CHAPTER 13

THE SEMIELLIPTIC FUNCTION $(b/a)\sqrt{a^2-x^2}$ AND ITS RECIPROCAL

The $(b/a)\sqrt{a^2-x^2}$ function is closely associated with the geometry of the ellipse, which is addressed in Section 13:14. The semicircle corresponds to the special $b = a$ instance and its geometry is the subject of Section 13:15. Whenever the b/a multiplier is of no importance, as in Sections 13:6, 13:10, and 13:11, it is omitted.

13:1 NOTATION

With $|x| < |a|$, a cartesian graph of the function pair $(\pm b/a)\sqrt{a^2-x^2}$ versus x is an ellipse and therefore *semielliptic function* is an appropriate name for $(b/a)\sqrt{a^2-x^2}$, with *semicircular function* being apposite for $\sqrt{a^2-x^2}$. Following the convention explained in Section 11:1, the notation $(b/a)[a^2-x^2]^{½}$ is equivalent to $(\pm b/a)\sqrt{a^2-x^2}$.

The parameters a and b, both positive, are the *semiaxes*, the larger being the *major semiaxis* (or "semimajor axis") and the smaller the *minor semiaxis*. Primarily, concern will be for the case in which $a \geq b$ and the ellipse to which this relates will be termed a *horizontal ellipse*, on account of the orientation of its major axis when the function is graphed. Conversely, when $b > a$, we speak of a *vertical ellipse*. The ratio, b/a, of the semiaxes of a horizontal ellipse is represented by k' in discussions of the elliptic family of functions [Chapters 61−63]. The quantity known as the *eccentricity*, or sometimes as the *ellipticity*, is

13:1:1
$$k = \sqrt{1-(k')^2} = \frac{\sqrt{a^2-b^2}}{a}$$

Eccentricities lie in the range $0 \leq k < 1$ for horizontal ellipses and for the functions that describe them. Zero eccentricity corresponds to a semicircle. Eccentricities of unity or more correspond to other functions and the entire class – the conic sections – is addressed in Section 15:15.

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13:2 BEHAVIOR

The functions $(b/a)\sqrt{a^2-x^2}$ and $a/\left(b\sqrt{a^2-x^2}\right)$ acquire real values only in the domain $-a \le x \le a$ of their argument. The range of the semielliptic function is between zero and b, whereas its reciprocal lies between $1/b$ and infinity. Figure 13-1 shows one graph of the reciprocal $a/\left(b\sqrt{a^2-x^2}\right)$ and two graphs of $(b/a)\sqrt{a^2-x^2}$, one of the latter corresponding to a vertical semiellipse with b greater than a, and the other to the more canonical horizontal semiellipse with b less than a.

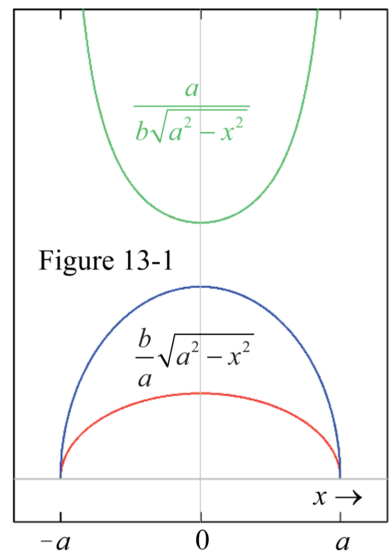


Figure 13-1

13:3 DEFINITIONS

The algebraic operations of squaring [Chapter 10] and taking the square root [Chapter 11], together with arithmetic operations, fully define the semielliptic function and its reciprocal.

Multiplying two related square-root functions [Chapter 11] is another route to the definition of a semielliptic function:

13:3:1
$$\sqrt{\frac{b}{a}x+b}\sqrt{\frac{-b}{a}x+b}=\frac{b}{a}\sqrt{a^2-x^2}$$

A parametric definition of the $\text{f}(x)=f=(\pm b/a)\sqrt{a^2-x^2}$ function pair is in terms of two trigonometric functions [Chapter 32]:

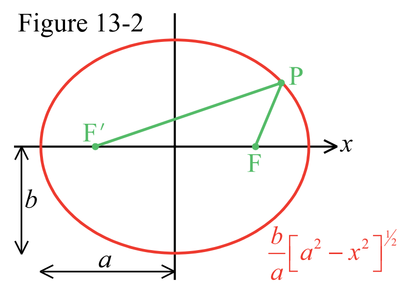
13:3:2
$$f=b\sin(t), \quad x=a\cos(t)$$

An ellipse may be defined geometrically in two distinct ways. One of these is explained in Section 15:15; the other is illustrated in Figure 13-2. The ellipse is defined as the locus of all points P such that the sum of the distances from P to two fixed points F and F′ obeys the simple relationship

13:3:3
$$\text{PF}+\text{PF}'=\text{a constant}$$

Each of the fixed points is termed a *focus* of the ellipse. The *interfocal separation*, the distance FF′ between the two foci, must, of course, be less than the constant in 13:3:3. If both foci lie on the x-axis, equidistant from the origin, then the ellipse is our standard horizontal ellipse, the foci have the coordinates $\left(\pm\sqrt{a^2-b^2},0\right)$, the interfocal separation is $2ka$, and the constant in equation 13:3:3 is $2a$.

Figure 13-2



13:4 SPECIAL CASES

When $b=a$, the semielliptical function becomes the semicircular function $\sqrt{a^2-x^2}$ and the common value of the two semiaxes is known as the *radius*. The geometrical definition of a circle is as the locus of all points lying at a constant distance a from a fixed point, the *center* of the circle. Other geometric properties of the semicircle are described in Section 13:15.

As $b \rightarrow 0$, the ellipse degenerates towards a straight line segment of length $2a$.

13:5 INTRARELATIONSHIPS

The $\mathrm{f}(x)=(b/a)\sqrt{a^2-x^2}$ function is an even function, $\mathrm{f}(-x)=\mathrm{f}(x)$, as is its reciprocal. The formula

13:5:1
$$\mathrm{f}(vx)=\frac{b}{(v/a)}\sqrt{\left(\frac{a}{v}\right)^2-x^2}$$

shows that the multiplication of the argument by a constant creates another semielliptic function, one semiaxis being rescaled, the other remaining unchanged. Multiplying the argument by $v=a/b$ creates a semicircle of radius b.

The inverse function [Section 0:3] of the horizontal $(b/a)\sqrt{a^2-x^2}$ function is the vertical $(a/b)\sqrt{b^2-x^2}$ function. These two semielliptic functions are sometimes said to be *conjugates* of each other.

13:6 EXPANSIONS

A binomial expansion of the semicircular function

13:6:1
$$\sqrt{a^2-x^2}=a\left[1-\frac{x^2}{2a^2}-\frac{x^4}{8a^4}-\frac{x^6}{16a^6}-\frac{5x^8}{128a^8}-\frac{7x^{10}}{256a^{10}}-\cdots\right]=a\sum_{j=0}^{\infty}\binom{\frac{1}{2}}{j}\left(\frac{-x^2}{a^2}\right)^j=a\sum_{j=0}^{\infty}\frac{(\frac{-1}{2})_j}{(1)_j}\left(\frac{x^2}{a^2}\right)^j$$

is valid provided $-a\le x\le a$. Each coefficient in the series is expressible either as a binomial coefficient [Chapter 6] or as a ratio of Pochhammer polynomials [Chapter 18], or indeed in several other ways. The similar expansion of the reciprocal semicircular function

13:6:2
$$\frac{1}{\sqrt{a^2-x^2}}=\frac{1}{a}\left[1+\frac{x^2}{2a^2}+\frac{3x^4}{8a^4}+\frac{5x^6}{16a^6}+\frac{35x^8}{128a^8}+\frac{63x^8}{256a^8}+\cdots\right]=\frac{1}{a}\sum_{j=0}^{\infty}\binom{\frac{-1}{2}}{j}\left(\frac{-x^2}{a^2}\right)^j=\frac{1}{a}\sum_{j=0}^{\infty}\frac{(\frac{1}{2})_j}{(1)_j}\left(\frac{x^2}{a^2}\right)^j$$

is restricted to $-a<x<a$.

Trigonometric substitution creates the series [see equations 32:6:2 and 33:6:2]

13:6:3
$$\sqrt{a^2-x^2}=a\sin(\theta)=a\left[\theta-\frac{\theta^3}{3!}+\frac{\theta^5}{5!}-\frac{\theta^7}{7!}+\cdots\right]$$

13:6:4
$$\frac{1}{\sqrt{a^2-x^2}}=\frac{\csc(\theta)}{a}=\frac{1}{a}\left[\frac{1}{\theta}+\frac{\theta}{6}+\frac{7\theta^3}{360}+\frac{31\theta^5}{15120}+\cdots\right]$$

$$\theta=\arccos\left(\frac{x}{a}\right)$$

See Section 21:15 for expansion of the semielliptic function and its reciprocal in terms of Legendre polynomials.

13:7 PARTICULAR VALUES

The *a* and *b* parameters are positive in the following table:

	$x=-a$	$x=0$	$x=\sqrt{a^2-b^2}$	$x=a$
$\frac{b}{a}\sqrt{a^2-x^2}$	0	b	$\frac{b^2}{a}$	0
$\frac{a}{b\sqrt{a^2-x^2}}$	∞	$\frac{1}{b}$	$\frac{a}{b^2}$	∞

13:8 NUMERICAL VALUES

It is straightforward to calculate values of the semielliptic function and its reciprocal. For example, *Equator*'s power function routine (keyword **power**) may be used with $v=\pm0.5$ after first using the variable construction feature [Appendix, Section C:4] with $k=b^2$, $w=-b^2/a^2$, and $p=2$ to adjust the argument.

13:9 LIMITS AND APPROXIMATIONS

As x approaches a, the semielliptic function comes to approximate a square-root function,

13:9:1 $$\frac{b}{a}\sqrt{a^2-x^2}\approx b\sqrt{\frac{2(a-x)}{a}} \qquad x\to a>0$$

The limiting form as x approaches $-a$ is $b\sqrt{2(a+x)/a}$.

13:10 OPERATIONS OF THE CALCULUS

Throughout this section, the argument x is restricted to a range between $-a$ and a, a being positive. The following formulas describe differentiation and indefinite integration:

13:10:1 $$\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{a^2-x^2}=\frac{-x}{\sqrt{a^2-x^2}}$$

13:10:2 $$\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{\sqrt{a^2-x^2}}=\frac{x}{\sqrt{\left(a^2-x^2\right)^3}}$$

13:10:3 $$\int_0^x\sqrt{a^2-t^2}\,\mathrm{d}t=\frac{x\sqrt{a^2-x^2}}{2}+\frac{a^2}{2}\arcsin\left(\frac{x}{a}\right)$$

13:10:4 $$\int_0^x\frac{\mathrm{d}t}{\sqrt{a^2-t^2}}=\arcsin\left(\frac{x}{a}\right)$$

13:10:5 $$\int_0^x t\sqrt{a^2-t^2}\,\mathrm{d}t=\frac{a^3-\sqrt{(a^2-x^2)^3}}{3}$$

and

13:10:6 $$\int_x^a\frac{1}{t\sqrt{a^2-t^2}}\mathrm{d}t=\frac{1}{a}\mathrm{arsech}\left(\frac{x}{a}\right) \qquad 0<x<a$$

The last two integrals are simple examples of a general class of indefinite integral $\int t^n\left(\sqrt{a^2-t^2}\right)^m\mathrm{d}t$, where n is any integer and m is an odd integer. Such integrals evaluate to algebraic expressions, or may contain an arcsin [Chapter 35] or arsech [Chapter 31] term. Gradshteyn and Ryzhik [Section 2.27] list more than fifty such integrals, including some general formulas.

Among other important integrals are

13:10:7
$$\int_0^x \frac{\mathrm{d}t}{\sqrt{a^2-t^2}\sqrt{a^2-k^2t^2}} = \frac{1}{a}\mathrm{F}\left\{k,\ \arcsin\left(\frac{x}{a}\right)\right\}$$

and

13:10:8
$$\int_0^x \frac{\sqrt{a^2-k^2t^2}}{\sqrt{a^2-t^2}}\mathrm{d}t = a\,\mathrm{E}\left\{k,\ \arcsin\left(\frac{x}{a}\right)\right\}$$

which serve as definitions of the incomplete elliptic integrals [Chapter 62] of the first and second kinds.

Semidifferentiation or semiintegration [Section 12:14], with lower limit zero, leads to:

13:10:9
$$\frac{\mathrm{d}^{1/2}}{\mathrm{d}x^{1/2}}\sqrt{a^2-x^2} = \sqrt{\frac{a}{2\pi}}\left[2\mathrm{E}(k,\varphi) - \mathrm{F}(k,\varphi) - \frac{2x-a}{\sqrt{2ax}}\right]$$

13:10:10
$$\frac{\mathrm{d}^{1/2}}{\mathrm{d}x^{1/2}}\frac{1}{\sqrt{a^2-x^2}} = \frac{\sqrt{a/2\pi}}{a+x}\left[\frac{2\mathrm{E}(k,\varphi)}{a-x} - \frac{\mathrm{F}(k,\varphi)}{a} + \sqrt{\frac{2}{ax}}\right]$$

13:10:11
$$\frac{\mathrm{d}^{-1/2}}{\mathrm{d}x^{-1/2}}\sqrt{a^2-x^2} = \sqrt{\frac{8a}{\pi}}\frac{x}{3}\left[2\mathrm{E}(k,\varphi) + \left(\frac{a}{x}-1\right)\mathrm{F}(k,\varphi) - \frac{2x-a}{\sqrt{2ax}}\right]$$

13:10:12
$$\frac{\mathrm{d}^{-1/2}}{\mathrm{d}x^{-1/2}}\frac{1}{\sqrt{a^2-x^2}} = \sqrt{\frac{2}{\pi a}}\mathrm{F}(k,\varphi)$$

$$0<x<a \qquad k=\sqrt{\frac{a+x}{2a}} \qquad \varphi = \arcsin\left(\sqrt{\frac{2x}{a+x}}\right)$$

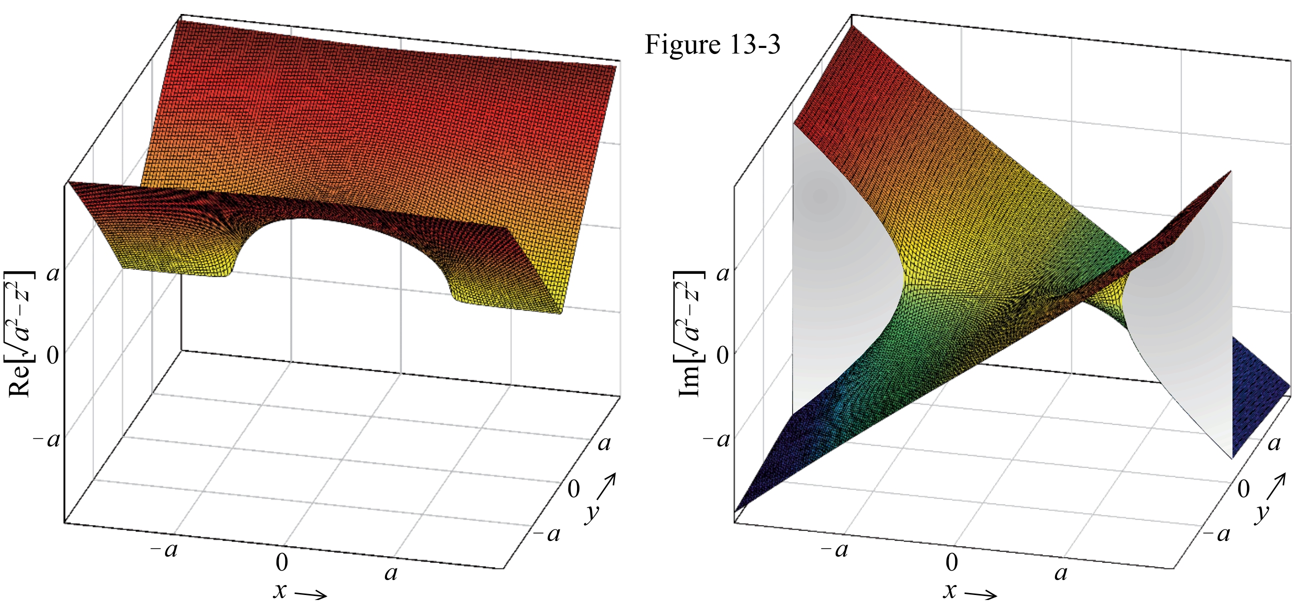
13:11 COMPLEX ARGUMENT

Figure 13-3 shows the real and imaginary components of the semicircular function of complex argument:

13:11:1
$$\sqrt{a^2-(x+iy)^2} = \sqrt{\frac{A+a^2-x^2+y^2}{2}} - i\,\mathrm{sgn}(xy)\sqrt{\frac{A-a^2+x^2-y^2}{2}}$$

where $A=\sqrt{(a^2+x^2+y^2)^2-4a^2x^2}$ and sgn is the signum function [Chapter 8].

Figure 13-3

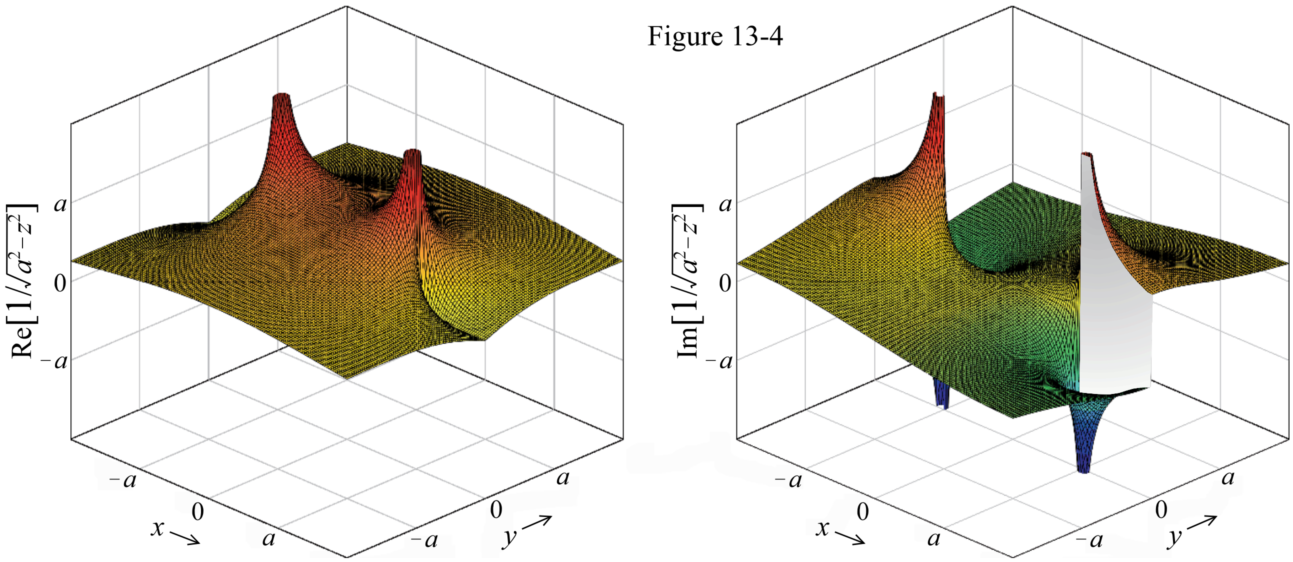


The corresponding parts of the reciprocal semicircular function are

13:11:2 $$\frac{1}{\sqrt{a^2-(x+iy)^2}}=\sqrt{\frac{A+a^2-x^2+y^2}{2A^2}}+i\,\text{sgn}(xy)\sqrt{\frac{A-a^2+x^2-y^2}{2A^2}}$$

as illustrated in Figure 13-4. Note the poles on the real axis at $x=\pm a$.

Figure 13-4



If the argument is purely imaginary, the semielliptic function becomes real

13:11:3 $$(b/a)\sqrt{a^2-(iy)^2}=(b/a)\sqrt{y^2+a^2}$$

and corresponds to a vertical semihyperbolic function, as described in the following chapter.

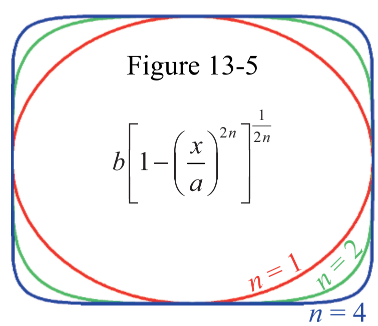
13:12 GENERALIZATIONS

The *root-quadratic function* $\sqrt{ax^2+bx+c}$ is a generalization of the semielliptic function. See Section 15:15 for the conditions under which the root-quadratic function becomes semielliptic.

The elliptic function may be regarded as the special $n=1$ case of the more general function

13:12:1 $$b\left[1-\left(\frac{x}{a}\right)^{2n}\right]^{\frac{1}{2n}} \qquad n=1,2,3,\cdots$$

The curves obtained by plotting these functions, the n = 1, 2, and 4 cases of which are included in Figure 13-5, have been called *superellipses*. As $n\to\infty$ the curve approaches a rectangle.



13:13 COGNATE FUNCTIONS

The functions of Chapter 14 are related to the semielliptic function in essentially the same way that the functions of Chapters 28−31 are related to those of Chapters 32−35. That is, members of one group of functions can be obtained from the other by replacement of x by ix, with perhaps a minor adjustment, such as a sign change. The word "modified" is used when this replacement is applied to Bessel functions [Chapters 52–54] to generate the functions of Chapters 49–51 and, in that sense, the functions of Chapter 14 are "modified semielliptic functions".

Like the standard semielliptic function pair $(\pm b/a)\sqrt{a^2-x^2}$, the function pair

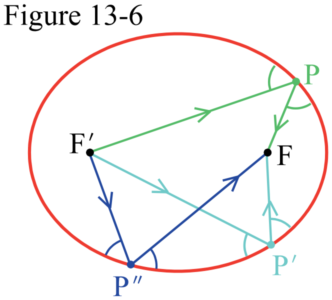
13:13:1 $$\frac{(a^2-b^2)x \pm ab\sqrt{2a^2+2b^2-4x^2}}{a^2+b^2}$$

describes an ellipse, centered at the origin, the major and minor semiaxes being of lengths a and b respectively. This ellipse, however, is oriented so that its major axis is at 45° to the x-axis. Note that it may be resolved into a linear function $(a^2-b^2)x/(a^2+b^2)$ plus a standard horizontal ellipse with semiaxes of lengths $\sqrt{(a^2+b^2)/2}$ and $\sqrt{2/(a^2+b^2)}$. A similar resolution is possible for an ellipse located anywhere in the cartesian plane and with any orientation.

13:14 RELATED TOPIC: geometric properties of the ellipse

The two foci, F′ and F, of the horizontal ellipse $(b/a)[a^2-x^2]^{1/2}$ are located on the x-axis at $x=\pm\sqrt{a^2-b^2}$. If P is any point on the ellipse, the lines PF′ and PF make equal angles with the ellipse, as Figure 13-6 indicates. This means that any wave motion radiating from point F′ will be reflected from the ellipse and arrive at point F from a variety of directions. The radiation is said to have been "focused" at F. As explained in Section 13-3, the path lengths via points P, P′ and P″ are all equal (to $2a$), and so the journey times are also equal and the radiation arrives in synchrony. This property of the ellipse is maintained in its three-dimensional counterpart, the *ellipsoid*, and is exploited in furnace design and in several acoustic and optical devices.

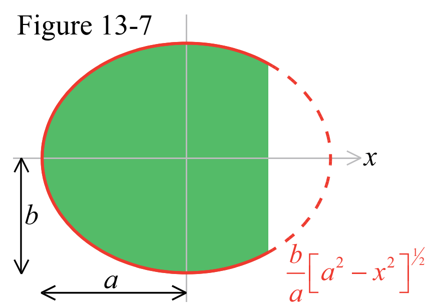
Figure 13-6



With a and b positive, the total area of an ellipse is πab. The area of the segment of an ellipse defined by the abscissal range $-a$ to x, and shown shaded in Figure 13-7 is

13:14:1 $$\begin{matrix}\text{green}\\ \text{area}\end{matrix} = \frac{2b}{a}\int_{-a}^{x}\sqrt{a^2-t^2}\,\mathrm{d}t = ab\left[\frac{x}{a^2}\sqrt{a^2-x^2}+\arcsin\left(\frac{x}{a}\right)+\frac{\pi}{2}\right]$$

Figure 13-7



The length of the curved portion of the boundary of the shaded region is

13:14:2 $$2\int_{-a}^{x}\sqrt{1+\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{b}{a}\sqrt{a^2-t^2}\right)^2}\,\mathrm{d}t = 2a\left[\mathrm{E}(k)+\mathrm{E}\left(k,\arcsin\left(\frac{x}{a}\right)\right)\right]$$

where k is the eccentricity of the ellipse, $\sqrt{1-(b/a)^2}$. The entire perimeter of the ellipse has a length of $4a\mathrm{E}(k)$. $\mathrm{E}(k)$ denotes the complete elliptic integral of the second kind of modulus k and $\mathrm{E}(k,\varphi)$ denotes the incomplete elliptic integral of the second kind of modulus k and amplitude φ. These functions are addressed in Chapters 61 and 62 respectively.

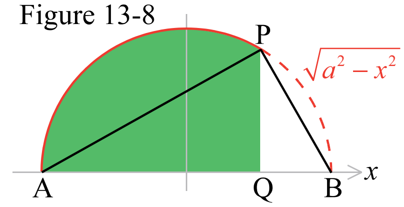
Together with the parabola [Chapter 11] and the hyperbola [Chapter 14], the ellipse and the circle constitute *curves of second degree*, also known as *conic sections*; their shared properties are the subject of Section 15:15.

13:15 RELATED TOPIC: geometry of the semicircle

Formula 13:10:3 shows that the area, shaded in Figure 13-8, of a segment of a semicircle is

13:15:1 $$\frac{a^2}{2}\left[\frac{\pi}{2}+\arcsin\left(\frac{x}{a}\right)\right]+\frac{x}{2}\sqrt{a^2-x^2}$$

Figure 13-8



irrespective of the sign of x. The total area of the semicircle is $\pi a^2/2$. The length of the curved boundary of the shaded region is

13:15:2 $$a\left[\frac{\pi}{2}+\arcsin\left(\frac{x}{a}\right)\right]$$

the total semicircumference being of length πa.

An important property of a semicircle is that the angle APB in Figure 13-8 is a right angle, for any point P on the perimeter. Hence the triangles APB, AQP and PQB are similar and right-angled, permitting pythagorean and trigonometric relationships [Chapters 32−34] to be applied.