CHAPTER 18

THE POCHHAMMER POLYNOMIALS $(x)_n$

There is little interest in the Pochhammer polynomials in their own right; however, their simple recursion properties enable these functions to play a valuable role in the algebra of other functions, especially the hypergeometric functions discussed in Section 18:14.

18:1 NOTATION

These polynomial functions, in which x is the argument and n the degree, were studied in 1730 by Stirling and later by Appell, who used the symbol (x,n). The name "Pochhammer polynomial" recognizes Leo August Pochhammer (German mathematician, 1841−1920) who introduced the now conventional $(x)_n$ notation. Alternative names are *shifted factorial function*, *rising factorial*, and *upper factorial*. The alternative overbarred symbol $x^{\overline{n}}$ is occasionally encountered.

18:2 BEHAVIOR

The Pochhammer polynomial is defined for all real x and all nonnegative integer n values (though see Section 18:12 for a generalization to negative n). In common with other polynomials, it has an unrestricted range when n is odd, but a semiinfinite range for $n = 2, 4, 6, \cdots$.

Figure 18-1 shows graphs of early members of the Pochhammer polynomial family; note that $(x)_n$ has exactly $\mathrm{Int}\left(\frac{n-1}{2}\right)$ maxima, $\mathrm{Int}\left(\frac{n}{2}\right)$ minima and n zeros, the latter occurring at $x = 0, -1, -2, \cdots, (1-n)$.

18:3 DEFINITIONS

The Pochhammer polynomial is defined by the n-fold product

18:3:1 $$(x)_n = x(x+1)(x+2)\cdots(x+n-1) = \prod_{j=0}^{n-1}(x+j)$$

K.B. Oldham et al., *An Atlas of Functions, Second Edition*,
DOI 10.1007/978-0-387-48807-3_19, © Springer Science+Business Media, LLC 2009

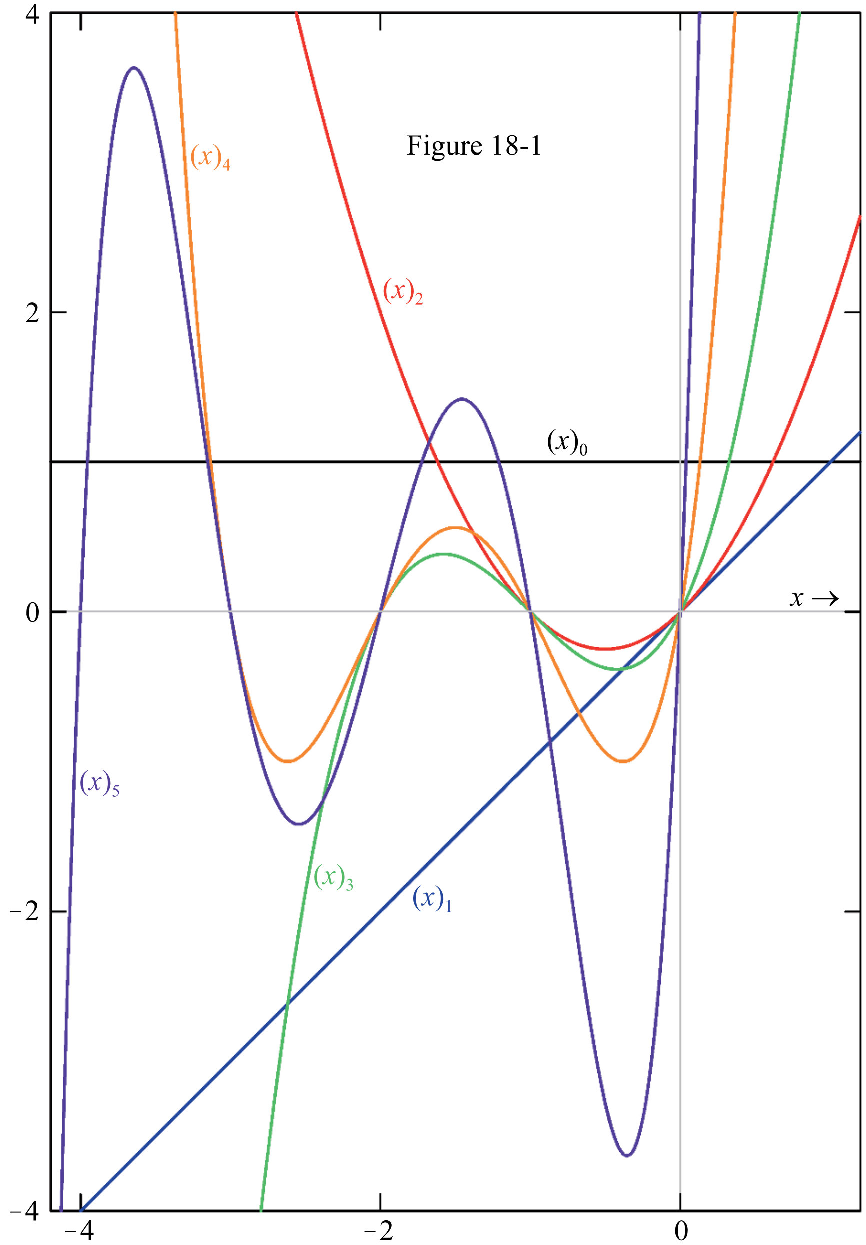


Figure 18-1

Empty products are generally interpreted as unity and this is the case for the Pochhammer polynomial of zero degree:

18:3:2 $$(x)_0 = 1$$

An equivalent definition

18:3:3 $$(x)_n = n!\binom{x+n-1}{n}$$

expresses $(x)_n$ in terms of a factorial function and a binomial coefficient [Chapters 2 and 6, respectively]. Equation 18:12:1 can also serve as a definition.

A generating function [Section 0:3] for the Pochhammer polynomial is

18:3:4 $$\frac{1}{(1-t)^{\nu}} = \sum_{n=0}^{\infty} (\nu)_n \frac{t^n}{n!}$$

and it is also generated by repeatedly differentiating a power of which $-\nu$ is the exponent:

18:3:5 $$x^{\nu} \frac{d^n}{dx^n} x^{-\nu} = (\nu)_n \left(\frac{-1}{x}\right)^n$$

18:4 SPECIAL CASES

$(x)_0$	$(x)_1$	$(x)_2$	$(x)_3$	$(x)_4$	$(x)_5$	$(x)_6$
1	x	x^2+x	x^3+3x^2+2x	$x^4+6x^3+11x^2+6x$	$x^5+10x^4+35x^3+50x^2+24x$	$x^6+15x^5+85x^4+225x^3+274x^2+120x$

18:5 INTRARELATIONSHIPS

Pochhammer polynomials obey the reflection formula

18:5:1 $$(-x)_n = (-)^n (x-n+1)_n$$

Equivalently

18:5:2 $$\left(\frac{n-1}{2} - x\right)_n = (-)^n \left(x - \frac{n-1}{2}\right)_n$$

which explains the even or odd symmetry about $x = (1-n)/2$ evident in Figure 18-1.

The argument-duplication formulas

18:5:3 $$(2x)_n = \begin{cases} 2^n (x)_{n/2} (x+\tfrac{1}{2})_{n/2} & n = 0,2,4,\cdots \\ 2^n (x)_{(n+1)/2} (x+\tfrac{1}{2})_{(n-1)/2} & n = 1,3,5,\cdots \end{cases}$$

have analogs in expressions for $(3x)_n$, $(4x)_n$, and generally for $(mx)_n$, where m is a positive integer. Equation 18:5:3 may be reformulated into a degree-duplication formula

18:5:4 $$(x)_{2n} = 4^n \left(\frac{x}{2}\right)_n \left(\frac{1+x}{2}\right)_n$$

and similarly

18:5:5 $$(x)_{2n+1} = 4^n x \left(\frac{1+x}{2}\right)_n \left(1+\frac{x}{2}\right)_n = 2^{2n+1} \left(\frac{x}{2}\right)_{n+1} \left(\frac{1+x}{2}\right)_n$$

Similar formulas for $(x)_{3n}$, $(x)_{3n+1}$, $(x)_{3n+2}$, $(x)_{4n}$, etc. may be derived readily.

Simple recursion formulas exist for both the argument

18:5:6 $$(x+1)_n = \left[1+\frac{n}{x}\right](x)_n$$

and the degree

18:5:7 $$(x)_{n+1} = [n+x](x)_n = x(x+1)_n$$

of Pochhammer polynomial functions. There are many useful formulas expressing the quotient of two Pochhammer polynomials:

18:5:8
$$\frac{(x)_n}{(x)_m} = \begin{cases} (x+m)_{n-m} & n \ge m \\ \dfrac{1}{(x+n)_{m-n}} & n \le m \end{cases}$$

18:5:9
$$\frac{(x+m)_n}{(x)_n} = \frac{(x+n)_m}{(x)_m} \qquad m = 0,1,2,\cdots$$

18:5:10
$$\frac{(x-m)_n}{(x)_n} = \frac{(x-m)_m}{(x-m+n)_m} = \frac{(1-x)_m}{(1-n-x)_m} \qquad m = 0,1,2,\cdots$$

Addition formulas exist for both the argument and the degree of a Pochhammer polynomial. The expression

18:5:11
$$(x+y)_n = \sum_{j=0}^{n} \binom{n}{j} (x)_j (y)_{n-j}$$

which closely resembles the binomial theorem [equation 6:14:1], is known as *Vandermonde's theorem* (Alexandre-Théophile Vandermonde, French violinist and mathematician, 1735 – 1795). The rule

18:5:12
$$(x)_{n+m} = (x)_n (x+n)_m$$

is a simple consequence of definition 18:3:1.

18:6 EXPANSIONS: Stirling numbers of the first kind

Of course, the Pochhammer polynomial is expansible as the product 18:3:1. As a sum, its expansion involves the absolute values of the numbers $\mathrm{S}_n^{(m)}$, known as the *Stirling numbers of the first kind.*

18:6:1 $$(x)_n = (-)^n \sum_{m=1}^{n} \mathrm{S}_n^{(m)} (-x)^m = \sum_{m=0}^{n} \left|\mathrm{S}_n^{(m)}\right| x^m$$

These numbers are negative whenever $n+m$ is odd and $0 < m < n$. Figure 18-2 shows the absolute values of early Stirling numbers of the first kind and more can be calculated via the recursion formula

18:6:2
$$\mathrm{S}_{n+1}^{(m)} = \mathrm{S}_n^{(m-1)} - n\mathrm{S}_n^{(m)} \qquad n = 0,1,2,\cdots \quad m = 1,2,3,\cdots$$

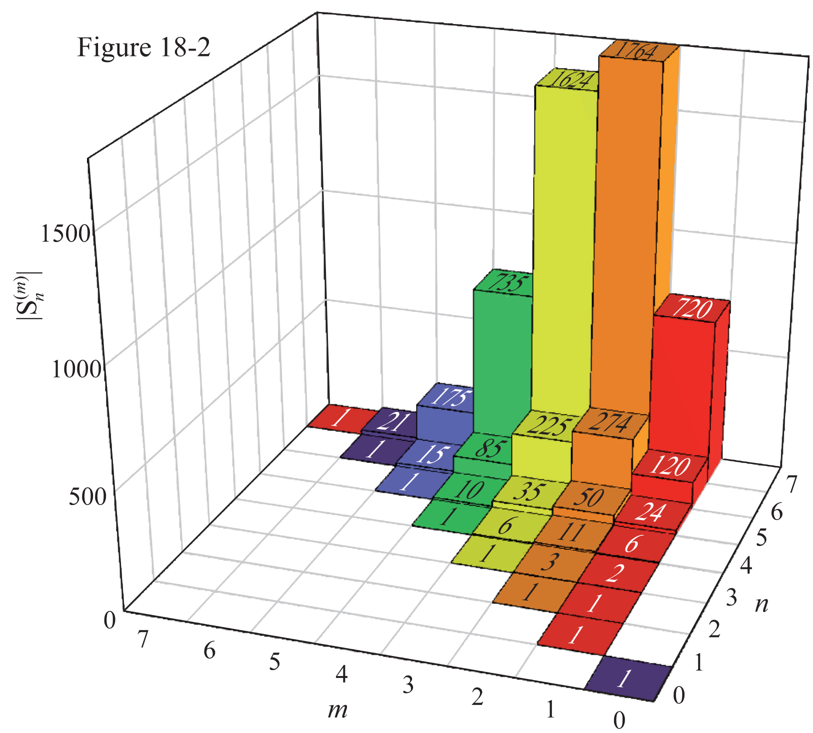
This formula is the basis of *Equator*'s Stirling number of the first kind (keyword **Snum**) routine. The numbers satisfy the following summations

18:6:3
$$\sum_{m=1}^{n} \mathrm{S}_n^{(m)} = 0 \qquad n = 2,3,4,\cdots$$

18:6:4
$$\sum_{m=0}^{n} \left|\mathrm{S}_n^{(m)}\right| = n! \qquad n = 0,1,2,\cdots$$

It is sometimes useful to expand a

Figure 18-2



reciprocal Pochhammer polynomial as partial fractions [Section 17:13]. The result is

18:6:5 $$\frac{1}{(x)_n}=\sum_{j=0}^{n-1}\frac{(-1)^j}{j!(n-j-1)!}\frac{1}{x+j}=(n-1)!\sum_{j=0}^{n-1}\binom{n-1}{j}\frac{(-1)^j}{x+j}$$

18:7 PARTICULAR VALUES

	$(-n)_n$	$\left(\frac{1-n}{2}\right)_n$	$(-m)_n$ $m=1,2,\cdots,n-1$	$\left(\frac{-1}{2}\right)_n$	$(0)_n$	$\left(\frac{1}{2}\right)_n$	$(1)_n$	$(2)_n$	$(n)_n$
$n=0$	1	1	1	1	1	1	1	1	1
$n=1,3,5,\cdots$	$-n!$	0	0	$\frac{-(2n)!}{4^n(2n-1)n!}$	0	$\frac{(2n)!}{4^n n!}$	$n!$	$(n+1)!$	$\frac{(2n)!}{2n!}$
$n=2,4,6,\cdots$	$n!$	$\frac{(-)^{n/2}(n!)^2}{4^n(\frac{n}{2}!)^2}$	0	$\frac{-(2n)!}{4^n(2n-1)n!}$	0	$\frac{(2n)!}{4^n n!}$	$n!$	$(n+1)!$	$\frac{(2n)!}{2n!}$

As the table shows, the Pochhammer polynomial of an integer can be expressed as a factorial function or as the quotient of two factorials

18:7:1 $$(1)_n=n! \qquad (2)_n=(n+1)! \qquad (3)_n=\frac{(n+2)!}{2} \qquad (m)_n=\frac{(n+m-1)!}{(m-1)!}$$

Similarly, the Pochhammer polynomial of half an odd integer is related to double-factorials [Section 2:13]

18:7:2 $$(1/2)_n=\frac{(2n-1)!!}{2^n} \qquad (3/2)_n=\frac{(2n+1)!!}{2^n} \qquad (m/2)_n=\frac{(2n+m-2)!!}{2^n(m-2)!!} \qquad m=1,3,5,\cdots$$

18:8 NUMERICAL VALUES

Equator can provide accurate values of $(x)_n$ by its Pochhammer polynomial routine (keyword **Poch**).

18:9 LIMITS AND APPROXIMATIONS

As $x\to+\infty$, $(x)_n$ approaches $+\infty$ smoothly and rapidly. As x becomes increasingly negative, $(x)_n$ passes through $(n-1)$ extrema before heading rapidly towards $+\infty$, if n is even, or $-\infty$ if n is odd. By use of equation 18:12:1, the limiting behavior of the Pochhammer polynomial can be deduced from those of the gamma function, as discussed in Section 43:9. Thus, when n is large, x remaining modest, the asymptotic expansion

18:9:1 $$(x)_n\sim\frac{n^{x-1}n!}{\Gamma(x)}\left[1+\frac{x(x-1)}{2n}+\frac{x(x-1)(x-2)(3x-1)}{24n^2}+\cdots\right] \qquad n\to\infty$$

holds and shows, for example, that

18:9:2 $$(\tfrac{1}{2})_n \to \frac{n!}{\sqrt{\pi n}} \qquad n\to\infty$$

On the other hand, the Stirling approximation [equation 43:6:6], coupled with 18:12:1, leads to

18:9:3 $$(\tfrac{1}{2})_n \to \sqrt{2}\left(\frac{n}{e}\right)^n \qquad n\to\infty$$

The coexistence of limits 18:9:2 and 18:9:3 provides an interesting link between what are probably the three most important irrational numbers: π, e, and $\sqrt{2}$.

For large n, and x close to $-n/2$, the Pochhammer polynomial approximates a sine function [Chapter 32].

18:9:4 $$(x)_n \approx 2\left(\frac{n}{2e}\right)^n \sin(\pi x) \qquad x+\frac{n}{2} << \sqrt{n} \quad \text{large positive } n$$

The development of this sinusoidal behavior is evident in Figure 18-1, even for n as small as 4.

18:10 OPERATIONS OF THE CALCULUS

Linear operators such as differentiation and indefinite integration may be applied term by term to all polynomials, including $(x)_n$. Differentiation and integration of the Pochhammer polynomial give

18:10:1 $$\frac{\mathrm{d}}{\mathrm{d}x}(x)_n = (x)_n\sum_{j=0}^{n-1}\frac{1}{x+j} = (x)_n\left[\psi(n+x)-\psi(x)\right]$$

18:10:2 $$\int_0^x (t)_n\,\mathrm{d}t = \sum_{j=1}^{n+1}\left|\mathrm{S}_n^{(j-1)}\right|\frac{x^j}{j}$$

The ψ function is the digamma function [Chapter 44] and $\mathrm{S}_n^{(j-1)}$ represents a Stirling number from Section 18:6.

18:11 COMPLEX ARGUMENT

If values of the Pochhammer polynomial with complex argument are needed, which they seldom are, they are available by combining equation 18:6:1 and 17:11:1.

18:12 GENERALIZATIONS

Pochhammer polynomials may be expressed as a ratio of two gamma functions [Chapter 43]

18:12:1 $$(x)_n = \frac{\Gamma(n+x)}{\Gamma(x)}$$

This representation opens the door to a generalization in which the degree n is not necessarily an integer.

A less profound generalization is to maintain n as an integer, but allow it to adopt negative values. This is possible by basing the definition of such Pochhammer polynomials on recursion 18:5:7 and leads to the conclusion that

18:12:2 $$(x)_{-1} = \frac{1}{x-1}$$

and generally

18:12:3 $$(x)_{-n} = \frac{1}{(x-1)(x-2)\cdots(x-n)} = \frac{(-)^n x}{[x-n](-x)_n}$$

18:13 COGNATE FUNCTIONS

Factorial functions [Chapter 2], binomial coefficients [Chapter 6], the gamma function [Chapter 43] and the (complete) beta function [Section 43:13] are all closely related to the Pochhammer polynomial.

A *factorial polynomial*, as defined by Tuma [Section 1.03] is

18:13:1 $$x_h^{(n)} = x(x-h)(x-2h)\cdots(x-nh+h)$$

but another function given the same name is

18:13:2 $$x^{[n]} = x(x-1)(x-2)\cdots(x-n+1)$$

This latter function also goes by the names *falling factorial* and *lower factorial* and may be symbolized $x^{\underline{n}}$ or, unfortunately, $(x)_n$. Yet another confusing symbolism, due to Kramp, is

18:13:3 $$x^{n/c} = x(x+c)(x+2c)\cdots(x+nc-c)$$

None of the notations in this paragraph is employed in the *Atlas*.

18:14 RELATED TOPIC: hypergeometric functions

Pochhammer polynomials occur in the coefficients of the special kind of power series known as a *hypergeometric function*. The most general representation of such a function is as the sum

18:14:1 $$\sum_{j=0}^{\infty} \frac{(a_1)_j (a_2)_j (a_3)_j \cdots (a_K)_j}{(c_1)_j (c_2)_j (c_3)_j \cdots (c_L)_j} x^j$$

where x is the argument, a_1, a_2, ⋯, a_K are prescribed *numeratorial parameters*, and c_1, c_2, ⋯, c_L are prescribed *denominatorial parameters.* Any real number is permissible as a parameter, except that nonpositive integers are problematic. If such an integer is one of the a parameter, series 18:14:1 will generally terminate, thus representing a polynomial. The only circumstance in which a nonpositive integer is legitimate as a denominatorial c parameter, is if another nonpositive integer of smaller magnitude (that is, a less negative integer) occurs in the numerator. In such cases the series terminates. Of course, the same Pochhammer term may not be in both the numerator and the denominator: they would cancel.

The argument x may have either sign but its permissible range is determined by the *numeratorial order* K and the *denominatorial order* L. These K and L orders are nonnegative integers, usually small ones. If $L > K$, the hypergeometric series necessarily converges for all finite values of x. If $L = K$, convergence is generally limited to the argument range $|x| < 1$. If $L < K$ the series diverges (unless it terminates) for all nonzero arguments, but it may nevertheless usefully represent a function asymptotically for small values of $|x|$ [37:6:5 provides an example].

The name “hypergeometric function” arises because 18:14:1 can be regarded as an extension of the geometric series (equation 1:6:4 or 6:14:9), to which it reduces when $L = K = 0$. Choosing suitable values of the a’s and c’s often gives rise to well-known functions when L and K are small. As well, a number of generic functions, such as the Kummer function [Chapter 47] the Gauss hypergeometric function [Chapter 60], and the Claisen functions [equation 18:14:5] are instances of hypergeometric functions in which the a’s and c’s are largely unrestricted. The so-called *generalized hypergeometric function*, or *extended hypergeometric function*, often denoted ${}_pF_q(a_1,\cdots,a_p;c_1,\cdots,c_q;x)$ is a hypergeometric function in which one of the denominatorial parameters is constrained to be unity:

18:14:2 $$ {}_pF_q\left(a_1,a_2,a_3,\cdots,a_p;c_1,c_2,c_3,\cdots,c_q;x\right)=\sum_{j=0}^{\infty}\frac{(a_1)_j(a_2)_j(a_3)_j\cdots(a_p)_j}{(c_1)_j(c_2)_j(c_3)_j\cdots(c_q)_j(1)_j}x^j $$

so that $p = K$ but q and L differ by unity. Other notations include

18:14:3 $$ {}_pF_q\left(\begin{matrix}a_1,a_2,a_3,\cdots,a_p\\c_1,c_2,c_3,\cdots,c_q\end{matrix}\Big|x\right) \qquad\text{and}\qquad \left[x\;\frac{a_1-1,a_2-1,a_3-1,\cdots,a_K-1}{c_1-1,c_2-1,c_3-1,\cdots,c_L-1}\right] $$

Some of these notations imply a phantom denominatorial $(1)_j$. In this *Atlas*, we adopt no special notation for hypergeometric functions, preferring to spell out the series explicitly as in 18:14:1. If a $(1)_j$ is present in the denominator, it is shown there.

As the tables in this section attest, a very large fraction of the functions discussed in the *Atlas* may be expressed hypergeometrically. Moreover, in the terminology of Section 43:14, almost all of these functions may be synthesized from a basis function, such as the ones listed in equations 43:14:1−4. Do not be misled into imagining that the only hypergeometric functions are those in the tables. In fact, subject to possible limitations on the argument x, almost any assignment of a's and c's leads to a valid hypergeometric function. It is just that most such assignments do not correspond to functions that have been glorified by special names and symbols.

Hypergeometric functions in which $L = K$ have the common feature of being amenable to synthesis, ultimately from one or other of the $1/(1\pm x)$ functions. Table 18-1 lists examples of $L = K = 1$ hypergeometric functions, while Table 18-2 similarly lists $L = K = 2$ hypergeometrics. There is a plethora of functions that are expressible as $L = K = 2$ hypergeometric functions; entries in Table 18-2 have been chosen as representative, rather than exhaustive. See Section 60:4 for details of the ways in which an associated Legendre function may be represented as a Gauss hypergeometric function; that is, formulated as an $L = K = 2$ hypergeometric. $L = K = 3$ cases, include the class of *Claisen functions*, important in hydrodynamics and described by

Table 18-1

a	c	$\sum_{j=0}^{\infty}\frac{(a)_j}{(c)_j}x^j$	$\sum_{j=0}^{\infty}\frac{(a)_j}{(c)_j}(-x)^j$
ν	1	$(1-x)^{-\nu}$	$(1+x)^{-\nu}$
ν	2	$\frac{1-(1-x)^{1-\nu}}{(1-\nu)x}$	$\frac{(1+x)^{1-\nu}-1}{(1-\nu)x}$
$1+\nu$	ν	$\frac{\nu+(1-\nu)x}{\nu(1-x)^2}$	$\frac{\nu-(1-\nu)x}{\nu(1+x)^2}$
1	2	$\frac{-\ln(1-x)}{x}$	$\frac{\ln(1+x)}{x}$
1	3	$\frac{2}{x^2}\left[x+(1-x)\ln(1-x)\right]$	$\frac{2}{x^2}\left[(1+x)\ln(1+x)-x\right]$
1	$\frac{3}{2}$	$\frac{\arcsin\left(\sqrt{x}\right)}{\sqrt{x(1-x)}}$	$\frac{\operatorname{arsinh}\left(\sqrt{x}\right)}{\sqrt{x(1+x)}}$
1	$\frac{1}{2}$	$\frac{1}{1-x}+\frac{\sqrt{x}\arcsin\left(\sqrt{x}\right)}{\sqrt{(1-x)^3}}$	$\frac{1}{1+x}-\frac{\sqrt{x}\operatorname{arsinh}\left(\sqrt{x}\right)}{\sqrt{(1+x)^3}}$
$-\frac{1}{2}$	$\frac{1}{2}$	$1-\sqrt{x}\operatorname{artanh}\left(\sqrt{x}\right)$	$1+\sqrt{x}\arctan\left(\sqrt{x}\right)$
$\frac{1}{2}$	$\frac{3}{2}$	$\frac{\operatorname{artanh}(\sqrt{x})}{\sqrt{x}}$	$\frac{\arctan(\sqrt{x})}{\sqrt{x}}$
ν	$1+\nu$	$\nu\Phi(x,1,\nu)$	$\nu\Phi(-x,1,\nu)$
a	c	$\frac{(c-1)\mathrm{B}(c-1,a-c+1,x)}{(1-x)^{a-c+1}x^{c-1}}$	

Table 18-2

a_1	a_2	c_1	c_2	$\sum_{j=0}^{\infty}\frac{(a_1)_j(a_2)_j}{(c_1)_j(c_2)_j}x^j$	$\sum_{j=0}^{\infty}\frac{(a_1)_j(a_2)_j}{(c_1)_j(c_2)_j}(-x)^j$
$v-\frac{1}{2}$	v	1	$2v$	$\left[\left(1+\sqrt{1-x}\right)/2\right]^{1-2v}$	$\left[\left(1+\sqrt{1+x}\right)/2\right]^{1-2v}$
$-v$	$\frac{1}{2}-v$	½	1	$\frac{1}{2}\left[\left(1+\sqrt{x}\right)^{2v}+\left(1-\sqrt{x}\right)^{2v}\right]$	$(1+x)^v\cos\left\{2v\arctan\left(\sqrt{x}\right)\right\}$
$-v$	v	½	1	$\cos\left\{2v\arcsin\left(\sqrt{x}\right)\right\}$	$\frac{1}{2}\left[\left(\sqrt{1+x}+\sqrt{x}\right)^{2v}+\left(\sqrt{1+x}-\sqrt{x}\right)^{2v}\right]$
$\frac{1}{2}-v$	$1-v$	1	3⁄2	$\frac{\left(1+\sqrt{x}\right)^{2v}-\left(1-\sqrt{x}\right)^{2v}}{4v\sqrt{x}}$	$\frac{(1-x)^{-v}}{2v\sqrt{x}}\sin\left\{2v\arctan(\sqrt{x})\right\}$
$\frac{1}{2}-v$	$\frac{1}{2}+v$	1	3⁄2	$\frac{\sin\left\{2v\arcsin\left(\sqrt{x}\right)\right\}}{2v\sqrt{x}}$	$\frac{\left(\sqrt{1+x}+\sqrt{x}\right)^{2v}-\left(\sqrt{1+x}-\sqrt{x}\right)^{2v}}{4v\sqrt{x}}$
$1-v$	$1+v$	1	3⁄2	$\frac{\sin\left\{2v\arcsin\left(\sqrt{x}\right)\right\}}{2v\sqrt{x(1-x)}}$	$\frac{\left(\sqrt{1+x}+\sqrt{x}\right)^{2v}-\left(\sqrt{1+x}-\sqrt{x}\right)^{2v}}{4v\sqrt{x(1+x)}}$
1	3⁄2	2	2	$\frac{-4}{x}\left[\ln\left(\frac{\sqrt{x}}{2}\right)+\text{arcosh}\left(\frac{1}{\sqrt{x}}\right)\right]$	$\frac{-4}{x}\left[\ln\left(\frac{\sqrt{x}}{2}\right)+\text{arsinh}\left(\frac{1}{\sqrt{x}}\right)\right]$
1	1	2	2	$-\left[\text{diln}(1-x)\right]/x$	$\left[\text{diln}(1+x)\right]/x$
½	½	1	3⁄2	$\left[\arcsin\left(\sqrt{x}\right)\right]/\sqrt{x}$	$\left[\text{arsinh}\left(\sqrt{x}\right)\right]/\sqrt{x}$
½	½	3⁄2	2	$\frac{2}{x}\left[\sqrt{x}\arcsin\left(\sqrt{x}\right)-1+\sqrt{1-x}\right]$	$\frac{2}{x}\left[\sqrt{x}\,\text{arcsinh}\left(\sqrt{x}\right)+1-\sqrt{1+x}\right]$
$-n$	n+1	1	1	$\text{P}_n(1-2x)$	$\text{P}_n(1+2x)$
$-n$	n	½	1	$\text{T}_n(1-2x)$	
$-n$	$\frac{1}{2}-n$	½	1	$(1-x)^n\text{T}_n\left\{(1+x)/(1-x)\right\}$	
$-n$	n+2	1	3⁄2	$\frac{1}{n+1}\text{U}_n(1-2x)$	$\frac{1}{n+1}\text{U}_n(1+2x)$
n	n+v	1	$\frac{1}{2}+\frac{v}{2}$	$\frac{n!}{(v)_n}\text{C}_n^{(v/2)}(1-2x)$	$\frac{n!}{(v)_n}\text{C}_n^{(v/2)}(1+2x)$
½	½	1	1	$\frac{2}{\pi}\text{K}\left(\sqrt{x}\right)$	$\frac{2}{\pi\sqrt{1+x}}\text{K}\left(\sqrt{\frac{x}{1+x}}\right)$
−½	½	1	1	$\frac{2}{\pi}\text{E}\left(\sqrt{x}\right)$	$\frac{2\sqrt{1+x}}{\pi}\text{E}\left(\sqrt{\frac{x}{1+x}}\right)$
a	b	1	c	$\text{F}(a,b;c;x)$	$\left(1+x\right)^{-a}\text{F}\left(a,c-b;c;\frac{x}{1+x}\right)$

Table 18-3

c	$\sum_{j=0}^{\infty}\frac{1}{(c)_j}x^j$	$\sum_{j=0}^{\infty}\frac{1}{(c)_j}(-x)^j$
$-\frac{1}{2}$	$1-2x-2\sqrt{\pi x^3}\exp(x)\mathrm{erf}\left(\sqrt{x}\right)$	$1+2x-4\sqrt{x^3}\,\mathrm{daw}\left(\sqrt{x}\right)$
$\frac{1}{2}$	$1+\sqrt{\pi x}\exp(x)\mathrm{erf}\left(\sqrt{x}\right)$	$1-2\sqrt{x}\,\mathrm{daw}\left(\sqrt{x}\right)$
1	$\exp(x)$	$\exp(-x)$
$\frac{3}{2}$	$\sqrt{\frac{\pi}{4x}}\exp(x)\mathrm{erf}\left(\sqrt{x}\right)$	$\frac{1}{\sqrt{x}}\,\mathrm{daw}\left(\sqrt{x}\right)$
2	$\frac{\exp(x)-1}{x}$	$\frac{1-\exp(-x)}{x}$
$\frac{5}{2}$	$\frac{3}{4x^2}\left[\sqrt{\pi x}\exp(x)\mathrm{erf}\left(\sqrt{x}\right)-2x\right]$	$\frac{3}{2x}\left[1-\frac{1}{\sqrt{x}}\mathrm{daw}\left(\sqrt{x}\right)\right]$
3	$\frac{2}{x^2}\left[\exp(x)-1-x\right]$	$\frac{2}{x^2}\left[x-1+\exp(-x)\right]$
n	$\frac{(n-1)!}{x^{n-1}}\left[\exp(x)-\mathrm{e}_{n-2}(x)\right]$	$\frac{(n-1)!x}{(-x)^n}\left[\mathrm{e}_{n-2}(-x)-\exp(-x)\right]$
v	$\Gamma(v)\exp(x)\gamma\mathrm{n}(v-1,x)$	$\Gamma(v)\exp(-x)\gamma\mathrm{n}(v-1,-x)$

Table 18-4

a	c_1	c_2	$\sum_{j=0}^{\infty}\frac{(a)_j}{(c_1)_j(c_2)_j}x^j$	$\sum_{j=0}^{\infty}\frac{(a)_j}{(c_1)_j(c_2)_j}(-x)^j$
$-v$	1	1	$\mathrm{L}_v(x)$	$\mathrm{L}_v(-x)$
$-n$	1	μ	$\frac{n!}{(\mu)_n}\mathrm{L}_n^{(\mu-1)}(x)$	$\frac{n!}{(\mu)_n}\mathrm{L}_n^{(\mu-1)}(-x)$
$-n$	$\frac{1}{2}$	1	$\frac{(-1)^n}{4^n\left(\frac{1}{2}\right)_n}\mathrm{H}_{2n}\left(\sqrt{x}\right)$	
$\frac{1}{2}$	$\frac{3}{2}$	2		$\left\{\sqrt{\pi}\left[\sqrt{x}+\mathrm{ierfc}\left(\sqrt{x}\right)-1\right]\right\}/x$
v	$\frac{1}{2}$	1	$\frac{\Gamma\left(v+\frac{1}{2}\right)}{2^{1-v}\sqrt{\pi}}\exp\left(\frac{x}{2}\right)\sum_{\pm}\mathrm{D}_{-2v}\left(\pm\sqrt{2x}\right)$	$\frac{\Gamma(1-v)}{2^v\sqrt{2x}}\exp\left(\frac{-x}{2}\right)\sum_{\pm}\mathrm{D}_{2v-1}\left(\pm\sqrt{2x}\right)$
$\frac{1}{2}$	1	$\frac{3}{2}$	$\frac{1}{\sqrt{x}}\exp(x)\mathrm{daw}\left(\sqrt{x}\right)$	$\sqrt{\frac{\pi}{4x}}\,\mathrm{erf}\left(\sqrt{x}\right)$
$\frac{3}{2}$	1	$\frac{5}{2}$	$\frac{3\exp(x)}{2x}\left[1-\frac{\mathrm{daw}\left(\sqrt{x}\right)}{\sqrt{x}}\right]$	$\frac{3}{2x}\left[\sqrt{\frac{\pi}{4x}}\,\mathrm{erf}\left(\sqrt{x}\right)-\exp(-x)\right]$
1	2	2	$[-\mathrm{Ein}(-x)]/x$	$[\mathrm{Ein}(x)]/x$
v	1	μ	$\mathrm{M}(\nu,\mu,x)$	$\exp(x)\mathrm{M}(\mu-\nu,\mu,x)$
v	1	$v+1$		$v\,x^{-v}\gamma(v,x)$

18:14:4
$$\sum_{j=0}^{\infty}\frac{(a_1)_j(a_2)_j(a_3)_j}{(c_1)_j(c_2)_j(1)_j}x^j$$

of which an example is

18:14:5
$$\sum_{j=0}^{\infty}\frac{(2\mu)_j\left(2v-2\mu-\frac{1}{2}\right)_j\left(v-\frac{1}{2}\right)_j}{(v)_j(2v-1)_j(1)_j}x^j=\left[\mathrm{F}(\mu,v-\mu-\tfrac{1}{2},\nu,x\right]^2$$

Please refer to the Symbol Index for the meaning of any unfamiliar symbol. Equation 18:14:6 provides a non-Claisen example of a $L=K=3$ hypergeometric function.

Table 18-5

c_1	c_2	$\sum_{j=0}^{\infty}\frac{1}{(c_1)_j(c_2)_j}x^j$	$\sum_{j=0}^{\infty}\frac{1}{(c_1)_j(c_2)_j}(-x)^j$
$-\frac{1}{2}$	$\frac{1}{2}$	$1-2\pi x\,\mathrm{l}_{-1}\left(2\sqrt{x}\right)$	$1+2\pi x\mathrm{h}_{-1}\left(2\sqrt{x}\right)$
$\frac{1}{2}$	1	$\cosh\left(2\sqrt{x}\right)$	$\cos\left(2\sqrt{x}\right)$
$\frac{1}{2}$	$\frac{3}{2}$	$\frac{\pi}{2}\mathrm{l}_{-1}\left(2\sqrt{x}\right)$	$\frac{\pi}{2}\mathrm{h}_{-1}\left(2\sqrt{x}\right)$
$\frac{2}{3}$	1	$\frac{3^{2/3}\Gamma(\frac{2}{3})}{2}\left[\frac{\mathrm{Bi}\left(3^{2/3}x^{1/3}\right)}{\sqrt{3}}+\mathrm{Ai}\left(3^{2/3}x^{1/3}\right)\right]$	
$\frac{3}{4}$	$\frac{5}{4}$		$\frac{\sqrt{\pi}}{2x^{1/4}}\left[\frac{\sin\left(2\sqrt{x}+\frac{\pi}{4}\right)}{\sqrt{2}}-\mathrm{Gres}\left(\sqrt{2}x^{1/4}\right)\right]$
1	1	$\mathrm{I}_0\left(2\sqrt{x}\right)$	$\mathrm{J}_0\left(2\sqrt{x}\right)$
1	$\frac{4}{3}$	$\frac{\Gamma(\frac{1}{3})}{(24x)^{1/3}}\left[\frac{\mathrm{Bi}\left(3^{2/3}x^{1/3}\right)}{\sqrt{3}}-\mathrm{Ai}\left(3^{2/3}x^{1/3}\right)\right]$	
1	$\frac{3}{2}$	$\frac{\sinh\left(2\sqrt{x}\right)}{2\sqrt{x}}$	$\frac{\sin\left(2\sqrt{x}\right)}{2\sqrt{x}}$
1	μ	$\Gamma(\mu)x^{(1-\mu)/2}\mathrm{I}_{\mu-1}\left(2\sqrt{x}\right)$	$\Gamma(\mu)x^{(1-\mu)/2}\mathrm{J}_{\mu-1}\left(2\sqrt{x}\right)$
$\frac{3}{2}$	$\frac{3}{2}$	$\frac{\pi}{4\sqrt{x}}\mathrm{l}_0\left(2\sqrt{x}\right)$	$\frac{\pi}{4\sqrt{x}}\mathrm{h}_0\left(2\sqrt{x}\right)$
$\frac{5}{4}$	$\frac{7}{4}$		$\frac{3\sqrt{\pi}}{8x^{3/4}}\left[\mathrm{Fres}\left(\sqrt{2}x^{1/4}\right)-\frac{\cos\left(2\sqrt{x}+\frac{\pi}{4}\right)}{\sqrt{2}}\right]$
$\frac{3}{2}$	2	$\frac{1}{x}\sinh^2\left(\sqrt{x}\right)$	$\frac{1}{x}\sin^2\left(\sqrt{x}\right)$
$\frac{3}{2}$	μ	$\frac{\sqrt{\pi}\,\Gamma(\mu)}{2}x^{(1-2\mu)/4}\,\mathrm{l}_{(2\mu-3)/2}\left(2\sqrt{x}\right)$	$\frac{\sqrt{\pi}\,\Gamma(\mu)}{2}x^{(1-2\mu)/4}\mathrm{h}_{(2\mu-3)/2}\left(2\sqrt{x}\right)$

18:14:6 $$\sum_{j=0}^{\infty}\frac{(1)_j\left(\frac{3}{2}\right)_j\left(\frac{3}{2}\right)_j}{(2)_j(2)_j\left(\frac{5}{2}\right)_j}x^j=\frac{12}{x}\left[1+\ln\left(\frac{2}{1+\sqrt{1-x}}\right)-\frac{\arcsin\left(\sqrt{x}\right)}{\sqrt{x}}\right]\qquad 0<x\le 1$$

The exponential function is the prototype $L = K+1$ hypergeometric function

18:14:7 $$\exp(\pm x)=\sum_{j=0}^{\infty}\frac{1}{(1)_j}(\pm x)^j$$

All other hypergeometric functions that have one more denominatorial than numeratorial parameter may be synthesized from it. Tables 18-3 and 18-4 respectively are listings of some examples of $L = K+1 = 1$ and $L = K+1 = 2$ hypergeometric functions. An example of an $L = K+1 = 3$ hypergeometric is

18:14:8 $$\sum_{j=0}^{\infty}\frac{(2)_j(2)_j}{(1)_j(3)_j(3)_j}(-x)^j=\frac{4}{x^2}\left[\mathrm{Ein}(x)+\exp(-x)-1\right]$$

The starting point for the synthesis of $L = K+2$ hypergeometric functions is the zero-order modified Bessel function $\mathrm{I}_0\left(2\sqrt{x}\right)$ or the corresponding (circular) Bessel function $\mathrm{J}_0\left(2\sqrt{x}\right)$. Examples of $L = K+2 = 3$ hypergeometrics are assembled in Tables 18-5 and 18-6. There are rather few instances of $L = K+2 = 4$ hypergeometrics, but one is

18:14:9 $$\sum_{j=0}^{\infty}\frac{(\nu)_j\left(\nu+\frac{1}{2}\right)_j}{(1)_j(1+\nu-\mu)_j(\mu+\nu)_j(2\nu)_j}x^j=\frac{\Gamma(\mu+\nu)\Gamma(1+\nu-\mu)}{(x/4)^{(2\nu-1)/2}}\mathrm{I}_{\mu+\nu-1}\left(\sqrt{x}\right)\mathrm{I}_{\nu-\mu}\left(\sqrt{x}\right)$$

Table 18-6

a	c_1	c_2	c_3	$\sum_{j=0}^{\infty}\frac{(a)_j}{(c_1)_j(c_2)_j(c_3)_j}x^j$	$\sum_{j=0}^{\infty}\frac{(a)_j}{(c_1)_j(c_2)_j(c_3)_j}(-x)^j$
$\frac{1}{2}$	$1-\nu$	1	$1+\nu$	$\nu\pi\csc(\nu\pi)\mathrm{I}_{-\nu}\left(\sqrt{x}\right)\mathrm{I}_{\nu}\left(\sqrt{x}\right)$	$\nu\pi\csc(\nu\pi)\mathrm{J}_{-\nu}\left(\sqrt{x}\right)\mathrm{J}_{\nu}\left(\sqrt{x}\right)$
$\frac{1}{2}$	1	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{1}{2\sqrt{x}}\mathrm{Shi}\left(2\sqrt{x}\right)$	$\frac{1}{2\sqrt{x}}\mathrm{Si}\left(2\sqrt{x}\right)$
$\frac{3}{4}$	1	$\frac{3}{2}$	$\frac{7}{4}$		$\frac{3\sqrt{\pi}}{4x^{3/4}}\mathrm{S}\left(\sqrt{2}\,x^{1/4}\right)$
1	$\frac{3}{2}$	2	2	$\frac{1}{x}\mathrm{Chin}\left(2\sqrt{x}\right)$	$\frac{1}{x}\mathrm{Cin}\left(2\sqrt{x}\right)$
2	1	$\frac{5}{2}$	3	$\frac{3}{x^2}\left[\frac{\sqrt{x}\sinh\left(2\sqrt{x}\right)}{2}-\sinh^2\left(\sqrt{x}\right)\right]$	
ν	$\frac{1}{2}$	1	$1+\nu$		$\frac{2\nu}{(4x)^{\nu}}\left[\mathrm{C}(2\nu,0)-\mathrm{C}\left(2\nu,2\sqrt{x}\right)\right]$
ν	1	$\nu+\frac{1}{2}$	2ν		$\left(\frac{4}{x}\right)^{(2\nu-1)/2}\left[\Gamma\left(\nu+\frac{1}{2}\right)\mathrm{J}_{(2\nu-1)/2}\left(\sqrt{x}\right)\right]^2$

For some obscure reason, hypergeometric functions in which the denominatorial order exceeds the numeratorial order by 3 seldom correspond to named functions, one a rare exception appearing in equation 53:11:3 and another being

18:14:10 $$\sum_{j=0}^{\infty}\frac{1}{\left(\frac{1}{3}\right)_j\left(\frac{2}{3}\right)_j(1)_j}x^j=\frac{1}{3}\exp\left(3x^{1/3}\right)+\frac{2}{3}\exp\left(-3/2x^{1/3}\right)\cos\left(3\sqrt{3}x^{1/3}/2\right)$$

which is an example of a Mittag-Leffler function [Section 45:14]. In contrast, named cases of $L = K+4 = 4$ hypergeometric functions are quite abundant, an instance being the Kelvin function [Chapter 55]

18:14:11 $$\sum_{j=0}^{\infty}\frac{1}{\left(\frac{1}{2}\right)_j\left(\frac{1}{2}\right)_j(1)_j(1)_j}(-x)^j=\mathrm{ber}\left(4x^{1/4}\right)$$

Table 18-7

a	$\sum_{j=0}^{\infty}(a)_j x^j$	$\sum_{j=0}^{\infty}(a)_j(-x)^j$
$-n$	$n!(-x)^n\mathrm{e}_n\left(\frac{-1}{x}\right)$	$n!x^n\mathrm{e}_n\left(\frac{1}{x}\right)$
$-\frac{1}{2}$	$1-\sqrt{x}\,\mathrm{daw}\left(\frac{1}{\sqrt{x}}\right)$	$1+\frac{\sqrt{\pi x}}{2}\exp\left(\frac{1}{x}\right)\mathrm{erfc}\left(\frac{1}{\sqrt{x}}\right)$
$\frac{1}{2}$	$\frac{2}{\sqrt{x}}\mathrm{daw}\left(\frac{1}{\sqrt{x}}\right)$	$\sqrt{\frac{\pi}{x}}\exp\left(\frac{1}{x}\right)\mathrm{erfc}\left(\frac{1}{\sqrt{x}}\right)$
1	$\frac{1}{x}\exp\left(\frac{-1}{x}\right)\mathrm{Ei}\left(\frac{1}{x}\right)$	$\frac{-1}{x}\exp\left(\frac{1}{x}\right)\mathrm{Ei}\left(\frac{-1}{x}\right)$
$\frac{3}{2}$		$\frac{2\sqrt{\pi}}{x}\exp\left(\frac{1}{x}\right)\mathrm{ierfc}\left(\frac{1}{\sqrt{x}}\right)$
v		$x^{-v}\exp\left(\frac{1}{x}\right)\Gamma\left(1-v,\frac{1}{x}\right)$

Table 18-8

a_1	a_2	c	$\sum_{j=0}^{\infty}\frac{(a_1)_j(a_2)_j}{(c)_j}x^j$	$\sum_{j=0}^{\infty}\frac{(a_1)_j(a_2)_j}{(c)_j}(-x)^j$
$\frac{1}{2}-v$	$\frac{1}{2}+v$	1	$\sqrt{\frac{\pi}{x}}\exp\left(\frac{-1}{2x}\right)\mathrm{I}_v\left(\frac{1}{2x}\right)$	$\frac{1}{\sqrt{\pi x}}\exp\left(\frac{1}{2x}\right)\mathrm{K}_v\left(\frac{1}{2x}\right)$
$\frac{-n}{2}$	$\frac{1-n}{2}$	1	$2-\left(\frac{\sqrt{x}}{2}\right)^n\mathrm{H}_n\left(\frac{1}{\sqrt{x}}\right)$	
$\frac{1}{6}$	$\frac{5}{6}$	1	$\sqrt{\pi}\left(\frac{3}{4x}\right)^{1/6}\exp\left(\frac{-1}{2x}\right)\mathrm{Bi}\left\{\left(\frac{3}{4x}\right)^{2/3}\right\}$	$\sqrt{\pi}\left(\frac{48}{x}\right)^{1/6}\exp\left(\frac{1}{2x}\right)\mathrm{Ai}\left\{\left(\frac{3}{4x}\right)^{2/3}\right\}$
v	$v+\frac{1}{2}$	1		$\left(\frac{2}{x}\right)^v\exp\left(\frac{1}{2x}\right)\mathrm{D}_{-2v}\left(\sqrt{\frac{2}{x}}\right)$
v	μ	1		$\frac{1}{x^v}\mathrm{U}\left(v,1+v-\mu;\frac{1}{x}\right)$

Examples of hypergeometric functions of the $L = K-1 = 0$ and $L = K-1 = 1$ families are listed in Tables 18-7 and 18-8. Of course, these correspond to asymptotic series. With even worse convergence properties are the $L = K-2 = 0$ hypergeometrics of which a few are shown in Table 18-9. The series

18:14:12 $$\sum_{j=0}^{\infty} \frac{\left(v-\tfrac{1}{2}\right)_j \left(\tfrac{1}{2}\right)_j \left(v+\tfrac{1}{2}\right)_j}{(1)_j} x^j = \frac{2}{\sqrt{x}} \mathrm{I}_v\left(\frac{1}{\sqrt{x}}\right) \mathrm{K}_v\left(\frac{1}{\sqrt{x}}\right)$$

is an example of an $L = K-2 = 1$ hypergeometric function. Two important $L = K-2 = 2$ hypergeometric functions occur in Section 53:6.

Table 18-9

a_1	a_2	$\sum_{j=0}^{\infty}(a_1)_j(a_2)_j(-x)^j$
v	$\tfrac{1}{2}$	$\frac{\sqrt{\pi}\Gamma(1-v)}{x^{(1+2v)/4}}\left[\mathrm{h}_{(1-2v)/2}\left(\frac{2}{\sqrt{x}}\right) - \mathrm{Y}_{(1-2v)/2}\left(\frac{2}{\sqrt{x}}\right)\right]$
$\tfrac{1}{4}$	$\tfrac{3}{4}$	$\frac{2\sqrt{\pi}}{x^{1/4}}\mathrm{Fres}\left(\frac{2}{x^{1/4}}\right)$
$\tfrac{1}{2}$	1	$\frac{2}{\sqrt{x}}\mathrm{fi}\left(\frac{2}{\sqrt{x}}\right)$
$\tfrac{3}{4}$	$\tfrac{5}{4}$	$\frac{8\sqrt{\pi}}{x^{3/4}}\mathrm{Gres}\left(\frac{2}{x^{1/4}}\right)$
1	$\tfrac{3}{2}$	$\frac{4}{x}\mathrm{gi}\left(\frac{2}{\sqrt{x}}\right)$
v	$v+\tfrac{1}{2}$	$\left(\frac{4}{x}\right)^v\left[\cos\left(\frac{2}{\sqrt{x}}\right)\mathrm{S}\left(1-2v,\frac{2}{\sqrt{x}}\right) - \sin\left(\frac{2}{\sqrt{x}}\right)\mathrm{C}\left(1-2v,\frac{2}{\sqrt{x}}\right)\right]$

Let G_j denote the following abbreviation

18:14:13 $$G_j = \frac{(a_1+j)(a_2+j)(a_3+j)\cdots(a_K+j)}{(c_1+j)(c_2+j)(c_3+j)\cdots(c_L+j)}$$

then any hypergeometric function is given by

18:14:14 $$1 \pm G_0 x + G_0 G_1 x^2 \pm G_0 G_1 G_2 x^3 + \cdots (\pm)^J G_0 G_1 G_2 \cdots G_{J-1} x^J + R_J$$

where R_J is the remainder if the summation is halted after the Jth term. Ignoring R_J, a convenient method of calculating the hypergeometric function is via the concatenation

18:14:15 $$\left(\left(\cdots\left(G_{J-1}x \pm 1\right)G_{J-2}x \pm \cdots \pm 1\right)G_1 x \pm 1\right)G_0 x + 1$$

In discussing the general properties of hypergeometric functions, use will be made of a collapsed notation exemplified by the replacement of $(a_1)_j(a_2)_j\cdots(a_K)_j$ by $\left(a_{1\to K}\right)_j$. Likewise $\left(a_{1\to K}+1\right)_j$ implies the K-fold product $(a_1+1)_j(a_2+1)_j\cdots(a_K+1)_j$.

The recursion relation

18:14:16
$$\sum_{j=0}^{\infty}\frac{(a_{1\to K}+1)_j}{(c_{1\to L}+1)_j}(\pm x)^j = \frac{\pm c_{1\to L}}{a_{1\to K}x}\left[-1+\sum_{j=0}^{\infty}\frac{(a_{1\to K})_j}{(c_{1\to L})_j}(\pm x)^j\right]$$

is satisfied by any hypergeometric function. Furthermore, any hypergeometric function can be split into two others with an inflated parameter set:

18:14:17
$$\sum_{j=0}^{\infty}\frac{(a_{1\to K}+1)_j}{(c_{1\to L}+1)_j}(\pm x)^j = \sum_{j=0}^{\infty}\frac{\left(\frac{1}{2}a_{1\to K}\right)_j\left(\frac{1}{2}+\frac{1}{2}a_{1\to K}\right)_j}{\left(\frac{1}{2}c_{1\to L}\right)_j\left(\frac{1}{2}+\frac{1}{2}c_{1\to L}\right)_j}\left(\frac{x^2}{4^{L-K}}\right)^j \pm \frac{a_{1\to K}x}{c_{1\to L}}\sum_{j=0}^{\infty}\frac{\left(\frac{1}{2}+\frac{1}{2}a_{1\to K}\right)_j\left(1+\frac{1}{2}a_{1\to K}\right)_j}{\left(\frac{1}{2}+\frac{1}{2}c_{1\to L}\right)_j\left(1+\frac{1}{2}c_{1\to L}\right)_j}\left(\frac{x^2}{4^{L-K}}\right)^j$$

Of course, this result may become invalid if it creates new denominatorial parameters that are nonnegative integers. Replacing x in this formula by ix shows that a hypergeometric function of imaginary argument has real and imaginary parts that are themselves hypergeometric functions.

Embodying the fractional calculus [Section 12:14], a formula of very wide applicability is

18:14:18
$$\frac{d^\nu}{dx^\nu}\left\{x^\mu\sum_{j=0}^{\infty}\frac{(a_{1\to K})_j}{(c_{1\to L})_j}(\pm x)^j\right\} = \frac{\Gamma(\mu+1)x^{\mu-\nu}}{\Gamma(\mu-\nu+1)}\sum_{j=0}^{\infty}\frac{(\mu+1)_j(a_{1\to K})_j}{(\mu-\nu+1)_j(c_{1\to L})_j}(\pm x)^j$$

where ν and μ are not necessarily integers. This formula is invalid if either μ or $\mu-\nu$ is a negative integer; if they are *both* negative integers, it fails if ν is negative. Examples of the $\mu = 0$ version include semidifferentiation

18:14:19
$$\frac{d^{1/2}}{dx^{1/2}}\left\{\sum_{j=0}^{\infty}\frac{(a_{1\to K})_j}{(c_{1\to L})_j}(\pm x)^j\right\} = \frac{1}{\sqrt{\pi x}}\sum_{j=0}^{\infty}\frac{(1)_j(a_{1\to K})_j}{(\frac{1}{2})_j(c_{1\to L})_j}(\pm x)^j$$

semiintegration

18:14:20
$$\frac{d^{-1/2}}{dx^{-1/2}}\left\{\sum_{j=0}^{\infty}\frac{(a_{1\to K})_j}{(c_{1\to L})_j}(\pm x)^j\right\} = 2\sqrt{\frac{x}{\pi}}\sum_{j=0}^{\infty}\frac{(1)_j(a_{1\to K})_j}{(\frac{3}{2})_j(c_{1\to L})_j}(\pm x)^j$$

and integration

18:14:21
$$\int_0^x\left\{\sum_{j=0}^{\infty}\frac{(a_{1\to K})_j}{(c_{1\to L})_j}(\pm t)^j\right\}dt = x\sum_{j=0}^{\infty}\frac{(1)_j(a_{1\to K})_j}{(2)_j(c_{1\to L})_j}(\pm x)^j$$

The formula for ordinary differentiation

18:14:22
$$\frac{d}{dx}\left\{\sum_{j=0}^{\infty}\frac{(a_{1\to K})_j}{(c_{1\to L})_j}(\pm x)^j\right\} = \frac{a_{1\to K}}{c_{1\to L}}\sum_{j=0}^{\infty}\frac{(2)_j(a_{1\to K})_j}{(1)_j(c_{1\to L})_j}(\pm x)^j$$

also follows from 18:14:18, but only after a preliminary step based on recursion 18:14:16. Notice that all the formulas 18:14:16−18:14:22 maintain the $L-K$ difference. Laplace transformation, however, decreases this difference

18:14:23
$$\int_0^\infty\left\{\sum_{j=0}^{\infty}\frac{(a_{1\to K})_j}{(c_{1\to L})_j}(\pm t)^j\right\}\exp(-st)dt = \mathfrak{L}\left\{\sum_{j=0}^{\infty}\frac{(a_{1\to K})_j}{(c_{1\to L})_j}(\pm t)^j\right\} = \frac{1}{s}\sum_{j=0}^{\infty}\frac{(1)_j(a_{1\to K})_j}{(c_{1\to L})_j}(\pm s)^j$$

worsening the convergence properties of the hypergeometric function.

Specific to the hypergeometric 1:1 functions, are the reflection formula

18:14:24
$$\sum_{j=0}^{\infty}\frac{(a)_j}{(c)_j}x^j = \frac{\Gamma(c)\Gamma(a-c+1)}{\Gamma(a)}\frac{(1-x)^{c-a-1}}{x^{c-1}} - \frac{c-1}{a-c+1}\sum_{j=0}^{\infty}\frac{(a)_j}{(a-c+2)_j}(1-x)^j$$

and the following rule

18:14:25 $$\sum_{j=0}^{\infty}\frac{(a)_j}{(c)_j}x^j=\frac{(c-a)_n}{(c)_n}\left(\frac{x}{x-1}\right)^n\sum_{j=0}^{\infty}\frac{(a)_j}{(c+n)_j}x^j+\frac{1}{1-x}\sum_{j=0}^{n-1}\frac{(1-c)_j}{(a-c+1)_j}\left(\frac{x}{x-1}\right)^j$$

which permits the denominatorial parameter to be incremented by an integer, at the expense of an additional polynomial function.