CHAPTER 16

THE CUBIC FUNCTION $x^3 + ax^2 + bx + c$

The cubic function is a polynomial function of degree 3, and accordingly the general properties of polynomials [Chapter 17] are applicable. Cubic functions find frequent application in data interpolation, a topic addressed in Section 16:14.

16:1 NOTATION

The most general formulation of a cubic function is $a_3x^3 + a_2x^2 + a_1x + a_0$, with four *coefficients*. However, it is a simple matter to factor out the leading coefficient and accordingly this chapter mostly addresses the function

16:1:1 $$\mathrm{f}(x) = x^3 + ax^2 + bx + c$$

The following quantities, that we term *parameters*, are important in determining the properties of the cubic function.

16:1:2 $$P = \frac{a^2 - 3b}{9}$$

16:1:3 $$Q = \frac{ab}{6} - \frac{c}{2} - \frac{a^3}{27}$$

and

16:1:4 $$D = P^3 - Q^2$$

The last, or sometimes its negative, is known as the *discriminant* of the cubic function.

16:2 BEHAVIOR

Irrespective of the values of its coefficients, the range and domain of the cubic function are unrestricted. A cartesian graph of the cubic function $f = \mathrm{f}(x) = x^3 + ax^2 + bx + c$ has inversion symmetry [Section 14:15] through the point with rectangular coordinates

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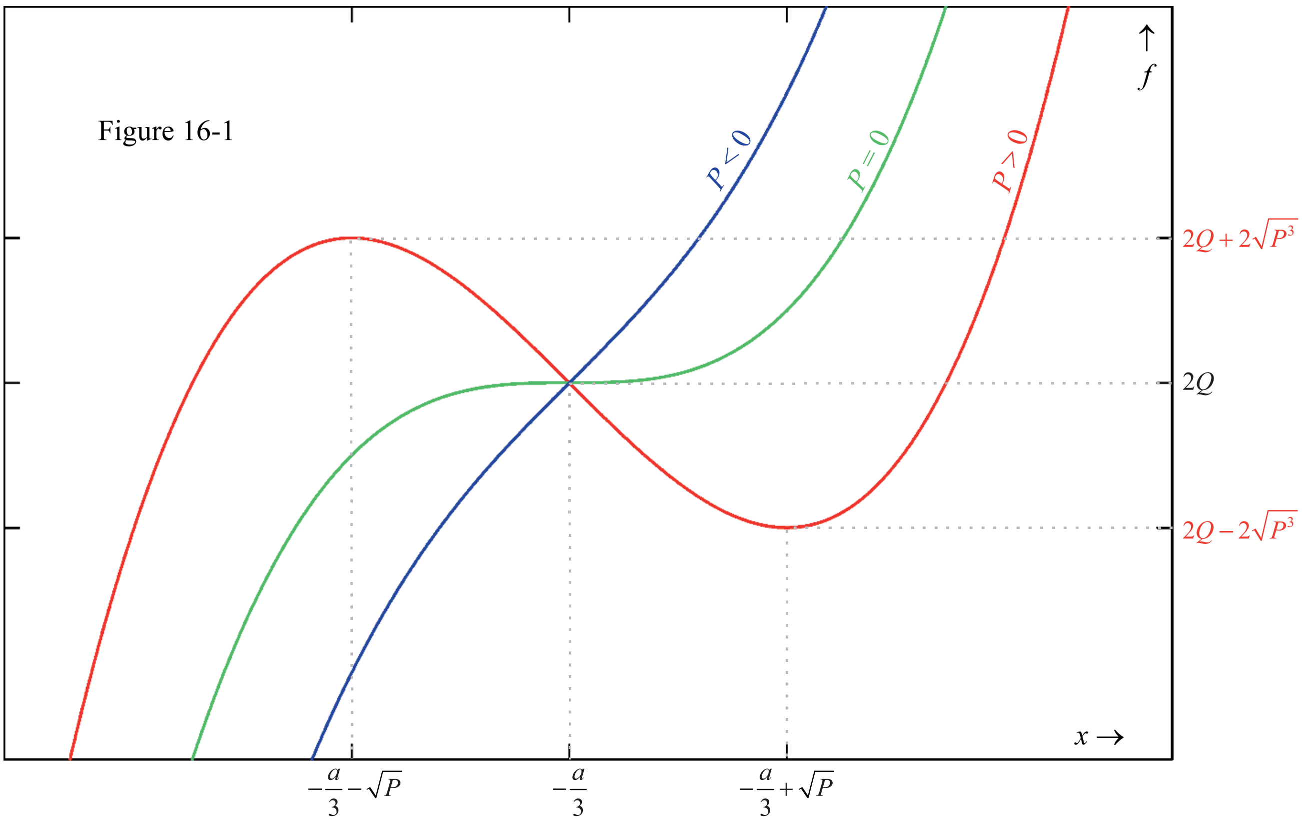

16:2:1 $$(x, f) = \left(\frac{-a}{3}, 2Q\right)$$

which is also a point of inflection, as Figure 16-1 illustrates. This diagram shows graphs of the cubic function for representative values of P and demonstrates that the sign of this parameter determines whether the cubic function possesses extrema. If a^2 exceeds $3b$, so that P is positive, then there is a maximum at $x = -(a/3) - \sqrt{P}$ and a minimum at $x = -(a/3) + \sqrt{P}$. Otherwise the cubic function is a *monotonic function*; that is, its slope never changes sign.

Figure 16-1 does not locate $f = 0$, but it is clear that its location will determine the number of real zeros that the cubic function possesses. In fact, the existence of three distinct zeros requires that both P and D be positive.

16:3 DEFINITIONS

Writing the cubic functions as the *concatenation*

16:3:1 $$\mathrm{f}(x) = c + x\big(b + x(a + x)\big)$$

confirms that the arithmetic operations of addition and multiplication suffice to define the cubic function.

The product of three linear functions creates a cubic function:

16:3:2 $$(x - r_0)(x - r_1)(x - r_{-1}) = x^3 - (r_0 + r_1 + r_{-1})x^2 + (r_0 r_1 + r_0 r_{-1} + r_1 r_{-1})x - r_0 r_1 r_{-1}$$

but not all cubic functions can be defined in this way, unless two of the r's are sometimes allowed to assume complex values. The r quantities, the *zeros* of the cubic function, are addressed in Section 16:7. *Every* cubic function may, however, be defined as the product of a linear function and a quadratic function:

16:3:3 $$(x-r)\left(x^2+(a+r)x-\frac{c}{r}\right)=x^3+ax^2+bx+c$$

with r being the real zero, or one of the real zeros.

16:4 SPECIAL CASES

The cubic equation $\mathrm{f}(x)=x^3+ax^2+bx+c$ factors straightforwardly when the c coefficient, or both of the other coefficients, or any one of the parameters [Section 16:1] equals zero; thus;

16:4:1 $$c=0 \qquad \mathrm{f}(x)=x\left[x^2+ax+b\right]$$

16:4:2 $$a=b=0 \qquad \mathrm{f}(x)=\left[x+c^{1/3}\right]\left[x^2-c^{1/3}x+c^{2/3}\right]$$

16:4:3 $$P=0 \qquad \mathrm{f}(x)=\left[y-\sqrt[3]{2Q}\right]\left[y^2+\sqrt[3]{2Q}\,y+\sqrt[3]{4Q^2}\right] \qquad y=x+\frac{a}{3},\quad 2Q=\frac{a^3}{27}-c$$

16:4:4 $$Q=0 \qquad \mathrm{f}(x)=y\left[y^2-3P\right] \qquad y=x+\frac{a}{3},\quad 3P=\frac{a^2}{9}-\frac{3c}{a}$$

16:4:5 $$D=0 \qquad \mathrm{f}(x)=\left[y+\sqrt{P}\right]^2\left[y-2\sqrt{P}\right] \qquad y=x+\frac{a}{3},\quad \sqrt{P}=\frac{\sqrt{a^2-3b}}{3}$$

By $\sqrt[3]{2Q}$ we imply the *real* cube root of $2Q$.

If all the coefficients are zero, the cubic reduces to a power function $\mathrm{f}(x) = x^3$. If all the parameters are zero, reduction occurs to another power function: $\mathrm{f}(x) = [x + (a/3)]^3$.

16:5 INTRARELATIONSHIPS

The cubic function $\mathrm{f}(x)=x^3+ax^2+bx+c$ obeys the reflection formula

16:5:1 $$\mathrm{f}\left(\frac{-a}{3}-x\right)=-4Q-\mathrm{f}\left(\frac{-a}{3}+x\right)$$

where the parameter Q is defined in 16:1:2.

Setting $y = x + (a/3)$ converts one cubic function to another:

16:5:2 $$x^3+ax^2+bx+c=y^3-3Py-2Q$$

This transformation represents a simplification because the new argument appears only twice in the new formulation. A further contraction to a form in which there is a single appearance of the argument is also possible. The form of the new argument depends on the sign of the P parameter and, if P is positive, also on the magnitude of $y/\sqrt{P}$. For negative P

16:5:3 $$\mathrm{f}(x)=2\left[\sqrt{(-P)^3}\sinh(t)-Q\right] \qquad \text{where} \qquad t=3\,\mathrm{arsinh}\left(\frac{3x+a}{6\sqrt{-P}}\right) \qquad P<0$$

When P is positive and larger than $[(3x + a)/6]^2$

16:5:4 $$\mathrm{f}(x)=2\left[\sqrt{P^3}\cos(\theta)-Q\right] \qquad \text{where} \qquad \theta=3\arccos\left(\frac{3x+a}{6\sqrt{P}}\right) \qquad P>\frac{(3x+a)^2}{36}$$

whereas if P is positive but smaller than $[(3x + a)/6]^2$

16:5:5
$$\mathrm{f}(x) = 2\left[\sqrt{P^3}\,\mathrm{sgn}\left(\frac{3x+a}{6\sqrt{P}}\right)\cosh(t) - Q\right] \qquad \text{where} \qquad t = 3\,\mathrm{arcosh}\left(\left|\frac{3x+a}{6\sqrt{P}}\right|\right)$$

The keys to deriving these formulas lie in equations 28:5:6, 32:5:5, and 28:5:5.

The inverse function of the cubic is multivalued if $P > 0$, but can be shown from 16:5:3 to be

16:5:6
$$\frac{-a}{3} + 2\sqrt{-P^3}\,\sinh\left\{\frac{1}{3}\mathrm{arsinh}\left(\frac{x+2Q}{2\sqrt{-P^3}}\right)\right\} \qquad P < 0$$

for a cubic with a negative P parameter.

16:6 EXPANSIONS

The expansions discussed in Section 17:6 apply, but they are of little utility for the cubic function.

16:7 PARTICULAR VALUES

The cubic function $x^3 + ax^2 + bx + c$ has an inflection at $x = -a/3$, irrespective of the other two coefficients. As Figure 16-1 shows, a maximum and a minimum are exhibited only if the parameter P is positive.

Equator's notation for the three zeros of the cubic functions is $\mathrm{r}_3(a,b,c,n)$, with $n = 0, \pm 1$. If any one of c, P, Q, or D are zero, or if both a and b are zero, then the zeros may be found straightforwardly from the special-case equations in Section 16:4. Otherwise the zeros are calculable by the procedure outlined in the following paragraph. One of these zeros, $\mathrm{r}_3(a,b,c,0)$ will be real invariably, but the other two, $\mathrm{r}_3(a,b,c,+1)$ and $\mathrm{r}_3(a,b,c,-1)$, will be complex (or imaginary) unless both P and D are positive. When two complex zeros exist, they always occur as a *conjugate pair*; that is, they have identical real parts and their imaginary parts are equal in magnitude but opposite in sign.

One real zero and two complex zeros exist when $P < 0$, irrespective of the value of D; they are:

16:7:1
$$\left.\begin{aligned} \mathrm{r}_3(a,b,c,0) &= \frac{-a}{3} + 2\sqrt{-P}\sinh(t) \\ \mathrm{r}_3(a,b,c,\pm 1) &= \frac{-a}{3} - \sqrt{-P}\sinh(t) \pm i\sqrt{-3P}\cosh(t) \end{aligned}\right\} \quad t = \frac{1}{3}\mathrm{arsinh}\left(\frac{Q}{\sqrt{(-P)^3}}\right)$$

For $P > 0$ and $D > 0$, there are three real zeros:

16:7:2
$$\mathrm{r}_3(a,b,c,n) = \frac{-a}{3} + 2\sqrt{P}\cos\left(\frac{2n\pi + \arccos\left(Q/\sqrt{P^3}\right)}{3}\right) \qquad n = 0, \pm 1$$

For $P > 0$ and $D < 0$, there is one real zero and two complex zeros:

16:7:3
$$\left.\begin{aligned} \mathrm{r}_3(a,b,c,0) &= \frac{-a}{3} + u + v \\ \mathrm{r}_3(a,b,c,\pm 1) &= \frac{-a}{3} - \frac{u+v}{2} \pm i\sqrt{3}\,\frac{u-v}{2} \end{aligned}\right\} \quad \left\{\begin{aligned} u &= \sqrt[3]{Q - \sqrt{-D}} \\ v &= \sqrt[3]{Q + \sqrt{-D}} \end{aligned}\right.$$

By $\sqrt[3]{}$ we imply the *real* cube root. These formulas originate from equations 16:5:3−5 and are used by *Equator* in its cubic zeros routine (keyword **r3**), though simpler methods are employed for the special cases enumerated in Section 16:4. *Equator* treats the zeros as a complex-valued quadrivariate function

16:7:4 $$\mathrm{r}_3(a,b,c,n) = \mathrm{Re}\left[\mathrm{r}_3(a,b,c,n)\right] + i\,\mathrm{Im}\left[\mathrm{r}_3(a,b,c,n)\right] \qquad n = -1, 0, +1$$

and outputs both real and imaginary parts, the latter being 0 whenever the zero is real. By default, *Equator* generates a short table giving all three zeros. The algorithm is exact but, because large losses of significance can occasionally occur, check answers carefully if precision is an issue.

16:8 NUMERICAL VALUES

These are easily calculated, for example through *Equator*'s cubic function routine (keyword **cubic**).

16:9 LIMITS AND APPROXIMATIONS

The cubic function is dominated by its x^3 term when its argument is of large magnitude.

There is seldom a need to approximate a cubic function; on the contrary, cubic functions are themselves often used to approximate more complicated functions, as explained in Section 16:14.

16:10 OPERATIONS OF THE CALCULUS

As with all polynomials, the operations of the calculus may be carried out on the cubic function term by term:

16:10:1 $$\frac{\mathrm{d}}{\mathrm{d}x}(x^3 + ax^2 + bx + c) = 3x^2 + 2ax + b$$

16:10:2 $$\int_0^x (t^3 + at^2 + bt + c)\,\mathrm{d}t = \frac{3x^4 + 4ax^3 + 6bx^2 + 12cx}{12}$$

16:10:3 $$\int_0^\infty (t^3 + at^2 + bt + c)\exp(-st)\,\mathrm{d}t = \mathfrak{L}\left\{t^3 + at^2 + bt + c\right\} = \frac{cs^3 + bs^2 + 2as + 6}{s^4}$$

Integrals of $\mathrm{f}(t)/\left(t^3 + at^2 + bt + c\right)$ can often be evaluated by following the procedure described in Section 17:13. Indefinite integrals of such functions as $1/\sqrt{t^3 + at^2 + bt + c}$ or $t/\sqrt{t^3 + at^2 + bt + c}$ are the subject of Section 62:14.

16:11 COMPLEX ARGUMENT

When the argument is $z = x + iy$, the real and imaginary parts of the cubic function are

16:11:1 $$z^3 + az^2 + bz + c = \left[x^3 + a(x^2 - y^2) + bx + c - 3xy^2\right] + iy\left[3x^2 - y^2 + 2axy + by\right]$$

The cubic function of complex argument encounters no poles or other discontinuities, other than at infinity.

16:12 GENERALIZATIONS: including zeros of the quartic function

A cubic function is a polynomial of degree 3. As such, it is the third member in a hierarchy of which the linear function and quadratic functions are lower members and the quartic, quintic, etc. are higher members. The *Atlas* treats these higher members as a general family in the next chapter. However, one aspect of *quartic* functions is addressed here because the properties of cubic functions are relevant.

If the coefficients of the quartic

16:12:1 $$x^4+a_3x^3+a_2x^2+a_1x+a_0$$

are real, the quartic's zeros may be calculated by first finding any zero r of the so-called *cubic resolvent function*

16:12:2 $$x^3+ax^2+bx+c \qquad \text{where} \quad \begin{cases} a=-a_2 \\ b=a_1a_3-4a_0 \\ c=4a_0a_2-a_1^2-a_0a_3^2 \end{cases}$$

using the method of Section 16:7. Then the four zeros of 16:12:1 can usually be found from *Ferrari's solution* [Lodovico Ferrari, Italian mathematician, 1522−1565]

16:12:3 $$\mathrm{r}_4(a_3,a_2,a_1,a_0)=\begin{Bmatrix} (-a_3+s\pm q)/4 \\ \text{and} \\ (-a_3-s\pm p)/4 \end{Bmatrix} \qquad \text{where} \quad \begin{cases} s=\sqrt{4r-4a_2+a_3^2} \\ q=\sqrt{2a_3^2-4a_2-4r+(8a_2a_3-2a_3^3-16a_1)/s} \\ p=\sqrt{2a_3^2-4a_2-4r-(8a_2a_3-2a_3^3-16a_1)/s} \end{cases}$$

If s should equal zero, the term $(8a_2a_3-2a_3^3-16a_1)/s$ is to be replaced by $8\sqrt{r^2-4a_0}$.

Ferrari's solution is the basis of *Equator*'s quartic zeros routine (keyword **r4**). However, if the chosen cubic zero leads to a small value of s, serious precision loss may occur. To counter this, *Equator* selects the cubic zero that leads to the largest s.

16:13 COGNATE FUNCTIONS

The *reciprocal cubic function* $1/(x^3+ax^2+bx+c)$ is of some interest and provides an exemplary model of reciprocal polynomial functions [Section 17:13] in general.

Partial fractionation is a method of expanding reciprocal polynomials. Equation 16:3:3 may be used to suggest the splitting of the reciprocal cubic as follows

16:13:1 $$\mathrm{f}(x)=\frac{1}{x^3+ax^2+bx+c}=\frac{1}{(x-r)\left(x^2+(a+r)x-\dfrac{c}{r}\right)}=\frac{\alpha}{x-r}+\frac{\beta x+\gamma}{x^2+(a+r)x-\dfrac{c}{r}}$$

where r is a real zero, the constants α, β, and γ being initially unknown. They may be determined, however, by first multiplying 16:13:1 by the cubic to remove the denominators, which leads to

16:13:2 $$1=[\alpha+\beta]x^2+[\alpha(a+r)-\beta r+\gamma]x+\left[\alpha\frac{c}{r}-\gamma r\right]$$

This is an identity and therefore one may equate coefficients of like terms from each side of equation 16:13:2. The three simultaneous equations $\alpha+\beta=0$, $\alpha(a+r)-\beta r+\gamma=0$, and $(-\alpha c/r)-\gamma r=1$ that emerge may then be solved, leading eventually to

16:13:3 $$\mathrm{f}(x)=\frac{r}{r^2+ar-\dfrac{c}{r}}\left[\frac{1}{x-r}-\frac{x+a+2r}{x^2+(a+r)x-\dfrac{c}{r}}\right]=\frac{-r}{br+2c}\left[\frac{1}{x-r}-\frac{x+a+2r}{x^2+(a+r)x-\dfrac{c}{r}}\right]$$

The second equality is a consequence of $r^3+ar^2+br+c=0$. Further expansion of the final term into the sum of two reciprocal linear terms is possible, but these may be complex.

Partial fractionation is often used as a prelude to an operation of the calculus. It permits, for example, the integral of a reciprocal cubic function to be evaluated, via equation 16:13:3, with the aid of formulas 7:10:4, 15:10:4, and 15:10:5. It is used abundantly in Laplace inversion [Section 26:15].

16:14 RELATED TOPICS: the sliding cubic and the cubic spline

Technologists and engineers commonly collect extensive lists of values f of a function f(x) without knowing the form of the relationship between f(x) and its argument x. The table shows fragments of such a list. A frequent need is to present these data graphically, or use them to estimate a value of the function at an argument where no measurement was made. Two situations arise in this setting. In the first, the tabular data are regarded as exact and the problem is one of *interpolation*. In this case, the task is the selection of a relationship that is satisfied locally or globally, that relationship then being assumed to apply equally well between measurement points. In the second scenario, error is assumed to contaminate the f data and a (usually rather simple) relationship is sought that does not exactly reproduce the measured f_j values, but comes close. Such a procedure is known as *regression*; Section 7:14 is devoted to the simplest kind of regression, in which the function to which the data are fitted is a straight line, and the use of more complicated fitting functions is explored in Section 17:14.

x	f(x)
x_0	f_0
x_1	f_1
$\vdots$	$\vdots$
x_{j-1}	f_{j-1}
x_j	f_j
x_{j+1}	f_{j+1}
x_{j+2}	f_{j+2}
$\vdots$	$\vdots$
x_{J-1}	f_{J-1}
x_J	f_J

Polynomials are commonly used for both interpolation and regression. The remainder of this section addresses two ways in which *piecewise-cubic functions* are employed in interpolation. The first, which provides a satisfactory interpolation without undue complexity, is the *sliding cubic* or *Lagrange four-point interpolate*. The idea is that a cubic function is fitted so as to pass through a quartet of adjacent data pairs: (x_{j-1},f_{j-1}), (x_j,f_j), (x_{j+1},f_{j+1}), and (x_{j+2},f_{j+2}), but is used to represent the data only between the middle two points of the quartet. The cubic that has this property is, for $x_j\le x\le x_{j+1}$,

16:14:1 $$\hat{\mathrm{f}}_j(x)=\sum_{k-j=-1}^{+2}\frac{(x-x_l)(x-x_m)(x-x_n)}{(x_k-x_l)(x_k-x_m)(x_k-x_n)}f_k$$

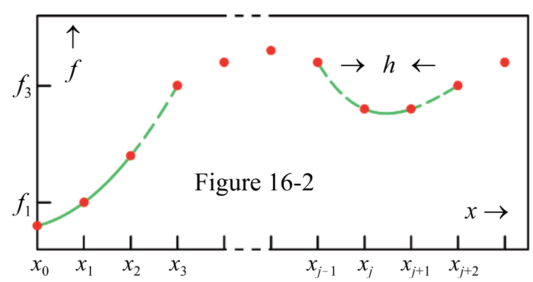

Figure 16-2

where l, m, and n are the three integers other than k from the set $(j-1, j, j+1, j+2)$. In the common case, illustrated in Figure 16-2, in which the data are evenly spaced so that $x_{j+2}-x_{j+1}=x_{j+1}-x_j=x_j-x_{j-1}=h$, equation 16:14:1 becomes

16:14:2 $$\hat{\mathrm{f}}_j(x)=a_{3j}\left(\frac{x-x_j}{h}\right)^3+a_{2j}\left(\frac{x-x_j}{h}\right)^2+a_{1j}\frac{x-x_j}{h}+a_{0j}$$

$$\begin{cases} a_{3j}=\frac{1}{6}f_{j+2}-\frac{1}{2}f_{j+1}+\frac{1}{2}f_j-\frac{1}{6}f_{j-1} \\ a_{2j}=\frac{1}{2}f_{j+1}-f_j+\frac{1}{2}f_{j-1} \\ a_{1j}=-\frac{1}{6}f_{j+2}+f_{j+1}-\frac{1}{2}f_j-\frac{1}{3}f_{j-1} \\ a_{0j}=f_j \end{cases}$$

This is the interpolating cubic fitted over $x_{j-1} \le x \le x_{j+2}$ but used only for the argument range from x_j to x_{j+1}, as shown at the right-hand side in Figure 16-2. As x reaches and passes x_{j+1}, the cubic "slides" to a next quartet. Of course, no quartet is available for the end regions $x_0 \le x \le x_1$ or $x_{J-1} \le x \le x_J$; and so the interpolating cubics from the penultimate internodal zones $x_1 \le x \le x_0$ and $x_{J-2} \le x \le x_{J-1}$ are taken to apply to the end zones too. This is illustrated at the left in Figure 16-2.

The curve produced by the sliding cubic interpolation is continuous, but there is a small (and often visually undetectable) discontinuity in slope at each node. This defect is overcome in the *cubic spline* which not only has no discontinuity in the slope (that is, in the first derivative of f) at the nodal points, but no discontinuity in the second derivative either! The equation describing the interpolated spline between the nodes x_j and x_{j+1} is the cubic function

16:14:3
$$\hat{\mathrm{f}}_j(x) = a_{3j}\left(\frac{x-x_j}{h}\right)^3 + a_{2j}\left(\frac{x-x_j}{h}\right)^2 + a_{1j}\frac{x-x_j}{h} + a_{0j} \qquad \begin{cases} a_{3j} = \frac{1}{6}g_{j+1} - \frac{1}{6}g_j \\ a_{2j} = \frac{1}{2}g_j \\ a_{1j} = f_{j+1} - f_j - \frac{1}{3}g_j - \frac{1}{6}g_{j+1} \\ a_{0j} = f_j \end{cases}$$

when the data are separated evenly by h. The g terms are proportional to the second derivatives of the spline at its nodes; for example

16:14:4
$$g_j = h^2 \left.\frac{\mathrm{d}^2\hat{\mathrm{f}}}{\mathrm{d}x^2}\right|_{x=x_j}$$

These terms are unknown a priori; however, the recursion formula

16:14:5
$$g_{j+1} = f_{j+1} - 2f_j + f_{j-1} - 4g_j - g_{j-1}$$

interrelates three consecutive g values. There are $J-1$ recursions linking the g_0, g_1, g_2, $\cdots$, g_{J-1}, g_J terms. These recursion equations may be solved simultaneously if g_0 and g_J are taken to be zero. Thereby a *natural cubic spline* may be created. A natural spline is one that is linear at its extremities. The description of splines given by Chapra and Canale [pages 495–505] is very readable and their book provides formulas for unequally spaced data. Hamming [Section 20.9], another excellent source, discusses "unnatural" splines and shows how to set up a tridiagonal matrix to solve the simultaneous equations.

There is a heavy computational burden in the creation of a cubic spline but the result is extremely smooth. Because the fitting is "global", there is a disconcerting dependence of the shape at one end of the fitted curve upon data at the other end. Such an effect is, of course, entirely absent in the sliding cubic and other "local" interpolations.