CHAPTER 9

THE HEAVISIDE u($x-a$) AND DIRAC δ($x-a$) FUNCTIONS

The functions of this chapter occur primarily as multipliers of other functions, which they thereby modulate. The Dirac function is the derivative of the Heaviside function

9:0:1
$$\delta(x-a)=\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{u}(x-a)$$

Strictly speaking, the Dirac function is not a function at all, because it violates conditions that respectable functions obey; nevertheless the great utility of this "function", notably in the fields of classical and quantum mechanics, warrants its inclusion in the *Atlas*.

9:1 NOTATION

The names of these functions recognize the achievements of two English innovators, Oliver Heaviside (electrical engineer, 1859 – 1925) and Paul Adrian Maurice Dirac (nuclear physicist, 1902 – 1984).

Synonyms of "Heaviside function" include *unit-step function*, *Heaviside theta function*, and *Heaviside's step function*; the symbols $\theta(x-a)$, $\mathrm{H}(x-a)$, and $\mathrm{S}_a(x)$ are encountered. The Dirac function has the alternative names *unit-impulse function*, *impulse function*, *delta function*, and *Dirac's delta function*. The last variant stresses the distinction from Kronecker's delta function [Section 9:13].

The salient property of each of these functions occurs when its argument is zero, and this is the reason for the unusual representation of the argument of these functions. In truth, these functions are bivariate, but it is only the difference, $x-a$, between the two variables that affects the functions' values. This also explains why the notations $\mathrm{u}(x)$ and $\delta(x)$ replace $\mathrm{u}(x-a)$ and $\delta(x-a)$ when a is zero.

9:2 BEHAVIOR

As illustrated in Figure 9-1, the Heaviside function $\mathrm{u}(x-a)$ adopts the value zero when x is less than a and the

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value unity when $x > a$, so that the alternative name "unit-step function" is, indeed, appropriate. The value of $\mathrm{u}(x-a)$, when $x = a$, is usually regarded as ½, but this is seldom of consequence.

The Dirac function cannot be graphed; $\delta(x-a)$ is zero for all x, except at $x = a$, where it is infinite.

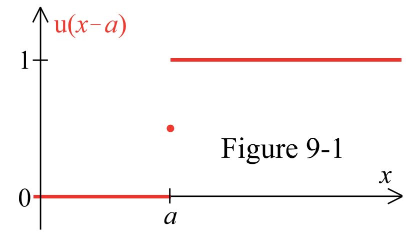

Figure 9-1

9:3 DEFINITIONS

The Heaviside function is defined by

9:3:1 $$\mathrm{u}(x-a) = \begin{cases} 0 & x < a \\ \tfrac{1}{2} & x = a \\ 1 & x > a \end{cases}$$

and therefore the effect of multiplying any function $\mathrm{f}(x)$ by $\mathrm{u}(x-a)$ is to nullify the function for arguments less than a, but to preserve $\mathrm{f}(x)$ unchanged for $x > a$.

Equation 9:0:1 provides one definition of the Dirac function. It may be defined as a limit in several ways, including

9:3:2 $$\delta(x-a) = \lim_{\nu\to\infty}\left[\sqrt{\frac{\nu}{\pi}}\exp\left\{-\nu(x-a)^2\right\}\right]$$

and

9:3:3 $$\delta(x-a) = \lim_{\varepsilon\to 0}\left[\frac{1}{2\varepsilon}\mathrm{sech}^2\left(\frac{x-a}{\varepsilon}\right)\right]$$

in terms of functions discussed in Chapters 27 and 29. The Dirac function may also be considered as the limiting case of a pulse function [Section 1:13] in which the pulse width is progressively diminished, while preserving the product (pulse width)×(pulse height) equal to unity. All these definitions – and many others – describe a function peaked at $x = a$ that, as the limit is approached, becomes infinitely high and infinitesimally wide but whose area remains constant and equal to unity. Figure 9-2 illustrates progress towards the limit in the case of definition 9:3:3.

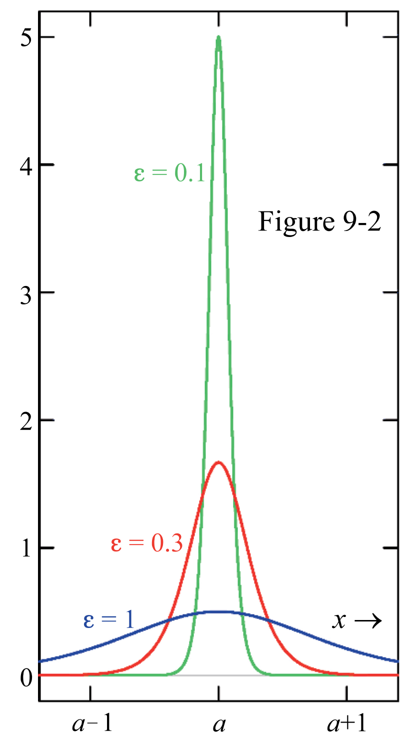

Figure 9-2

Yet another representation is as the definite integral

9:3:4 $$\delta(x-a) = \int_{-\infty}^{\infty}\cos\{2\pi(x-a)t\}\,\mathrm{d}t$$

9:4 SPECIAL CASES

The signum function [Chapter 8] is an adaptation of the Heaviside function

9:4:1 $$\mathrm{sgn}(x) = 2\mathrm{u}(x) - 1$$

9:5 INTRARELATIONSHIPS

The Heaviside function satisfies the reflection formula

9:5:1 $$\mathrm{u}(a-x)=1-\mathrm{u}(x-a)$$

whereas, for the Dirac function

9:5:2 $$\delta(a-x)=\delta(x-a)$$

Other intrarelationships obeyed by the Dirac function include the multiplication property

9:5:3 $$\delta\{v(x-a)\}=\frac{\delta(x-a)}{v} \qquad v>0$$

and

9:5:4 $$\delta(x^2-a^2)=\frac{1}{2|a|}\left[\delta(x-a)+\delta(x+a)\right]$$

9:6 EXPANSIONS

There are none.

9:7 PARTICULAR VALUES

	$x<a$	$x=a$	$x>a$
u(x−a)	0	½	1
δ(x−a)	0	∞	0

9:8 NUMERICAL VALUES

Values of these functions require no computation.

9:9 LIMITS AND APPROXIMATIONS

The discontinuous Heaviside and Dirac functions may be approximated by continuous functions in many ways. For example

9:9:1 $$\mathrm{u}(x-a)\approx\frac{1+\tanh\{v(x-a)\}}{2} \qquad \text{very large } v$$

or

9:9:2 $$\delta(x-a)\approx\frac{v\exp\left\{-v^2(x-a)^2\right\}}{\sqrt{\pi}} \qquad \text{very large } v$$

9:10 OPERATIONS OF THE CALCULUS

As equation 9:0:1 states, the derivative of the Heaviside function is the Dirac function. If the Dirac function is itself differentiated, the unit-moment function $\delta^{(1)}(x-a)$, mentioned in Section 9:12, results. Integration of the two functions yields:

9:10:1 $$\int_{x_0}^{x} \mathrm{u}(t-a)\,\mathrm{d}t = [x-a]\mathrm{u}(x-a) \qquad x_0 < a < x$$

9:10:2 $$\int_{x_0}^{x} \delta(t-a)\,\mathrm{d}t = \begin{cases} 1 & x_0 < a < x \\ 0 & a < x_0 \quad \text{or} \quad a > x \end{cases}$$

The integration of the product of an arbitrary function f(x) with the functions of this chapter produces interesting and useful results:

9:10:3 $$\int_{x_0}^{x} \mathrm{u}(t-a)\mathrm{f}(t)\,\mathrm{d}t = \int_{a}^{x} \mathrm{f}(t)\,\mathrm{d}t \qquad x_0 < a < x$$

9:10:4 $$\int_{x_0}^{x} \delta(t-a)\mathrm{f}(t)\,\mathrm{d}t = \mathrm{u}(x-a)\mathrm{f}(a) \qquad x_0 < a < x$$

A special case of the last equation constitutes what is known as the *sifting property*

9:10:5 $$\int_{-\infty}^{\infty} \delta(t-a)\mathrm{f}(t)\,\mathrm{d}t = \mathrm{f}(a)$$

of the Dirac function: multiplying a function f(x) by $\delta(x-a)$ and integrating “sifts out” the value of f at $x = a$.

Another way in which the Dirac function finds use is in being “convolved” with another function. The notation f(x)✻g(x) represents the *convolution* of the f and g functions, defined by

9:10:6 $$\mathrm{f}(x) \ast \mathrm{g}(x) = \int_{-\infty}^{\infty} \mathrm{f}(t)\mathrm{g}(x-t)\,\mathrm{d}t = \int_{-\infty}^{\infty} \mathrm{f}(x-t)\mathrm{g}(t)\,\mathrm{d}t$$

The convolution of two Dirac functions obeys the rule

9:10:7 $$\delta(x-a) \ast \delta(x-b) = \delta(x-a-b)$$

Laplace transformation of the Heaviside and Dirac functions leads to an exponentially decaying function [Chapter 26] of the dummy variable

9:10:8 $$\int_{0}^{\infty} \mathrm{u}(t-a)\exp(-st)\,\mathrm{d}t = \mathfrak{L}\{\mathrm{u}(t-a)\} = \frac{\exp\{-as\}}{s} \qquad a > 0$$

9:10:9 $$\int_{0}^{\infty} \delta(t-a)\exp(-st)\,\mathrm{d}t = \mathfrak{L}\{\delta(t-a)\} = \exp\{-as\} \qquad a > 0$$

The latter transform exemplifies the sifting property.

9:11 COMPLEX ARGUMENT

A complex number is zero only if its real and imaginary parts are *both* zero. Thus the Dirac function of complex argument $\delta(x+iy-a-bi)$ is nonzero only when $x = a$ and $y = b$. It obeys

9:11:1 $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\delta(x+iy-a-ib)\,\mathrm{d}x\,\mathrm{d}y=1$$

as well as other relations that parallel its behavior as a function of a real argument.

9:12 GENERALIZATIONS

The functions $\mathrm{u}(x-a)$, $\delta(x-a)$ and $\delta^{(1)}(x-a)$ may be regarded as the $v=0$, $v=1$, and $v=2$ cases of a continuum of functions defined by the differintegral [Section 12:14]

9:12:1 $$\frac{\mathrm{d}^v}{\mathrm{d}x^v}\mathrm{u}(x-a) \qquad a>0$$

which evaluates to

9:12:2 $$\frac{\mathrm{d}^v}{\mathrm{d}x^v}\mathrm{u}(x-a)=\mathrm{u}(x-a)\frac{(x-a)^{-v}}{\Gamma(1-v)}$$

The gamma function, Γ, is addressed in Chapter 43. When $v=1,2,3,\cdots$, the gamma function $\Gamma(1-v)$ is infinite, so that formula 9:12:2 evaluates to zero, except at $x=a$.

The $v=2$ case of 9:12:1, the *unit-moment function*, may be defined by limiting operations analogous to 9:3:2 and 9:3:3, for example

9:12:3 $$\delta^{(1)}(x-a)=\lim_{\varepsilon\to 0}\left[\frac{-1}{\varepsilon^2}\mathrm{sech}^2\left(\frac{x-a}{\varepsilon}\right)\tanh\left(\frac{x-a}{\varepsilon}\right)\right]$$

Alternatively, it may be considered as the limit of two pulse functions: first a positive-going pulse, immediately followed by a negative-going replica. As the pulses individually approach Dirac functions, the second on the heels of the first, the combination becomes $\delta^{(1)}(x-a)$. The unit-moment function, which is alternatively symbolized $\delta'(x-a)$, has a sifting property too, but it sifts out the derivative of the $\mathrm{f}(x)$ function:

9:12:4 $$\int_{-\infty}^{\infty}\delta^{(1)}(t-a)\mathrm{f}(t)\,\mathrm{d}t=-\frac{\mathrm{df}}{\mathrm{d}t}(a)$$

9:13 COGNATE FUNCTIONS

A *window function* contains a segment of some function $\mathrm{f}(x)$, whose definition is preserved within the argument range $x_0<x<x_1$, but which is zero otherwise. Two Heaviside functions switch the function on and off in the following formula

9:13:1 $$\mathrm{f}(x)\left[\mathrm{u}(x-x_0)-\mathrm{u}(x-x_1)\right] \qquad x_1>x_0$$

that implements the windowing. The pulse function [Section 1:13] is the simplest example.

The Dirac function is a bivariate function of two variables, x and a, each of which can adopt any real value; it is zero unless these two variables are equal. The *Kronecker function* or *Kronecker delta function*, $\delta_{n,m}$, (Leopold Kronecker, German mathematician, 1823 −1891) is an analogous bivariate function but its two variables are restricted to integer values. It is defined by

9:13:2
$$\delta_{n,m} = \begin{cases} 0 & n \neq m \\ 1 & n = m \end{cases}$$

The *comb function* or *shah function*, comb(x,P), generates a sequence of Dirac functions whenever x is a multiple of P. For example, comb(x,1) has the property of being zero except for integer x, when it is infinite.

9:14 RELATED TOPIC: Green's functions

One application of the Dirac function arises in the context of *Green's function*, named for the English applied mathematician George Green (1793 − 1841). Of course, Green did not use the term "Dirac function": he died long before Dirac's birth.

The concept of a Green function is at once simple, yet profound [see, for example, Morse and Feshbach, Chapter 7]. These functions are employed in studies of physical situations in which a source of something (radiation, heat, electric field, diffusing chemicals, etc.) makes its presence felt at some remote site. The source may be of any shape and it may be of constant or varying intensity. The idea is that the source may be dissected into an infinite array of Dirac functions, in space and/or time; then, knowing the remote effect of one of these elements, suitable integrations will lead to knowledge of the effect of the source as a whole. The Green function is the contribution of one such element. Consider the example of a uniform cartesian plane containing a source, one element of which is at location (x', y') emitting diffusant at time t'. The effect of that element at some other point (x, y), at some subsequent time t, is given by a Green function, which takes the form

9:14:1
$$\frac{Q(x',y',t')}{4\pi\kappa[t-t']}\exp\left\{\frac{-[x-x']^2-[y-y']^2}{4\kappa[t-t']}\right\}$$

if the plane is infinite in both spatial dimensions. κ is a characteristic constant, the diffusivity. The Q term is the intensity of the source element. The emission from that element is represented by

9:14:2
$$Q(x',y',t')\delta(x-x')\delta(y-y')\delta(t-t')\,\mathrm{d}x'\,\mathrm{d}y'\,\mathrm{d}t'$$

and involves three Dirac functions.