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Thermally Induced Rate-Dependence of Hysteresis in Nonclassical Nonlinear Acoustics

Vitalyi Gusev[1] and Vincent Tournat

Université du Maine, Avenue Olivier Massaien, 72085 Le Mans, Cedex 09, France.
e-mail: vitali.goussev@univ-lemans.fr; vincent.tournat@univ-lemans.fr.
[1]To whom correspondence should be addressed.

Abstract
Contribution of hysteretic mechanical elements to the stress/strain relationship of microinhomogeneous material is analyzed within the framework of the Preisach–Arrhenius model where the transitions between the different mechanical states of the individual elements in addition to acoustic loading can be induced by thermal fluctuations. The model provides an explanation of why with increasing wave amplitude a transition from a behavior, which is quasi-independent of wave amplitude, to another, characterized by the dominance of nonclassical hysteretic quadratic nonlinearity, takes place in microinhomogeneous materials. Analytical evaluation of the Preisach–Arrhenius model for the acoustic hysteresis confirms the expectation that thermal relaxation effects are capable of recovering the dependence of the nonlinear acoustic properties of the material on acoustic wave frequency. The theory predicts the boundaries for an intermediate interval of frequencies where hysteretic quadratic nonlinearity dominates in microinhomogeneous materials. Outside this interval (at sufficiently low or sufficiently high frequencies) the nonlinearity significantly diminishes. However the width of the frequency interval for the hysteretic quadratic nonlinearity depends on the acoustic wave amplitude and increases with the increasing wave amplitude. The low-frequency cutoff of the interval diminishes with increasing wave amplitude and the high-frequency cutoff increases. As a result, if the system manifests linearity or quasinonhysteretic nonlinearity at sufficiently low acoustic amplitudes, sooner or later with increasing wave amplitude it will manifest hysteretic quadratic nonlinearity.

Keywords: Dispersion of nonlinearity, hysteretic nonlinearity, microinhomogeneous materials, nonclassical nonlinearity, Preisach–Arrhenius model, rate-dependent hysteresis, thermal relaxation

1. Introduction

The objective of nonlinear acoustics is the evaluation of material nonlinearity, that is to say, of a deviation of the material mechanical behavior from the Hooke's law, by application of low-amplitude (acoustic) strain waves. Typical amplitude values of periodic strain waves do not exceed 10^{-5} and the nonlinear contribution to the material stress/strain relationship is small. Currently there exists a consensus, according to which the nonlinear mechanical properties of microinhomogeneous materials (such as rocks, polycrystalline metals, and ceramics, e.g.) are dominated by nonclassical hysteretic nonlinearity,[1–4] as opposed to the nonlinearity of the interatomic interactions and the kinematic nonlinearity.[5,6] Hysteretic nonlinearity is understood phenomenologically in terms of the nonlinear hysteretic motion of the mesoscopic mechanical elements such as dislocations, intergrain contacts, or defects, for example, with the dimensions exceeding interatomic distances but significantly smaller than the acoustic wavelength.[1,2] As a mathematical tool for the description of hysteresis in nonlinear mechanical properties, the Preisach–Mayergoyz (PM) model of hysteresis[7–10] can be applied. Even in its simplest formulation the PM model explains what is, perhaps, the best known and the most common manifestation of the hysteretic nonlinearity, that is to say, the shift of the resonance frequency of a solid microinhomogeneous bar proportional to the wave amplitude in the bar.[1–4,11] However, the PM model does not explain either experimentally observed dependence of the nonlinear phenomena on frequency[12,13] or the absence of the hysteretic quadratic nonlinearity at very low amplitudes of the acoustic loading.[14–16]

We note here that the Preisach (Preisach–Mayergoyz) formalism[7–10] attributes hysteresis in the nonlinear stress/strain relationship to combined behavior of individual bistable (two-level) hysteretic mechanical units, sometimes referred to as hysterons.[17,18] The transitions (Barkhausen jumps[10]) between two possible states (i.e., energy levels) are assumed to take place instantaneously and exactly at some critical levels of varying stress (strain). For different individual mechanical elements, the levels are different. This model of the hysteretic nonlinearity is essentially dispersionless, that is to say, it is frequency-independent, because there are no characteristic scales of either time or length in the model. The PM model does not take into account that hysteresis is always a dynamic phenomenon. If thermal fluctuations are taken into account in the description of the mesoscopic elements, then there will be no hysteresis in the static limit because the thermal fluctuations are always pushing the system to a unique equilibrium state. In quasistatic experiments, hysteresis will appear at frequencies for which thermal fluctuations have insufficient time to force the system during a wave period in a state having free energy at its absolute minimum value. Instead, the system will be in a state in which its free energy is in a local minimum, that is to say, in a metastable state. Consequently, the nonlinear mesoscopic mechanical elements, in reality, are nonhysteretic in the static limit and hysteretic only in their dynamic behavior. In the theory of magnetism, the Preisach–Mayergoyz model is considered as a zero-temperature limit for rate-independent hysteresis,[10] because the thermal fluctuations are not included and because the stress/strain relation depends only on the sign of the strain rate but not on its magnitude.

The Preisach–Arrhenius (PA) model for the description of thermally activated relaxation, or "after-effect," in magnetic materials[10,18] takes into account that the transitions between the energy levels of the system can be thermally activated and that the probability of the transition is controlled by the Boltzmann factor $\exp(-\Delta E/k_B T)$, where ΔE is the difference in energy levels, or some activation energy, k_B is the Boltzmann constant, and T is the absolute temperature. The thermally controlled transition is not instantaneous statistically. Rather, there is a characteristic time scale for each individual mechanical element that can be estimated by $\tau_0 \exp(\Delta E / k_B T)$ as defined by the Arrhenius formula for the transition time, where τ_0 is some characteristic attempt time associated with the Barkhausen jump between the energy levels.[10] Consequently, dispersion in the acoustic nonlinearity is expected in the Preisach–Arrhenius model. In the rate-independent approximation assumed by the PM model the external action on the system remains nearly unchanged during the time needed to complete the Barkhausen jump. Thus, the external field creates the conditions for the system instability and spontaneous (or thermally initiated) Barkhausen jumps from one local energy minimum to the next. Therefore, when appreciable variations of an external action take place during individual Barkhausen jumps, then rate-independence no longer applies. At high frequency and weak acoustic wave amplitude the characteristic time for a thermally stimulated transition to occur can significantly exceed the acoustic wave period. Thus, the individual elements have insufficient time to modify their state even when loading makes it for some time allowed by energy considerations.[10]

The acoustic wave affects the system through the modulation of the difference ΔE between the energy levels, and in doing so, renders the thermally activated relaxation processes amplitude dependent. Qualitatively speaking, the Preisach–Arrhenius model describes nonlinear temperature-dependent relaxation of the hysteretic mechanical elements. Consequently it might be expected that the nonlinearity of the system is due not only to the intrinsic nonlinearity of the bistable hysteretic elements but also due to the nonlinearity of the relaxation process.

2. Preisach–Arrhenius Model for Acoustic Response of Microinhomogeneous Media

There exists a consensus that microinhomogeneous materials may contain some mechanical elements that are mesoscopic (with the dimensions exceeding the atomic scale but significantly smaller than the acoustic wavelength) and hysteretic (such as reversible Griffith cracks[8] or contacts with adhesion, e.g.). The hysteresis in the behavior of an individual mechanical element might be imagined in the simplest way as being related to the possibility for the element to be in different states under the same mechanical loading. In which state the mechanical element is actually a function of the acoustic loading history. Both in the Preisach–Mayergoyz[7–10] and the Preisach–Arrhenius[10,18,19] models it is assumed that the mechanical elements have two states (two energy levels) and that the contribution σ' of an element to the macroscopic stress in material depends on its state. This phenomenological description assumes that the free-energy of the material, which possesses multiple local minima reflecting the complexity of the mutual interactions among the system's components, can be represented as a linear superposition of two-level bi-stable contributions.[10] In the PM theory the transition of an element from state 1 to state 2 takes place when $\partial\varepsilon/\partial t > 0, \quad \varepsilon = \varepsilon_2$, and the inverse transition takes place when $\partial\varepsilon/\partial t < 0, \quad \varepsilon = \varepsilon_1 < \varepsilon_2$ (Figure 21.1). The difference between the critical switching strains ε_2 and ε_1 ($\varepsilon_2 \neq \varepsilon_1$) is at the origin of the hysteretic nature of these elements. If the notation $f(\varepsilon_1, \varepsilon_2)$ is introduced

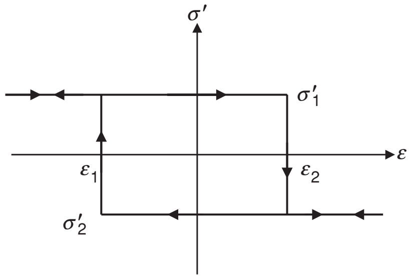

Fig. 21.1. Contribution σ' of an individual mechanical element to stress in the framework of the Preisach–Mayergoyz model. Arrowheads indicate direction of strain variation in time.

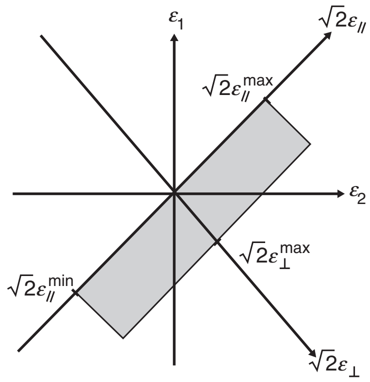

Fig. 21.2. Presentation of mechanical element distribution at Preisach–Mayergoyz plane $(\varepsilon_2, \varepsilon_1)$, where ε_2 and ε_1 are the critical strain values for switching the elements between the levels. A distribution, limited in PM plane by $\varepsilon_{\perp} \equiv (\varepsilon_2 - \varepsilon_1)/2 \le \varepsilon_{\perp}^{\max}$ and $\varepsilon_{//}^{\min} \le \varepsilon_{//} \equiv (\varepsilon_2 + \varepsilon_1)/2 \le \varepsilon_{//}^{\max}$, is presented in gray as an example.

to represent the distribution function of the elements in the plane $(\varepsilon_2, \varepsilon_1)$ then the contribution of all the elements to the stress is given as

$$\sigma = \int_{-\infty}^{\varepsilon_2} d\varepsilon_1 \int_{\varepsilon_1}^{\infty} d\varepsilon_2 \sigma'(\varepsilon_1, \varepsilon_2, \varepsilon) f(\varepsilon_1, \varepsilon_2). \tag{21.1}$$

Here $f(\varepsilon_1, \varepsilon_2) d\varepsilon_1 d\varepsilon_2$ is the number of the elements with critical strains belonging to the intervals $(\varepsilon_1, \varepsilon_1 + d\varepsilon_1)$ and $(\varepsilon_2, \varepsilon_2 + d\varepsilon_2)$ of the PM plane $(\varepsilon_2, \varepsilon_1)$. Due to the assumed condition $\varepsilon_2 > \varepsilon_1$ the integration in the PM plane is in the sector to the right of the diagonal $\varepsilon_2 = \varepsilon_1$ (Figure 21.2). The arguments of the function $\sigma'(\varepsilon_1, \varepsilon_2, \varepsilon)$ indicate that, in general, the contribution of an element to the total stress depends on its position in the PM plane and the loading history as it is presented in Figure 21.1. An important feature of the PM model is that hysteresis in the mechanical behavior of the individual elements exists independently of the magnitude of the strain rate, because the transitions at critical levels ε_2 and ε_1 are assumed to be instantaneous. It is assumed that the transition $1 \Rightarrow 2$ always happens when the strain $\varepsilon(\partial\varepsilon/\partial t > 0)$ exceeds ε_2 independently of how fast ε returns back to the region $\varepsilon < \varepsilon_2$. From a physics point of view, in the PM model the acoustic loading not only creates the conditions for the transition but also induces the change of the state.

The physical nature of $\sigma'(\varepsilon_1, \varepsilon_2, \varepsilon)$ in the Preisach–Arrhenius model is very different. In fact, the acoustic field is no longer the only physical actor that can induce transitions between the states 1 and 2. There are also thermal fluctuations that statistically can cause the same transitions. In the PA model, the transition from state 1 to state 2, for example, can occur during a finite time interval and at values of ε that do not strictly satisfy the conditions $\varepsilon = \varepsilon_2 \quad (\partial\varepsilon/\partial t > 0)$. At finite values of the temperature, the elements can overcome the energy barrier by thermal activation at lower strains as long as there is a second (local) energy minimum in which they can jump. Qualitatively speaking, thermal fluctuations accelerate the transitions below the critical level of strain ε_2. At the same time, above the critical strain ε_2, thermal fluctuations induce inverse transitions (from state 2 to state 1), which are completely forbidden in the zero-temperature model. The picture of the inverse transitions $2 \to 1$ near the critical strain ε_1 is similar.

In the Arrhenius model for thermally initiated transitions, the transition time τ_{12} from level 1 to level 2 is equal to $\tau_{12} = \tau_0 \exp[d(\varepsilon_2 - \varepsilon)/k_B T]$, where d measures the variation of energy difference ΔE_{12} between states 1 and 2 caused by an applied unit strain (deformation potential). Accordingly the transition time τ_{12} diminishes exponentially with increasing strain when the applied strain exceeds the critical level ε_2. Similarly, the time τ_{21} of the inverse transition is $\tau_{21} = \tau_0 \exp[d(\varepsilon - \varepsilon_1)/k_B T]$. The transition times τ_{12} and τ_{21} control the probabilities W_1 and W_2 to find the element in states 1 and 2, respectively,

$$\begin{aligned} \partial W_1/\partial t &= -W_1/\tau_{12} + W_2/\tau_{21}, \\ \partial W_2/\partial t &= \quad W_1/\tau_{12} - W_2/\tau_{21}, \\ W_1 + W_2 &= 1. \end{aligned} \tag{21.2}$$

These equations are sufficient to describe the variation of stress in response to acoustical loading. Actually the average level of $\sigma'(\varepsilon_1, \varepsilon_2, \varepsilon)$ in the absence of the acoustic wave does not contribute to dynamic stress in Eq. (21.1). Thus it is useful to evaluate the variations of $\sigma'(\varepsilon_1, \varepsilon_2, \varepsilon)$ relative to the average level $(\sigma'_1 + \sigma'_2)/2$, where σ'_1 and σ'_2 are the contributions to stress when the element is in positions 1 and 2, respectively. Then the contributions of states 1 and 2 to stress that can be modified by acoustic excitation are described as $(\sigma'_1 - \sigma'_2)/2 = \Delta\sigma'(\varepsilon_1, \varepsilon_2)$ and $(\sigma'_2 - \sigma'_1)/2 = -\Delta\sigma'(\varepsilon_1, \varepsilon_2)$, respectively. Taking into account the probabilities of finding the element in the corresponding states, the wave-dependent contribution $\sigma''(\varepsilon_1, \varepsilon_2, \varepsilon)$ to $\sigma'(\varepsilon_1, \varepsilon_2, \varepsilon)$ can be presented as

$$\begin{aligned} \sigma''(\varepsilon_1, \varepsilon_2, \varepsilon) &= \Delta\sigma'(\varepsilon_1, \varepsilon_2) W_1 - \Delta\sigma'(\varepsilon_1, \varepsilon_2) W_2 \\ &= \Delta\sigma'(\varepsilon_1, \varepsilon_2)(W_1 - W_2) \equiv \Delta\sigma'(\varepsilon_1, \varepsilon_2) Q. \end{aligned} \tag{21.3}$$

The relations (21.2) lead to a single equation describing the dynamics of the function Q, which has been introduced in Eq. (21.3) to characterize the asymmetry of the element distribution between the two levels,

$$\partial Q/\partial t + (1/\tau_{21} + 1/\tau_{12}) Q = (1/\tau_{21} - 1/\tau_{12}). \tag{21.4}$$

An obvious but important conclusion based on Eq. (21.4) is the absence of the hysteresis in the contribution of an element to stress under the static conditions. For $\partial/\partial t \to 0$ (zero frequency of the acoustic action) the solution of Eq. (21.4) is

$$Q_0 = -\tanh\left[d\left(\varepsilon - \frac{\varepsilon_1 + \varepsilon_2}{2}\right)/k_B T\right]. \tag{21.5}$$

Thus, in contrast to the PM model the hysteresis in the PA model is a dynamic phenomenon due to the finite rate of acoustic loading (compare the solutions in Figure 21.1 and in Figure 21.3).

For the following analysis the characteristic strain $\varepsilon_0 = k_B T/d$, which provides a scale for the amplitude of acoustic loading necessary for significant (e times) modification of the relaxation times τ_{12} and τ_{21}, is introduced. All the strains are normalized to this level ($\varepsilon/\varepsilon_0 \equiv \varepsilon, \quad \varepsilon_{1,2}/\varepsilon_0 \equiv \varepsilon_{1,2}$). Two new variables $\varepsilon_{//} = (\varepsilon_2 + \varepsilon_1)/2$ and $\varepsilon_\perp = (\varepsilon_2 - \varepsilon_1)/2$ are then introduced. Qualitatively speaking $\left|\varepsilon_{//}\right|$ characterizes the average energy of the mechanical element (from the acoustics point of view), and $\varepsilon_\perp$ characterizes the separation of the energy levels 1 and 2 in the absence of acoustic loading. On the other hand, $\varepsilon_{//}$ and $\varepsilon_\perp$ have a clear geometrical interpretation: with reference to the diagonal $\varepsilon_2 = \varepsilon_1$ in the PM plane, they are proportional to the coordinates measured along that line and the direction orthogonal to it, respectively[10] (Figure 21.2).

In the introduced notations, Eq. (21.4) takes the form

$$\partial Q/\partial\theta + (2/F)\exp(-\varepsilon_\perp)\cosh(\varepsilon(t) - \varepsilon_{//})Q = -(2/F)\exp(-\varepsilon_\perp)\sinh(\varepsilon(t) - \varepsilon_{//}). \tag{21.6}$$

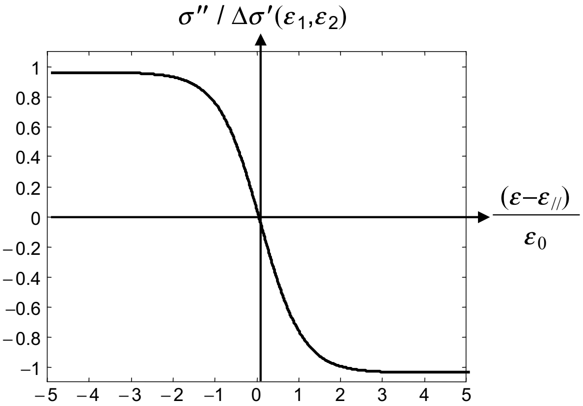

Fig. 21.3. Contribution σ'' of an individual mechanical element to stress in the framework of the Preisach–Arrhenius model in the case of infinitely low frequency of acoustic action. In accordance with Eqs. (21.3) and (21.5) the element behaves in response to strain variation as a two-level but a nonhysteretic unit.

Here the time is normalized to the period T_A of acoustic loading ($\theta = t/T_A$), and the parameter $F = \tau_0/T_A$ is the normalized frequency of the acoustic action. The integral relation (21.1) for the evaluation of the stress becomes

$$\sigma = -\varepsilon_0^2 \int_0^{\infty} d\varepsilon_\perp \int_{-\infty}^{\infty} d\varepsilon_{//} \Delta\sigma'(\varepsilon_\perp, \varepsilon_{//}) f(\varepsilon_\perp, \varepsilon_{//}) Q(\varepsilon_\perp, \varepsilon_{//}, \varepsilon(t)). \tag{21.7}$$

To investigate the acoustic properties of the Preisach–Arrhenius model, Eq. (21.6) is integrated. The exact solution subjected to the conditions of periodicity ($Q(\theta + 1) = Q(\theta)$) is

$$Q = -\frac{\int_\theta^{\theta+1} d\theta' g_s(\theta') \exp\left[-\int_{\theta'}^{\theta+1} g_c(\theta'')d\theta''\right]}{1 - \exp\left[-\int_\theta^{\theta+1} g_c(\theta'')d\theta''\right]}, \tag{21.8}$$

where $g_s = (2/F)\exp(-\varepsilon_\perp)\sinh(\varepsilon(\theta) - \varepsilon_{//})$, $g_c = (2/F)\exp(-\varepsilon_\perp)\cosh(\varepsilon(\theta) - \varepsilon_{//})$.

The formulae (21.7) and (21.8) with an appropriate modeling of the distributions $\Delta\sigma'(\varepsilon_\perp, \varepsilon_{//})$ and $f(\varepsilon_\perp, \varepsilon_{//})$ are sufficient for the description of the acoustic response of materials in the frame of the PA model. Here the results of the analysis are presented for the simplest variation of $\Delta\sigma'(\varepsilon_\perp, \varepsilon_{//})$ and $f(\varepsilon_\perp, \varepsilon_{//})$ in the PM plane $(\varepsilon_\perp, \varepsilon_{//})$. For this purpose the product $\Delta\sigma'(\varepsilon_\perp, \varepsilon_{//}) f(\varepsilon_\perp, \varepsilon_{//})$ is estimated by its characteristic value $(\Delta\sigma' f)_0$ and the extent of the element distribution in the PM plane is assumed to be limited by the boundaries $0 \le \varepsilon_\perp \le \varepsilon_\perp^{\max}, \varepsilon_{//}^{\min} \le \varepsilon_{//} \le \varepsilon_{//}^{\max} (\varepsilon_{//}^{\min} < 0, \quad \varepsilon_{//}^{\max} > 0)$ (Figure 21.2). It is worth mentioning that the assumption $\Delta\sigma'(\varepsilon_\perp, \varepsilon_{//}) f(\varepsilon_\perp, \varepsilon_{//}) \approx const$ is rather common in the applications of the Preisach–Mayergoyz model to acoustics, because only a small area of the PM plane with the dimensions $\propto \varepsilon_A \varepsilon_A/2$ (where ε_A is the amplitude of the acoustic wave) interacts with sound in the PM model.[2,3,9] In this case the details of the $\Delta\sigma' f$ distribution outside this small area play no role. In the Preisach–Arrhenius model the situation is different because the acoustic wave perturbs the relaxation of all the elements of the PM plane and it may appear of considerable relevance (in particular, for the case of low-frequency acoustic loading) that the distribution of the elements is somehow limited (i.e., $\left|\Delta\sigma' f\right|$ diminishes when $\varepsilon_\perp \to \infty$ and $\left|\varepsilon_{//}\right| \to \infty$).

In Figure 21.4 the results of the numerical evaluation of the hysteresis stress/strain loops predicted by Eq. (21.7) and Eq. (21.8) are presented[19] for the particular case of a sinusoidal strain variation and a homogeneous element distribution inside the rectangular area $\varepsilon_\perp \le \varepsilon_\perp^{\max} = 10, \; -10 = \varepsilon_{//}^{\min} \le \varepsilon_{//} \le \varepsilon_{//}^{\max} = 10$. Modification of the hysteresis loop with increasing wave amplitude at intermediate nondimensional frequency $F = 1$ is demonstrated in Figure 21.4a. The transformation of an elliptical loop, which is typical for linear hysteresis in a stress/strain relationship, to a nonelliptical loop, which is typical of nonlinear hysteresis, with increasing wave amplitude, is clearly seen. Figure 21.4b demonstrates the opening of the hysteresis loop with increasing frequency, indicating the dynamic nature of hysteresis phenomena captured by the Preisach–Arrhenius model.

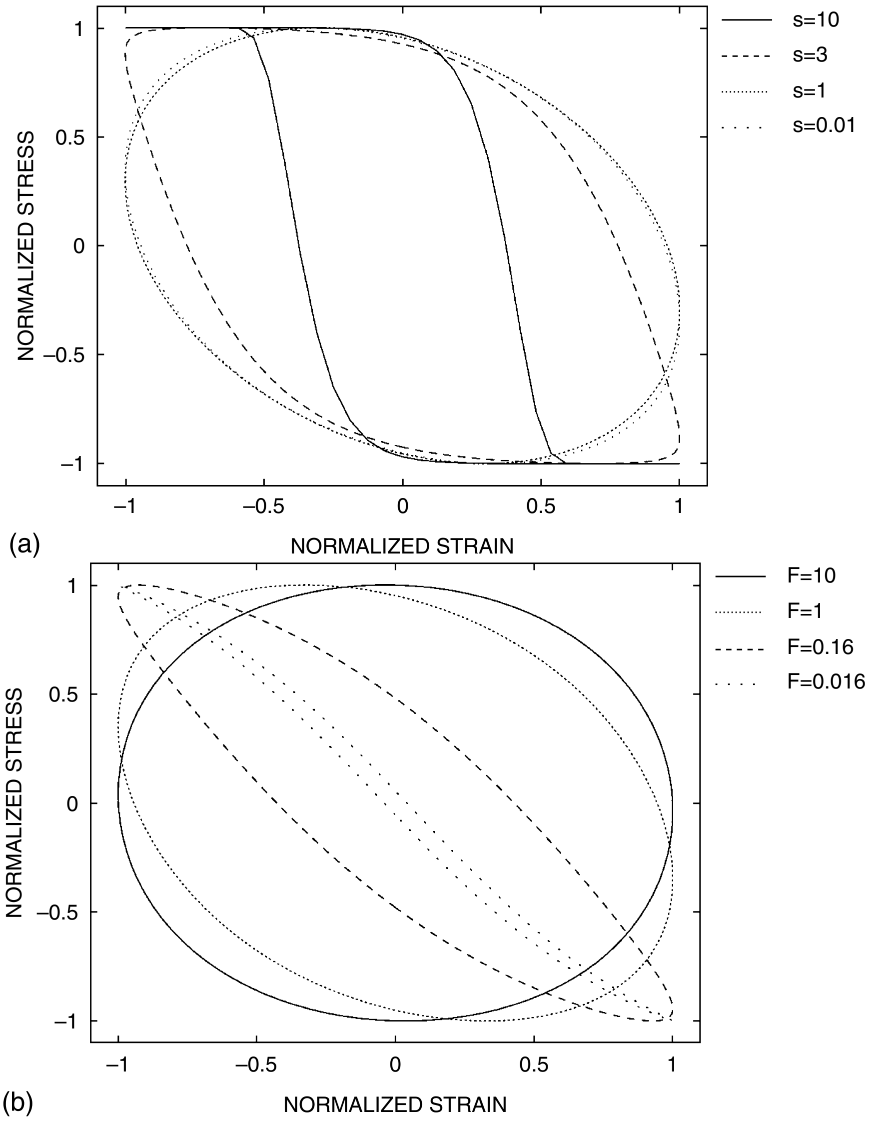

Fig. 21.4. Numerically obtained normalized stress/strain hysteretic dependences in the case of homogeneous element distribution inside the rectangular $\varepsilon_{\perp} \le 10,\ -10 \le \varepsilon_{//} \le 10$. The path of the material state variation is directed clockwise along the loops. Modification of the hysteresis loop with increasing wave amplitude at fixed frequency $F = 1$ (a). Modification of the hysteresis loop with increasing frequency for the fixed wave amplitude $\varepsilon_A = 1$ (b).

3. Transition from Rate-Dependent to Rate-Independent Hysteresis

From the qualitative analysis of the validity limits of the Preisach–Mayergoyz model (presented in Section 1) it could be concluded that the PM regime should be located between the quasiequilibrium and the quasifrozen limits of the Preisach–Arrhenius model. From physical considerations, the PM regime is absent at very low frequencies, because there are nearly no hysteresis phenomena. In fact, an element has sufficient time both during loading and unloading to assume the same equilibrium configuration (see Figure 21.4b). At very high frequencies, the role of hysteresis is expected to be nearly negligible because the elements have no time to switch from one level to another. The numerical analysis of Section II has also confirmed that the transition from linear to nonlinear hysteresis tends to occur with increasing wave amplitude (see Figure 21.4a). These qualitative arguments are supported by the analytical estimates of the nonlinear contribution to the elastic modulus, which can be obtained[19] in the frame of the mathematical formalism presented in Eqs. (21.7) and (21.8). The so-called secant modulus[20] $\langle E\rangle \equiv \sigma(\varepsilon = \varepsilon_A)/\varepsilon_A$, which is one of the possible forms of presenting the modulus defect, was estimated analytically under the assumption of the infinite extension of the homogeneous distribution of the elements in the PM plane (in other words, $\varepsilon_{\perp}^{\max} \to \infty$, $\varepsilon_{//}^{\max} \to \infty$, $\varepsilon_{//}^{\min} \to -\infty$), and by approximating the sinusoidal strain variation in the acoustic wave by a sawtooth profile.

The analysis has demonstrated that the linear decrease of the modulus defect with the acoustic wave amplitude $\langle E\rangle \propto -\varepsilon_A$, which is characteristic of rate-independent hysteresis in the frame of the PM model, can be realized only at high amplitudes of the acoustic loading ($\varepsilon_A \gg 1$). However, the latter should be in the region of the homogeneity of the elements' distribution (formally $\varepsilon_{\perp}^{\max} \to \infty$, $\varepsilon_{//}^{\max} \to \infty$, $\varepsilon_{//}^{\min} \to -\infty$, when $\varepsilon_A \gg 1$). Three different frequency regimes can be identified.

In the high-frequency regime, determined by the inequality $F \gg F_H \equiv \exp(2\varepsilon_A)/(4\varepsilon_A)$, the contribution to the modulus (which, in the following, is normalized by $(\Delta\sigma' f)_0 \varepsilon_0^2$) is very small

$$|\langle E\rangle| \approx \left[1/(4F\varepsilon_A^2)\right][\ln(F/F_H)/(F/F_H)] \ll 1. \tag{21.9}$$

The significant values of $\langle E\rangle$ with the dominant contribution, which is linear in strain, have been found only in the intermediate frequency regimes $\exp(\varepsilon_A)/(4\varepsilon_A) \equiv F_I \ll F \ll F_H \equiv \exp(2\varepsilon_A)/(4\varepsilon_A)$ and $\exp(-\varepsilon_A/2)/(4\varepsilon_A) \equiv F_L \ll F \ll F_I \equiv \exp(\varepsilon_A)/(4\varepsilon_A)$, where the secant modulus varies as $\langle E\rangle \approx -4\varepsilon_A + [\ln(4F\varepsilon_A)]^2/\varepsilon_A$ and $\langle E\rangle \approx -\varepsilon_A + 2\ln(4F\varepsilon_A)$, respectively. Linear dependence of the modulus on strain amplitude disappears in the low-frequency regime defined by the inequality $F \ll F_L \equiv \exp(-\varepsilon_A/2)/(4\varepsilon_A)$, where the dependence of the modulus on the strain amplitude is logarithmically weak

$$\langle E\rangle \approx -2\ln[1/(4F\varepsilon_A)]. \tag{21.10}$$

The obtained estimates correlate with the qualitative expectations. First, the Preisach–Mayergoyz regime, in which $\langle E\rangle \propto -\varepsilon_A$, has been recovered as a particular case of the Preisach–Arrhenius model. It is predicted that the PM regime can be obtained for $\varepsilon_A \gg 1$ in a wide frequency interval

$$\exp(-\varepsilon_A/2)/(4\varepsilon_A) \equiv F_L \ll F \ll F_H \equiv \exp(2\varepsilon_A)/(4\varepsilon_A). \tag{21.11}$$

Note that for $\varepsilon_A \gg 1$ we have $F_L \ll 1$, whereas $F_H \gg 1$. The theory predicts that acoustic nonlinearity grows with increasing frequency of a high-amplitude excitation ($\varepsilon_A \gg 1$) in the low-frequency domain $F \ll F_L$, that it weakly (logarithmically) depends on the frequency in the intermediate domain $F_L \ll F \ll F_H$ of quadratic hysteretic nonlinearity, and that it falls in the high-frequency domain $F \gg F_H$.

Second, in accordance with the derived formulae in transition from the low-frequency regime $F \ll F_L$ to the intermediate-frequency regime $F_L \ll F \ll F_H$, the dominant contribution to the secant modulus changes from being nearly strain independent to having a linear dependence on the strain. So, for a material loaded by high-amplitude acoustic waves, the critical frequency F_L can be identified as a transition frequency from the regime where its elements behave as if they were in quasiequilibrium (Figure 21.3), to the regime where they behave as bi-stable units (Figure 21.1).

Third, in accordance with the derived formulae, in the transition from the intermediate-frequency regime $F_L \ll F \ll F_H$ to the high-frequency regime $F \gg F_H$, there is a significant diminishing in secant modulus magnitude that is accompanied by the disappearance of the contribution which is linear in strain amplitude. Consequently, the critical frequency F_H can be identified as a transition frequency from the regime where the mesoscopic mechanical elements behave as bi-stable units, to the regime where they behave as quasifrozen ones.

In accordance with the obtained results, if the dominant contribution to the modulus defect in experiment is linear in wave amplitude, this necessitates the strong inequality $s_A \gg 1$. In other words, the dimensional acoustic strain amplitude should significantly exceed the characteristic strain $s_0 = k_B T/d$ of the material. In this limiting case, the theory predicts that the dispersion of the nonlinearity, which is accompanied by the deviation from the $\langle E \rangle \propto -\varepsilon_A$ law, might be expected in the frequency ranges $F \leq F_L$ and $F \geq F_H$.

It should be also noted that the obtained results correlate well with the experimentally observed dependence of the modulus defect on the wave amplitude.[14] For the comparison it should be taken into account that in the high-amplitude regime the dependence of the critical frequencies on the wave amplitude is exponentially strong [see Eq. (21.11)]. For example, if for the initial amplitude of the acoustic excitation with $\varepsilon_A \gg 1$ the system is in the low-frequency regime $F \ll F_L$, then with increasing ε_A the characteristic frequency $F_L \equiv \exp(-\varepsilon_A/2)/(4\varepsilon_A)$ diminishes and sooner or later the opposite condition $F_L \ll F$ will be fulfilled. This corresponds to the transition of the system with increasing wave amplitude from the low-frequency quasilinear regime (21.10) to the intermediate-frequency regime characterized by $\langle E \rangle \propto -\varepsilon_A$ typical of PM model.

If for the initial amplitude of the acoustic excitation with $\varepsilon_A \gg 1$ the system is in the high-frequency regime $F \gg F_H$, then with increasing ε_A the characteristic frequency $F_H \equiv \exp(2\varepsilon_A)/(4\varepsilon_A)$ increases and sooner or later the opposite condition $F \ll F_H$ will be fulfilled. This corresponds to the transition of the system with increasing wave amplitude from the high-frequency quasifrozen regime (21.9) to the intermediate frequency regime characterized by $\langle E \rangle \propto -\varepsilon_A$ typical of PM model.

Taking into account that the PA model naturally describes quasilinear behavior of the microinhomogeneous material at weak amplitudes of acoustic loading ($\varepsilon_A \ll 1$; see Figure 21.4a), it can be also concluded that the developed theory predicts the transition from the amplitude-independent modulus defect to the law $\langle E \rangle \propto -\varepsilon_A$ (typical of hysteretic quadratic nonlinearity) with acoustic amplitude increasing from $\varepsilon_A \ll 1$ to $\varepsilon_A \gg 1$.

4. Discussion

Prior to drawing any conclusion from the work presented above,it should be clearly stated that the thermal relaxation Preisach–Arrhenius model does not include all the effects producing rate-dependence of the hysteresis. See, for comparison, the description of rate-dependent hysteretic phenomena in magnetism.[10] The rate-dependence may also appear due to the fact that the acoustic field cannot, in principle, transform the state of a mechanical element infinitely fast.[10,21] In other words, an individual mechanical element cannot change its configuration instantaneously either due to direct effect of the acoustic field or due to thermal fluctuations. In the Preisach–Arrhenius model, the finite transition time appears only statistically in averaging over all the elements, whereas each of the elements still exhibits instantaneous transitions as in the zero-temperature (PM) model. To introduce finite transition times for the individual elements, either a micromechanical model of the transition between the different states should be formulated,[10,22] or the finite transition times could be introduced phenomenologically as a temperature-independent relaxation process.[21] Surely, the generalized theoretical model of hysteresis should include a correct description of the time evolution of both the transitions caused by thermal fluctuations and of those directly induced by the acoustic forces. The development of a generalized model would be highly desirable for the quantitative interpretation of the experiments,[12,13,15,16,23] where the dependence of the acoustic nonlinearity of the microinhomogeneous materials on frequency has been observed.

5. Conclusions

The evaluation of the Preisach–Arrhenius model for the acoustic hysteresis demonstrates that thermal effects are capable of inducing a dependence on wave frequency of the nonlinear acoustic properties of microinhomogeneous materials. Thermal effects can also lead to an amplitude-dependent behavior of the material which differs from that predicted by the Preisach–Mayergoyz model in several important aspects. The Preisach–Arrhenius model of rate-dependent acoustic hysteresis also explains the possible transition in acoustic behavior of microinhomogeneous materials with increasing wave amplitude from a linear one to another characterized by dominance of the hysteretic quadratic nonlinearity. From the physics point of view this is due to the fact that the higher the amplitude of the material mechanical loading, the more difficult for the thermal fluctuations to retain the system in a unique quasiequilibrium state.

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