CHAPTER 1

THE CONSTANT FUNCTION *c*

Lacking dependence on even a single variable, the constant function is the simplest, and an almost trivial, function.

1:1 NOTATION

Constants are also known as *invariants* and are represented by a variety of symbols, mostly letters drawn from early members of the Latin and Greek alphabets. In this chapter, we mostly employ c to represent an arbitrary constant.

1:2 BEHAVIOR

Figure 1-1 is a graphical representation of the constant function $\mathrm{f}(x) = c$, a horizontal line extending to $x = \pm\infty$, reflecting the fact that f takes the same value for all x.

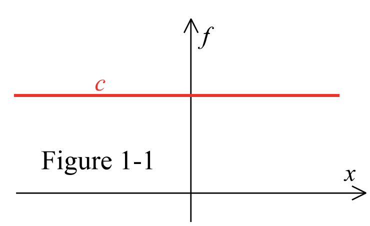


Figure 1-1

1:3 DEFINITIONS

The constant function is defined for all values of its argument x and has the same value, c, irrespective of x.

1:4 SPECIAL CASES

When c is zero, the constant function is sometimes termed the *zero function*. Likewise, the function $\mathrm{f}(x) = c = 1$ is sometimes known as the unit function or *unity function*.

K.B. Oldham et al., *An Atlas of Functions, Second Edition*,
DOI 10.1007/978-0-387-48807-3_2, © Springer Science+Business Media, LLC 2009

1:5 INTRARELATIONSHIPS

Being relations between function values at different values of the argument, intrarelationships are of no consequence for the constant function.

1:6 EXPANSIONS

A constant may be represented as a finite sum by utilizing the formulas for an *arithmetic series*:

1:6:1
$$c = \alpha + (\alpha+\delta) + (\alpha+2\delta) + \cdots + (\alpha+J\delta) = \sum_{j=0}^{J} (\alpha + j\delta)$$
$$\alpha = \frac{c}{J+1} - \frac{J\delta}{2} \qquad \text{or} \qquad c = (J+1)\left(\alpha + \frac{J\delta}{2}\right)$$

a *geometric series*:

1:6:2
$$c = \alpha + \alpha\beta + \alpha\beta^2 + \cdots + \alpha\beta^J = \sum_{j=0}^{J} \alpha\beta^j$$
$$\alpha = c\frac{\beta-1}{\beta^{J+1}-1} \qquad \text{or} \qquad c = \alpha\frac{\beta^{J+1}-1}{\beta-1}$$

or an *arithmetic-geometric series*:

1:6:3
$$c = \alpha + \beta(\alpha+\delta) + \beta^2(\alpha+2\delta) + \cdots + \beta^J(\alpha+J\delta) = \sum_{j=0}^{J} \beta^j(\alpha + j\delta)$$
$$\alpha = \frac{c(\beta-1) - J\delta\beta^{J+1} + \delta\beta(\beta^J-1)/(\beta-1)}{\beta^{J-1}-1} \qquad \text{or} \qquad c = \left(\beta^{J+1}-1\right)\left[\beta(\alpha+\delta)-\alpha\right] + J(\beta-1)\beta^{J+1}\delta$$

In these formulas β and δ are arbitrary and J may be any positive integer.

Any constant greater than ½ may be expanded as the infinite geometric sum

1:6:4
$$c = 1 + \left(\frac{c-1}{c}\right) + \left(\frac{c-1}{c}\right)^2 + \left(\frac{c-1}{c}\right)^3 + \cdots = \sum_{j=1}^{\infty}\left(\frac{c-1}{c}\right)^j \qquad c > \tfrac{1}{2}$$

or as the infinite product

1:6:5
$$c = \left[1 + \left(\frac{c-1}{c}\right)\right]\left[1 + \left(\frac{c-1}{c}\right)^2\right]\left[1 + \left(\frac{c-1}{c}\right)^4\right]\cdots = \prod_{j=0}^{\infty}\left[1 + \left(\frac{c-1}{c}\right)^{2^j}\right] \qquad c > \tfrac{1}{2}$$

A constant is expansible as the infinite continued fraction

1:6:6
$$c = \frac{\alpha}{\beta+}\frac{\alpha}{\beta+}\frac{\alpha}{\beta+}\cdots$$

in the variety of ways indicated in the table, which lists three alternative assignments of the terms α and β, any one of which validates expansion 1:6:6.

α	β	constraint
c	$1-c$	$-1 \le c < 1$
1	$\frac{1-c^2}{c}$	$0 < c^2 < 1$
c^2+c	1	$c \ge -\tfrac{1}{2}$

1:7 PARTICULAR VALUES

Certain constants occur frequently in the theory of functions. Four of these – *Archimedes's constant*, *Catalan's constant*, the *base of natural logarithms* and *Euler's constant* – are important irrational numbers. There are many formulations of these four constants other than the ones we present here; see Gradshteyn and Ryzhik [Chapter 0] for some of these.

Archimedes (Archimedes of Syracuse, Greek philosopher, 287 – 212 BC) himself was content merely to bracket his constant by (223/71) < π < (22/7). It was the sixteenth-century Frenchman François Viète ("Vieta") who discovered the first formula

1:7:1 $$\pi = \frac{2}{\sqrt{\tfrac{1}{2}} \times \sqrt{\tfrac{1}{2}+\tfrac{1}{2}\sqrt{\tfrac{1}{2}}} \times \sqrt{\tfrac{1}{2}+\tfrac{1}{2}\sqrt{\tfrac{1}{2}+\tfrac{1}{2}\sqrt{\tfrac{1}{2}}}} \times \cdots} = 3.1415\ 92653\ 58979$$

for *Archimedes's constant*, also known simply as *pi*. It may also be defined by the infinite sum

1:7:2 $$\pi = 4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \cdots = 4\sum_{j=0}^{\infty} \frac{(-1)^j}{2j+1} = 3.1415\ 92653\ 58979$$

discovered by Gregory (James Gregory, Scottish mathematician, 1638 – 1675), as the infinite product

1:7:3 $$\pi = 2 \times \frac{4}{3} \times \frac{16}{15} \times \frac{36}{35} \times \frac{64}{63} \times \cdots = 2\prod_{j=1}^{\infty} \frac{j^2}{j^2 - \frac{1}{4}} = 3.1415\ 92653\ 58979$$

and in numerous other ways. The definition of *Catalan's constant* (Eugène Charles Catalan, Belgian mathematician 1814 – 1894) is similar to 1:7:2

1:7:4 $$G = 1 - \frac{1}{9} + \frac{1}{25} - \frac{1}{49} + \cdots = \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)^2} = 0.91596\ 55941\ 77219$$

The *base of natural logarithms* may be defined as a sum of all reciprocal factorial functions [Chapter 2]

1:7:5 $$e = 1 + \frac{1}{1} + \frac{1}{1\times 2} + \frac{1}{1\times 2\times 3} + \cdots = \sum_{j=0}^{\infty} \frac{1}{j!} = 2.7182\ 81828\ 45905$$

or by the limit operation

1:7:6 $$e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n = 2.7182\ 81828\ 45905$$

A limit operation also defines *Euler's constant*

1:7:7 $$\gamma = \lim_{n\to\infty}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln(n)\right) = \lim_{n\to\infty}\left(\sum_{j=1}^{n} \frac{1}{j} - \ln(n)\right) = 0.57721\ 56649\ 01533$$

The latter is also known as *Mascheroni's constant* (Lorenzo Mascheroni, Italian priest, 1750 – 1800) and is often denoted by C. Confusingly, authors who employ C to represent Euler's constant may use γ to represent e^C.

Also of widespread occurrence throughout the *Atlas* is the *Gauss's constant*

1:7:8 $$g = \frac{1}{\mathrm{mc}\left(1, \sqrt{2}\right)} = 0.83462\ 68416\ 74073$$

where mc denotes the common, or arithmeticogeometric, mean [Section 61:14]. It is related to the *ubiquitous constant U* through $Ug = 1/\sqrt{2}$. Other named constants are Apéry's constant Z [Section 3:7] and the golden section υ [Section 23:14].

A very important family of constants are the *integers*, $\cdots,-3,-2,-1,0,1,2,3,\cdots$ and especially the *natural numbers*, $1,2,3,\cdots$ discussed in Section 1:14. Other families that occur principally in coefficients of series expansions are the factorials $n!$ [Chapter 2], Bernoulli numbers B_n [Chapter 4], and Euler numbers E_n [Chapter 5]. Fibonacci numbers are discussed in Section 23:14.

1:8 NUMERICAL VALUES

Equator provides values of the constants π, G, e, γ, g, Z, and υ, exact to 15 digits. Simply type the corresponding keyword, which is **pi**, **catalan**, **ebase**, **euler**, **gauss**, **apery**, or **golden**. These keywords may be freely used in "constructing" the variable(s) of any other *Equator* function, as explained in Appendix C. As well as these seven mathematical constants, many physical constants are available through *Equator*: see Appendix A for these.

1:9 LIMITS AND APPROXIMATIONS

Approximations are seldom needed for constants, but approximations as fractions are available through *Equator*'s rational approximation routine (keyword **rational**) [Section 8:13].

1:10 OPERATIONS OF THE CALCULUS

Differentiation gives

1:10:1 $$\frac{\mathrm{d}}{\mathrm{d}x}c = 0$$

while indefinite and definite integration produce

1:10:2 $$\int_0^x c\,\mathrm{d}t = cx$$

and

1:10:3 $$\int_{x_0}^{x_1} c\,\mathrm{d}t = c(x_1 - x_0)$$

respectively. The result

1:10:4 $$\int_0^\infty c\,\exp(-st)\,\mathrm{d}t = \mathcal{L}\{c\} = \frac{c}{s}$$

describes the Laplace transformation of a constant.

The results of semidifferentiation and semiintegration [Section 12:14] with a lower limit of zero are

1:10:5 $$\frac{\mathrm{d}^{1/2}}{\mathrm{d}x^{1/2}}c = \frac{c}{\sqrt{\pi x}}$$

and

1:10:6 $$\frac{\mathrm{d}^{-1/2}}{\mathrm{d}x^{-1/2}}c = 2c\sqrt{\frac{x}{\pi}}$$

Differintegration [Section 12:14] with a lower limit of zero yields

1:10:7 $$\frac{d^{\nu}}{dx^{\nu}}c = \frac{cx^{-\nu}}{\Gamma(1-\nu)}$$

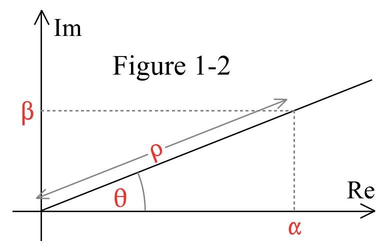
where Γ is the gamma function [Chapter 43]. In fact, equations 1:10:1, 1:10:2, 1:10:5, and 1:10:6 are the $\nu = 1, -1$, ½ and −½ instances of 1:10:7.

1:11 COMPLEX ARGUMENT

A *complex constant* can be expressed in terms of two real constants in either rectangular or polar notation

1:11:1 $$c = \begin{cases} \alpha + i\beta & \text{where} \quad \alpha = \rho\cos(\theta) \quad \text{and} \quad \beta = \rho\sin(\theta) \\ \rho\exp(i\theta) & \text{where} \quad \rho = \sqrt{\alpha^2+\beta^2} \quad \text{and} \quad \theta = \arctan(\beta/\alpha) + \pi[1-\text{sgn}(\alpha)]/2 \end{cases}$$

with $i = \sqrt{-1}$. The names *real part*, *imaginary part*, *modulus*, and *phase* are accorded to α, β, ρ, and θ. Figure 1-2 shows how α, β, ρ and θ are related. The expression $c = \alpha + i\beta$ is the more useful in formulating the rules for the addition or subtraction of two complex constants:



1:11:2 $$c_1 \pm c_2 = (\alpha_1 + i\beta_1) \pm (\alpha_2 + i\beta_2) = (\alpha_1 \pm \alpha_2) + i(\beta_1 \pm \beta_2)$$

whereas $c = \rho\exp(i\theta)$ is the more convenient to formulate the multiplication

1:11:3 $$c_1c_2 = [\rho_1\exp(i\theta_1)][\rho_2\exp(i\theta_2)] = \rho_1\rho_2\exp\{i(\theta_1+\theta_2)\}$$

or division

1:11:4 $$\frac{c_1}{c_2} = \frac{\rho_1\exp(i\theta_1)}{\rho_2\exp(i\theta_2)} = \frac{\rho_1}{\rho_2}\exp\{i(\theta_1-\theta_2)\}$$

of two complex numbers, or in the raising of a complex number to a real power

1:11:5 $$c^{\nu} = [\rho\exp(i\theta)]^{\nu} = \rho^{\nu}\exp(i\nu\theta)$$

If ν is not an integer, this exponentiation operation gives rise to a multivalued complex number [see, for example, Section 13:14]. The raising of a real number to a complex-valued power is handled by the expression

1:11:6 $$\nu^{\alpha+i\beta} = \nu^{\alpha}\exp\{i\beta\ln(\nu)\}$$

provided that ν is positive.

The inverse Laplace transform of the constant c is a Dirac function [Chapter 9], of magnitude c, located at the origin

1:11:7 $$\int_{\alpha-i\infty}^{\alpha+i\infty} c\frac{\exp(ts)}{2\pi i}ds = \mathfrak{G}\{c\} = c\delta(t)$$

1:12 GENERALIZATIONS

A constant is a member of the polynomial function family, other members of which are discussed in Chapters 19−25. The constant function is the special $b = 0$ case of the linear function discussed in Chapter 7.

1:13 COGNATE FUNCTIONS

Whereas the constant function has the same value for all x, the related *pulse function* is zero at values of the argument outside a "window" of width h, and is a nonzero constant, c, within this window. The concept of a general "window function" is discussed in Section 9:13. The pulse function in Figure 1-3 takes the value c in the range $a-(h/2) < x < a+(h/2)$ but equals zero elsewhere. The value of the a parameter establishes the location of the pulse, while c and h are termed the *pulse height* and *pulse width* respectively. The pulse function may be represented by

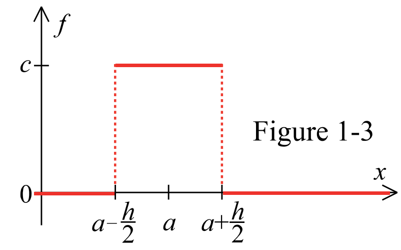


Figure 1-3

1:13:1 $$c\left[\mathrm{u}\left(x-a+\frac{h}{2}\right)-\mathrm{u}\left(x-a-\frac{h}{2}\right)\right]$$

in terms of the Heaviside function [Chapter 9].

The addition of a number of pulse functions, having various locations, heights, and widths, produces a function whose map consists of horizontal straight line segments. Such a function, known as a *piecewise-constant function*, may be used to approximate a more complicated or incompletely known function. It is the approximation recorded, for example, whenever a varying quantity is measured by a digital instrument.

1:14 RELATED TOPIC: the natural numbers

The *natural numbers*, $1,2,3,\cdots$ are ubiquitous in mathematics and science. We record here several results for finite sums of their powers:

1:14:1 $$1+2+3+\cdots+n=\sum_{j=1}^{n} j=\frac{n(n+1)}{2} \qquad n=1,2,3,\cdots$$

1:14:2 $$1^2+2^2+3^2+\cdots+n^2=\sum_{j=1}^{n} j^2=\frac{n(n+1)(2n+1)}{6} \qquad n=1,2,3,\cdots$$

1:14:3 $$1^3+2^3+3^3+\cdots+n^3=\sum_{j=1}^{n} j^3=\frac{n^2(n+1)^2}{4} \qquad n=1,2,3,\cdots$$

Similarly, the sums of fourth and fifth powers of the first n natural numbers are $n(n+1)(2n+1)(3n^2+3n-1)/30$ and $n^2(n+1)^2(2n^2+2n-1)/12$, respectively. The general case is

1:14:4 $$1^m+2^m+3^m+\cdots+n^m=\sum_{j=1}^{n} j^m=\frac{\mathrm{B}_{m+1}(n+1)-\mathrm{B}_{m+1}}{m+1} \qquad n,m=1,2,3,\cdots$$

where B_m denotes a Bernoulli number [Chapter 4] and $\mathrm{B}_m(x)$ denotes a Bernoulli polynomial [Chapter 19]. If m is not an integer, summation 1:14:4 may be evaluated generally by equation 12:5:5. The sum of the reciprocals of the first n natural numbers is

1:14:5 $$\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}=\sum_{j=1}^{n}\frac{1}{j}=\gamma+\psi(n+1) \qquad n=1,2,3,\cdots$$

where γ is Euler's constant [Section 1:7] and $\psi(x)$ denotes the digamma function [Chapter 44]. When continued indefinitely, the sum 1:14:5 defines the divergent *harmonic series*.

The corresponding expressions when the signs alternate are

1:14:6 $$1-2+3-4+\cdots\pm n=-\sum_{j=1}^{n}(-)^j j=\begin{cases}(n+1)/2 & n=1,3,5,\cdots\\ -n/2 & n=2,4,6,\cdots\end{cases}$$

1:14:7 $$1^2-2^2+3^2-4^2+\cdots\pm n^2=-\sum_{j=1}^{n}(-)^j j^2=\begin{cases}n(n+1)/2 & n=1,3,5,\cdots\\ -n(n+1)/2 & n=2,4,6,\cdots\end{cases}$$

1:14:8 $$1^3-2^3+3^3-4^3+\cdots\pm n^3=-\sum_{j=1}^{n}(-)^j j^3=\begin{cases}(2n^3+3n^2-1)/4 & n=1,3,5,\cdots\\ -n^2(2n+3)/4 & n=2,4,6,\cdots\end{cases}$$

1:14:9 $$1^m-2^m+3^m-4^m+\cdots\pm n^m=-\sum_{j=1}^{n}(-)^j j^m=-\frac{\mathrm{E}_m(0)}{2}-\frac{(-)^n\mathrm{E}_m(n+1)}{2} \qquad n,m=1,2,3,\cdots$$

where $\mathrm{E}_m(x)$ denotes an Euler polynomial [Chapter 20], and

1:14:10 $$\frac{1}{1}-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots\pm\frac{1}{n}=-\sum_{j=1}^{n}\frac{(-1)^j}{j}=\begin{cases}\psi(n+1)-\psi\left(\dfrac{n+1}{2}\right) & n=1,3,5,\cdots\\ \psi(n+1)-\psi\left(\dfrac{n}{2}+1\right) & n=2,4,6,\cdots\end{cases}$$

Note that, whereas the $n=\infty$ version of the harmonic series 1:14:5 does not converge, series 1:14:10 approaches the limit ln(2) as $n\to\infty$.

The numbers 2,4,6,⋯ are called the *even numbers*. Sums of their powers are easily found by using the identity

1:14:11 $$2^m+4^m+6^m+\cdots+n^m=2^m\left[1^m+2^m+3^m+\cdots+\left(\frac{n}{2}\right)^m\right] \qquad n=2,4,6,\cdots$$

in conjunction with equations 1:14:1−1:14:5. Likewise, use of these equations, together with the identity

1:14:12 $$1^m+3^m+5^m+\cdots+n^m=\left[1^m+2^m+3^m+\cdots+n^m\right]-2^m\left[1^m+2^m+3^m+\cdots+\left(\frac{n-1}{2}\right)^m\right] \qquad n=1,3,5,\cdots$$

permits sums of powers of the *odd numbers*, 1,3,5,⋯, to be evaluated.

For the *infinite* sums $\sum j^{-\nu}$ where *j* runs from 1 to ∞, see Chapter 3. The same chapter also addresses the related infinite sums $\sum(-)^j j^{-\nu}$, $\sum(2j-1)^{-\nu}$, and $\sum(-)^j(2j-1)^{-\nu}$. For other sums of numerical series, see Sections 44:14 and 64:6.