6 Multidimensional and Longitudinal Poverty: an Integrated Fuzzy Approach

Gianni Betti, Bruno Cheli, Achille Lemmi, Vijay Verma

Gianni Betti, Achille Lemmi, Vijay Verma
Dipartimento di Metodi Quantitativi, University of Siena
Bruno Cheli
Dipartimento di Statistica e Matematica applicata all'Economia, University of Pisa

6.1 Introduction

This Chapter is a contribution to the analysis of deprivation seen as a multi-dimensional condition, and in the longitudinal context. Multi-dimensionality involves both monetary and diverse non-monetary aspects – the former as the situation, either absolute or relative to the average standard, of low income, and the latter as a lack of access to other resources, facilities, social interactions and even individual attributes determining life-style. Persistence and movement over time is an equally important aspect of the intensity of deprivation, requiring longitudinal study at the micro level and in the aggregate.

Most of the methods designed for the analysis of poverty share two limitations: i) they are unidimensional, i.e. they refer to only one proxy of poverty, namely low income or consumption expenditure; ii) they need to dichotomize the population into the *poor* and the *non-poor* by means of the so called *poverty line*.

Nowadays many authors recognize that poverty is a complex phenomenon that cannot be reduced solely to monetary dimension. This leads to the need for a multidimensional approach that consists in extending the analysis to a variety of non-monetary indicators of living conditions. If multidimensional analyses are increasingly feasible as the available information increases, it was the development of multidimensional approaches that in turn stimulated the surveying of a variety of aspects of living conditions.

By contrast, however, little attention has been devoted to the second limitation of the traditional approach, i.e. the rigid poor/non-poor dichotomy, with the consequence that most of the literature on poverty measurement continues to be based on the use of poverty thresholds. Yet it is undisputable that such a clear-cut division causes a loss of information and removes the nuances that exist between the two extremes of substantial welfare on the one hand and distinct material hardship on the other. In other words, poverty should be considered as a *matter of degree* rather than as an attribute that is simply present or absent for individuals in the population.

An early attempt to incorporate this concept at methodological level (and in a multidimensional framework) was made by Cerioli and Zani (1990) who drew inspiration from the theory of Fuzzy Sets initiated by Zadeh (1965).

Given a set X of elements $x \in X$, any fuzzy subset A of X is defined as follows: $A = \{x, \mu_A(x)\}$, where $\mu_A(x)$: $X \rightarrow [0,1]$ is called the *membership function* (*m.f.*) in the fuzzy subset A. The value $\mu_A(x)$ indicates the degree of membership of *x* in *A*. Thus $\mu_A(x) = 0$ means that x does not belong to *A*, whereas $\mu_A(x) = 1$ means that *x* belongs to *A* completely. When $0 < \mu_A(x) < 1$, *x* partially belongs to *A* and its degree of membership in *A* increases in proportion to the proximity of $\mu_A(x)$ to 1.

Cerioli and Zani's original proposal was later developed by Cheli and Lemmi (1995) giving origin to the so called *Totally Fuzzy and Relative* (TFR) approach. Both methods have been applied by a number of authors subsequently, with a preference for the TFR version[1], and in parallel the same TFR method was refined by Cheli (1995) who also used it to analyze poverty in fuzzy terms in the dynamic context represented by two consecutive panel waves.

From this point on, the methodological implementation of this approach has developed in two directions, with somewhat different emphasis despite their common orientation and framework. The first of these is typified by the contributions of Cheli and Betti (1999) and Betti, Cheli and Cambini (2004), focusing more on the time dimension, in particular utilizing the tool of transition matrices. The second, with the contributions of Betti and Verma (1999, 2002, 2004) and Verma and Betti (2002), has focused more on capturing the multi-dimensional aspects, developing the concepts of "manifest" and "latent" deprivation to reflect the intersection and union of different dimensions.

In this Chapter we draw on the state-of-the-art of these developments, to integrate them in the form of, what may be called, an "Integrated Fuzzy and Relative" (IFR) approach to the analysis of poverty and deprivation.

[1] For instance, Chiappero Martinetti (2000), Clark and Quizilbash (2002) and Lelli (2001) use the TFR method in order to analyze poverty or well-being according to Sen's capability approach.

The concern of the chapter is primarily methodological. We re-examine two important aspects introduced by the use of fuzzy measures, namely:

(i) the choice of membership functions i.e. quantitative specification of individuals' or households' degrees of poverty and deprivation, given the level and distribution of income and other aspects of living conditions of the population; and

(ii) the choice of rules for the manipulation of the resulting fuzzy sets, rules defining their complements, intersection, union and aggregation.

In relation to (i), we note the relationship of the proposed fuzzy monetary measure with the Lorenz curve and the Gini coefficient. Certain conceptual and theoretical aspects concerning fuzzy set logic and operations pertinent for the definition of multidimensional measures of deprivation are then clarified, and utilized in the construction of a number of such measures.

The need for (ii) arises because, for longitudinal analysis of poverty using the fuzzy set approach, we need *joint membership functions* covering more than one time period, which have to be constructed on the basis of the series of cross-sectional membership functions over those time periods. We propose a general rule for the construction of fuzzy set intersections, that is, a rule for the construction of longitudinal poverty measures from a sequence of cross-sectional measures. On the basis of the results obtained, various fuzzy poverty measures over time can be constructed as consistent generalizations of the corresponding conventional (dichotomous) measures. Examples are rates of any-time, persistent and continuous poverty, distribution of persons and poverty spells according to duration, rates of exit and re-entry into the state of poverty, etc.

6.2 Income poverty

Diverse "conventional" measures of monetary poverty and inequality are well-known and are not discussed here. Here we will focus on only the most commonly used measure, namely the proportion of a population classified as "poor" in purely relative terms on the following lines. To dichotomize the population into the "poor" and the "non-poor" groups, each person i is assigned the equivalised income y_i of the person's household. Persons with equivalised income below a certain threshold or poverty line (such as 60% of the median equivalised income as adopted by Eurostat) are considered to be poor, and the others as non-poor. The conventional income poverty rate (the head count ratio, H) is the proportion of the population below the poverty line.

Apart from the various methodological choices involved in the construction of conventional poverty measures, the introduction of fuzzy measures brings in *additional* factors on which choices have to be made. As noted, these concern at least two aspects: the choice of *membership functions*; and the choice of *rules for manipulation* of the resulting fuzzy sets. To be meaningful both these choices must meet some basic logical and substantive requirements. It is also desirable that they be useful in the sense of elucidating aspects of the situation not captured (or not captured as adequately) by the conventional approach.

We begin with the issue of choice of the poverty membership function (m.f.). In the conventional head count ratio H, the m.f. may be seen as $\mu(y_i)=1$ *if* $y_i < z$, $\mu(y_i)=0$ *if* $y_i \geq z$, where y_i is equivalised income of individual i, and z is the poverty line. In order to move away from the poor/non-poor dichotomy, Cerioli and Zani (1990) proposed the introduction of a transition zone (z_1-z_2) between the two states, a zone over which the m.f. declines from 1 to 0 linearly.

In the TFR approach, Cheli and Lemmi (1995) define the m.f. as the distribution function of income, normalized (linearly transformed) so as to equal 1 for the poorest and 0 for the richest person in the population. The mean of m.f. defined in this way is always 0.5, by definition. It is desirable, however, to make this mean represent the average level of poverty or deprivation in the population, just as H in the conventional approach.

In order to make this mean equal to some specified value (such as 0.1) so as to facilitate comparison with the conventional poverty rate, Cheli (1995) takes the m.f. as normalized distribution function, raised to some power $\alpha \geq 1$:

$$\mu_i = (1-F_i)^\alpha = \left(\sum_{\gamma=i+1}^{n} w_\gamma \Big/ \sum_{\gamma=2}^{n} w_\gamma \right)^\alpha ; \quad \mu_n = 0 \tag{6.1}$$

where F_i is the income distribution function and w_γ is the sample weight of individual of rank γ (1 to n) in the ascending income distribution.

Increasing the value of this exponent implies giving more weight to the poorer end of the income distribution: empirically, large values of the m.f. would then be concentrated at that end, making the propensity to income poverty sensitive to the *location* of the poorer persons in the income distribution. Beyond that, the choice of the value of α is essentially arbitrary, or at best based on some external consideration: this is unavoidable since any method for the quantification of the extent of poverty is inevitably based on the arbitrary choice of some parameter (Hagenaars 1986). Later Cheli and Betti (1999) and Betti and Verma (1999) have chosen the parameter α so that the mean of the m.f. is equal to the head count ratio computed for the official poverty line. In this way we avoid explicit choice of α, by adapting to the political choice which is implicit in the poverty line. Moreover, comparison between the conventional and fuzzy measures is facilitated. Betti and Verma (1999) have used a somewhat refined version of the above formulation (6.1) in order to define what they called the Fuzzy Monetary indicator (FM):

$$\mu_i = FM_i = (1-L_i)^\alpha = \left(\sum_{\gamma=i+1}^{n} w_\gamma y_\gamma \Big/ \sum_{\gamma=2}^{n} w_\gamma y_\gamma \right)^\alpha ; \quad \mu_n = 0 \tag{6.2}$$

where y_γ is the equivalised income and L_i represents the value of the Lorenz curve of income for individual i. In other terms, $(1-L_i)$ represents the *share of the total equivalised income* received by all individuals who are less poor than the person concerned. It varies from 1 for the poorest, to 0 for the richest individual. $(1-L_i)$ can be expected to be a more sensitive indicator of the actual disparities in income, compared to $(1-F_i)$ which is simply the *proportion of individuals* less poor than the person concerned. It may be noted that while the mean of $(1-F_i)$ values is ½ by definition, the mean of $(1-L_i)$ values equals $(1+G)/2$, where G is the Gini coefficient of the distribution.

Here we propose a new measure which combines the TFR approach of Cheli and Lemmi (1995) and the approach of Betti and Verma (1999) into an "Integrated Fuzzy and Relative" (IFR) approach, which takes into account both the *proportion of individuals* less poor than the person concerned, and the *share of the total equivalised income* received by all individuals less poor than the person concerned. We define this measure as:

$$\begin{aligned} \mu_i &= FM_i = (1-F_i)^{\alpha-1}[1-L_i] = \\ &= \left(\sum_{\gamma=i+1}^{n} w_\gamma \Big/ \sum_{\gamma=2}^{n} w_\gamma \right)^{\alpha-1} \left(\sum_{\gamma=i+1}^{n} w_\gamma y_\gamma \Big/ \sum_{\gamma=2}^{n} w_\gamma y_\gamma \right); \quad \mu_n = 0 \end{aligned} \tag{6.3}$$

where, again, parameter α may be chosen so that the mean of these measures, FM, equals the head count ratio H:

$$FM = \frac{\alpha + G_\alpha}{\alpha(\alpha+1)} = H \tag{6.4}$$

It is important to note that the Fuzzy Monetary (FM) measure as defined above is expressible in terms of the generalized Gini measures G_α, which is a generalization of the standard Gini coefficient (for $\alpha=1$). In the continuous case it is defined as:

$$G_\alpha = \alpha(\alpha+1) \int_0^1 (1-F)^{\alpha-1} (F-L)\, dF . \tag{6.5}$$

This measure weights the distance $(F - L)$ between the line of perfect equality and the Lorenz curve by a function of the individual's position in the income distribution, giving more weight to its poorer end.

6.3 Non-monetary deprivation ("Fuzzy Supplementary")

In addition to the level of monetary income, the standard of living of households and persons can be described by a host of indicators. Quantification of and putting together diverse indicators of deprivation involves a number of steps, models and assumptions. Specifically, decisions are required with regard to assigning numerical values to the ordered categories, weighting the score to construct composite indicators, choosing their appropriate distributional form, and scaling the resulting measures in a meaningful way.

Choice and grouping of indicators

Firstly, from the large set which may be available, a selection has to be made of indicators which are substantively meaningful and useful for the analysis of deprivation. This is a substantive as well as a statistical question. Secondly, it is useful to identify the underlying dimensions and to group the indicators accordingly. Taking into account the manner in which different indicators cluster together adds to the richness of the analysis; ignoring such dimensionality can result in misleading conclusions (Whelan et al. 2001).

Assigning numerical values to ordered categories

Individual items indicating non-monetary deprivation often take the form of simple "yes/no" dichotomies (such as the presence or absence of enforced lack of certain goods or facilities), or sometimes ordered polytomies. Perhaps the simplest scheme for assigning numerical values to categories is by assuming that the ranking of the categories represents an equally-spaced metric variable (Cerioli and Zani 1990). An alternative which has been proposed is replacing the simple ranking of the categories with their distribution function in the population (Cheli and Lemmi 1995).

Weighting for constructing composite measures

When aggregating several indicators at macro level, an early attempt to choose an appropriate weighting system was made by Ram (1982), using principal component analysis, which was also adopted by Maasoumi and Nickelsburg (1988). For the construction of fuzzy measures, however, it is necessary to weight and aggregate items at the *micro level.* Nolan and Whelan (1996) adopted factor analysis for this purpose. In order also to give more weight to more widespread items, Cerioli and Zani (1990) specified the weight of any item as a function of the proportion deprived of the item. Another very important principle that the weighting system should satisfy is that of avoiding redundancy, that is, limiting the influence of those indicators that are highly correlated. To this effect, Betti and Verma (1999) proposed the item weights to comprise two factors. The first factor is determined by the variable's power to differentiate among individuals in the population, that is, by its dispersion: this may be taken as proportional to the coefficient of variation of deprivation score for the variable concerned. The second factor is taken as a function of the correlation of any item with other items, in such a manner that it is not affected by the introduction of variables entirely uncorrelated with the item concerned, but is reduced proportionately to the number of highly correlated variables present.

Functional form of the distribution

Of course, the numerical values for composite indicators of deprivation as obtained above may be directly used as fuzzy degrees of membership, as has been done by a number of authors. Betti and Verma (1999) proposed instead to treat the non-monetary scores in a way entirely analogous to that for monetary poverty measures, described in the previous section. On the basis of this approach, the function corresponding to equation (6.2) would be:

$$\mu_i = FS_i = \left(1-F_{(S),i}\right)^{\alpha_S-1}\left(1-L_{(S),i}\right);\ \alpha_S \geq 1, \tag{6.6}$$

where $F_{(S),i}$ represents the distribution function of the overall supplementary deprivation (S) evaluated for individual i, and $L_{(S),i}$ the value of the Lorenz curve of S for individual i, and α_S is a parameter to be determined[2].

[2] The above approach to combining diverse indicators of non-monetary deprivation treats them as additive. The same methodology can be applied for constructing separate measures for different dimensions of non-monetary deprivation, such as those concerning life-style, housing or the environment (Eurostat 2002; Nolan and Whelan 1996). In either case, alternative forms of aggregation are also possible, such as adding the scores separately within dimensions of deprivation, and then aggregating the dimension-specific scores using some other methodology such as fuzzy intersections and unions.

Scaling of the measures

Strictly, the scale of the deprivation measures so constructed remains arbitrary. From a substantive point of view, Betti and Verma (1999) propose to determine α_s so as to make the overall non-monetary deprivation rate numerically identical to the *monetary poverty* rate H.

This completes the specification of the fuzzy m.f. of deprivation.

6.4 Fuzzy set operations appropriate for the analysis of poverty and deprivation

6.4.1 Multidimensional measures

In the previous sections we have considered poverty as a fuzzy state and defined measures of its degree in different dimensions, namely: monetary, overall non-monetary, and possibly concerning particular aspects of life. In multidimensional analysis it is of interest to know the extent to which deprivation in different dimensions tends to overlap for individuals. Similarly, in longitudinal analysis it would be of interest to know the extent to which the state of poverty or deprivation persists over time for the person concerned. Such analyses require the specification of rules for the manipulation of fuzzy sets.

As a concrete example, consider deprivation in two dimensions: monetary poverty and supplementary (overall non-monetary) deprivation that we denote by *m* and *s* respectively, each of them being characterized by two opposite states - labeled as 0 (non-deprived) and 1 (deprived) - that correspond to a pair of sets forming a fuzzy partition. Any individual i belongs *to a certain degree* to each of the four sets: the two cross-sectional sets m and s, and their complements. Since fuzzy sets 0 and 1 are complementary, having defined the degree of membership in one as FM_i or FS_i, it is straightforward (and necessary) to calculate the membership in its complementary set as $(1 - FM_i)$ or $(1 - FS_i)$, respectively.

In the conventional approach, a joint analysis of monetary and non-monetary deprivation (both seen as dichotomous characteristics) is carried out by assigning each individual to one (and only one) of the four sets representing the intersections $m \cap s$ ($m = 0,1$; $s = 0,1$). This can be viewed as individual membership functions in the four sets such that, for a given individual, the membership equals 1 in one of the sets and equals 0 in the three remaining sets. For any particular set, the mean value of the individual membership functions is simply the proportion of individuals in the category corresponding to that set.

Viewed in this way, these "degrees of membership" in the four cross-sectional sets sum to 1 for any individual. In a similar way, we view these fuzzy memberships of an individual to form "fuzzy partitions", which must sum to 1 over the four sets[3]. More precisely, denoting by μ_{ims} the degree of membership in $m \cap s$ (m $\in$ [0,1] ; s $\in$ [0,1]) of individual i, the marginal constraints specified in Table 6.1 must be satisfied. The quantity μ_{ims} represents a measure of the extent to which the individual is affected by the particular combination of states (m,s).

Table 6.1. Situation of a generic individual i seen in fuzzy terms: membership functions for the four intersection sets and for the marginals

		Non-monetary deprivation (s)		
	poverty status	non-poor (0)	poor (1)	total
Monetary deprivation (m)	non-poor (0)	μ_{i00}	μ_{i01}	$1 - FM_i$
	poor (1)	μ_{i10}	μ_{i11}	FM_i
	total	$1 - FS_i$	FS_i	1

6.4.2 Definition of poverty measures according to both monetary and non-monetary dimensions

Our main goal is to find a specification of μ_{ims} that is *the most appropriate to our purpose of analyzing poverty and deprivation.* In this respect, a most important consideration is the following.

Fuzzy set operations are a generalization of the corresponding crisp set operations in the sense that the former reduce to (exactly reproduce) the latter when the fuzzy membership functions, being in the whole range [0,1], are reduced to a {0,1} dichotomy. There is, however, *more than one way in which the fuzzy set operations can be formulated*, each representing an equally valid generalization of the corresponding crisp set operations. The choice among alternative formulations has to be made primarily on substantive grounds: some options are more appropriate (meaningful, convenient) than others, depending on the context and objectives of the appli-

[3] If for each unit in the population, its membership μ_i in a certain set is decomposed into components μ_{ik} such that $\mu_i = \Sigma_k \mu_{ik}$, then the μ_{ik} values constitute m.f.'s corresponding to fuzzy partitions of the original set.

cation. While the rules of fuzzy set operations cannot be discussed in this Chapter in any detail, it is essential to clarify their application *specifically for the study of poverty and deprivation.*

Since fuzzy sets are completely specified by their membership functions, any operation with them (such as union, intersection, complement or aggregation) is defined in terms of the membership functions of the original fuzzy sets involved. As an example, membership μ_{i11} of Table 6.1 is a function of FM_i and FS_i and might be more precisely written as μ_{i11} (FM_i, FS_i). However in the following discussion it will be convenient to use the following simplified notation: (a,b) for the membership functions of two sets for individual i (subscript i can be dropped when not essential), where $a=FM_i$ and $b=FS_i$; also $s_1=\min(a,b)$ and $s_2=\max(a,b)$. We also denote by $\bar{a}=1-a$, $a \cap b$ and $a \cup b$ the basic set operations of complementation, intersection and union, respectively[4]. Table 6.2 shows three commonly-used groups of rules – termed *Standard, Algebraic* and *Bounded* (Klir and Yuan, 1995) – specifying fuzzy intersection and union. Such rules are "permissible" in the sense that they satisfy certain essential requirements such as reducing to the corresponding crisp set operations with dichotomous variables, satisfying the required boundary conditions, being monotonic and commutative, etc.

For our application, a most important observation is that the Standard fuzzy operations provide the *largest* (the most loose, the weakest) intersection and by contrast the *smallest* (the most tight or the strongest) union among all the permitted forms. It is for this reason that they have been labeled as i_{max} and u_{min} in Table 6.2. *It is this factor which makes it inappropriate to use the Standard set operations uniformly throughout in our application to poverty analysis.* In fact, if the Standard operation were applied to all the four intersections of Table 6.1, their sum would exceed 1 and the marginal constraints would not be satisfied[5].

Now it can easily be verified that the Algebraic form, applied to all the four intersections, is the *only one* which satisfies the marginal constraints. But despite this numerical consistency, we *do not* regard the Algebraic form to give results which, for our particular application, would be generally acceptable on intuitive or substantive grounds. In fact, if we take the liberty of viewing the fuzzy propensities as probabilities, then the Algebraic product rule $a \cap b$ as the joint probability, $a \cap b = a*b$ implies zero correlation between the two forms of deprivation, which is clearly at

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[4] This is a short-hand notation for the following. If, for example, a and b refer to an individual's degrees of memberships in sets A and B respectively, then we write the person's degree of membership of set $A \cap B$ as $a \cap b$.

[5] For details, see Betti and Verma (2004).

variance with the high positive correlation we expect in the real situation for *similar* states. The rule therefore seems to provide an unrealistically *low* estimate for the resulting membership function for the intersection of two similar states. The Standard rules, giving higher overlaps (intersections) are more realistic for (a,b) representing similar states.

Table 6.2. Some basic forms of fuzzy operations

	Intersection	Union
Type of operation	$a \cap b$	$a \cup b$
S (standard)	min(a,b)=i_{max}	max(a,b)=u_{min}
A (algebraic)	a*b	a+b-a*b
B (bounded)	max(0,a+b-1)	min(1,a+b)

By contrast, in relation to *dissimilar* states $(\bar{a}, b)$ and $(a, \bar{b})$ (lack of correspondence between deprivations in two dimensions), it appears that the Algebraic rule (and hence also the Standard rule) tends to give *unrealistically high* estimates for the resulting membership function for the intersection. The reasoning similar to the above applies: in real situations, we expect large negative correlations (hence reduced intersections) between *dissimilar* states in the two dimensions of deprivation. In fact, it can be seen by considering some particular numerical values for $(\bar{a}, b)$ or $(a, \bar{b})$ that Bounded rule, for instance, gives much more realistic results for dissimilar states.

Given the preceding considerations, the specification of the fuzzy intersection $a \cap b$ that appears to be the most reasonable for our particular application and that satisfies the above mentioned marginal constraints is of a "Composite" type as follows (Betti and Verma 2004):

- For sets representing *similar states* - such as the presence (or absence) of both types of deprivation - the Standard operations (which provide larger intersections than Algebraic operations) are used.
- For sets representing *dissimilar states* - such as the presence of one type but the absence of the other type of deprivation - we use the Bounded operations (which provide smaller intersections than Algebraic operations).

By applying this composite intersection the marginal constraints of Table 6.1 are specified as shown in Table 6.3. Note that with this operation the propensity to the deprived in at least one of the two dimensions equals

$\max(FM_i, FS_i)$, which can be viewed as any of the three entirely equivalent forms:

- as the complement of cell "0-0" in Table 6.3, or
- as the sum of the membership functions in the other three cells, or
- as the union of (FM_i, FS_i) under the Standard fuzzy set operations.

Table 6.3. Joint measures of deprivation according to the Betti-Verma Composite operation

		Non-monetary deprivation (s)		
	poverty status	non-poor (0)	poor (1)	total
Monetary deprivation (m)	non-poor (0)	$\min(1-FM_i, 1-FS_i)$ $=1-\max(FM_i, FS_i)$	$\max(0, FS_i-FM_i)$	$1-FM_i$
	poor (1)	$\max(0, FM_i-FS_i)$	$\min(FM_i, FS_i)$	FM_i
	total	$1-FS_i$	FS_i	1

Figure 6.1 illustrates the Composite set operations graphically. *Such a representation is fundamental to the development and illustration of the methodology presented in this Chapter.* In the figure, the degree of membership in the "universal set" X is represented by a rectangle of unit length, and the individual's memberships on the two subsets (say, $0 \le a \le 1$, $0 \le b \le 1$, and their complements) have been placed within it. Different forms of fuzzy set operations (Table 6.2) are reproduced by different placements of the subset memberships within the rectangle for X. The figure shows intersections; fuzzy set unions can be similarly represented. The Standard form, appropriate for *similar* sets, is represented by placing the two memberships (a,b) on the *same base*, so that their intersection is min(a,b), and union is max(a,b). In the Bounded form, appropriate for *dissimilar* sets, the two sets are placed at the *opposite ends* of X, thus their intersection is max(0, a+b-1) and union is min(1, a+b), exactly as required from Table 6.2. It can be seen that the Algebraic form is represented by placing membership (b) symmetrically over memberships (a) and (non-a), i.e. each of the two receiving a proportionate share of (b), respectively a*b and (1-a)*b. Hence a*b is the intersection, while the union is (a+b-a*b). Generally, by moving one set membership higher than the other within X, the overlap (intersection) is reduced, and the underlay (union) increased.

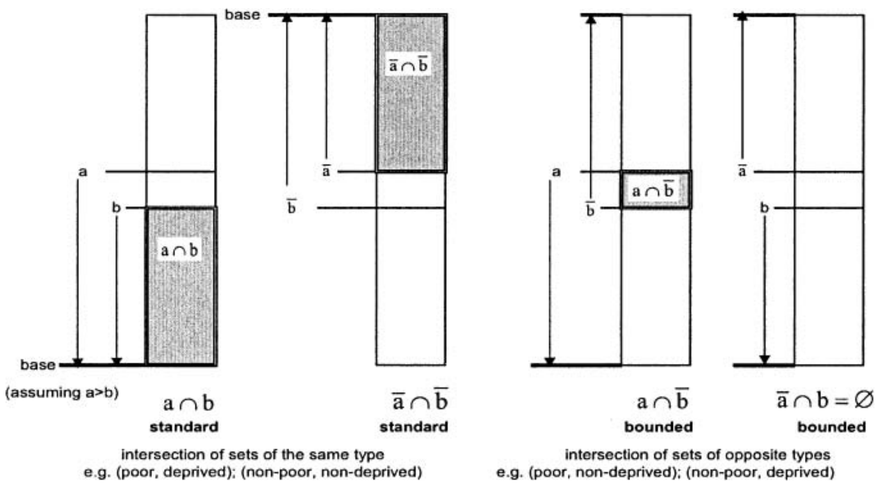

Fig. 6.1. The Composite fuzzy set operations: a graphical representation of intersections

6.4.3 Income poverty and non-monetary deprivation in combination: Manifest and Latent deprivation

The two measures – FM_i the propensity to income poverty, and FS_i the overall non-monetary deprivation propensity – may be combined to construct composite measures which indicate the extent to which the two aspects of income poverty and non-monetary deprivation overlap for the individual concerned. These measures are as follows.

M_i Manifest deprivation,

representing the propensity to both income poverty and non-monetary deprivation simultaneously.

L_i Latent deprivation,

representing the individual being subject to at least one of the two, income poverty and/or non-monetary deprivation.

Once the propensities to income poverty (FM_i) and non-monetary deprivation (FS_i) have been defined at the individual level (i), the corresponding combined measures are obtained in a straightforward way, using the Composite set operations. These individual propensities can then be averaged to produce the relevant rates for the population. The Manifest deprivation propensity of individual i is the intersection (the smaller) of the two (similar) measures FM_i and FS_i:

$$M_i = \min(FM_i, FS_i). \tag{6.7}$$

Similarly, the Latent deprivation propensity of individual i is the complement of the intersection indicating the absence of both types of deprivation:

$$L_i = 1 - \min(\overline{FM}_i, \overline{FS}_i) = \max(FM_i, FS_i) \tag{6.8}$$

which turns out to be simply the union (the larger) of the two measures FM_i and FS_i under the Standard operation.

From empirical experience (Betti and Verma 2002; Betti et al. 2005a), it appears that the degree of overlap between income poverty and non-monetary deprivation at the level of individual persons tends to be *higher* in *poorer* areas, and lower in richer areas. A useful indicator in this context is the Manifest deprivation index defined as a percentage of Latent deprivation index; in theory, this ratio varies from 0 to 1. When there is no overlap (i.e., when the subpopulation subject to income poverty is entirely different from the subpopulation subject to non-monetary deprivation), Manifest deprivation rate and hence the above mentioned ratio equals 0. When there is complete overlap, i.e., when each individual is subject to exactly the same degree of income poverty and of non-monetary deprivation ($FM_i = FS_i$), the Manifest and Latent deprivation rates are the same and hence the above mentioned ratio equals 1.

6.5 On longitudinal analysis of poverty conceptualized as a fuzzy state

6.5.1 Longitudinal application of the Composite fuzzy operation

The procedure developed above to represent multi-dimensional aspects of deprivation extends directly to the representation of its longitudinal aspects: in mathematical terms the two are in fact identical. This can be seen from Table 6.4 which shows persistence and transitions in the state of poverty over two time periods.

In place of the two dimensions of deprivation (monetary and non-monetary), here we have fuzzy sets representing the state of poverty at two times. Persistent poverty (row 2 of Table 6.4), for instance, corresponds to "manifest" deprivation defined in the previous section, and "ever in poverty" (row 5) to "latent" deprivation. Similarly, the propensity to exit from poverty (row 3) is given by the intersection of sets representing two dissimilar states, namely set "poor" at time 1 and set "non-poor" at time 2; to these, the Bounded operations apply.

Table 6.4. Longitudinal measures of interest over two time periods for individual i

	Measure	Membership function	Description
1	Never in poverty	$\bar{a}_i \cap \bar{b}_i = 1 - \max(a_i, b_i)$	Poverty at *neither* of the two years
2	Persistent in poverty	$a_i \cap b_i = \min(a_i, b_i)$	Poverty at *both* of the years
3	Exiting from poverty	$a_i \cap \bar{b}_i = \max(0, a_i - b_i)$	Poverty at time 1, but non-poverty at time 2
4	Entering into poverty	$\bar{a}_i \cap b_i = \max(0, b_i - a_i)$	Non-poverty at time 1, but poverty at time 2
5	Ever in poverty	$a_i \cup b_i = \max(a_i, b_i)$	Poverty at *at least one* of the two years

6.5.2 The general procedure

We need procedures which can handle in a consistent and realistic manner the analysis of poverty at any number of time periods (and also for any number of dimensions of deprivation).

Let, for a series of cross-sections (1,...t,...T), each person's propensity to be in poverty (i.e. the person's membership in the set "poor") be given as $(\mu_1, \mu_2, ..., \mu_T)$, $\mu_t \in [0,1]$. We also define the complements of the above at each time, i.e. the membership function (m.f.) in the set "non-poor" as $\bar{\mu}_t = 1 - \mu_t$. The above cross-sectional measures generate 2^T longitudinal sequences of length T, in which any element t can take one of two values, μ_t and its complement $\bar{\mu}_t = (1 - \mu_t)$.

Figure 6.2 provides an example of one such sequence. An individual's propensities to poverty (and their complements, propensities to non-poverty) over 6 time periods are represented. Given these cross-sectional propensities (degrees of membership), we need rules to specify the joint membership function (j.m.f.) for any specified longitudinal sequence of particular states, for example of sets "poor" at times (1, 4 and 5), and of sets "non-poor" at the remaining times (2, 3 and 6). These sets of interest are represented by shaded rectangles in Figure 6.2. Note that, as in the case of Figure 6.1, sets representing the same state (e.g., "poverty") are placed on the same base, and those representing the opposite state (e.g., "non-poverty") are placed at the opposite end.

The figure immediately gives the required intersection, i.e. the individual's joint membership for the particular longitudinal sequence: it is simply the overlap (if any) between the smallest of the memberships in the "poor" set, and the largest of the memberships in the "non-poor" set. Clearly, the time-ordering of the various cross-sections is entirely irrelevant in this conceptualization. The result can be seen more clearly by ordering the cross-sections according to the size of the memberships, as shown at the right in the figure.

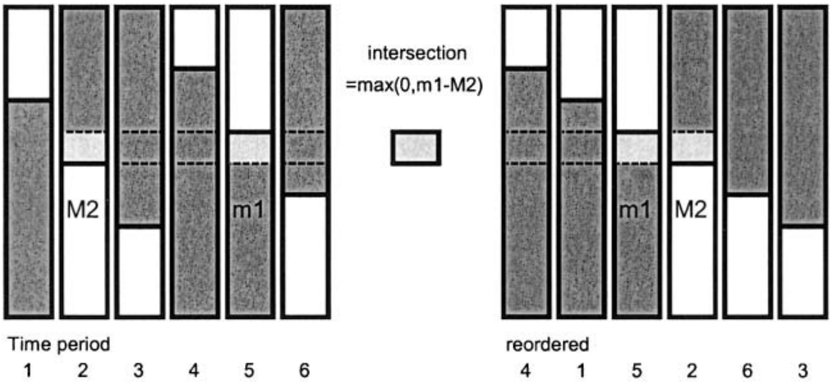

Fig. 6.2. Example of degrees of membership for a longitudinal sequence of "poor" and "non-poor" sets

Returning to the general case, let S(1,2,....,T) be a *particular pattern* of T "poor" and "non-poor" sets for which the j.m.f. is required. Let the elements (cross-sectional sets) of this pattern be grouped into two parts: $S_1 = (\ldots, t_1, \ldots)$, $S_2 = (\ldots, t_2, \ldots)$, where t_1 indicates any of T_1 elements of the same type (say, "poor") in the first group, and t_2 any of T_2 elements of the group of the opposite type ("non-poor"), with $T_1 + T_2 = T$. Let: $m_1 = \min(\ldots, \mu_{t_1}, \ldots)$; $M_2 = \max(\ldots, \mu_{t_2}, \ldots)$. The required j.m.f. for the particular pattern of interest is given by the following: [6]

$$\mathrm{JMF} = \max(0, m_1 - M_2). \tag{6.9}$$

Different types of longitudinal measures correspond to, or can be simply derived from, different patterns S. A number of applications are described

[6] Note that this is the intersection of m.f.'s of opposite types, m_1 and $\overline{M}_2 = 1 - M_2$, using the Bounded operator $\mathrm{JMF} = \max(0, m_1 + \overline{M}_2 - 1)$. Note also that throughout we use symbols μ to represent propensities of the same type (e.g. propensity to "poverty"); it is the type of cross-sectional sets of interest which are different in the two groups (e.g. S_1 "poor", S_2 "non-poor").

later. As an example, for the propensity to be poor at time 1, non-poor at time 2, and then re-entering poverty at time 3, we have:

$$S_1 = (1,3),\ S_2 = (2),\ JMF = \max\left(0, \min(\mu_1, \mu_3) - \mu_2\right). \qquad (6.10)$$

On the basis of the above, we formulate a general procedure in the following terms. Consider any sequence of cross-sectional propensities to poverty or deprivation. It can always be expressed in the form: $(\ldots, \mu_{t_1}, \ldots), (\ldots, \mu_{t_2}, \ldots)$, where t_1 indicates T_1 elements of the same type in one group, and t_2 indicates T_2 elements of the opposite type in the other group.

(i) Sort the elements into two groups by type, for instance all T_1 elements of one type followed by all T_2 elements of the other type.

(ii) Construct the intersection for each group involving elements of the same type using the Standard operator.

(iii) Finally, construct the intersection of the two results of the above operation using the Bounded operator (equation 6.9).

Since the temporal order of cross-sectional propensities is immaterial in the construction of their intersection using this rule, we may view the application of this rule as being without memory. More precisely perhaps, we may designate it as a procedure "without chronology": the outcome depends on the whole "history" (i.e., the specified type of cross-sectional sets in the time sequence t=1 to T, and the associated membership functions); but it does not depend on the actual chronology, the temporal sequence, of those cross-sections[7].

Marginal constraints

As noted, a set of T cross-sections yields 2^T longitudinal sequences. In the conventional analysis, these represent 2^T exhaustive and non-overlapping classes, with each individual unit belonging to one and only one of these, i.e. having some particular pattern (k) of poverty and non-poverty over the T years. Population totals or proportions over any grouping of these patterns are clearly additive. The same consistency must also hold under fuzzy conceptualization.

This condition is ensured by marginal constraints. The above procedure satisfies all the required marginal constraints (Betti et al. 2005b), as can be noted from the following. By definition, all the marginal constraints involved are expressed by successive applications of the following relationship:

[7] This procedure has certain similarities with that proposed by Betti, Cheli and Cambini (2004). However, the present procedure is more general and more consistent.

$$I_{t-1} = I_t + \bar{I}_t,\ t = T \text{ to } 1, (\text{with } I_0 = 1) \tag{6.11}$$

Here I_t is the joint membership of an individual in a particular longitudinal sequence of length t. As before, let (S_1, S_2) represent the two groups of terms of opposing types in the sequence. I_t and $\bar{I}_t$ differ only in that for one of the time periods in the sequence, the cross sectional sets considered are of opposite types. In other words, $\bar{I}_t$ is the degree of joint membership for the sequence obtained from I_t by replacing any particular term in one of the groups (say S_1) with complement of that term in the other group (S_2). I_{t-1} is the degree of joint membership obtained by removing that term altogether, giving a sequence of only (t-1) terms. Equation (6.11) states that $(I_t, \bar{I}_t)$ are fuzzy partitions of I_{t-1}; that of course is exactly what is meant by "marginal constraints". It can be seen that $\bar{I}_t$ is non-zero only if the term moved from S_1 is the smallest term in that set (otherwise, $I_{t-1} = I_t$). In this case, let $m_1^{(2)}$ be the second largest value in S_1 (i.e. the smallest value left after the move). It can be seen that:

$$I_t = \max(0, m_1 - M_2),$$

$$\bar{I}_t = \max\left(0, m_1^{(2)} - \max(m_1, M_2)\right),$$

$$I_{t-1} = \max\left(0, m_1^{(2)} - M_2\right),$$

which satisfies the required marginal constraint (6.11).

6.6 Application to specific situations

In this section we describe some important applications of the above rule for the construction of fuzzy intersections defining longitudinal measures.

6.6.1 Persistence of poverty

Analysis of the persistence of poverty over time requires the specification of j.m.f.'s of the type:

$$I_T = \mu_1 \cap \mu_2 \ldots\ldots\ldots \cap \mu_T\ ,$$

$$U_T = \mu_1 \cup \mu_2 \ldots\ldots\ldots \cup \mu_T\ ,$$

where the first expression is the intersection of a series of T cross-sectional m.f.'s for any individual unit, and the second expression is their union. I_T represents the individual's propensity to be poor at all T periods. U_T is the propensity to be poor at *at least one* of the T periods. Since all sets $\mu_1,.....,\mu_T$ are of the same type (all being propensities to "poverty" rather than to "non-poverty"), the Standard operations apply:

$$I_T = \min(\mu_1, \mu_2, ..., \mu_t, ...\mu_T)$$

$$U_T = \max(\mu_1, \mu_2, ..., \mu_t, ...\mu_T).$$

The complement of U_T, $\overline{U_T} = (1 - U_T)$, is the propensity to be *never poor*[8].

The propensity to be poor in exactly t out of T years is the sum of j.m.f.'s over all sequences with t cross-sectional sets of the type "poor" and the remaining (T–t) of the type "non-poor". For any particular sequence of this type, rearrange the sets such that the *first* t terms are of the type "poor". Hence the j.m.f. for the particular sequence is:

$$JMF=\max[0, \min(\mu_1, \mu_2, \ldots, \mu_t) - \max(\mu_{t+1}, \mu_{t+2}, \ldots, \mu_T)],$$

which is non-zero only for one sequence in which the first group contains the t largest memberships. With [t] denoting the ordered sequence of decreasing μ values, the required j.m.f. becomes:

Poor (exactly t out of T years): $\mu_{[t]} - \mu_{[t+1]}$,

and by simple addition:

Poor (at least t out of T years): $\mu_{[t]}$, the t^{th} largest value.

If we define "persistent" poverty, for instance, as the propensity to be poor over at least a majority of the T years, i.e. over at least t years, with t=int(T/2)+1, the smallest integer being strictly larger than (T/2), the required propensity to persistent poverty is the $[\text{int}(T/2)+1]^{th}$ largest value in the sequence $(\mu_1,......\mu_T)$.

With the conventional poor/non-poor dichotomy, any individual spends some specified number of years between 0 and T in the state of poverty during the interval T. With poverty treated as a matter of degree, any particular individual is seen as contributing to the *whole distribution*, from 0 to T, of the number of years spent in poverty. Over an interval of T years

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[8] The same result is obtained by considering intersection of non-poor sets:

$$\overline{U_T} = \min(\overline{\mu}_1, \overline{\mu}_2, ..., \overline{\mu}_t, ...\overline{\mu}_T) = 1 - \max(\mu_1, \mu_2, ..., \mu_T) = 1 - U_T.$$

the proportion of the time spent in poverty by the ith individual is (with $\mu_{[T+1]} = 0$):

$$t_i = \sum_{t=1}^{T} t \cdot \left(\mu_{[t]} - \mu_{[t+1]}\right) / T = \sum_{t=1}^{T} \mu_t \Big/ T,$$

i.e. simply the mean over the T periods of an individual's cross-sectional propensities to poverty.

6.6.2 Rates of exit and re-entry

Consider for instance the following. Given the state of poverty at time 1, and also at a later time (t–1), what is the proportion exiting from poverty at time t=2, 3, …? Given the state of poverty at time 1, but of non-poverty at a later time (t–1), what is the proportion which has re-entered poverty at time t=3, 4, ….?

In conventional analysis, the above rates are computed simply from the count of persons in various states. Consider for instance individuals poor at times t and (t–1). For exit rate at time t, the numerator is the count of persons poor at times 1 and (t–1), but non-poor at time t; the denominator is the count of all persons who are poor at times 1 and (t–1). Similarly for persons poor at time 1, non poor at (t–1) but poor again at t, the re-entry rate numerator is the count of persons poor at time 1, non-poor at time (t–1), but poor again at time t; the denominator is the count of persons who are poor at time 1 and non-poor at time (t–1). The construction of these measures using fuzzy m.f.'s is also straightforward. With μ_t as a person's propensity to poverty at time t, the person's contribution of these rates is as follows.

Exit rate:

Numerator $\left(\mu_1 \cap \mu_{t-1}\right) \cap \bar{\mu}_t = \max\left[0, \min\left(\mu_1, \mu_{t-1}\right) - \mu_t\right]$

Denominator $\left(\mu_1 \cap \mu_{t-1}\right) = \min\left(\mu_1, \mu_{t-1}\right)$.

Re-entry rate:

Numerator

$$\mu_1 \cap \bar{\mu}_{t-1} \cap \mu_t = \left(\mu_1 \cap \mu_t\right) \cap \bar{\mu}_{t-1} = \max\left[0, \min\left(\mu_1, \mu_t\right) - \mu_{t-1}\right]$$

Denominator $\mu_1 \cap \bar{\mu}_{t-1} = \max\left[0, \mu_1 - \mu_{t-1}\right]$.

The corresponding rates for the population are computed by simply averaging the above individual contributions.

6.7 Concluding remarks

When poverty is viewed as a matter of degree in contrast to the conventional poor/non-poor dichotomy, that is, as a fuzzy state, two additional aspects are introduced into the analysis.

(i) The choice of membership functions i.e. quantitative specification of individuals' or households' degrees of poverty and deprivation.

(ii) And the choice of rules for the manipulation of the resulting fuzzy sets, rules defining their complements, intersections, union and aggregation. Specifically, for longitudinal analysis of poverty using the fuzzy set approach, we need *joint membership functions* covering more than one time period, which have to be constructed on the basis of the series of cross-sectional *membership functions* over those time periods.

This Chapter has discussed approaches and procedures for constructing fuzzy measures of income poverty and of combining them with similarly constructed measures of non-monetary deprivation using the fuzzy set approach.

In fact, the procedures for combining fuzzy measures in multiple dimensions at a given time are identical, in formal terms, to the procedures for combining fuzzy cross-sectional measures over multiple time periods. We have proposed a general rule for the construction of fuzzy set intersections, that is, for the construction of a longitudinal poverty measure from a sequence of cross-sectional measures under fuzzy conceptualization. This general rule is meant to be applicable to any sequence of "poor" and "non-poor" sets, and it satisfies all the marginal constraints. On the basis of the results obtained, various fuzzy poverty measures over time can be constructed as consistent generalizations of the corresponding conventional (dichotomous) measures.

Numerical results of these procedures applied to measures of multidimensional poverty and deprivation, and to combinations of such measures have been presented elsewhere.

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