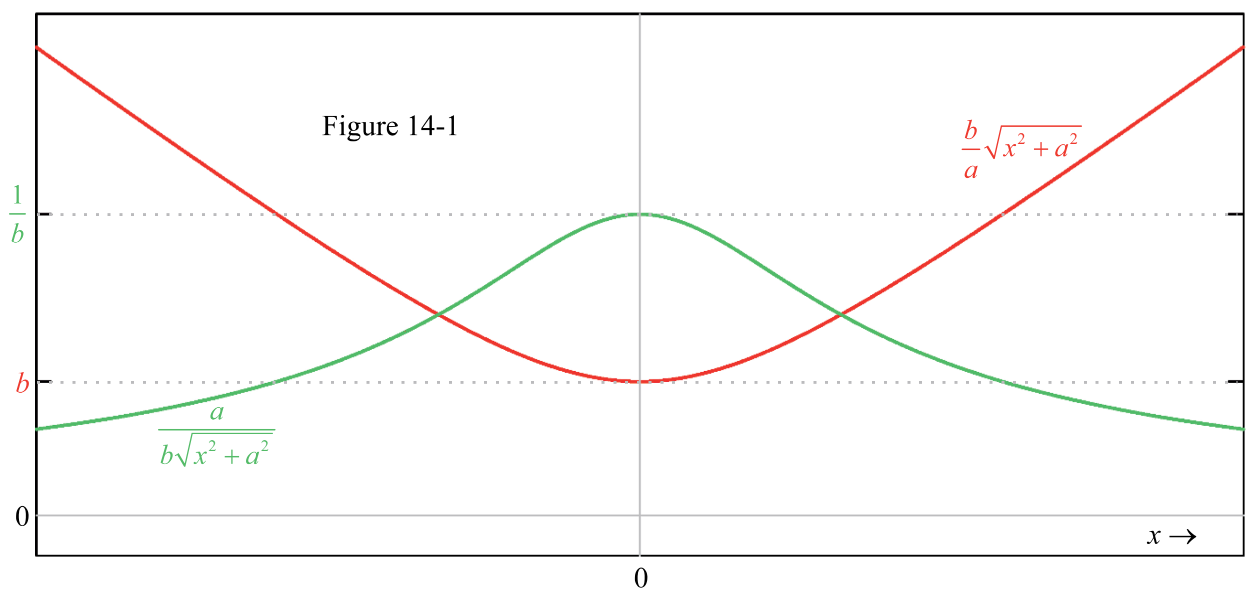
CHAPTER
14

THE SEMIHYPERBOLIC FUNCTIONS $(b/a)\sqrt{x^2 \pm a^2}$ AND THEIR RECIPROCALS

The functions of this chapter are closely associated with the geometry of the hyperbola, a topic addressed in Section 14:14. The graphical representations of the $(b/a)\sqrt{x^2+a^2}$ and $(b/a)\sqrt{x^2-a^2}$ functions are interconvertible by scaling and rotation operations; these, and other operations, are the subject of Section 14:15.

14:1 NOTATION

The $(b/a)\sqrt{x^2+a^2}$ function, shown in Figure 14-1, corresponds to one-half of a hyperbola, and the $(b/a)\sqrt{x^2-a^2}$ function, illustrated in Figure 14-2, corresponds to two-quarters of a different hyperbola. For this reason, these two functions are called *semihyperbolic functions*. The constants a and b are the *parameters*; they are



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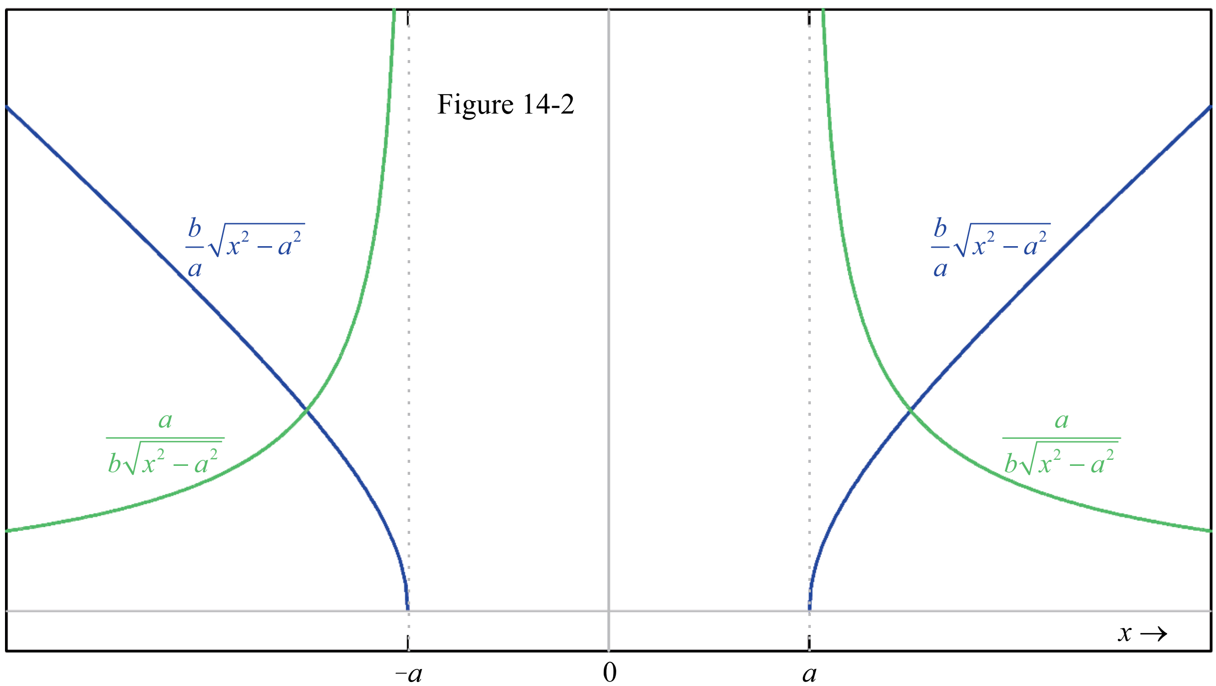
regarded as positive throughout the chapter. The adjectives "vertical" and "horizontal" will be used to distinguish between the $(b/a)\sqrt{x^2+a^2}$ and $(b/a)\sqrt{x^2-a^2}$ functions, in recognition of their graphical orientation. These two functions are said to be *conjugates* of each other.

14:2 BEHAVIOR

The red curve in Figure 14-1 depicts a typical vertical semihyperbolic function. It accepts any argument x; its range lies between b and ∞. The reciprocal vertical semihyperbolic function, shown in green in the same figure, is also defined for all x; it adopts values between zero and $1/b$.

The behaviors of the horizontal semihyperbolic function and its reciprocal are evident in Figure 14-2. Neither of these functions adopts real values in the $-a < x < a$ gap. Outside this forbidden zone, both functions adopt positive values ranging between zero and infinity.



14:3 DEFINITIONS

The algebraic operations of squaring [Chapter 10] and taking the square root [Chapter 11], together with arithmetic operations, fully define both varieties of semihyperbolic function and their reciprocals.

One way of defining a horizontal semihyperbolic function is as the product of two closely related square-root functions [Chapter 11]:

14:3:1 $$\sqrt{\frac{bx}{a}+b}\sqrt{\frac{bx}{a}-b} = \frac{b}{a}\sqrt{x^2-a^2}$$

but no corresponding definition (from real functions) exists for the vertical version.

The semihyperbolic functions, of both the vertical and horizontal varieties, are expansible hypergeometrically [Section 18:14], as in equations 14:6:1 and 14:6:3. The same is true of the reciprocal semihyperbolic functions, whose expansions are given in 14:16:2 and 14:16:3. These expansions open the way to definition via synthesis [Section 43:14] from simpler functions.

A parametric definition [Section 0:3] of the vertical semihyperbolic function is in terms of the hyperbolic sine and cosine functions [Chapter 28]:

14:3:2 $$\mathrm{f} = b\cosh(t), \quad x = a\sinh(t): \qquad \mathrm{f}(x) = \frac{b}{a}\sqrt{x^2 + a^2}$$

The roles are reversed for the horizontal version

14:3:3 $$\mathrm{f} = b\sinh(t), \quad x = a\cosh(t): \qquad \mathrm{f}(x) = \frac{b}{a}\sqrt{x^2 - a^2}$$

A hyperbola, and hence the semihyperbolic functions, may be defined geometrically in two distinct ways. One of these is detailed in Section 14:14, the other in Section 15:15.

14:4 SPECIAL CASES

When $b = a$, the horizontal and vertical semihyperbolic functions become $\sqrt{x^2 - a^2}$ and $\sqrt{x^2 + a^2}$ respectively. Shown in Figure 14-3, they are termed *rectangular semihyperbolic functions*. The horizontal rectangular semihyperbolic function may be transformed into its vertical cohort on rotation about the origin by an angle of π/2. This can be established by setting θ = π/2 in the formulas [Section 14:15]

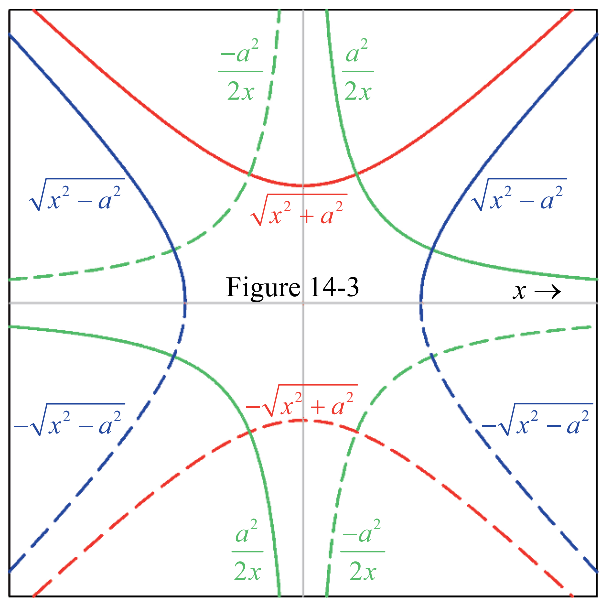


Figure 14-3

14:4:1 $$x_n = x_o\cos(\theta) - f_o\sin(\theta)$$
$$f_n = f_o\cos(\theta) + x_o\sin(\theta)$$

for rotation counterclockwise through an angle θ about the origin. Here the subscript "$_o$" denotes an old (pre-rotation) coordinate, whereas "$_n$" signifies the new (post-rotation) equivalent. More interesting than rotation by a right-angle, however, is the effect of rotation by an angle of π/4 applied to the horizontal rectangular semihyperbolic function, $f_o = \sqrt{x_o^2 - a^2}$. Then, because $\cos(\pi/4) = \sin(\pi/4) = 1/\sqrt{2}$, one finds

14:4:2 $$\left.\begin{aligned} x_n &= \frac{x_o - f_o}{\sqrt{2}} \\ f_n &= \frac{f_o + x_o}{\sqrt{2}} \end{aligned}\right\} x_n f_n = \frac{x_o^2 - f_o^2}{2} = \frac{a^2}{2} \qquad \text{whence} \qquad f_n = \frac{a^2}{2x_n}$$

Thus the rotated function, which could be considered a *diagonal semihyperbolic function*, is a special case of a reciprocal linear function, as in Chapter 7, or an integer power function [Chapter 10]. A further rotation of $a^2/2x$ by π/4 produces $\sqrt{x^2 + a^2}$. Still more rotations by 45° lead successively to the various functions shown in Figure 14-3, all of which are branches of rectangular semihyperbolic functions.

14:5 INTRARELATIONSHIPS

Semihyperbolic functions are even functions, as are their reciprocals

14:5:1 $$\mathrm{f}(-x) = \mathrm{f}(x) \qquad \mathrm{f}(x) = \frac{b}{a}\sqrt{x^2 \pm a^2} \quad \text{or} \quad \frac{a}{b\sqrt{x^2 \pm a^2}}$$

Multiplication of the argument of a semihyperbolic function f(x) by a constant leads to another semihyperbolic function

14:5:2 $$\mathrm{f}(\nu x) = \frac{b}{a}\sqrt{(\nu x)^2 \pm a^2} = \frac{b}{(a/\nu)}\sqrt{x^2 \pm (a/\nu)^2}$$

the b parameter being unaffected. In Section 14:15, this is termed an "argument scaling operation".

Apart from an interchange of the a and b parameters, the inverse function [Section 0:3] of a semihyperbolic function is its conjugate

14:5:3 $$\mathrm{F}(x) = \frac{a}{b}\sqrt{x^2 \mp b^2} \qquad \text{where} \qquad \mathrm{F}\big(\mathrm{f}(x)\big) = x \quad \text{and} \quad \mathrm{f}(x) = \frac{b}{a}\sqrt{x^2 \pm a^2}$$

with interchanged parameters.

14:6 EXPANSIONS

The horizontal semihyperbolic function may be expanded binomially

14:6:1 $$\frac{b}{a}\sqrt{x^2 - a^2} = b\left[\frac{x}{a} - \frac{a}{2x} - \frac{a^3}{8x^3} - \frac{a^5}{16x^5} - \frac{5a^7}{128x^7} - \cdots\right] = b\sum_{j=0}^{\infty}(-)^j\binom{\frac{1}{2}}{j}\frac{a^{2j-1}}{x^{2j-1}} = \frac{bx}{a}\sum_{j=0}^{\infty}\frac{(\frac{-1}{2})_j}{(1)_j}\left(\frac{x^2}{a^2}\right)^{-j}$$

Of course, this expansion is invalid in the region $|x| < a$, where the real function does not exist. There are several alternative ways of expressing the coefficients of such series, in addition to the binomial coefficient [Chapter 6] or Pochhammer polynomials [Chapter 18] employed here. The similar expansion of the reciprocal horizontal semihyperbolic function

14:6:2 $$\frac{a}{b\sqrt{x^2 - a^2}} = \frac{1}{b}\left[\frac{a}{x} + \frac{a^3}{2x^3} + \frac{3a^5}{8x^5} + \frac{5a^7}{16x^7} + \frac{35a^9}{128x^9} + \cdots\right] = \frac{1}{b}\sum_{j=0}^{\infty}(-)^j\binom{\frac{-1}{2}}{j}\frac{a^{1+2j}}{x^{1+2j}} = \frac{a}{bx}\sum_{j=0}^{\infty}\frac{(\frac{1}{2})_j}{(1)_j}\left(\frac{x^2}{a^2}\right)^{-j}$$

is again restricted to $-a < x < a$. However, for the vertical semihyperbolic function, and its reciprocal, there are no restrictions because alternative binomial expansions exist. The first version of each equation below is applicable when $|x| \le a$, the second when $|x| > a$.

14:6:3 $$\frac{b}{a}\sqrt{x^2 + a^2} = \begin{cases} b\left[1 + \frac{x^2}{2a^2} - \frac{x^4}{8a^4} + \frac{x^6}{16a^6} - \frac{5x^8}{128a^8} + \frac{7x^{10}}{256a^{10}} - \cdots\right] = b\sum_{j=0}^{\infty}\binom{\frac{1}{2}}{j}\frac{x^{2j}}{a^{2j}} = b\sum_{j=0}^{\infty}\frac{(\frac{-1}{2})_j}{(1)_j}\left(\frac{-x^2}{a^2}\right)^{j} \\ b\left[\frac{x}{a} + \frac{a}{2x} - \frac{a^3}{8x^3} + \frac{a^5}{16x^5} - \frac{5a^7}{128x^7} + \frac{7a^9}{256x^9} - \cdots\right] = b\sum_{j=0}^{\infty}\binom{\frac{1}{2}}{j}\frac{a^{2j-1}}{x^{2j-1}} = \frac{bx}{a}\sum_{j=0}^{\infty}\frac{(\frac{-1}{2})_j}{(1)_j}\left(\frac{-x^2}{a^2}\right)^{-j} \end{cases}$$

14:6:4
$$\frac{a}{b\sqrt{x^2+a^2}} = \begin{cases} \frac{1}{b}\left[1-\frac{x^2}{2a^2}+\frac{3x^4}{8a^4}-\frac{5x^6}{16a^6}+\frac{35x^8}{128a^8}-\frac{63x^{10}}{256a^{10}}+\cdots\right] = \frac{1}{b}\sum_{j=0}^{\infty}\binom{-\frac{1}{2}}{j}\frac{x^{2j}}{a^{2j}} = \frac{1}{b}\sum_{j=0}^{\infty}\frac{(\frac{1}{2})_j}{(1)_j}\left(\frac{-x^2}{a^2}\right)^j \\ \frac{1}{b}\left[\frac{a}{x}-\frac{a^3}{2x^3}+\frac{3a^5}{8x^5}-\frac{5a^7}{16x^7}+\frac{35a^9}{128x^9}-\frac{63a^{11}}{256x^{11}}+\cdots\right] = \frac{1}{b}\sum_{j=0}^{\infty}\binom{-\frac{1}{2}}{j}\frac{a^{2j+1}}{x^{2j+1}} = \frac{a}{bx}\sum_{j=0}^{\infty}\frac{(\frac{1}{2})_j}{(1)_j}\left(\frac{-x^2}{a^2}\right)^{-j} \end{cases}$$

More rapidly convergent series may result when hyperbolic functions are substituted

14:6:5
$$\frac{b}{a}\sqrt{x^2-a^2} = b\left[t+\frac{t^3}{3!}+\frac{t^5}{5!}+\frac{t^7}{7!}+\cdots\right]$$

14:6:6
$$\frac{a}{b\sqrt{x^2-a^2}} = \frac{1}{b}\left[\frac{1}{t}-\frac{t}{6}+\frac{7t^3}{360}-\frac{31t^5}{15120}+\cdots\right] \qquad x = a\cosh(t)$$

14:6:7
$$\frac{b}{a}\sqrt{x^2+a^2} = b\left[1+\frac{t^2}{2!}+\frac{t^4}{4!}+\frac{t^6}{6!}+\cdots\right]$$

14:6:8
$$\frac{a}{b\sqrt{x^2+a^2}} = \frac{1}{b}\left[1-\frac{t^2}{2}+\frac{5t^4}{24}-\frac{61t^6}{720}+\cdots\right] \qquad x = a\sinh(t)$$

See Chapter 28 and 29 for the bases of these equations.

14:7 PARTICULAR VALUES

	$x=-\infty$	$x=-\sqrt{a^2+b^2}$	$x=-a$	$x=0$	$x=a$	$x=\sqrt{a^2+b^2}$	$x=\infty$
$\frac{b}{a}\sqrt{x^2-a^2}$	$+\infty$	$\frac{b^2}{a}$	0	undef	0	$\frac{b^2}{a}$	$+\infty$
$\frac{b}{a}\sqrt{x^2+a^2}$	$+\infty$	$\frac{b\sqrt{2a^2+b^2}}{a}$	$\sqrt{2}\,b$	b	$\sqrt{2}\,b$	$\frac{b\sqrt{2a^2+b^2}}{a}$	$+\infty$

14:8 NUMERICAL VALUES

These are readily calculated, for example by *Equator*'s x^ν power function routine (keyword **power**) with $\nu = \pm 1/2$, after the variable construction feature [Appendix, Section C:4] is first used with $w = b^2/a^2$, $p = 2$, and $k = \pm b^2$.

14:9 LIMITS AND APPROXIMATIONS

Both semihyperbolic functions approach the linear functions $x = \pm bx/a$ as $x \to \pm\infty$. These lines are known as the *asymptotes* of the corresponding hyperbolas. Specifically

14:9:1
$$\left.\begin{matrix} \lim\limits_{x\to\infty}\left\{(b/a)\sqrt{x^2 \pm a^2}\right\} \\ \lim\limits_{x\to-\infty}\left\{(-b/a)\sqrt{x^2 \pm a^2}\right\} \end{matrix}\right\} = \frac{bx}{a}$$

and

14:9:2
$$\left.\begin{matrix} \lim\limits_{x\to-\infty}\left\{(b/a)\sqrt{x^2 \pm a^2}\right\} \\ \lim\limits_{x\to\infty}\left\{(-b/a)\sqrt{x^2 \pm a^2}\right\} \end{matrix}\right\} = \frac{-bx}{a}$$

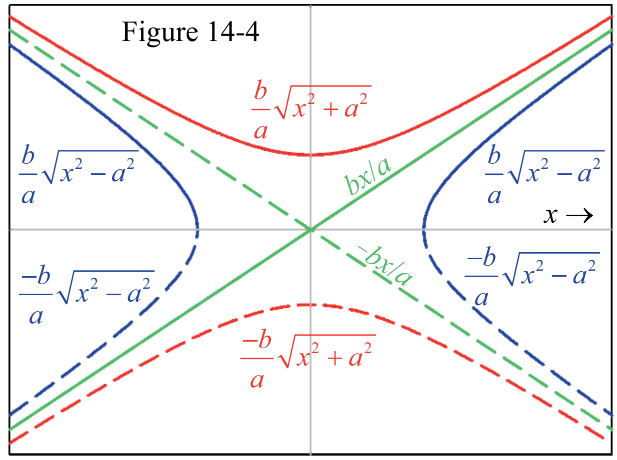


Figure 14-4 shows these asymptotes and also nicely illustrates the relationships between the four semihyperbolic functions diagrammed there.

Correspondingly, as $x \to \pm\infty$, the reciprocal semihyperbolic functions approach zero as reciprocal linear functions $\pm a/bx$.

Near its positive apex, $x = a$, the horizontal semihyperbolic function approximates a square-root function:

14:9:3
$$\frac{b}{a}\sqrt{x^2 - a^2} \approx \sqrt{\frac{2b^2}{a}}\sqrt{x-a} \qquad x-a \text{ small}$$

14:10 OPERATIONS OF THE CALCULUS

The b/a multiplier will be omitted in this section. Formulas for differentiation and integration are

14:10:1
$$\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{x^2 \pm a^2} = \frac{x}{\sqrt{x^2 \pm a^2}}$$

14:10:2
$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{\sqrt{x^2 \pm a^2}} = \frac{-x}{\sqrt{\left(x^2 \pm a^2\right)^3}}$$

14:10:3
$$\int_0^x \sqrt{t^2 + a^2}\,\mathrm{d}t = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\mathrm{arsinh}\left(\frac{x}{a}\right)$$

14:10:4
$$\int_a^x \sqrt{t^2 - a^2}\,\mathrm{d}t = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\mathrm{arcosh}\left(\frac{x}{a}\right)$$

14:10:5
$$\int_0^x \frac{1}{\sqrt{t^2 + a^2}}\mathrm{d}t = \mathrm{arsinh}\left(\frac{x}{a}\right)$$

14:10:6
$$\int_a^x \frac{1}{\sqrt{t^2 - a^2}}\mathrm{d}t = \mathrm{arcosh}\left(\frac{x}{a}\right)$$

The last two integrals serve as definitions of the inverse hyperbolic sine and cosine functions [Chapter 31]. A long list of indefinite integrals of the form $\int t^n(t^2 \pm a^2)^{m+½}\mathrm{d}t$, where n and m are integers, will be found in Gradshteyn and Ryzhik [Section 2.27]; one example, generating a function from Chapter 35, is

14:10:7 $$\int_a^x \frac{\sqrt{t^2-a^2}}{t}\,\mathrm{d}t = \sqrt{x^2-a^2} - a\,\mathrm{arcsec}\left(\frac{x}{a}\right)$$

Pages 219 − 284 of the same book are an invaluable source of hundreds of results, such as that required in14:14:6, for the integration of functions involving several terms of the form $\sqrt{\pm x \pm c}$.

The Laplace transform of the reciprocal vertical semihyperbolic function is given by

14:10:8 $$\int_0^\infty \frac{1}{\sqrt{t^2+a^2}} \exp(-st)\mathrm{d}t = \mathfrak{L}\left\{\frac{1}{\sqrt{t^2+a^2}}\right\} = \frac{\pi}{2}\left[\mathrm{h}_0(as) - \mathrm{Y}_0(as)\right]$$

where the functions appearing in the transform are the Struve function [Chapter 57] and the Neumann function [Chapter 54] of zero order.

14:11 COMPLEX ARGUMENT

With imaginary argument, a horizontal semihyperbolic function becomes an imaginary vertical semihyperbolic function

14:11:1 $$(b/a)\sqrt{(iy)^2 - a^2} = (ib/a)\sqrt{y^2+a^2}$$

The converse is true only in part, because the vertical semihyperbolic function becomes a real semielliptic function [Chapter 13] only for a range of magnitudes of its imaginary argument

14:11:2 $$(b/a)\sqrt{(iy)^2+a^2} = \begin{cases} (b/a)\sqrt{a^2-y^2} & |y| < a \\ (ib/a)\sqrt{y^2-a^2} & |y| > a \end{cases}$$

With $z = x + iy$, the real and imaginary parts of the vertical semihyperbolic function of complex argument are given by

14:11:3 $$\frac{b}{a}\sqrt{z^2+a^2} = \frac{b}{\sqrt{2}a}\sqrt{x^2-y^2+a^2+\sqrt{A+B}} + \frac{ib\,\mathrm{sgn}(xy)}{\sqrt{2}a}\sqrt{\sqrt{A+B}-x^2+y^2-a^2}$$

where $A = a^4 + (x^2+y^2)^2$ and $B = 2a^2(x^2-y^2)$; sgn is the signum function [Chapter 8] equal to ±1 according to the sign of its argument, or to zero if its argument is zero. The corresponding formula for the horizontal semihyperbolic function of complex argument is

14:11:4 $$\frac{b}{a}\sqrt{z^2-a^2} = \frac{b}{\sqrt{2}a}\sqrt{\sqrt{A-B}+x^2-y^2-a^2} + \frac{ib\,\mathrm{sgn}(xy)}{\sqrt{2}a}\sqrt{-x^2+y^2+a^2+\sqrt{A-B}}$$

14:12 GENERALIZATIONS

Semihyperbolic functions are instances of the root-quadratic function discussed in Section 15:13. They are also conic sections [Section 15:15].

14:13 COGNATE FUNCTIONS

For n = 3, 4, 5, ⋯, the functions $(b/a)\left[x^n \pm a^n\right]^{1/n}$ have shapes very similar to hyperbolas, especially if n is even. The straight line $\mathrm{f}(x) = bx/a$ is an asymptote for all these functions, as is $\mathrm{f}(x) = -bx/a$ if n is even.

14:14 RELATED TOPIC: geometry of the hyperbola

There are two distinct geometric definitions of a hyperbola, one of which is addressed in Section 15:15. The second, illustrated in Figure 14-5, is based on two points, F and F′, each of which is termed a *focus* of the hyperbola. A hyperbola is defined as the locus of all points P such that the distance from P to the more remote focus exceeds that to the nearer focus by a constant:

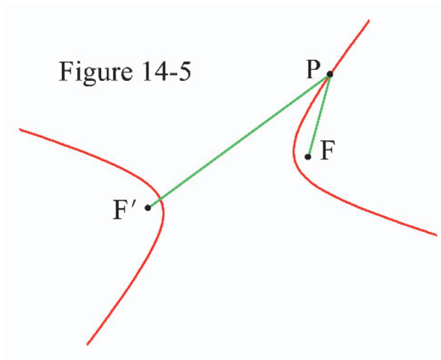


Figure 14-5

14:14:1 $$|\mathrm{PF}' - \mathrm{PF}| = \text{a constant} = 2a$$

The eccentricity k of the hyperbola, which necessarily exceeds unity and equals $\sqrt{2}$ for a rectangular hyperbola, is defined as

14:14:2 $$\frac{|\mathrm{FF}'|}{|\mathrm{PF}' - \mathrm{PF}|} = k > 1$$

where FF′ is the *interfocal separation*, the distance between the two foci, equal to $2ka$. The two parameters of the hyperbola (sometimes called its *semiaxes*) are a, defined in 14:14:1 and b, given by

14:14:3 $$b = a\sqrt{k^2 - 1} \qquad \text{whence} \qquad k = \frac{\sqrt{a^2 + b^2}}{a}$$

The b parameter may have a magnitude smaller than, equal to, or greater than a. As Figure 14-5 shows, the hyperbola has two branches, separated from each other (by $2a$ at their closest approach). The definition in this paragraph covers both branches equally.

If the two foci are equidistant from the origin and on a line perpendicular to the x-axis through the origin, then the equations

14:14:4 $$\mathrm{f}(x) = \frac{b}{a}\sqrt{x^2 + a^2} \qquad \text{and} \qquad \mathrm{f}(x) = \frac{-b}{a}\sqrt{x^2 + a^2}$$

describe the upper and lower branches of the hyperbola respectively. This is the hyperbola that we call a vertical hyperbola. In view of the distinction [Section 12:1] between the symbols $\sqrt{t}$ and $t^{½}$, the vertical hyperbola in its entirety is described by $(b/a)[x^2 + a^2]^{½}$.

If the two foci lie on the x-axis, equidistant from the origin, then the hyperbola is described as a horizontal hyperbola and it is described by the formula $(b/a)[x^2 - a^2]^{½}$. The upper half of *each* branch of this hyperbola is described by $(b/a)\sqrt{x^2 - a^2}$ while the lower half of each branch is covered by the formula $(-b/a)\sqrt{x^2 - a^2}$.

Conjugate hyperbolas, that is, vertical and horizontal hyperbolas sharing the same a and b parameters, also share the same asymptotes.

The area enclosed by the rightmost branch of a horizontal hyperbola, $\pm(b/a)\sqrt{x^2 - a^2}$ and the ordinate $\mathrm{f}(x) = x$, as illustrated in Figure 14-6, may be evaluated by recourse to integral 14:10:4 and is

14:14:5 $$\begin{matrix}\text{shaded}\\ \text{area}\end{matrix} = 2\int_a^x \frac{b}{a}\sqrt{t^2 - a^2}\,\mathrm{d}t = \frac{bx}{a}\sqrt{x^2 - a^2} - ab\,\mathrm{arcosh}\left(\frac{x}{a}\right)$$

The curved perimeter of the shaded area bounded by graphs of the function $\mathrm{f}(x) = \pm(b/a)\sqrt{x^2 - a^2}$ has a length given in terms of incomplete elliptic integrals F and E [Chapter 62] by

14:14:6 $$2\int_a^x \sqrt{1 + \left(\frac{\mathrm{d}\mathrm{f}}{\mathrm{d}t}\right)^2}\,\mathrm{d}t = 2k\int_a^x \sqrt{\frac{t^2 - (a/k)^2}{t^2 - a^2}}\,\mathrm{d}t = 2ka\left[\frac{k^2 - 1}{k^2}\mathrm{F}\left(\frac{1}{k}, \varphi\right) - \mathrm{E}\left(\frac{1}{k}, \varphi\right) + \frac{x}{a}\sin(\varphi)\right]$$

where $k = \sqrt{a^2 + b^2}/a$ and $\varphi = \arctan\{(k/b)\sqrt{x^2 - a^2}\}$.

Also shown in Figure 14-6 are the asymptotes OS and OT of the hyperbola and the tangent MN to the hyperbolic branch at an arbitrary point P (M and N being the points at which the tangent meets the asymptotes, as depicted in Figure 14-6). Two remarkable properties of the hyperbola are that P bisects the line MN, so that MP = PN, and that the area of the triangle MNO equals $|ab|$ independent of the position of P on the hyperbola.

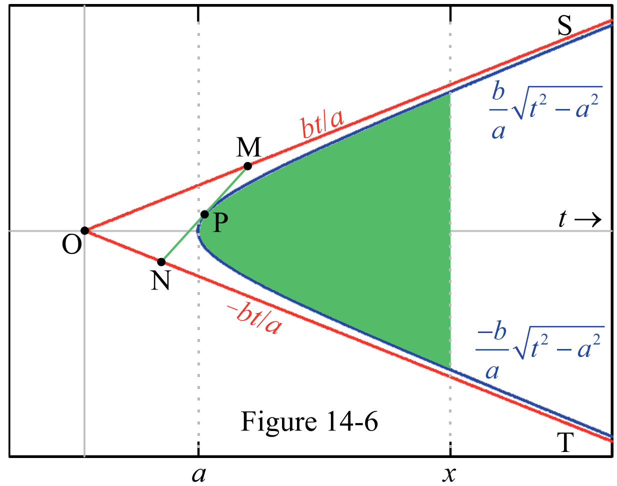


Figure 14-6

14:15 RELATED TOPIC: graphical operations

Because two- or three-dimensional graphs are generally helpful in appreciating the properties of functions, many are scattered throughout this *Atlas*.

For a univariate function f(x), the common graphical representation is as a cartesian graph in which the argument x and the value f of the function at that argument serve as the rectangular coordinates (x, f); Figures 14-1 and 14-2 are examples. Beyond mere visualization, graphs can be useful in revealing relationships between functions; for example, in Section 14:4 it is shown how an operation – rotation about the origin in that case – could convert a rectangular semihyperbolic function into the simpler $a^2/2x$ function. In this section we catalog five operations that change the shape, the location, or the orientation of a graph, and show how this affects the formula of the function. The original function (x_o, f_o) transforms to a new function (x_n, f_n) on subjection to some specified operation. Note that the axes are treated as fixed; it is the function that changes. Figure 14-7 shows a fragment of a representative function, in black. In each of five other colors is shown the result of a specified operation.

Perhaps the simplest operation is *scaling*, of which there are two versions. In *function scaling*, all function values are multiplied by a *scaling factor*, here λ. The equations describing function scaling are $x_n = x_o$ and $f_n = \lambda f o$. The result is a function that has been altered by expansion or contraction of its vertical dimension by a factor of λ, as illustrated in red in Figure 14-7, for the λ = 2 case. There is also *argument scaling* in which it is x that is multiplied by a scaling factor v, leading to $f_n = \mathrm{f}(vx_o)$. This stretches or compresses the curve horizontally, but is not illustrated in Figure 14-7.

Translation affects the location of a function without changing its shape, size or orientation. The equation pair

14:15:1 $$x_n = x_o + x_P \quad \text{and} \quad f_n = f_o + f_P$$

describes the operation. Here (x_P, f_P) are the coordinates of the point P which corresponds, in the new location, to the old origin, as illustrated in blue in Figure 14-7 for the $x_P = 3, f_P = 1$ case.

By *rotation about a point* R is meant that every point on the original graph retains its original distance from point R, but the line joining the two points rotates counterclockwise through an angle θ. The transformation equations are

14:15:2 $$x_n = x_R - (x_R - x_o)\cos(\theta) + (f_R - f_o)\sin(\theta) \quad \text{and} \quad f_n = f_R - (f_R - f_o)\cos(\theta) - (x_R - x_o)\sin(\theta)$$

For the point $(x_R, f_R) = (-5/4, 5/4)$ and θ = 135°, the result of this operation is shown in green. Commonly, point R is the origin, in which case $x_R = f_R = 0$ and the equation pair 14:15:3 reduces to equations 14:4:1. The operations described in this section may be applied sequentially; one application of this concept establishes a relationship between the vertical semihyperbolic function and the horizontal semihyperbolic function:

14:15:3 $$\frac{b}{a}\sqrt{x^2-a^2} \xrightarrow[\lambda=a/b]{\text{scale with}} \sqrt{x^2-a^2} \xrightarrow[\text{with } \theta=\pi/2]{\text{rotate about origin}} \sqrt{x^2+a^2} \xrightarrow[\lambda=b/a]{\text{scale with}} \frac{b}{a}\sqrt{x^2+a^2}$$

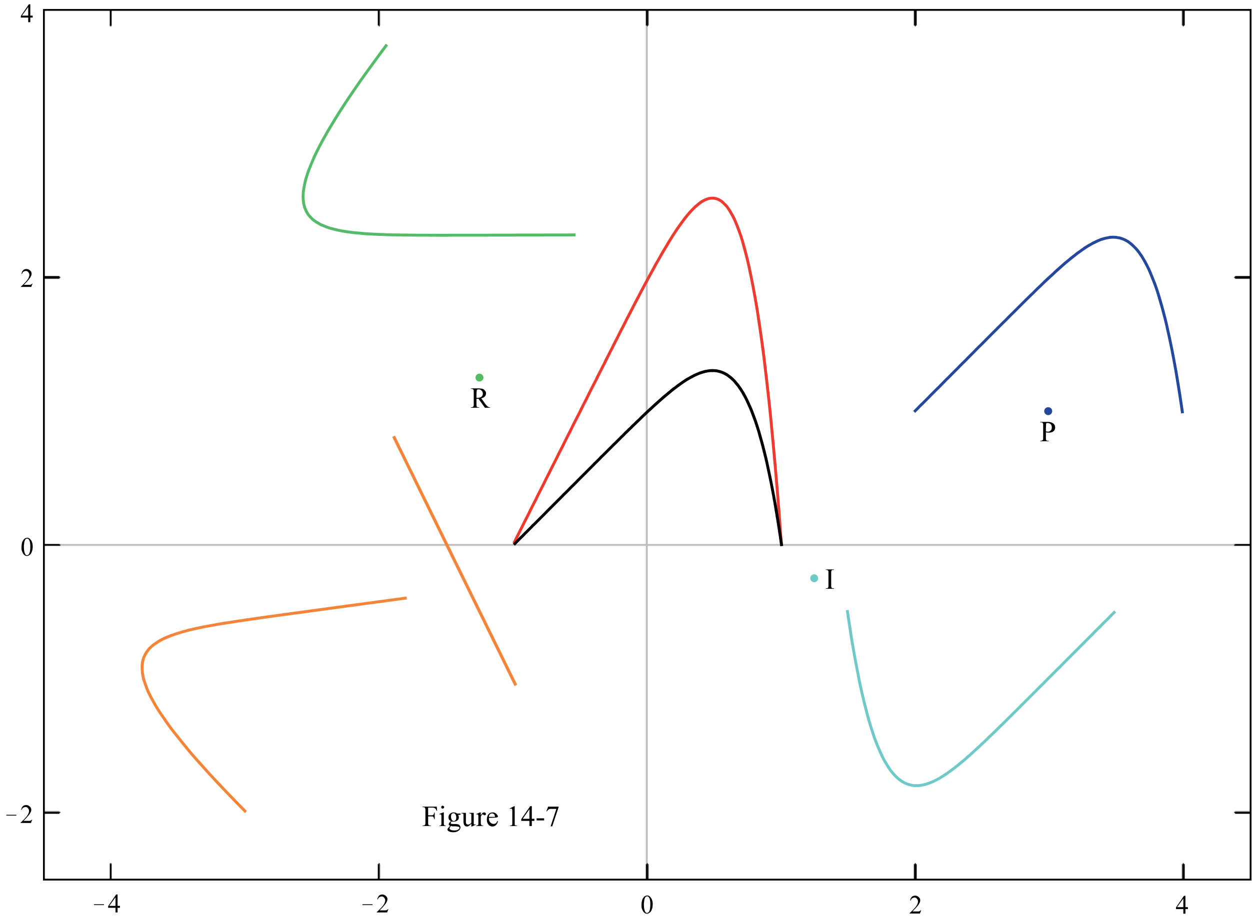


Figure 14-7

Reflection in the line bx+c implies that, from each point on the original graph, a perpendicular is dropped onto the line and then extrapolated to a new point that is as far from the line as was the line from the original point. The formulas governing this operation, are

14:15:4 $$x_n = \frac{(1-b^2)x_o + 2b(f_o - c)}{1+b^2} \quad \text{and} \quad f_n = \frac{(b^2-1)f_o + 2bx_o + 2c}{b^2+1}$$

With the line $f = -2x-3$ serving as the "mirror", the transformation is illustrated in orange. Reflection in the line $x = 0$ simply alters the sign of the argument, $(x_n, f_n) = (-x_o, f_o)$; a characteristic of *even* functions is that reflection in the line $x = 0$ leaves the function unchanged. The property of being unaffected by reflection is termed *mirror symmetry*. Reflection in the line $f = x$ causes an interchange of the function's value with its argument, $(x_n, f_n) = (f_o, x_o)$; that is, it generates the *inverse function* [Section 0:3].

The final operation that will be mentioned is named *inversion* though, confusingly, this is unconnected with inverse functions. *Inversion through a point* I means constructing the straight line that joins each point on the original graph to I, extrapolating this line and then creating a new point on the extrapolate an equal distance beyond. The formulas

14:15:5 $$x_n = 2x_I - x_o \quad \text{and} \quad f_n = 2f_I - f_o$$

describe the operation of inversion, in the present sense. The result of an inversion through point $(x_I, f_I) = (5/4, -1/4)$ is shown in turquoise in Figure 14-7. Inversion through the origin changes the sign of both coordinates $(x_n, f_n) = (-x_o, -f_o)$; a characteristic of *odd* functions is that they are unchanged by inversion through the origin. *Inversion symmetry* is the name given to the property of being unaffected by inversion.