n,m)$ is a more informative symbolism. A large, possibly an infinite, number of weights might be needed in a complete solution that matches the boundary conditions of a physical problem, but we shall not pursue any particular application.

To summarize, the equation

59:14:8 $$\mathrm{F}(\phi,\theta)=\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\mathrm{P}_n^{(m)}\big(\cos(\phi)\big)\big[w_1(n,m)\sin(m\theta)+w_2(n,m)\cos(m\theta)\big]$$

provides a general solution to Laplace's equation on the surface of a sphere. The components of this solution, that is the terms $\sin(m\theta)\,\mathrm{P}_n^{(m)}\big(\cos(\phi)\big)$ and $\cos(m\theta)\,\mathrm{P}_n^{(m)}\big(\cos(\phi)\big)$, possibly with normalizing multipliers, are known as *surface harmonics* or sometimes "spherical harmonics".