# 3 An Axiomatic Approach to Multidimensional Poverty Measurement via Fuzzy Sets

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## 3.1 Introduction

Poverty has been in existence for many years and continues to exist in a large number of countries in the World. Therefore, targeting of poverty alleviation remains an important policy issue in many countries. To understand the threat that the problem of poverty poses it is necessary to know the dimension of poverty and the process through which it seems to be deepened. In this context an important question is: how to measure the poverty level of a society and its changes.

In a pioneering contribution, Sen (1976) conceptualized the poverty measurement problem as involving two exercises: (i) the identification of the poor and (ii) aggregation of the characteristics of the poor into an overall indicator that quantifies the extent of poverty. In the literature, the income method has been used mostly to solve the first problem. It requires specification of a poverty line representing the income necessary for a subsistence standard of living. A person is said to be poor if his income falls below the poverty line. On the aggregation issue, Sen (1976) criticized two crude indicators of poverty, the head count ratio (the proportion of persons with incomes below the poverty line) and the income gap ratio (the difference between the poverty line and the average income of the poor, expressed as a proportion of the poverty line), because they remain unaltered under a redistribution of income between two poor persons and the former also does not change if a poor person becomes poorer due to a reduction in his income. Sen (1976) also characterized axiomatically a more sophisticated index of poverty[2].

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[1] I am grateful to Sabina Alkire and Jacques Silber for bringing some important references to my attention and making them available to me.

[2] Several contributions suggested alternatives and variations of the Sen index. See, for example, Takayama (1979), Blackorby and Donaldson (1980), Kakwani (1980a, 1980b), Clark et al. (1981), Chakravarty (1983a, 1983b, 1983c, 1997), Thon (1983), Foster et al. (1984), Haagenars (1987) and Shorrocks (1995).

However, the well-being of a population, and hence its poverty, which is a manifestation of insufficient well-being, is a multidimensional phenomenon and should therefore depend on both monetary and non-monetary attributes or components. It is certainly true that with a higher income or consumption budget a person may be able to improve the position of some of his non-monetary attributes of well-being. But it may happen that markets for certain non-monetary attributes do not exist. One such example is a public good like flood control or malaria prevention program in an underdeveloped country. Therefore, it has often been argued that income as the sole attribute of well-being is inappropriate and should be supplemented by other attributes, e.g., housing, literacy, life expectancy at birth, nutritional status, provision of public goods etc.

We can provide further justifications for viewing the poverty measurement problem from a multidimensional perspective. In the basic needs approach, advocated by development economists, development is regarded as an improvement in the array of human needs, not just as growth of income alone (Streeten 1981). There is a debate about the importance of low income as a determinant of under nutrition (Lipton and Ravallion 1995) and often it is argued that the population's failure to achieve a desirable nutritional status should be regarded as an indicator of poverty (Osmani 1992). In the capability-functioning approach, where a functioning is what a person "succeeds in doing with the commodities and characteristics at his or her command" (Sen 1985, p.10) and capabilities indicate a person's freedom with respect to functionings (Sen 1985, 1992), poverty is regarded as a problem of functioning failure. Functionings here are closely approximated by attributes like literacy, life expectancy, clothing, attending social activities etc. The living standard is then viewed in terms of the set of available capabilities of the person to function. An example of a multidimensional index of poverty in terms of functioning failure is the human poverty index suggested by the UNDP (1997). It aggregates the country level deprivations in the living standard of a population for three basic dimensions of life, namely, decent living standard, educational attainment rate and life expectancy at birth. Chakravarty and Majumder (2005) axiomatized a generalized version of the human poverty index using failures in an arbitrary number of dimensions of life.

In view of the above, in contrast to the income method, it has often been assumed in the literature that each person is characterized by a vector of basic need attributes (see, for example, Sen 1987, 1992; Ravallion 1996; Bourguignon and Chakravarty 1999, 2003; Atkinson 2003), and a direct method of identification of poor checks if the person has "minimally acceptable levels" (Sen 1992, p. 139) of different basic needs. Therefore, the direct method views poverty from a multidimensional perspective, more precisely, in terms of shortfalls of attribute quantities from respective threshold levels. These threshold levels are determined independently of the attribute distributions. A person is said to be poor with respect to an attribute if his consumption of the attribute falls below its minimally acceptable level. "In an obvious sense the direct method is superior to the income method, since the former is not based on particular assumptions of consumer behavior which may or may not be accurate" (Sen 1981, p. 26). If direct information on different attributes is not available, one can adopt the income method, "so that the income method is at most a second best" (Sen 1981, p. 26).

While the direct and income methods differ substantially in certain respects, they have one feature in common: each individual in the population must be counted as either poor or non-poor. The prospect of an intermediate situation is not considered by them. However, it is often impossible to acquire sufficiently detailed information on income and consumption of different basic needs and hence the poverty status of a person is not always clear cut. For instance, the respondents may be unwilling to provide exact information on income and consumption levels. There can be a wide range of threshold limits for basic needs which co-exist in reasonable harmony. The likelihood that relevant information is missing suggests that there is a degree of ambiguity in the concept of poverty. Now, if there is some ambiguity in a concept, "then a precise representation of that ambiguous concept must *preserve* that ambiguity" (Sen 1997, p. 121). Zadeh (1965) introduced the notion of fuzzy set with a view to tackling problems in which indefiniteness arising from a sort of ambiguity plays a fundamental role. Thus, given that the concept of poverty itself is vague, the poverty status of a person is intrinsically fuzzy. This shows that a fuzzy set approach to poverty measurement is sufficiently justifiable.

Fuzzy set theory –based approaches to the measurement of poverty has gained considerable popularity recently (see, for example, Cerioli and Zani 1990; Blaszczak-Przybycinska 1992; Dagum et al. 1992; Pannuzi and Quaranta 1995; Shorrocks and Subramanian 1994; Cheli and Lemmi 1995; Balestrino 1998; Betti and Verma 1998; Qizilbash 2002)[3].

However, a rigorous discussion on desirable axioms for a multidimensional poverty index in a fuzzy environment has not been carried out in the literature. The purpose of this Chapter is to fill this gap. We also investi-

[3] For applications of fuzzy set to inequality measurement, see Basu (1987) and Ok (1995). Fuzzy set theory is also helpful in analyzing the valuations of functioning vectors and capability sets (see, for example, Balestrino 1994; Balestrino and Chiappero Martinetti 1994; Chiappero Martinetti 1994, 1996, 2004; Casini and Bernetti 1996; Baliamoune 2003; Alkire 2005).

gate how a variety of multidimensional poverty indices suggested recently (see, for example, Chakravarty et al. 1998; Bourguignon and Chakravarty 1999, 2003; Tsui 2002 ) can be reformulated in a fuzzy structure. These are referred to as fuzzy multidimensional poverty indices.

The Chapter is organized as follows. The next section begins by defining a fuzzy membership function that determines a person's poverty status in a dimension. A characterization of a particular membership function is also presented in this section Sect. 3.3 offers appropriate fuzzy reformulations of the axioms for a multidimensional poverty index. Sect. 3.4 shows how the conventional multidimensional poverty indices can be extended in a fuzzy framework. Finally, Sect. 3.5 concludes.

## 3.2 Fuzzy Membership Function

We begin by assuming that for a set of $n$-persons, the *ith* person possesses an $k$-vector $(x_{i1}, x_{i2}, \ldots, x_{ik}) = x_i \in R_+^k$ of attributes, where $R_+^k$ is the non-negative orthant of the $k$-dimensional Euclidean space $R^k$. The *jth* coordinate of the vector $x_i$ specifies the quantity of attribute $j$ possessed by person $i$. The vector $x_i$ is the *ith* row of an $n \times k$ matrix $X \in M^n$, where $M^n$ is the set of all $n \times k$ matrices whose entries are non-negative real numbers. The *jth* of column $x_{.j}$ of $X \in M^n$ gives the distribution of attribute $j$ ($j = 1, 2, \ldots, k$) among the $n$ persons. Let $M = \bigcup_{n \in N} M^n$, where $N$ is the set of all positive integers. For any $X \in M$, we write $n(X)$ (or $n$) for the associated population size.

In the conventional set up, the poverty status of person $i$ for attribute $j$ may be represented by a dichotomous function $\mu^*(x_{ij})$, which maps $x_{ij}$ into either zero or one, depending on whether he is non-poor or poor in the attribute, that is, whether $x_{ij} \geq z_j$ or $x_{ij} < z_j$, where $z_j > 0$ is the minimally acceptable or threshold level of attribute $j$. To allow for fuzziness in the poverty status, we consider a more general membership function $\mu_j : R_+^1 \to [0,1]$ for attribute $j$, where $\mu(x_{ij})$ indicates the degree of confidence in the statement that person $i$ with consumption level $x_{ij}$ of attribute $j$ is possibly poor with respect to the attribute. Thus, $\mu_j$ is a generalized characteristic function, that is, one which varies uniformly between zero and one, rather than assuming just two values of zero and one (Zadeh

1965; Chakravarty and Roy 1985). We assume here that $\mu_j$ depends on $x_{ij}$ only. One can also consider a more general formulation where $\mu_j$ depends on the entire distribution (Cheli and Lemmi 1995). Since $\mu_j^*$ declares the poverty status of a person in dimension *j* unambiguously, we refer to it as a crisp membership function.

Now, let $m_j > 0$ be the quantity of attribute *j* at or above which a person is regarded as non-poor with certainty with respect to the attribute, that is, if $x_{ij} \geq m_j$, then person *i* is certainly non-poor in dimension *j*. (See Cerioli and Zani 1990 and Shorrocks and Subramanian 1994 for a similar assumption in the context of income based fuzzy poverty measurement). For instance, for life expectancy $m_j$ can be taken as the age level 60. Likewise, for the income dimension it can be the level of mean per capita income. We assume here that $m_j$ coincides with one of the $x_{ij}$ values. For example, if a person with the mean level of attribute *j*, $\eta_j$, is considered as certainly non-poor in the attribute, then $m_j$ can be taken as the minimum value of $x_{ij}$ which is at least as large as $\eta_j$. That is, $m_j$ =min$\{x_{ij}\}$, where $i \in \{1,2,\ldots,n\}$ and $x_{ij}$ - $\eta_j \geq 0$. Thus, we can say that the poverty extent of $x_{ij}$, as measured by $\mu_j$, is zero if $x_{ij} \geq m_j$, that is, $\mu_j(x_{ij}) = 0$ if $x_{ij} \geq m_j$. Similarly if $x_{ij} = 0$ then the poverty level associated with $x_{ij}$ is maximal, and hence $\mu_j(0) = 1$. Furthermore, a reasonable presumption is that a rise in $x_{ij}$ decreases the possibility of person *i's* being poor in attribute *j*. Hence $\mu_j$ is assumed to be decreasing over $(0, m_j)$. It is also assumed to be continuous. The above properties of $\mu_j$ can now be summarized as follows:

$$\begin{aligned} \mu_j(x_{ij}) &= 1 \quad \text{if} \quad x_{ij} = 0, \\ \mu_j(x_{ij}) &= 0 \quad \text{if} \quad x_{ij} \geq m_j. \end{aligned} \tag{3.1}$$

It is decreasing over the interval $(0, m_j)$ and continuous everywhere. We write $\mu$ for the vector $(\mu_1, \mu_2, \ldots, \mu_k)$. Let *A* be the set of vectors of membership functions of the form $\mu$.

An example of a suitable fuzzy membership function for attribute $j$ is:

$$\mu_j(x_{ij}) = \begin{cases} 1 & if \quad x_{ij} = 0 \\ \left(\dfrac{m_j - x_{ij}}{m_j}\right)^{\theta_j} & if \quad x_{ij} \in (0, m_j) \\ 0 & if \quad x_{ij} \geq m_j \end{cases} \tag{3.2}$$

where $\theta_j \geq 1$ is a parameter.

It satisfies all the conditions laid down (3.1). It is an individualistic function in the sense that it depends only on $x_{ij}$ and treats $m_j$ as a parameter.

Given $\mu_j$, let $S_{\mu_j}(X)$ (or, simply $S_{\mu_j}$) be the set of persons who are possibly poor in dimension $j$ in $X \in M^n$, where $n \in N$ is arbitrary, that is:

$$S_{\mu_j}(X) = \{i \in \{1,2,\ldots,n\} \mid \mu_j(x_{ij}) > 0\} \tag{3.3}$$

Attribute $j$ will be called possibly meager or certainly non-meager for person $i$ according as $i \in S_{\mu_j}(X)$ or $i \notin S_{\mu_j}(X)$. Person $i$ is referred to as certainly non-poor if $x_{ij} \geq m_j$ for all $j = 1,2,\ldots k$, that is, if $i \notin S_{\mu_j}(X)$ for all $j$.

It will now be worthwhile to characterize a fuzzy membership function. Such a characterization exercise will enable us to understand the membership function in a more elaborate way through the axioms used in the exercise. The following axioms are proposed for a general membership function $\mu_j : R^1_+ \to [0,1]$ for attribute $j$.

$(A1)$ Homogeneity of Degree Zero: $\mu_j$ is homogeneous of degree zero.

$(A2)$ Linear Decreasingness: For any $x_{ij} \in [0, m_j)$ and $c_{ij} \in [0, m_j - x_{ij})$, $\mu_j(x_{ij}) - \mu_j(x_{ij} + c_{ij}) = \dfrac{c_{ij}}{m_j}$.

$(A3)$ Continuity: $\mu_j$ is continuous on its domain.

$(A4)$ Maximality: $\mu_j(0) = 1$.

$(A5)$ Independence of Non-meager Attribute Quantities: For all $x_{ij} \geq m_j$, $\mu_j(x_{ij}) = k$, where $k$ is a constant.

$(A1)$ ensures that $\mu_j$ remains unaltered under equi-proportionate variations in quantities of attribute $j$. $(A2)$ makes a specific assumption about the decreasing of the membership function. It says that the extent of reduction in the membership function resulting from an increase in $x_{ij}$ by $c_{ij}$ is the fraction $c_{ij}/m_j$. It is weaker than the decreasing assumption of the membership function over $[0, m_j)$. A membership function may as well decrease non-linearly. For instance, if $\theta_j > 1$, $\mu_j$ in (3.2) decreases at an increasing rate. $(A3)$ means that $\mu_j$ should vary in a continuous manner with respect to variations in attribute quantities. $(A4)$ specifies that $\mu_j$ should achieve its maximal value 1 when the level of the attribute is zero. Finally, $(A5)$ shows insensitivity of $\mu_j$ to the attribute quantities of the persons who are certainly non-poor in the attribute through the assumption that the value of the membership function on $[m_j, \infty)$ is a constant. Thus, instead of assuming that the membership function takes on the value zero on $[m_j, \infty)$, we derive it as an implication of more primitive axioms.

**Proposition 1**: The only membership function that satisfies axioms $(A1)-(A5)$ is:

$$\mu_j(x_{ij}) = \begin{cases} 1 & if \quad x_{ij} = 0 \\ \left(\dfrac{m_j - x_{ij}}{m_j}\right) & if \quad x_{ij} \in (0, m_j) \\ 0 & if \quad x_{ij} \geq m_j \end{cases} \tag{3.4}$$

**Proof:** In view of $(A1)$, we have $\mu_j(x_{ij}) = \mu_j(\frac{x_{ij}}{m_j})$. Hence $(A2)$ becomes:

$$\mu_j(\frac{x_{ij}}{m_j}) - \mu_j(\frac{x_{ij} + c_{ij}}{m_j}) = \frac{c_{ij}}{m_j}$$

Since in the above equation, $x_{ij} \in [0, m_j)$ is arbitrary, we can interchange the roles of $x_{ij}$ and $c_{ij}$ in it and derive that:

$$\mu_j(\frac{c_{ij}}{m_j}) - \mu_j(\frac{c_{ij}+x_{ij}}{m_j}) = \frac{x_{ij}}{m_j}$$

These two equations imply that:

$$\mu_j(\frac{x_{ij}}{m_j}) - \mu_j(\frac{c_{ij}}{m_j}) = \frac{c_{ij}}{m_j} - \frac{x_{ij}}{m_j} .$$

Letting $c_{ij}$ =0 in the above expression , we get:

$$\mu_j(\frac{x_{ij}}{m_j}) = \mu_j(0) - \frac{x_{ij}}{m_j},$$

from which in view of $(A4)$ it follows that:

$$\mu_j(\frac{x_{ij}}{m_j}) = \frac{m_j - x_{ij}}{m_j}.$$

Applying $(A1)$ to the above form of $\mu_j$ and using $(A3)$, we note that $\mu_j(m_j) = 0$. This along with $(A5)$ reveals that $k = 0$. Hence $\mu_j(x_{ij}) = 0$ for all $x_{ij} \geq m_j$. This establishes the necessity part of the proposition. The sufficiency is easy to check. Δ

Proposition 1 thus characterizes axiomatically the linear sub-case of the membership function in (3.2).

## 3.3 Properties for a Fuzzy Multidimensional Poverty Index

In this section we lay down the postulates for a fuzzy multidimensional poverty index $P: M \times A \to R^1$. For all $n \in N$, the restriction of $P$ on $M^n \times A$ is denoted by $P^n$. For any $X \in M^n, \mu \in A, P^n(X;\mu)$ gives the extent of possible poverty (poverty, for short) level associated with $X$.

Sen (1976) suggested two basic postulates for an income poverty index. These are: (i) the monotonicity axiom, which requires poverty to increase under a reduction in the income of a poor person, and (ii) the transfer axiom, which demands that poverty should increase if there is a transfer of income from a poor person to anyone who is richer. Following Sen (1976) several other axioms have been suggested in the literature. (See, for example, Sen 1979; Foster 1984; Foster et al. 1984; Donaldson and Weymark 1986; Seidl 1988; Chakravarty 1990; Foster and Shorrocks 1991; Zheng 1997). Multidimensional generalizations of different postulates proposed for an income poverty index have been introduced, among others, by

Chakravarty et al. (1998), Bourguignon and Chakravarty (1999, 2003) and Tsui (2002). The axioms we suggest below for an arbitrary $P$ are fuzzy variants of the axioms presented in Chakravarty et al. (1998), Bourguignon and Chakravarty (1999, 2003) and Tsui (2002).

**Focus (FOC):** For all $n \in N; X, \hat{X} \in M^n; \mu \in A;$ if $S_{\mu_j}(X) = S_{\mu_j}(\hat{X})$, $1 \le j \le k$ and $x_{ij} = \hat{x}_{ij}$ for all $i \in S_{\mu_j}(X), 1 \le j \le k$, then:

$P^n(X;\mu) = P^n(\hat{X};\mu)$.

**Normalization (NOM):** For all $n \in N; X \in M^n; \mu \in A; j \in \{1,2,\ldots,k\}$, if $S_{\mu_j}(X) = \phi$, the empty set, then $P^n(X;\mu) = 0$.

**Monotonicity (MON):** For all $n \in N; X, \hat{X} \in M^n; \mu \in A$; if $x_{rl} = \hat{x}_{rl}$ for all $r \in \{1,2,\ldots,n\} - \{i\}, \ell \in \{1,\ldots,k\}$; $x_{il} = \hat{x}_{il}$ for all $\ell \in \{1,\ldots,k\} - \{j\}$ and $x_{ij} > \hat{x}_{ij}$ where $i \in S_{\mu_j}(\hat{X})$, then $P^n(X;\mu) < P^n(\hat{X};\mu)$.

**Transfers Principle (TRP):** For all $n \in N; X, \hat{X} \in M^n; \mu \in A$, if X is obtained from $\hat{X}$ by multiplying $\hat{X}_p$ by a bistochastic matrix $B$ and $B\hat{X}_p$ is not a permutation of the rows of $\hat{X}_p$, then $P^n(X;\mu) < P^n(\hat{X};\mu)$, where $\hat{X}_p$ is the matrix of attribute quantities of possibly the poor in $\hat{X}$, given that the bundles of attributes of the rich remain unaffected[4].

**Principle of Population (POP):** For all $n \in N; X \in M^n; \mu \in A$, $P^n(X;\mu) = P^{hn}(\hat{X};\mu)$ where $\hat{X}$ is the $h$ – fold replication of $X$.

**Symmetry (SYM):** For all $n \in N; X \in M^n; \mu \in A$:

$P^n(X;\mu) = P^n(\Pi X;\mu)$, where $\Pi$ is an $n \times n$ permutation matrix.

**Subgroup Decomposability (SUD):** For $X^1, X^2, \ldots, X^h \in M$ and $\mu \in A$:

[4] An $n \times n$ matrix is called a bistochastic matrix if its entries are non-negative and each of its rows and columns sums to one. A bistochastic matrix is called a permutation matrix if there is exactly one positive entry in each row and column.

$P^n(X^1, X^2, \ldots, X^h; \mu) = \sum_{i=1}^{h} \frac{n_i}{n} P^{n_i}(X^i; \mu)$, where $n_i$ is the population size corresponding $X^i$ and $n = \sum_{i=1}^{h} n_i$.

**Continuity (CON):** For all $n \in N$; $\mu \in A$; $P^n(X; \mu)$ is continuous on $M^n$.

**Increasingness in Membership Functions (IMF):** For all $n \in N; X \in M^n$, $\mu, \mu' \in A$ if $\mu_h = \mu'_h$ for all $h \in \{1, \ldots, k\} - \{j\}, S_{\mu_j}(X) = S_{\mu'_j}(X)$ and $\mu_j(x_{ij}) > \mu'_j(x_{ij})$ for all $i \in S_{\mu_j}$, then $P^n(X; \mu') < P^n(X; \mu)$.

**Non-poverty Growth (NPG):** For all $n \in N; X \in M^n; \mu \in A$ if $\hat{X}$ is obtained from $X$ by adding a certainly non-poor person to the society, then $P^{n+1}(\hat{X}; \mu) < P^n(X; \mu)$.

**Scale Invariance (SCI)**: For all $n \in N$; $X \in M^n$; $\mu \in A$:

$P^n(X\Omega; \mu) = P^n(X; \mu)$, where $\Omega$ is the diagonal matrix:

diag $(\omega_1, \omega_2, \ldots, \omega_k)$, $\omega_j > 0$ for all $j = 1, \ldots, k$.

**FOC,** which has a similar spirit to $(A5)$, states that, given the population size, the poverty index depends only on the attribute quantities of the persons who are possibly poor in different dimensions. Thus, if a person is certainly non-poor with respect to an attribute, then giving him more of this attribute does not change the intensity of poverty, even if he is possibly poor in the other attributes. Clearly, **FOC** rules out trade off between the two attributes of a person who is possibly poor with respect to one but certainly non-poor with respect to the other. Thus, if life expectancy and composite good are the two attributes, more life expectancy in the domain in which it is certainly non-meager is of no use if the composite good is possibly meager. This, however, does not exclude the possibility of a trade off if both the attributes are possibly meager for a person. **NOM** is a cardinality property of the poverty index. It says that if all persons in a society are certainly non-poor, then the index value is zero. According to **MON**, poverty decreases if the condition of a poor improves. **MON** includes the possibility that the beneficiary may become certainly non-poor in the dimension concerned.

To understand **TRP**, let us recall a result from the literature on inequality measurement. Of two income distributions $u$ and $v$ of a given total over a given population size $n$, where $u$ is not a permutation of $v$ ,the former can be obtained from the latter through a sequence of rank preserving progressive transfers transferring incomes from the better off persons to those who are worse off if and only if $u = vB$ for some bistochastic matrix $B$ of order $n$ (Kolm 1969; Dasgupta et al. 1973). In the multidimensional context, Kolm (1977) showed that the distribution of a set of attributes summarized by some matrix $X$ is more equal than another matrix $\hat{X}$ (whose rows are not identical) if and only if $X = E\hat{X}$ , where $E$ is some bistochastic matrix and $X$ cannot be derived from $\hat{X}$ by permutation of the rows of $\hat{X}$ . Intuitively, multiplication of $\hat{X}$ by a bistochastic matrix makes the resulting distribution less concentrated. Following Kolm (1977), the analogous property applied to the set of possibly poor persons is **TRP**. It simply says that there is less possible poverty under $X$ than under $\hat{X}$ if the former is obtained from the latter by redistributing the attributes of the possibly poor using some bistochastic transformation.

Under **POP**, if an attribute matrix is replicated several times, then poverty remains unchanged. Since by replication we can transform two different sized matrices into the same size, **POP** is helpful for inter-temporal and interregional poverty comparisons. **SYM** demands anonymity. Any characteristic other than the quantities in different dimensions under consideration, for instance, the names of the individuals, is immaterial to the measurement of poverty. **CON**, which is similar to $(A3)$, ensures that minor changes in attribute quantities will not give rise to an abrupt jump in the value of the poverty index. Therefore, a continuous poverty index will not be oversensitive to minor observational errors on basic need quantities.

**SUD** says that if a population is divided into several subgroups, say $h$, defined along ethnic, geographical or other lines, then the overall poverty is the population share weighted average of subgroup poverty levels. The contribution of subgroup $i$ to overall poverty is $n_i P^{n_i}(X^i;\mu)/n$ and overall poverty will precisely fall by this amount if poverty in subgroup $i$ is eliminated.

$(n_i P^{n_i}(X^i;\mu)/\ nP^n(X;\mu))100$ is the percentage contribution of subgroup $i$ to total poverty. Each of these statistics is useful to policy-makers because they become helpful for isolating subgroups of the population that are more susceptible to poverty (see Anand 1997; Chakravarty 1983a; Foster et al. 1984; Foster and Shorrocks 1991).

Between two identical communities, the one with higher membership function of an attribute should have a higher poverty because of higher possibility of individuals' being poor in that dimension. This is what **IMF** demands. A poverty index will be called $\mu$ **-monotonic** if it satisfies **IMF**. According to **NPG** poverty should decrease if a person who is certainly rich joins the society. Thus, under **FOC**, **NPG** says that the poverty index is a decreasing function of the population size (see Kundu and Smith 1983; Subramanian 2002; Chakravarty et al. 2005). Finally, **SCI,** which parallels $A1$, means that the poverty index is invariant under scale transformations of attribute quantities, that is, it is homogeneous of degree zero. Hence it should be independent of the units of measurement of attributes. Thus, if life expectancy is measured in months instead of in years, the level of poverty remains unchanged.

We will now consider a property which takes care of the essence of multidimensional measurement through correlation between attributes. By taking into account the association of attributes, as captured by the degree of correlation between them, this property also underlines the difference between single and multidimensional poverty measurements. To illustrate the property, consider the two-person two-attribute case, where both the attributes are possibly meager for these persons. Suppose that $x_{11} > x_{21}$ and $x_{12} < x_{22}$. Now, consider a switch of attribute 2 between the two persons. This switch increases the correlation between the attributes because person 1 who had more of attribute 1 has now more of attribute 2 too and that is why we refer to it as a correlation increasing switch between two possibly poor persons. Formally, we have:

**Definition 1**: For any $n \geq 2; X \in M^n; \mu \in A; j,h \in \{1,2,\ldots..,k\}$, suppose that for some $i,t \in S_{\mu_j}(X) \bigcap S_{\mu_h}(X)$, $x_{ij} < x_{tj}$ and $x_{th} < x_{ih}$. $\hat{X}$ is then said to be obtained from *X* by a correlation increasing switch between two possibly poor persons if $(i)\hat{x}_{ij} = x_{tj}, (ii)\hat{x}_{tj} = x_{ij}, (iii)\hat{x}_{rj} = x_{rj}$ for all $r \neq i,t$ and $(iv)\hat{x}_{rs} = x_{rs}$ for all $s \neq j$ and for all *r*.

If the two attributes are substitutes, that is, if one attribute compensates for the lack of another for a person who is possibly poor in both dimensions, then the switch should increase poverty. This is because the richer of the possibly poor is getting even better in the attributes which correspond to the similar aspect of poverty after the rearrangement. After the switch the poorer person is less able to compensate the lower quantity of one attribute by the quantity of the other. Indeed, the switch just defined does not modify the marginal distribution of each attribute but it reduces the extent to which the lack of one attribute may be compensated by the availability of the other. An analogous argument will establish that poverty should decrease under a correlation increasing switch if the two attributes are complements. (For more detailed arguments along this line, see Atkinson and

Bourguignon 1980; Bourguignon and Chakravarty 2003). We state this principle formally for substitutes as:

**Increasing Poverty Under Correlation Increasing Switch (IPC):** For all $n \in N; \mu \in A; X \in M^n$, if $\hat{X}$ is obtained from $X$ by a correlation increasing switch between two possibly poor persons, then $P^n(X;\mu) < P^n(\hat{X};\mu)$ if the two attributes are substitutes.

The corresponding property which demands poverty to decrease under such a switch when the attributes are complements is denoted by **DPC**. If a poverty index does not change under a correlation increasing switch, then it treats the attributes as "independents".

## 3.4 The Subgroup Decomposable Fuzzy Multidimensional Poverty

### 3.4.1 Poverty Indices

The objective of this section is to discuss the subgroup decomposable family of fuzzy multidimensional poverty indices. The necessity for a subgroup decomposable index arose from practical considerations. The use of such an index allows policy-makers to design effective, consistent national and regional anti-poverty policies.

Repeated application of **SUD** shows that we can write a subgroup decomposable index as:

$$P^n(X;\mu) = \frac{1}{n}\sum_{i=1}^{n} p(x_i;\mu) \tag{3.4}$$

where $n \in N; X \in M^n$ and $\mu \in A$ are arbitrary. Since $p(x_i;\mu)$ depends only on person $i$'s consumption of the attributes, we call it "individual poverty function". If we define $p(x_i;\mu)$ as the weighted average of grades of membership of individual $i$ across dimensions, that is, if $p(x_i;\mu) = \sum_{j=1}^{k} \delta_j \mu_j(x_{ij})$, where $0 < \delta_j < 1$ and $\sum_{j=1}^{k} \delta_j = 1$, then $P^n$ in (3.4) becomes:

$$P^n(X;\mu) = \frac{1}{n}\sum_{j=1}^{k} \delta_j \sum_{i \in S_{\mu j}} \mu_j(x_{ij}) \tag{3.5}$$

The weight $\delta_j$ may be assumed to reflect the importance that we attach in our aggregation to dimension $j$. It may also be assumed as reflecting the importance that the government assigns for alleviating poverty for that dimension. Since $\sum_{i \in S_{\mu j}} \mu_j(x_{ij})$ gives the cardinality of the fuzzy set of the poor in the $jth$ attribute (Dubois and Prade 1980, p. 30), $P^n$ in (3.5) is a weighted average of the proportions of possibly poor persons across dimensions. If $\mu_i$ coincides with the crisp membership function $\mu_j^*$, then the index in (3.5) becomes a weighted average of the proportions of persons who are poor in different dimensions.

Alternatively we may interpret the formula as follows. $\mu_j(x_{ij})$ can be regarded as the extent of deprivation felt by person $i$ for being included in the set of persons who are possibly poor in attribute $j$. As his quantity of consumption of the attribute increases, deprivation decreases and $\mu_j(m_j) = 0$ shows the absence of this feeling at the level $m_j$. Therefore, $P^n$ is the population average of the weighted average of dimension –wise individual deprivations.

Defining $\frac{1}{n} \sum_{i \in S_{\mu j}} \mu_j(x_{ij})$ as the possible poverty level associated with attribute $j$ and denoting it by $P^n(x._j; \mu_j)$, we can rewrite $P^n$ in (3.5) in a more compact way as:

$$P^n(X, \mu) = \sum_{j=1}^{k} \delta_j P_j^n(x._j; \mu_j) \tag{3.6}$$

This shows that $P^n(X; \mu)$ can also be viewed as a weighted average of attribute-wise (possible) poverty values. We refer to this property as "Factor Decomposability". The percentage contribution of dimension $j$ to total fuzzy poverty is $\left(\delta_j P^n(x._j; \mu_j) / P^n(X; \mu)\right)100$. The elimination of poverty for the $jth$ dimension will lower community poverty by the amount $\delta_j P^n(x._j; \mu_j)$.

We can use the two decomposability postulates to construct a two-way poverty profile and to calculate each attribute's poverty within each subgroup. This type of micro breakdown will help us to identify simultaneously the population subgroup(s) as well as attribute(s) for which poverty levels are severe and formulate appropriate antipoverty policies.

It will now be worthwhile to examine the behavior of $P^n$ given by (3.5) with respect to the axioms stated in Sect. 3.3. These axioms conveniently translate into constraints on the form of $\mu_j$. Evidently, $P^n$ in (3.5) is focused, normalized, monotonic, symmetric, population replication invariant, $\mu$-monotonic, continuous and correctly responsive to non-poverty growth. It satisfies **SCI** if and only if for each $j$, $\mu_j$ is homogeneous of degree zero, a condition fulfilled by the form given in (3.2). It is transfer preferring, that is, **TRP** holds if and only if $\mu_j$ is strictly convex over $(0, m_j)$, $1 \le j \le k$, (see Marshall and Olkin 1979, p. 433). This means that the decline in the possibility of poverty with increase in quantities of attributes is greatest at the lowest levels of the attribute. The membership function defined in (3.2) satisfies the convexity condition if $\theta_j \ge 2$. Finally, because of additivity across attributes it remains unchanged under a correlation increasing switch. We summarize these observations on the behavior of $P^n$ as follows:

**Proposition 2**: The subgroup decomposable fuzzy multidimensional poverty index given by (3.5) satisfies the **Focus**, **Normalization**, **Monotonicity**, **Principle of Population, Symmetry**, **Continuity**, **Increasingness in Membership Functions** and **Non-Poverty Growth** axioms. It fulfills the **Scale Invariance** axiom if and only if the membership functions for different attributes are homogeneous of degree zero. It meets the **Transfers Principle** axiom if and only if for each j, $\mu_j$ is strictly convex on the relevant part of the domain. Finally, it remains unchanged under a correlation increasing switch between two possibly poor persons.

To illustrate the general formula in (3.5), suppose that the membership function is of the form (3.2). In this case the index is:

$$P_\theta^n(X;\mu) = \frac{1}{n}\sum_{j=1}^{k}\delta_j \sum_{i \in S_{\mu_j}} \left(1 - \frac{x_{ij}}{m_j}\right)^{\theta_j} \tag{3.7}$$

where $\theta = (\theta_1, \theta_2, \ldots, \theta_k)$, which reflect different perceptions of poverty. This is a fuzzy counterpart to the multidimensional generalization of the Foster – Greer – Thorbecke (FGT) (1984) index considered by Chakravarty et al. (1998) and Bourguignon and Chakravarty (2003). For a given $X$, $P_\theta^n$ increases as $\theta_j$ increases, $1 \le j \le k$. For $\theta_j = 1$, for all $j$, $P_\theta^n$ becomes:

$$P_\theta^n(X;\mu) = \frac{1}{n}\sum_{j=1}^{k} \delta_j H_j I_j \tag{3.8}$$

where $I_j$ is the average of the grades of membership of the persons in $S_{\mu_j}$ when $\theta_j = 1$, that is, $I_j = \sum_{i\in S_{\mu_j}} (m_j - x_{ij})/q_j m_j$, with $q_j$ being the cardinality of $S_{\mu_j}$ and $H_j = q_j / n$ is the fuzzy head-count ratio in dimension $j$. Thus, for a given $H_j$, an increase in $I_j$, say, due to a reduction of $x_{ij}$, increases the index.

If $\theta_j = 2$ for all $j$, $P_\theta^n$ can be written as:

$$P_\theta^n(X;\mu) = \sum_{j=1}^{k} \delta_j H_j \left[I_j^2 + (1-I_j)^2 C_j^2\right] \tag{3.9}$$

where $C_j^2 = \sum_{i\in S_{\mu_j}} (x_{ij} - \rho_j)^2 / q_j \rho_j^2$ is the squared coefficient of variation of the distribution of attribute $j$ among those for whom it is possibly meager, with $\rho_j = \sum_{i\in S_{\mu_j}} x_{ij} / q_j$ being the mean of the distribution. Now, $C_j^2$ is an index of inequality of the concerned distribution. Clearly, given $I_j$ and $H_j$, $P_\theta^n$ in (3.9) reduces as $C_j$ reduces, say through a transfer from a less possibly poor to a more possibly poor. Thus, the decomposition in (3.9) shows that the poverty index is related in a positive monotonic way with the inequality levels of the possibly poor in different dimensions.

An alternative of interest arises from the following specification of the membership function:

$$\mu_j(x_{ij}) = 1 - \left(\frac{x_{ij}}{m_j}\right)^{c_j} \tag{3.10}$$

where for all j, $1 \le j \le k, c_j \in (0,1)$. It satisfies all the conditions laid down in (3.1) along with homogeneity of degree zero and strict convexity. The associated poverty index is:

$$P_c^n\ (X;\mu) = \frac{1}{n}\sum_{j=1}^{k} \delta_j \sum_{i\in S_{\mu_j}} \left[1 - \left(\frac{x_{ij}}{m_j}\right)^{c_j}\right] \tag{3.11}$$

where $c=(c_1,c_2,\ldots,c_m)$. This index is a fuzzy version of the multidimensional extension of the subgroup decomposable single dimensional Chakravarty (1983b) index suggested by Chakravarty et al. (1998). Given $X$, $P_c^n$ is increasing in $c_j$ for all $j$. For $c_j=1$, the index coincides with the particular case of $P_\theta^n$ when $\theta_j=1,\ 1\le j\le k$. On the other hand as $c_j\to 0$ for all $j$, $P_c^n\to 0$. As $c_j$ decreases over the interval (0, 1), $P_c^n$ becomes more sensitive to transfers lower down the scale of distribution along dimension $j$.

We may also consider a logarithmic formulation of the membership function that fulfils all conditions stated in (3.1):

$$\mu_j(x_{ij})=\frac{\log\left(1+e^{\lambda_j(m_j-x_{ij})/m_j}\right)-\log 2}{\log\left(1+e^{\lambda_j}\right)-\log 2} \tag{3.12}$$

where $\lambda_j>0$ is a parameter. The corresponding additive poverty index turns out to be:

$$P_\lambda^n(X;\mu)=\frac{1}{n}\sum_{j=1}^{k}\delta_j\sum_{i\in S_{\mu_j}}\frac{\log\left(1+e^{\lambda_j(m_j-x_{ij})/m_j}\right)-\log 2}{\log\left(1+e^{\lambda_j}\right)-\log 2} \tag{3.13}$$

where $\lambda$ is the parameter vector $(\lambda_1,\lambda_2,\ldots,\lambda_k)$. $P_\lambda^n$ can be regarded as a fuzzy sister of the multidimensional generalization of the Watts (1967) poverty index characterized by Tsui (2002). The parameter $\lambda_j$ determines the curvature of the poverty contour. An increase in $\lambda_j$ for any $j$ makes the fuzzy poverty contour more convex to the origin. If $\lambda_j\to 0$ for all $j$, then $P_\lambda^n\to 0$. In the particular case when $\theta_j=\lambda_j=1$ for all $j$, the ranking of two attribute matrices $X,\hat{X}\in M^n$ by $P_\theta^n$ will be the same as that generated by $P_\lambda^n$. Since $P_\lambda^n$ is transfer preferring for all $\lambda_j>0$, it satisfies TRP even in this case. But $P_\theta^n$ does not fulfill TRP here.

There can be simple non-additive formulations of fuzzy multidimensional extensions of single dimensional subgroup decomposable indices.

They satisfy **SUD** but not factor decomposability. Assuming that $\theta_j$ in (3.2) is constant across attributes, say equal to $\beta$, one such index can be:

$$P_{\alpha\beta}^{n}(X;\mu)=\frac{1}{n}\left[\sum_{j=1}^{k}\sum_{i=1}^{n} a_j\mu_j(x_{ij})\right]^{\frac{\alpha}{\beta}}=\frac{1}{n}\left[\sum_{j=1}^{k}\sum_{i\in S_{\mu_j}} a_j\left(\frac{m_j-x_{ij}}{m_j}\right)^{\beta}\right]^{\frac{\alpha}{\beta}} \tag{3.14}$$

where $a_j > 0$ for all $j$ and $\alpha$ is a positive parameter. $P^{n}_{\alpha,\beta}$ is the fuzzy counterpart to the multidimensional version of the FGT index suggested by Bourguignon and Chakravarty (2003). The interpretation of this index is quite straightforward. The membership functions in various dimensions are first aggregated into a composite membership using a particular value of $\beta$ and the coefficients $a_j$. Multidimensional fuzzy poverty is then defined as the average of that composite membership value, raised to the power $\alpha$, over the whole population. $P^{n}_{\alpha,\beta}$ satisfies **IPC** or **DPC** depending on whether $\alpha$ is greater or less than $\beta$. For $\alpha = 1$, it becomes the weighted sum of order $\beta$ of the membership grades and for a given $X \in M^n$, it is increasing in $\beta$.

We may suggest an alternative to (3.14) using the membership function in (3.11). This form is defined by:

$$T_c^{n}(X;\mu)=\frac{1}{n}\sum_{i=1}^{n}\left(1-\prod_{j=1}^{k}(1-\mu_j)\right)=\frac{1}{n}\sum_{i=1}^{n}\left(1-\prod_{j=1}^{k}\left(\frac{\hat{x}_{ij}}{m_j}\right)^{c_j}\right) \tag{3.15}$$

where $\hat{x}_{ij}=\min(x_{ij},m_j)$. This is a fuzzy translation of the multidimensional generalization of the Chakravarty (1983b) index developed by Tsui (2002). In (3.15) for each person complements from unity of the grades of membership along various dimensions are subjected to a product transformation which is then averaged over persons after subtracting from its maximum value, that is, 1. Since $T_c^n$ is unambiguously decreasing under a correlation increasing switch between two possibly poor persons, it treats the concerned attributes unambiguously as complements, that is, it satisfies **DPC**.

Given a membership fraction $\mu_j$, there will be a corresponding multidimensional fuzzy poverty index that meets all the postulates considered in Sect. 3.2. These indices will differ only in the manner in which we use $\mu_j$ to aggregate membership grades of different persons along different dimensions in an overall indicator.

## 3.5 Conclusions

This Chapter has explored the problem of replacing the traditional crisp view of poverty status with a fuzzy structure which allows membership of poverty set or the possibility of poverty in different dimensions of life to take any value in the interval [ 0, 1]. An attempt has been made to establish how standard multidimensional poverty indices might be translated into the fuzzy framework. Suggestions have been made for suitable fuzzy analogues of axioms for a multidimensional poverty index, such as, **Focus**, **Monotonicity**, **Transfers Principle** and **Continuity**. We have also added a condition which requires poverty to increase if the possibility of poverty shifts upward along any dimension.

We will now make a comparison of our index with some existing indices. Assuming that the individual well-being depends only on income, Cerioli and Zani (1990) suggested the use of the arithmetic average of grades of membership of different individuals as a fuzzy poverty index. It "represents the proportions of individuals "belonging" in a fuzzy sense to the poor subset" (Cerioli and Zani 1990, p. 282). Clearly, this index is similar in nature to $P^n$ given by (3.5).

In a multidimensional framework, Cerioli and Zani (1990) introduced a transition zone $x_j^L < x_{ij} \leq x_{ij}^H$ for attribute j over which the membership function declines from 1 to 0 linearly:

$$\mu_j\left(x_{ij}\right)=\begin{cases} 1 & \text{if} \quad x_{ij} \leq x_j^{(L)} \\ \dfrac{x_j^{(H)}-x_{ij}}{x_j^{(H)}-x_j^{(L)}} & \text{if} \quad x_{ij} \in \left(x_j^{(L)}, x_j^{(H)}\right] \\ 0 & \text{if} \quad x_{ij} > x_j^{(H)} \end{cases} \tag{3.16}$$

They then defined the membership function for person $i$ as $\dfrac{\sum_{j=1}^{k} \mu_j(x_{ij}) w_j}{\sum_{i=1}^{k} w_i}$, where $w_1, w_2, \ldots, w_k$ represent a system of weights.

In what has been called the "Totally Fuzzy and Relative" approach, Cheli and Lemmi (1995) defined the membership function for attribute $j$ as the distribution function $F(x_{ij})$, normalized (linearly transformed) so as to equal 1 for the poorest and 0 for the richest person in the population:

$$\mu_j(x_{ij}) = \begin{cases} 1 & if \quad x_{ij} = x_j^{(s)} \\ \mu_j\left(x_j^{(l-1)}\right) + \dfrac{F(x_j^{(l)}) - F(x_j^{(l-1)})}{1 - F(x_j^{(l)})} & if \quad x_{ij} = x_j^{(l)} \\ 0 & if \quad x_{ij} = x_j^{(1)} \end{cases} \tag{3.17}$$

where $x_j^{(1)}, x_j^{(2)}, \ldots, x_j^{(s)}$ are modalities of dimension $j$ in increasing order with respect to the risk of poverty connected to them.

An alternative specification of the membership function for person $i$ arises if we replace $\mu_j$ in (16) by $\mu_j$ in (3.17). In either case, as Cerioli and Zani (1990) and Cheli and Lemmi (1995) suggested, under appropriate specification of weights, we can take:

$$C^n = \frac{\sum_{i=1}^{n} \sum_{j=1}^{k} \mu_j(x_{ij}) w_j}{n \sum_{i=1}^{k} w_i} \tag{3.18}$$

as an indicator of poverty. Cerioli and Zani (1990) chose $w_j = \log(1/p_j)$, where $p_j$ is the proportion of persons with *jth* poverty symptoms, and Cheli and Lemmi (1995) preferred to use $w_j = \log\left(n / \sum_{i=1}^{n} \mu_j(x_{ij})\right)$. $C^n$ indicates the cardinality of the fuzzy subset of the poor as a proportion of the population size.

An important difference between $P^n$ in (3.5) and $C^n$ is that while $P^n$ is subgroup decomposable, $C^n$ is not. This is because $C^n$ depends on different kinds of rank orders. Precisely, because of this a poverty index based on a Gini type inequality index or welfare function is not subgroup decomposable. Examples are the Sen (1976), Kakwani (1980b) and Thon (1983) indices.

A rank preserving transfer of some quantity of an attribute from a possibly poor to a worse off person will not change the rank orders of the modalities in the concerned dimension. Therefore, satisfaction of the **Transfers Principle** by the general index $C^n$ will depend on the assumption about the weight system. Likewise, a rank preserving reduction in the quantity of an attribute will not change the rank orders of the modalities. Hence a similar argument holds concerning fulfillment of **Monotonicity.** However, $C^n$ is normalized, symmetric, scale invariant (under appropriate choices of modalities) and responds correctly to non-poverty growth. It is continuous for the membership function in (3.16). Continuity for the membership function in (3.17) will hold if $F$ is continuous. To check whether it is population replication invariant, concrete specification of the weight sequence is necessary.

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