Chapter 8

Fixed Points

8.1 The Brouwer Fixed Point Theorem

Many questions in optimization and analysis reduce to solving a nonlinear equation $h(x) = 0$, for some function $h : \mathbf{E} \to \mathbf{E}$. Equivalently, if we define another map $f = I - h$ (where I is the identity map), we seek a point x in $\mathbf{E}$ satisfying $f(x) = x$; we call x a *fixed point* of f.

The most potent fixed point existence theorems fall into three categories: "geometric" results, devolving from the Banach contraction principle (which we state below), "order-theoretic" results (to which we briefly return in Section 8.3), and "topological" results, for which the prototype is the theorem of Brouwer forming the main body of this section. We begin with Banach's result.

Given a set $C \subset \mathbf{E}$ and a continuous *self map* $f : C \to C$, we ask whether f has a fixed point. We call f a *contraction* if there is a real constant $\gamma_f < 1$ such that

$$\|f(x) - f(y)\| \leq \gamma_f \|x - y\| \text{ for all } x, y \in C. \tag{8.1.1}$$

Theorem 8.1.2 (Banach contraction) *Any contraction on a closed subset of* $\mathbf{E}$ *has a unique fixed point.*

Proof. Suppose the set $C \subset \mathbf{E}$ is closed and the function $f : C \to C$ satisfies the contraction condition (8.1.1). We apply the Ekeland variational principle (7.1.2) to the function

$$z \in \mathbf{E} \mapsto \begin{cases} \|z - f(z)\| & \text{if } z \in C \\ +\infty & \text{otherwise} \end{cases}$$

at an arbitrary point x in C, with the choice of constants

$$\epsilon = \|x - f(x)\| \text{ and } \lambda = \frac{\epsilon}{1 - \gamma_f}.$$

This shows there is a point v in C satisfying

$$\|v - f(v)\| < \|z - f(z)\| + (1 - \gamma_f)\|z - v\|$$

for all points $z \neq v$ in C. Hence v is a fixed point, since otherwise choosing $z = f(v)$ gives a contradiction. The uniqueness is easy. □

What if the map f is not a contraction? A very useful weakening of the notion is the idea of a *nonexpansive* map, which is to say a self map f satisfying

$$\|f(x) - f(y)\| \leq \|x - y\| \text{ for all } x, y$$

(see Exercise 2). A nonexpansive map on a nonempty compact set or a nonempty closed convex set may not have a fixed point, as simple examples like translations on $\mathbf{R}$ or rotations of the unit circle show. On the other hand, a straightforward argument using the Banach contraction theorem shows this cannot happen if the set is nonempty, compact, *and* convex. However, in this case we have the following more fundamental result.

Theorem 8.1.3 (Brouwer) *Any continuous self map of a nonempty compact convex subset of* $\mathbf{E}$ *has a fixed point.*

In this section we present an "analyst's approach" to Brouwer's theorem. We use the two following important analytic tools concerning $C^{(1)}$ (continuously differentiable) functions on the closed unit ball $B \subset \mathbf{R}^n$.

Theorem 8.1.4 (Stone–Weierstrass) *For any continuous map* $f : B \to \mathbf{R}^n$*, there is a sequence of* $C^{(1)}$ *maps* $f_r : B \to \mathbf{R}^n$ *converging uniformly to* f.

An easy exercise shows that, in this result, if f is a self map then we can assume each f_r is also a self map.

Theorem 8.1.5 (Change of variable) *Suppose that the set* $W \subset \mathbf{R}^n$ *is open and that the* $C^{(1)}$ *map* $g : W \to \mathbf{R}^n$ *is one-to-one with* ∇g *invertible throughout* W*. Then the set* $g(W)$ *is open with measure*

$$\int_W |\det \nabla g|.$$

We also use the elementary topological fact that the open unit ball $\operatorname{int} B$ is *connected*; that is, it cannot be written as the disjoint union of two nonempty open sets.

The key step in our argument is the following topological result.

Theorem 8.1.6 (Retraction) *The unit sphere* S *is not a* $C^{(1)}$ **retract** *of the unit ball* B*; that is, there is no* $C^{(1)}$ *map from* B *to* S *whose restriction to* S *is the identity.*

Proof. Suppose there is such a retraction map $p : B \to S$. For real t in $[0,1]$, define a self map of B by $p_t = tp + (1-t)I$. As a function of the variables $x \in B$ and t, the function $\det \nabla p_t(x)$ is continuous and hence strictly positive for small t. Furthermore, p_t is one-to-one for small t (Exercise 7).

If we denote the open unit ball $B \setminus S$ by U, then the change of variables theorem above shows for small t that $p_t(U)$ is open with measure

$$\nu(t) = \int_U \det \nabla p_t. \tag{8.1.7}$$

On the other hand, by compactness, $p_t(B)$ is a closed subset of B, and we also know $p_t(S) = S$. A little manipulation now shows we can write U as a disjoint union of two open sets:

$$U = (p_t(U) \cap U) \cup (p_t(B)^c \cap U). \tag{8.1.8}$$

The first set is nonempty, since $p_t(0) = tp(0) \in U$. But as we observed, U is connected, so the second set must be empty, which shows $p_t(B) = B$. Thus the function $\nu(t)$ defined by equation (8.1.7) equals the volume of the unit ball B for all small t.

However, as a function of $t \in [0,1]$, $\nu(t)$ is a polynomial, so it must be constant. Since p is a retraction we know that all points x in U satisfy $\|p(x)\|^2 = 1$. Differentiating implies $(\nabla p(x))p(x) = 0$, from which we deduce $\det \nabla p(x) = 0$, since $p(x)$ is nonzero. Thus $\nu(1)$ is zero, which is a contradiction. □

Proof of Brouwer's theorem. Consider first a $C^{(1)}$ self map f on the unit ball B. Suppose f has no fixed point. A straightforward exercise shows there are unique functions $\alpha : B \to \mathbf{R}_+$ and $p : B \to S$ satisfying the relationship

$$p(x) = x + \alpha(x)(x - f(x)) \quad \text{for all } x \text{ in } B. \tag{8.1.9}$$

Geometrically, $p(x)$ is the point where the line extending from the point $f(x)$ through the point x meets the unit sphere S. In fact p must then be a $C^{(1)}$ retraction, contradicting the retraction theorem above. Thus we have proved that any $C^{(1)}$ self map of B has a fixed point.

Now suppose the function f is just continuous. The Stone–Weierstrass theorem (8.1.4) implies there is a sequence of $C^{(1)}$ maps $f_r : B \to \mathbf{R}^n$ converging uniformly to f, and by Exercise 4 we can assume each f_r is a self map. Our argument above shows each f_r has a fixed point x^r. Since B is compact, the sequence (x^r) has a subsequence converging to some point x in B, which it is easy to see must be a fixed point of f. So any continuous self map of B has a fixed point.

Finally, consider a nonempty compact convex set $C \subset \mathbf{E}$ and a continuous self map g on C. Just as in our proof of Minkowski's theorem (4.1.8), we may as well assume C has nonempty interior. Thus there is a *homeomorphism* (a continuous onto map with continuous inverse) $h : C \to B$ (see Exercise 11). Since the function $h \circ g \circ h^{-1}$ is a continuous self map of B, our argument above shows this function has a fixed point x in B, and therefore $h^{-1}(x)$ is a fixed point of g. □

Exercises and Commentary

Good general references on fixed point theory are [68, 174, 83]. The Banach contraction principle appeared in [7]. Brouwer proved the three-dimensional case of his theorem in 1909 [49] and the general case in 1912 [50], with another proof by Hadamard in 1910 [89]. A nice exposition of the Stone–Weierstrass theorem may be found in [16], for example. The Change of variable theorem (8.1.5) we use can be found in [177]; a beautiful proof of a simplified version, also sufficient to prove Brouwer's theorem, appeared in [118]. Ulam conjectured and Borsuk proved their result in 1933 [17].

1. **(Banach iterates)** Consider a closed subset $C \subset \mathbf{E}$ and a contraction $f : C \to C$ with fixed point x^f. Given any point x_0 in C, define a sequence of points inductively by

$$x_{r+1} = f(x_r) \quad \text{for } r = 0, 1, \ldots.$$

   (a) Prove $\lim_{r,s\to\infty} \|x_r - x_s\| = 0$. Since $\mathbf{E}$ is *complete*, the sequence (x_r) converges. (Another approach first shows (x_r) is bounded.) Hence prove in fact x_r approaches x^f. Deduce the Banach contraction theorem.

   (b) Consider another contraction $g : C \to C$ with fixed point x^g. Use part (a) to prove the inequality

$$\|x^f - x^g\| \le \frac{\sup_{z\in C} \|f(z) - g(z)\|}{1 - \gamma_f}.$$

2. **(Nonexpansive maps)**

   (a) If the $n \times n$ matrix U is orthogonal, prove the map $x \in \mathbf{R}^n \to Ux$ is nonexpansive.

   (b) If the set $S \subset \mathbf{E}$ is closed and convex then for any real λ in the interval $[0, 2]$ prove the *relaxed projection*

$$x \in \mathbf{E} \mapsto (1 - \lambda)x + \lambda P_S(x)$$

   is nonexpansive. (Hint: Use the nearest point characterization in Section 2.1, Exercise 8(c).)

(c) **(Browder–Kirk [51, 112])** Suppose the set $C \subset \mathbf{E}$ is compact and convex and the map $f : C \to C$ is nonexpansive. Prove f has a fixed point. (Hint: Choose an arbitrary point x in C and consider the contractions

$$z \in C \mapsto (1-\epsilon)f(z) + \epsilon x$$

for small real $\epsilon > 0$.)

(d)* In part (c), prove the fixed points form a nonempty compact convex set.

3. **(Non-uniform contractions)**

(a) Consider a nonempty compact set $C \subset \mathbf{E}$ and a self map f on C satisfying the condition

$$\|f(x) - f(y)\| < \|x - y\| \text{ for all distinct } x, y \in C.$$

By considering $\inf \|x - f(x)\|$, prove f has a unique fixed point.

(b) Show the result in part (a) can fail if C is unbounded.

(c) Prove the map $x \in [0,1] \mapsto xe^{-x}$ satisfies the condition in part (a).

4. In the Stone–Weierstrass theorem, prove that if f is a self map then we can assume each f_r is also a self map.

5. Prove the interval $(-1,1)$ is connected. Deduce the open unit ball in $\mathbf{R}^n$ is connected.

6. In the Change of variable theorem (8.1.5), use metric regularity to prove the set $g(W)$ is open.

7. In the proof of the Retraction theorem (8.1.6), prove the map p is Lipschitz, and deduce that the map p_t is one-to-one for small t. Also prove that if t is small then $\det \nabla p_t$ is strictly positive throughout B.

8. In the proof of the Retraction theorem (8.1.6), prove the partition (8.1.8), and deduce $p_t(B) = B$.

9. In the proof of the Retraction theorem (8.1.6), prove $\nu(t)$ is a polynomial in t.

10. In the proof of Brouwer's theorem, prove the relationship (8.1.9) defines a $C^{(1)}$ retraction $p : B \to S$.

11. **(Convex sets homeomorphic to the ball)** Suppose the compact convex set $C \subset \mathbf{E}$ satisfies $0 \in \operatorname{int} C$. Prove that the map $h : C \to B$ defined by

$$h(x) = \begin{cases} \gamma_C(x)\|x\|^{-1}x & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

(where γ_C is the gauge function we defined in Section 4.1) is a homeomorphism.

12. * **(A nonclosed nonconvex set with the fixed point property)** Let Z be the subset of the unit disk in $\mathbf{R}^2$ consisting of all lines through the origin with rational slope. Prove every continuous self map of Z has a fixed point.

13. * **(Change of variable and Brouwer)** A very simple proof may be found in [118] of the formula

$$\int (f \circ g)|\nabla g| = \int f$$

when the function f is continuous with bounded support and the function g is differentiable, equaling the identity outside a large ball. Prove any such g is surjective by considering an f supported outside the range of g (which is closed). Deduce Brouwer's theorem.

14. ** **(Brouwer and inversion)** The central tool of the last chapter, the Surjectivity and metric regularity theorem (7.1.5), considers a function h whose *strict* derivative at a point satisfies a certain surjectivity condition. In this exercise, which comes out of a long tradition, we use Brouwer's theorem to consider functions h which are merely *Fréchet* differentiable. This exercise proves the following result.

Theorem 8.1.10 *Consider an open set $U \subset \mathbf{E}$, a closed convex set $S \subset U$, and a Euclidean space $\mathbf{Y}$, and suppose the continuous function $h : U \to \mathbf{Y}$ has Fréchet derivative at the point $x \in S$ satisfying the surjectivity condition*

$$\nabla h(x) T_S(x) = \mathbf{Y}.$$

Then there is a neighbourhood V of $h(x)$, a continuous, piecewise linear function $F : \mathbf{Y} \to \mathbf{E}$, and a function $g : V \to \mathbf{Y}$ that is Fréchet differentiable at $h(x)$ and satisfies $(F \circ g)(V) \subset S$ and

$$h((F \circ g)(y)) = y \quad \textit{for all } y \in V.$$

Proof. We can assume $x = 0$ and $h(0) = 0$.

(a) Use Section 4.1, Exercise 20 (Properties of the relative interior) to prove $\nabla h(0)(\mathbf{R}_+ S) = \mathbf{Y}$.

(b) Deduce that there exists a basis $y_1, y_2, \ldots, y_n$ of $\mathbf{Y}$ and points $u_1, u_2, \ldots, u_n$ and $w_1, w_2, \ldots, w_n$ in S satisfying

$$\nabla h(0) u_i = y_i = -\nabla h(0) w_i \quad \text{for } i = 1, 2, \ldots, n.$$

(c) Prove the set

$$B_1 = \Big\{ \sum_1^n t_i y_i \;\Big|\; t \in \mathbf{R}^n,\ \sum_1^n |t_i| \le 1 \Big\}$$

and the function F defined by

$$F\Big(\sum_1^n t_i y_i\Big) = \sum_1^n \big(t_i^+ u_i + (-t_i)^+ w_i\big)$$

satisfy $F(B_1) \subset S$ and $\nabla(h \circ F)(0) = I$.

(d) Deduce there exists a real $\epsilon > 0$ such that $\epsilon B_{\mathbf{Y}} \subset B_1$ and

$$\|h(F(y)) - y\| \le \frac{\|y\|}{2} \quad \text{whenever } \|y\| \le 2\epsilon.$$

(e) For any point v in the neighbourhood $V = (\epsilon/2) B_{\mathbf{Y}}$, prove the map

$$y \in V \mapsto v + y - h(F(y))$$

is a continuous self map of V.

(f) Apply Brouwer's theorem to deduce the existence of a fixed point $g(v)$ for the map in part (e). Prove $\nabla g(0) = I$, and hence complete the proof of the result.

(g) If x lies in the interior of S, prove F can be assumed linear.

(Exercise 9 (Nonexistence of multipliers) in Section 7.2 suggests the importance here of assuming h continuous.)

15. * **(Knaster–Kuratowski–Mazurkiewicz principle [114])** In this exercise we show the equivalence of Brouwer's theorem with the following result.

Theorem 8.1.11 (KKM) *Suppose for every point x in a nonempty set $X \subset \mathbf{E}$ there is an associated closed subset $M(x) \subset X$. Assume the property*

$$\operatorname{conv} F \subset \bigcup_{x \in F} M(x)$$

holds for all finite subsets $F \subset X$. Then for any finite subset $F \subset X$ we have

$$\bigcap_{x \in F} M(x) \neq \emptyset.$$

Hence if some subset $M(x)$ is compact we have

$$\bigcap_{x \in X} M(x) \neq \emptyset.$$

(a) Prove that the final assertion follows from the main part of the theorem using Theorem 8.2.3 (General definition of compactness).

(b) **(KKM implies Brouwer)** Given a continuous self map f on a nonempty compact convex set $C \subset \mathbf{E}$, apply the KKM theorem to the family of sets

$$M(x) = \{y \in C \mid \langle y - f(y), y - x \rangle \leq 0\} \quad \text{for } x \in C$$

to deduce f has a fixed point.

(c) **(Brouwer implies KKM)** With the hypotheses of the KKM theorem, assume $\cap_{x \in F} M(x)$ is empty for some finite set F. Consider a fixed point z of the self map

$$y \in \operatorname{conv} F \mapsto \frac{\sum_{x \in F} d_{M(x)}(y) x}{\sum_{x \in F} d_{M(x)}(y)}$$

and define $F' = \{x \in F \mid z \notin M(x)\}$. Show $z \in \operatorname{conv} F'$ and derive a contradiction.

16. ** **(Hairy ball theorem [140])** Let S_n denote the Euclidean sphere

$$\{x \in \mathbf{R}^{n+1} \mid \|x\| = 1\}.$$

A *tangent vector field* on S_n is a function $w : S_n \to \mathbf{R}^{n+1}$ satisfying $\langle x, w(x) \rangle = 0$ for all points x in S_n. This exercise proves the following result.

Theorem 8.1.12 *For every even n, any continuous tangent vector field on S_n must vanish somewhere.*

Proof. Consider a nonvanishing continuous tangent vector field u on S_n.

(a) Prove there is a nonvanishing $C^{(1)}$ tangent vector field on S_n, by using the Stone–Weierstrass theorem (8.1.4) to approximate u by a $C^{(1)}$ function p and then considering the vector field

$$x \in S_n \mapsto p(x) - \langle x, p(x) \rangle x.$$

(b) Deduce the existence of a positively homogeneous $C^{(1)}$ function $w : \mathbf{R}^{n+1} \to \mathbf{R}^{n+1}$ whose restriction to S_n is a *unit norm* $C^{(1)}$ tangent vector field: $\|w(x)\| = 1$ for all x in S_n.

Define a set

$$A = \{x \in \mathbf{R}^{n+1} \mid 1 < 2\|x\| < 3\}$$

and use the field w in part (b) to define functions $w_t : \mathbf{R}^{n+1} \to \mathbf{R}^{n+1}$ for real t by

$$w_t(x) = x + tw(x).$$

(c) Imitate the proof of Brouwer's theorem to prove the measure of the image set $w_t(A)$ is a polynomial in t when t is small.

(d) Prove directly the inclusion $w_t(A) \subset \sqrt{1+t^2}A$.

(e) For any point y in $\sqrt{1+t^2}A$, apply the Banach contraction theorem to the function $x \in kB \mapsto y - tw(x)$ (for large real k) to deduce in fact

$$w_t(A) = \sqrt{1+t^2}A \quad \text{for small } t.$$

(f) Complete the proof by combining parts (c) and (e). □

(g) If f is a continuous self map of S_n where n is even, prove either f or $-f$ has a fixed point.

(h) **(Hedgehog theorem)** Prove for even n that any nonvanishing continuous vector field must be somewhere *normal*: $|\langle x, f(x)\rangle| = \|f(x)\|$ for some x in S_n.

(i) Find examples to show the Hairy ball theorem fails for all odd n.

17. * **(Borsuk–Ulam theorem)** Let S_n denote the Euclidean sphere

$$\{x \in \mathbf{R}^{n+1} \mid \|x\| = 1\}.$$

We state the following result without proof.

Theorem 8.1.13 (Borsuk–Ulam) *For any positive integers $m \leq n$, if the function $f : S_n \to \mathbf{R}^m$ is continuous then there is a point x in S_n satisfying $f(x) = f(-x)$.*

(a) If $m \leq n$ and the map $f : S_n \to \mathbf{R}^m$ is continuous and odd, prove f vanishes somewhere.

(b) Prove any odd continuous self map f on S_n is surjective. (Hint: For any point u in S_n, consider the function

$$x \in S_n \mapsto f(x) - \langle f(x), u \rangle u$$

and apply part (a).)

(c) Prove the result in part (a) is equivalent to the following result:

Theorem 8.1.14 *For positive integers $m < n$ there is no continuous odd map from S_n to S_m.*

(d) **(Borsuk–Ulam implies Brouwer [178])** Let B denote the unit ball in $\mathbf{R}^n$, and let S denote the boundary of $B \times [-1, 1]$:

$$S = \{(x, t) \in B \times [-1, 1] \mid \|x\| = 1 \text{ or } |t| = 1\}.$$

(i) If the map $g : S \to \mathbf{R}^n$ is continuous and odd, use part (a) to prove g vanishes somewhere on S.

(ii) Consider a continuous self map f on B. By applying part (i) to the function

$$(x, t) \in S \mapsto (2 - |t|)x - tf(tx),$$

prove f has a fixed point.

18. ** **(Generalized Riesz lemma)** Consider a smooth norm $|||\cdot|||$ on $\mathbf{E}$ (that is, a norm which is continuously differentiable except at the origin) and linear subspaces $U, V \subset \mathbf{E}$ satisfying $\dim U > \dim V = n$. Denote the unit sphere in U (in this norm) by $S(U)$.

(a) By choosing a basis $v_1, v_2, \ldots, v_n$ of V and applying the Borsuk–Ulam theorem (see Exercise 17) to the map

$$x \in S(U) \mapsto (\langle \nabla |||\cdot|||(x), v_i \rangle)_{i=1}^n \in \mathbf{R}^n,$$

prove there is a point x in $S(U)$ satisfying $\nabla |||\cdot|||(x) \perp V$.

(b) Deduce the origin is the nearest point to x in V (in this norm).

(c) With this norm, deduce there is a unit vector in U whose distance from V is equal to one.

(d) Use the fact that any norm can be uniformly approximated arbitrarily well by a smooth norm to extend the result of part (c) to arbitrary norms.

(e) Find a simpler proof when $V \subset U$.

19. ** **(Riesz implies Borsuk)** In this question we use the generalized Riesz lemma, Exercise 18, to prove the Borsuk–Ulam result, Exercise 17(a). To this end, suppose the map $f : S_n \to \mathbf{R}^n$ is continuous and odd. Define functions

$$\begin{aligned} u_i : S_n &\to \mathbf{R} \text{ for } i = 1, 2, \ldots, n+1 \\ v_i : \mathbf{R}^n &\to \mathbf{R} \text{ for } i = 1, 2, \ldots, n \end{aligned}$$

by $u_i(x) = x_i$ and $v_i(x) = x_i$ for each index i. Define spaces of continuous odd functions on S_n by

$$\begin{aligned} U &= \text{span}\,\{u_1, u_2, \ldots . u_{n+1}\} \\ V &= \text{span}\,\{v_1 \circ f, v_2 \circ f, \ldots, v_n \circ f\} \\ \mathbf{E} &= U + V, \end{aligned}$$

with norm $\|u\| = \max u(S_n)$ (for u in $\mathbf{E}$).

(a) Prove there is a function u in U satisfying $\|u\| = 1$ and whose distance from V is equal to one.

(b) Prove u attains its maximum on S_n at a unique point y.

(c) Use the fact that for any function w in $\mathbf{E}$, we have

$$(\nabla \| \cdot \|(u))w = w(y)$$

to deduce $f(y) = 0$.

8.2 Selection and the Kakutani–Fan Fixed Point Theorem

The Brouwer fixed point theorem in the previous section concerns functions from a nonempty compact convex set to itself. In optimization, as we have already seen in Section 5.4, it may be convenient to broaden our language to consider *multifunctions* Ω from the set to itself and seek a *fixed point*—a point x satisfying $x \in \Omega(x)$. To begin this section we summarize some definitions for future reference.

We consider a subset $K \subset \mathbf{E}$, a Euclidean space $\mathbf{Y}$, and a multifunction $\Omega : K \to \mathbf{Y}$. We say Ω is *USC* at a point x in K if every open set U containing $\Omega(x)$ also contains $\Omega(z)$ for all points z in K close to x.

Thus a multifunction Ω is USC if for any sequence of points (x_n) approaching x, any sequence of elements $y_n \in \Omega(x_n)$ is eventually close to $\Omega(x)$. If Ω is USC at every point in K we simply call it *USC*. On the other hand, as in Section 5.4, we say Ω is *LSC* if, for every x in K, every neighbourhood V of any point in $\Omega(x)$ intersects $\Omega(z)$ for all points z in K close to x.

We refer to the sets $\Omega(x)$ $(x \in K)$ as the *images* of Ω. The multifunction Ω is a *cusco* if it is USC with nonempty compact convex images. Clearly such multifunctions are *locally bounded*: any point in K has a neighbourhood whose image is bounded. Cuscos appear in several important optimization contexts. For example, the Clarke subdifferential of a locally Lipschitz function is a cusco (Exercise 5).

To see another important class of examples we need a further definition. We say a multifunction $\Phi : \mathbf{E} \to \mathbf{E}$ is *monotone* if it satisfies the condition

$$\langle u - v, x - y\rangle \geq 0 \ \text{ whenever } u \in \Phi(x) \text{ and } v \in \Phi(y).$$

In particular, any (not necessarily self-adjoint) positive semidefinite linear operator is monotone, as is the subdifferential of any convex function. One multifunction *contains* another if the graph of the first contains the graph of the second. We say a monotone multifunction is *maximal* if the only monotone multifunction containing it is itself. The subdifferentials of closed proper convex functions are examples (see Exercise 16). Zorn's lemma (which lies outside our immediate scope) shows any monotone multifunction is contained in a maximal monotone multifunction.

Theorem 8.2.1 (Maximal monotonicity) *Maximal monotone multifunctions are cuscos on the interiors of their domains.*

Proof. See Exercise 16. □

Maximal monotone multifunctions in fact have to be single-valued *generically*, that is on sets which are "large" in a topological sense, specifically on a dense set which is a "G_δ" (a countable intersection of open sets)—see Exercise 17.

Returning to our main theme, the central result of this section extends Brouwer's theorem to the multifunction case.

Theorem 8.2.2 (Kakutani–Fan) *If the set $C \subset \mathbf{E}$ is nonempty, compact and convex, then any cusco $\Omega : C \to C$ has a fixed point.*

Before we prove this result, we outline a little more topology. A *cover* of a set $K \subset \mathbf{E}$ is a collection of sets in $\mathbf{E}$ whose union contains K. The cover is *open* if each set in the collection is open. A *subcover* is just a subcollection of the sets which is also a cover. The following result, which we state as a theorem, is in truth the definition of compactness in spaces more general than $\mathbf{E}$.

Theorem 8.2.3 (General definition of compactness) *Any open cover of a compact set in $\mathbf{E}$ has a finite subcover.*

Given a finite open cover $\{O_1, O_2, \ldots, O_m\}$ of a set $K \subset \mathbf{E}$, a *partition of unity subordinate to* this cover is a set of continuous functions $p_1, p_2, \ldots, p_m : K \to \mathbf{R}_+$ whose sum is identically equal to one and satisfying $p_i(x) = 0$ for all points x outside O_i (for each index i).. We outline the proof of the next result, a central topological tool, in the exercises.

Theorem 8.2.4 (Partition of unity) *There is a partition of unity subordinate to any finite open cover of a compact subset of $\mathbf{E}$.*

Besides fixed points, the other main theme of this section is the idea of a *continuous selection* of a multifunction Ω on a set $K \subset \mathbf{E}$, by which we mean a continuous map f on K satisfying $f(x) \in \Omega(x)$ for all points x in K. The central step in our proof of the Kakutani–Fan theorem is the following "approximate selection" theorem.

Theorem 8.2.5 (Cellina) *Given any compact set $K \subset \mathbf{E}$, suppose the multifunction $\Omega : K \to \mathbf{Y}$ is USC with nonempty convex images. Then for any real $\epsilon > 0$ there is a continuous map $f : K \to \mathbf{Y}$ which is an "approximate selection" of Ω :*

$$d_{G(\Omega)}(x, f(x)) < \epsilon \ \ \textit{for all points } x \textit{ in } K. \tag{8.2.6}$$

Furthermore the range of f is contained in the convex hull of the range of Ω.

Proof. We can assume the norm on $\mathbf{E} \times \mathbf{Y}$ is given by

$$\|(x, y)\|_{\mathbf{E}\times\mathbf{Y}} = \|x\|_{\mathbf{E}} + \|y\|_{\mathbf{Y}} \ \text{ for all } x \in \mathbf{E} \text{ and } y \in \mathbf{Y}$$

(since all norms are equivalent—see Section 4.1, Exercise 2). Now, since Ω is USC, for each point x in K there is a real δ_x in the interval $(0, \epsilon/2)$ satisfying

$$\Omega(x + \delta_x B_{\mathbf{E}}) \subset \Omega(x) + \frac{\epsilon}{2} B_{\mathbf{Y}}.$$

Since the sets $x + (\delta_x/2)\mathrm{int}\, B_{\mathbf{E}}$ (as the point x ranges over K) comprise an open cover of the compact set K, there is a finite subset $\{x_1, x_2, \ldots, x_m\}$ of K with the sets $x_i + (\delta_i/2)\mathrm{int}\, B_{\mathbf{E}}$ comprising a finite subcover (where δ_i is shorthand for δ_{x_i} for each index i).

Theorem 8.2.4 shows there is a partition of unity $p_1, p_2, \ldots, p_m : K \to \mathbf{R}_+$ subordinate to this subcover. We now construct our desired approximate selection f by choosing a point y_i from $\Omega(x_i)$ for each i and defining

$$f(x) = \sum_{i=1}^{m} p_i(x) y_i \quad \text{for all points } x \text{ in } K. \tag{8.2.7}$$

Fix any point x in K and define the set $I = \{i \mid p_i(x) \neq 0\}$. By definition, x satisfies $\|x - x_i\| < \delta_i/2$ for each i in I. If we choose an index j in I maximizing δ_j, the triangle inequality shows $\|x_j - x_i\| < \delta_j$, whence we deduce the inclusions

$$y_i \in \Omega(x_i) \subset \Omega(x_j + \delta_j B_{\mathbf{E}}) \subset \Omega(x_j) + \frac{\epsilon}{2} B_{\mathbf{Y}}$$

for all i in I. In other words, for each i in I we know $d_{\Omega(x_j)}(y_i) \le \epsilon/2$. Since the distance function is convex, equation (8.2.7) shows $d_{\Omega(x_j)}(f(x)) \le \epsilon/2$. Since we also know $\|x - x_j\| < \epsilon/2$, this proves inequality (8.2.6). The final claim follows immediately from equation (8.2.7). □

Proof of the Kakutani–Fan theorem. With the assumption of the theorem, Cellina's result above shows for each positive integer r there is a continuous self map f_r of C satisfying

$$d_{G(\Omega)}(x, f_r(x)) < \frac{1}{r} \quad \text{for all points } x \text{ in } C.$$

By Brouwer's theorem (8.1.3), each f_r has a fixed point x^r in C, which therefore satisfies

$$d_{G(\Omega)}(x^r, x^r) < \frac{1}{r} \quad \text{for each } r.$$

Since C is compact, the sequence (x^r) has a convergent subsequence, and its limit must be a fixed point of Ω because Ω is closed by Exercise 3(c) (Closed versus USC). □

In the next section we describe some variational applications of the Kakutani–Fan theorem. But we end this section with an *exact* selection theorem parallel to Cellina's result but assuming an LSC rather than a USC multifunction.

Theorem 8.2.8 (Michael) *Given any closed set $K \subset \mathbf{E}$, suppose the multifunction $\Omega : K \to \mathbf{Y}$ is LSC with nonempty closed convex images. Then given any point $(\bar{x}, \bar{y})$ in $G(\Omega)$, there is a continuous selection f of Ω satisfying $f(\bar{x}) = \bar{y}$.*

We outline the proof in the exercises.

Exercises and Commentary

Many useful properties of cuscos are summarized in [27]. An excellent general reference on monotone operators is [153]. The topology we use in this section can be found in any standard text (see [67, 106], for example). The Kakutani–Fan theorem first appeared in [109] and was extended in [74]. Cellina's approximate selection theorem appears, for example, in [4, p. 84]. One example of the many uses of the Kakutani–Fan theorem is establishing equilibria in mathematical economics. The Michael selection theorem appeared in [137].

1. **(USC and continuity)** Consider a closed subset $K \subset \mathbf{E}$ and a multifunction $\Omega : K \to \mathbf{Y}$.

   (a) Prove the multifunction

   $$x \in \mathbf{E} \mapsto \begin{cases} \Omega(x) & \text{for } x \in K \\ \emptyset & \text{for } x \notin K \end{cases}$$

   is USC if and only if Ω is USC.

   (b) Prove a function $f : K \to \mathbf{Y}$ is continuous if and only if the multifunction $x \in K \mapsto \{f(x)\}$ is USC.

   (c) Prove a function $f : \mathbf{E} \to [-\infty, +\infty]$ is lower semicontinuous at a point x in $\mathbf{E}$ if and only if the multifunction whose graph is the epigraph of f is USC at x.

2. * **(Minimum norm)** If the set $U \subset \mathbf{E}$ is open and the multifunction $\Omega : U \to \mathbf{Y}$ is USC, prove the function $g : U \to \mathbf{Y}$ defined by

   $$g(x) = \inf\{\|y\| \mid y \in \Omega(x)\}$$

   is lower semicontinuous.

3. **(Closed versus USC)**

   (a) If the multifunction $\Phi : \mathbf{E} \to \mathbf{Y}$ is closed and the multifunction $\Omega : \mathbf{E} \to \mathbf{Y}$ is USC at the point x in $\mathbf{E}$ with $\Omega(x)$ compact, prove the multifunction

   $$z \in \mathbf{E} \mapsto \Omega(z) \cap \Phi(z)$$

   is USC at x.

(b) Hence prove that any closed multifunction with compact range is USC.

(c) Prove any USC multifunction with closed images is closed.

(d) If a USC multifunction has compact images, prove it is locally bounded.

4. **(Composition)** If the multifunctions Φ and Ω are USC prove their composition $x \mapsto \Phi(\Omega(x))$ is also.

5. * **(Clarke subdifferential)** If the set $U \subset \mathbf{E}$ is open and the function $f : U \to \mathbf{R}$ is locally Lipschitz, use Section 6.2, Exercise 12 (Closed subdifferentials) and Exercise 3 (Closed versus USC) to prove the Clarke subdifferential $x \in U \mapsto \partial_\circ f(x)$ is a cusco.

6. ** **(USC images of compact sets)** Consider a given multifunction $\Omega : K \to \mathbf{Y}$.

   (a) Prove Ω is USC if and only if for every open subset U of $\mathbf{Y}$ the set $\{x \in K \mid \Omega(x) \subset U\}$ is open in K.

   Now suppose K is compact and Ω is USC with compact images. Using the general definition of compactness (8.2.3), prove the range $\Omega(K)$ is compact by following the steps below.

   (b) Fix an open cover $\{U_\gamma \,|\, \gamma \in \Gamma\}$ of $\Omega(K)$. For each point x in K, prove there is a finite subset Γ_x of Γ with

   $$\Omega(x) \subset \bigcup_{\gamma \in \Gamma_x} U_\gamma.$$

   (c) Construct an open cover of K by considering the sets

   $$\left\{ z \in K \,\middle|\, \Omega(z) \subset \bigcup_{\gamma \in \Gamma_x} U_\gamma \right\}$$

   as the point x ranges over K.

   (d) Hence construct a finite subcover of the original cover of $\Omega(K)$.

7. * **(Partitions of unity)** Suppose the set $K \subset \mathbf{E}$ is compact with a finite open cover $\{O_1, O_2, \ldots, O_m\}$.

   (i) Show how to construct another open cover $\{V_1, V_2, \ldots, V_m\}$ of K satisfying $\mathrm{cl}\, V_i \subset O_i$ for each index i. (Hint: Each point x in K lies in some set O_i, so there is a real $\delta_x > 0$ with $x + \delta_x B \subset O_i$; now take a finite subcover of $\{x + \delta_x \mathrm{int}\, B \,|\, x \in K\}$ and build the sets V_i from it.)

(ii) For each index i, prove the function $q_i : K \to [0,1]$ given by

$$q_i = \frac{d_{K \backslash O_i}}{d_{K \backslash O_i} + d_{V_i}}$$

is well-defined and continuous, with q_i identically zero outside the set O_i.

(iii) Deduce that the set of functions $p_i : K \to \mathbf{R}_+$ defined by

$$p_i = \frac{q_i}{\sum_j q_j}$$

is a partition of unity subordinate to the cover $\{O_1, O_2, \ldots, O_m\}$.

8. Prove the Kakutani–Fan theorem is also valid under the weaker assumption that the images of the cusco $\Omega : C \to \mathbf{E}$ always intersect the set C using Exercise 3(a) (Closed versus USC).

9. ** **(Michael's theorem)** Suppose all the assumptions of Michael's theorem (8.2.8) hold. We consider first the case with K compact.

(a) Fix a real $\epsilon > 0$. By constructing a partition of unity subordinate to a finite subcover of the open cover of K consisting of the sets

$$O_y = \{x \in \mathbf{E} \mid d_{\Omega(x)}(y) < \epsilon\} \ \text{ for } y \text{ in } Y,$$

construct a continuous function $f : K \to Y$ satisfying

$$d_{\Omega(x)}(f(x)) < \epsilon \ \text{ for all points } x \text{ in } K.$$

(b) Construct a sequence of continuous functions $f_1, f_2, \ldots : K \to Y$ satisfying

$$\begin{aligned} d_{\Omega(x)}(f_i(x)) &< 2^{-i} \ \text{ for } i = 1, 2, \ldots \\ \|f_{i+1}(x) - f_i(x)\| &< 2^{1-i} \ \text{ for } i = 1, 2, \ldots \end{aligned}$$

for all points x in K. (Hint: Construct f_1 by applying part (a) with $\epsilon = 1/2$; then construct f_{i+1} inductively by applying part (a) to the multifunction

$$x \in K \mapsto \Omega(x) \cap (f_i(x) + 2^{-i} B_{\mathbf{Y}})$$

with $\epsilon = 2^{-i-1}$.

(c) The functions f_i of part (b) must converge uniformly to a continuous function f. Prove f is a continuous selection of Ω.

(d) Prove Michael's theorem by applying part (c) to the multifunction

$$\hat{\Omega}(x) = \begin{cases} \Omega(x) & \text{if } x \neq \bar{x} \\ \{\bar{y}\} & \text{if } x = \bar{x}. \end{cases}$$

(e) Now extend to the general case where K is possibly unbounded in the following steps. Define sets $K_n = K \cap nB_{\mathbf{E}}$ for each $n = 1, 2, \ldots$ and apply the compact case to the multifunction $\Omega_1 = \Omega|_{K_1}$ to obtain a continuous selection $g_1 : K_1 \to \mathbf{Y}$. Then inductively find a continuous selection $g_{n+1} : K_{n+1} \to \mathbf{Y}$ from the multifunction

$$\Omega_{n+1}(x) = \begin{cases} \{g_n(x)\} & \text{for } x \in K_n \\ \Omega(x) & \text{for } x \in K_{n+1} \setminus K_n \end{cases}$$

and prove the function defined by

$$f(x) = g_n(x) \ \text{ for } x \in K_n, \ n = 1, 2, \ldots$$

is the required selection.

10. **(Hahn–Katetov–Dowker sandwich theorem)** Suppose the set $K \subset \mathbf{E}$ is closed.

(a) For any two lower semicontinuous functions $f, g : K \to \mathbf{R}$ satisfying $f \geq -g$, prove there is a continuous function $h : K \to \mathbf{R}$ satisfying $f \geq h \geq -g$ by considering the multifunction $x \mapsto [-g(x), f(x)]$. Observe the result also holds for extended-real-valued f and g.

(b) **(Urysohn lemma)** Suppose the closed set V and the open set U satisfy $V \subset U \subset K$. By applying part (i) to suitable functions, prove there is a continuous function $f : K \to [0, 1]$ that is identically equal to one on V and to zero on U^c.

11. **(Continuous extension)** Consider a closed subset K of $\mathbf{E}$ and a continuous function $f : K \to \mathbf{Y}$. By considering the multifunction

$$\Omega(x) = \begin{cases} \{f(x)\} & \text{for } x \in K \\ \mathrm{cl}\,(\mathrm{conv}\, f(K)) & \text{for } x \notin K, \end{cases}$$

prove there is a continuous function $g : \mathbf{E} \to \mathbf{Y}$ satisfying $g|_K = f$ and $g(\mathbf{E}) \subset \mathrm{cl}\,(\mathrm{conv}\, f(K))$.

12. * **(Generated cuscos)** Suppose the multifunction $\Omega : K \to \mathbf{Y}$ is locally bounded with nonempty images.

(a) Among those cuscos containing Ω, prove there is a unique one with minimal graph, given by

$$\Phi(x) = \bigcap_{\epsilon>0} \operatorname{cl}\operatorname{conv}(\Omega(x+\epsilon B)) \quad \text{for } x \in K.$$

(b) If K is nonempty, compact, and convex, $\mathbf{Y} = \mathbf{E}$, and Ω satisfies the conditions $\Omega(K) \subset K$ and

$$x \in \Phi(x) \Rightarrow x \in \Omega(x) \quad \text{for } x \in K,$$

prove Ω has a fixed point.

13. * **(Multifunctions containing cuscos)** Suppose the multifunction $\Omega : K \to \mathbf{Y}$ is closed with nonempty convex images, and the function $f : K \to \mathbf{Y}$ has the property that $f(x)$ is a point of minimum norm in $\Omega(x)$ for all points x in K. Prove Ω contains a cusco if and only if f is locally bounded. (Hint: Use Exercise 12 (Generated cuscos) to consider the cusco generated by f.)

14. * **(Singleton points)** For any subset D of $\mathbf{Y}$, define

$$s(D) = \inf\{r \in \mathbf{R} \mid D \subset y + rB_{\mathbf{Y}} \text{ for some } y \in \mathbf{Y}\}.$$

Consider an open subset U of $\mathbf{E}$.

(a) If the multifunction $\Omega : U \to \mathbf{Y}$ is USC with nonempty images, prove for any real $\epsilon > 0$ the set

$$S_\epsilon = \{x \in U \mid s(\Omega(x)) < \epsilon\}$$

is open. By considering the set $\cap_{n>1} S_{1/n}$, prove the set of points in U whose image is a singleton is a G_δ.

(b) Use Exercise 5 (Clarke subdifferential) to prove that the set of points where a locally Lipschitz function $f : U \to \mathbf{R}$ is strictly differentiable is a G_δ. If U and f are convex (or if f is regular throughout U), use Rademacher's theorem (in Section 6.2) to deduce f is generically differentiable.

15. **(Skew symmetry)** If the matrix $A \in \mathbf{M}^n$ satisfies $0 \neq A = -A^T$, prove the multifunction $x \in \mathbf{R}^n \mapsto x^T A x$ is maximal monotone, yet is not the subdifferential of a convex function.

16. ** **(Monotonicity)** Consider a monotone multifunction $\Phi : \mathbf{E} \to \mathbf{E}$.

(a) **(Inverses)** Prove Φ^{-1} is monotone.

(b) Prove Φ^{-1} is maximal if and only if Φ is.

(c) **(Applying maximality)** Prove Φ is maximal if and only if it has the property

$$\langle u - v, x - y \rangle \geq 0 \text{ for all } (x, u) \in G(\Phi) \Rightarrow v \in \Phi(y).$$

(d) **(Maximality and closedness)** If Φ is maximal, prove it is closed with convex images.

(e) **(Continuity and maximality)** Supposing Φ is everywhere single-valued and *hemicontinuous* (that is, continuous on every line in $\mathbf{E}$), prove it is maximal. (Hint: Apply part (c) with $x = y + tw$ for w in $\mathbf{E}$ and $t \downarrow 0$ in $\mathbf{R}$.)

(f) We say Φ is *hypermaximal* if $\Phi + \lambda I$ is surjective for some real $\lambda > 0$. In this case, prove Φ is maximal. (Hint: Apply part (c) and use a solution $x \in \mathbf{E}$ to the inclusion $v + \lambda y \in (\Phi + \lambda I)(x)$.) What if just Φ is surjective?

(g) **(Subdifferentials)** If the function $f : \mathbf{E} \to (\infty, +\infty]$ is closed, convex, and proper, prove ∂f is maximal monotone. (Hint: For any element ϕ of $\mathbf{E}$, prove the function

$$x \in \mathbf{E} \mapsto f(x) + \|x\|^2 + \langle \phi, x \rangle$$

has a minimizer, and deduce ∂f is hypermaximal.)

(h) **(Local boundedness)** By completing the following steps, prove Φ is locally bounded at any point in the core of its domain.

  (i) Assume $0 \in \Phi(0)$ and $0 \in \operatorname{core} D(\Phi)$, define a convex function $g : \mathbf{E} \to (\infty, +\infty]$ by

$$g(y) = \sup\{\langle u, y - x \rangle \mid x \in B,\ u \in \Phi(x)\}.$$

  (ii) Prove $D(\Phi) \subset \operatorname{dom} g$.

  (iii) Deduce g is continuous at zero.

  (iv) Hence show $|g(y)| \leq 1$ for all small y, and deduce the result.

(j) **(Maximality and cuscos)** Use parts (d) and (h), and Exercise 3 (Closed versus USC) to conclude that any maximal monotone multifunction is a cusco on the interior of its domain.

(k) **(Surjectivity and growth)** If Φ is surjective, prove

$$\lim_{\|x\| \to \infty} \|\Phi(x)\| = +\infty.$$

(Hint: Assume the maximality of Φ, and hence of Φ^{-1}; deduce Φ^{-1} is a cusco on $\mathbf{E}$, and now apply Exercise 6 (USC images of compact sets).)

17. ** **(Single-valuedness and maximal monotonicity)** Consider a maximal monotone multifunction $\Omega : \mathbf{E} \to \mathbf{E}$ and an open subset U of its domain, and define the minimum norm function $g : U \to \mathbf{R}$ as in Exercise 2.

    (a) Prove g is lower semicontinuous. An application of the Baire category theorem now shows that any such function is generically continuous.

    (b) For any point x in U at which g is continuous, prove $\Omega(x)$ is a singleton. (Hint: Prove $\|\cdot\|$ is constant on $\Omega(x)$ by first assuming $y, z \in \Omega(x)$ and $\|y\| > \|z\|$, and then using the condition

    $$\langle w - y, x + ty - x\rangle \geq 0 \text{ for all small } t > 0 \text{ and } w \in \Omega(x + ty)$$

    to derive a contradiction.)

    (c) Conclude that any maximal monotone multifunction is generically single-valued.

    (d) Deduce that any convex function is generically differentiable on the interior of its domain.

8.3 Variational Inequalities

At the very beginning of this book we considered the problem of minimizing a differentiable function $f : \mathbf{E} \to \mathbf{R}$ over a convex set $C \subset \mathbf{E}$. A necessary optimality condition for a point x_0 in C to be a local minimizer is

$$\langle \nabla f(x_0), x - x_0 \rangle \geq 0 \text{ for all points } x \text{ in } C, \tag{8.3.1}$$

or equivalently

$$0 \in \nabla f(x_0) + N_C(x_0).$$

If the function f is convex instead of differentiable, the necessary and sufficient condition for optimality (assuming a constraint qualification) is

$$0 \in \partial f(x_0) + N_C(x_0),$$

and there are analogous nonsmooth necessary conditions.

We call problems like (8.3.1) "variational inequalities". Let us fix a multifunction $\Omega : C \to \mathbf{E}$. In this section we use the fixed point theory we have developed to study the *multivalued variational inequality*

$VI(\Omega, C)$: Find points x_0 in C and y_0 in $\Omega(x_0)$ satisfying $\langle y_0, x - x_0 \rangle \geq 0$ for all points x in C.

A more concise way to write the problem is this:

$$\text{Find a point } x_0 \text{ in } C \text{ satisfying } 0 \in \Omega(x_0) + N_C(x_0). \tag{8.3.2}$$

Suppose the set C is closed, convex, and nonempty. Recall that the projection $P_C : \mathbf{E} \to C$ is the (continuous) map that sends points in $\mathbf{E}$ to their unique nearest points in C (see Section 2.1, Exercise 8). Using this notation we can also write the variational inequality as a fixed point problem:

$$\text{Find a fixed point of } P_C \circ (I - \Omega) : C \to C. \tag{8.3.3}$$

This reformulation is useful if the multifunction Ω is single-valued, but less so in general because the composition will often not have convex images.

A more versatile approach is to define the (multivalued) *normal mapping* $\Omega_C = (\Omega \circ P_C) + I - P_C$, and repose the problem as follows:

$$\text{Find a point } \bar{x} \text{ in } \mathbf{E} \text{ satisfying } 0 \in \Omega_C(\bar{x}). \tag{8.3.4}$$

Then setting $x_0 = P_C(\bar{x})$ gives a solution to the original problem. Equivalently, we could phrase this as follows:

$$\text{Find a fixed point of } (I - \Omega) \circ P_C : \mathbf{E} \to \mathbf{E}. \tag{8.3.5}$$

As we shall see, this last formulation lets us immediately use the fixed point theory of the previous section.

The basic result guaranteeing the existence of solutions to variational inequalities is the following.

Theorem 8.3.6 (Solvability of variational inequalities) *If the subset C of $\mathbf{E}$ is compact, convex, and nonempty, then for any cusco $\Omega : C \to \mathbf{E}$ the variational inequality $VI(\Omega, C)$ has a solution.*

Proof. We in fact prove *Theorem 8.3.6 is equivalent to the Kakutani–Fan fixed point theorem (8.2.2).*

When Ω is a cusco its range $\Omega(C)$ is compact—we outline the proof in Section 8.2, Exercise 6. We can easily check that the multifunction $(I - \Omega) \circ P_C$ is also a cusco because the projection P_C is continuous. Since this multifunction maps the compact convex set $\mathrm{conv}\,(C - \Omega(C))$ into itself, the Kakutani–Fan theorem shows it has a fixed point, which, as we have already observed, implies the solvability of $VI(\Omega, C)$.

Conversely, suppose the set $C \subset \mathbf{E}$ is nonempty, compact, and convex. For any cusco $\Omega : C \to C$, the Solvability theorem (8.3.6) implies we can solve the variational inequality $VI(I - \Omega, C)$, so there are points x_0 in C and z_0 in $\Omega(x_0)$ satisfying

$$\langle x_0 - z_0, x - x_0 \rangle \geq 0 \ \text{ for all points } x \text{ in } C.$$

Setting $x = z_0$ shows $x_0 = z_0$, so x_0 is a fixed point. □

An elegant application is von Neumann's minimax theorem, which we proved by a Fenchel duality argument in Section 4.2, Exercise 16. Consider Euclidean spaces $\mathbf{Y}$ and $\mathbf{Z}$, nonempty compact convex subsets $F \subset \mathbf{Y}$ and $G \subset \mathbf{Z}$, and a linear map $A : \mathbf{Y} \to \mathbf{Z}$. If we define a function $\Omega : F \times G \to \mathbf{Y} \times \mathbf{Z}$ by $\Omega(y, z) = (-A^* z, Ay)$, then it is easy to see that a point (y_0, z_0) in $F \times G$ solves the variational inequality $VI(\Omega, F \times G)$ if and only if it is a *saddlepoint*:

$$\langle z_0, Ay \rangle \leq \langle z_0, Ay_0 \rangle \leq \langle z, Ay_0 \rangle \ \text{ for all } y \in F,\ z \in G.$$

In particular, by the Solvability of variational inequalities theorem, there exists a saddlepoint, so

$$\min_{z \in G} \max_{y \in F} \langle z, Ay \rangle = \max_{y \in F} \min_{z \in G} \langle z, Ay \rangle.$$

Many interesting variational inequalities involve a noncompact set C. In such cases we need to impose a growth condition on the multifunction to guarantee solvability. The following result is an example.

Theorem 8.3.7 (Noncompact variational inequalities) *If the subset C of $\mathbf{E}$ is nonempty, closed, and convex, and the cusco $\Omega : C \to \mathbf{E}$ is* **coercive**, *that is, it satisfies the condition*

$$\liminf_{\|x\|\to\infty,\ x\in C} \inf \langle x, \Omega(x) + N_C(x)\rangle > 0, \tag{8.3.8}$$

then the variational inequality $VI(\Omega, C)$ has a solution.

Proof. For any large integer r, we can apply the solvability theorem (8.3.6) to the variational inequality $VI(\Omega, C \cap rB)$ to find a point x_r in $C \cap rB$ satisfying

$$\begin{aligned} 0 &\in \Omega(x_r) + N_{C\cap rB}(x_r) \\ &= \Omega(x_r) + N_C(x_r) + N_{rB}(x_r) \\ &\subset \Omega(x_r) + N_C(x_r) + \mathbf{R}_+ x_r \end{aligned}$$

(using Section 3.3, Exercise 10). Hence for all large r, the point x_r satisfies

$$\inf \langle x_r, \Omega(x_r) + N_C(x_r)\rangle \leq 0.$$

This sequence of points (x_r) must therefore remain bounded, by the coercivity condition (8.3.8), and so x_r lies in $\operatorname{int} rB$ for large r and hence satisfies $0 \in \Omega(x_r) + N_C(x_r)$, as required. □

A straightforward exercise shows in particular that the growth condition (8.3.8) holds whenever the cusco Ω is defined by $x \in \mathbf{R}^n \mapsto x^T A x$ for a matrix A in $\mathbf{S}^n_{++}$.

The most important example of a noncompact variational inequality is the case when the set C is a closed convex cone $S \subset \mathbf{E}$. In this case $VI(\Omega, S)$ becomes the *multivalued complementarity problem*:

$$\begin{array}{l} \text{Find points } x_0 \text{ in } S \text{ and } y_0 \text{ in } \Omega(x_0) \cap (-S^-) \\ \text{satisfying } \langle x_0, y_0\rangle = 0. \end{array} \tag{8.3.9}$$

As a particular example, we consider the dual pair of abstract linear programs (5.3.4) and (5.3.5):

$$\inf\{\langle c, z\rangle \mid Az - b \in H,\ z \in K\} \tag{8.3.10}$$

(where $\mathbf{Y}$ is a Euclidean space, the map $A : \mathbf{E} \to \mathbf{Y}$ is linear, the cones $H \subset \mathbf{Y}$ and $K \subset \mathbf{E}$ are closed and convex, and b and c are given elements of $\mathbf{Y}$ and $\mathbf{E}$ respectively), and

$$\sup\{\langle b, \phi\rangle \mid A^*\phi - c \in K^-,\ \phi \in -H^-\}. \tag{8.3.11}$$

As usual, we denote the corresponding primal and dual optimal values by p and d. We consider a corresponding variational inequality on the space $\mathbf{E} \times \mathbf{Y}$:

$$VI(\Omega, K \times (-H^-)) \ \text{ with } \ \Omega(z, \phi) = (c - A^*\phi, Ax - b). \tag{8.3.12}$$

Theorem 8.3.13 (Linear programming and variational inequalities) *Any solution of the above variational inequality (8.3.12) consists of a pair of optimal solutions for the linear programming dual pair (8.3.10) and (8.3.11). The converse is also true, providing there is no duality gap* ($p = d$).

We leave the proof as an exercise.

Notice that the linear map appearing in the above example, namely $M : \mathbf{E} \times \mathbf{Y} \to \mathbf{E} \times \mathbf{Y}$ defined by $M(z, \phi) = (-A^*\phi, Az)$, is monotone. We study monotone complementarity problems further in Exercise 7.

To end this section we return to the complementarity problem (8.3.9) in the special case where $\mathbf{E}$ is $\mathbf{R}^n$, the cone S is $\mathbf{R}^n_+$, and the multifunction Ω is single-valued: $\Omega(x) = \{F(x)\}$ for all points x in $\mathbf{R}^n_+$. In other words, we consider the following problem:

Find a point x_0 in $\mathbf{R}^n_+$ satisfying $F(x_0) \in \mathbf{R}^n_+$ and $\langle x_0, F(x_0)\rangle = 0$.

The lattice operation $\wedge$ is defined on $\mathbf{R}^n$ by $(x \wedge y)_i = \min\{x_i, y_i\}$ for points x and y in $\mathbf{R}^n$ and each index i. With this notation we can rewrite the above problem as the following *order complementarity problem*.

$OCP(F)$: Find a point x_0 in $\mathbf{R}^n_+$ satisfying $x_0 \wedge F(x_0) = 0$.

The map $x \in \mathbf{R}^n \mapsto x \wedge F(x) \in \mathbf{R}^n$ is sometimes amenable to fixed point methods.

As an example, let us fix a real $\alpha > 0$, a vector $q \in \mathbf{R}^n$, and an $n \times n$ matrix P with nonnegative entries, and define the map $F : \mathbf{R}^n \to \mathbf{R}^n$ by $F(x) = \alpha x - Px + q$. Then the complementarity problem $OCP(F)$ is equivalent to finding a fixed point of the map $\Phi : \mathbf{R}^n \to \mathbf{R}^n$ defined by

$$\Phi(x) = \frac{1}{\alpha}(0 \vee (Px - q)), \tag{8.3.14}$$

a problem that can be solved iteratively (see Exercise 9).

Exercises and commentary

A survey of variational inequalities and complementarity problems may be found in [93]. The normal mapping Ω_C is especially well studied when the multifunction Ω is single-valued with affine components and the set C is polyhedral. In this case the normal mapping is piecewise affine (see [164]). More generally, if we restrict the class of multifunctions Ω we wish to consider in the variational inequality, clearly we can correspondingly restrict the versions of the Kakutani–Fan theorem or normal mappings we study. Order complementarity problems are studied further in [26]. The Nash equilibrium theorem (Exercise 10(d)), which appeared in [147], asserts the existence of a Pareto efficient choice for n individuals consuming from n associated convex sets with n associated joint cost functions.

1. Prove the equivalence of the various formulations (8.3.2), (8.3.3), (8.3.4) and (8.3.5) with the original variational inequality $VI(\Omega, C)$.

2. Use Section 8.2, Exercise 4 (Composition) to prove the multifunction

$$(I - \Omega) \circ P_C$$

in the proof of Theorem 8.3.6 (Solvability of variational inequalities) is a cusco.

3. Consider Theorem 8.3.6 (Solvability of variational inequalities). Use the function

$$x \in [0,1] \mapsto \begin{cases} \dfrac{1}{x} & \text{if } x > 0 \\ -1 & \text{if } x = 0 \end{cases}$$

to prove the assumption in the theorem—that the multifunction Ω is USC—cannot be weakened to Ω closed.

4. * **(Variational inequalities containing cuscos)** Suppose the set $C \subset \mathbf{E}$ is nonempty, compact, and convex, and consider a multifunction $\Omega : C \to \mathbf{E}$.

   (a) If Ω contains a cusco, prove the variational inequality $VI(\Omega, C)$ has a solution.

   (b) Deduce from Michael's theorem (8.2.8) that if Ω is LSC with nonempty closed convex images then $VI(\Omega, C)$ has a solution.

5. Check the details of the proof of von Neumann's minimax theorem.

6. Prove Theorem 8.3.13 (Linear programming and variational inequalities).

7. **(Monotone complementarity problems)** Suppose the linear map $M : \mathbf{E} \to \mathbf{E}$ is monotone.

   (a) Prove the function $x \in \mathbf{E} \mapsto \langle Mx, x\rangle$ is convex.

   For a closed convex cone $S \subset \mathbf{E}$ and a point q in $\mathbf{E}$, consider the optimization problem

$$\inf\{\langle Mx + q, x\rangle \mid Mx + q \in -S^-,\ x \in S\}. \tag{8.3.15}$$

   (b) If the condition $-q \in \operatorname{core}(S^- + MS)$ holds, use the Fenchel duality theorem (3.3.5) to prove problem (8.3.15) has optimal value zero.

(c) If the cone S is polyhedral, problem (8.3.15) is a convex "quadratic program": when the optimal value is finite, it is known that there is no duality gap for such a problem and its (Fenchel) dual, and that both problems attain their optimal value. Deduce that when S is polyhedral and contains a point x with $Mx+q$ in $-S^-$, there is such a point satisfying the additional complementarity condition $\langle Mx+q, x\rangle = 0$.

8. * Consider a compact convex set $C \subset \mathbf{E}$ satisfying $C = -C$ and a continuous function $f : C \to \mathbf{E}$. If f has no zeroes, prove there is a point x on the boundary of C satisfying $\langle f(x), x\rangle < 0$. (Hint: For positive integers n, consider $VI(f + I/n, C)$.)

9. **(Iterative solution of OCP [26])** Consider the order complementarity problem $OCP(F)$ for the function F that we defined before equation (8.3.14). A point x^0 in $\mathbf{R}^n_+$ is *feasible* if it satisfies $F(x^0) \geq 0$.

   (a) Prove the map Φ in equation (8.3.14) is *isotone*: $x \geq y$ implies $\Phi(x) \geq \Phi(y)$ for points x and y in $\mathbf{R}^n$.

   (b) Suppose the point x^0 in $\mathbf{R}^n_+$ is feasible. Define a sequence (x^r) in $\mathbf{R}^n_+$ inductively by $x^{r+1} = \Phi(x^r)$. Prove this sequence decreases monotonically: $x_i^{r+1} \leq x_i^r$ for all r and i.

   (c) Prove the limit of the sequence in part (b) solves $OCP(F)$.

   (d) Define a sequence (y^r) in $\mathbf{R}^n_+$ inductively by $y^0 = 0$ and $y^{r+1} = \Phi(y^r)$. Prove this sequence increases monotonically.

   (e) If $OCP(F)$ has a feasible solution, prove the sequence in part (d) converges to a limit $\bar{y}$ which solves $OCP(F)$. What happens if $OCP(F)$ has no feasible solution?

   (f) Prove the limit $\bar{y}$ of part (e) is the *minimal* solution of $OCP(F)$: any other solution x satisfies $x \geq \bar{y}$.

10. * **(Fan minimax inequality [74])** We call a real function g on a convex set $C \subset \mathbf{E}$ *quasiconcave* if the set $\{x \in C \,|\, g(x) \geq \alpha\}$ is convex for all real α.

    Suppose the set $C \subset \mathbf{E}$ is nonempty, compact, and convex.

    (a) If the function $f : C \times C \to \mathbf{R}$ has the properties that the function $f(\cdot, y)$ is quasiconcave for all points y in C and the function $f(x, \cdot)$ is lower semicontinuous for all points x in C, prove *Fan's inequality*:

$$\min_y \sup_x f(x,y) \leq \sup_x f(x,x).$$

(Hint: Apply the KKM theorem (Section 8.1, Exercise 15) to the family of sets

$$\{y \in C \mid f(x,y) \leq \beta\} \quad \text{for } x \in C,$$

where β denotes the right hand side of Fan's inequality.)

(b) If the function $F : C \to \mathbf{E}$ is continuous, apply Fan's inequality to the function $f(x,y) = \langle F(y), y - x \rangle$ to prove the variational inequality $VI(F,C)$ has a solution.

(c) Deduce Fan's inequality is equivalent to the Brouwer fixed point theorem.

(d) **(Nash equilibrium)** Define a set $C = C_1 \times C_2 \times \ldots \times C_n$, where each set $C_i \subset \mathbf{E}$ is nonempty, compact, and convex. For any continuous functions $f_1, f_2, \ldots, f_n : C \to \mathbf{R}$, if each function

$$x_i \in C_i \mapsto f_i(y_1, \ldots, x_i, \ldots, y_n)$$

is convex for all elements y of C, prove there is an element y of C satisfying the inequalities

$$f_i(y) \leq f_i(y_1, \ldots, x_i, \ldots, y_n) \quad \text{for all } x_i \in C_i,\ i = 1, 2, \ldots, n.$$

(Hint: Consider the function

$$f(x,y) = \sum_i (f_i(y) - f_i(y_1, \ldots, x_i, \ldots, y_n))$$

and apply Fan's inequality.)

(e) **(Minimax)** Apply the Nash equilibrium result from part (d) in the case $n = 2$ and $f_1 = -f_2$ to deduce the Kakutani minimax theorem (Section 4.3, Exercise 14).

11. **(Bolzano–Poincaré–Miranda intermediate value theorem)** Consider the box

$$J = \{x \in \mathbf{R}^n \mid 0 \leq x_i \leq 1 \text{ for all } i\}.$$

We call a continuous map $f : J \to \mathbf{R}^n$ *reversing* if it satisfies the condition

$$f_i(x) f_i(y) \leq 0 \quad \text{whenever } x_i = 0,\ y_i = 1, \text{ and } i = 1, 2, \ldots, n.$$

Prove any such map vanishes somewhere on J by completing the following steps:

(a) Observe the case $n = 1$ is just the classical intermediate value theorem.

(b) For all small real $\epsilon > 0$, prove the function $f^\epsilon = f + \epsilon I$ satisfies for all i

$$x_i = 0 \text{ and } y_i = 1 \;\Rightarrow\; \begin{cases} \text{either} & f_i^\epsilon(y) > 0 \text{ and } f_i^\epsilon(x) \le 0 \\ \text{or} & f_i^\epsilon(y) < 0 \text{ and } f_i^\epsilon(x) \ge 0. \end{cases}$$

(c) ¿From part (b), deduce there is a function $\widetilde{f}^\epsilon$, defined coordinatewise by $\widetilde{f}_i^\epsilon = \pm f_i^\epsilon$, for some suitable choice of signs, satisfying the conditions (for each i)

$$\widetilde{f}_i^\epsilon(x) \le 0 \text{ whenever } x_i = 0 \quad \text{and}$$
$$\widetilde{f}_i^\epsilon(x) > 0 \text{ whenever } x_i = 1.$$

(d) By considering the variational inequality $VI(\widetilde{f}^\epsilon, J)$, prove there is a point x^ϵ in J satisfying $\widetilde{f}^\epsilon(x^\epsilon) = 0$.

(e) Complete the proof by letting ϵ approach zero.

12. **(Coercive cuscos)** Consider a multifunction $\Omega : \mathbf{E} \to \mathbf{E}$ with nonempty images.

(a) If Ω is a coercive cusco, prove it is surjective.

(b) On the other hand, if Ω is monotone, use Section 8.2, Exercise 16 (Monotonicity) to deduce Ω is hypermaximal if and only if it is maximal. (We generalize this result in Exercise 13 (Monotone variational inequalities).)

13. ** **(Monotone variational inequalities)** Consider a continuous function $G : \mathbf{E} \to \mathbf{E}$ and a monotone multifunction $\Phi : \mathbf{E} \to \mathbf{E}$.

(a) Given a nonempty compact convex set $K \subset \mathbf{E}$, prove there is point x_0 in K satisfying

$$\langle x - x_0, y + G(x_0) \rangle \ge 0 \quad \text{for all } x \in K,\; y \in \Phi(x)$$

by completing the following steps:

(i) Assuming the result fails, show the collection of sets

$$\{x \in K \mid \langle z - x, w + G(x) \rangle < 0\} \quad \text{for } z \in K,\; w \in \Phi(z)$$

is an open cover of K.

(ii) For a partition of unity $p_1, p_2, \ldots, p_n$ subordinate to a finite subcover $K_1, K_2, \ldots K_n$ corresponding to points $z_i \in K$ and $w_i \in \Phi(z_i)$ (for $i = 1, 2, \ldots, n$), prove the function

$$f(x) = \sum_i p_i(x) z_i$$

is a continuous self map of K.

(iii) Prove the inequality

$$\begin{aligned}\langle f(x) - x, \textstyle\sum_i p_i(x)w_i + G(x)\rangle \\ &= \sum_{i,j} p_i(x)p_j(x)\langle z_j - x, w_i + G(x)\rangle \\ &< 0\end{aligned}$$

by considering the terms in the double sum where $i = j$ and sums of pairs where $i \neq j$ separately.

(iv) Deduce a contradiction with part (ii).

(b) Now assume G satisfies the growth condition

$$\lim_{\|x\|\to\infty} \|G(x)\| = +\infty \quad \text{and} \quad \liminf_{\|x\|\to\infty} \frac{\langle x, G(x)\rangle}{\|x\|\|G(x)\|} > 0.$$

(i) Prove there is a point x_0 in $\mathbf{E}$ satisfying

$$\langle x - x_0, y + G(x_0)\rangle \geq 0 \quad \text{whenever } y \in \Phi(x).$$

(Hint: Apply part (a) with $K = nB$ for $n = 1, 2, \ldots$.)

(ii) If Φ is maximal, deduce $-G(x_0) \in \Phi(x_0)$.

(c) Apply part (b) to prove that if Φ is maximal then for any real $\lambda > 0$, the multifunction $\Phi + \lambda I$ is surjective.

(d) **(Hypermaximal $\Leftrightarrow$ maximal)** Using Section 8.2, Exercise 16 (Monotonicity), deduce a monotone multifunction is maximal if and only if it is hypermaximal.

(e) **(Resolvent)** If Φ is maximal then for any real $\lambda > 0$ and any point y in $\mathbf{E}$ prove there is a unique point x satisfying the inclusion

$$y \in \Phi(x) + \lambda x.$$

(f) **(Maximality and surjectivity)** Prove a maximal Φ is surjective if and only if it satisfies the growth condition

$$\lim_{\|x\|\to\infty} \inf \|\Phi(x)\| = +\infty.$$

(Hint: The "only if" direction is Section 8.2, Exercise 16(k) (Monotonicity); for the "if" direction, apply part (e) with $\lambda = 1/n$ for $n = 1, 2, \ldots$, obtaining a sequence (x_n); if this sequence is unbounded, apply maximal monotonicity.)

14. * **(Semidefinite complementarity)** Define $F : \mathbf{S}^n \times \mathbf{S}^n \to \mathbf{S}^n$ by

$$F(U, V) = U + V - (U^2 + V^2)^{1/2}.$$

For any function $G : \mathbf{S}^n \to \mathbf{S}^n$, prove $U \in \mathbf{S}^n$ solves the variational inequality $VI(G, \mathbf{S}^n_+)$ if and only if $F(U, G(U)) = 0$. (Hint: See Section 5.2, Exercise 11.)

Monotonicity via convex analysis

Many important properties of monotone multifunctions can be derived using convex analysis, without using the Brouwer fixed point theorem (8.1.3). The following sequence of exercises illustrates the ideas. Throughout, we consider a monotone multifunction $\Phi : \mathbf{E} \to \mathbf{E}$. The point $(u, v) \in \mathbf{E} \times \mathbf{E}$ is *monotonically related* to Φ if $\langle x - u, y - v \rangle \geq 0$ whenever $y \in \Phi(x)$: in other words, appending this point to the graph of Φ does not destroy monotonicity. Our main aim is to prove a central case of the *Debrunner-Flor extension theorem* [59]. The full theorem states that if Φ has range contained in a nonempty compact convex set $C \subset \mathbf{E}$, and the function $f : C \to \mathbf{E}$ is continuous, then there is a point $c \in C$ such that the point $(f(c), c)$ is monotonically related to Φ. For an accessible derivation of this result from Brouwer's theorem, see [154]: the two results are in fact equivalent (see Exercise 19).

We call a convex function $\mathcal{H} : \mathbf{E} \times \mathbf{E} \to (\infty, +\infty]$ *representative* for Φ if all points $x, y \in \mathbf{E}$ satisfy $\mathcal{H}(x, y) \geq \langle x, y \rangle$, with equality if $y \in \Phi(x)$. Following [79], the *Fitzpatrick function* $\mathcal{F}_\Phi : \mathbf{E} \times \mathbf{E} \to [-\infty, +\infty]$ is defined by

$$\mathcal{F}_\Phi(x, y) = \sup\{\langle x, v \rangle + \langle u, y \rangle - \langle u, v \rangle \mid v \in \Phi(u)\},$$

while [171, 150] the *convexified representative* $\mathcal{P}_\Phi : \mathbf{E} \times \mathbf{E} \to [-\infty, +\infty]$ is defined by

$$\mathcal{P}_\Phi(x, y) = \inf \Big\{ \sum_{i=1}^m \lambda_i (x_i, y_i) \,\Big|\, m \in \mathbf{N},\ \lambda \in \mathbf{R}_+^m, \\ \sum_{i=1}^m \lambda_i (x_i, y_i, 1) = (x, y, 1),\ y_i \in \Phi(x_i)\ \forall i \Big\}.$$

These constructions are explored extensively in [30, 43, 172].

15. **(Fitzpatrick representatives)**

    (a) Prove the Fitzpatrick function $\mathcal{F}_\Phi$ is closed and convex.

    (b) Prove $\mathcal{F}_\Phi(x, y) = \langle x, y \rangle$ whenever $y \in \Phi(x)$.

    (c) Prove $\mathcal{F}_\Phi$ is representative providing Φ is maximal.

    (d) Find an example where $\mathcal{F}_\Phi$ is not representative.

16. **(Convexified representatives)** Consider points $x \in \mathbf{E}$ and $y \in \Phi(x)$.

    (a) Prove $\mathcal{P}_\Phi(x, y) \leq \langle x, y \rangle$.

    Now consider any points $u, v \in \mathbf{E}$.

(b) Prove $\mathcal{P}_\Phi(u,v) \geq \langle u,y\rangle + \langle x,v\rangle - \langle x,y\rangle$.

(c) Deduce $\mathcal{P}_\Phi(x,y) = \langle x,y\rangle$.

(d) Deduce $\mathcal{P}_\Phi(x,y) + \mathcal{P}_\Phi(u,v) \geq \langle u,y\rangle + \langle x,v\rangle$.

(e) Prove $\mathcal{P}_\Phi(u,v) \geq \langle u,v\rangle$ if $(u,v) \in \operatorname{conv} G(\Phi)$ and is $+\infty$ otherwise.

(f) Deduce that convexified representatives are indeed both convex and representative.

(g) Prove $\mathcal{P}_\Phi^* = \mathcal{F}_\Phi \leq \mathcal{F}_\Phi^*$.

17. * **(Monotone multifunctions with bounded range)** Suppose that the monotone multifunction $\Phi : \mathbf{E} \to \mathbf{E}$ has bounded range $R(\Phi)$, and let $C = \operatorname{cl}\operatorname{conv} R(\Phi)$. Apply Exercise 16 to prove the following properties.

(a) Prove the convexity of the function $f : \mathbf{E} \to [-\infty, +\infty]$ defined by
$$f(x) = \inf\{\mathcal{P}_\Phi(x,y) \mid y \in C\}.$$

(b) Prove that the function $g = \inf_{y\in C} \langle \cdot, y\rangle$ is a continuous concave minorant of f.

(c) Apply the Sandwich theorem (Exercise 13 in Section 3.3) to deduce the existence of an affine function α satisfying $f \geq \alpha \geq g$.

(d) Prove that the point $(0, \nabla\alpha)$ is monotonically related to Φ.

(e) Prove $\nabla\alpha \in C$.

(f) Given any point $x \in \mathbf{E}$, show that Φ is contained in a monotone multifunction Φ' with x in its domain and $R(\Phi') \subset C$.

(g) Give an alternative proof of part (f) using the Debrunner-Flor extension theorem.

(h) Extend part (f) to monotone multifunctions with unbounded ranges, by assuming that the point x lies in the set $\operatorname{int}\operatorname{dom} f - \operatorname{dom}\delta_C^*$. Express this condition explicitly in terms of C and the domain of Φ.

18. ** **(Maximal monotone extension)** Suppose the monotone multifunction $\Phi : \mathbf{E} \to \mathbf{E}$ has bounded range $R(\Phi)$.

(a) Use Exercise 17 and Zorn's lemma to prove that Φ is contained in a monotone multifunction Φ' with domain $\mathbf{E}$ and range contained in $\operatorname{cl}\operatorname{conv} R(\Phi)$.

(b) Deduce that if Φ is in fact maximal monotone, then its domain is $\mathbf{E}$.

(c) Using Exercise 16 (Local boundedness) in Section 8.2, prove that the multifunction $\Phi'' : \mathbf{E} \to \mathbf{E}$ defined by

$$\Phi''(x) = \bigcap_{\epsilon>0} \operatorname{cl}\operatorname{conv}\Phi'(x+\epsilon B)$$

is both monotone and a cusco.

(d) Prove that a monotone multifunction is a cusco on the interior of its domain if and only if it is maximal monotone.

(e) Deduce that Φ is contained in a maximal monotone multifunction with domain $\mathbf{E}$ and range contained in $\operatorname{cl}\operatorname{conv} R(\Phi)$.

(f) Apply part (e) to Φ^{-1} to deduce a parallel result.

19. ** **(Brouwer via Debrunner-Flor)** Consider a nonempty compact convex set $D \subset \operatorname{int} B$ and a continuous self map $g : D \to D$. By applying the Debrunner-Flor extension theorem in the case where $C = B$, the multifunction Φ is the identity map, and $f = g \circ P_D$ (where P_D is the nearest point projection), prove that g has a fixed point.

In similar fashion one may establish that the sum of two maximal monotone multifunctions S and T is maximal assuming the condition $0 \in \operatorname{core}(\operatorname{dom} T - \operatorname{dom} S)$. One commences with the *Fitzpatrick inequality* that

$$\mathcal{F}_T(x, x^*) + \mathcal{F}_S(x, -x^*) \geq 0,$$

for all x, x^* in $\mathbf{E}$. This and many other applications of representative functions are described in [30].