CHAPTER 15

THE QUADRATIC FUNCTION $ax^2 + bx + c$ AND ITS RECIPROCAL

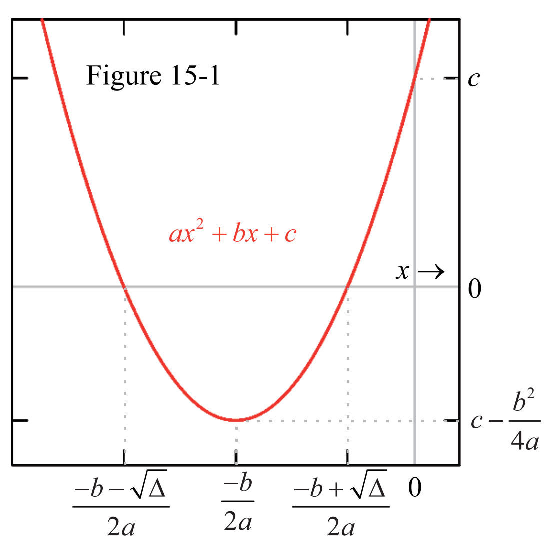
As a cartesian graph, the shape of $ax^2 + bx + c$ is that of a parabola and, in this respect, the quadratic function resembles the square-root function of Chapter 11. The root-quadratic function, addressed in Section 15:13 and 15:14, may also adopt the shape of a parabola.

15:1 NOTATION

The constants a, b, and c, that, together with the argument x, compose the quadratic function, are called *coefficients*. In the graphs of this chapter, a is taken to be positive, though the formulas are valid for either sign. The sign of the quantity

15:1:1 $$\Delta = b^2 - 4ac$$

known as the *discriminant* of the quadratic function, influences several of the function's properties. Some authors define the discriminant as the negative of the quantity specified in 15:1:1, as it was in the first edition of this *Atlas*.



15:2 BEHAVIOR

Irrespective of the values of its coefficients, the quadratic function adopts real values for any real argument; however, it has a limited range, extending (for positive a) only over $-\Delta/4a \le ax^2 + bx + c \le \infty$. At $x = -b/2a$, the function experiences an extremum: a minimum or a maximum according as a is positive or negative. It is the sign of the discriminant that determines whether the quadratic function adopts the value zero. In drawing Figure 15-1, both a and Δ are treated as positive, so that the quadratic function crosses the x-axis twice.

K.B. Oldham et al., *An Atlas of Functions, Second Edition*,
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The behavior of the reciprocal quadratic function is even more affected by the sign of the discriminant. If Δ is negative, the $1/(ax^2+bx+c)$ function is a contiguous function, adopting values between zero and $-4a/\Delta$, as illustrated in blue in Figure 15-2. However, when the discriminant is positive, the reciprocal quadratic function has the three branches, shown in red in the figure, with discontinuities at $x=(\sqrt{\Delta}-b)/2a$ and $x=-(\sqrt{\Delta}+b)/2a$.

Both the quadratic function and its reciprocal have mirror symmetry about the line $x=-b/2a$, irrespective of the value of the discriminant.

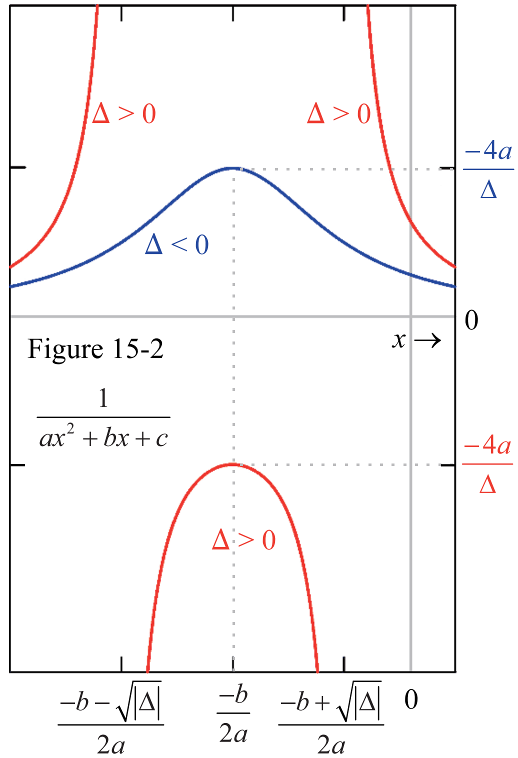


Figure 15-2

15:3 DEFINITIONS

Writing the quadratic functions as $c+x(b+ax)$ shows that the operations of multiplication and addition suffice to provide a definition. It may also be defined as the product of two linear functions:

15:3:1 $$ax^2+bx+c=\left(ax+\frac{b-\sqrt{\Delta}}{2}\right)\left(x+\frac{b+\sqrt{\Delta}}{2a}\right)$$

The cartesian graph of the $f=ax^2+bx+c$ function is a parabola with its *focus* at the point $(x, f) = (-b/2a,\ c+(b^2-1)/4a)$ and its *directrix* as the horizontal line $f=c-(b^2-1)/4a$. Thus the function may be defined by recourse to the definition of a parabola given in Section 11:3. Yet another definition is as the inverse function of a translated [Section 14:15] square-root function [Chapter 11]

15:3:2 $$\mathrm{F}(x)=ax^2+bx+c \qquad \mathrm{f}(x)=\sqrt{\frac{x}{a}+\frac{\Delta}{4a^2}}-\frac{b}{2a} \qquad \mathrm{F}\big(\mathrm{f}(x)\big)=\mathrm{f}\big(\mathrm{F}(x)\big)=x$$

If the discriminant is positive, the reciprocal quadratic function can be defined as the difference between two reciprocal linear functions [Chapter 7]:

15:3:3 $$\frac{1}{ax^2+bx+c}=\frac{1}{\sqrt{\Delta}\,x+\dfrac{b\sqrt{\Delta}-\Delta}{2a}}-\frac{1}{\sqrt{\Delta}\,x+\dfrac{b\sqrt{\Delta}+\Delta}{2a}} \qquad \Delta>0$$

15:4 SPECIAL CASES

A linear function [Chapter 7] is the special $a=0$ case of the quadratic function. When $b=2\sqrt{ac}$, so that the discriminant is zero, the quadratic function reduces to $[\sqrt{a}\,x+\sqrt{c}\,]^2$, a square function [Chapter 10].

When the discriminant is zero, the reciprocal quadratic function has the unusual property of encountering a infinite discontinuity of the $+\infty|+\infty$ type at $x=-\sqrt{c/a}$.

In the special case when $b^2=4(ac-\pi^2)$, equation 15:10:4 shows the total area under the $1/(ax^2+bx+\mathrm{c})$ curve to be unity. In this circumstance, one can rewrite the normalized formula as

15:4:1
$$\frac{1}{ax^2+bx+c}=\frac{\pi/a}{\pi\left[\{\pi/a\}^2+\{x+(b/2a)\}^2\right]}$$

This is the equation that describes a Lorenz distribution [see the table in Section 27:14], which is therefore a special case of the reciprocal quadratic function.

15:5 INTRARELATIONSHIPS

Both the quadratic function and its reciprocal obey the reflection formula

15:5:1
$$\mathrm{f}\left(x+\frac{b}{2a}\right)=\mathrm{f}\left(-x-\frac{b}{2a}\right)\qquad \mathrm{f}=ax^2+bx+c \quad\text{or}\quad \frac{1}{ax^2+bx+c}$$

The sum or difference of two quadratic functions is generally another quadratic function, while their product is invariably a quartic function [Section 16:13]. Provided that the discriminant, Δ, of the denominatorial function is positive, the quotient of two quadratic functions can be expressed in terms of a constant and two reciprocal linear functions, as follows:

15:5:2
$$\frac{a'x^2+b'x+c'}{ax^2+bx+c}=\frac{a'}{a}-\frac{a'r_+^2+b'r_++c'}{\sqrt{\Delta}(x-r_+)}+\frac{a'r_-^2+b'r_-+c'}{\sqrt{\Delta}(x-r_-)}\qquad r_\pm=\left(-b\pm\sqrt{\Delta}\right)/2a$$

where r_+ and r_- are the zeros [Section 15:7] of the denominatorial quadratic function.

15:6 EXPANSIONS

Trinomial expansions [Section 6:12] for the reciprocal quadratic function exist, though they are of limited utility:

15:6:1
$$\frac{1}{ax^2+bx+c}=\begin{cases}\dfrac{1}{c}-\dfrac{b}{c^2}x+\dfrac{b^2-ac}{c^3}x^2-\dfrac{b^3-2abc}{c^4}x^3+\dfrac{b^4-3ab^2c+a^2c^2}{c^5}x^4-\cdots=\dfrac{1}{c}\sum\limits_{j=0}^{\infty}\left(\dfrac{ax^2+bx}{-c}\right)^j\\[2ex] \dfrac{1}{ax^2}-\dfrac{b}{a^2x^3}+\dfrac{b^2-ac}{a^3x^4}-\dfrac{b^3-2abc}{a^4x^5}+\dfrac{b^4-3ab^2c+a^2c^2}{a^5x^6}-\cdots=\dfrac{1}{ax^2}\sum\limits_{j=0}^{\infty}\left(\dfrac{bx+c}{-ax^2}\right)^j\end{cases}$$

The first expansion is valid for small argument (that is, when $|ax^2+bx|<|c|$), the second for large argument (when $|ax^2|>|bx+c|$).

15:7 PARTICULAR VALUES

The zeros of the ax^2+bx+c function, and the discontinuities of its reciprocal, are given by the well-known formula

15:7:1
$$r_\pm=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

To preserve significance it is better (if b is positive), to calculate r_- first and then r_+ as c/ar_-. There is a double zero at $-b/2a$ if the discriminant vanishes, and the zeros are complex if the discriminant is negative. *Equator* treats the zeros as the quadrivariate function

15:7:2 $$\mathrm{r}_2(a,b,c,n)=\frac{-b+n\sqrt{b^2-4ac}}{2a}=\mathrm{Re}[\mathrm{r}_2(a,b,c,n)]+i\,\mathrm{Im}[\mathrm{r}_2(a,b,c,n)] \qquad n=-1,+1$$

and its quadratic zeros routine (keyword **r2**) outputs both the real and imaginary parts, the latter being 0 unless the discriminant is negative. By default, *Equator* generates a short table giving both zeros.

The following table is applicable whether the discriminant is positive or negative, but the a coefficient is assumed positive.

		$x=-\infty$	$x=\frac{-b-\sqrt{\lvert\Delta\rvert}}{2a}$	$x=\frac{-b}{2a}$	$x=0$	$x=\frac{-b+\sqrt{\lvert\Delta\rvert}}{2a}$	$x=\infty$
ax^2+bx+c	$\Delta>0$	$+\infty$	0	$\frac{-\Delta}{4a}$ (minimum)	c	0	$+\infty$
	$\Delta<0$	$+\infty$	$-\Delta/2a$		c	$-\Delta/2a$	$+\infty$
$\frac{1}{ax^2+bx+c}$	$\Delta>0$	0	$+\infty\vert-\infty$	$\frac{-4a}{\Delta}$ (maximum)	$1/c$	$-\infty\vert+\infty$	0
	$\Delta<0$	0	$-2a/\Delta$		$1/c$	$-2a/\Delta$	0

15:8 NUMERICAL VALUES

These are easily calculated, for example with *Equator*'s quadratic function routine (keyword **quadratic**).

15:9 LIMITS AND APPROXIMATIONS

Limiting expressions for the reciprocal quadratic function could be derived from 15:6:1, but they are seldom used.

15:10 OPERATIONS OF THE CALCULUS

The following formulas address the differentiation and integration of the quadratic function and its reciprocal

15:10:1 $$\frac{\mathrm{d}}{\mathrm{d}x}(ax^2+bx+c)=2ax+b$$

15:10:2 $$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{ax^2+bx+c}\right)=\frac{-2ax-b}{(ax^2+bx+c)^2}$$

15:10:3 $$\int_0^x(at^2+bt+c)\mathrm{d}t=\frac{2ax^3+3bx^2+6cx}{6}$$

15:10:4 $$\int_{-b/2a}^{x}\frac{1}{at^2+bt+c}\mathrm{d}t=\begin{cases}\dfrac{-2}{\sqrt{\Delta}}\mathrm{artanh}\left(\dfrac{2ax+b}{\sqrt{\Delta}}\right) & \Delta>0 \qquad x<\dfrac{\sqrt{\Delta}-b}{2a}\\[2ex] \dfrac{2}{\sqrt{-\Delta}}\arctan\left(\dfrac{2ax+b}{\sqrt{-\Delta}}\right) & \Delta<0\end{cases}$$

The 15:10:4 integral is infinite if $\Delta = 0$, but if the lower limit is changed to zero, it equals $2/(2ax+b)$. Another important integral is

15:10:5
$$\int_0^x \frac{t}{at^2+bt+c}\mathrm{d}t = \frac{1}{2a}\ln\left(\frac{ax^2+bx+c}{c}\right) - \begin{cases} \dfrac{b}{a\sqrt{\Delta}}\operatorname{artanh}\left(\dfrac{x\sqrt{\Delta}}{bx+2c}\right) & \Delta > 0 \\ \dfrac{b}{a\sqrt{-\Delta}}\arctan\left(\dfrac{x\sqrt{-\Delta}}{bx+2c}\right) & \Delta < 0 \end{cases}$$

and many related integrals of the general form $\int t^n(at^2+bt+c)^m\,\mathrm{d}t$, where n and m are integers, will be found listed by Gradshteyn and Ryzhik [Section 2.17].

The Laplace transform of the quadratic function is straightforward

15:10:6
$$\int_0^\infty (at^2+bt+c)\exp(-st)\,\mathrm{d}t = \mathfrak{L}\left\{at^2+bt+c\right\} = \frac{2a+bs+cs^2}{s^3}$$

but that of its reciprocal is elaborate

15:10:7
$$\mathfrak{L}\left\{\frac{1}{at^2+bt+c}\right\} = \begin{cases} \dfrac{\exp(b/2a)}{\sqrt{\Delta}}\left[\exp\left(\dfrac{\sqrt{\Delta}}{2a}\right)\mathrm{Ei}\left(\dfrac{\sqrt{\Delta}+b}{2a}\right) - \exp\left(\dfrac{-\sqrt{\Delta}}{2a}\right)\mathrm{Ei}\left(\dfrac{\sqrt{\Delta}-b}{2a}\right)\right] & \Delta > 0 \\ \dfrac{2}{b}+\dfrac{s}{a}\exp\left(\dfrac{bs}{2a}\right)\mathrm{Ei}\left(\dfrac{-bs}{2a}\right) & \Delta = 0 \\ \dfrac{2}{\sqrt{-\Delta}}\left[\left\{\dfrac{\pi}{2}-\mathrm{Si}\left(\dfrac{\sqrt{-\Delta}}{2a}s\right)\right\}\cos\left(\dfrac{\sqrt{-\Delta}}{2a}s\right)+\mathrm{Ci}\left(\dfrac{\sqrt{-\Delta}}{2a}s\right)\sin\left(\dfrac{\sqrt{-\Delta}}{2a}s\right)\right] & \Delta < 0 \end{cases}$$

and involves functions from Chapters 26, 32, 36, and 37.

15:11 COMPLEX ARGUMENT

The real and imaginary parts of the quadratic function and its reciprocal when the argument is $z = x+iy$ are

15:11:1
$$az^2+bz+c = \left[a(x^2-y^2)+bx+c\right]+i\left[2axy+by\right]$$

and

15:11:2
$$\frac{1}{az^2+bz+c} = \frac{a(x^2-y^2)+bx+c}{\left[a(x^2+y^2)+bx+c\right]^2+y^2\Delta} - i\frac{2axy+by}{\left[a(x^2+y^2)+bx+c\right]^2+y^2\Delta}$$

Important inverse Laplace transforms include

15:11:3
$$\mathfrak{I}\left\{\frac{1}{as^2+bs+c}\right\} = \frac{\exp(r_+t)-\exp(r_-t)}{a(r_+-r_-)} = \begin{cases} (2/\sqrt{\Delta})\exp(-bt/2a)\sinh(\sqrt{\Delta}t/2a) & \Delta > 0 \\ (t/a)\exp(-bt/2a) & \Delta = 0 \\ (2/\sqrt{-\Delta})\exp(-bt/2a)\sin(\sqrt{-\Delta}t/2a) & \Delta < 0 \end{cases}$$

where, $r_\pm$ are given in 15:7:1 and, as before, $\Delta = b^2-4ac$.

15:12 GENERALIZATIONS

A quadratic function is a member of all the polynomial families [Chapters 17–24].

15:13 COGNATE FUNCTION: the root-quadratic function

The *root-quadratic function* $\sqrt{ax^2+bx+c}$ and its reciprocal are functions of some importance; they are clearly generalizations of the functions addressed in Chapters 11, 13, and 14. Some unifying properties of the root-quadratic function are presented in Section 15:15.

Several valuable integrals involving the reciprocal root-quadratic function are

15:13:1
$$\int_{-b/2a}^{x}\frac{1}{\sqrt{at^2+bt+c}}\mathrm{d}t=\begin{cases}\dfrac{-1}{\sqrt{-a}}\operatorname{arcsin}\left(\dfrac{b+2ax}{\sqrt{\Delta}}\right) & \Delta>0 \quad a<0 \quad \dfrac{b-\sqrt{\Delta}}{-2a}\le x\le\dfrac{b+\sqrt{\Delta}}{-2a}\\[2ex] \dfrac{1}{\sqrt{a}}\operatorname{arsinh}\left(\dfrac{2ax+b}{\sqrt{-\Delta}}\right) & \Delta<0 \quad a>0\end{cases}$$

15:13:2
$$\int_{x_0}^{x}\frac{1}{\sqrt{at^2+bt+c}}\mathrm{d}t=\frac{1}{\sqrt{a}}\operatorname{arcosh}\left(\frac{2ax+b}{\sqrt{\Delta}}\right)\qquad \Delta>0 \qquad a>0 \qquad x_0=\frac{\sqrt{\Delta}-b}{2a}$$

15:13:3
$$\int_{-2c/b}^{x}\frac{1}{t\sqrt{at^2+bt+c}}\mathrm{d}t=\begin{cases}\dfrac{1}{\sqrt{-c}}\operatorname{arcsin}\left(\dfrac{bx+2c}{x\sqrt{\Delta}}\right) & \Delta>0 \quad c<0 \quad \dfrac{-2c}{b+\sqrt{\Delta}}\le x\le\dfrac{-2c}{b-\sqrt{\Delta}}\\[2ex] \dfrac{-1}{\sqrt{c}}\operatorname{arsinh}\left(\dfrac{bx+2c}{x\sqrt{-\Delta}}\right) & \Delta<0 \quad c>0\end{cases}$$

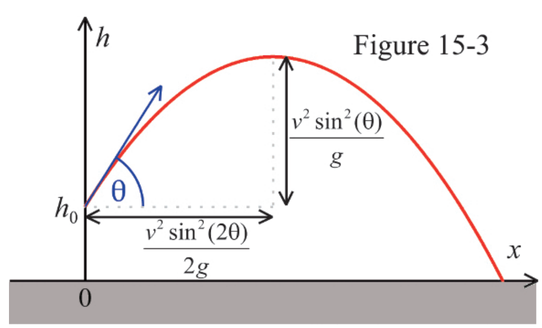
and

15:13:4
$$\int_{x_0}^{x}\frac{1}{t\sqrt{at^2+bt+c}}\mathrm{d}t=\frac{-1}{\sqrt{c}}\operatorname{arcosh}\left(\frac{bx+2c}{x\sqrt{\Delta}}\right)\qquad \Delta>0 \qquad c>0 \qquad x_0=\frac{2c}{\sqrt{\Delta}-b}$$

Others are given by Gradshteyn and Ryzhik [Section 2.26] and by Jeffrey [Section 4.3.4].

15:14 RELATED TOPIC: the trajectory of a projectile

Heavy projectiles journey through the air following a parabolic course that is best described by a quadratic function. Neglecting the effect of air resistance, the object travels with a constant speed in the horizontal direction, while experiencing a constant acceleration (or force) vertically downwards. If the projectile is launched from a height h_0 with an initial velocity v at an angle θ to the horizontal, the equation describing its trajectory gives its height h, at a distance x downrange, as



15:14:1
$$h=\frac{-g\sec^2(\theta)}{2v^2}x^2+x\tan(\theta)+h_0$$

where g is the gravitational acceleration [see Appendix, Section A:6]. As Figure 15-3 will confirm, the greatest height is attained by launching at θ = π/2, but the greatest range requires that θ = π/4. The projectile remains airborne for the time interval

15:14:2 $$\frac{v}{g}\left[\sin(\theta)+\sqrt{\frac{2gh_0}{v^2}+\sin^2(\theta)}\right]$$

Multiply this expression by $v\cos(\theta)$ to find the total range.

15:15 RELATED TOPIC: conic sections

Figure 15-4 shows that there is a unity between the geometries of the functions discussed in Chapters 11, 13, and 14 that is not apparent when these geometries – those of the horizontal parabola, ellipse, and hyperbola – are described by the canonical formulations used in their respective chapters. However, if the curves are moved along the x-axis, so that one of the foci falls at $x = 0$, the three horizontal geometries come to be described by the single root-quadratic equation

15:15:1 $$\mathrm{f}(x)=\left[f_0^{\,2}+2kf_0x+(k^2-1)x^2\right]^{1/2}$$

Whereas three *different* equations are normally used to describe the ellipse, the parabola and the hyperbola, this

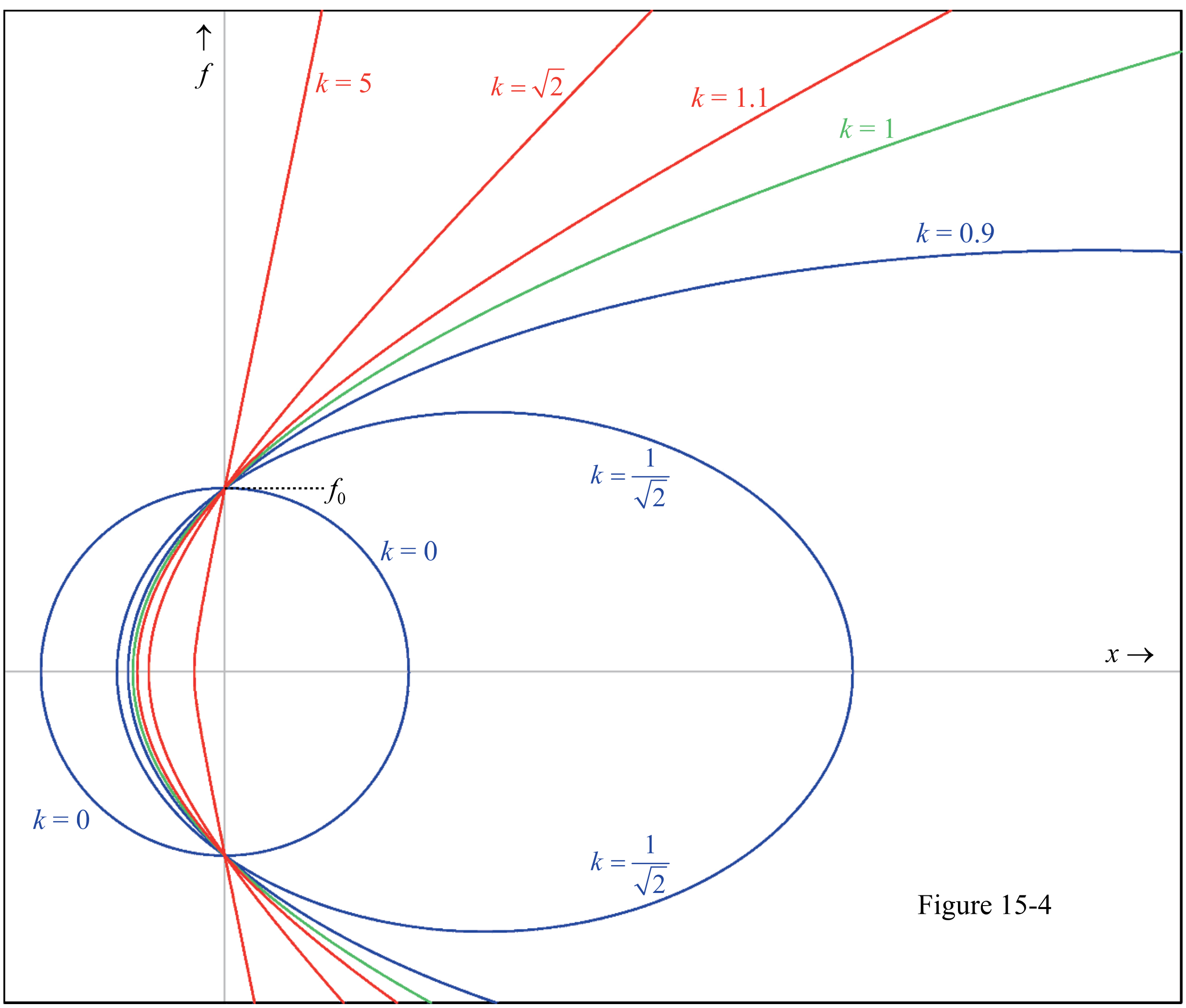


Figure 15-4

equation accommodates all three! If the eccentricity k lies between 0 and 1, equation 15:15:1 describes an ellipse; if $k = 1$, the equation is that of a parabola; and if k exceeds 1, a hyperbola is represented. The quantity f_0 is the value of the function at $x = 0$, that is, at the focus. Figure 15-4 was drawn using a range of k values, but a single f_0. The circle is the $k = 0$ case and $k = \sqrt{2}$ corresponds to a rectangular hyperbola [Section 14:4]. The $k = 1$ parabola separates the elliptical curves from those corresponding to hyperbolas (for clarity only one branch of each hyperbola is shown, though the equation describes both). For large k, the hyperbola is virtually a pair of straight lines, which is evident from equation 15:15:1 because, when k is so large that $k^2 - 1 \approx k^2$, the equation becomes $\mathrm{f}(x) \approx \pm(kx + f_0)$.

Collectively, all these curves are called *conic sections*, or simply *conics*, because each can be generated by intersecting a cone with an appropriately oriented plane. More formally, they are known as *curves of the second degree*. Equation 15:15:1 can be considered the defining equation of any *horizontal conic*. Conics possess certain features in common. With the exception of the parabola, they each have two axes of mirror symmetry: one is the x-axis, the other being the line $x = kf_0/(1-k^2)$. In general they have two foci, with an interfocal separation of $2kf_0/(1-k^2)$, but this is zero for the circle and infinite for the parabola. Both foci lie on the x-axis with one at $x = 0$. In the context of Figure 15-4, the second focus of the ellipses lies to the right of the origin whereas it lies to the left for the hyperbolas.

By rewriting equation 15:15:1 as the square root of the product of two linear functions

15:15:2 $$\mathrm{f}(x) = \left[\{f_0 + (k-1)x\}\{f_0 + (k+1)x\}\right]^{1/2}$$

one may identify the domain of the real function as

15:15:3 $$\frac{-f_0}{1+k} \le x \le \frac{f_0}{1-k} \qquad \text{when} \qquad 0 \le k \le 1$$

15:15:4 $$-\infty \le x \le \frac{-f_0}{k-1} \quad \text{and} \quad \frac{-f_0}{k+1} \le x \le +\infty \qquad \text{when} \qquad k \ge 1$$

This conforms with the property that the ellipse is a contiguous curve, whereas each hyperbola has two branches (the left-hand branches are not shown in Figure 15-4).

Whereas equation 15:15:1 serves as a definition only of horizontal conics, there is a geometric definition that applies to a conic anywhere in the cartesian plane. Let F be a point in Figure 15-5 that will serve as a focus of the conic, and DD″ be a straight line, called the *directrix*, positioned anywhere in the plane and with any orientation. The conic is uniquely defined once the locations of the point and the line are selected, and a nonnegative constant k is chosen. Then the conic is defined as the locus of all points P such that

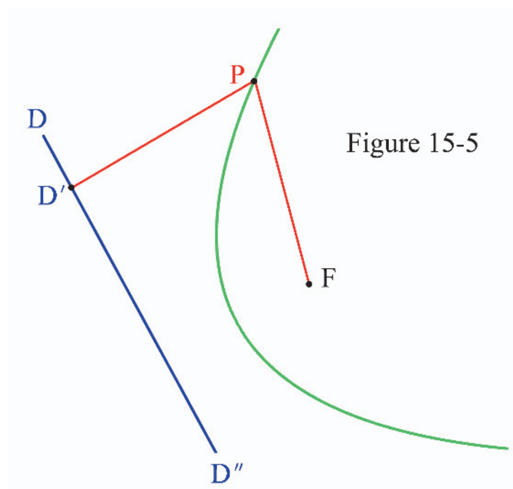


Figure 15-5

15:15:5 $$\frac{\mathrm{PF}}{\mathrm{PD'}} = k$$

where D′ is the nearest point on the directrix to P. The constant k is, of course, the *eccentricity*, so that

15:15:6 $$\text{If } \frac{\mathrm{PF}}{\mathrm{PD'}} \begin{cases} < 1, & \text{the conic is an ellipse} \\ = 1, & \text{the conic is a parabola} \\ > 1, & \text{the conic is a hyperbola} \end{cases}$$

So this simple property serves as a definition of all three types of curve. If the conic obeys equation 15:15:1, and the focus in question is that positioned at the origin, then the equation of the directrix is $x = -f_0/k$. Since, apart from the special cases of the circle and the parabola, each conic has two foci, so it has two directrices.