CHAPTER 12

THE NONINTEGER POWERS x^ν

Relationships in science and engineering frequently involve fractional exponents. In this chapter we address the power function x^ν, where ν is real but otherwise unrestricted. As is clarified in Section 12:2, the function x^ν is often undefined as a real number when x is negative.

12:1 NOTATION

The symbol x^ν represents the argument x raised to power ν; both these quantities are real except in Section 12:11. The symbols $x^{-\nu}$ and $1/x^\nu$ represent identical functions.

When ν equals $1/n$, where $n = 2, 3, 4, \cdots$, the symbol $\sqrt[n]{x}$ sometimes replaces $x^{1/n}$, though this symbolism is seldom used in the *Atlas* [but see Section 16:4 for the distinction we draw there between $x^{1/3}$ and $\sqrt[3]{x}$]. The names *square root* of x, *cube root* of x, *fourth root* of x and n^{th} *root* of x are given to $x^{1/2}$, $x^{1/3}$, $x^{1/4}$, and $x^{1/n}$.

12:2 BEHAVIOR

With both x and ν restricted to real values, it is instructive to examine the behavior of the x^ν function in the context of quadrants, as defined in Section 0:2. In all quadrants other than the first, radically different behaviors are exhibited according to the properties of the number ν.

The power function x^ν exists in the first quadrant irrespective of the magnitude and rationality of ν. It acquires the value unity at $x = 1$ for all ν. If ν is positive, x^ν is zero at $x = 0$ and increases indefinitely as x takes ever-larger positive values, as is evident from several of the examples illustrated in Figure 12-1. If ν is negative, x^ν is infinite at $x = 0$, but adopts finite values that diminish as x increases, approaching zero in the $x \to \infty$ limit.

If ν is irrational (that is incapable, like π or $-\pi/4$, of being expressed as a ratio m/n of two integers), then x^ν exists only in the first quadrant as illustrated by the black curves in Figure 12-1.

There are three classes of rational numbers, according as the parities of m/n are even/odd, odd/odd, or odd/even. Of course, there is no even/even class because such fractions can always be reduced to one of the other classes by sufficient divisions of m and n by 2.

K.B. Oldham et al., *An Atlas of Functions, Second Edition*,
DOI 10.1007/978-0-387-48807-3_13, © Springer Science+Business Media, LLC 2009

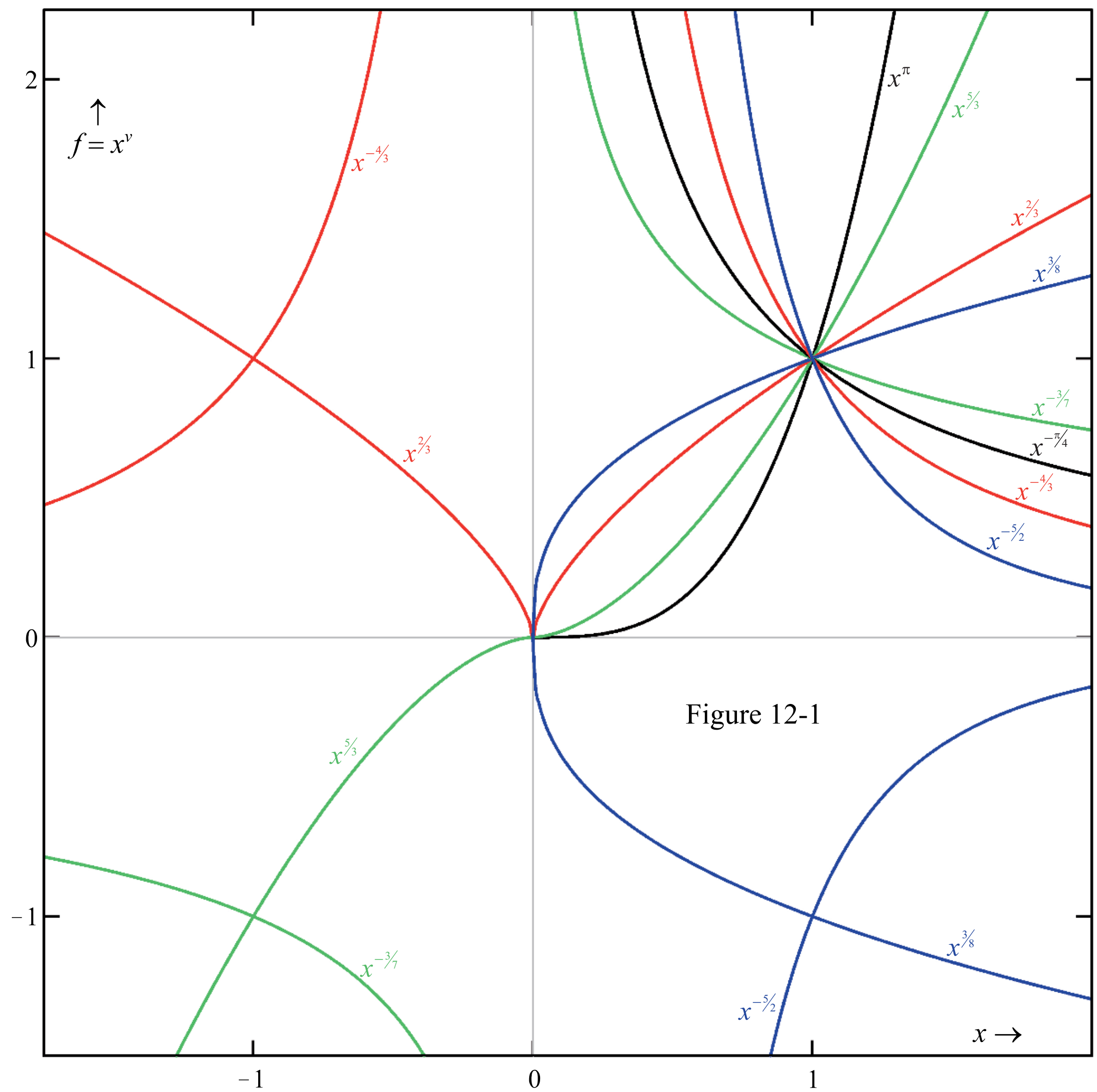

Figure 12-1

When ν is a rational number of the even/odd class, such as $\nu = m/n = 2/3$ or $-4/3$, the x^ν function exists in the first and second quadrants only. The function itself is even in these cases and takes only nonnegative values, as illustrated by the red curves in Figure 12-1.

It is the first and third quadrants that are occupied by the x^ν function when ν is a rational number, positive or negative, of the odd/odd class. The power function is then an odd function, conforming to the $(-x)^\nu = -x^\nu$ rule, as exemplified by the green curves in the figure.

When ν falls in the odd/even class, x^ν is defined as a real number only for nonnegative x and is two-valued, with plus-or-minus options (except perhaps at $x = 0$). For example $2^{3/8} = \pm 1.2968\cdots$. The blue curves in Figure 12-1 show the $\nu = 3/8$ and $-5/2$ examples, which lie in the first and fourth quadrants only.

The behavior of the power function x^ν close to $x = 0$ is of interest. Notice that, though the function itself may be continuous there, the slope of x^ν suffers a discontinuity if $0 < \nu < 1$. This is true irrespective of the parities of m and n, though not all instances are illustrated in Figure 12-1.

12:3 DEFINITIONS

When ν is the reciprocal of a positive integer n greater than zero, the definition of x^ν relies on the concept of an inverse function [Section 0:3]. Thus the n^{th} root, $x^{1/n}$ is defined as the real number f such that

12:3:1 $$f^n = x \qquad \text{whence} \qquad f = x^{1/n} \qquad n = 2,3,4,\cdots$$

Thus there is a plus-or-minus option in $x^{1/n}$ if n is even, but positivity is mandatory if n is odd. Any other rational, noninteger power of x is then defined by raising $x^{1/n}$ to the appropriate integer power, positive or negative:

12:3:2 $$\left(x^{1/n}\right)^m \qquad \text{where} \qquad \nu = \frac{m}{n} \qquad m = \pm1,\pm2,\pm3,\cdots$$

The plus-or-minus option is thereby lost if m is even, but not otherwise.

When ν is irrational, x^ν require definition as a limit. Let m_1/n_1, m_2/n_2, m_3/n_3, ··· be progressively better rational approximations to ν. Then

12:3:3 $$\lim_{j\to\infty}\left\{x^{m_j/n_j}\right\} = x^\nu$$

provides the required definition. For example, x^π could be defined as the limit of the sequence x^3, $x^{31/10}$, $x^{314/100}$, $x^{3142/1000}$, $x^{31416/10000}$,···, or some similar sequence.

In view of equation 12:10:8, the function x^ν may be defined as the result of the differintegration operation [Section 12:14] applied to the function $f(x) = x$

12:3:4 $$x^\nu = \Gamma(1+\nu)\frac{d^{1-\nu}}{dx^{1-\nu}}x \qquad x > 0$$

for all ν except negative integers. Γ is the (complete) gamma function [Chapter 43].

12:4 SPECIAL CASES

Powers with exponents $\nu = \pm1$ and $\nu = \pm\tfrac{1}{2}$ are addressed in Chapters 7 and 11. Cases in which ν is an integer are the subject of Chapter 10.

12:5 INTRARELATIONSHIPS

No reflection formula holds when ν is irrational or a member of the odd/even family of rational powers. Otherwise the power function is either even or odd according to the parity of m:

12:5:1 $$(-x)^{m/n} = (-)^m x^{m/n} \qquad m = \pm1,\pm2,\pm3,\cdots \qquad n = 1,3,5,\cdots$$

The following *laws of exponents* apply for all values of μ and ν:

12:5:2 $$x^\mu x^\nu = x^{\mu+\nu} \qquad \frac{x^\mu}{x^\nu} = x^{\mu-\nu} \qquad \text{and} \qquad \left(x^\mu\right)^\nu = x^{\mu\nu}$$

but care is needed to ensure that all quantities remain real.

Useful for large J, the series of powers of the natural numbers has a sum given asymptotically by

12:5:3 $$1+2^\nu+\cdots+J^\nu = \sum_{j=1}^{J} j^\nu \sim \zeta(-\nu) - \sum_{k=0}^{\infty}\frac{(-\nu)_{k-1}B_k}{k!J^{k-1-\nu}} = \zeta(-\nu) + \frac{J^{1+\nu}}{1+\nu} + \frac{J^\nu}{2} + \frac{\nu J^{\nu-1}}{12} - \frac{\left(\nu^3-3\nu^2+2\nu\right)J^{\nu-3}}{720} + \cdots$$

and involves functions from Chapters 2, 3, 4, and 18. The summation is invalid for $\nu = 0$ or -1, and terminates if

v is a positive integer. Finite and infinite sums of the related series

12:5:4 $$u^v + (1+u)^v + (2+u)^v + (3+u)^v \cdots = \sum_{j=0} (j+u)^v$$

and

12:5:5 $$u^v + (1+u)^v x + (2+u)^v x^2 + (3+u)^v x^3 \cdots = \sum_{j=0} (j+u)^v x^j$$

may, in favorable circumstances, be obtained by exploiting the properties of Hurwitz and Lerch functions [Chapter 64 and Section 64:12].

12:6 EXPANSIONS

The x^v function may be expanded as a power series either binomially [Section 6:14] in $(x-1)$

12:6:1 $$x^v = 1 + v(x-1) + \frac{v(v-1)}{2!}(x-1)^2 + \frac{v(v-1)(v-2)}{3!}(x-1)^3 + \cdots = \sum_{j=0}^{\infty} \binom{v}{j} (x-1)^j$$

or via the Euler transformation [Section 10:13], in $(x-1)/x$

12:6:2 $$x^v = \frac{1}{x} + (1+v)\frac{x-1}{x^2} + \left(1 + \tfrac{3}{2}v + \tfrac{1}{2}v^2\right)\frac{(x-1)^2}{x^3} + \cdots = \frac{1}{x-1}\sum_{k=1}^{\infty}\left(\frac{x-1}{x}\right)^k \sum_{j=0}^{k-1}\binom{k-1}{j}\binom{v}{j}$$

Equation 53:14:4 provides an expansion of x^v in terms of Bessel functions.

12:7 PARTICULAR VALUES

	$x=-\infty$	$x=-1$	$x=0$	$x=1$	$x=\infty$
irrational, $v<0$	undef	undef	$+\infty$	1	0
irrational, $v>0$	undef	undef	0	1	$+\infty$
even/odd, $v<0$	0	1	$+\infty \mid +\infty$	1	0
even/odd, $v>0$	$+\infty$	1	0	1	$+\infty$
odd/odd, $v<0$	0	-1	$-\infty \mid +\infty$	1	0
odd/odd, $v>0$	$-\infty$	-1	0	1	$+\infty$
odd/even, $v<0$	undef	undef	$\pm\infty$	$+1$	0
odd/even, $v>0$	undef	undef	0	± 1	$\pm\infty$

In the table, in which the colors are keyed to Figure 12-1, "undef" means that x^v is not defined as a real function for the argument in question. The particular values arising when $x = e = 2.71828\cdots$ might also be mentioned, for then $x^v = \exp(v)$ [Chapter 26].

12:8 NUMERICAL VALUES

Most computer languages provide numerical access to noninteger powers via the coding *x*^*v* or, occasionally, *x*∗∗*v*. Alternatively, the simple algorithm

12:8:1 $$x^v = \exp\{v\ln(x)\}$$

involving functions from Chapters 26 and 25, may be used when *x* is positive.

If *v* is an integer, *Equator*'s bivariate power function routine (keyword **power**) returns a correctly signed value of x^v, for either sign of *x*. Moreover, *Equator* provides values of x^v for any positive values of *x*, whatever value *v* might have. But when *v* is a rational number of the odd/even class, only the positive option of x^v is returned, this being the principal value [Section 0:0]. When *x* is negative and *v* is an noninteger, *Equator* returns the message "complex" even though, in some cases, there exists a real number that would be an appropriate answer. The example $(-32)^{0.2} = -2$ is such an instance. To obtain a real answer, in a case such as this, type the *v* input in the format "integer/integer"; a real answer will be provided by *Equator* whenever one exists.

Equator's complex number raised to a real power routine (keyword **compower**) [Section 10:11] allows either real or complex numbers to be raised to a real power, integer or noninteger, positive or negative. The output is generally a complex number, unlike that of the power function routine. The significance of the output is described in Section 12:11.

Equator's variable construction feature (Appendix, Section C:4) permits the use of a power as the argument of another function.

12:9 LIMITS AND APPROXIMATIONS

The table in Section 12:7 shows the limits approached by x^v as $x \to 0$, $x \to +\infty$ or $x \to -\infty$.

When *x* is close to unity and the magnitude of *v* is not too large, the linear approximation

12:9:1 $$x^v \approx 1 - v + vx$$

is good, but

12:9:2 $$x^v \approx \frac{x+1+v(x-1)}{x+1-v(x-1)}$$

is better.

For small *v*, the approximation

12:9:3 $$x^v = \exp\{v\ln(x)\} \approx 1 + v\ln(x)$$

is useful.

12:10 OPERATIONS OF THE CALCULUS

Differentiation gives

12:10:1 $$\frac{\mathrm{d}}{\mathrm{d}x}x^v = vx^{v-1}$$

Indefinite integration of the power function requires different limits according to the value of *v*:

12:10:2 $$\int_0^x t^v\,\mathrm{d}t = \frac{x^{v+1}}{v+1} \qquad v > -1 \qquad x > 0$$

12:10:3 $$\int_1^x t^\nu \,\mathrm{d}t = \ln(x) \qquad \nu = -1 \qquad \text{all } x$$

12:10:4 $$\int_x^\infty t^\nu \,\mathrm{d}t = \frac{-x^{1+\nu}}{1+\nu} \qquad \nu < -1 \qquad x > 0$$

The reflection property [equation 12:5:1] may be used to adapt 12:10:2 or 12:10:4 to negative x. A general expression for integration of the product of a power function with a linear function raised to a power is

12:10:5 $$\int_0^x t^\nu (bt+c)^\mu \,\mathrm{d}t = \begin{cases} \dfrac{c^{1+\nu+\mu}}{b^{1+\nu}} \mathrm{B}\left(1+\nu, -1-\nu-\mu, \dfrac{bx}{bx+c}\right) & b>0,\ 0<bx<c \\ \dfrac{c^{1+\nu+\mu}}{(-b)^{1+\nu}} \mathrm{B}\left(1+\nu, 1+\mu, \dfrac{-bx}{c}\right) & b<0,\ 0<-bx<c \end{cases} \quad c>0 \quad \nu>-1$$

the integral being in terms of the trivariate incomplete beta function [Chapter 58].

Formulas for semidifferentiation and semiintegration of the power function, with lower limit zero, involve the gamma function [Chapter 43]:

12:10:6 $$\frac{\mathrm{d}^{1/2}}{\mathrm{d}x^{1/2}} x^\nu = \frac{\Gamma(1+\nu)}{\Gamma(\frac{1}{2}+\nu)} x^{\nu-1/2} \qquad \nu>-1,\ x>0$$

12:10:7 $$\frac{\mathrm{d}^{-1/2}}{\mathrm{d}x^{-1/2}} x^\nu = \frac{\Gamma(1+\nu)}{\Gamma(\frac{3}{2}+\nu)} x^{\nu+1/2} \qquad \nu>-1,\ x>0$$

These are just the $\mu = \pm\frac{1}{2}$ cases of the general rule for differintegration [Section 12:14] of a power function:

12:10:8 $$\frac{\mathrm{d}^\mu}{\mathrm{d}x^\mu} x^\nu = \frac{\Gamma(1+\nu)}{\Gamma(1-\mu+\nu)} x^{\nu-\mu} \qquad \nu>-1,\ x>0$$

and equations 12:10:1 and 12:10:2 display the $\mu = \pm 1$ instances. The corresponding formula for differintegration with a lower limit of $-\infty$ is

12:10:9 $$\left.\frac{\mathrm{d}^\mu}{\mathrm{d}x^\mu} x^\nu\right|_{-\infty} = \frac{\Gamma(\mu-\nu)}{\Gamma(-\nu)} (-x)^{\nu-\mu} \qquad \nu<\mu,\ x<0$$

The transformation

12:10:10 $$\int_0^\infty \mathrm{f}(t) t^{\nu-1} \mathrm{d}t$$

creates a possibly-complex-valued function of the ν variable, known as the *Mellin transform* (Robert Hjalmar Mellin, Finnish mathematician, 1854 – 1933) of the function f(t). A tabulation of over 250 Mellin transforms is given by Erdélyi, Magnus, Oberhettinger, and Tricomi [*Tables of Integral Transforms*, Volume 1, Chapter 6], together with some general formulas and a listing of inverse Mellin transforms. Thus the following definite integrals may be regarded as Mellin transforms, as may that in 12:10:16.

12:10:11 $$\int_0^\infty \frac{t^\nu}{(bt+c)^\mu} \mathrm{d}t = \frac{c^{1+\nu-\mu}}{b^{1+\nu}} \mathrm{B}(1+\nu,\ \mu-\nu-1) \qquad -1<\nu<\mu-1$$

12:10:12 $$\int_0^\infty \frac{t^\nu}{t^2+a^2} \mathrm{d}t = \frac{\pi a^{\nu-1}}{2} \sec\left(\frac{\nu\pi}{2}\right) \qquad -1<\nu<1$$

12:10:13 $$\int_0^\infty t^v \ln(1+a^2t)\mathrm{d}t = \frac{-\pi}{(1+v)a^{2+2v}}\csc(v\pi) \qquad -2<v<-1$$

12:10:14 $$\int_0^\infty t^v \sin(\omega t)\mathrm{d}t = \frac{\Gamma(1+v)}{\omega^{1+v}}\cos\left(\frac{v\pi}{2}\right) \qquad -2<v<0 \qquad v\neq -1$$

12:10:15 $$\int_0^\infty t^v \arctan(t)\,\mathrm{d}t = \frac{\pi}{2(1+v)}\csc\left(\frac{v\pi}{2}\right) \qquad -2<v<-1$$

The Γ function in 12:10:14 and the B function in 12:10:11 are the complete gamma function [Chapter 43] and the (bivariate) complete beta function [Section 43:13]. Note that the domain of v required to validate these transforms is often severely restricted.

The following Laplace transforms involve the complete [Chapter 43] and incomplete [Chapter 45] gamma functions

12:10:16 $$\int_0^\infty t^v \exp(-st)\mathrm{d}t = \mathfrak{L}\{t^v\} = \frac{\Gamma(v+1)}{s^{v+1}} \qquad v>-1$$

12:10:17 $$\int_0^\infty (bt+c)^v \exp(-st)\,\mathrm{d}t = \mathfrak{L}\{(bt+c)^v\} = \frac{b^v}{s^{v+1}}\exp\left(\frac{cs}{b}\right)\Gamma\left(v+1,\frac{cs}{b}\right) \qquad v>-1$$

12:11 COMPLEX ARGUMENT

If z is complex and v real, different portions of the chain of equalities

12:11:1 $$(x+iy)^v = z^v = [\rho\exp(i\theta)]^v = \rho^v\exp(iv\theta) = \rho^v[\cos(\theta)+i\sin(\theta)]^v = \rho^v[\cos(v\theta)+i\sin(v\theta)]$$

are identified by different authorities as *de Moivre's theorem* (Abraham de Moivre, French mathematician, 1667 − 1754). Here ρ is the modulus $|z|$ of the complex variable, equal to $\sqrt{x^2+y^2}$, and θ is its phase, equal to arctan(y/x) when x is positive and to π + arctan(y/x) when x is negative. In implementing this formula, one must recognize that θ may be augmented by $2k\pi$, where k is any integer, without changing its import.

If v is real and equal to m/n, the complex power z^v may be represented as $\left(z^{1/n}\right)^m$, that is, as the complex variable $z^{1/n}$ raised to an integer power, as in Section 10:11. Hence, in the case of a rational v, ascribing significance to z^v devolves into identifying the properties of $z^{1/n}$, the so-called *nth root* of the complex z. By de Moivre's theorem:

12:11:2 $$z^{1/n} = \left|\rho^{1/n}\right|\left[\cos\left(\frac{\theta}{n}\right)+i\sin\left(\frac{\theta}{n}\right)\right]$$

To illustrate this formula, consider the $n = 3$ case when the complex variable z has real and imaginary parts of 0.94 and 1.29 respectively. Its modulus and phase are $\rho = \sqrt{(0.94)^2+(1.29)^2} = 1.60$ and $\theta = \arctan(1.29/0.94) = 0.941 = 54^\circ$. Thus, in Figure 12-2, the point labeled Z represents z. One calculates $|\rho^{1/3}| = 1.17$ and $\theta/3 = 18^\circ$. The following possibilities must be considered:

12:11:3 $$z^{1/3} = 1.17\left[\cos\left(18^\circ+\frac{k}{3}360^\circ\right)+i\sin\left(18^\circ+\frac{k}{3}360^\circ\right)\right] \qquad k=0,\pm1,\pm2,\cdots$$

but it turns out that, whatever value is chosen for k, there are only three distinct answers, that we may take to arise from the choices $k = 0, \pm1$. They are:

12:11:4 $$z^{1/3} = \begin{cases} 1.17\left[\cos\left(18^{\circ}\right) + i\sin\left(18^{\circ}\right)\right] \\ \qquad = 1.11 + 0.36i \\ 1.17\left[\cos\left(18^{\circ}+120^{\circ}\right) + i\sin\left(18^{\circ}+120^{\circ}\right)\right] \\ \qquad = -0.87 + 0.78i \\ 1.17\left[\cos\left(18^{\circ}-120^{\circ}\right) + i\sin\left(18^{\circ}-120^{\circ}\right)\right] \\ \qquad = -0.24 - 1.14i \end{cases}$$

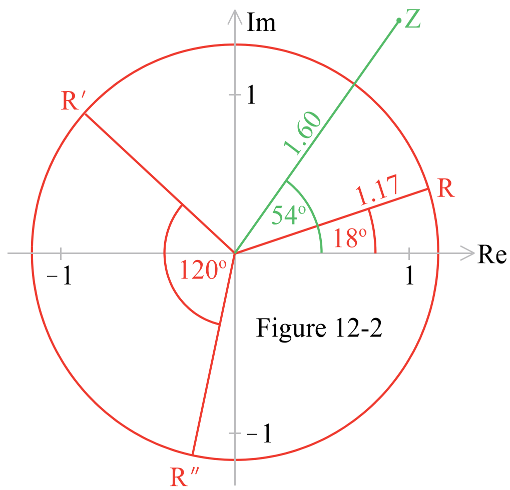

Figure 12-2

and they are represented on the diagram by the points R, R′, and R″, which are equally spaced around a circle of radius 1.17.

It is generally true that there are n complex roots, not only of $z^{1/n}$ but also of $z^{m/n}$. Two or one of these will be real if z is real, depending on whether n is even or odd. When ν is irrational, the range of k must be restricted so that $-\pi < \theta + 2k\pi/\nu \le \pi$; the number of roots will lie between $(2\pi/\nu)-1$ and $(2\pi/\nu)+1$. *Equator*'s complex number raised to a real power routine (keyword **compower**) may be used to evaluate the real and imaginary parts of $(x+iy)^{\nu}$; however, it provides only one such pair of numbers, the principal value, defined as $\exp\{\nu\ln(x)\}$ [Section 25:11]. Note that, when the input to this routine has $y = 0$, the principal value of x^{ν} is returned by *Equator*. However, this is not always what might be expected. For example, $(-8)^{1/3}$ is not -2, but $1+\sqrt{3}\,i$.

Some important inverse Laplace transforms involving the arbitrary power function are

12:11:5 $$\int_{\alpha-i\infty}^{\alpha+i\infty} s^{\nu}\frac{\exp(ts)}{2\pi i}\,\mathrm{d}s = \mathfrak{I}\left\{s^{\nu}\right\} = \frac{1}{\Gamma(-\nu)t^{\nu+1}} \qquad \nu<0$$

12:11:6 $$\int_{\alpha-i\infty}^{\alpha+i\infty} (s+a)^{\nu}\frac{\exp(ts)}{2\pi i}\,\mathrm{d}s = \mathfrak{I}\left\{(s+a)^{\nu}\right\} = \frac{\exp(-at)}{\Gamma(-\nu)t^{\nu+1}} \qquad \nu<0$$

12:11:7 $$\int_{\alpha-i\infty}^{\alpha+i\infty} \left(\sqrt{s}+\sqrt{a}\right)^{\nu}\frac{\exp(ts)}{2\pi i}\,\mathrm{d}s = \mathfrak{I}\left\{(\sqrt{s}+\sqrt{a})^{\nu}\right\} = \frac{-\nu\exp(at/2)}{\sqrt{2^{1+\nu}\pi t^{2+\nu}}}\mathrm{D}_{\nu-1}\left(\sqrt{2at}\right) \qquad \nu<0$$

12:11:8 $$\int_{\alpha-i\infty}^{\alpha+i\infty} s^{\nu}\exp(-as)\frac{\exp(ts)}{2\pi i}\,\mathrm{d}s = \mathfrak{I}\left\{s^{\nu}\exp(-as)\right\} = \frac{\mathrm{u}(t-a)}{\Gamma(-\nu)(t-a)^{\nu+1}} \qquad \nu<0$$

12:11:9 $$\int_{\alpha-i\infty}^{\alpha+i\infty} s^{\nu}\exp\left(-a\sqrt{s}\right)\frac{\exp(ts)}{2\pi i}\,\mathrm{d}s = \mathfrak{I}\left\{s^{\nu}\exp\left(-a\sqrt{s}\right)\right\} = \frac{\exp(-a^{2}/8t)}{\sqrt{2^{2\nu+1}\pi t^{\nu+1}}}\mathrm{D}_{2\nu-1}\left(\frac{a}{\sqrt{2t}}\right) \qquad \nu<0$$

12:11:10 $$\int_{\alpha-i\infty}^{\alpha+i\infty} s^{\nu}\exp(-a/s)\frac{\exp(ts)}{2\pi i}\,\mathrm{d}s = \mathfrak{I}\left\{s^{\nu}\exp(-a/s)\right\} = \left(\frac{a}{t}\right)^{(\nu+1)/2}\mathrm{J}_{-\nu-1}\left(2\sqrt{at}\right) \qquad \nu<0$$

12:11:11 $$\int_{\alpha-i\infty}^{\alpha+i\infty} s^{\nu}\exp(a/s)\frac{\exp(ts)}{2\pi i}\,\mathrm{d}s = \mathfrak{I}\left\{s^{\nu}\exp(a/s)\right\} = \left(\frac{a}{t}\right)^{(\nu+1)/2}\mathrm{I}_{-\nu-1}\left(2\sqrt{at}\right) \qquad \nu<0$$

Among the functions appearing in these inverse transforms are the parabolic cylinder function D_{ν} of Chapter 46, the Bessel function J_{ν} [Chapter 53], and the modified Bessel function I_{ν} [Chapter 50]. Some important special cases of 12:11:5 are summarized in the following panel, in which g represents Gauss's constant.

$\frac{1}{s^{1/4}}$	$\frac{1}{\sqrt{s}}$	$\frac{1}{s^{3/4}}$	$\frac{1}{s}$	$\frac{1}{\sqrt{s^3}}$	$\frac{1}{s^2}$	$\frac{1}{s^{5/2}}$	$\frac{1}{s^3}$	$\frac{1}{s^{7/2}}$	$\frac{1}{s^n}$	$\frac{1}{s^{n+\frac{1}{2}}}$
$\frac{1}{\sqrt{g}(2\pi t)^{3/4}}$	$\frac{1}{\sqrt{\pi t}}$	$\sqrt{g}\left(\frac{2}{\pi t}\right)^{1/4}$	1	$2\sqrt{\frac{t}{\pi}}$	t	$\frac{4}{3}\sqrt{\frac{t^3}{\pi}}$	$\frac{t^2}{2}$	$\frac{8t^{5/2}}{15\sqrt{\pi}}$	$\frac{t^{n-1}}{(n-1)!}$	$\frac{4^n n! t^{n-\frac{1}{2}}}{(2n)!\sqrt{\pi}}$

12:12 GENERALIZATIONS

The functions $(bx+c)^{\nu}$ and $(ax^2+bx+c)^{\nu}$ generalize x^{ν} and their properties may sometimes be deduced by appropriate substitutions. The important $\nu = \pm\frac{1}{2}$ instances of these two generalizations are discussed in Chapter 11 and Section 15:15.

12:13 COGNATE FUNCTIONS

There is a multitude of functions of the forms $f(x^{\nu})$ and $[f(x)]^{\nu}$, involving arbitrary powers, though the $\nu = \pm\frac{1}{2}$ instances are the most important noninteger cases. The $(a^2-x^2)^{\pm 1/2}$ and $(x^2 \pm a^2)^{\pm 1/2}$ functions are the subjects of the next two chapters.

12:14 RELATED TOPIC: the fractional calculus

It is conventional to represent the operations of double, triple, and n-fold differentiation of the function $f(x)$ by the notation

12:14:1
$$\frac{d^2}{dx^2}f(x) \qquad \frac{d^3}{dx^3}f(x) \qquad \text{and} \qquad \frac{d^n}{dx^n}f(x)$$

Inasmuch as integration and differentiation are, to an extent, inverse processes, it is not unreasonable to use a symbolism similar to those in 12:14:1, but with negative superscripts, to represent single and multiple integrations. Thus one may define

12:14:2
$$\frac{d^{-1}}{dx^{-1}}f(x) \equiv \int_0^x f(x')\,dx' \qquad \frac{d^{-2}}{dx^{-2}}f(x) \equiv \int_0^x\int_0^{x'} f(x'')\,dx''\,dx' \qquad \text{etc.}$$

In this way, we have created a unified notation

12:14:3
$$\frac{d^{\mu}}{dx^{\mu}}f(x)$$

that encompasses repeated differentiation when the order μ is 2, 3, $\cdots$, n, $\cdots$, and single or multiple integration when $\mu = -1, -2, \cdots, -n, \cdots$. Also, of course, the $\mu = 1$ and $\mu = 0$ versions can represent single differentiation and $f(x)$ itself, so that all integer values are covered. The term *differintegration* is used to describe the hybrid operation: *differintegral* describes the resulting function.

The mission of the fractional calculus is to extend the meaning of 12:14:3 to include noninteger values of μ and find utility for the resulting operation. There are several ways to define a differintegral so that reduction occurs to established definitions of differentiations and integration when μ is an integer. Probably the most general of these is the limit definition

12:14:4 $$\frac{d^\mu}{dx^\mu}f(x) \equiv \lim_{J\to\infty}\left\{\left(\frac{J}{x}\right)^\mu \sum_{j=0}^{J-1}\frac{(-\mu)_j}{j!}f\left(\frac{J-j}{J}x\right)\right\}$$

due to Grünwald and written here in terms of the Pochhammer polynomial [Chapter 18]. Of more utility, however, is the definition as an integral transform, originating in the work of Riemann and Liouville (Joseph Liouville, French mathematician, 1809 – 1882):

12:14:5 $$\frac{d^\mu}{dx^\mu}f(x) \equiv \frac{1}{\Gamma(-\mu)}\int_0^x \frac{f(x')}{(x-x')^{1+\mu}}dx' \qquad \mu < 0$$

which, however, applies only to negative orders of differintegration. Here Γ is the gamma function [Chapter 43]. Extension to positive orders relies on classical differentiation through the formula

12:14:6 $$\frac{d^\mu}{dx^\mu}f(x) \equiv \frac{d^n}{dx^n}\left\{\frac{d^{\mu-n}}{dx^{\mu-n}}f(x)\right\} \qquad n > \mu > 0$$

where n is any integer greater than μ, so that the embraced quantity in 12:14:6 has a negative order and can be defined by 12:14:5. For other definitions, see Oldham and Spanier [Chapter 3].

In formulating the equations above, a lower limit of zero was assumed and this is the most usual choice. Other alternatives may be selected, however. It may seem counterintuitive to refer to a lower limit in the context of differentiation but, in fact, a lower limit must be specified for *all* instances of differintegration, except when $\mu = 0, 1, 2, 3, \cdots$. The most general case has an arbitrary lower limit, say a. An appropriate notation, and the corresponding *Riemann-Liouville definitions* are

12:14:7 $$\left.\frac{d^\mu}{dt^\mu}f(t)\right|_a^x \equiv \begin{cases} \dfrac{1}{\Gamma(-\mu)}\displaystyle\int_a^x \frac{f(t)}{(x-t)^{1+\mu}}dt & \mu < 0 \\ \dfrac{1}{\Gamma(n-\mu)}\dfrac{d^n}{dx^n}\displaystyle\int_a^x \frac{f(t)}{(x-t)^{1+\mu-n}}dt & n > \mu > 0 \end{cases}$$

A popular choice of lower limit is $-\infty$ and these instances, termed *Weyl differintegrals*, are the subject of Section 64:14.

There are many practical applications of the fractional calculus [Hilfer]. In some fields, μ is an adjustable parameter; in others it is fixed. In the latter case, the most important values of μ are $\pm\frac{1}{2}$, these operations being known as *semidifferentiation* or *semiintegration*, and collectively as *semidifferintegration*. Frequently in the *Atlas*, examples of zero-lower-limit semidifferintegrals are included in Section 10 of the chapter devoted to the target function, and sometime Weyl differintegrals and more general results are listed too [as in Section 10 of the present chapter]. In Section 43:14, it is explained how the fractional calculus serves to provide a facile method of "synthesizing" one function from another. The important *composition rule*

12:14:8 $$\frac{d^\nu}{dx^\nu}\left\{\frac{d^\mu}{dx^\mu}f(x)\right\} = \frac{d^{\nu+\mu}}{dx^{\nu+\mu}}f(x)$$

of which 12:14:6 is an instance, plays the crucial role in these syntheses. There are some exceptions to the composition "rule" (for example when $f(x) = x^2$, $\mu = 3$ and $\nu = -2$), but it applies in most instances.