4

Resonance

4.1 The Method of Delaunay's Lunar Theory

Delaunay was the first astronomer to use the mechanics of Hamilton and Jacobi to obtain the approximated solution of the equations of motion of a celestial body. His lunar theory [22] is a pioneer work in many respects. We credit Delaunay with the introduction of the set of angle–action variables ℓ, g, h, L, G, H in which the Lagrange equations for the variation of the orbital elements under a perturbation are canonical. His theory of the motion of the Moon is not a collection of clever tricks, as other theories in the old Celestial Mechanics. Having obtained the variation equations in canonical form, his problem was to find the solutions of the differential equations defined by the Hamiltonian

$$H = H_0(J) + \varepsilon \sum_{h \in \mathbf{D}} A_h(J) \cos{(h|\theta)}, \tag{4.1}$$

where the canonical variables are $J \equiv (J_1, \ldots, J_N)$ and $\theta \equiv (\theta_1, \ldots, \theta_N)$, ε is a small parameter and $\mathbf{D} \subset \mathbf{Z}^N$. The technique adopted by Delaunay is methodologically very clear. He defined an *operation* and performed it, successively, almost 500 times. This operation starts with the choice of one argument $(h_1|\theta)$ in (4.1) and the consideration of the dynamical system defined by the abridged Hamiltonian

$$\mathcal{F}_1 = H_0(J) + \varepsilon A_{h_1}(J) \cos{(h_1|\theta)}. \tag{4.2}$$

This system is integrable, since the angles θ_i are present only through the linear combination $(h_1|\theta)$. The main step of one Delaunay operation is to obtain a particular solution of this selected system and to use this solution to derive a canonical transformation leading to the elimination of the term $A_{h_1}(J) \cos{(h_1|\theta)}$ from the given Hamiltonian. (In fact, it is a transformation leading to the substitution of this term by others with much smaller coefficients.) To obtain the solution of the dynamical system defined by $\mathcal{F}$, we introduce the Jacobian generating function

$$S(\theta, J^*) \stackrel{\text{def}}{=} (\theta|J^*) + \Sigma(\theta, J^*), \tag{4.3}$$

where Σ is a function of order $\mathcal{O}(\varepsilon)$, and consider the Hamilton–Jacobi equation

$$E_1 = H_0\left(\frac{\partial S}{\partial \theta}\right) + \varepsilon A_{h_1}\left(\frac{\partial S}{\partial \theta}\right)\cos(h_1|\theta). \tag{4.4}$$

The functions of $\partial S/\partial\theta_i$, on the right-hand side of this equation, may be expanded about $\partial S/\partial\theta_i = J_i^*$ and (4.4) becomes

$$E_1 = H_0(J^*) + \sum_{i=1}^{N} \frac{\partial H_0(J^*)}{\partial J_i^*}\frac{\partial \Sigma}{\partial \theta_i} + \varepsilon A_{h_1}(J^*)\cos(h_1|\theta) + \mathcal{O}(\varepsilon^2). \tag{4.5}$$

At variance with the standard Hamilton–Jacobi theory, we do not look for a complete solution of the equation. We assume $E_1 = H_0(J^*)$, and seek a suitable particular solution of the partial differential equation for Σ:

$$0 = \sum_{i=1}^{N} \frac{\partial H_0(J^*)}{\partial J_i^*}\frac{\partial \Sigma}{\partial \theta_i} + \varepsilon A_{h_1}(J^*)\cos(h_1|\theta) + \mathcal{O}(\varepsilon^2). \tag{4.6}$$

If the higher-order terms are neglected, we have the immediate particular solution

$$\Sigma = -\frac{\varepsilon A_{h_1}(J^*)\sin(h_1|\theta)}{(h_1|\nu^*)}, \tag{4.7}$$

where $\nu^* \equiv (\nu_1^*, \nu_2^*, \cdots, \nu_N^*)$ and

$$\nu_i^* = \frac{\partial H_0(J^*)}{\partial J_i^*}. \tag{4.8}$$

Once we have obtained a first-order solution of the dynamical system spanned by $\mathcal{F}_1$, we go back to the given Hamiltonian H and perform the transformation of the variables generated by the function S:

$$\theta_i^* = \frac{\partial S}{\partial J_i^*} = \theta_i + \frac{\partial \Sigma}{\partial J_i^*}, \qquad J_i = \frac{\partial S}{\partial \theta_i} = J_i^* + \frac{\partial \Sigma}{\partial \theta_i}. \tag{4.9}$$

To complete the exposition of a Delaunay operation, we write the full Hamiltonian as

$$H = \mathcal{F}_1 + \Delta\mathcal{F}. \tag{4.10}$$

Hence, according to (4.5)–(4.7), when the above variable change is done, $\mathcal{F}_1$ becomes

$$\mathcal{F}_1^*(\theta^*, J^*) = H_0(J^*) + \mathcal{O}(\varepsilon^2), \tag{4.11}$$

that is, E_1 plus the higher-order terms of (4.6), which were neglected when (4.7) was obtained. With the same change, the additional part $\Delta\mathcal{F}(\theta, J)$ is transformed into $\Delta\mathcal{F}(\theta^*, J^*)+\mathcal{O}(\varepsilon^2)$. (The function $\Delta\mathcal{F}$ is the same as before.)

The result of the Delaunay operation is, then, a new Hamiltonian

$$H^* = H_0(J^*) + \Delta\mathcal{F}(\theta^*, J^*) + \mathcal{O}(\varepsilon^2) \tag{4.12}$$

differing formally from the given one, in only two respects:

(a.) the term $\varepsilon A_{h_1} \cos(h_1|\theta)$ has disappeared;
(b.) new terms of order $\mathcal{O}(\varepsilon^2)$ were added.

In this way, performing as many operations as necessary, we may expect to eliminate from H all periodic terms of order $\mathcal{O}(\varepsilon)$. Indeed, as shown in the previous chapter, all these operations can be performed at one stroke, by finding the function S generating a transformation that eliminates all periodic terms of order $\mathcal{O}(\varepsilon)$.

We may also expect to eliminate, with a second sequence of operations, those terms of order $\mathcal{O}(\varepsilon^2)$, after that, the terms of order $\mathcal{O}(\varepsilon^3)$, and so on. In reality, as discussed in Sect. 3.12, this is not so. The combination of the arguments $(h|\theta)$ in the transformation of H tends to enlarge the set of values of h (the maximum of $|h|$ increases). Thus, values of h for which $(h|\nu^*)$ is too small can be reached (Poincaré Theorem) and the Delaunay theory, as well as the theories of the previous chapter (with the exception of Kolmogorov's) cannot be extended indefinitely. Only a finite number of operations can be done and the non-resonance condition $(h|\nu^*) \neq 0$ must be verified for all $h \in \mathbf{D}$, and for all h generated in the calculations. Otherwise, the theory needs to be modified as discussed thereafter.

We may also consider the case where one or more values $h \in \mathbf{D}$ are already such that $(h|\nu^*) \equiv 0$. This case happens when $H_0(J)$ is degenerate, that is, when H_0 does not depend on all components of J. One essential degeneracy of this kind appears in Celestial Mechanics where H_0 depends only on the Delaunay variable L and on the variable Λ, the canonical conjugate to the time t:

$$H_0 = -\frac{\mu^2}{2L^2} + \Lambda. \tag{4.13}$$

In this case, the Delaunay theory does not allow one to get rid of the terms independent of both the time t and the mean anomaly ℓ (conjugate to L). In the particular problem of the motion of the Moon, periodic terms of this kind do not exist in the given perturbation (see the discussion in Sect. 3.9) and the theory developed by Delaunay allowed all periodic terms of order $\mathcal{O}(\varepsilon)$ to be eliminated.

4.2 Introduction of the Square Root of the Small Parameter

Let us consider, in this section, the equations of the Delaunay theory in the case where one resonance exists. Let us assume that

$$(h_1|\nu^*) = 0 \tag{4.14}$$

for some $h_1 \in \mathbf{D}$ and some point $J^* \in \mathcal{O}$ ($\mathcal{O}$ is the open set of $\mathbf{R}^N$ under study). We may continue as in the previous section up to equation (4.7). However, in this case, the resonance $(h_1|\nu^*) = 0$ happens at one point of $\mathcal{O}$. At such a point, the first term in the right-hand side of (4.6) vanishes and the equation becomes singular. If we do not get rid of this singularity and continue calculating as before, the divisor appearing in the result will become null when the exact resonance is reached. To study this problem, we will perform the same sequence of calculations as in the previous section, but keeping in explicit form some second-order terms.

For the sake of simplicity, we will only consider, here, the simplest case of only one degree of freedom, in which case the resonance assumption given by (4.14) becomes, simply,

$$\nu_1^* = 0. \tag{4.15}$$

Let us introduce again the generating function as

$$S(\theta_1, J_1^*) = \theta_1 J_1^* + \Sigma(\theta_1, J_1^*)$$

and let us expand the function

$$H_0(J_1) = H_0\left(\frac{\partial S}{\partial \theta_1}\right) = H_0\left(J_1^* + \frac{\partial \Sigma}{\partial \theta_1}\right).$$

Then

$$H_0(J_1) = H_0(J_1^*) + \nu_1^* \frac{\partial \Sigma}{\partial \theta_1} + \frac{1}{2}\,\nu_{11}^* \left(\frac{\partial \Sigma}{\partial \theta_1}\right)^2 + \cdots,$$

where we have introduced

$$\nu_1^* = \frac{\mathrm{d}H_0(J_1^*)}{\mathrm{d}J_1^*}, \qquad \nu_{11}^* = \frac{\mathrm{d}^2 H_0(J_1^*)}{\mathrm{d}J_1^{*2}}. \tag{4.16}$$

In the same way, we expand

$$R_1(\theta_1, J_1) \stackrel{\text{def}}{=} A_{h_1}(J_1)\cos\theta_1$$

to obtain

$$R_1(\theta_1, J_1) = R_1(\theta_1, J_1^*) + \frac{\partial R_1(\theta_1, J_1^*)}{\partial J_1^*}\frac{\partial \Sigma}{\partial \theta_1} + \cdots.$$

When these expansions are substituted into the Hamilton–Jacobi equation (4.4), we obtain

$$E_1 = H_0(J_1^*) + \nu_1^* \frac{\partial \Sigma}{\partial \theta_1} + \frac{1}{2}\,\nu_{11}^* \left(\frac{\partial \Sigma}{\partial \theta_1}\right)^2 + \cdots + \varepsilon R_1(\theta_1, J_1^*) + \varepsilon \frac{\partial R_1}{\partial J_1^*}\frac{\partial \Sigma}{\partial \theta_1} + \cdots \tag{4.17}$$

and (4.6), correspondingly, becomes

$$\nu_1^* \frac{\partial \Sigma}{\partial \theta_1} + \frac{1}{2}\nu_{11}^* \left(\frac{\partial \Sigma}{\partial \theta_1}\right)^2 + \cdots + \varepsilon R_1(J_1^*, \theta_1) + \varepsilon \frac{\partial R_1}{\partial J_1^*}\frac{\partial \Sigma}{\partial \theta_1} + \cdots = 0. \quad (4.18)$$

Let us, now, investigate the algebraic inversion of this equation. This is done with the help of some classical results of Weierstrass' implicit functions theory. However, instead of making an application of the theory itself, we prefer, here, to adapt it to the present problem.

Equation (4.18) may be written in a more compact form as

$$\mathcal{F}(\sigma, \varepsilon) = a_{01}\varepsilon + a_{10}\sigma + a_{20}\sigma^2 + \sum_i \sum_j a_{ij}\sigma^i \varepsilon^j = 0, \quad (4.19)$$

where

$$\sigma = \frac{\partial \Sigma}{\partial \theta_1} \quad (4.20)$$

and the a_{ij} have obvious meanings. When the resonance condition

$$a_{10} = \nu_1^* = 0$$

holds, the leading terms in the expansion of $\mathcal{F}(\sigma, \varepsilon)$ are $a_{01}\varepsilon$ and $a_{20}\sigma^2$. Therefore, the only possibility of having $\mathcal{F}(\sigma, \varepsilon) = 0$, identically, with $a_{01} \neq 0$ and $a_{20} \neq 0$, is that the solution $\sigma(\varepsilon)$ has, at the origin, an algebraic critical point of order 2. Then, we may write

$$\sigma = b_1\sqrt{\varepsilon} + b_2\varepsilon + b_3\varepsilon\sqrt{\varepsilon} + \cdots. \quad (4.21)$$

Since $\sqrt{\varepsilon}$ has two branches, we have two solutions forming a system of two algebraic functions, each corresponding to one branch of $\sqrt{\varepsilon}$. It is worth emphasizing that, when the series written in (4.19) is convergent in a neighborhood of the origin, the fundamental theorem on algebraic functions can be used to prove the convergence of the solutions given by (4.21).

4.2.1 Garfinkel's Abnormal Resonance

One hypothesis implicitly considered above and in this whole chapter is $\nu_{11}^* \neq 0$. The case $\nu_{11}^* = 0$ was called, by Garfinkel, *abnormal*. In such a case, $a_{10} = a_{20} = 0$ and the leading terms of the expansion of $\mathcal{F}(\sigma, \varepsilon)$ are $a_{01}\varepsilon$ and $a_{30}\sigma^3$. Therefore, the origin is an algebraic critical point of order 3 and we have to use the cube root of ε instead of the square root in the series expansion of $\sigma(\varepsilon)$.

4.3 Delaunay Theory According to Poincaré

Poincaré considered Delaunay theory in the first part of his chapter on Bohlin's theory ([80], Chap. XIX). He considered the one-degree-of-freedom problem

with a disturbing potential formed by the term $\varepsilon R_1 = \varepsilon A_1 \cos\theta_1$ only. In this section, we present the complete Delaunay theory for the canonical equations defined by the Hamiltonian

$$H = H_0(J_1) + \sum_{k=1}^{\infty} \varepsilon^k H_{2k}(\theta_1, J_1). \tag{4.22}$$

One may note that the subscripts were modified to indicate the order of the terms in $\sqrt{\varepsilon}$.

The initial calculations are the same as in the previous section. Since we know that, in the neighborhood of the resonance, Σ may be expanded in a power series in $\sqrt{\varepsilon}$, we consider the canonical transformation

$$(\theta_1, J_1) \Rightarrow (\alpha, E)$$

defined by the Jacobian generating function

$$S = \theta_1 J_1^* + \sum_{k=1}^{n} \varepsilon^{k/2} S_k(\theta_1, E), \tag{4.23}$$

where J_1^* is the solution of the equation giving the exact resonance:

$$\nu_1(J_1^*) = \left(\frac{\mathrm{d}H_0}{\mathrm{d}J_1}\right)_{J_1 = J_1^*} = 0. \tag{4.24}$$

Poincaré considered, separately, the case $\nu_1^* = 0$ and the general case $\nu_1^* \neq 0$ (but close to zero). The consideration of the case $\nu_1^* \neq 0$ is, however, not necessary and is not done here.

The equations of the canonical transformation are

$$\alpha = \frac{\partial S}{\partial E}, \qquad J_1 = \frac{\partial S}{\partial \theta_1} \tag{4.25}$$

and the transformed Hamiltonian is assumed to have a main part

$$\varepsilon E + H^*(E)$$

independent of α, and a remainder $\mathcal{R}_{n+1}$ divisible by $\varepsilon^{(n+1)/2}$.

The solution is given by the integral

$$E = \text{const} \tag{4.26}$$

and the quadrature

$$\alpha = \int \frac{\partial}{\partial E}(H^* + \varepsilon E)\,\mathrm{d}t. \tag{4.27}$$

Since the transformation is conservative, we have

$$H(\theta_1, J_1) = \varepsilon E + H^*(E) + \mathcal{R}_{n+1}(\alpha, E). \tag{4.28}$$

Taking into account the canonical transformation generated by S, this equation becomes

$$H\left(\theta_1, \frac{\partial S}{\partial \theta_1}\right) = \varepsilon E + H^*(E) + \mathcal{R}_{n+1}. \tag{4.29}$$

To identify both sides of (4.29) according to the powers of $\sqrt{\varepsilon}$, we need the power-series expansions of H_k and H^*. These expansions are identical to those performed in Poincaré theory (see Sects. 3.2.1 and 3.2.2). We have

$$H_0 = G_{0,0} + \varepsilon G_{0,2} + \varepsilon^{3/2} G_{0,3} + \cdots + \varepsilon^{n/2} G_{0,n} + \cdots \tag{4.30}$$

$$H_k = G_{k,k} + \varepsilon^{1/2} G_{k,k+1} + \varepsilon G_{k,k+2} + \cdots + \varepsilon^{n/2} G_{k,n} + \cdots \tag{4.31}$$

and

$$H^*(E) = \sum_{k=0}^{n} \varepsilon^{k/2} H_k^*(E). \tag{4.32}$$

All remaining terms are at least of order $\varepsilon^{(n+1)/2}$. Since $\nu_1^* = 0$, then $G_{0,1} = 0$ and $G_{0,k} = \mathcal{E}_k$ (see 3.15). The functions $G_{k,j}$ are defined by (3.22). In particular, $G_{k,k} = H_k(\theta_1, J_1^*)$.

The identification in the powers of the small parameter is made simple by the fact that ε is always explicit in the formulas and that all other quantities are finite. Thus, we have

$$\begin{gathered}
H_0(J_1^*) = H_0^*, \\
0 = H_1^*, \\
\frac{1}{2}\nu_{11}^* \left(\frac{\partial S_1}{\partial \theta_1}\right)^2 + H_2(\theta_1, J_1^*) = H_2^* + E, \\
\nu_{11}^* \frac{\partial S_1}{\partial \theta_1}\frac{\partial S_2}{\partial \theta_1} + G_{2,3} + \mathcal{E}_3' = H_3^*, \\
\cdots\cdots \\
\nu_{11}^* \frac{\partial S_1}{\partial \theta_1}\frac{\partial S_k}{\partial \theta_1} + G_{2,k+1} + G_{4,k+1} + \cdots + \mathcal{E}_{k+1}' = H_{k+1}^*, \\
\cdots\cdots \\
\nu_{11}^* \frac{\partial S_1}{\partial \theta_1}\frac{\partial S_{n-1}}{\partial \theta_1} + G_{2,n} + G_{4,n} + \cdots + \mathcal{E}_n' = H_n^*.
\end{gathered} \tag{4.33}$$

(The functions $\mathcal{E}_k'$ are those defined implicitly by (3.20).) All remaining terms have at least $\varepsilon^{(n+1)/2}$ as a factor and are supposed to be grouped with the remainder $\mathcal{R}_{n+1}$.

As in the theories of the previous chapter, the first equation gives H_0^* and says that it is the value of the function H_0 at $J_1 = J_1^*$. Thus H_0^* is, now, just a number (it does not depend on the new variables α, E). The second

equation says that $H_1^* = 0$. The third equation is the fundamental equation of Delaunay theory (the *Delaunay* or *Delaunay–Poincaré equation*):

$$\frac{1}{2}\,\nu_{11}^* \left(\frac{\partial S_1}{\partial \theta_1}\right)^2 + H_2(\theta_1, J_1^*) - H_2^* = E. \tag{4.34}$$

This equation is indeterminate while H_2^* is not fixed. This indetermination is overcome by introducing the averaging rule

$$H_2^* = < H_2(\theta_1, J_1^*) >, \tag{4.35}$$

where $< \cdots >$ stands for the average over the angle θ_1. Therefore, we have

$$\frac{1}{2}\,\nu_{11}^* \left(\frac{\partial S_1}{\partial \theta_1}\right)^2 + H_{2(K)} = E, \tag{4.36}$$

where

$$H_{2(K)}(J_1^*) = H_2(\theta_1, J_1^*) - < H_2(\theta_1, J_1^*) > . \tag{4.37}$$

Taking into account that the functions $G_{2,k+1}, G_{4,k+1}, \cdots$ and $\mathcal{E}'_{k+1}$ are completely known when the functions $S_1, S_2, \cdots, S_{k-1}$ are known, the generic or *homological* form of (4.33) (for $k \geq 2$) is

$$\nu_{11}^* \frac{\partial S_1}{\partial \theta_1}\frac{\partial S_k}{\partial \theta_1} + \Psi_{k+1}^*(\theta_1, E) = H_{k+1}^*(E), \tag{4.38}$$

where Ψ_{k+1} represents known functions. At variance with the fundamental Delaunay–Poincaré equation, the homological equation is linear and it is sufficient to obtain particular solutions of it.

4.3.1 First-Approximation Solution

When a complete integral of the fundamental equation is known, the generating function

$$S_{(1)} = \theta_1 J_1^* + \sqrt{\varepsilon}\, S_1(\theta_1, E)$$

defines a canonical transformation leading to a transformed Hamiltonian independent of α, except for terms factored by, at least, $\varepsilon^{3/2}$.

From the equations of the canonical transformation we have

$$J_1 \;=\; J_1^* + \sqrt{\varepsilon}\,\frac{\partial S_1}{\partial \theta_1} + \mathcal{O}(\varepsilon) = J_1^* \pm \sqrt{\frac{2\varepsilon}{\nu_{11}^*}(E - H_{2(K)})} + \mathcal{O}(\varepsilon), \tag{4.39}$$

$$\alpha \;=\; \sqrt{\varepsilon}\,\frac{\partial S_1}{\partial E} + \mathcal{O}(\varepsilon) = \pm\frac{\partial}{\partial E}\int \sqrt{\frac{2\varepsilon}{\nu_{11}}(E - H_{2(K)})}\;\mathrm{d}\theta_1 + \mathcal{O}(\varepsilon). \tag{4.40}$$

The last equation, combined with (4.27) (which is reduced, at this order, to $\alpha = \int \varepsilon\,\mathrm{d}t$), gives

$$t - t_0 = \int \frac{\pm \mathrm{d}\theta_1}{\sqrt{2\varepsilon\nu_{11}(E - H_{2(K)})}} + \mathcal{O}(\varepsilon) \tag{4.41}$$

showing that the time scale of resonant phenomena is inversely proportional to $\sqrt{\varepsilon}$, that is, the frequencies associated with the resonance are proportional to $\sqrt{\varepsilon}$.

Equations (4.39) and (4.41) are the formal solutions of order $\mathcal{O}(\sqrt{\varepsilon})$ of the problem of Delaunay, in the presence of one resonance, in one degree of freedom.

4.4 Garfinkel's Ideal Resonance Problem

Let us use the Delaunay theory to obtain a complete solution of the Ideal Resonance Problem. This problem, thoroughly studied by Garfinkel [37], is defined as the problem of obtaining a formal solution of order $\varepsilon^{n/2}$ of the canonical equations defined by the Hamiltonian

$$H = H_0(J_1) - \varepsilon A(J_1)\cos\theta_1 \tag{4.42}$$

in the neighborhood of the value J_1^* for which $\nu_1 = \mathrm{d}H_0/\mathrm{d}J_1 = 0$. The disturbing term has not, here, the same form $2\varepsilon A(J_1)\sin^2(\theta_1/2)$ considered in Garfinkel's work, but the two forms are equivalent.

This Hamiltonian system has two equilibrium solutions, viz. $\theta_1 = 0$ and $\theta_1 = \pi$ whose stability depends on the sign of $A^*\nu_{11}^*$ ($A^* = A(J_1^*)$). Without loss of generality, we assume that $A^*\nu_{11}^* > 0$ and the stable equilibrium is at $\theta_1 = 0$; otherwise, it is enough to change θ_1 into $\theta_1' + \pi$ so that the system satisfies this assumption.

The fundamental equation corresponding to the Hamiltonian of (4.42) is

$$\frac{1}{2}\,\nu_{11}^* \left(\frac{\partial S_1}{\partial \theta_1}\right)^2 - A^*\cos\theta_1 = E \tag{4.43}$$

or

$$\frac{\partial S_1}{\partial \theta_1} = \pm\sqrt{\frac{2}{\nu_{11}^*}(E + A^*\cos\theta_1)}, \tag{4.44}$$

where we take into account that $H_2^* = < -A^*\cos\theta_1 > = 0$.

We may note that this fundamental equation is nothing but the Hamilton–Jacobi equation of the simple pendulum. However, at variance with the conventional simple pendulum, the "inverse mass" ν_{11}^* may be either positive or negative. The solutions of the simple pendulum given in Sect. B.1 apply without modification. We just have to take care of the sign differences between the cases $\nu_{11}^* < 0$ and $\nu_{11}^* > 0$.

The homological equation is (see 4.38)

$$\frac{\partial S_k}{\partial \theta_1} = \frac{1}{\nu_{11}^*}\left(\frac{\partial S_1}{\partial \theta_1}\right)^{-1}(H_{k+1}^* - \Psi_{k+1}^*) \qquad (k = 2, \cdots, n), \tag{4.45}$$

where Ψ^*_{k+1} is a polynomial in the derivatives of $S_1, S_2, \cdots, S_{k-1}$ whose coefficients are constants or derivatives of H_2. For instance

$$\Psi^*_3 = \frac{\partial S_1}{\partial \theta_1}\left[\frac{1}{6}\,\nu^*_{111}\left(\frac{\partial S_1}{\partial \theta_1}\right)^2 + \frac{\partial H_2}{\partial J^*_1}\right] \tag{4.46}$$

and

$$\begin{aligned}\Psi^*_4 \;=\; & \frac{1}{2}\,\nu^*_{11}\left(\frac{\partial S_2}{\partial \theta_1}\right)^2 + \frac{1}{2}\,\nu^*_{111}\left(\frac{\partial S_1}{\partial \theta_1}\right)^2\frac{\partial S_2}{\partial \theta_1} + \frac{1}{24}\,\nu^*_{1111}\left(\frac{\partial S_1}{\partial \theta_1}\right)^4 \\ & + \frac{\partial H_2}{\partial J^*_1}\frac{\partial S_2}{\partial \theta_1} + \frac{1}{2}\,\frac{\partial^2 H_2}{\partial J^{*2}_1}\left(\frac{\partial S_1}{\partial \theta_1}\right)^2.\end{aligned}$$

From the previous equations, we may write

$$H_2(\theta_1, J^*_1) = -A^*\cos\theta_1 = E - \frac{1}{2}\,\nu^*_{11}\left(\frac{\partial S_1}{\partial \theta_1}\right)^2 \tag{4.47}$$

and

$$\frac{\partial^k H_2}{\partial J^{*k}_1} = -\frac{\mathrm{d}^k A^*}{\mathrm{d}J^{*k}_1}\cos\theta_1 = \frac{1}{A^*}\,\frac{\mathrm{d}^k A^*}{\mathrm{d}J^{*k}_1}\left[E - \frac{1}{2}\,\nu^*_{11}\left(\frac{\partial S_1}{\partial \theta_1}\right)^2\right], \tag{4.48}$$

that is, H_2 and its derivatives may be written as polynomials in the first derivative of S_1. Therefore, $\Psi^*_3, \Psi^*_4, \cdots, \Psi^*_{n-1}$ may be, successively, written as polynomials in the first derivative of S_1:

$$\Psi^*_{k+1} = \sum_{k'=0}^{k+1} C_{k,k'}\left(\frac{\partial S_1}{\partial \theta_1}\right)^{k'}, \tag{4.49}$$

where $C_{k,k'} = 0$ when k and k' have the same parity; then, (4.45) may be written as

$$\frac{\partial S_k}{\partial \theta_1} = \frac{1}{\nu^*_{11}}\left(\frac{\partial S_1}{\partial \theta_1}\right)^{-1}\left[H^*_{k+1} - \sum_{k'=0}^{k+1} C_{k,k'}\left(\frac{\partial S_1}{\partial \theta_1}\right)^{k'}\right]. \tag{4.50}$$

To avoid the singularity at the libration boundaries, where $\partial S_1/\partial\theta_1 = 0$, H^*_{k+1} may be chosen to be such that the coefficient of $(\partial S_1/\partial\theta_1)^{-1}$ in (4.50) vanishes:

$$H^*_{k+1} = C_{k,0}.$$

(One may note that $H^*_k = 0$ for all k odd because of the parity rule of the coefficients $C_{k,k'}$.) The homological equation then becomes

$$\frac{\partial S_k}{\partial \theta_1} = -\frac{1}{\nu^*_{11}}\sum_{k'=1}^{k+1} C_{k,k'}\left(\frac{\partial S_1}{\partial \theta_1}\right)^{k'-1}. \tag{4.51}$$

In particular, for $k = 2$, we have

$$\frac{\partial S_2}{\partial \theta_1} = -\frac{E}{A^* \nu_{11}^*} \frac{\mathrm{d}A^*}{\mathrm{d}J_1^*} - \left(\frac{\nu_{111}^*}{6\nu_{11}^*} - \frac{1}{2A^*} \frac{\mathrm{d}A^*}{\mathrm{d}J_1^*} \right) \left(\frac{\partial S_1}{\partial \theta_1} \right)^2. \tag{4.52}$$

Once S is known, we may construct the formal solutions of the Ideal Resonance Problem. To order $\mathcal{O}(\varepsilon)$, they are:

$$\begin{aligned} J_1 &= J_1^* + \sqrt{\varepsilon} \frac{\partial S_1}{\partial \theta_1} + \varepsilon \frac{\partial S_2}{\partial \theta_1} \\ \alpha \ &= \varepsilon(t - t_0) \ = \ \sqrt{\varepsilon} \frac{\partial S_1}{\partial E} + \varepsilon \frac{\partial S_2}{\partial E}. \end{aligned} \tag{4.53}$$

4.4.1 Garfinkel–Jupp–Williams Integrals

The integration of (4.51), for all k, involves the integrals

$$\mathcal{I}_k = \int \left(\frac{\partial S_1}{\partial \theta_1} \right)^k d\theta_1 = \int \left(\frac{2}{\nu_{11}^*} (E + A^* \cos \theta_1) \right)^{k/2} \mathrm{d}\theta_1,$$

which can be calculated by means of recurrence formulas [36]. Differentiating (4.43) with respect to θ_1, we obtain

$$\frac{\partial S_1}{\partial \theta_1} \frac{\partial}{\partial \theta_1} \left(\frac{\partial S_1}{\partial \theta_1} \right) = -\frac{A^*}{\nu_{11}^*} \sin \theta_1. \tag{4.54}$$

Hence,

$$\frac{\partial}{\partial \theta_1} \left(\frac{\partial S_1}{\partial \theta_1} \right)^k = -\ k \left(\frac{\partial S_1}{\partial \theta_1} \right)^{k-2} \frac{A^*}{\nu_{11}^*} \sin \theta_1, \tag{4.55}$$

$$\frac{\partial^2}{\partial \theta_1^2} \left(\frac{\partial S_1}{\partial \theta_1} \right)^k = k(k-2) \left(\frac{\partial S_1}{\partial \theta_1} \right)^{k-4} \left(\frac{A^*}{\nu_{11}^*} \right)^2 \sin^2 \theta_1 - k \left(\frac{\partial S_1}{\partial \theta_1} \right)^{k-2} \frac{A^*}{\nu_{11}^*} \cos \theta_1.$$

The trigonometric functions may be eliminated with the help of (4.43) giving

$$\begin{aligned} \frac{\partial^2}{\partial \theta_1^2} \left(\frac{\partial S_1}{\partial \theta_1} \right)^k = \ & -\frac{k^2}{4} \left(\frac{\partial S_1}{\partial \theta_1} \right)^k + \ k(k-1) \frac{E}{\nu_{11}^*} \left(\frac{\partial S_1}{\partial \theta_1} \right)^{k-2} \\ & + k(k-2) \frac{A^{*2} - E^2}{\nu_{11}^{*2}} \left(\frac{\partial S_1}{\partial \theta_1} \right)^{k-4} \end{aligned}$$

whose integration, with respect to θ_1, followed by the use of (4.55) and the definition of $\mathcal{I}_k$, yields

$$\frac{k}{4} \mathcal{I}_k = \left(\frac{\partial S_1}{\partial \theta_1} \right)^{k-2} \frac{A^*}{\nu_{11}^*} \sin \theta_1 + (k-1) \frac{E}{\nu_{11}^*} \mathcal{I}_{k-2} + (k-2) \frac{A^{*2} - E^2}{\nu_{11}^{*2}} \mathcal{I}_{k-4}. \tag{4.56}$$

(The integration constant is chosen to be such that $\mathcal{I}_k = 0$ at $\theta_1 = 0$.) Thus, all integrals are known when we know a sequence of four of them.

For even k, the integrals are elementary and are the same no matter whether the motion is a libration or a circulation:

$$\mathcal{I}_0 = \int \mathrm{d}\theta_1 = \theta_1,$$

$$\mathcal{I}_2 = \int \left(\frac{\partial S_1}{\partial \theta_1}\right)^2 \mathrm{d}\theta_1 = \frac{2E}{\nu_{11}^*}\,\theta_1 + \frac{2A^*}{\nu_{11}^*}\sin\theta_1. \tag{4.57}$$

For odd k, the integrals are elliptic and we have to consider separately the cases where $E\nu_{11}^* > A^*\nu_{11}^*$ (circulation), $\mid E \mid < \mid A^* \mid$ (libration) and $E = A^*$ (asymptotic motion). This will be done in the forthcoming sections.

4.4.2 Circulation ($E\nu_{11}^* > A^*\nu_{11}^* > 0$)

Let us calculate the solutions of the Ideal Resonance Problem in the case of circulations. The first step is to calculate the Garfinkel–Jupp–Williams integrals necessary to generate the solutions at all orders. To complete the set of four integrals necessary to span the whole set, we need two of them with odd values of k. They are

$$\mathcal{I}_{-1} = \pm\sqrt{\frac{2\nu_{11}^*}{E+A^*}}\;\mathcal{F}\left(\frac{\theta_1}{2},\kappa\right),$$

and

$$\mathcal{I}_1 = \pm\sqrt{\frac{8}{\nu_{11}^*}\,(E+A^*)}\;\mathcal{E}\left(\frac{\theta_1}{2},\kappa\right),$$

where $\mathcal{F}(\frac{\theta_1}{2},\kappa)$ and $\mathcal{E}(\frac{\theta_1}{2},\kappa)$ are incomplete elliptic integrals[1] of the first and second kind, respectively, of modulus

$$\kappa = \sqrt{\frac{2A^*}{E+A^*}} \qquad (0 < \kappa < 1).$$

Double signs were used in front of the square roots to stress that these functions have two branches each corresponding to a distinct family of circulations.

The solutions of the equations for S_k ($k = 1$ and $k = 2$) are

$$S_1 = \mathcal{I}_1 = \pm\frac{4}{\kappa}\sqrt{\frac{A^*}{\nu_{11}^*}}\;\mathcal{E}\left(\frac{\theta_1}{2},\kappa\right) \tag{4.58}$$

and

$$S_2 = -\frac{E}{A^*\nu_{11}^*}\,\frac{\mathrm{d}A^*}{\mathrm{d}J_1^*}\,\mathcal{I}_0 - \frac{\nu_{111}^*}{6\nu_{11}^*}\,\mathcal{I}_2 + \frac{1}{2A^*}\,\frac{\mathrm{d}A^*}{\mathrm{d}J_1^*}\,\mathcal{I}_2,$$

[1] The slight change in the usual notation for the elliptic integrals made here ($\mathcal{F}$ and $\mathcal{E}$ instead of F and E) is necessary to avoid confusion with other functions in the book. $\mathbb{K}$ and $\mathbb{E}$ are the corresponding complete elliptic integrals.

or

$$S_2 = -\frac{E\nu_{111}^*}{3\nu_{11}^{*2}}\,\theta_1 + \left(\frac{\mathrm{d}A^*}{\mathrm{d}J_1^*} - \frac{A^*\nu_{111}^*}{3\nu_{11}^*}\right)\frac{\sin\theta_1}{\nu_{11}^*}. \tag{4.59}$$

From (4.58) and (4.59) we have, respectively,

$$\frac{\partial S_1}{\partial E} = \pm\frac{\kappa}{\sqrt{A^*\nu_{11}^*}}\,\mathcal{F}\left(\frac{\theta_1}{2},\kappa\right), \qquad \frac{\partial S_2}{\partial E} = -\frac{\nu_{111}^*\theta_1}{3\nu_{11}^{*2}}.$$

Therefore,

$$t = t_0 \pm \frac{\kappa}{\sqrt{\varepsilon A^*\nu_{11}^*}}\,\mathcal{F}\left(\frac{\theta_1}{2},\kappa\right) - \frac{\nu_{111}^*}{3\nu_{11}^{*2}}\,\theta_1 + \mathcal{O}(\sqrt{\varepsilon}), \tag{4.60}$$

where the upper sign corresponds to prograde circulations and the lower one to retrograde circulations. $\theta_1 = 0$ when $t = t_0$.

The period of the circulations is given, to this order, by

$$T = \frac{2\kappa}{\sqrt{\varepsilon A^*\nu_{11}^*}}\,\mathbb{K}(\kappa) \mp \frac{2\pi\nu_{111}^*}{3\nu_{11}^{*2}} + \mathcal{O}(\sqrt{\varepsilon}), \tag{4.61}$$

where $\mathbb{K}(\kappa)$ is the complete elliptic integral of the first kind of modulus κ.

The use of Jacobian functions is necessary to write the complete solution, as well as, for instance, to give the explicit form of the time law $\theta = \theta(t)$. Inverting the elliptic integral in (4.60) we obtain

$$\theta_1 = \pm 2\,\mathrm{am}\left(\tau + \frac{\sqrt{\varepsilon A^*\nu_{11}^*}}{\kappa}\,\frac{\nu_{111}^*}{3\nu_{11}^{*2}}\,\theta_1\right) + \mathcal{O}(\varepsilon),$$

where *am* is the Jacobian *amplitude*, and

$$\tau = \frac{\sqrt{\varepsilon A^*\nu_{11}^*}}{\kappa}\,(t - t_0). \tag{4.62}$$

To the given order of approximation, we may still write

$$\theta_1 = \pm\,2\,\mathrm{am}\,\tau + 4\,\frac{\sqrt{\varepsilon A^*\nu_{11}^*}}{\kappa}\,\frac{\nu_{111}^*}{3\nu_{11}^{*2}}\,\mathrm{am}\,\tau\,\mathrm{dn}\,\tau + \mathcal{O}(\varepsilon), \tag{4.63}$$

where *dn* is the Jacobian *delta amplitude* elliptic function.

The variation of the action J_1, to the same order of approximation, is

$$\begin{aligned} J_1 \;=\; & J_1^* \pm \frac{2}{\kappa}\sqrt{\frac{\varepsilon A^*}{\nu_{11}^*}}\,\sqrt{1 - \kappa^2\sin^2\frac{\theta_1}{2}} - \frac{\varepsilon E\nu_{111}^*}{3\nu_{11}^{*2}} \\ & + \varepsilon\left(\frac{\mathrm{d}A^*}{\mathrm{d}J_1^*} - \frac{A^*\nu_{111}^*}{3\nu_{11}^*}\right)\frac{\cos\theta_1}{\nu_{11}^*} + \mathcal{O}(\varepsilon\sqrt{\varepsilon}), \end{aligned} \tag{4.64}$$

where the upper sign corresponds to motions above the libration zone ($J_1 > J_1^*$) and the lower sign to motions below the libration zone ($J_1 < J_1^*$). One should be aware that the relationship between the double signs in (4.63) and (4.64) is not always the same. When $\nu_{11}^* > 0$, the upper (resp. lower) sign in one of them corresponds to the upper (resp. lower) sign in the other (the circulations above the libration zone are prograde and the circulations below the libration zone are retrograde). When $\nu_{11}^* < 0$, we have to consider that the second of them carries the sign of $\partial S_1/\partial\theta_1$ (which is proportional to $\sqrt{A^*/\nu_{11}^*}$), while the first of them carries the sign of $\partial S_1/\partial E$ (which is proportional to $\nu_{11}^*\sqrt{A^*/\nu_{11}^*}$; written as $\sqrt{A^*\nu_{11}^*}$). Then, when $\nu_{11}^* < 0$, these two partial derivatives have opposite signs and, to the upper sign in one of the equations, corresponds the lower sign in the other (the circulations above the libration zone are retrograde and the circulations below the libration zone are prograde).

In the inner limit $\kappa \to 1$, we have $\mathbb{K} \to \infty$ and, thus, $T \to \infty$. The outer limit $\kappa \to 0$ corresponds to $E \to \infty$. From (4.56) and (4.57), it is evident that, for k even, $\mathcal{I}_k$ has a leading term in $E^{k/2}$; thus, for $\kappa \to 0$, the series giving the function S is divergent, meaning that this theory does not allow us to study the motion far of the resonance; it is only valid in the region of *deep* resonance where $\kappa > \mathcal{O}(\sqrt{\varepsilon})$ and where the general theories of the previous chapter would fail because of the small divisor ν_1^*.

4.4.3 Libration ($|E| < |A^*|$)

The basic equations for librations and circulations are the same. However, elliptic integrals must be treated in a different way since, now, $\kappa > 1$. We need to use the reciprocal modulus transformation

$$\sin\zeta = \kappa \sin\frac{\theta_1}{2} \tag{4.65}$$

(see Sect. B.1.2) and the solutions describing the librations are obtained from those describing the circulations by means of the well-known relations

$$\kappa\mathcal{F}\left(\frac{\theta_1}{2},\kappa\right) = \mathcal{F}\left(\zeta,\frac{1}{\kappa}\right) \tag{4.66}$$

and

$$\kappa\,\mathcal{E}\left(\frac{\theta_1}{2},\kappa\right) = \kappa^2\,\mathcal{E}\left(\zeta,\frac{1}{\kappa}\right) - (\kappa^2-1)\,\mathcal{F}\left(\zeta,\frac{1}{\kappa}\right). \tag{4.67}$$

We thus have

$$\mathcal{I}_{-1} = \sqrt{\frac{\nu_{11}^*}{A^*}}\,\mathcal{F}\left(\zeta,\frac{1}{\kappa}\right)$$

and

$$\mathcal{I}_1 = 4\sqrt{\frac{A^*}{\nu_{11}^*}}\left[\,\mathcal{E}\left(\zeta,\frac{1}{\kappa}\right) + \beta\mathcal{F}\left(\zeta,\frac{1}{\kappa}\right)\right],$$

where

$$\beta = \frac{1-\kappa^2}{\kappa^2} \tag{4.68}$$

and

$$\zeta = \arcsin\left(\kappa \sin\frac{\theta_1}{2}\right) = \arcsin\sqrt{\frac{A^*(1-\cos\theta_1)}{E+A^*}}. \tag{4.69}$$

The integrals $\mathcal{I}_0$ and $\mathcal{I}_2$ are the same as before.

The solution of the equation for $k=1$ is again

$$S_1 = \mathcal{I}_1,$$

(with the new value of $\mathcal{I}_1$); for $k=2$, the solution is the same as for circulations. We also have

$$\frac{\partial S_1}{\partial E} = \frac{1}{\sqrt{A^*\nu_{11}^*}}\,\mathcal{F}\left(\zeta, \frac{1}{\kappa}\right).$$

Substitution of these results into (4.53) gives, now, the time law

$$t = t_0 + \frac{1}{\sqrt{\varepsilon A^*\nu_{11}^*}}\,\mathcal{F}\left(\zeta, \frac{1}{\kappa}\right) - \frac{\nu_{111}^*}{3\nu_{11}^{*2}}\,\theta_1 + \mathcal{O}(\sqrt{\varepsilon}), \tag{4.70}$$

where we assume $\theta_1 = 0$ and $\dot{\theta}_1 > 0$ (or $\zeta = 0$ and $\dot{\zeta} > 0$) at $t = t_0$.

The period of the librations is the time for θ_1 to perform a complete oscillation between the boundaries of the libration. We may first note that the term proportional to θ_1 does not contribute to the period since the angle θ_1 will be brought back to the initial value without completing one revolution; this term only says that θ_1 is faster in one direction than in another (if $\nu_{111}^* > 0$, it is faster when θ_1 grows). We have to consider, then, only the contribution of the term involving the elliptic integral, whose calculation is the same as for the simple pendulum:

$$T = \frac{4}{\sqrt{\varepsilon A^*\nu_{11}^*}}\,\mathbb{K}\left(\frac{1}{\kappa}\right) + \mathcal{O}(\sqrt{\varepsilon}). \tag{4.71}$$

The inversion of the elliptic integral in (4.70) gives, now,

$$\kappa \sin\frac{\theta_1}{2} = \sin\zeta = \operatorname{sn}\left\{\sqrt{\varepsilon A^*\nu_{11}^*}\left[(t-t_0) + \left(\frac{\nu_{111}^*}{3\nu_{11}^{*2}}\right)\theta_1\right]\right\} + \mathcal{O}(\varepsilon), \tag{4.72}$$

where *sn* is the Jacobian *sine amplitude* elliptic function with modulus $1/\kappa$. An iteration over θ_1 is necessary to complete the inversion of (4.70).

The analog of (4.64), in this case, is

$$J_1 = J_1^* \pm \frac{2}{\kappa}\sqrt{\frac{\varepsilon A^*}{\nu_{11}^*}}\cos\zeta - \frac{\varepsilon E\nu_{111}^*}{3\nu_{11}^{*2}} + \varepsilon\left(\frac{\mathrm{d}A^*}{\mathrm{d}J_1^*} - \frac{A^*\nu_{111}^*}{3\nu_{11}^*}\right)\frac{\cos\theta_1}{\nu_{11}^*} + \mathcal{O}(\varepsilon\sqrt{\varepsilon}). \tag{4.73}$$

In all equations before the last one, we have not used double signs since the two branches of the square roots meet at the boundary of the libration and are parts of the same solution. However, as in the case of circulations, when $\nu_{11}^* < 0$, $\sqrt{A^*/\nu_{11}^*}$ and $\sqrt{A^*\nu_{11}^*}$ must be considered with different signs. Hence, a double sign was included in the last equation, the positive sign holding when $\nu_{11}^* > 0$ and the negative one when $\nu_{11}^* < 0$.

The inner limit $\kappa \to \infty$ ($\frac{1}{\kappa} \to 0$) corresponds to $E \to -A^*$, that is, to the stable equilibrium point. The outer limit $\kappa \to 1$ corresponds to the separatrix (see below).

4.4.4 Asymptotic Motions ($E = A^*$)

When $\kappa = 1$ ($E = A^*$), (4.44) becomes, simply,

$$\frac{\partial S_1}{\partial \theta_1} = \pm \sqrt{\frac{2A^*}{\nu_{11}^*}\,(1 + \cos\theta_1)} = \pm \sqrt{\frac{4A^*}{\nu_{11}^*}}\,\cos\frac{\theta_1}{2};$$

the corresponding integral is only pseudo-elliptic and gives

$$S_1 = \pm \sqrt{\frac{8A^*}{\nu_{11}^*}\,(1 - \cos\theta_1)} = \pm \sqrt{\frac{16A^*}{\nu_{11}^*}}\,\sin\frac{\theta_1}{2}.$$

The derivative $\partial S_1/\partial E$ needs some special consideration since, now, E is a constant. This derivative may be obtained by calculating $\partial^2 S_1/\partial\theta_1\partial E$ from (4.44), then making $E = A^*$, and integrating with respect to θ_1. Then

$$\frac{\partial S_1}{\partial E} = \pm \int \sqrt{\frac{1}{4A^*\nu_{11}^*}}\,\sec\frac{\theta_1}{2}\,\mathrm{d}\theta_1 = \pm \sqrt{\frac{1}{A^*\nu_{11}^*}}\,\ln\tan\left(\frac{\pi}{4} + \frac{\theta_1}{4}\right)$$

($-\pi < \theta_1 < \pi$). The formal solution of order $\mathcal{O}(\varepsilon)$, for this particular choice of the integration constant, is

$$J_1 = J_1^* \pm \sqrt{\frac{4A^*\varepsilon}{\nu_{11}^*}}\,\cos\frac{\theta_1}{2} - \frac{A^*\varepsilon\nu_{111}^*}{3\nu_{11}^{*2}}\,(1 + \cos\theta_1) + \frac{\varepsilon}{\nu_{11}^*}\frac{\mathrm{d}A^*}{\mathrm{d}J_1^*}\cos\theta_1 + \mathcal{O}(\varepsilon\sqrt{\varepsilon}), \tag{4.74}$$

$$t = t_0 \pm \sqrt{\frac{1}{\varepsilon A^*\nu_{11}^*}}\,\ln\tan\left(\frac{\pi}{4} + \frac{\theta_1}{4}\right) - \frac{\nu_{111}^*}{3\nu_{11}^{*2}}\,\theta_1 + \mathcal{O}(\sqrt{\varepsilon}). \tag{4.75}$$

In these two equations, each choice in the double signs corresponds to one of the separatrices. They are to be chosen in accordance with the same rules used for circulations: upper or lower separatrix in the double sign of (4.74) and prograde or retrograde motion in the double sign of (4.75). The terms coming from the derivatives of S_2 introduce an asymmetric correction to the height of the pendulum separatrices and on the asymptotic motions on them.

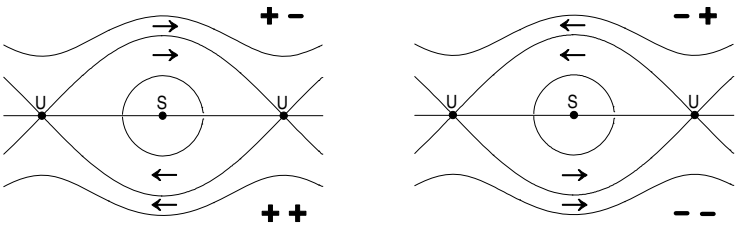

Fig. 4.1. Solutions of the Ideal Resonance Problem for diverse sign choices. *Left*: $\nu_{11}^* > 0$. *Right*: $\nu_{11}^* < 0$

4.5 Angle–Action Variables of the Ideal Resonance Problem

The angle–action variables of the Ideal Resonance Problem may be easily calculated using the one-degree-of freedom formulas of Sect. 2.1.1. We just have to pay attention to the need of some notation changes, since J_1 was already used to denote the actions in the undisturbed ($\varepsilon = 0$) problem. We will calculate the new angle w_1 and the new action

$$\Lambda_1 = \pm\frac{1}{2\pi}\oint (J_1 - J_1^*)d\theta_1, \tag{4.76}$$

in the two regimes of periodic motion: circulation and libration. The introduction of J_1^* in the function under the integral sign has the effect of adding a constant to the definition given by (2.6); this can always be done, since actions are defined except for an arbitrary additive constant.

4.5.1 Circulation

From (4.76) and (4.64), we have

$$\Lambda_1 = \pm\frac{4}{\kappa\pi}\sqrt{\frac{\varepsilon A^*}{\nu_{11}^*}}\,\mathbb{E}(\kappa) \mp \frac{\varepsilon E \nu_{111}^*}{3\nu_{11}^{*2}} + \mathcal{O}(\varepsilon\sqrt{\varepsilon}), \tag{4.77}$$

where the sign in front of the integral is to be fixed in accordance with the rules stated in Sect. 2.1.2. It is positive when $\dot\theta_1 > 0$ and negative when $\dot\theta_1 < 0$. Combining this rule with the double sign of S_1, there are four possible sign combinations: As a rule of thumb, the first of the double signs is + when $\nu_{11}^* > 0$ and − when $\nu_{11}^* < 0$ and the second one is + for retrograde motions and − for prograde motions. (See Fig. 4.1.)

The calculation of w_1 gives

$$w_1 = \pm\frac{\pi\mathcal{F}(\theta_1/2,\kappa)}{\mathbb{K}(\kappa)} - \frac{\sqrt{\varepsilon A^*\nu_{11}^*}\nu_{111}^*\pi}{3\kappa\nu_{11}^{*2}\mathbb{K}(\kappa)}\left(\theta_1 - \frac{\pi\mathcal{F}(\theta_1/2,\kappa)}{\mathbb{K}(\kappa)}\right) + \mathcal{O}(\varepsilon). \tag{4.78}$$

or $w_1 = \dot{w}_1(t - t_0)$, where

$$\dot{w}_1 = \frac{2\pi}{T} = \frac{\pi\sqrt{\varepsilon A^* \nu_{11}^*}}{\kappa \mathbb{K}(\kappa)} \left(1 \pm \frac{\pi \nu_{111}^* \sqrt{\varepsilon A^* \nu_{11}^*}}{3\nu_{11}^{*2} \kappa \mathbb{K}(\kappa)}\right) + \mathcal{O}(\varepsilon\sqrt{\varepsilon}), \tag{4.79}$$

the double signs corresponding to prograde or retrograde circulations as in (4.60).

4.5.2 Libration

We continue as before, just taking into account that, in the libration regime, $\mathcal{I}_1$ is not the same as for a circulation. In this case, the contribution of some terms of $J_1 - J_1^*$ vanishes, since θ_1 oscillates in a bounded interval returning to the initial value after one libration period, without performing a complete rotation. The first approximation of the angle–action variables of the libration is, thus, the same as in the simple pendulum (with just a different constant factor and a double sign in Λ_1):

$$\begin{aligned} \Lambda_1 &= \pm\frac{8}{\pi}\sqrt{\frac{\varepsilon A^*}{\nu_{11}^*}}\left[\mathbb{E}\left(\frac{1}{\kappa}\right) + \beta\mathbb{K}\left(\frac{1}{\kappa}\right)\right] + \mathcal{O}(\varepsilon^{3/2}) \\ &= \pm\frac{2}{\kappa^2}\sqrt{\frac{\varepsilon A^*}{\nu_{11}^*}}\left(1 + \frac{1}{8\kappa^2} + \cdots\right) + \mathcal{O}(\varepsilon^{3/2}) \end{aligned} \tag{4.80}$$

and

$$w_1 = \frac{\pi \mathcal{F}(\zeta, \kappa^{-1})}{2\mathbb{K}(\kappa^{-1})} - \frac{\pi \nu_{111}^* \sqrt{\varepsilon A^* \nu_{11}^*}}{6\nu_{11}^{*2} \mathbb{K}(\kappa^{-1})}\,\theta_1 + \mathcal{O}(\varepsilon) \tag{4.81}$$

or, $w_1 = \dot{w}_1(t - t_0)$, where

$$\dot{w}_1 = \frac{2\pi}{T} = \frac{\pi\sqrt{\varepsilon A^* \nu_{11}^*}}{2\mathbb{K}(\kappa^{-1})} + \mathcal{O}(\varepsilon\sqrt{\varepsilon}). \tag{4.82}$$

The inversion of (4.81) gives

$$\sin\zeta = \mathrm{sn}\left(\frac{2\mathbb{K}}{\pi}\, w_1 + \sqrt{\varepsilon A^* \nu_{11}^*}\,\frac{\nu_{111}^* \theta_1}{3\nu_{11}^{*2}} + \mathcal{O}(\varepsilon)\right) \tag{4.83}$$

or

$$\sin\zeta = \mathrm{sn}\left(\frac{2\mathbb{K}}{\pi} w_1\right) + \frac{\pi}{2\mathbb{K}}\frac{\mathrm{d}}{\mathrm{d}w_1}\,\mathrm{sn}\left(\frac{2\mathbb{K}}{\pi} w_1\right)\sqrt{\varepsilon A^* \nu_{11}^*}\,\frac{\nu_{111}^* \theta_1}{3\nu_{11}^{*2}} + \mathcal{O}(\varepsilon). \tag{4.84}$$

All elliptic functions and integrals have modulus κ^{-1}. The elliptic function may be replaced by its Fourier expansion[2]

[2] See [17], Sect. 908.

$$\operatorname{sn}\left(\frac{2\mathbb{K}}{\pi}w_1\right) = \frac{\pi\kappa}{\mathbb{K}}\sum_{j=0}^{\infty}\operatorname{csch}\left[\left(j+\frac{1}{2}\right)\chi(\kappa^{-1})\right]\sin(2j+1)w_1,$$

where $\chi(\kappa^{-1}) = \dfrac{\pi\mathbb{K}(\sqrt{1-\kappa^{-2}})}{\mathbb{K}(\kappa^{-1})}$ (see B.31). We also know that

$$\theta_1 = 2\arcsin\left(\frac{1}{\kappa}\sin\zeta\right)$$

and some iterations are needed to obtain the expansion of θ_1 at a given order. Here, it is useful to recall that

$$\operatorname{csch}\left[\left(j+\frac{1}{2}\right)\chi\right] = 2(\mathrm{e}^{-\chi})^{j+\frac{1}{2}}\left\{1-(\mathrm{e}^{-\chi})^{2j+1}\right\}^{-1}$$

and that $\lim_{\kappa^{-1}\to 0}\chi(\kappa^{-1}) = \infty$.

In an analogous way, we may use (4.84) to obtain similar expansions for $\cos\zeta$:

$$\cos\zeta = \operatorname{cn}\left(\frac{2\mathbb{K}}{\pi}w_1\right) + \frac{\pi}{2\mathbb{K}}\frac{\mathrm{d}}{\mathrm{d}w_1}\operatorname{cn}\left(\frac{2\mathbb{K}}{\pi}w_1\right)\sqrt{\varepsilon A^*\nu_{11}^*}\,\frac{\nu_{111}^*\theta_1}{3\nu_{11}^{*2}} + \mathcal{O}(\varepsilon) \quad (4.85)$$

and

$$\operatorname{cn}\left(\frac{2\mathbb{K}}{\pi}w_1\right) = \frac{\pi\kappa}{\mathbb{K}}\sum_{j=0}^{\infty}\operatorname{sech}\left[\left(j+\frac{1}{2}\right)\chi(\kappa^{-1})\right]\cos(2j+1)w_1$$

where *cn* is the Jacobian *cosine amplitude* elliptic function with modulus $1/\kappa$. We also recall that

$$\operatorname{sech}\left[\left(j+\frac{1}{2}\right)\chi\right] = 2\left(\mathrm{e}^{-\chi}\right)^{j+\frac{1}{2}}\left\{1+(\mathrm{e}^{-\chi})^{2j+1}\right\}^{-1}.$$

This series may be substituted into (4.73) to obtain J_1.

4.5.3 Small-Amplitude Librations

When the amplitude of the librations is small, that is, when $\kappa^{-1} \sim 0$, we may consider only the leading terms of the Taylor expansions of the elliptic integrals in powers of κ^{-1} and, thus, obtain

$$\theta_1 = \frac{2}{\kappa}\sin w_1 + \mathcal{O}(\kappa^{-3}),$$

$$J_1 = J_1^* \pm \frac{2}{\kappa}\sqrt{\frac{\varepsilon A^*}{\nu_{11}^*}}\cos w_1 + \mathcal{O}(\sqrt{\varepsilon}\kappa^{-3}).$$

To obtain θ_1 and J_1 as functions of the action Λ_1, we need to invert (4.80) with respect to κ^{-1}:

$$\frac{1}{\kappa} = \sqrt{\frac{|\Lambda_1|}{2}} \left(\frac{\nu_{11}^*}{\varepsilon A^*}\right)^{\frac{1}{4}} \left(1 - \frac{|\Lambda_1|}{16}\sqrt{\frac{\nu_{11}^*}{8A^*}} + \cdots\right) + \mathcal{O}(\varepsilon^{\sim 1}). \tag{4.86}$$

It is also useful to introduce the libration frequency

$$\dot{w}_1 = \frac{2\pi}{T} = \sqrt{\varepsilon A^* \nu_{11}^*}\left(1 - \frac{1}{4\kappa^2} + \cdots\right) = \sqrt{\varepsilon A^* \nu_{11}^*} - \frac{1}{8}\nu_{11}^* \Lambda_1 + \cdots. \tag{4.87}$$

An easy calculation allows us to obtain

$$\theta_1 = \sqrt{\frac{2\Lambda_1 \nu_{11}^*}{\dot{w}_1}} \sin w_1 + \mathcal{O}\left(\kappa^{-3}\right); \tag{4.88}$$

$$J_1 = J_1^* \pm \sqrt{\frac{2\Lambda_1 \dot{w}_1}{\nu_{11}^*}} \cos w_1 + \mathcal{O}\left(\sqrt{\varepsilon}\kappa^{-3}\right). \tag{4.89}$$

These equations give, at the lower order of approximation, θ_1, J_1 as functions of the angle–action variables of Garfinkel's Ideal Resonance Problem. We recall that Λ_1 and ν_{11}^* can be either positive or negative, but their product or quotient is always positive. $\dot{w}_1$ is always positive. The sign in front of the square root of (4.89) is positive or negative according to the sign of ν_{11}^*. The calculation of terms of higher orders requires more work, but it does not present any difficulty. (See Sect. 8.8.1.)

4.6 Morbidelli's Successive Elimination of Harmonics

The central idea of Delaunay's lunar theory has been explored by Morbidelli [76] and used to study the overlap of resonances in the phase space of the dynamical system defined by the Hamiltonian

$$H = H_0(J) + \varepsilon \sum_{h \in \mathbf{D}} A_h(J) \cos\left(h|\theta\right). \tag{4.90}$$

Morbidelli's successive elimination of harmonics starts with the choice of an argument $(h_1|\theta)$ of H and the consideration of the system defined by the abridged Hamiltonian

$$\mathcal{F}_1 = H_0(J) + \varepsilon A_{h_1}(J) \cos\left(h_1|\theta\right), \tag{4.91}$$

where $h_1 \equiv (h_{1(1)}, h_{1(2)}, \cdots, h_{1(N)}) \in \mathbf{Z}^N$ This system is integrable. However, at variance with Delaunay theory, the non-resonance condition $(h_1|\nu) \neq 0$ is not assumed; on the contrary, the term to start the procedure is selected from among the most important resonant terms in the domain of the phase

space under study. It is chosen in the set of resonant terms, by its topological consequences. For instance, we may define the resonance strength of a term by its width – defined as the maximum separation between the two branches of the separatrices. From the equations of Sect. 4.4.4, we have

$$\Delta J_{\rm sep} = 4\sqrt{\left|\frac{\varepsilon A_{h_1}}{\tilde{\nu}_{11}}\right|}, \tag{4.92}$$

where $\tilde{\nu}_{11}$ is the second derivative of $H_0(J)$ with respect to the action J_1' conjugate to $(h_1|\theta)$. It is easy to see[3] that $\tilde{\nu}_{11} \sim \mathcal{O}(|h_1^2|)$. Therefore, the most important resonances are those with higher A_{h_1} and lower $|h_1|$.

Once the term h_1 is selected, we change variables through a Lagrangian extended point transformation where we impose $\theta_1' = (h_1|\theta)$. Let it be, for example,

$$\begin{array}{lll} \theta_1' = (h_1|\theta) & \qquad & J_1' = J_1/h_{1(1)} \\ \theta_\varrho' = \theta_\varrho & & J_\varrho' = J_\varrho - (h_{1(\varrho)}/h_{1(1)})J_1 \end{array} \tag{4.93}$$

($\varrho = 2, \cdots, N$). Then, $\mathcal{F}_1$ becomes

$$\mathcal{F}_1 = H_0(J(J')) + \varepsilon A_{h_1}(J(J'))\cos\theta_1'. \tag{4.94}$$

This is the Hamiltonian of the Ideal Resonance Problem and we may construct its angle–action variables w_1, Λ_1 (see Sect. 4.5). Hence,

$$\theta_1' = \theta_1'(w_1, \Lambda_1; J_\varrho') \qquad J_1' = J_1'(w_1, \Lambda_1; J_\varrho'). \tag{4.95}$$

Since the given system has N degrees of freedom, we have to extend this transformation of one pair of variables to the whole set, which is done by imposing $J_\varrho' = \Lambda_\varrho$ and by using one of the algorithms of Sect. 2.4.4:

$$w_\varrho = \theta_\varrho' + \Xi_\varrho(w_1, \Lambda), \tag{4.96}$$

where, for instance,

$$\Xi_\varrho = \int_0^{w_1} \left(\frac{\partial\theta_1'}{\partial w_1}\frac{\partial J_1'}{\partial \Lambda_\varrho} - \frac{\partial J_1'}{\partial w_1}\frac{\partial\theta_1'}{\partial\Lambda_\varrho} \right) {\rm d}w_1 \tag{4.97}$$

(Henrard-Lemaitre transformation).

Once we have completed the transformation, we go back to the given Hamiltonian H and perform the canonical transformation $(\theta', J') \Rightarrow (w, \Lambda)$. $\mathcal{F}_1$ will become a function of Λ only, and the remaining terms of (4.90), not included in $\mathcal{F}_1$, will be periodic functions of the angles w. They may be expanded in Fourier series so that, instead of H, we have a new Hamiltonian

$$H^* = H_0^*(\Lambda) + \varepsilon \sum_{h\in\mathbf{D}^*} A_h^*(\Lambda)\cos(h|w). \tag{4.98}$$

[3] From (4.93), we obtain $\tilde{\nu}_{11} = \sum_{j=1}^N \sum_{k=1}^N \nu_{jk} h_{1(j)} h_{1(k)}$.

This completes one Delaunay–Morbidelli operation. We may, then, restart the procedure by choosing a new term in H^*:

$$\mathcal{F}_2 = H_0^*(\Lambda) + \varepsilon A_{h_2}^*(\Lambda)\cos(h_2|w). \tag{4.99}$$

We then introduce $w_1' = (h_2|w)$ and new angles w_ϱ' and momenta Λ' through a Lagrange point transformation; we construct new angle–action variables ψ_1, K_1 and complete the transformation to include the other degrees of freedom. We thus get a new H^{**} and continue as before.

We have, purposely, presented the Delaunay–Morbidelli operation without stressing that the angle–action variables are not globally valid and are not computed in the same way in circulations and librations. In fact, Morbidelli's successive elimination of harmonics is not meant to construct formal solutions (the chains of elliptic functions and integrals would make it impossible), but to map the geometry of the resonances in a given domain of the phase space. One important point is that, in H^*, the angles are w. The w_ϱ differ from the given θ_ϱ by the quantity Ξ_ϱ, which is small: the corresponding frequencies in $H_0(J)$ and $H_0^*(\Lambda)$ are of the same order. w_1 is the uniform angle associated with the libration (or circulation), and has the frequency of this motion. Thus, new resonances may appear in H^*, involving w_1 and some of the w_ϱ not appearing in the given H. The best known examples are the so-called secondary resonances in the Kirkwood gaps of the asteroid belt (see [77]). These gaps appear near initial conditions corresponding to asteroids with an orbital period commensurable with Jupiter's period. The motion of an asteroid inside the gap is a libration about a periodic orbit; the libration frequency may be approximately known by selecting the main term with the critical combination of the two longitudes, and using the Ideal Resonance Problem. The Hamiltonian H^* shows new critical terms in which the libration frequency is a multiple of the frequency of motion of the perihelion (one of the θ_ϱ). The overlap of these secondary resonances may be studied taking, in turn, each of these terms in H^* to compose the abridged Hamiltonian $\mathcal{F}_2$.

To circumvent the difficulties due to elliptic functions and integrals, it is possible to construct numerically all transformations mentioned in this section. We may use the direct techniques described in Sect. 2.2 to construct the angle–action variables. In such case, the result will not be written as formal functions, but as functions defined by a table or computer code allowing them to be known.

4.6.1 An Example

Let us consider an application of Morbidelli's elimination algorithm to the Hamiltonian function obtained at the end of Sect. 3.8. We discard terms of the order $\mathcal{O}(\varepsilon^2)$ and adopt the notation θ, J (without stars) for angles and actions. Also, for practical reasons, we interchange the subscripts 2 and 3 in the variables and adopt the particular value $A_0 = \alpha J_2$ for the secular

term of the perturbation. We also assume that α, B_1, L_0 and M_1 are positive constants. Hence,

$$H(\theta, J) \;=\; H_0 + \varepsilon H_2$$

where

$$H_0 \;=\; -\frac{1}{2J_1^2} + \nu_3(J_3 - 2J_1) \tag{4.100}$$

and

$$H_2 \;=\; \alpha J_2 + L_0\sqrt{-J_2}\cos\theta_2 + B_1\cos\theta_1 + M_1\sqrt{-J_2}\cos(\theta_1+\theta_2). \tag{4.101}$$

The action J_3 is a constant (since the angle θ_3 is absent from the Hamiltonian) and the exact resonance value of J_1 is defined by

$$\nu_1^* = \nu_1(J_1^*) = \left.\frac{\partial H_0}{\partial J_1}\right|_{J_1=J_1^*} \stackrel{\text{def}}{=} 0; \tag{4.102}$$

that is

$$J_1^* = \frac{1}{\sqrt[3]{2\nu_3}}. \tag{4.103}$$

We also have

$$\nu_{11}^* = -6\nu_3\sqrt[3]{2\nu_3} = -3(2\nu_3)^{4/3}. \tag{4.104}$$

We recall that the example of Sect. 3.8 is founded on the asteroidal three-body problem and B_1 is, there, a quantity of the order of the orbital eccentricity of the disturbing planet.

In the neighborhood of $J_1 = J_1^*$, the Hamiltonian given by (4.101) has two resonant terms: $\varepsilon B_1\cos\theta_1$ and $\varepsilon M_1\sqrt{-J_2}\cos(\theta_1+\theta_2)$. Let us consider the Ideal Resonance Problems (IRPs) which they, separately, define:

$$\begin{aligned}\mathcal{F}_{1(a)} &= H_0(J) + \varepsilon B_1\cos\theta_1\\ \mathcal{F}_{1(b)} &= H_0(J) + \varepsilon M_1\sqrt{-J_2}\cos(\theta_1+\theta_2).\end{aligned} \tag{4.105}$$

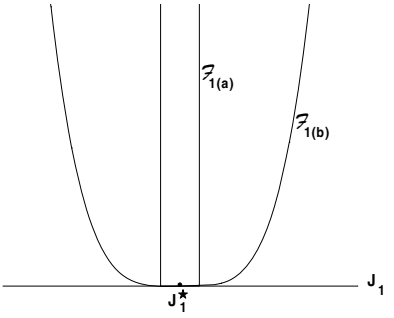

Fig. 4.2. Separatrices of the two IRPs of (4.105)

The widths (maximum libration amplitudes) of these resonances are, respectively,

$$\Delta J_{\text{sep(a)}} = 4\sqrt{\frac{-\varepsilon B_1}{\nu_{11}^*}}$$

$$\Delta J_{\text{sep(b)}} = 4\sqrt{\frac{-\varepsilon M_1}{\nu_{11}^*}}\sqrt[4]{-J_2}.$$

Figure 4.2 shows the locus of the separatrices of the two considered IRPs, in the plane $J_1, |J_2|$. If $|B_1| \ll |M_1|$, the strip corresponding to the resonance of $\mathcal{F}_{1(a)}$ is narrow (as shown in the figure) and the hierarchy of the two considered harmonics is well established. It is then possible to start the elimination of harmonics with the largest one, $\mathcal{F}_{1(b)}$.

Following the recipe given above, we perform, initially, the point transformation

$$\begin{array}{ll} \theta_2' = \theta_1 & J_2' = J_1 - J_2 - J_1^* \\ \theta_1' = \theta_1 + \theta_2 & J_1' = J_2; \end{array} \tag{4.106}$$

$\mathcal{F}_1$ becomes

$$\mathcal{F}_1 = \mathcal{F}_{1(b)} = H_0(J(J')) + \varepsilon M_1\sqrt{-J_1'}\cos\theta_1'. \tag{4.107}$$

Let us consider the small-amplitude librations of this one-degree-of-freedom system about the libration center $J_1'^* = -J_2'$. They are given by (see 4.88 and 4.89):

$$\theta_1' = \sqrt{\frac{2\Lambda_1\nu_{11}^*}{\dot{w}_1}}\sin w_1 \tag{4.108}$$

$$J_1' = -J_2' - \sqrt{\frac{2\Lambda_1\dot{w}_1}{\nu_{11}^*}}\cos w_1, \tag{4.109}$$

where w_1, Λ_1 are the angle–action variables of the IRP defined by $\mathcal{F}_1$, ν_{11}^* is a known number and

$$\dot{w}_1 = \sqrt{-\varepsilon\nu_{11}^* M_1}\sqrt[4]{J_2'} - \frac{1}{8}\nu_{11}^*\Lambda_1. \tag{4.110}$$

In order to have $\theta_1' = 0$ at the libration center, we assumed $M_1 > 0$ (we recall that $\nu_{11}^* < 0$ and $\Lambda_1 < 0$). The next step in Morbidelli's algorithm is to complete the canonical transformation $(\theta_1', \theta_2', J_1', J_2') \Rightarrow (w_1, w_2, \Lambda_1, \Lambda_2)$ through

$$\begin{aligned} \theta_2' &= w_2 - \Xi_2(w_1, \Lambda_1, \Lambda_2) \\ J_2' &= \Lambda_2, \end{aligned}$$

where

$$\Xi_2 = \int_0^{w_1}\left(\frac{\partial\theta_1'}{\partial w_1}\frac{\partial J_1'}{\partial\Lambda_2} - \frac{\partial J_1'}{\partial w_1}\frac{\partial\theta_1'}{\partial\Lambda_2}\right)\mathrm{d}w_1.$$

We note that θ_1', J_1' depend on J_2', that is, on Λ_2, also through $\dot{w}_1$. The derivatives are

$$\frac{\partial \theta_1'}{\partial w_1} = \sqrt{\frac{2\Lambda_1 \nu_{11}^*}{\dot{w}_1}}\cos w_1,$$

$$\frac{\partial J_1'}{\partial w_1} = \sqrt{\frac{2\Lambda_1 \dot{w}_1}{\nu_{11}^*}}\sin w_1,$$

$$\frac{\partial \theta_1'}{\partial \Lambda_2} = \frac{\nu_{11}^*}{8\Lambda_2^{3/4}}\sqrt{\frac{-2\varepsilon M_1 \Lambda_1}{\dot{w}_1^3}}\sin w_1,$$

$$\frac{\partial J_1'}{\partial \Lambda_2} = -1 - \frac{1}{8\Lambda_2^{3/4}}\sqrt{\frac{-2\varepsilon M_1 \Lambda_1}{\dot{w}_1}}\cos w_1.$$

Hence,

$$\Xi_2 = -\int_0^{w_1}\left(\frac{\partial \theta_1'}{\partial w_1} + \frac{|\Lambda_1|\sqrt{-\varepsilon M_1 \nu_{11}^*}}{4\Lambda_2^{3/4}\dot{w}_1}\right)\mathrm{d}w_1,$$

or

$$\Xi_2 = -\theta_1' - \frac{\Lambda_1}{8\Lambda_2}, \tag{4.111}$$

where, for the sake of simplicity, we kept $\dot{w}_1$ restricted to its first approximation. The transformation is, now, complete and may be used to transform the given Hamiltonian.

With the new variables, $\mathcal{F}_1$ may depend only on the actions. The substitution of variables in $\mathcal{F}_1$ is cumbersome and the cancellation of periodic terms, in higher orders, is only partially achieved because of the many simplifications introduced. However, a shortcut exists. We know that, if we denote by $\widehat{\mathcal{F}}_1(\Lambda_1, \Lambda_2)$ the result of the transformation, by definition,

$$\dot{w}_1 = \frac{\partial \widehat{\mathcal{F}}_1}{\partial \Lambda_1}$$

or

$$\widehat{\mathcal{F}}_1 = \int \dot{w}_1 \,\mathrm{d}\Lambda_1.$$

The problem with this shortcut is that the integration introduces an arbitrary additive function of Λ_2, for whose derivation, the direct transformation is necessary. Since this additive function cannot depend on Λ_1 and all involved functions are polynomials in $\sqrt{-\Lambda_1}$, we need just transform the parts of $\widehat{\mathcal{F}}_1$ independent of Λ_1 to obtain it. Hence

$$\widehat{\mathcal{F}}_1(\Lambda_1, \Lambda_2) = \sqrt{-\varepsilon \nu_{11}^* M_1}\ \sqrt[4]{\Lambda_2}\Lambda_1 - \frac{1}{16}\nu_{11}^*\Lambda_1^2 + \varepsilon M_1\sqrt{\Lambda_2}, \tag{4.112}$$

where the two first terms resulted from the integration of $\dot{w}_1$ and the last one from a direct calculation. The constant terms (depending on J_3 and J_1^*) do not need to be taken into account since they do not contribute to the equations. The terms of H not considered in $\mathcal{F}_{1(b)}$ need to be written with

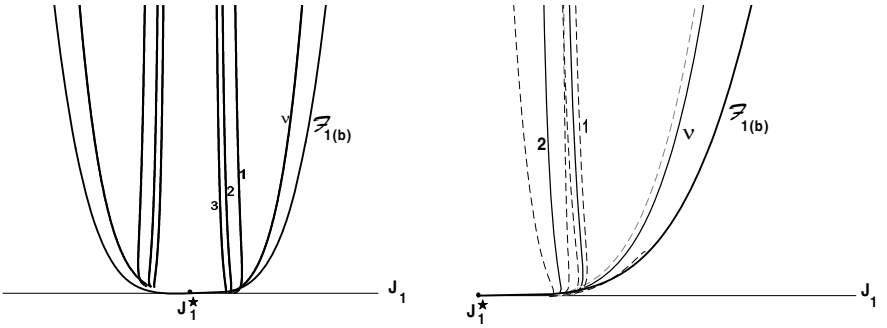

Fig. 4.3. Secular (ν) and secondary ($k = 1, 2, 3$) resonances

the new variables. The results are Fourier expansions in the angle $w_2 + kw_1$. The new $\widehat{H}_0$ is

$$\widehat{H}_0 = \widehat{\mathcal{F}}_1 - \varepsilon\alpha\Lambda_2. \tag{4.113}$$

To do a new Delaunay–Morbidelli operation, we have to select a new resonant periodic term to add to $\widehat{H}_0$. Let us, first, search the resonance locus of the main terms. To do this, we need the expressions for $\dot{w}_1$ and $\dot{w}_2$:

$$\dot{w}_1 = \sqrt{-\varepsilon\nu_{11}^* M_1}\ \sqrt[4]{\Lambda_2} - \frac{1}{8}\nu_{11}^*\Lambda_1 \tag{4.114}$$

$$\dot{w}_2 = \frac{\partial\widehat{H}_0}{\partial\Lambda_2} = \sqrt{-\varepsilon\nu_{11}^* M_1}\,\frac{\Lambda_1}{4\Lambda_2^{3/4}} + \frac{1}{2}\frac{\varepsilon M_1}{\sqrt{\Lambda_2}} - \varepsilon\alpha. \tag{4.115}$$

When numerical values are given to ε, ν_{11}^*, M_1 and α, the locus of the curves $\dot{w}_2 \pm k\dot{w}_1 = 0$ is easily found. It is convenient to show these curves in the plane $J_1, |J_2|$ instead of the plane Λ_1, Λ_2. The transformation $\Lambda_1, \Lambda_2 \Rightarrow J_1, J_2$, however, depends on w_1. It is, then, necessary to fix the value of w_1. We follow the same practice usual in resonant asteroid dynamics, and fix it at the boundaries of the librations of the action J_1' conjugate to the critical angle θ_1'. Thus, we assume $|\cos w_1| = 1$. As a consequence, to each point in the plane (Λ_1, Λ_2) we obtain two points in the plane (J_1, J_2), one on each side of the vertical line $J_1 = J_1^*$. Figure 4.3 (*left*) shows the lines falling inside the boundary of the libration domain. They are: the secular resonance $\dot{w}_2 = 0$ (indicated by ν following astronomers' classical notation); and the secondary resonances $\dot{w}_2 + k\dot{w}_1 = 0$ with $k > 0$ (the lines $k = 1, 2, 3$ are shown)[4].

To each of the resonances in Fig. 4.3 (*left*) there corresponds one libration zone defined by the separatrices of the Ideal Resonance Problem obtained when the corresponding perturbative term is selected and added to $\widehat{H}_0$. Let us introduce the new set of canonical variables

[4] For a Lie series study of secular and secondary resonances, see Sects. 9.4.5 and 9.4.6.

$$\begin{array}{ll} \varphi_1 = w_2 + k w_1 & K_1 = \Lambda_2 \\ \varphi_2 = w_1 & K_2 = \Lambda_1 - k\Lambda_2 \end{array} \tag{4.116}$$

and study the Ideal Resonance Problems

$$\mathcal{F}_{2(k)} = \widehat{H}_0(K(\Lambda)) + \mathcal{A}_k(K)\cos\varphi_1 \qquad (k = 0, 1, 2, 3), \tag{4.117}$$

where the coefficients $\mathcal{A}_k(K)$ come from the expansion of those terms of $H(\theta, J)$ whose coefficients are B_1 and L_0.

Figure 4.3 (*right*) shows the separatrices of the IRPs corresponding to $k = 0, 1, 2$. The secular resonance $k = 0$ (ν) and the secondary resonance $k = 1$ are isolated in this figure, while the secondary resonances $k = 2$ and $k = 3$ (shown only in Fig. 4.3 *left*) are very close and overlap each other. (The secondary resonance $k = 3$ is inside the resonance zone of the secondary resonance $k = 2$.) If Fig. 4.3 (*right*) were the result of an exact numerical calculation, $\mathcal{F}_{2(0)}$ and $\mathcal{F}_{2(1)}$ could be considered as good candidates for elimination of further harmonics. On the contrary, because of the overlap of their libration domains, the isolated consideration of $\mathcal{F}_{2(2)}$ or $\mathcal{F}_{2(3)}$ would be unrealistic. However, Fig. 4.3 (*right*) is the result of analytical approximations valid only in a small neighborhood of $J_1 = J_1^*$, and we have to restrict our analysis to it. The motions in this neighborhood are far from the resonance lines of Fig. 4.3 (*left*) and we may use the original Delaunay operation of Sect. 4.1 to get rid of the harmonic remaining in the Hamiltonian. Maybe, in the case of the harmonic $k = 2$, given the broadness of its resonant zone, we should consider the expression of the circulations given by the Ideal Resonance Problem, since that given by the classical Delaunay operation assumes that the resonance is very far and do not influence the solution.

One important remark yet to be made concerns the numerical choice of the coefficients appearing in the Hamiltonian. To obtain Fig. 4.3 (*right*), we had to consider $L_0 \ll M_1$ and neglect the term $\varepsilon B_2 \cos\theta_1$. Otherwise, the libration zones of the $\mathcal{F}_{2(k)}$ would be so broad that they would overlap over almost the whole region shown in the figures. In that case, it would no longer be possible to select one domain in the plane for further studies with the technique discussed here. These limitations may not, however, be considered as a weak point. On the contrary, allowing us to map the overlap of resonances, Morbidelli's successive elimination of harmonics clearly shows the extreme limits where approximate regular solutions can exist.

The given example used the heavy analytical machinery of Garfinkel's Ideal Resonance Problem with the aim of allowing the reader to have a step-by-step view of the technique. But one should take advantage of the possibility of direct numerical construction of the transformations leading to particular angle–action variables, as discussed in Sect. 2.2, to have exact calculations and, as a consequence, an exact chart of resonances and libration domains, at every step.