11

Introduction to Advanced Dynamics

11.1. Introduction

At the age of 19, recognized for his extraordinary mathematical abilities, Joseph Louis de Lagrange (1736–1813) was appointed professor of geometry and mechanics at the Royal Artillery School at Turin, Italy, his birthplace. Here he developed his method of variations, invented earlier by Euler (in 1744) who later named it the calculus of variations. Lagrange left Turin in 1766 to become director of the Berlin Academy of Sciences until 1787 when, at the invitation of King Louis XVI of France, he was appointed to the Paris Academy of Sciences.[††] Shortly thereafter his most celebrated work, *Mécanique Analytique*, appeared in 1788, nearly a century after the appearance of Newton's *Principia*. Therein, Lagrange sets down an energy based approach for *dynamics—the analysis of motion*.

Inspired and strongly influenced by his senior contemporaries D'Alembert (1717–1785) and Euler (1707–1783), Lagrange linked the classical concepts and postulates of others in an invariant formulation of the equations of classical mechanics, now known as *Lagrange's equations*. The method begins with construction of a single scalar function of the total kinetic and potential energies, called the Lagrangian function, and for general dynamical systems it employs the method of virtual work to identify the nonconservative generalized forces. Although Lagrange's analytical mechanics embraces the theories of Newton and Euler, as it must, but in terms of work and energy, we shall see that it does not explicitly identify specific concepts of momentum, moment of momentum, center of mass, and rigidity. With these classical concepts in hand, Lagrange's method provides a systematic scheme for the formulation of the equations of motion and

[††] Lagrange's life and times are sketched in the translators' "Introduction" in Lagrange's *Analytical Mechanics* cited in the chapter References. See also Truesdell's *Essays*.

their first integrals for any multidegree of freedom dynamical system consisting of any number of particles and rigid bodies.

Although a detailed study of Lagrange's analytical dynamics is beyond the scope of this Introduction, still, we can accomplish a great deal. Our objective is to derive Lagrange's equations of motion for all sorts of classical (holonomic) dynamical systems, both conservative and nonconservative, consisting of a particle, a system of particles, a rigid body, several connected rigid bodies, in fact, any combination of these objects. First, various kinds of system constraints are discussed. Then, Lagrange's equations of motion for a particle are formulated and illustrated in some applications. Their straightforward extension for a system of particles follows. Hamilton's principle of stationary action, a method based upon the calculus of variations, is introduced, and Lagrange's equations are then derived from this principle without mention of any specific dynamical system. A number of examples are exhibited along the way.

11.2. Generalized Coordinates, Degrees of Freedom, and Constraints

We begin with a description of degrees of freedom and system constraints. Recall from Chapter 2 that the number of degrees of freedom of a dynamical system is the number of *independent* coordinates required to uniquely specify the location and orientation of all material points of the system relative to an assigned reference frame. A rigid disk free to move in the xy-plane, for example, has three degrees of freedom; two coordinates (x_p, y_p) specify the location of any disk point P and one coordinate θ provides the angle of rotation of the disk about its normal axis, say. If P is constrained to move on a specified path $y_p = f(x_p)$, only two coordinates x_p and θ are independent and hence the disk now has two degrees of freedom. In general, if there are c independent kinematical constraint equations relating the n coordinates, there remain $n - c = d$ independent coordinates, i.e. degrees of freedom.

The number of degrees of freedom is strictly a property of the system; it is independent of the particular coordinates used to uniquely specify the configuration of the system. Imagine, for example, that the dynamical system requires m Cartesian coordinates x_k, $k = 1, 2, \ldots, m$, to uniquely specify its configuration in a Cartesian frame ψ at an instant t, and these m coordinates are related by r kinematical equations of constraint. Then $d = m - r$. Suppose, on the other hand, that the x_k coordinates are related to another set of p *generalized coordinates* q_l, $l = 1, 2, \ldots, p$, that uniquely specify the system configuration in ψ at the instant t so that, in general, $x_k = x_k(q_1, q_2, \ldots, q_p, t) \equiv x_k(q_l, t)$, say. These equations describe the transformation from the set of ordinary coordinates x_k to the set of generalized coordinates q_l for a fixed t. If the new coordinates q_k are related by s kinematical equations of constraint, then, regardless of the particular set of coordinates used to specify the configuration in ψ at time t, $d = m - r = p - s$, the number of degrees of freedom of the system is the same. It may not be possible, however, to solve the constraint equations and thus eliminate the dependent variables. This is discussed next.

11.2.1. Holonomic Constraints

Kinematical constraints are classified as either holonomic or nonholonomic. First consider a system described by p generalized coordinates q_l related by s algebraic, kinematical constraint equations

$$f_j(q_l, t) \equiv f_j(q_1, q_2, \ldots, q_p, t) = 0, \qquad j = 1, 2, \ldots, s < p. \tag{11.1}$$

Kinematical constraints of this kind, or any that can be recast in this form as discussed later, are called *holonomic constraints.* In principle, these constraint equations for s coordinates can be solved in terms of the remaining $p - s = d$ coordinates and the time t, thus retaining only as many generalized coordinates as there are degrees of freedom. The elimination of s variables among the p generalized coordinates by use of (11.1), however, generally is quite awkward. In most cases, we try to find a set of generalized coordinates that describe the constrained system without our actually having to use the constraint equations.

For illustration, consider a simple pendulum whose bob is supported by a rigid rod of length ℓ and negligible mass, and pivoted at a support O. The general configuration of the bob is specified by the three Cartesian coordinates (x, y, z) in a reference frame $\psi = \{O; \mathbf{i}_k\}$. Now, suppose that the motion is confined to the vertical xy-plane in ψ. The two obvious constraint equations of the form (11.1) are $f_1(x, y, z, t) = z = 0$ and $f_2(x, y, z, t) = x^2(t) + y^2(t) - \ell^2 = 0$ relating x and y, and therefore this system has $3 - 2 = 1$ degree of freedom. On the other hand, for the plane motion of the pendulum, we also may write $x = \ell \cos q$, $y = \ell \sin q$, relating each of the Cartesian coordinates to the single generalized coordinate angle $q \equiv \theta(t)$ which completely describes the single degree of freedom motion of the pendulum without violating the constraints and without our having actually to use the constraint equations.

The aforementioned holonomic constraints do not depend *explicitly* on the time t, so these are further classified as *scleronomic constraints.* Holonomic constraints that depend explicitly on time are called *rheonomic constraints.* A holonomic dynamical system, therefore, is respectively classified as *rheonomic* if one or more constraints are time-dependent, or *scleronomic* when all constraints are time-independent. Suppose that a particle P at $\mathbf{x} = \hat{\mathbf{x}}(X, Y)$ in the frame $\Phi = \{O; \mathbf{I}_k\}$ is constrained by forces to move on an inclined plane of slope b that is moving toward the right with a speed $v(t)$, a prescribed function of t. Then the coordinates of P must satisfy the holonomic constraint $f(X, Y, t) = Y - b(X - \int_0^t v(t)dt) = 0$, and now the position vector of P may be written as an explicit function $\mathbf{x} = \mathbf{x}(X, t)$ of only one independent coordinate and the time. This is an example of a rheonomic constraint of the type (11.1) in which $q_1 = X,\ q_2 = Y$. The holonomic constraint on the motion of a pendulum suspended from a moving support is another example. Sometimes a holonomic equation of constraint may be an inequality $f(q_l, t) \le 0$

restricting the values of the generalized coordinates. We shall not encounter these kinds of bounded constraints in our studies here.

11.2.2. Nonholonomic Constraints

Kinematical constraints that are not of the form (11.1), or cannot be recast in that form, are called *nonholonomic constraints.* These kinds of constraints are expressible only in terms of differentials of the generalized coordinates and time in the form

$$\sum_{k=1}^{p} a_{jk}(q_l, t)dq_k(t) + b_j(q_l, t)dt = 0, \qquad j = 1, 2, \ldots, s < p, \qquad (11.2)$$

where a_{jk} and b_j are certain functions of the p generalized coordinates q_l and the time t. Clearly, nonholonomic constraints are characterized by their being nonintegrable; otherwise, upon integration they would reduce to holonomic constraints of the type (11.1). As a consequence of their nonintegrability, the nonholonomic constraint equations (11.2) cannot be used to reduce the number of generalized coordinates. Therefore, nonholonomic systems always require more coordinates to specify the configuration of the system than there are degrees of freedom.

A circular disk of radius a situated in the vertical plane and rolling without slipping along a curved path $\mathscr{C}$ in the horizontal plane is described by two nonholonomic constraint equations. Suppose the point of rolling contact has Cartesian coordinates (x, y) in the frame $\Phi = \{O; \mathbf{i}_k\}$. Let $d\sigma$ denote the elemental arc length along $\mathscr{C}$, and let ϕ denote the angle between the tangent vector to $\mathscr{C}$ and the x-axis. Then for rolling without slipping, we have $d\sigma = a d\theta$, the elemental arc length traced by the rim of the disk rotating through an angle $d\theta$. Thus, constraint equations of the kind (11.2) describing rolling contact without slip may be written as

$$dx - a\cos\phi d\theta = 0, \qquad dy - a\sin\phi d\theta = 0. \qquad (11.3)$$

These are two independent constraint equations in four coordinates x, y, ϕ, and θ that cannot be used to reduce the number of generalized coordinates; nevertheless, the system has $4 - 2 = 2$ degrees of freedom. Because neither of these equations can be expressed as an exact differential of the form

$$df(x, y, \phi, \theta) = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial \phi}d\phi + \frac{\partial f}{\partial \theta}d\theta = 0, \qquad (11.4)$$

they are not integrable. If this were true, the constraints (11.3) could be recast as a holonomic constraint $f(x, y, \phi, \theta) = 0$ of the type (11.1). The difficulty arises because the value of the angle of rotation θ cannot be specified until the path or the length of the path along which the disk has rolled in reaching the point (x, y) is known. When the path is specified by the additional constraint that the disk roll parallel to the y-axis so that $\phi = \pi/2$ in (11.3), the system becomes holonomic with the integrable constraints $dx = 0$ and $dy = a d\theta = d\sigma$ that yield

$x = x_0,\ y = \sigma = y_0 + a\theta$, where $x_0,\ y_0$ are constants for which the origin O may be chosen so that these vanish.

A kinematical constraint given as a differential relation among the generalized coordinates may be holonomic or nonholonomic. This may prove quite difficult to decide. To assess its type, the relation must be tested for its integrability to determine the existence of integrals of the differential equations (11.2). The case $p = 2$ in (11.2) is exceptional, because a differential relation between two variables is always integrable, though not always exactly in closed form, and therein lies the difficulty. If relation (11.2) for any fixed $s < p$ is integrable, then that differential constraint is holonomic, otherwise not. If any one of the s constraints is not integrable, the system is nonholonomic. To illustrate the test procedure, let us consider a single differential, scleronomic constraint relation among $p = 3$ generalized coordinates (q_1, q_2, q_3); say, $a_{11}dq_1 + a_{12}dq_2 + a_{13}dq_3 = 0$, in which the coefficients $a_{jk} = a_{jk}(q_1, q_2, q_3)$ are certain nonzero, differentiable functions of $q_k(t)$. Then solving this relation for dq_3 and writing $C_k(q_1, q_2, q_3) \equiv -a_{1k}/a_{13}$, $k = 1, 2$, we form the differential relation

$$dq_3 = C_1(q_1, q_2, q_3)dq_1 + C_2(q_1, q_2, q_3)dq_2. \tag{11.5}$$

If there is a holonomic condition relating the three variables so that $q_3 = q_3(q_1, q_2)$, then in accordance with (11.5) we must have $C_1(q_1, q_2, q_3) = \partial q_3/\partial q_1$, $C_2(q_1, q_2, q_3) = \partial q_3/\partial q_2$, and hence the integrability condition $\partial^2 q_3/\partial q_2 \partial q_1 = \partial^2 q_3/\partial q_1 \partial q_2$, that is,

$$\frac{\partial C_1}{\partial q_2} + \frac{\partial C_1}{\partial q_3} C_2 = \frac{\partial C_2}{\partial q_1} + \frac{\partial C_2}{\partial q_3} C_1 \tag{11.6}$$

must be satisfied identically for all values of q_1 and q_2. If this result is an identity, then (11.5) is integrable and hence holonomic; but integration of the constraint to obtain $f(q_1, q_2, q_3)$ may not be apparent. We learn that an integral exists, but it is not revealed. On the other hand, if (11.6) yields a relation $q_3 = q_3(q_1, q_2)$, then we test this relation to see whether or not $\partial q_3/\partial q_1 = C_1(q_1, q_2, q_3)$ and $\partial q_3/\partial q_2 = C_2(q_1, q_2, q_3)$. If these hold, then the relation $q_3 = q_3(q_1, q_2)$ yields the desired holonomic constraint equation $f(q_1, q_2, q_3) = 0$. See Problem 11.1 for an equivalent alternative method.

Exercise 11.1. Show that the following differential, scleronomic constraint is holonomic, and determine its algebraic form (11.1):

$$(\sin^2 y - e^{2x} - z)e^x dx + (z - \sin^2 y - e^x \sin y)\cos y\, dy + e^x dz = 0. \qquad \square$$

This concludes the discussion of kinematical constraints. In this book, only holonomic constraints, mainly of the scleronomic type and usually so evident as to require no special attention, are encountered. The further study of nonholonomic constraints is left for advanced study. See the texts listed in the References.

11.3. Lagrange's Equations of Motion for a Particle

Our immediate objective is to derive the first fundamental form of Lagrange's equations of motion for a particle. Later, however, independent of the specific nature of the dynamical system, it is shown that the same energy based equations of motion hold for far more complex, multidegree of freedom dynamical systems consisting of several particles and rigid bodies. For simplicity, however, let us begin with a particle P with position vector $\mathbf{x}(P, t) = x_k \mathbf{i}_k$ in a Cartesian reference frame $\Phi = \{O; \mathbf{i}_k\}$. The use of rectangular Cartesian coordinates x_k, as we know, is not always convenient; so, a more suitable set of independent generalized coordinates q_k , say, that also may serve to specify uniquely and more naturally the configuration of P in Φ is introduced. Then the coordinates x_k are certain functions of these generalized coordinates q_k and perhaps time t. For example, if cylindrical coordinates $(r, \phi, \hat{z})$ are chosen as the generalized coordinates $q_1 = r,\ q_2 = \phi$, and $q_3 = \hat{z}$ to describe the motion of a particle in a frame ψ that has a specified motion $\zeta(t)$ along the z-axis in Φ, these are related to the regular Cartesian coordinates (x, y, z) in Φ by the coordinate transformation relations

$$x = r\cos\phi, \qquad y = r\sin\phi, \qquad z = \hat{z} + \zeta(t) = z(\hat{z}, t).$$

This typical sort of change of variables may be written as $x_k = x_k(r, \phi, \hat{z}, t) = x_k(q_1, q_2, q_3, t)$; or briefly, $x_k = x_k(q_j, t)$ in which $q_j = q_j(t)$. We thus introduce, more generally,

$$\mathbf{x} = \mathbf{x}(q_j, t), \tag{11.7}$$

in which the q_j are independent generalized coordinates, the actual number of which will depend on the number of degrees of freedom, hence the number of holonomic constraints, if any be imposed.

11.3.1. Two Useful Identities

We now derive two important identities relating partial derivatives that arise in the formulation of Lagrange's equations. Since $q_j = q_j(t)$ are functions of time t, differentiation of (11.7) with respect to time gives the velocity vector

$$\dot{\mathbf{x}} = \frac{\partial \mathbf{x}}{\partial q_j}\dot{q}_j + \frac{\partial \mathbf{x}}{\partial t}. \tag{11.8}$$

Here and throughout this chapter the summation rule (see Chapter 3) applies to twice repeated indices, unless explicitly noted otherwise. The quantities $\dot{q}_j$ are named the *generalized velocity components*, or briefly the *generalized velocities*. In view of (11.8), the particle's velocity vector is a function of the independent variables q_k, $\dot{q}_k$, and t; namely, $\dot{\mathbf{x}} = \dot{\mathbf{x}}(q_k, \dot{q}_k, t)$. Notice, however, that $\partial\mathbf{x}/\partial q_j$ and $\partial\mathbf{x}/\partial t$ are independent of the generalized velocities $\dot{q}_j$. Consequently, recalling the definition (3.2) of the Kronecker delta and noting that $\partial\dot{q}_j/\partial\dot{q}_k = \delta_{jk}$, we obtain from (11.8) our first identity, the *rule of cancellation of the dots*:

$$\frac{\partial \dot{\mathbf{x}}}{\partial \dot{q}_k} = \frac{\partial \mathbf{x}}{\partial q_k}. \tag{11.9}$$

Further, since $\partial \mathbf{x}/\partial q_k$ are functions of q_k and t alone, we have

$$\frac{d}{dt}\left(\frac{\partial \mathbf{x}}{\partial q_k}\right) = \frac{\partial^2 \mathbf{x}}{\partial q_j \partial q_k}\dot{q}_j + \frac{\partial^2 \mathbf{x}}{\partial t \partial q_k}.$$

By (11.8), however,

$$\frac{\partial \dot{\mathbf{x}}}{\partial q_k} = \frac{\partial^2 \mathbf{x}}{\partial q_k \partial q_j}\dot{q}_j + \frac{\partial^2 \mathbf{x}}{\partial q_k \partial t}.$$

We shall require that $\mathbf{x}(q_k, t)$ has continuous second partial derivatives with respect to q_k and t in the domain considered. Then the last two expressions are identical, and hence follows our second identity, the *rule for interchange of derivatives*:

$$\frac{d}{dt}\left(\frac{\partial \mathbf{x}}{\partial q_k}\right) = \frac{\partial \dot{\mathbf{x}}}{\partial q_k} = \frac{\partial}{\partial q_k}\left(\frac{d\mathbf{x}}{dt}\right). \tag{11.10}$$

11.3.2. First Fundamental Form of the Lagrange Equations of Motion

We are now prepared to derive Lagrange's equations for a particle, based on its kinetic energy. In the Lagrangian theory, however, it is customary to denote the kinetic energy by T so that in an arbitrary motion of the particle

$$T \equiv K = \frac{1}{2}m\dot{\mathbf{x}} \cdot \dot{\mathbf{x}}. \tag{11.11}$$

Because $\dot{\mathbf{x}} = \dot{\mathbf{x}}(q_k, \dot{q}_k, t)$ in (11.8), $T = T(q_k, \dot{q}_k, t)$ as well. Hence, from (11.11),

$$\frac{\partial T}{\partial q_k} = m\dot{\mathbf{x}} \cdot \frac{\partial \dot{\mathbf{x}}}{\partial q_k}, \qquad \frac{\partial T}{\partial \dot{q}_k} = m\dot{\mathbf{x}} \cdot \frac{\partial \dot{\mathbf{x}}}{\partial \dot{q}_k} = m\dot{\mathbf{x}} \cdot \frac{\partial \mathbf{x}}{\partial q_k}, \tag{11.12}$$

wherein the rule of cancellation of the dots (11.9) is introduced. Further, differentiation of the second equation in (11.12) with respect to t and use of rule (11.10) for the interchange of derivatives yields

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) = m\ddot{\mathbf{x}} \cdot \frac{\partial \mathbf{x}}{\partial q_k} + m\dot{\mathbf{x}} \cdot \frac{\partial \dot{\mathbf{x}}}{\partial q_k}.$$

Therefore, use of this relation and the first equation in (11.12) yields the result

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} = m\ddot{\mathbf{x}} \cdot \frac{\partial \mathbf{x}}{\partial q_k}. \tag{11.13}$$

Thus far, no laws of motion have been imposed; so (11.13), except for the presence of the mass m, is essentially a purely kinematical result. Now introduce on the right-hand side of (11.13) the Newton–Euler equation of motion $\mathbf{F} = m\ddot{\mathbf{x}}$ for a particle of mass m acted upon by a force $\mathbf{F} = \mathbf{F}(q_k, \dot{q}_k, t)$, where $\mathbf{x}$ is its position vector in an *inertial* reference frame $\Phi = \{O; \mathbf{i}_k\}$, and define the *generalized forces* $Q_k = Q_k(q_k, \dot{q}_k, t)$ by the relation

$$Q_k \equiv \mathbf{F} \cdot \frac{\partial \mathbf{x}}{\partial q_k}. \tag{11.14}$$

Then (11.13) yields the *first fundamental form of Lagrange's equations*,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} = Q_k, \tag{11.15}$$

for the unconstrained ($k = 1, 2, 3$) *or at most holonomic constrained* ($1 \leq k < 3$) *motion of a particle.* The force $\mathbf{F}(q_j, \dot{q}_j, t)$, hence the generalized forces $Q_k = Q_k(q_j, \dot{q}_j, t)$, includes all sorts of applied forces, such as conservative, nonconservative, time varying driving forces, and forces of constraint.

11.4. The Generalized Forces and Virtual Work

The generalized forces $Q_k(q_k, \dot{q}_k, t)$ may be found from the virtual work done by the total force $\mathbf{F}(q_k, \dot{q}_k, t)$ acting on a particle over a small *virtual* (*imaginary*) *displacement* $\delta\mathbf{x}(q_k)$ corresponding to arbitrary small virtual increments δq_k in the generalized coordinates during which *time is held fixed* and the *applied forces do not change.* The virtual increments δq_k, also called virtual displacements, must respect the kinematic constraints, any moving constraints being momentarily halted with time. By (11.7), the virtual displacement vector $\delta\mathbf{x}$ is given by

$$\delta\mathbf{x} = \frac{\partial \mathbf{x}}{\partial q_k}\delta q_k. \tag{11.16}$$

In the presence of any holonomic constraints (11.1), the virtual displacements must respect the corresponding constraint conditions $\delta f_j = (\partial f_j/\partial q_k)\delta q_k = 0$, valid for both scleronomic and rheonomic systems. This is not the same as the differential of f_j for "real" displacements dq_k for which $df_j = (\partial f_j/\partial q_k)dq_k + (\partial f_j/\partial t)dt = 0$. Here df_j is the infinitesimal change in $f_j(q_k, t)$ when both q_k and t are varied, whereas $\delta f_j(q_k, t)$ is the infinitesimal change in $f_j(q_k, t)$ when only q_k are varied. These are the same only for scleronomic constraints. The nonholonomic constraints given in (11.2) for real displacements are replaced by $\sum_{k=1}^{p} a_{jk}\delta q_k = 0$ for virtual displacements in both scleronomic and rheonomic systems; however, nonholonomic constraints are not encountered in this text.

The *virtual work* $\delta\mathscr{W}$ done by the total force acting on the particle over its virtual displacement $\delta\mathbf{x}$ is defined by

$$\delta\mathscr{W} = \mathbf{F} \cdot \delta\mathbf{x}. \tag{11.17}$$

Substitution of (11.16) and the use of (11.14) yields, equivalently,

$$\delta\mathscr{W} = Q_k\delta q_k. \tag{11.18}$$

This is the virtual work done by the generalized forces frozen in time, i.e. treated as constants, and acting over the generalized virtual displacements that satisfy the constraints. The generalized forces may be found from these results.

If the virtual work done by the total force **F** in a virtual displacement compatible with the constraints vanishes in (11.17), by (11.18), the generalized total forces Q_k also are workless. Moreover, for holonomic systems, the δq_k may be independently chosen, and hence, by (11.18), $Q_k = 0$, the corresponding generalized total forces must vanish. Consider the part **P** of the total force **F** with corresponding generalized forces $Q_k^P = \mathbf{P} \cdot \partial \mathbf{x}/\partial q_k$, defined in accordance with (11.14), which does no virtual work. Then $Q_k^P = 0$. Hence, nontrivial generalized forces arise only from those applied forces that do virtual work.* Consequently, workless forces of constraint contribute nothing to Lagrange's equations (11.15) for the motion of a particle.

Example 11.1. Apply Lagrange's equations (11.15) to derive the equations of unconstrained motion of a particle in cylindrical coordinates.

Solution. The three independent generalized coordinates and generalized velocity components for the unconstrained motion of a particle in terms of cylindrical coordinates are defined by

$$(q_1, q_2, q_3) = (r, \phi, z), \qquad (\dot{q}_1, \dot{q}_2, \dot{q}_3) = (\dot{r}, \dot{\phi}, \dot{z}). \tag{11.19a}$$

It should be noted that the generalized coordinates and velocities are not the respective physical scalar components of either the actual position vector $\mathbf{x} = r\mathbf{e}_r + z\mathbf{e}_z$ or the velocity vector $\mathbf{v} = \dot{r}\mathbf{e}_r + r\dot{\phi}\mathbf{e}_\phi + \dot{z}\mathbf{e}_z$ in cylindrical coordinates. The kinetic energy function in cylindrical coordinates is given by

$$T = \frac{1}{2}m\mathbf{v}\cdot\mathbf{v} = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2 + \dot{z}^2). \tag{11.19b}$$

Hence, with (11.19a) and (11.15) in mind, we use (11.19b) to first derive

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{r}}\right) - \frac{\partial T}{\partial r} &= \frac{d}{dt}(m\dot{r}) - mr\dot{\phi}^2 = m(\ddot{r} - r\dot{\phi}^2), \\
\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\phi}}\right) - \frac{\partial T}{\partial \phi} &= \frac{d}{dt}(mr^2\dot{\phi}) = m\frac{d}{dt}(r^2\dot{\phi}), \\
\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{z}}\right) - \frac{\partial T}{\partial z} &= \frac{d}{dt}(m\dot{z}) = m\ddot{z}.
\end{aligned} \tag{11.19c}$$

To complete the formulation of the equations of motion from the Lagrange equations (11.15), we next consider the generalized forces in (11.15). The virtual work

* We may think of the result (11.18) in the space of the q_ks as a generalized scalar product $\mathbf{Q}^P \cdot \delta\mathbf{q}$ that vanishes if and only if $\mathbf{Q}^P$ is "perpendicular" to $\delta\mathbf{q}$. Because the only vector perpendicular to every vector is the zero vector, $\mathbf{Q}^P \cdot \delta\mathbf{q} = 0$ for all $\delta\mathbf{q} \neq \mathbf{0}$ implies that $\mathbf{Q}^P = \mathbf{0}$ for all workless holonomic constraints.

(11.17) done by the total force $\mathbf{F} = F_r\mathbf{e}_r + F_\phi\mathbf{e}_\phi + F_z\mathbf{e}_z$ in the virtual displacement $\delta\mathbf{x} = \delta r\mathbf{e}_r + r\delta\phi\mathbf{e}_\phi + \delta z\mathbf{e}_z$ is given by

$$\delta\mathscr{W} = F_r\delta r + rF_\phi\delta\phi + F_z\delta z. \tag{11.19d}$$

By (11.18), the virtual work done by the generalized forces $(Q_1, Q_2, Q_3) = (Q_r, Q_\phi, Q_z)$ acting over the generalized virtual displacements $(\delta q_1, \delta q_2, \delta q_3) = (\delta r, \delta\phi, \delta z)$ is

$$\delta\mathscr{W} = Q_r\delta r + Q_\phi\delta\phi + Q_z\delta z. \tag{11.19e}$$

Since there are no constraints, the virtual work relations in (11.19d) and (11.19e) must be equal *for all* arbitrary virtual displacements $\delta r, \delta\phi, \delta z$, and hence the generalized force components are given by

$$Q_r = F_r, \qquad Q_\phi = rF_\phi, \qquad Q_z = F_z. \tag{11.19f}$$

Notice that $[Q_\phi] = [FL]$ has dimensional units of torque. We thus see that the generalized forces need not have dimensional units of force, as do Q_r and Q_z.

Use of the two sets of results (11.19c) and (11.19f) in the Lagrange equations (11.15) now yields the familiar equations of unconstrained motion for a particle in terms of its cylindrical coordinates:

$$m(\ddot{r} - r\dot{\phi}^2) = F_r, \qquad \frac{m}{r}\frac{d}{dt}(r^2\dot{\phi}) = F_\phi, \qquad m\ddot{z} = F_z. \tag{11.19g}$$

These agree with equations (6.4) based on Newton's second law. □

Exercise 11.2. (a) Recall the position vector in cylindrical coordinates and note that $\mathbf{e}_r$ and $\mathbf{e}_\phi$ are known functions of ϕ alone. Apply the definition (11.14) to derive the generalized force components (11.19f). (b) In view of (11.9), the generalized forces (11.14) also may be written as

$$Q_k = \mathbf{F} \cdot \frac{\partial\mathbf{v}}{\partial\dot{q}_k}. \tag{11.20}$$

Use this result to derive the generalized forces in (11.19f). This is the simpler of these two methods, because the basis vectors depend on the generalized coordinates q_k, not the generalized velocities $\dot{q}_k$. □

Exercise 11.3. Apply Lagrange's equations (11.15) to derive the equations of unconstrained motion of a particle in spherical coordinates (r, θ, ϕ). Find the generalized forces (a) by use of the method of virtual work, and (b) by application of both (11.14) and (11.20). Compare the results with (6.5). □

We have seen that the generalized force components corresponding to a workless applied force $\mathbf{P}$, including any workless force of holonomic constraint, must vanish. The same result follows readily from (11.20). To see this, consider a force

$\mathbf{P}$ that does no work in the real motion. This force is perpendicular to the particle's path, and hence to its velocity vector $\mathbf{v} = \mathbf{v}(\dot{q}_k, q_k, t) = v(\dot{q}_k, q_k, t)\mathbf{t}(q_k, t)$, where $\mathbf{t}(q_k, t) = d\mathbf{x}(q_k, t)/ds$ is the unit tangent vector. By (11.20), since $\mathbf{P} \cdot \mathbf{t}(q_k, t) = 0$, the corresponding generalized forces $Q_k^P = \partial v(\dot{q}_j, q_j, t)/\partial \dot{q}_k \mathbf{P} \cdot \mathbf{t}(q_j, t) \equiv 0$. For a specific illustration, let the reader consider the workless force of constraint $\mathbf{P} = T(\theta)\mathbf{n}(\theta)$ that acts on the bob of a simple pendulum having a single degree of freedom with $q_1 = \theta$, the usual angular placement. Note that $\mathbf{v} = l\dot{\theta}\mathbf{t}(\theta)$, and thus confirm that the corresponding generalized force $Q_1^P = Q_\theta^P = 0$. The gravitational part $\mathbf{W}$ of the total force, however, does virtual work $\mathbf{W} \cdot \delta\mathbf{x} = \mathbf{W} \cdot l\delta\theta\mathbf{t} = Q_\theta^W \delta\theta$ from which $Q_\theta^W = -mgl\sin\theta$.

11.5. The Work–Energy Principle for Scleronomic Systems

For holonomic systems of scleronomic type, the first integral of Lagrange's equations with respect to the generalized displacements is the familiar work–energy principle (7.36). To demonstrate this result, let us consider only those dynamical systems for which (11.7) is not an explicit function of t so that $\partial\mathbf{x}/\partial t \equiv \mathbf{0}$ in (11.8); then,

$$\mathbf{x} = \mathbf{x}(q_r(t)), \qquad \dot{\mathbf{x}} = \frac{\partial \mathbf{x}}{\partial q_k}\dot{q}_k. \tag{11.21}$$

This is a special case for which all of the previous results apply.

First, recall (7.21) and introduce the second relation in (11.21) to obtain

$$\mathscr{W} = \int_{\mathscr{C}} \mathbf{F} \cdot d\mathbf{x} = \int_{\mathscr{C}} \mathbf{F} \cdot \frac{\partial \mathbf{x}}{\partial q_k} dq_k, \tag{11.22}$$

in which $\mathbf{F} = \mathbf{F}(\mathbf{x}(q_r))$ and $\mathscr{C}$ is the particle's path. Then use of (11.14) in which $Q_k = Q_k(q_r)$ delivers *the work done by the generalized forces acting over the generalized displacements*:

$$\mathscr{W} = \int_{\mathscr{C}} Q_k dq_k. \tag{11.23}$$

The kinetic energy (11.11) may be cast in terms of the generalized velocities by use of the second expression in (11.21) to obtain

$$T = \frac{1}{2} M_{jk}(q_r)\dot{q}_j\dot{q}_k. \tag{11.24}$$

Remember that repeated indices are summed over their range, and note that $T(q_r, \dot{q}_r)$ is not an explicit function of time, which is generally the case when there are no moving constraints. Hence, (11.24) is a homogeneous function of degree 2 wherein the symmetric coefficient matrix $M_{jk} = M_{kj}$, which need not have physical dimensions of mass, is a function of the generalized coordinates only:

$$M_{jk}(q_r) \equiv m \frac{\partial \mathbf{x}}{\partial q_j} \cdot \frac{\partial \mathbf{x}}{\partial q_k}. \tag{11.25}$$

It thus follows from (11.24) that

$$\frac{\partial T}{\partial \dot{q}_k} \dot{q}_k = 2T. \tag{11.26}$$

Finally, we consider

$$\frac{dT(\dot{q}_r, q_r)}{dt} = \frac{\partial T}{\partial \dot{q}_k} \ddot{q}_k + \frac{\partial T}{\partial q_k} \dot{q}_k = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k} \dot{q}_k\right) - \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) \dot{q}_k + \frac{\partial T}{\partial q_k} \dot{q}_k.$$

Use of (11.26) in the first time derivative on the far right-hand side yields

$$\frac{dT(\dot{q}_r, q_r)}{dt} = \left(\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k}\right) \dot{q}_k. \tag{11.27}$$

This is equivalent to our multiplying the k^{th} Lagrange equation (11.15) by $\dot{q}_k$ and adding the results, as indicated by the right-hand sum of (11.27), to obtain

$$\frac{dT(\dot{q}_r, q_r)}{dt} = Q_k \dot{q}_k. \tag{11.28}$$

Integration of this equation in accord with (11.23) yields *the work–energy principle for scleronomic systems*:

$$\mathscr{W} = \Delta T, \tag{11.29}$$

as a first integral of the Lagrange equations (11.15). Moreover, based on (7.38), this result also provides an expression for the *mechanical power*: $\mathscr{P} = dT/dt = d\mathscr{W}/dt$. Notice that if T were to depend explicitly on time, additional terms would arise in (11.24) and (11.27), and the work–energy equation (11.29) would not hold.

Exercise 11.4. Begin with (11.11) for the kinetic energy of a particle and derive (11.26). □

Exercise 11.5. Begin with (11.7) and (11.8) and show that the kinetic energy for a particle has the general explicit time dependent form

$$T(\dot{q}_r, q_r, t) = A_{ij}(q_r, t)\dot{q}_i \dot{q}_j + B_j(q_r, t)\dot{q}_j + C(q_r, t), \tag{11.30}$$

in which $A_{ij} = A_{ji}$. Identify the coefficient functions and thus establish that for scleronomic systems, $B_j(q_r, t) = 0$, $C(q_r, t) = 0$, and $2A_{ij}(q_r, t) = M_{ij}(q_r)$ defined by (11.25). Clearly, the general form of the time dependent kinetic energy function in generalized coordinates is far more complex than its scleronomic form. □

11.6. Conservative Scleronomic Dynamical Systems

When the total force is conservative, there exists a potential energy function $\hat{V}(\mathbf{x}(q_k)) \equiv V(q_k)$ that depends only on the generalized coordinates such that $\mathbf{F} = -\boldsymbol{\nabla}\hat{V}$. Then, in accordance with (11.14),

$$Q_k = -\boldsymbol{\nabla}\hat{V} \cdot \frac{\partial \mathbf{x}}{\partial q_k} = -\frac{\partial \hat{V}}{\partial \mathbf{x}} \cdot \frac{\partial \mathbf{x}}{\partial q_k},$$

and hence the *conservative generalized forces* are given by

$$Q_k = -\frac{\partial V}{\partial q_k}. \tag{11.31}$$

The same relation holds if a part of the total force is workless and the remaining part is conservative. The generalized forces for the workless part vanish and the conservative part yields (11.31). In all such cases, the dynamical system is called conservative. If the system also is scleronomic, it follows from (11.23) that *the work done by conservative generalized forces of a scleronomic dynamical system is equal to the decrease in the total potential energy*:

$$\mathscr{W} = -\Delta V(q_r). \tag{11.32}$$

This is equivalent to (7.45). Moreover, by (11.29), *the total energy for a conservative, scleronomic dynamical system is constant throughout the motion*:

$$T(\dot{q}_k, q_k) + V(q_k) = E, \ \text{a constant.} \tag{11.33}$$

Exercise 11.6. For conservative forces, the virtual work (11.18) may be expressed as $\delta\mathscr{W} = -\delta V$. Begin with this relation and show conversely that for unconstrained or holonomic systems (11.31) follows. □

11.7. Lagrange's Equations for General Conservative Systems

Let us return to the first fundamental form of Lagrange's equations (11.15) in which $T = T(\dot{q}_r, q_r, t)$ and all the Q_ks are derivable from a potential energy function in accord with (11.31). Now, introduce the *Lagrangian function* $L(\dot{q}_r, q_r, t)$ defined by

$$L(\dot{q}_r, q_r, t) \equiv T(\dot{q}_r, q_r, t) - V(q_r), \tag{11.34}$$

called briefly the *Lagrangian*. The potential energy does not depend on the generalized velocities, so $\partial V/\partial \dot{q}_k \equiv 0$. Therefore, upon substitution of (11.31) in (11.15) and introduction of the Lagrangian (11.34), we obtain the classical form of *Lagrange's equations of motion for a general conservative dynamical system*:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0, \tag{11.35}$$

with $k = 1, 2, 3$ for a single, unconstrained particle.

This result also holds more generally for holonomic systems in which the potential function $V = V(q_r, t)$ may depend on time. The virtual work $\delta \mathscr{W} = -\delta V(q_r, t) = Q_k \delta q_k$ yields $Q_k(q_r, t) = -\partial V(q_r, t)/\partial q_k$. Therefore, with the Lagrangian defined by $L(\dot{q}_r, q_r, t) \equiv T(\dot{q}_r, q_r, t) - V(q_r, t)$, (11.15) again transforms to (11.35). The principle of conservation of energy, however, does not hold for a time dependent potential energy function. See the discussion regarding (7.79).

Thus, to obtain the equations of motion for a general conservative dynamical system, we need only determine the Lagrangian function (11.34) and apply (11.35). Lagrange's method, like the work–energy method, provides no information about the inessential workless forces of constraint.

Example 11.2. Derive Lagrange's equations of motion for the simple pendulum shown in Fig. 6.15, page 138.

Solution. The motion of the pendulum is restricted to the vertical plane and a rigid wire of negligible mass constrains the motion to a circle, so the scleronomic constraints are $z = 0$ and $r = \ell$. The tension in the string is workless, and the weight of the bob is a conservative force for which $V = mg\ell(1 - \cos\theta)$. Hence, this scleronomic system is conservative with one degree of freedom described by $q_1 = \theta$. The kinetic energy of the bob is $T = \frac{1}{2}m\ell^2\dot{\theta}^2$. Therefore, the Lagrangian (11.34) is

$$L = T - V = \frac{1}{2}m\ell^2\dot{\theta}^2 - mg\ell(1 - \cos\theta), \tag{11.36a}$$

and hence with

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{d}{dt}(m\ell^2\dot{\theta}) = m\ell^2\ddot{\theta}, \qquad \frac{\partial L}{\partial \theta} = -mg\ell\sin\theta, \tag{11.36b}$$

the Lagrange equations (11.35) yield the differential equation of motion for the pendulum bob:

$$m\ell^2\ddot{\theta} + mg\ell\sin\theta = 0. \tag{11.36c}$$

The inessential, workless string tension of constraint, an incidental result of the Newton–Euler law in (6.67b), does not explicitly enter the argument. Of course, since only derivatives of V with respect to the generalized coordinates appear in Lagrange's equations (11.35), any constant in the potential energy function also is unimportant. Thus, in the present problem we could have written, for example, $V = -mg\ell\cos\theta$. □

Example 11.3. Derive the equations of motion for a particle of mass m moving in a plane under a central force $\mathbf{F} = -(\mu m/r^2)\mathbf{e}_r$, where μ is a constant and cylindrical coordinates are used.

Solution. The holonomic constraint $z = 0$ is evident, and the velocity vector is $\mathbf{v} = \dot{r}\mathbf{e}_r + r\dot{\phi}\mathbf{e}_\phi$. Hence, the kinetic energy is given by

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2). \tag{11.37a}$$

The central force is conservative with potential energy

$$V = -\mu m/r, \tag{11.37b}$$

as shown in (7.62) with $\mu \equiv MG$. Thus, the Lagrangian (11.34) for this conservative scleronomic dynamical system is

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) + \frac{\mu m}{r}. \tag{11.37c}$$

The generalized coordinates are $(q_1, q_2) = (r, \phi)$, the generalized velocity components are $(\dot{q}_1, \dot{q}_2) = (\dot{r}, \dot{\phi})$, and with (11.35) in mind we use (11.37c) to obtain

$$\frac{\partial L}{\partial \dot{r}} = m\dot{r}, \qquad \frac{\partial L}{\partial \dot{\phi}} = mr^2\dot{\phi}, \qquad \frac{\partial L}{\partial r} = mr\dot{\phi}^2 - \frac{\mu m}{r^2}, \qquad \frac{\partial L}{\partial \phi} = 0. \tag{11.37d}$$

Lagrange's equations (11.35) thus yield the equations of motion for this conservative system with two degrees of freedom:

$$\ddot{r} - r\dot{\phi}^2 + \frac{\mu}{r^2} = 0, \qquad \frac{d}{dt}(r^2\dot{\phi}) = 0. \tag{11.37e}$$

□

11.8. Second Fundamental Form of Lagrange's Equations

Suppose that some of the generalized forces are conservative with a total potential energy $V(q_k)$. Let the remaining generalized forces that are not workless be denoted by $Q_k^N = Q_k^N(\dot{q}_r, q_r, t)$; these are called *nonconservative generalized forces*. The dynamical system, in this case, is called *nonconservative*. In the absence of any constraints, with $Q_k = -\partial V/\partial q_k + Q_k^N$, and use of (11.34), (11.15) may be rewritten to obtain *the second fundamental form of Lagrange's equations*:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = Q_k^N. \tag{11.38}$$

If it happens that the nature of a force is uncertain, the ambiguous force may be treated as a "nonconservative force" in the formulation of (11.38). Moreover, generalized forces of rheonomic constraints that are specified functions of time alone do no virtual work; for, with time held fixed, the corresponding virtual displacements are zero. Hence, these nonconservative forces do not enter the Lagrange equations of motion. On the other hand, these same nonconservative forces generally do work in the real motion of the system. This is illustrated in a subsequent example.

Example 11.4. Derive the equation of motion for a particle P that falls from rest in a Stokes medium.

Solution. The kinetic energy of P is $T = \frac{1}{2}m\dot{y}^2$, its gravitational potential energy is $V = -mgy$, and the nonconservative Stokes force is $\mathbf{F} = -c\mathbf{v} = -c\dot{y}\mathbf{j}$, where $\mathbf{j}$ is the downward direction of the motion. Let the reader apply the method of virtual work to find Q_1^N. Here we consider (11.20). Accordingly, with $q_1 = y$, the nonconservative generalized force in (11.38) is $Q_1^N = \mathbf{F}\cdot\partial\mathbf{v}/\partial\dot{q}_1 = -c\dot{y}\mathbf{j}\cdot\mathbf{j} = -c\dot{y}$. Now form the Lagrangian $L = T - V = \frac{1}{2}m\dot{y}^2 + mgy$, and apply (11.38) to obtain the equation of motion $m\ddot{y} - mg = -c\dot{y}$. That is, with $v = \dot{y}$ and $\nu = c/m$, we recover (6.34a): $dv/dt = g - \nu v$. □

Example 11.5. (a) Derive the Lagrange equations of motion for a heavy bead of mass m that slides freely in a smooth circular tube of radius a, as the tube spins with constant angular speed $\dot{\phi} = \omega$ about its fixed vertical axis, as shown in the diagram for Problem 6.66. Obtain the first integral of the equation of motion, and find the tangential constraint force normal to the plane of the tube. (b) Relax the rheonomic constraint, treat $\phi(t)$ as an independent generalized coordinate, and derive the Lagrange equations of motion for the bead.

Solution of (a). Introduce the spherical coordinates (a, ϕ, θ) of the bead in its constrained motion referred to a moving, spherical reference frame $\psi = \{O; \mathbf{e}_r, \mathbf{e}_\phi, \mathbf{e}_\theta\}$ in which $\mathbf{e}_k$ are in the directions of their increasing coordinates, and note that the angle between $\mathbf{k}$ and $\mathbf{e}_r$ in the problem diagram is $\pi - \theta$. The workless scleronomic constraint is $r = a$, constant, and the working rheonomic constraint is $\dot{\phi} = \omega$, a constant, that is, $\phi = \phi_0 + \omega t$. Notice, alternatively, that a rheonomic constraint $\dot{\phi} = \omega(t)$, a specified function of t, also is holonomic with $\phi = \phi_0 + \int_{t_0}^{t}\omega(t)dt$. In either instance, ϕ is a specified function of time; it is not a generalized coordinate. The radial component of the nonconservative force $\mathbf{F} = -N\mathbf{e}_r + R\mathbf{e}_\phi$ exerted by the tube on the mass m is workless, but due to the moving tube constraint the component R perpendicular to the plane of the tube is not, so the system is nonconservative. Nevertheless, the *virtual work* done by this force in the virtual displacement $\delta\mathbf{x} = a\delta\theta\mathbf{e}_\theta + a\sin\theta\delta\phi\mathbf{e}_\phi$ compatible with the constraint $\delta\phi = \omega\delta t \equiv 0$ vanishes, $\delta\mathscr{W} = \mathbf{F}\cdot\delta\mathbf{x} = Ra\sin\theta\delta\phi = 0$, because the moving constraint in the virtual displacement is frozen. The bead has one degree of freedom described by $q_1 = \theta$ with the corresponding generalized force $Q_\theta^N = 0$.

The gravitational potential energy of m is given by

$$V = mga(1 - \cos\theta). \tag{11.39a}$$

The absolute velocity of the bead is $\mathbf{v} = a\dot{\phi}\sin\theta\mathbf{e}_\phi + a\dot{\theta}\mathbf{e}_\theta$, and hence its total kinetic energy is

$$T = \frac{1}{2}ma^2(\dot{\phi}^2\sin^2\theta + \dot{\theta}^2). \tag{11.39b}$$

Now introduce $\dot{\phi} = \omega$, a constant, form the Lagrangian function (11.34),

$$L = \frac{1}{2}ma^2(\omega^2\sin^2\theta + \dot{\theta}^2) - mga(1 - \cos\theta), \tag{11.39c}$$

and thus derive

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = ma^2\ddot{\theta}, \qquad \frac{\partial L}{\partial \theta} = ma^2\omega^2\sin\theta\cos\theta - mga\sin\theta. \tag{11.39d}$$

Collecting the results in (11.38), we reach the Lagrange equation of motion for the bead:

$$\ddot{\theta} + \sin\theta\left(\frac{g}{a} - \omega^2\cos\theta\right) = 0. \tag{11.39e}$$

Notice that the result is independent of the mass m.

Equation (11.39e) written as $\ddot{\theta} = \dot{\theta}d\dot{\theta}/d\theta = f(\theta)$ has an easy first integral given by

$$\dot{\theta}^2 = \dot{\theta}_0^2 + \omega^2(\sin^2\theta - \sin^2\theta_0) + \frac{2g}{a}(\cos\theta - \cos\theta_0), \tag{11.39f}$$

in which $\dot{\theta}_0 = \dot{\theta}(0)$ and $\theta_0 = \theta(0)$ denote the initial values. For small angular hoop speeds the term in ω^2 may be neglected to obtain from (11.39e) and (11.39f) the equation of motion and its first integral for the finite amplitude oscillations of the bead as an equivalent simple pendulum.

The constraint force R is not determined; it is inconsequential to the determination of the general motion of the bead by Lagrange's method. Nevertheless, we can find R easily by writing the moment of momentum equation about the z-axis. The bead is at the instantaneous distance $a\sin\theta$ from this axis, and its momentum in the direction perpendicular to the plane of the tube is $ma\omega\sin\theta$. Therefore, the moment of momentum of the bead about the axis of rotation is $h_z = ma^2\omega\sin^2\theta$. The constraint driving torque relation about the spin axis is given by $\dot{h}_z = Ra\sin\theta$; and hence

$$R = 2ma\omega\dot{\theta}\cos\theta. \tag{11.39g}$$

Solution of (b). Now consider a different situation in which we ignore the rheonomic constraint and treat $\phi(t)$ as an independent variable. The bead now has two degrees of freedom described by $(q_1, q_2) = (\theta, \phi)$ with the corresponding nonconservative generalized forces (Q_θ^N, Q_ϕ^N). The virtual work done by these forces is given by $\delta \mathscr{W} = \mathbf{F} \cdot \delta \mathbf{x} = Ra \sin\theta \delta\phi = Q_\theta^N \delta\theta + Q_\phi^N \delta\phi$ for all $\delta\theta$ and $\delta\phi$, and hence

$$Q_\theta^N = 0, \qquad Q_\phi^N = Ra \sin\theta. \tag{11.39h}$$

The Lagrangian $L = \frac{1}{2} ma^2(\dot{\phi}^2 \sin^2\theta + \dot{\theta}^2) - mga(1 - \cos\theta)$ is the same as (11.39c) in which $\omega = \dot{\phi}$. In addition to (11.39d), we now have

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = ma^2 \ddot{\phi} \sin^2\theta + 2ma^2 \dot{\phi}\dot{\theta} \sin\theta \cos\theta, \qquad \frac{\partial L}{\partial \phi} = 0. \tag{11.39i}$$

Assembling the results (11.39d), (11.39h), and (11.39i) in (11.38), we obtain the θ-equation (11.39e), as before, and the new ϕ-equation:

$$ma\ddot{\phi} \sin\theta + 2ma\dot{\phi}\dot{\theta} \cos\theta = R. \tag{11.39j}$$

If $R(\dot{\theta}, \dot{\phi}, \theta, \phi, t)$ is some specified function, (11.39j) is a nonlinear differential equation for $\phi(t)$ coupled with (11.39e) in which $\omega = \dot{\phi}(t)$. On the other hand, for a specified function $\phi(t)$, and with $\theta(t)$ determined by (11.39e), (11.39j) is an equation that determines $R(t)$. In particular, for $\dot{\phi} = \omega$, a constant, (11.39j) gives $R = 2ma\omega\dot{\theta} \cos\theta$, which is the same nonconservative, rheonomic constraint force obtained in (11.39g). The original Lagrange method eliminates the need to determine the inconsequential working rheonomic constraint force, which may be found by other methods, if needed. □

Exercise 11.7. Apply Lagrange's equations (11.38) to derive the equation of motion for the forced vibration of the system shown in Fig. 6.20, page 152, for a linear viscous damper and a linear elastic spring. □

11.9. Lagrange's Equations for a System of Particles

Lagrange's equations for a single particle are equivalent to the Newton–Euler equations of motion; they contain no new principles. Unlike the Newton–Euler approach, however, the Lagrangian method never involves workless forces of holonomic constraint, the sometimes laborious calculation of accelerations is avoided, and for conservative systems the equations of motion are readily derived from a single scalar Lagrangian function. A bit more effort is required to determine nonconservative generalized forces, but the procedure is straightforward. In all, Lagrange's method often is easier to apply because only generalized coordinates and velocities are involved. On the other hand, the physics of the analysis is not always apparent. Subsequent developments, however, will shed light on the physical interpretation of certain terms that arise in the Lagrange formulation.

The aforementioned advantages are significantly magnified in applications of Lagrange's method to more complex systems. Recall that the equation of motion (5.41) for a system of particles "determines" only the motion of the center of mass. Finding the motions of the individual particles requires formulation of a vector equation of motion (5.39) for each particle, so it is necessary to include all of the forces that act on each particle separately, including mutual forces exerted by all of the other particles, and to introduce all of their accelerations. This procedure, even for simple problems, is cumbersome. The Lagrangian approach avoids these difficulties.

So far, the Lagrangian theory is strictly valid only for a single particle. We shall see that precisely the same equations and analytical structure emerge for a system of N particles. Without constraints, this system has $3N$ degrees of freedom, and hence $3N$ generalized coordinates are required to describe its configuration. When the constraints are *holonomic*, some of the generalized coordinates may be eliminated. In this case the number of degrees of freedom n is fewer than $3N$, and the remaining generalized coordinates are independent variables. In the following description, all functions of the n independent generalized variables $q_1, q_2, \ldots, q_n$ and their time derivatives are abbreviated by use of q_r and $\dot{q}_r$ alone, as done previously for a single particle for which $n \leq 3$. For example, $f(\dot{q}_r, q_r, t) \equiv f(\dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n, q_1, q_2, \ldots, q_n, t)$.

Consider a system of N particles. The position vector (11.7) for the i^{th} particle is written as $\mathbf{x}_i = \mathbf{x}_i(q_r, t)$ in frame $\Phi = \{O; \mathbf{i}_k\}$, $\dot{\mathbf{x}}_i = (\partial \mathbf{x}_i / \partial q_k)\dot{q}_k + \partial \mathbf{x}_i / \partial t$ coincides with (11.8), and corresponding rules of the form (11.9) and (11.10) hold as well. Here we suppose that the functions $\mathbf{x}_i(q_r, t)$ have continuous second partial derivatives in the domain of interest so that $\partial^2 \mathbf{x}_i / \partial q_j \partial q_k = \partial^2 \mathbf{x}_i / \partial q_k \partial q_j$ and $\partial^2 \mathbf{x}_i / \partial q_j \partial t = \partial^2 \mathbf{x}_i / \partial t \partial q_j$. The total kinetic energy is defined by $T(\dot{q}_r, q_r, t) \equiv \sum_{i=1}^{N} \frac{1}{2} m_i \dot{\mathbf{x}}_i \cdot \dot{\mathbf{x}}_i$, in accordance with (8.50). Therefore, upon retracing the steps starting from (11.11), introducing the equation of motion $\mathbf{F}_i = m_i \ddot{\mathbf{x}}_i$ (no sum) for the total force on the i^{th} particle, and noting that $Q_k(\dot{q}_r, q_r, t) \equiv \sum_{i=1}^{N} \mathbf{F}_i(\dot{\mathbf{x}}_j, \mathbf{x}_j, t) \cdot \partial \mathbf{x}_i(q_r, t)/\partial q_k$, the reader will find that Lagrange's first fundamental form (11.15) holds for a general holonomic system of N particles having $k = n$ degrees of freedom. Clearly, collinear, mutual internal forces between pairs of particles of the system contribute nothing to the total force and they are workless. And workless forces of holonomic constraint of any sort contribute nothing to the generalized forces, so these forces of constraint do not enter Lagrange's equations for a system of particles.

For scleronomic systems, we can say more. In this case, the total kinetic energy is given by (11.24) in which the symmetric matrix $M_{jk}(q_r) \equiv \sum_{i=1}^{N} m_i \partial \mathbf{x}_i / \partial q_j \cdot \partial \mathbf{x}_i / \partial q_k$, with $j, k = 1, 2, \ldots, n \leq 3N$ for a system having n degrees of freedom, and the work–energy principle (11.29) thus holds for a system of particles, as shown differently in (8.60). For conservative forces $\mathbf{F}_i(\mathbf{x}_j(q_k)) = -\partial V_i / \partial \mathbf{x}_i$ (no sum), we recall (11.31) in which $V(q_r) = \sum_{i=1}^{N} V_i(\mathbf{x}_j(q_k))$ is the total potential

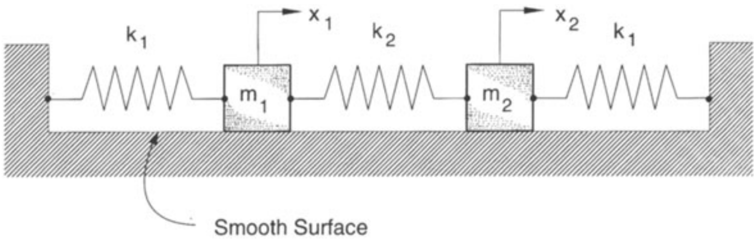


Figure 11.1. A conservative holonomic system of two particles.

energy derived from all of the conservative forces that act on the separate particles, both external and internal as described in (8.84). It is not necessary, however, to elaborate on these details in the construction of the total potential energy function. It thus follows from (11.29) and (11.32) that the principle of conservation of energy (11.33) holds for a system of particles under scleronomic constraints, which is a more precise description of (8.86).

The Lagrangian (11.34) for a general holonomic conservative system is now defined as the difference of the total kinetic and potential energies. It is readily seen that the second fundamental form of Lagrange's equations (11.38) holds for general holonomic constraints. We now explore an application of Lagrange's equations for a conservative system of two mass points and study the problem solution in a special case.

Example 11.6. (i) Derive the equations for the uniaxial motion of the spring-mass system shown in Fig. 11.1. The supporting surface is smooth and all springs are linearly elastic and unstretched initially. (ii) Determine the motion of the system for the special symmetric case when $m_1 = m_2 = m$ and $k_1 = k_2 = k$.

Solution of (i). The holonomic constraints for the uniaxial motion are evident. Let each mass m_1 and m_2 be displaced uniaxially an amount x_1 and x_2, respectively. The system has two degrees of freedom with independent generalized coordinates $(q_1, q_2) = (x_1, x_2)$. The total kinetic energy for the system of particles is

$$T = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 . \tag{11.40a}$$

Notice that this has the form (11.24) in which $\left[M_{jk}\right]$ =diag$[m_1, m_2]$ is a diagonal matrix . The spring forces are conservative, and all other forces are workless. In consequence, the system is conservative with total potential energy

$$V = \frac{1}{2}k_1x_1^2 + \frac{1}{2}k_2\,(x_2 - x_1)^2 + \frac{1}{2}k_1x_2^2. \tag{11.40b}$$

Therefore, the Lagrangian (11.34) is given by

$$L = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 - \frac{1}{2}k_1\left(x_1^2 + x_2^2\right) - \frac{1}{2}k_2(x_2 - x_1)^2, \tag{11.40c}$$

and hence

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) = m_1\ddot{x}_1, \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right) = m_2\,\ddot{x}_2, \tag{11.40d}$$

$$\frac{\partial L}{\partial x_1} = -k_1x_1 + k_2(x_2 - x_1), \qquad \frac{\partial L}{\partial x_2} = -k_1x_2 - k_2(x_2 - x_1). \tag{11.40e}$$

Use of (11.40d) and (11.40e) in (11.35) for $k = 1, 2$ delivers two coupled, ordinary, linear differential equations of motion for the conservative system:

$$m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2x_2 = 0, \qquad m_2\ddot{x}_2 + (k_1 + k_2)x_2 - k_2x_1 = 0. \tag{11.40f}$$

Solution of (ii). The general solution of this coupled system of equations may be readily obtained; however, to simplify the analysis and preserve the essential methodology, the motion will be determined for the special symmetrical case when $m_1 = m_2 = m$ and $k_1 = k_2 = k$. Then with $p^2 = k/m$, the equations in (11.40f) simplify to

$$\ddot{x}_1 + p^2\,(2x_1 - x_2) = 0, \qquad \ddot{x}_2 + p^2\,(2x_2 - x_1) = 0. \tag{11.40g}$$

The typical procedure for solving this class of problems is to assume a trial solution for x_1 and x_2 having the *same circular frequency* α *and initial phase* ϕ. We thus consider a trial solution of the form

$$x_1^T = C_1\sin(\alpha t + \phi), \qquad x_2^T = C_2\sin(\alpha t + \phi), \tag{11.40h}$$

in which C_1 and C_2 are constant amplitudes. Substitution of (11.40h) into (11.40g) yields a system of two homogeneous algebraic equations that determine α and the amplitude ratio C_1/C_2:

$$\begin{aligned} (2p^2 - \alpha^2)C_1 - p^2C_2 &= 0, \\ -p^2C_1 + (2p^2 - \alpha^2)C_2 &= 0. \end{aligned} \tag{11.40i}$$

For nontrivial amplitudes C_k, we must have

$$\det\begin{bmatrix} 2p^2 - \alpha^2 & -p^2 \\ -p^2 & 2p^2 - \alpha^2 \end{bmatrix} = 0. \tag{11.40j}$$

This leads to the quadratic equation $(2p^2 - \alpha^2)^2 - p^4 = 0$ with the following two solutions for the circular frequency in (11.40h):

$$\alpha_1 = p, \qquad \alpha_2 = \sqrt{3}p. \tag{11.40k}$$

Equation (11.40j) is called the *characteristic equation* for the system, and its positive roots (11.40k) are called *characteristic frequencies*. The latter also are known as *eigenfrequencies*, *normal mode*, or *natural frequencies*.

For each of these frequencies, the system (11.40i) determines a corresponding amplitude ratio and separate trial solutions of the type (11.40h). It is useful, therefore, to denote by C_{jk} the different amplitudes associated with each generalized coordinate x_j, frequency α_k, and phase angle ϕ_k. In particular, $x_1^T = C_{11}\sin(\alpha_1 t + \phi_1)$ and $x_1^T = C_{12}\sin(\alpha_2 t + \phi_2)$, and hence the general solution for x_1 is provided by their sum. Similarly for x_2. Hence, with the aid of (11.40h) and (11.40k), the general solution of the linear system (11.40g) is given by

$$\begin{aligned} x_1 &= C_{11}\sin(pt + \phi_1) + C_{12}\sin(\sqrt{3}pt + \phi_2), \\ x_2 &= C_{21}\sin(pt + \phi_1) + C_{22}\sin(\sqrt{3}pt + \phi_2). \end{aligned} \tag{11.40l}$$

When α_k is used in (11.40i), the former amplitudes C_1, C_2 are replaced by the respective amplitudes C_{1k}, C_{2k}. Successive use of (11.40k) in (11.40i) yields the following amplitude ratios corresponding to α_1 and α_2:

$$\frac{C_{11}}{C_{21}} = 1, \qquad \frac{C_{22}}{C_{12}} = -1. \tag{11.40m}$$

In consequence, the general solution (11.40l) may be written as

$$\begin{aligned} x_1 &= C_{11}\sin(pt + \phi_1) - C_{22}\sin(\sqrt{3}pt + \phi_2), \\ x_2 &= C_{11}\sin(pt + \phi_1) + C_{22}\sin(\sqrt{3}pt + \phi_2). \end{aligned} \tag{11.40n}$$

The four constants C_{11}, C_{22}, ϕ_1, and ϕ_2 are determined by assigned initial data. Suppose we specify the initial data so that $C_{22} = 0$, then the solution (11.40n) has the form

$$x_1 = x_2 = C_{11}\sin(pt + \phi_1). \tag{11.40o}$$

On the other hand, if we specify initial data so that $C_{11} = 0$, we obtain

$$-x_1 = x_2 = C_{22}\sin(\sqrt{3}pt + \phi_2). \tag{11.40p}$$

Each of these motions is described by a single amplitude, frequency, and phase. In general, a motion of a multidegree of freedom vibrating system that can be described by a single frequency is called a *mode*. The solutions (11.40o) and (11.40p) correspond to modes having the distinct natural frequencies p and $\sqrt{3}p$. In the case (11.40o), the masses move in the same direction with the same circular frequency p, initial phase ϕ_1, and amplitude C_{11}. In the case (11.40p), the masses move symmetrically, in opposite directions with the same circular frequency $\sqrt{3}p$, initial phase ϕ_2, and amplitude C_{22}.

Indeed, when the masses are equally displaced in the same direction an amount B and released from rest, the initial data $x_1(0) = x_2(0) = B,\ \dot{x}_1(0) = \dot{x}_2(0) = 0$

applied to the system (11.40n) provides four algebraic equations that are satisfied with $C_{22} = 0$, $\phi_1 = \pi/2$, and $B = C_{11}$. Thus, (11.40o) has the explicit form

$$x_1 = x_2 = C_{11} \cos pt. \tag{11.40q}$$

Similarly, when initially the masses are equally but oppositely displaced an amount D and released from rest so that $-x_1(0) = x_2(0) = D$ and $\dot{x}_1(0) = \dot{x}_2(0) = 0$, we find $C_{11} = 0$, $\phi_2 = \pi/2$, and $D = C_{22}$. So, the explicit solution is

$$-x_1 = x_2 = C_{22} \cos \sqrt{3} pt. \tag{11.40r}$$

The two natural frequencies of the system are the characteristic frequencies given by (11.40k). By introduction of new coordinates ξ_k, the most general motion of the system (11.40n) for arbitrary initial data may be viewed as the superposition of normal mode motions corresponding to these natural frequencies. Indeed, we see from (11.40n) that each of the coordinates ξ_k, which may be defined in terms of the physical coordinates x_k by

$$\begin{aligned} \xi_1 &\equiv \frac{1}{2}(x_1 + x_2) = C_{11} \sin(pt + \phi_1), \\ \xi_2 &\equiv \frac{1}{2}(x_1 - x_2) = -C_{22} \sin(\sqrt{3} pt + \phi_2), \end{aligned} \tag{11.40s}$$

has only one frequency. These are the natural modes of vibration of the system. The coordinates ξ_k are called *normal coordinates* and their corresponding modes given on the right-hand side of equations (11.40s) are known as the *normal modes of vibration*. It is now evident that the general motion (11.40n) is a superposition of normal mode motions described by $x_1 = \xi_1 + \xi_2$ and $x_2 = \xi_1 - \xi_2$. These relations uncouple the original equations of motion. Let the reader use these results to show that (11.40g) may be written as

$$\ddot{\xi}_1 + p^2 \xi_1 = 0, \qquad \ddot{\xi}_2 + 3p^2 \xi_2 = 0. \tag{11.40t}$$

These are the normal equations of motion for which the normal mode frequencies, now evident, are given in (11.40k). □

11.10. First Integrals of the Lagrange Equations

Two first integrals of the Lagrange equations of motion (11.35) for a conservative system that are equivalent to the three classical principles of conservation of momentum, moment of momentum, either of which may hold also for nonconservative systems with appropriate forces as described later, and energy are derived next. Presentation of the general energy principle for arbitrary generalized forces follows. Finally, a first integral of Lagrange's equations (11.15) with respect to time leads to the generalized impulse–momentum principle.

11.10.1. Ignorable Coordinates: An Easy First Integral of Lagrange's Equations

Let us consider a conservative dynamical system for which the Lagrangian function L is independent of a generalized coordinate q_e, say. Then $\partial L/\partial q_e \equiv 0$, and the corresponding Lagrange equation for q_e, in accord with (11.35) for a general conservative system, yields a first integral

$$\frac{\partial L}{\partial \dot{q}_e} = \gamma_e, \tag{11.41}$$

where γ_e is a constant of integration fixed by the initial conditions. The absentee coordinate q_e is called an *ignorable coordinate.* Since q_e is absent from L, equation (11.41) can be used to obtain $\dot{q}_e$ as a function of all of the other generalized coordinates and their velocities; and consequently $\dot{q}_e$ can then be removed from the remaining differential equations. In this way, a conservative dynamical system having n degrees of freedom and k ignorable coordinates can be reduced to a problem having $n - k$ degrees of freedom. The general reduction procedure for conservative systems is discussed in advanced texts; see Whittaker (Section 38), for example.

Notice in Example 11.3, page 509, that ϕ is absent from the Lagrangian in (11.37c); and, therefore, from the second and fourth equations in (11.37d), $mr^2\dot{\phi} = \gamma_\phi$, a constant, in accordance with (11.41), that is,

$$\dot{\phi} = \frac{\gamma}{r^2}, \tag{11.42a}$$

where $\gamma \equiv \gamma_\phi/m$, a constant. Now, use of this relation in the first equation in (11.37e) yields a differential equation in r alone:

$$\ddot{r} + \frac{\mu}{r^2} - \frac{\gamma^2}{r^3} = 0. \tag{11.42b}$$

In principle, with the solution $r = r(t)$ of this equation in hand, (11.42a) may be used to find $\phi(t)$. The result (11.42a) is the same as (7.72b) leading to Kepler's equal area rule—it is a reflection of the principle of conservation of moment of momentum, as shown in (7.72a). A generalized principle of momentum that includes this case is described later on.

It is seen in Example 11.5(b), page 512, that ϕ is an absentee coordinate for the *nonconservative problem* described by the additional equations in (11.39i), the first of which does not vanish; rather, it leads to (11.39j). Hence, (11.41) based on Lagrange's equations (11.35) for a general conservative system does not hold for the absentee coordinate ϕ in (11.39i), because the nonconservative part of the generalized force $Q_\phi^N \neq 0$ in (11.39h). On the other hand, depending on the nature of the generalized forces, it is quite possible that momenta may be conserved in a nonconservative system. Let us look more closely at the role of ignorable coordinates in relation to the familiar classical conservation principles.

11.10.2. Principle of Conservation of Generalized Momentum

The fundamental principles of conservation of momentum and moment of momentum arise as first integrals of the Newton–Euler equations of motion when a specific component $F_e \equiv \mathbf{F} \cdot \mathbf{e}$ of the total force $\mathbf{F}$, or $M_e^Q \equiv \mathbf{M}_Q \cdot \mathbf{e}$ of the total torque $\mathbf{M}_Q$ about an appropriate point Q, for a fixed direction $\mathbf{e}$ in an inertial frame Φ, vanishes. Consequently, the specific corresponding component $p_e \equiv \mathbf{p} \cdot \mathbf{e}$ of the momentum , or $h_e^Q \equiv \mathbf{h}_Q \cdot \mathbf{e}$ of the moment of momentum, is a constant throughout the motion in Φ. The total force $\mathbf{F}$, however, need not be conservative. We now show that these important fundamental principles are imbedded within the Lagrangian theory.

To motivate the principal idea involved, let us rewrite the second equation in (11.12) as $\partial T(\dot{q}_r, q_r, t)/\partial \dot{q}_k = \mathbf{p}(\dot{q}_r, q_r, t) \cdot \partial \mathbf{x}(q_r, t)/\partial q_k$; and, similarly, by (11.24), obtain $\partial T(\dot{q}_r, q_r)/\partial \dot{q}_k = M_{kj}(q_r)\dot{q}_j$ (sum on j). Notice that both relations have the form of a kind of general momentum component. Consequently, we are led to introduce the *generalized momenta* $p_k(\dot{q}_r, q_r, t)$ defined by

$$p_k(\dot{q}_r, q_r, t) \equiv \frac{\partial T(\dot{q}_r, q_r, t)}{\partial \dot{q}_k} = \frac{\partial L(\dot{q}_r, q_r, t)}{\partial \dot{q}_k}. \tag{11.43}$$

Use of (11.43) in (11.15) casts the Lagrange equations of motion in a somewhat familiar form:

$$\dot{p}_k(\dot{q}_r, q_r, t) = Q_k(\dot{q}_r, q_r, t) + \frac{\partial T(\dot{q}_r, q_r, t)}{\partial q_k} \equiv \mathscr{F}_k(\dot{q}_r, q_r, t), \tag{11.44}$$

in which the quantities $\partial T/\partial q_k$ are called *pseudoforces* (See Problem 11.12.) and $\mathscr{F}_k$, the totals of the corresponding generalized and pseudoforces, are named the *Lagrange forces.* The pseudoforces include the familiar inertial forces. Because the generalized force Q_k has the physical interpretation as either a force or a torque, by dimensional homogeneity, the pseudoforces and the Lagrange forces share a corresponding physical interpretation. With $Q_k = Q_k^N(\dot{q}_r, q_r, t) - \partial V(q_r)/\partial q_k$, (11.44) may be rewritten as

$$\dot{p}_k(\dot{q}_r, q_r, t) = Q_k^N(\dot{q}_r, q_r, t) + \frac{\partial L(\dot{q}_r, q_r, t)}{\partial q_k} \equiv \mathscr{F}_k(\dot{q}_r, q_r, t), \tag{11.45}$$

which also follows directly from (11.38) and (11.43). For a general conservative dynamical system, $Q_k^N = 0$ and (11.45) reduces to

$$\dot{p}_k(\dot{q}_r, q_r, t) = \frac{\partial L(\dot{q}_r, q_r, t)}{\partial q_k} = \mathscr{F}_k(\dot{q}_r, q_r, t). \tag{11.46}$$

Equation (11.44), or equivalently (11.45), is the Lagrange form of the Newton–Euler laws: *The time rate of change of a generalized momentum is equal to the corresponding Lagrange force:* $\dot{p}_k = \mathscr{F}_k$. *Hence, the generalized momentum* p_e *corresponding to a generalized coordinate–velocity pair* $(\dot{q}_e, q_e)$ *is constant throughout the motion if and only if the corresponding Lagrange force* $\mathscr{F}_e = 0$.

If a generalized coordinate q_e is ignorable, then $\partial L(\dot{q}_r, q_r, t)/\partial q_e \equiv 0$ and (11.45) yields the corresponding generalized momentum equation $\dot{p}_e = Q_e^N$. If the generalized force Q_e^N also vanishes, as it does when the system is conservative, then $\dot{p}_e = 0$, and hence

$$p_e(\dot{q}_r, q_r, t) = \frac{\partial L(\dot{q}_r, q_r, t)}{\partial \dot{q}_e} = \gamma_e, \text{ a constant.} \tag{11.47}$$

This is the *principle of conservation of generalized momentum*: *The generalized momentum p_e corresponding to an ignorable coordinate q_e for which $Q_e^N = 0$ is a constant throughout the motion.* In this case, (11.41) for a conservative system is the same as the more general momentum rule (11.47). If the system is not conservative, however, in order that (11.47) may hold for an ignorable coordinate q_e, the nonconservative part of the corresponding generalized force Q_e^N also must vanish. In particular, recall again equations (11.39h) and (11.39i), for which $\partial L/\partial \phi = 0$, but $Q_\phi^N \neq 0$; so, in this instance (11.47) does not hold.

Example 11.7. A nonconservative holonomic system having two degrees of freedom with generalized coordinates (q_1, q_2) and corresponding generalized forces $Q_1^N = -mb^2\nu\dot{q}_1$, $Q_2^N = 0$, has a Lagrangian function

$$L = \frac{1}{2}ma^2\sin^2 q_1 + mb^2\left(\dot{q}_2 + \frac{a}{b}\cos q_1\right)^2 + \frac{1}{2}mb^2(\dot{q}_1 + c)^2, \tag{11.48a}$$

in which a, b, c, and m are constants. Derive the Lagrange equations.

Solution. Notice that q_2 is ignorable *and* $Q_2^N = 0$. Therefore, we have immediately by (11.47) the corresponding momentum integral

$$p_2 = \frac{\partial L}{\partial \dot{q}_2} = 2mb^2\left(\dot{q}_2 + \frac{a}{b}\cos q_1\right) = \gamma_2, \text{ a constant.} \tag{11.48b}$$

Caution: We must continue to apply the Lagrange equations to (11.48a) in which all of the variables are considered independent. Equation (11.48b) is a partial solution of one of these equations that necessarily relates these variables; but it is not to be substituted into the Lagrangian, it is to be used in connection with the companion equation for q_1. The second of Lagrange's equations (11.38) yields

$$mb^2\ddot{q}_1 + 2mab\left(\dot{q}_2 + \frac{a}{b}\cos q_1\right)\sin q_1 - ma^2\sin q_1\cos q_1 = Q_1^N = -mb^2\nu\dot{q}_1. \tag{11.48c}$$

We now use (11.48b) to eliminate $\dot{q}_2$ from (11.48c) to obtain

$$\ddot{q}_1 + \nu\dot{q}_1 + \frac{a}{b}\sin q_1\left(\frac{\gamma_2}{mb^2} - \frac{a}{b}\cos q_1\right) = 0. \tag{11.48d}$$

The two equations (11.48b) and (11.48d), in principle, determine $q_1(t)$ and $q_2(t)$. □

Consider two additional easy examples. The Lagrangian for the conservative particle motion on the constraint-free path from B to C in Example 7.10, page 249, is given by $L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgy$, in which x is an ignorable generalized coordinate. Hence, by (11.47), we have $p_x = m\dot{x} = \gamma_x$, a constant, which is equivalent to the result obtained earlier from the principle of conservation of momentum. The Lagrangian $L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2 + \dot{z}^2) + mgz$ for the conservative spherical pendulum problem in Example 7.15, page 260, shows that ϕ is an ignorable coordinate, and hence by (11.47), $p_\phi = mr^2\dot{\phi} = \gamma_\phi$, a constant, as shown earlier in (7.83d) based on conservation of moment of momentum. In the Lagrange formulation, however, it is not necessary to recognize the specific application of the principles of conservation of momentum and moment of momentum; these emerge naturally and easily from the Lagrangian structure with ignorable coordinates.

11.10.3. The Principle of Conservation of Energy for Scleronomic Systems Revisited

The familiar principle of conservation of energy for a particle in (11.33) holds only when the kinetic energy is not an explicit function of time, always the case when the constraints are scleronomic. Here we show that this conservation law for a conservative, scleronomic system may be derived differently to reveal a more general analytical structure that leads to a first integral applicable to dynamical systems other than merely a single particle.

We begin with $L = L(\dot{q}_r, q_r)$, which does not involve t explicitly, observe that

$$\frac{d}{dt}L(\dot{q}_r, q_r) = \frac{\partial L}{\partial \dot{q}_r}\ddot{q}_r + \frac{\partial L}{\partial q_r}\dot{q}_r,$$

introduce (11.35) for a conservative system to write

$$\frac{d}{dt}L(\dot{q}_r, q_r) = \frac{\partial L}{\partial \dot{q}_r}\ddot{q}_r + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_r}\right)\dot{q}_r = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_r}\dot{q}_r\right),$$

and thus obtain

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_r}\dot{q}_r - L\right) = 0. \tag{11.49}$$

This yields the *Lagrangian form of the law of conservation of energy for a conservative, scleronomic dynamical system*:

$$\frac{\partial L}{\partial \dot{q}_r}\dot{q}_r - L = E, \quad \text{a constant.} \tag{11.50}$$

Finally, introduce the Lagrangian function $L(\dot{q}_r, q_r) = T(\dot{q}_r, q_r) - V(q_r)$ and recall (11.26), in which $T(\dot{q}_r, q_r)$ is given by (11.24) for a scleronomic system, to deduce from (11.50) the *principle of conservation of energy for a conservative, scleronomic dynamical system*:

$$T(\dot{q}_r, q_r) + V(q_r) = E, \text{ constant.} \tag{11.51}$$

The derivation of (11.51) involved only the time independence of the Lagrangian function $L(\dot{q}_r, q_r)$, the Lagrange equations (11.35) for a conservative system, and the general representation (11.24) for the kinetic energy $T(\dot{q}_r, q_r)$ in terms of a certain symmetric coefficient matrix $M_{jk}(q_r)$ to be determined by the system, and which does not involve t explicitly. Consequently, the result (11.51) may be applied to any conservative, scleronomic dynamical system. In particular, the two degree of freedom spring-mass system described by (11.40a) and (11.40b) is conservative with constant total energy given by

$$\frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 + \frac{1}{2}k_1(x_1^2 + x_2^2) + \frac{1}{2}k_2(x_2 - x_1)^2 = E,$$

which certainly is not evident from the equations of motion in (11.40f).

11.10.4. The General Energy Principle

Let us write each generalized force as the sum $Q_k = Q_k^C + Q_k^N$ of a conservative part $Q_k^C = -\partial V(q_r)/\partial q_k$ with total potential energy $V(q_r)$ and a nonconservative part $Q_k^N = Q_k^N(\dot{q}_r, q_r, t)$. Then (11.28) for a scleronomic system may be written as

$$\frac{d}{dt}(T(\dot{q}_r, q_r) + V(q_r)) = Q_k^N \dot{q}_k,$$

whose integration yields as a first integral of (11.15), hence also (11.38), *the general energy principle for a nonconservative scleronomic system*:

$$\Delta\mathscr{E} = \int_{\mathscr{C}} Q_k^N dq_k \equiv \mathscr{W}_N, \tag{11.52}$$

where $\mathscr{E}(\dot{q}_r, q_r) \equiv T(\dot{q}_r, q_r) + V(q_r)$ is the *total energy of the system. Accordingly, the change in the total energy of a scleronomic system is equal to the total work done by all of the nonconservative generalized forces. Thus, the total energy is conserved and* (11.51) *holds if and only if the nonconservative parts of all generalized forces are workless, i.e. trivially, when the scleronomic system is conservative.* This result also follows directly from the work–energy principle (11.29). The rule (11.52) is applicable to all scleronomic, multidegree of freedom dynamical systems. It subsumes all previous special results (7.80) for a particle, (8.93) for a system of particles, and (10.131) for a rigid body—in fact, it holds for any combination of these dynamical systems, as shown more precisely later on.

Exercise 11.8. Begin with (11.38) and show that

$$\Delta\left(\frac{\partial L}{\partial \dot{q}_k}\dot{q}_k - L\right) = \int_{\mathscr{C}} Q_k^N dq_k \equiv \mathscr{W}_N, \tag{11.53}$$

and thus derive the general energy principle (11.52). □

11.10.5. The Generalized Impulse–Momentum Principle

The impulse–momentum and torque–impulse principles for particles and rigid bodies were obtained in earlier chapters as first integrals of the Newton–Euler equations with respect to time. We are now able to derive an inclusive, generalized impulse–momentum principle as a first integral of Lagrange's equations for an arbitrary holonomic dynamical system having n degrees of freedom. First, however, it is important to recognize that an impulsive force acting on a single particle (or a separate body) has no *instantaneous effect* on the system as a whole unless that particle (or body) is connected to or has contact (impact) with another particle (or body). An impulsive force acting on one of two particles attached to the ends of a massless rigid rod, on the other hand, clearly affects the instantaneous motion of both particles of the system. Similarly, the motion of a system of two rigid rods situated in a plane and connected by a smooth hinge at one end is affected by a force suddenly applied at any point along either one. The configuration of an entire assembly of contacting billiard balls, initially at rest and impacted by another ball, is instantaneously affected. A glass jar full of marbles or pebbles that explodes upon striking a hard surface is another example where the motion of an entire system is affected by impulsive forces that act on all of the "particles" at the same moment. There are lots of other examples that may be built upon these. Therefore, in subsequent developments for a system having n degrees of freedom, it is important to bear in mind the limitations imposed by the nature of the system at the impulsive instant. Further, by extension of our earlier model of an instantaneous impulse, during an impulsive action, the generalized impulsive forces vary only with time; all other applied forces, like those due to gravity or attached springs, remain finite during the impulsive instant and contribute nothing to the effect of the impulse. The affected particles (or bodies) suffer an instantaneous change in their velocities, but no instantaneous change in their positions; therefore, the impulsive forces, without accounting in some fashion for deformation of the objects, do no work in the real motion. On the other hand, the instantaneous impulsive forces can do work in an admissible virtual motion during which time is frozen.

With these observations in mind, let us consider the integral of Lagrange's equations (11.15) with respect to time over the interval $[t_0, t]$ to obtain

$$\left.\frac{\partial T}{\partial \dot{q}_k}\right|_{t_0}^{t} - \int_{t_0}^{t} \frac{\partial T}{\partial q_k} dt = \mathscr{Q}_k(t_0; t), \tag{11.54}$$

wherein, by definition, the *impulse of the generalized force* is

$$\mathscr{Q}_k(t_0;t) \equiv \int_{t_0}^{t} Q_k dt. \tag{11.55}$$

In accordance with (11.43) the first term in (11.54) is the change $\Delta p_k \equiv p_k(t) - p_k(t_0)$ in the generalized momenta during the impulsive interval $[t_0, t]$:

$$\left.\frac{\partial T}{\partial \dot{q}_k}\right|_{t_0}^{t} = \Delta p_k. \tag{11.56}$$

During the impulsive instant the generalized coordinates q_k do not change and the generalized velocities $\dot{q}_k$ remain finite. Therefore, the quantities Δp_k and $\partial T(\dot{q}_r, q_r)/\partial q_k$ in (11.54) remain finite. In the limit as $\Delta t = t - t_0 \to 0$, $\Delta p_k \to \Delta p_k^*$, the instantaneous change in the generalized momenta, and the integral of the pseudoforces vanishes: $\text{limit}_{\Delta t\to 0}(\partial T/\partial q_k)\Delta t = 0$. (This is trivially satisfied for all t when all of the generalized coordinates are ignorable.) The instantaneous impulse of any finite generalized forces that act on the system likewise will vanish as $\Delta t \to 0$. The impulsive forces in an instantaneous impulse tend to infinity in such a way that the limit of (11.55) exists and is finite; its value is *the instantaneous impulse of the generalized forces* defined by

$$\mathscr{Q}_k^* \equiv \lim_{\Delta t\to 0} \mathscr{Q}_k(t_0;t) = \lim_{\Delta t\to 0} \int_{t_0}^{t} Q_k dt. \tag{11.57}$$

Thus, in the limit of (11.54) as $\Delta t \to 0$, we obtain the *generalized impulse–momentum principle:*

$$\mathscr{Q}_k^* = \Delta p_k^*; \tag{11.58}$$

the instantaneous impulse of the generalized force is equal to the instantaneous change in the corresponding generalized momentum of the system.

Notice from (11.44) that $\Delta p_k = \int_{t_0}^{t} \mathscr{T}_k dt$; that is, the impulse of the Lagrange force is equal to the change in the corresponding generalized momentum. It is easily seen that bounded quantities vanish in the limit as $\Delta t \to 0$ and this reduces to the generalized impulse–momentum principle (11.58). The generalized impulse–momentum principle includes both the impulse–momentum and torque-impulse principles of earlier studies, depending on the physical interpretation of the generalized momenta.

The generalized impulsive forces may be calculated from their virtual work $\delta \mathscr{W}^* = \mathscr{Q}_k^* \delta q_k$ done in an admissible general virtual displacement consistent with any holonomic constraints, as though they were ordinary fixed forces. Given the initial values of q_k and $\dot{q}_k$ prior to the impulsive action, the system of n algebraic equations (11.58) then relates the instantaneous impulsive forces $\mathscr{Q}_k^*$ and instantaneous generalized velocities $\dot{q}_k$ following the impulsive action. Therefore, these equations may be used to provide the initial conditions for the subsequent motion of the system. Since no specific form of the kinetic energy was introduced in the construction, the principles hold for all dynamical systems. This observation is reinforced later.

Example 11.8. Two particles of equal mass m are attached to the ends of a massless rigid rod of length ℓ initially oriented parallel to the y-axis of a frame $\Psi = \{O; \mathbf{i}_k\}$ and at rest on a smooth horizontal surface. An instantaneous impulsive normal force $\mathbf{P} = P\mathbf{i}$ acts on the particle closer to O. Determine the subsequent instantaneous generalized velocities of the system, find the instantaneous increase of the total energy of the system due to the impulse, and describe the subsequent motion for all time.

Solution. Following the impulse, the center of mass of the system is at (x^*, y^*) in Ψ and the rod makes an angle θ with the y-axis. The three generalized coordinates, therefore, are $(q_1, q_2, q_3) = (x^*, y^*, \theta)$. The total kinetic energy of the system in its subsequent general motion is given by (8.53); we find

$$T = \frac{1}{2}(2m)(\dot{x}^{*2} + \dot{y}^{*2}) + \frac{1}{2}\left(\frac{m\ell^2}{2}\right)\dot{\theta}^2. \tag{11.59a}$$

All of the generalized variables are ignorable, and hence $\partial T/\partial q_r \equiv 0$. Since the system is at rest initially, the instantaneous changes in the generalized momenta (11.56) are given by

$$\Delta p_1^* = 2m\dot{x}^*, \qquad \Delta p_2^* = 2m\dot{y}^*, \qquad \Delta p_3^* = \frac{m\ell^2}{2}\dot{\theta}. \tag{11.59b}$$

Notice that Δp_1^* and Δp_2^* are instantaneous changes in linear momenta due to an impulsive force, and Δp_3^* is the instantaneous change in the moment of momentum due to a torque–impulse.

Let $(\mathscr{Q}_1^*, \mathscr{Q}_2^*, \mathscr{Q}_3^*)$ denote the corresponding generalized impulsive forces. The position vector $\mathbf{x}_P$ of the point of application of the impulsive force $\mathbf{P} = P\mathbf{i}$ is given by $\mathbf{x}_P = (x^* + \frac{1}{2}\ell\sin\theta)\mathbf{i} + (y^* - \frac{1}{2}\ell\cos\theta)\mathbf{j}$ so its virtual displacement is $\delta\mathbf{x}_P = (\delta x^* + \frac{1}{2}\ell\delta\theta\cos\theta)\mathbf{i} + (\delta y^* + \frac{1}{2}\ell\delta\theta\sin\theta)\mathbf{j}$. Hence, at the impulsive instant at which $\theta = 0$, the virtual work of the applied forces and the generalized impulsive forces is given by

$$\delta\mathscr{W}^* = \mathscr{Q}_1^*\delta x^* + \mathscr{Q}_2^*\delta y^* + \mathscr{Q}_3^*\delta\theta = P\left(\delta x^* + \frac{1}{2}\ell\delta\theta\right), \tag{11.59c}$$

for all arbitrary virtual displacements. Therefore,

$$\mathscr{Q}_1^* = P, \qquad \mathscr{Q}_2^* = 0, \qquad \mathscr{Q}_3^* = \frac{P\ell}{2}. \tag{11.59d}$$

We thus see that $\mathscr{Q}_1^*$ is the instantaneous impulsive force while $\mathscr{Q}_3^*$ is its instantaneous moment about the center of mass. Use of (11.59b) and (11.59d) in (11.58) yields the instantaneous values of the generalized velocities $(\dot{x}_i^*, \dot{y}_i^*, \dot{\theta}_i)$:

$$\dot{x}_i^* = \frac{P}{2m}, \qquad \dot{y}_i^* = 0, \qquad \dot{\theta}_i = \frac{P}{m\ell}. \tag{11.59e}$$

The instantaneous increase in the total energy due to the impulse on the system, initially at rest, follows by substitution of (11.59e) into (11.59a): $T_i = P^2/2m$.

The values (11.59e) of the instantaneous translational velocity $\dot{\mathbf{x}}^* = \dot{x}_i^*\mathbf{i}$ of the center of mass and the instantaneous angular velocity $\boldsymbol{\omega} = \dot{\theta}_i\mathbf{k}$ of the system about the center of mass are the initial conditions for the subsequent motion of the system under no forces. It follows that the motion of the center of mass is uniform with velocity $\mathbf{v}^* = P/2m\mathbf{i}$, its initial value. Moreover, there are no applied torques, so the moment of momentum for the system about the center of mass is constant: $\mathbf{h}_C = m\ell^2\dot{\theta}/2\mathbf{k}$, and hence the angular velocity is constant, $\boldsymbol{\omega} = P/m\ell\mathbf{k}$. Alternatively, this being a conservative system with total kinetic energy (11.59a), and all of whose generalized coordinates are ignorable and whose generalized forces are zero, (11.47) yields the principles of conservation of generalized momenta:

$$p_1 = 2m\dot{x}^* = \gamma_1, \qquad p_2 = 2m\dot{y}^* = \gamma_2, \qquad p_3 = \frac{m\ell^2}{2}\dot{\theta} = \gamma_3, \tag{11.59f}$$

where the constants γ_k are determined by the initial values (11.59e). We thus find $\gamma_1 = P,\ \gamma_2 = 0,\ \gamma_3 = P\ell/2$, and hence, $\mathbf{v}^* = P/2m\mathbf{i}$ and $\boldsymbol{\omega} = P/m\ell\mathbf{k}$ for all t. □

11.11. Hamilton's Principle

Let us consider a holonomic system having n degrees of freedom. To compare two neighboring motions of the same system with n independent generalized coordinates $q_r^*(t)$ and $q_r(t)$ described in the *same time* between the *same end states* so that $q_r^*(t_1) = q_r(t_1)$ and $q_r^*(t_2) = q_r(t_2)$ at the respective arbitrary times t_1 and t_2, as shown in Fig. 11.2, suppose that the motions differ by only a small amount described by the generalized coordinate *variations* $\delta q_r(t) \equiv q_r^*(t) - q_r(t) = \varepsilon\eta_r(t)$, where ε is an arbitrary small parameter and $\eta_r(t)$ are n arbitrary, continuously differentiable functions. Then $\delta q_r(t_1) = \varepsilon\eta_r(t_1) = 0$ and $\delta q_r(t_2) = \varepsilon\eta_r(t_2) = 0$ must hold at the end states. The process of variation requires that only the independent generalized coordinates $q_r(t)$ are varied, not the time t; hence $\delta t = 0$. The parameter ε allows us to modify the functions $q_r(t)$ by arbitrarily small amounts, and because ε may decrease to zero, the variation of a function $q_r(t)$ thus constitutes an infinitesimal virtual change of that function.

Now consider a given function $f(q_r(t), t) \equiv f(q_1, q_2, \ldots, q_n, t)$ whose variation is caused by the variation $\delta q_r(t)$ of the independent variables $q_r(t)$, noting that the functional dependence is not altered by the variation. Then $\delta f(q_r, t) \equiv f(q_r^*, t) - f(q_r, t) = f(q_r + \varepsilon\eta_r, t) - f(q_r, t)$; and by the Taylor series

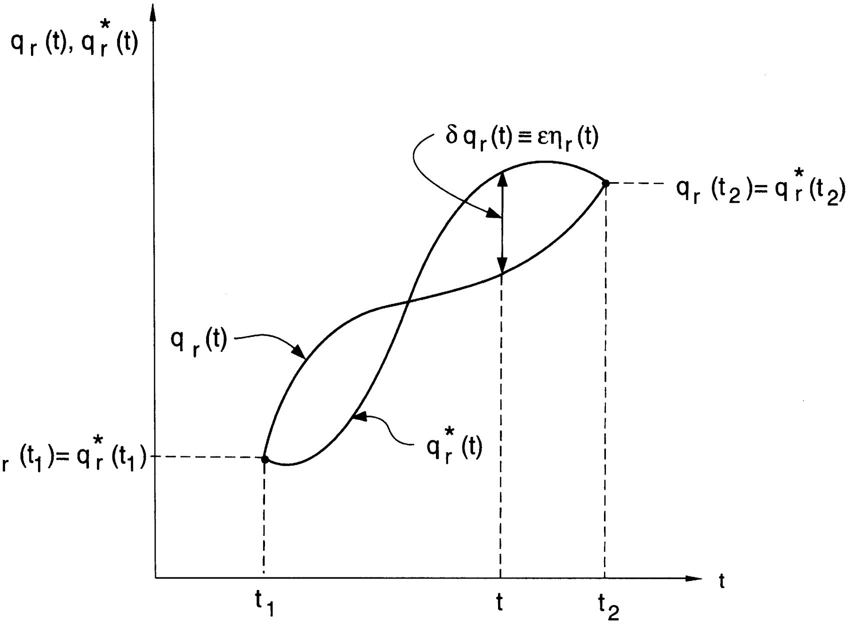


Figure 11.2. Comparison of the generalized coordinates for two motions having the same end states, for a system with n degrees of freedom.

approximation about $\varepsilon = 0$, we find $f(q_r + \varepsilon\eta_r, t) = f(q_r, t) + (\partial f/\partial q_r)\varepsilon\eta_r(t)$ to the first order in ε. The variation of the function is thus given by $\delta f(q_r, t) = (\partial f/\partial q_r)\delta q_r(t) = (\partial f/\partial q_r)\varepsilon\eta_r(t)$, sum on r. Therefore, the variation of a function with $\delta t = 0$ behaves like a differential in terms of the variation $\delta q_r(t)$ of its independent variables, the explicit time dependence in $f(q_r, t)$ playing no role.

11.11.1. Hamilton's Principle for a Conservative, Holonomic System

The *action* $\mathscr{A}$ for a general conservative, holonomic system is defined by the definite integral

$$\mathscr{A} \equiv \int_{t_1}^{t_2} L(\dot{q}_r(t), q_r(t), t)dt, \tag{11.60}$$

in which $L(\dot{q}_r, q_r, t) \equiv T(\dot{q}_r, q_r, t) - V(q_r)$ is the Lagrangian function for an otherwise unspecified dynamical system. Now suppose that the curves $q_r(t)$ in Fig. 11.2 represent the actual or natural motion of the system between arbitrarily assigned end states, and that δq_r defines an arbitrary infinitesimal variation in $q_r(t)$ in passing from the natural motion to a neighboring motion defined by the variables $q_r^*(t)$ between the same end states. This results in an infinitesimal variation $\delta\mathscr{A}$ in the action (11.60) defined by

$$\begin{aligned}\delta\mathscr{A} &= \delta\int_{t_1}^{t_2} L(\dot{q}_r, q_r, t)dt \equiv \int_{t_1}^{t_2} L(\dot{q}_r^*, q_r^*, t)dt - \int_{t_1}^{t_2} L(\dot{q}_r, q_r, t)dt \\ &= \int_{t_1}^{t_2} (L(\dot{q}_r^*, q_r^*, t) - L(\dot{q}_r, q_r, t))dt = \int_{t_1}^{t_2} \delta L(\dot{q}_r, q_r, t)dt;\end{aligned} \tag{11.61}$$

that is, the variation of the definite integral is equal to the integral of the variation of its integrand. Of course, the actual motion is not yet known, because the equations of motion have not been introduced and solved. To deduce Lagrange's equations for a general conservative, holonomic system without mention of the specific nature of the Lagrangian energy function, we introduce a variational principle due to Sir William Rowan Hamilton (1805–1865) applicable to both conservative and nonconservative, holonomic dynamical systems.

Hamilton's principle: *Among all possible motions between assigned end states of any holonomic dynamical system, the actual motion is the one for which the action is stationary; that is,*

$$\delta\mathscr{A} = 0. \tag{11.62}$$

First, let us consider a conservative dynamical system for which $\mathscr{A}$ is defined by (11.60), and apply the previous description of the variation of a function $f(q_r, t)$ to the Lagrangian function $L(\dot{q}_r, q_r, t)$ in the action integral (11.61) to obtain

$$\delta\mathscr{A} = \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial \dot{q}_r}\delta\dot{q}_r + \frac{\partial L}{\partial q_r}\delta q_r \right) dt, \tag{11.63}$$

repeated indices being summed. Since $\delta\dot{q}_r \equiv \dot{q}_r^* - \dot{q}_r$, we have $\delta\dot{q}_r = \dfrac{d}{dt}\delta q_r$, and hence the first term in (11.63) may be integrated by parts to obtain

$$\int_{t_1}^{t_2} \frac{\partial L}{\partial \dot{q}_r}\delta\dot{q}_r dt = \left.\frac{\partial L}{\partial \dot{q}_r}\delta q_r\right|_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_r}\right)\delta q_r dt. \tag{11.64}$$

In view of the null end conditions, however, the first of the right-hand terms vanishes, and hence use of this result in (11.63) and application of Hamilton's principle (11.62) yield the stationary action condition

$$\delta\mathscr{A} = \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q_r} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_r}\right)\right)\delta q_r dt = 0. \tag{11.65}$$

We shall assume that the integrand is a continuous function of t. For holonomic systems, $\delta q_r = \varepsilon\eta_r(t)$ are arbitrary infinitesimal quantities in which $\eta_r(t)$ are any continuously differentiable functions. Since (11.65) must vanish for all arbitrary variations δq_r that satisfy the end conditions $\delta q_r(t_1) = \delta q_r(t_2) = 0$, it follows that in the integrand sum all of the functions in the parentheses must vanish. To prove this, let us write the integral (11.65) briefly as $I \equiv \varepsilon \int_{t_1}^{t_2} \mathscr{L}_r(t)\eta_r(t)dt$. Now take all $\eta_r(t) = 0$ except any one you wish, $\eta_a(t)$, say. Now choose this function to vanish everywhere except on an arbitrary small interval $\tau - \alpha \leq t \leq \tau + \alpha$ around the point $t = \tau$ and on which, without loss of generality, we may choose $\eta_a(t) > 0$, say. For example, the function $\eta_a(t) = (t - \tau + \alpha)^2(t - \tau - \alpha)^2$ for $t \in [\tau - \alpha, \tau + \alpha]$ and $\eta_a(t) = 0$ elsewhere satisfies the conditions and is continuously differentiable. Within this interval the continuous function $\mathscr{L}_a(t) = \mathscr{L}_a(\tau)$, very nearly, and hence $I = \varepsilon\mathscr{L}_a(\tau)\int_{\tau-\alpha}^{\tau+\alpha} \eta_a(t)dt$, approximately, the error tending to zero as α tends to zero. Since the integral $\int_{\tau-\alpha}^{\tau+\alpha} \eta_a(t)dt > 0$, it follows that (11.65) holds only if $\mathscr{L}_a(\tau) = 0$. The point $t = \tau$, however, may be chosen as any point of the interval (t_1, t_2), and hence the continuous function $\mathscr{L}_a(t)$ must vanish for all $t \in [t_1, t_2]$. Because our choice of the a^{th} member of the sum in (11.65) was arbitrary, (11.65) holds only if every term $\mathscr{L}_r(t)$ of its integrand sum vanishes. Conversely, if all $\mathscr{L}_r(t) = 0$ holds, (11.65) is satisfied. Hence, $\delta\mathscr{A} = 0$ holds for arbitrary variations $\delta q_r(t)$ that satisfy the end conditions, if and only if the Lagrangian function $L(\dot{q}_r, q_r, t)$ for a general conservative, holonomic dynamical system satisfies Lagrange's equations,†

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_r}\right) - \frac{\partial L}{\partial q_r} = 0, \qquad r = 1, 2, \ldots, n. \tag{11.66}$$

These equations are the same as (11.35). The difference, however, is that they now apply to every conservative, holonomic dynamical system; it is only the Lagrangian function for a specific conservative, holonomic dynamical system that must be defined.

† For holonomic systems for which elimination of the constraints by direct substitution may be inconvenient or cumbersome, the Lagrange multiplier method may be used to derive a modified form of Lagrange's equations. For a system of $p < m$ holonomic constraints,

$$f_j(q_r, t) \equiv f_j(q_1, q_2, \ldots, q_m, t) = 0, \qquad j = 1, 2, \ldots, p, \tag{a}$$

(hence $n = m - p$ degrees of freedom), p Lagrange multipliers λ_j are introduced, and Hamilton's principle is then applied to a modified Lagrangian function $\hat{L}(\dot{q}_r, q_r, t) = L(\dot{q}_r, q_r, t) + \sum_{j=1}^{p} \lambda_j f_j(q_r, t)$ to obtain the modified Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_r}\right) - \frac{\partial L}{\partial q_r} - \sum_{j=1}^{p} \lambda_j \frac{\partial f_j}{\partial q_r} = 0, \qquad r = 1, 2, \ldots, m \tag{b}$$

subject to the constraints (a). The system of equations (a) and (b) determine the $m + p$ generalized variables q_r and multipliers λ_j. We shall find no need to apply this method in our studies here. Further discussion of these matters and extension of Hamilton's principle and Lagrange's equations to nonholonomic dynamical systems may be found in advanced works on analytical dynamics. See, for example, the books by Lanczos, Pars, Rosenberg, and Whittaker, among others listed in the chapter references.

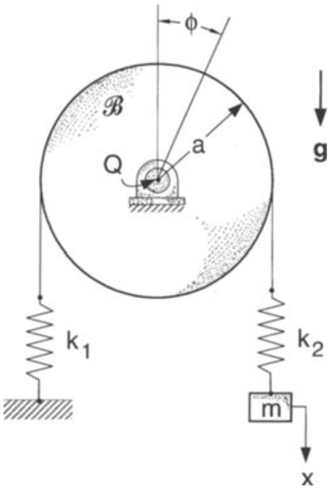


Figure 11.3. A conservative holonomic dynamical system of two bodies.

Example 11.9. A system of rigid bodies in its static equilibrium position is shown in Fig. 11.3. The homogeneous wheel $\mathscr{B}$ of radius a is free to rotate in a smooth bearing about Q, and the block of mass m is supported by a linear spring of stiffness k_2 attached to an inextensible cable that wraps around the wheel. The other end of the cable is fastened to a linear spring of modulus k_1 fixed to the machine foundation. There is no slip between the cable and the wheel when the block is displaced vertically and released. Derive the equations of motion for the system.

Solution. Because the elastic spring response is linear, we may consider the motion about the prestretched static equilibrium position of the system. In consequence, gravity has no further influence on the motion. This system has two degrees of freedom characterized by the independent generalized coordinates $(q_1, q_2) = (x, \phi)$ measured from the equilibrium state shown in Fig. 11.3. Due to the no slip and inextensibility constraints, the extension $\tilde{x}$ of the foundation spring is $\tilde{x} = a\phi$. The smooth bearing reaction force is workless, and the remaining forces that act on the system are conservative. Therefore, relative to the static equilibrium state, the Lagrangian function for the system is given by

$$L = T - V = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I\dot{\phi}^2 - \left[\frac{1}{2}k_1 a^2\phi^2 + \frac{1}{2}k_2(x - a\phi)^2\right], \qquad (11.67a)$$

where I is the moment of inertia of the wheel about its principal axis at Q. Notice in passing that the total kinetic energy in (11.67a) has the form (11.24) in which

$[M_{jk}]$ =diag$[m, I]$. With (11.66) in mind, we first determine

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x}, \qquad \frac{\partial L}{\partial x} = -k_2(x - a\phi), \tag{11.67b}$$

$$\frac{\partial L}{\partial \dot{\phi}} = I\dot{\phi}, \qquad \frac{\partial L}{\partial \phi} = -k_1 a^2\phi + ak_2(x - a\phi), \tag{11.67c}$$

and thereby obtain the following coupled pair of linear differential equations of motion for the system:

$$m\ddot{x} + k_2 x - k_2 a\phi = 0, \qquad I\ddot{\phi} + (k_1 + k_2)a^2\phi - k_2 ax = 0. \tag{11.67d}$$

The structure of these equations is similar to (11.40f) studied earlier. □

11.11.2. Hamilton's Principle for a Nonconservative, Holonomic System

Let us recall that the total work $\mathscr{W}$ done by a conservative system of forces is equal to the decrease in the total potential energy, and hence, for a conservative dynamical system, the integrand in (11.60) may be rewritten as $L = T + \mathscr{W}$. This suggests that for an arbitrary nonconservative, holonomic dynamical system having n degrees of freedom the action is appropriately defined by

$$\mathscr{A} \equiv \int_{t_1}^{t_2} \left[T(\dot{q}_r, q_r, t) + \mathscr{W}(\dot{q}_r, q_r, t)\right] dt, \tag{11.68}$$

where the total work $\mathscr{W}$ is defined by the sum of integrals in (11.23) in which $\mathscr{C} = \cup_{k=1}^{n} \mathscr{C}_k$ and each integral is over a path $\mathscr{C}_k$ for the generalized coordinate q_k corresponding to the generalized force Q_k, in the same time interval $[t_1, t]$. Hence, Hamilton's principle (11.62) applied to (11.68) yields the stationary action condition

$$\delta\mathscr{A} = \int_{t_1}^{t_2} \left[\delta T(\dot{q}_r, q_r, t) + \delta\mathscr{W}(\dot{q}_r, q_r, t)\right] dt = 0. \tag{11.69}$$

First, consider the variation $\delta\mathscr{W}$. In accordance with (11.23),

$$\delta\mathscr{W} = \sum_{k=1}^{n} \int_{q_k^*(t_1)}^{q_k^*(t)} Q_k(\dot{q}_r, q_r, t) dq_k - \sum_{k=1}^{n} \int_{q_k(t_1)}^{q_k(t)} Q_k(\dot{q}_r, q_r, t) dq_k, \tag{11.70}$$

in which $q_k^*(t_1) = q_k(t_1)$ for all k at the end point and $q_k^*(t) = q_k(t) + \delta q_k(t)$. Then, because $\delta q_k(t) = \varepsilon\eta_k(t)$ are infinitesimal quantities, (11.70) may be rewritten as

$$\delta\mathscr{W} = \sum_{k=1}^{n} \int_{q_k(t)}^{q_k(t)+\delta q_k(t)} Q_k(\dot{q}_r, q_r, t) dq_k = \sum_{k=1}^{n} \hat{Q}_k(\dot{q}_r, q_r, t)\delta q_k(t), \tag{11.71}$$

where $\hat{Q}_k(\dot{q}_r, q_r, t) = Q_k(\dot{q}_r, q_r, t) + \Delta Q_k(\varepsilon)$, the last term being an infinitesimal quantity of order ε. Therefore, to the first order in ε, (11.71) yields the variation $\delta \mathscr{W} = \sum_{k=1}^{n} Q_k \delta q_k \equiv Q_k \delta q_k$, the virtual work done by all of the generalized forces at the fixed time t.

Retracing the procedure used earlier to obtain the variation δL leading to (11.65) and now applied to $\delta T(\dot{q}_r, q_r, t)$ with vanishing end conditions, we find that Hamilton's principle (11.69), to the first order in ε, requires

$$\delta \mathscr{A} = \int_{t_1}^{t_2} \left(\frac{\partial T}{\partial q_r} - \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_r} \right) + Q_r \right) \delta q_r dt = 0, \tag{11.72}$$

for all $\delta q_r = \varepsilon \eta_r(t)$ such that $\delta q_r(t_1) = \delta q_r(t_2) = 0$, repeated indices being summed over $r = 1, 2, \ldots, n$. We shall assume that each integrand term in parentheses is a continuous function $\mathscr{L}_r(t)$ of time t—all being independent of ε. It then follows by our previous argument that each integrand function $\mathscr{L}_r(t)$ must vanish for all t. Consequently,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_r} \right) - \frac{\partial T}{\partial q_r} = Q_r, \qquad r = 1, 2, \ldots, n, \tag{11.73}$$

for every nonconservative, holonomic dynamical system of n degrees of freedom. These equations, while the same as (11.15), are now applicable to every nonconservative, holonomic dynamical system. Writing $Q_r(\dot{q}_r, q_r, t) = -\partial V(q_r)/\partial q_r + Q_r^N(\dot{q}_r, q_r, t)$ in terms of its conservative and nonconservative parts, we deduce from (11.73) the generalized form of Lagrange's equations (11.38) for nonconservative, holonomic dynamical systems.

Example 11.10. A rigid body shown in Fig. 11.4 is driven by a torque $\mu(t)$ about a fixed, principal body axis **k** in a smooth bearing at H. (i) Apply (11.73) to

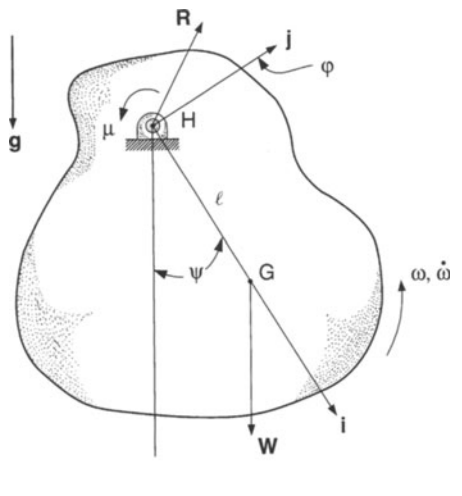


Figure 11.4. Torque driven rotation of a rigid body—a nonconservative holonomic dynamical system.

derive the equation of motion for the body. (ii) Repeat the derivation from (11.38). Show that the result has the familiar form of the equation of motion of a driven pendulum. (iii) Apply Euler's law to obtain the equation of motion.

Solution of (i). The system is holonomic with one degree of freedom described by $q_1 = \psi$; hence, (11.73) yields

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\psi}}\right) - \frac{\partial T}{\partial \psi} = Q_\psi. \tag{11.74a}$$

With the total kinetic energy of the body $T = \frac{1}{2} I \dot{\psi}^2$, where I is the principal moment of inertia about the body axis at H, (11.74a) becomes

$$I\ddot{\psi} = Q_\psi. \tag{11.74b}$$

We next determine the generalized force Q_ψ. The bearing reaction force $\mathbf{R}$ is workless, and the total external torque $\mathbf{M}_H$ about H is the sum of the gravitational torque $-W\ell \sin\psi \mathbf{k}$ and the applied driving torque $\boldsymbol{\mu} = \mu\mathbf{k}$. The virtual work $\delta \mathscr{W}$ done by the total torque in the virtual displacement $\delta\boldsymbol{\psi} \equiv \delta\psi\mathbf{k}$ is thus given by

$$\delta \mathscr{W} = \mathbf{M}_H \cdot \delta\boldsymbol{\psi} = (-W\ell \sin\psi + \mu)\delta\psi \equiv Q_\psi \delta\psi. \tag{11.74c}$$

Hence, $Q_\psi = -W\ell \sin\psi + \mu$, and (11.74b) yields the equation of motion:

$$I\ddot{\psi} + mg\ell \sin\psi = \mu(t). \tag{11.74d}$$

Solution of (ii). Application of the Lagrange equations (11.38) yields

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\psi}}\right) - \frac{\partial L}{\partial \psi} = Q_\psi^N. \tag{11.74e}$$

Again, the workless constraint force $\mathbf{R}$ need not be considered, and the gravitational force is conservative with total potential energy $V = mg\ell(1 - \cos\psi)$. Therefore, the Lagrangian is

$$L = T - V = \frac{1}{2} I \dot{\psi}^2 - mg\ell(1 - \cos\psi). \tag{11.74f}$$

The virtual work done by the nonconservative generalized force is $\delta \mathscr{W}_N = \boldsymbol{\mu} \cdot \delta\boldsymbol{\psi} = \mu\delta\psi = Q_\psi^N \delta\psi$. Hence, $Q_\psi^N = \mu$, and (11.74e) leads to (11.74d).

With $I = mR^2$ in terms of the radius of gyration R, (11.74d) may be written in the form of the equation of motion of a driven pendulum for which $p^2 \equiv g\ell/R^2$ and $\hat{\mu}(t) \equiv \mu(t)/I$; namely,

$$\ddot{\psi} + p^2 \sin\psi = \hat{\mu}(t). \tag{11.74g}$$

Solution of (iii). Euler's law for the rotation about a fixed principal axis at H requires $\mathbf{M}_H = I_H\dot{\boldsymbol{\omega}}$, wherein $\mathbf{M}_H = (\mu - W\ell\sin\psi)\mathbf{k}$ and $I_H\dot{\boldsymbol{\omega}} = I\ddot{\psi}\,\mathbf{k}$. This yields the equation of motion $I\ddot{\psi} = \mu - W\ell\sin\psi$, which is the same as (11.74d). □

Exercise 11.9. Begin with the action (11.68) and introduce from the start the decomposition of $\mathscr{W}$ into its conservative and nonconservative parts: $\mathscr{W}(\dot{q}_r, q_r, t) = \mathscr{W}_C(q_r) + \mathscr{W}_N(\dot{q}_r, q_r, t)$. Show that the action integral for the nonconservative system may be written as

$$\mathscr{A} \equiv \int_{t_1}^{t_2} (L(\dot{q}_r, q_r, t) + \mathscr{W}_N(\dot{q}_r, q_r, t))\,dt. \tag{11.75}$$

Then work out the details for $\delta\mathscr{A} = 0$ and thus derive (11.38). □

11.12. Additional Applications of the Lagrange Equations

Several additional applications of Lagrange's equations are investigated, starting with analysis of an experimental technique useful in engineering design for evaluation of the moment of inertia of a complex structured body. Then the finite amplitude oscillation of a rotating simple pendulum, for which generalized forces arise from the moving constraint, is studied. Next, we revisit the problem of the gyrocompass with torsional damping and conclude with analysis of the general motion of a spinning top about a fixed point.

11.12.1. A Problem in Engineering Design Analysis

The moment of inertia of a table assembly $\mathscr{T}$ shown in Fig. 11.5 in the horizontal plane is calibrated experimentally to have a principal value $I_O(\mathscr{T})$ about

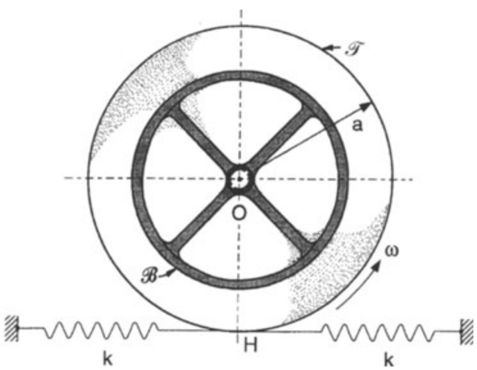


Figure 11.5. Experimental apparatus to determine the moment of inertia of a complex structured body.

its normal axis of rotation in a smooth bearing at O. Identical springs of stiffness k, initially unstretched, are attached symmetrically to the table in a tangential line at point H. The moment of inertia of another complex structured body $\mathscr{B}$ placed on the table in a specified orientation can be found experimentally by measuring the frequency $f_{\mathscr{S}}$ of small oscillations of the system $\mathscr{S} = \mathscr{T} \cup \mathscr{B}$ consisting of the table and the body. We thus derive an equation for the frequency of the system and thereby determine the moment of inertia $I_O(\mathscr{B})$ of a flywheel $\mathscr{B}$ placed centrally at O.

Let us consider a small angular placement $q_1 = \theta(t)$ of the system $\mathscr{S} = \mathscr{T} \cup \mathscr{B}$ from its initial, natural equilibrium state in the horizontal plane in Fig. 11.5. No work is done by the smooth bearing reaction forces; so, the system is conservative with total kinetic and potential energies given by

$$T = \frac{1}{2} I_O(\mathscr{S})\dot{\theta}^2(t), \qquad V = 2\left(\frac{1}{2}ka^2\theta^2\right) = ka^2\theta^2, \tag{11.76a}$$

where $I_O(\mathscr{S})$ is the moment of inertia of the system about the normal axis at O. Hence, $L = \frac{1}{2} I_O(\mathscr{S})\dot{\theta}^2(t) - ka^2\theta^2$, and Lagrange's equations (11.35) yield

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = I_O(\mathscr{S})\ddot{\theta} + 2ka^2\theta = 0. \tag{11.76b}$$

This is the equation for a simple harmonic oscillator whose small amplitude circular frequency $p_{\mathscr{S}}$ is defined by

$$p_{\mathscr{S}} = 2\pi f_{\mathscr{S}} = \sqrt{\frac{2ka^2}{I_O(\mathscr{S})}}, \tag{11.76c}$$

in which the measured frequency is $f_{\mathscr{S}}$ and the values for k and a are known. With $I_O(\mathscr{S}) = I_O(\mathscr{T}) + I_O(\mathscr{B})$ in (11.76c), we obtain from this data the moment of inertia $I_O(\mathscr{B})$ of the flywheel about its central axis:

$$I_O(\mathscr{B}) = \frac{ka^2}{2\pi^2 f_{\mathscr{S}}^2} - I_O(\mathscr{T}). \tag{11.76d}$$

In the event that $I_O(\mathscr{T})$ is not known or the system may need to be recalibrated, the frequency $f_{\mathscr{T}}$ of the table assembly alone may be measured to obtain by the same process $I_O(\mathscr{T}) = ka^2/2\pi^2 f_{\mathscr{T}}^2$. Thus, in terms of measurable data alone,

$$I_O(\mathscr{B}) = \frac{ka^2}{2\pi^2}\left(\frac{1}{f_{\mathscr{S}}^2} - \frac{1}{f_{\mathscr{T}}^2}\right). \tag{11.76e}$$

This example is typical of useful applications of dynamics in engineering design analysis.

11.12.2. Rotating Simple Pendulum

A rotating simple pendulum shown in the figure for Problem 6.47 consists of a bob of mass m constrained by a rigid wire of length l and negligible mass fastened to a smooth hinge O at r from the center C of a smooth table that rotates in the horizontal plane with a constant angular speed ω, as shown. Relative to an observer in the table frame, the pendulum oscillates with a finite amplitude angle β_0. First, we apply Lagrange's equations subject to the rheonomic constraint to derive the equation of motion of the bob relative to the table. We then relax the constraint, use Lagrange's equations to derive the equations of motion, and find exactly the nonconservative constraint tension in the wire as a function of the finite angular placement β alone. Finally, the angular placement and period of the finite amplitude motion of the bob are determined, thus solving the problem exactly and entirely.

11.12.2.1. Application of the Rheonomic Constraint

Introduce generalized coordinates $(q_1, q_2) = (\beta, \theta)$, where β is the angular placement of the pendulum relative to the table, and θ is the angular placement of the table in the ground frame. The rheonomic constraint yields $\theta = \omega t$, so the system has only one degree of freedom. The absolute velocity $\mathbf{v}_m$ of the pendulum bob referred to the moving frame $\varphi = \{O; \mathbf{i}_k\}$ is given by

$$\mathbf{v}_m = r\omega \sin\beta \mathbf{i} + (r\omega\cos\beta + l(\omega + \dot{\beta}))\mathbf{j}, \tag{11.77a}$$

and hence its total kinetic energy is

$$T = \frac{1}{2}m[r^2\omega^2 + 2rl\omega(\omega + \dot{\beta})\cos\beta + l^2(\omega + \dot{\beta})^2], \tag{11.77b}$$

where $\omega = \dot{\theta}$, a constant. The total potential energy $V = 0$; so $L = T$, which is independent of θ. The planar wire force $\mathbf{F} = -P\mathbf{i}$ on the bob does no work in the motion relative to the table, so it is evident that the generalized force $Q_\beta = 0$. The system is conservative; hence use of (11.77b) in (11.66) delivers the equation for the finite amplitude motion of the pendulum relative to the table:

$$\ddot{\beta} + \frac{r}{l}\omega^2 \sin\beta = 0. \tag{11.77c}$$

The wire constraining force, inconsequential to the bob's motion, is not determined by this analysis.

11.12.2.2. Equations of Motion with Relaxed Constraint

Now let us consider a new problem in which θ is treated as an independent variable in consideration of Lagrange's equations (11.73) to obtain

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\beta}}\right) - \frac{\partial T}{\partial \beta} = Q_\beta, \qquad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) = Q_\theta. \tag{11.77d}$$

The generalized forces Q_k are determined from the virtual work done by the wire constraining force $\mathbf{F} = -P\mathbf{i}$, due to the motion of the table frame. The virtual displacement may be read from (11.77a) by replacing ω with $\delta\theta$ and $\dot{\beta}$ with $\delta\beta$ to obtain $\delta\mathbf{x} = r\sin\beta\delta\theta\mathbf{i} + [(l + r\cos\beta)\delta\theta + l\delta\beta]\mathbf{j}$. Then

$$\delta\mathscr{W} = \mathbf{F}\cdot\delta\mathbf{x} = -rP\sin\beta\delta\theta = Q_\theta\delta\theta + Q_\beta\delta\beta, \tag{11.77e}$$

yields the generalized forces

$$Q_\theta = -rP\sin\beta, \qquad Q_\beta = 0. \tag{11.77f}$$

Since $Q_\theta \neq 0$, the absentee coordinate θ in (11.77b) is not ignorable. The wire tension does no work in the β-motion of the bob relative to the table, so it is evident that the generalized force Q_β should vanish, which it does. Moreover, notice that use of the constraint $\delta\theta = \omega\delta t = 0$ in (11.77e) shows only that $Q_\beta = 0$, but says nothing about Q_θ in the constrained case studied above. A neat alternative derivation of (11.77f) uses (11.20) for a particle, in which $\mathbf{F} = -P\mathbf{i}$ and $\dot{\mathbf{x}} = \mathbf{v}_m$ in (11.77a); namely,

$$Q_\beta = -P\mathbf{i}\cdot\frac{\partial\mathbf{v}_m}{\partial\dot{\beta}} = -P\mathbf{i}\cdot l\mathbf{j} = 0, \tag{11.77g}$$

$$Q_\theta = -P\mathbf{i}\cdot\frac{\partial\mathbf{v}_m}{\partial\dot{\theta}} = -P\mathbf{i}\cdot(r\sin\beta\mathbf{i} + (l + r\cos\beta)\mathbf{j}) = -Pr\sin\beta. \tag{11.77h}$$

Substituting (11.77b) and (11.77f) into (11.77d) and following some simplifications, we obtain the two equations

$$\ddot{\beta} + \frac{r}{l}\omega^2\sin\beta + \left(1 + \frac{r}{l}\cos\beta\right)\dot{\omega} = 0, \tag{11.77i}$$

$$ml(\ddot{\beta}(l + r\cos\beta) - r(2\omega + \dot{\beta})\dot{\beta}\sin\beta) + m(r^2 + l^2 + 2rl\cos\beta)\dot{\omega} = -Pr\sin\beta. \tag{11.77j}$$

These are two equations in three unknown quantities: P, β, and ω. Thus, suppose that ω is constant. Then (11.77i) reduces to (11.77c) which determines the oscillatory motion $\beta(t)$ of the bob relative to the table, and (11.77j) determines the wire tension force P acting on the bob in the moving table frame. The former is the primary equation of interest, readily derived without our having to find the inconsequential wire constraining force. On the other hand, it is important in engineering analysis that the nature of the forces that act on a dynamical system be known. A variety of methods are available to evaluate these. Here we continue with the case when ω is constant.

11.12.2.3. General and Exact Solution of the Equations of Motion

Integration of (11.77c) for the initial data $\dot{\beta}(0) = 0$ at $\beta(0) = \beta_0$ yields

$$\dot{\beta}^2 = 2\frac{r}{l}\omega^2(\cos\beta - \cos\beta_0). \tag{11.77k}$$

Now use of (11.77c) and (11.77k) in (11.77j) delivers the equation for the wire tension as an exact function of its placement β alone:

$$P(\beta) = ml\omega^2 \left[1 + \frac{r}{l}(3\cos\beta - 2\cos\beta_0) \pm 2\sqrt{2\frac{r}{l}(\cos\beta - \cos\beta_0)} \right], \quad (11.77l)$$

in which the $+$ sign corresponds to the case when $\beta(t)$ is increasing with time.

Further, integration of (11.77k) for increasing values of $\beta(t)$ determines the travel time as a function of the finite angular placement of the pendulum:

$$pt = \int_0^\beta \frac{d\beta}{\sqrt{2(\cos\beta - \cos\beta_0)}}, \qquad p \equiv \omega\sqrt{\frac{r}{l}}, \quad (11.77m)$$

where p, which depends on ω, characterizes the circular frequency of the pendulum in its small amplitude motion. The exact solution, therefore, is given by an elliptic integral of the first kind defined by (7.87d) with $k = \sin(\beta_0/2)$, $\sin(\beta/2) = k\sin\phi$, in accord with (7.87b). In consequence, the exact period of the oscillation is provided by (7.87f) and the motion may be read from (7.89b). Thus,

$$\beta(t) = 2\sin^{-1}[k\,\mathrm{sn}(pt)], \qquad \tau^* = \frac{4}{p}K(k), \quad (11.77n)$$

in which $K(k)$ is the complete elliptic integral of the first kind in (7.87e) and $\mathrm{sn}(pt)$ is the Jacobian elliptic sine function with properties (7.88h) and (7.88i). This concludes the fully exact solution for the motion (11.77n) and the wire tension (11.77l) of the rotating simple pendulum.

11.12.3. The Gyrocompass with Torsional Damping Revisited

Lagrange's equations are applied here to derive the equations for the small motion of the torsionally damped gyrocompass shown schematically in Fig. 10.12, page 450. The rotor has two degrees of freedom with generalized coordinates $(q_1, q_2) = (\theta, \alpha)$, where θ is the angular placement of the rotor about the $\mathbf{j}$-axis and α is the angular placement of the gimbal frame about the $\mathbf{k}$-axis of the gimbal reference frame $\varphi = \{C; \mathbf{i}_k\}$. With the aid of (10.81b) and (10.81d), the total kinetic energy of the rotor relative to its center of mass C is provided by (10.100):

$$T = \frac{1}{2}\boldsymbol{\omega}\cdot\mathbf{h}_C = \frac{1}{2}I_{11}\Omega^2\cos^2\lambda\sin^2\alpha + \frac{1}{2}I_{22}(\dot\theta + \Omega\cos\lambda\cos\alpha)^2 + \frac{1}{2}I_{11}(\dot\alpha + \Omega\sin\lambda)^2, \quad (11.78a)$$

where λ is the latitude angle and Ω is the Earth's very small angular rate of rotation, a rheonomic constraint. Here and below all infinitesimal terms of order Ω^2 are neglected. Equation (11.78a) thus simplifies to

$$T = \frac{1}{2}I_{22}(\dot\theta^2 + 2\Omega\dot\theta\cos\lambda\cos\alpha) + \frac{1}{2}I_{11}(\dot\alpha^2 + 2\Omega\dot\alpha\sin\lambda). \quad (11.78b)$$

The potential energy $V = 0$ and the supporting constraint forces are workless. In view of (10.81a) there is no torque about $\mathbf{j}$, so $\mathbf{M}_C \cdot \mathbf{j} = 0 = Q_\theta$, and hence θ is an ignorable coordinate. Therefore, the Lagrange equations (11.73) yield

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\alpha}}\right) - \frac{\partial T}{\partial \alpha} = Q_\alpha, \qquad \frac{\partial T}{\partial \dot{\theta}} = \gamma_\theta, \text{ a constant,} \tag{11.78c}$$

the last reflecting conservation of the generalized momentum $p_\theta = \gamma_\theta$, that is, the principle of conservation of moment of momentum about $\mathbf{j}$. This fact was not so apparent in our earlier discussion of this problem.

The damping torque about the $\mathbf{k}$-axis is defined by $\mathbf{M}_C \cdot \mathbf{k} = -2I_{11}\nu\dot{\alpha} = Q_\alpha$. Therefore, it follows from (11.78b) for a small compass drift angle α that the Lagrange equations (11.78c) for the damped gyrocompass, to the first order in α, yield

$$I_{22}(\dot{\theta} + \Omega\cos\lambda) = \gamma_\theta, \tag{11.78d}$$

$$\ddot{\alpha} + 2\nu\dot{\alpha} + p^2\alpha = 0, \tag{11.78e}$$

where p^2 is defined by (10.81j). By (11.78d), $\dot{\theta}$ is a constant: $\dot{\theta} = \dot{\theta}(0) \equiv \omega_0$ given by the initial data. Hence, $\theta(t) = \omega_0 t$. It is seen that (11.78e) is the same as (10.81l), whose general solution is provided in (10.81m).

11.12.4. Motion of a Symmetrical Top about a Fixed Point

A homogeneous rigid top (a body of revolution) of mass m rotates about a point O fixed on a rough horizontal surface in the ground frame $\Phi = \{O; \mathbf{I}_k\}$ in a gravitational field. The three independent Euler angles (ϕ, θ, ψ) introduced in Chapter 3, Fig. 3.14, page 209, characterize the general rotational orientation of the top, as shown in Fig. 11.6. The equations of motion will be derived by Lagrange's method. Afterwards, the physical nature of the top's motion is described.

11.12.4.1. Equations of Motion for the Top

To find the total kinetic energy of the top, we first determine its total angular velocity referred to the principal body frame $\varphi = \{O; \mathbf{i}_k\}$ in Fig. 11.6, wherein several reference frames are identified. The top turns about the $\mathbf{K}$-axis of the ground frame $0 = \Phi$ with angular velocity $\boldsymbol{\omega}_{10} = \dot{\phi}\mathbf{K}$, followed by a rotation about the $\mathbf{i}'$-axis of frame $1 = \{O; \mathbf{i}_k'\}$ with angular velocity $\boldsymbol{\omega}_{21} = \dot{\theta}\mathbf{i}'$, and finally, it spins about the $\mathbf{k}$-axis of the body frame $3 = \varphi$, with angular spin $\boldsymbol{\omega}_{32} = \dot{\psi}\mathbf{k}$. Hence, the total angular velocity of the top in Φ is $\boldsymbol{\omega} \equiv \boldsymbol{\omega}_{30} = \dot{\psi}\mathbf{k} + \dot{\theta}\mathbf{i}' + \dot{\phi}\mathbf{K}$. With $\mathbf{i}' = \cos\psi\mathbf{i} - \sin\psi\mathbf{j}$, $\mathbf{K} = \cos\theta\mathbf{k} + \sin\theta\mathbf{j}'$, $\mathbf{j}' = \sin\psi\mathbf{i} + \cos\psi\mathbf{j}$, the total angular velocity of the top referred to φ is given by

$$\boldsymbol{\omega} = (\dot{\theta}\cos\psi + \dot{\phi}\sin\theta\sin\psi)\mathbf{i} + (\dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi)\mathbf{j} + (\dot{\psi} + \dot{\phi}\cos\theta)\mathbf{k}. \tag{11.79a}$$

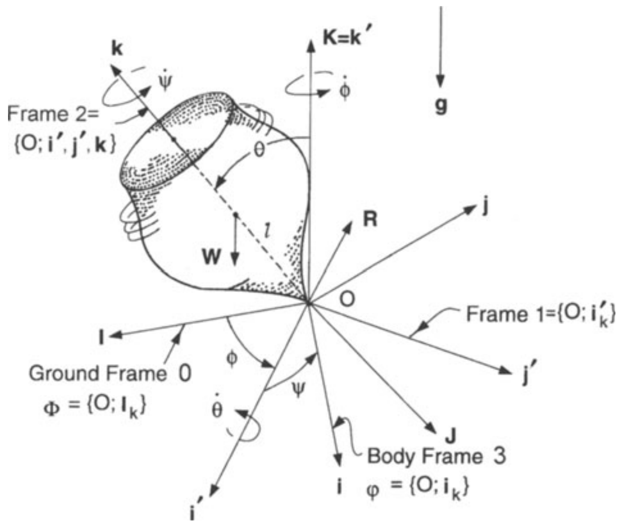


Figure 11.6. Symmetrical top rotating about a fixed point.

The homogeneous top is symmetrical about its **k**-axis with principal moments of inertia $I_{11} = I_{22}$. For future convenience, let $I_1 \equiv I_{11}$, $I_3 \equiv I_{33}$, and assume that $I_1 \neq I_3$. The rough surface assures that point O of the top does not slide on the surface. Then, by (10.102), the total kinetic energy relative to the fixed point O of the principal body frame φ is given by $T = \frac{1}{2}\boldsymbol{\omega} \cdot \mathbf{I}_O\boldsymbol{\omega} = \frac{1}{2}I_1(\omega_1^2 + \omega_2^2) + \frac{1}{2}I_3\omega_3^2$, which, by (11.79a), yields

$$T = \frac{1}{2}I_1(\dot\theta^2 + \dot\phi^2 \sin^2\theta) + \frac{1}{2}I_3(\dot\psi + \dot\phi\cos\theta)^2. \tag{11.79b}$$

The total potential energy is

$$V = mgl\cos\theta. \tag{11.79c}$$

The supporting force **R** at O is workless, and hence the holonomic system is conservative with Lagrangian function $L = T - V$:

$$L = \frac{1}{2}I_1(\dot\theta^2 + \dot\phi^2 \sin^2\theta) + \frac{1}{2}I_3(\dot\psi + \dot\phi\cos\theta)^2 - mgl\cos\theta, \tag{11.79d}$$

in which both ψ and ϕ are ignorable coordinates. Hence, in accordance with (11.41),

$$\frac{\partial L}{\partial\dot\psi} = I_3(\dot\psi + \dot\phi\cos\theta) = \alpha, \tag{11.79e}$$

$$\frac{\partial L}{\partial\dot\phi} = I_1\dot\phi\sin^2\theta + I_3(\dot\psi + \dot\phi\cos\theta)\cos\theta = \beta, \tag{11.79f}$$

are constants of the motion provided by initial conditions. These equations determine $\dot{\phi}$ and $\dot{\psi}$ as functions of θ; we find

$$\dot{\phi} = \frac{\beta - \alpha\cos\theta}{I_1 \sin^2\theta}, \qquad \dot{\psi} = \frac{\alpha}{I_3} - \cos\theta\left(\frac{\beta - \alpha\cos\theta}{I_1\sin^2\theta}\right). \tag{11.79g}$$

Finally, by (11.66), the third of Lagrange's equations $d(\partial L/\partial\dot{\theta})/dt - \partial L/\partial\theta = 0$ yields

$$I_1\ddot{\theta} + (I_3 - I_1)\dot{\phi}^2 \sin\theta\cos\theta + I_3\dot{\psi}\dot{\phi}\sin\theta - mgl\sin\theta = 0. \tag{11.79h}$$

This equation, upon integration with specified initial data, determines the motion $\theta(t)$ of the top in a vertical plane through the **K**-axis, and which rotates about this fixed spatial axis with variable angular speed $\dot{\phi}(t)$ given by (11.79g), as shown in Fig. 11.6.

11.12.4.2. General Solution of the Equations of Motion

The main problem now is to solve the nonlinear equation (11.79h) for $\theta(t)$, and then use the result in (11.79g) to obtain $\phi(t)$ and $\psi(t)$. This is possible, in principle, but not entirely in terms of elementary functions. To begin, recall (11.79e) and the component ω_3 in (11.79a) to obtain

$$I_3(\dot{\psi} + \dot{\phi}\cos\theta) = I_3\omega_3 = \alpha, \tag{11.79i}$$

i.e., the total angular spin of the top about its **k**-axis is constant: $\omega_3 = \alpha/I_3$.

Because the system is conservative, the total energy $T + V = E$, a constant. Therefore, with (11.79b) and (11.79c), the first integral of (11.79h) is thus given by

$$\frac{1}{2}I_1(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta) + \frac{1}{2}I_3(\dot{\psi} + \dot{\phi}\cos\theta)^2 + mgl\cos\theta = E, \tag{11.79j}$$

certainly not evident. With the aid of (11.79i), this reduces to

$$\frac{1}{2}I_1(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta) + mgl\cos\theta = E_0, \tag{11.79k}$$

in which the constant $E_0 \equiv E - \frac{1}{2}I_3\omega_3^2$. Use of the first expression in (11.79g) yields

$$\frac{1}{2}I_1\dot{\theta}^2 + V(\theta) = E_0, \tag{11.79l}$$

where

$$V(\theta) \equiv \frac{(\beta - \alpha\cos\theta)^2}{2I_1\sin^2\theta} + mgl\cos\theta. \tag{11.79m}$$

Equation (11.79l) has the appearance of an energy equation for a single degree of freedom system for which $V(\theta)$ is the *apparent potential energy*. Its integration delivers the travel time t in the motion $\theta(t)$:

$$t = \pm\sqrt{\frac{I_1}{2}}\int_{\theta_0}^{\theta}\frac{d\theta}{\sqrt{E_0 - V(\theta)}}. \tag{11.79n}$$

The appropriate sign is to be fixed in accordance with the initial conditions for which $\theta_0 = \theta(0)$. The inverse of this integral determines $\theta(t)$, and integration of (11.79g) yields $\phi(t)$ and $\psi(t)$. The three Euler angles (ϕ, θ, ψ) thus determine the orientation of the top at each instant and provide the complete formal solution of the problem. For brevity, however, we omit these details‡ and focus on some interesting qualitative aspects of the top's rotational motion.

11.12.4.3. Physical Characterization of the Motion

Let us return to the apparent energy equation (11.79l) and recall that the phase plane diagram $\dot{\theta}$ versus θ characterizes the curves of constant energy. These curves are closed and the motion is periodic if and only if there are exactly two values $\theta^* = \theta_k^*$, $k = 1, 2$, called the *turning points*, for which $\dot{\theta}(\theta^*) = 0$ in (11.79l). The turning points of the motion $\theta(t)$ are determined by

$$V(\theta^*) = \frac{(\beta - \alpha\cos\theta^*)^2}{2I_1\sin^2\theta^*} + mgl\cos\theta^* = E_0, \tag{11.79o}$$

wherein the energy constant E_0 is fixed by the initial data.

‡ By the introduction of a change of variable $x = \cos\theta$ for $-1 \le x \le 1$ it can be shown that the energy equation (11.79l) may be written as $\dot{x}^2 = f(x)$, where

$$f(x) \equiv \frac{2}{I_1}(E_0 - mglx)(1 - x^2) - \frac{1}{I_1^2}(\beta - \alpha x)^2. \tag{A}$$

This is a real cubic polynomial having three real roots x_k such that $-1 \le x_1 \le x_2 \le 1 \le x_3$, the root x_3 having no physical relevance. Since $f(x) = \dot{x}^2$ cannot be negative, x oscillates between the physically realizable values x_1 and x_2; that is, θ oscillates between the turning points θ_1 and θ_2 at which $\dot{\theta}(\theta) = 0$. (Note that in the text, we write $\theta_1^* = \theta_2 \le \theta_1 = \theta_2^*$.) With

$$\dot{x}^2 = \frac{2mgl}{I_1}(x - x_1)(x - x_2)(x - x_3), \tag{B}$$

we are led to an elliptic integral of the first kind. This eventually follows from (11.79n). Then it turns out that the inverse function for the nutational motion $\theta(t) \in [\theta_2, \theta_1]$ is determined precisely in terms of a Jacobian elliptic sine function of t; namely,

$$\cos\theta = x = x_1 + (x_2 - x_1)\mathrm{sn}^2[p(t - t_0)], \tag{C}$$

where $p = \sqrt{mgl(x_3 - x_1)/2I_3}$. Use of (C) in (11.79g) yields exact formal solutions for $\phi(t)$ and $\psi(t)$ in terms of elliptic integrals of the third kind with modulus $k = \sqrt{(x_2 - x_1)/(x_3 - x_1)}$. These and other analytical details and further discussion of the nutational–precessional paths of the motion traced by the unit vector $\mathbf{k}$ on the surface of a unit sphere centered at the fixed point O, as shown in Fig. 11.7, may be found in the texts by Greenwood, Marion, Rosenberg, Synge and Griffith, and Whittaker.

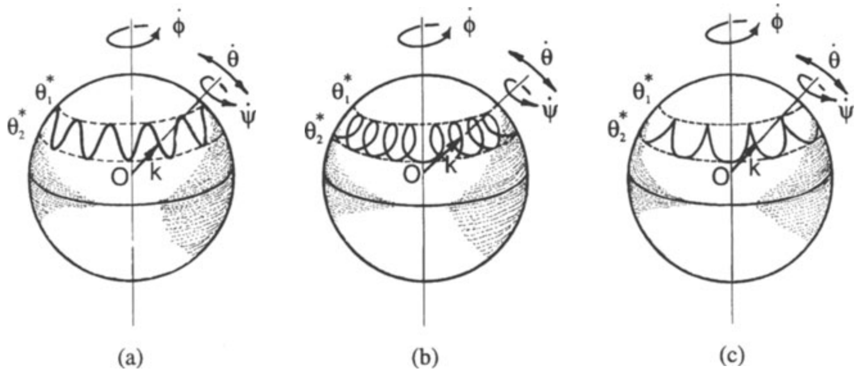


Figure 11.7. Geometrical description of a top's nutational–precessional motion.

In all real motions, (11.79l) and (11.79o) show that $E_0 = V(\theta^*) \geq V(\theta)$. Hence, the portion of the plane curve $y = V(\theta)$ of interest is situated below the horizontal line $y = E_0$. It is seen from (11.79m) that $V(\theta) \to +\infty$ at the extremes[§] $\theta = 0, \pi$; and hence the apparent potential energy function $y = V(\theta)$ must have a minimum value at some intermediate point θ_0^*, the point at which, by (11.79l), $\dot{\theta}$ has its greatest value. Consequently, the graph $y = V(\theta)$ is concave upward, somewhat like a skewed parabola; so the horizontal line $y = E_0$ must intersect this graph at precisely two points θ_1^* and θ_2^* for which (11.79o) holds. Therefore, the motion $\theta(t)$ is periodic; the top oscillates between the extreme angular positions θ_1^* and θ_2^* from the vertical spatial **K**-axis in Fig. 11.6, points at which $\dot{\theta}(\theta_k^*) = 0$. This oscillation phenomenon is called *nutation*. At the same time, by the first relation in (11.79g), the axis of the top in Fig. 11.6 turns as a function of θ about the vertical **K**-axis. This variable rotational motion $\phi(\theta(t))$, induced by the gravitational torque about O, is called *precession*. In the special case for which the line $y = E_0 = V(\theta_0^*)$, the minimum value of $V(\theta)$, $\theta(t)$ is restricted to the single fixed value θ_0^*; and the top, inclined at this fixed angle, turns about the **K**-axis with a constant angular speed $\dot{\phi}(\theta_0^*)$, so this motion is called a *steady precession*.

The essential features of the simultaneous nutational–precessional motion of the top may be visualized in Fig. 11.7 by tracking the end point of the unit vector **k** (the spin axis) on the surface of a unit sphere centered at O. The precessional rotation of the axis of the top as a function of the nutation angle $\theta(t)$ is described by $\dot{\phi}$ in (11.79g). The specific geometry depends on how the top is started, that is, it depends on the initial values of θ, $\dot{\theta}$, $\dot{\phi}$, and $\dot{\psi}$ (these determine α, β, E_0). The simplest undulatory case described above in which the top traces a sinusoidal-like

[§] The actual physical model is of no concern in the geometrical description of the function $V(\theta)$. The motion of a top (any body of revolution) as we commonly think of it, however, is restricted to values of $\theta < \pi/2$. For values of $\theta > \pi/2$, any body of revolution supported by a smooth ball joint at O on the axis of symmetry is known as a *gyroscopic pendulum*.

trajectory between two horizontal colatitude limit circles θ_1^* and θ_2^* is shown in Fig. 11.7a. In this case, $\dot{\phi}(t) > 0$ for all t during the motion; the **k**-axis of the top moves up and down (nutation) at the rate $\dot{\theta}$ as the top rotates (precession) about the fixed vertical **K**-axis at the rate $\dot{\phi}$. The looping motion shown in Fig. 11.7b is characterized by $\dot{\phi}$ increasing, then decreasing, over and over again. Therefore, there must be a value θ_l at which $\dot{\phi}(\theta_l) = 0$. The criterion for loops to occur, therefore, by (11.79g), is that $\theta_l = \cos^{-1}(\beta/\alpha)$ must lie in the interval between θ_1^* and θ_2^*. Finally, when initially the axis of the spinning top is fixed at an angle $\theta_0 = \theta_1^*$ and released with $\dot{\phi}(\theta_1^*) = 0$, the top at first falls to θ_2^*, but recovers and rises again to $\theta_1^* = \cos^{-1}(\beta/\alpha)$, and this nodding motion is repeated over and over. As the top falls, the gravitational torque induces a precession in the direction of the torque, so the axis of spin turns about **K** at the rate $\dot{\phi}$. This phenomenon leads to the cuspidal motion in Fig. 11.7(c). We shall not pause to explore the analytical details characterizing these geometrical properties; rather, we turn to the general problem of a steady precession for which $\theta(t) = \theta_0^*$, $\dot{\theta} = 0$, and $\dot{\phi}$ and $\dot{\psi}$ are constants.

11.12.4.4. Steady Precession and Stability of the Motion of a Top

Differentiation of (11.79l) with respect to θ yields the modified equation of motion for $\theta(t)$:

$$I_1\ddot{\theta} + \frac{dV(\theta)}{d\theta} = 0. \tag{11.79p}$$

The geometrical description of the apparent potential function, however, showed that $V(\theta)$ has a minimum at a point $\theta_0^* \in [\theta_1^*, \theta_2^*]$; hence, $dV(\theta)/d\theta = 0$ at $\theta = \theta_0^*$. (It is shown later that in fact $V(\theta)$ has a minimum at θ_0^*.) Hence, from (11.79p), the position θ_0^* is a relative equilibrium position in θ at which the top maintains a tilted position at an angle θ_0^* from the vertical axis, and the motion is a *steady precession* of the top about the vertical **K**-axis at a constant angular rate $\dot{\phi}_0 \equiv \dot{\phi}(\theta_0^*)$, that is,

$$\dot{\phi}_0 = \frac{\beta - I_3\omega_3\cos\theta_0^*}{I_1\sin^2\theta_0^*}, \tag{11.79q}$$

the top now having a constant spin $\dot{\psi}_0 \equiv \dot{\psi}(\theta_0^*)$ about its body axis **k**, in accordance with (11.79g). In addition to θ_0^*, both rates depend on the constants of the motion, β, ω_3, whose values must be appropriately chosen to support the steady precession in accordance with assigned initial data.

The point θ_0^* of steady precession, from (11.79m), is thus determined by

$$\left.\frac{dV}{d\theta}\right|_{\theta_0^*} = (I_3\omega_3\dot{\phi}_0 - I_1\dot{\phi}_0^2\cos\theta_0^* - mgl)\sin\theta_0^* = 0, \tag{11.79r}$$

where $\theta_1^* < \theta_0^* < \theta_2^*$. The same relation follows from (11.79h) in which $\theta(t) = \theta_0^*$.

The solutions $\theta_0^* = 0, \pi$ are considered separately later; otherwise, (11.79r) yields a quadratic equation for $\dot{\phi}_0(\theta_0^*)$:

$$(I_1 \cos\theta_0^*)\dot{\phi}_0^2 - I_3\omega_3\dot{\phi}_0 + mgl = 0, \tag{11.79s}$$

with solution

$$\dot{\phi}_0 = \frac{I_3\omega_3}{2I_1\cos\theta_0^*}\left(1 \pm \sqrt{1 - \frac{4mglI_1\cos\theta_0^*}{I_3^2\omega_3^2}}\right). \tag{11.79t}$$

By (11.79q), this is a transcendental equation for $\beta - I_3\omega_3\cos\theta_0^*$, and hence θ_0^*, whose solution depends on the constants β, ω_3. This is solvable for θ_0^* only by trial and error, when all constants are assigned. Of course, (11.79t) also is an equation for $\dot{\phi}_0$ that now depends only on ω_3 and θ_0^*; and for real values of $\dot{\phi}_0$, it is necessary that

$$I_3^2\omega_3^2 \geq 4mglI_1\cos\theta_0^*, \tag{11.79u}$$

in which case there are two speeds of steady precession. It follows that a steady precession at a fixed angle $\theta_0^* \in (\theta_1^*, \theta_2^*)$ is possible only when the total angular spin ω_3 is not less than its critical limit $\omega_3^c \equiv (2/I_3)(mglI_1\cos\theta_0^*)^{1/2}$ for which the corresponding critical steady precessional rate is

$$\dot{\phi}_0^c \equiv \frac{I_3\omega_3^c}{2I_1\cos\theta_0^*} = \frac{2mgl}{I_3\omega_3^c}. \tag{11.79v}$$

Therefore, consider a fast spinning top for which $\omega_3 \gg \omega_3^c$. Then use of the binomial theorem in the radicand of (11.79t) leads to the following two approximate speeds of steady precession for the fast top:

$$\dot{\phi}_0^+ = \frac{I_3\omega_3}{I_1\cos\theta_0^*}\left(1 - \frac{mglI_1\cos\theta_0^*}{I_3^2\omega_3^2}\right), \qquad \dot{\phi}_0^- = \frac{mgl}{I_3\omega_3}, \tag{11.79w}$$

where the signs correspond to those in (11.79t). The first, $\dot{\phi}_0^+$, is called the *fast precession*; it grows increasingly larger with ω_3. For sufficiently great values of ω_3 this further simplifies to $\dot{\phi}_0^+ = I_3\omega_3/(I_1\cos\theta_0^*)$, which is independent of the weight of the top. The second speed, $\dot{\phi}_0^-$, is called the *slow precession*; this is independent of θ_0^* and it goes toward zero as ω_3 grows increasingly great. The latter is the precession rate commonly observed in a fast spinning top (or gyroscope). Returning to (11.79s), we see that for $\theta_0^* = \pi/2$, $\dot{\phi}_0 = \dot{\phi}_0^-$ is an exact slow precession result. The foregoing results hold for $0 < \theta_0^* < \pi/2$, certainly the physical case for a top. More generally, however, for a gyroscopic pendulum $\pi/2 < \theta_0^* < \pi$ is possible. In this case, the radicand of (11.79t) is always positive, so there is no critical value of ω_3. The slow precession of the gyroscopic pendulum is the same as before, but its fast precession is greater and has the opposite direction.

Either of the speeds (11.79w) is possible provided that the top is started so that the initial data precisely satisfies (11.79s) and (11.79u). But it is not yet known whether these steady motions are stable. To investigate the infinitesimal stability of a steady precession, let $\theta(t) = \theta_0^* + \delta(t)$ and $\dot\phi(t) = \dot\phi_0 + \dot\varepsilon(t)$, where $\delta(t)$ and $\dot\varepsilon(t)$ are infinitesimal disturbances from the top's steady motion at θ_0^*, and assume that the constants β and ω_3 of the steady motion are unchanged by the disturbance. Only the change in $\theta(t)$ need be considered in the perturbation of (11.79p); and, by the first of (11.79g), $\dot\varepsilon$ is proportional to δ. Thus, with $dV(\theta)/d\theta = dV(\theta)/d\theta|_{\theta_0^*} + d^2V(\theta)/d\theta^2|_{\theta_0^*}\,\delta$ to the first order in δ, and use of (11.79r), we obtain from (11.79p) the equation for the perturbed motion:

$$I_1\ddot\delta + d^2V(\theta)/d\theta^2\big|_{\theta_0^*}\,\delta = 0. \tag{11.79x}$$

Therefore, for infinitesimal stability it is necessary that $d^2V(\theta)/d\theta^2\big|_{\theta_0^*} > 0$. This is a condition necessary in order that $V(\theta)$ shall have a minimum at θ_0^*, as required earlier. This is now confirmed analytically. With the aid of the first relation in (11.79g) and noting (11.79s), we find

$$\left.\frac{d^2V(\theta)}{d\theta^2}\right|_{\theta_0^*} = \frac{1}{I_1\dot\phi_0^2}\left[\left(mgl - I_1\dot\phi_0^2\cos\theta_0^*\right)^2 + I_1^2\dot\phi_0^4\sin^2\theta_0^*\right] > 0. \tag{11.79y}$$

Hence, the function $V(\theta)$ has a minimum at θ_0^*; and the arbitrarily small disturbance of the steady precession is oscillatory, and hence stable¶. The proportionality of $\dot\varepsilon$ and δ implies that ε also is periodic. Therefore, both values of $\dot\phi_0$ given by (11.79t) and subject to (11.79u) correspond to stable motions of steady precession.

Finally, let us consider a so-called *sleeping top* that merely spins about the vertical axis so that $\theta(t) = \theta_0^* = 0$ and $\dot\theta(t) = 0$. This steady motion is a solution of the equation of motion (11.79h) and for which, from (11.79e) and (11.79f), the constants of the motion satisfy $\beta = \alpha = I_3\omega_3$; and, by (11.79g), $\dot\phi(\theta_0^*) = \dot\phi_0 = \alpha/2I_1$. It is evident that if the total spin rate ω_3 of the top in its vertical position is not sufficiently great, any slight disturbance of the top will cause it to wobble, fall down, and roll to rest. So, our intuition suggests that there exists a critical total spin rate below which the motion of the top is unstable. To explore this, we put $\theta(t) = \theta_0^* + \delta(t) = \delta(t)$ and $\dot\phi = \dot\phi_0 + \dot\varepsilon$ into the equation of motion (11.79h),

¶ Alternatively, introducing both $\theta(t) = \theta_0^* + \delta(t)$ and $\dot\phi(t) = \dot\phi_0 + \dot\varepsilon(t)$ in (11.79g) and (11.79h), assuming that the moment of momentum $I_3\omega_3$ is unchanged and ultimately removing it by use of (11.79s), we eventually derive the two equations

$$\begin{gathered} I_1\dot\phi_0\sin\theta_0^*\dot\varepsilon + \left(I_1\dot\phi_0^2\cos\theta_0^* - mgl\right)\delta = 0, \\ I_1\dot\phi_0\ddot\delta + \sin\theta_0^*\left(mgl - I_1\dot\phi_0^2\cos\theta_0^*\right)\dot\varepsilon + I_1\dot\phi_0^3\sin^2\theta_0^*\delta = 0. \end{gathered} \tag{D}$$

Substitution of the first relation for $\dot\varepsilon$ into the second equation yields the incremental equation of motion for δ:

$$I_1^2\dot\phi_0^2\ddot\delta + \left(\left(mgl - I_1\dot\phi_0^2\cos\theta_0^*\right)^2 + I_1^2\dot\phi_0^4\sin^2\theta_0^*\right)\delta = 0, \tag{E}$$

in which the coefficient of δ is plainly positive and has the same form as (11.79y). The solution $\delta(t)$ is periodic. It follows from the first equation in (D) that the disturbance ε has the same period. Therefore, the motion of steady precession is stable, as shown more directly by (11.79x).

note that $\dot{\psi}(\theta_0^*) = (\alpha/2I_1)(2I_1/I_3 - 1)$, and thus obtain to the first order,

$$4I_1^2\ddot{\delta} + \left(I_3^2\omega_3^2 - 4I_1 mgl\right)\delta = 0. \tag{11.79z}$$

Consequently, the disturbance of the sleeping top is oscillatory, hence stable, provided that its total spin $\omega_3^s > \dfrac{2}{I_3}\sqrt{I_1 mgl}$. This value is just a bit greater than the critical rate $\omega_3^c = (2/I_3)(I_1 mgl \cos\theta_0^*)^{1/2}$ for a steady precession at an angle $\theta_0^* \neq 0$, which reduces to the former when $\theta_0^* = 0$. Hence, the sleeping top is stable provided its spin $\omega_3^s > \omega_3^c = (2/I_3)(I_1 mgl)^{1/2}$.

Exercise 11.10. Show that the vertical configuration of a "sleeping" gyroscopic pendulum for which $\theta(t) = \theta_0^* = \pi$ and $\dot{\theta}(t) = 0$ always is a stable configuration. □

This concludes‖ our study of the motion of a top about a fixed point. Note, however, that the Earth behaves like a top whose center revolves around the Sun, and similar top phenomena are characteristic of the motions of gyroscopes, spinning projectiles, bicycles, motorcycles, engine flywheels, propellers, jet engines, and more. So, with the beginnings sketched above, we have accomplished more than simply analyzing the motion of a child's toy. The reader is now equipped to explore by a variety of methods more advanced areas of gyrodynamics.

11.13. Introduction to the Theory of Vibrations

The equations of motion of many dynamical systems are nonlinear differential equations for which exact solutions are beyond reach of analysis, so either computational or analytical perturbation methods are applied to effect useful approximate solutions that shed light on important and interesting nonlinear phenomena. Discussion of various perturbation techniques may be found in books dedicated to this field. Here we focus on a fundamental perturbation method of linearization to study the theory of small vibrations of multidegree of freedom dynamical systems, a general method of approximate analysis based on a second order power series expansion of the motion about a stable, static equilibrium configuration of a nonlinear, holonomic system. The smallness approximation leads to a system of linearized differential equations for which the general theory of simultaneous linear equations is directly applicable. While certainly some interesting nonlinear phenomena and potentially useful information may be lost in our adopting only

‖ We have barely scratched the surface of a fascinating but difficult class of problems analyzed systematically and extensively by F. Klein and A. Sommerfeld in their treatise *Über die Theorie des Kreisels*, B.G. Teubner, Leipzig, 1910. See also the text by A. Gray, *A Treatise on Gyrostats and Rotational Motion*, Macmillan, London, 1918; Dover, New York, 1959.

a lowest order approximation in the final equations of motion, the linearization procedure offers useful insight into the first order physical nature of an otherwise complicated multidegree of freedom nonlinear system. Moreover, the theory provides a general framework within which many difficult problems may be solved.

11.13.1. Small Oscillations of a Simple Pendulum Revisited

In our earlier studies, linearization of the equations of motion for small vibrations of a specific dynamical system was applied after the general system of nonlinear equations was derived. This procedure, however, can be greatly simplified for conservative holonomic systems with scleronomic constraints. For these dynamical systems, the terms in the Lagrangian function can be expanded in power series at the outset of the problem formulation to retain all terms up to those quadratic in the variables. Then the equations of motion automatically will be linear in these variables. To illustrate the idea, consider the small motion of a pendulum about its stable equilibrium position. Recall that quadratic terms in θ and $\dot{\theta}$ are neglected in linearization of the equation of motion (11.36c). Since this equation is obtained by differentiation of the Lagrangian function, the linearization process may begin by our writing the potential and kinetic energies as quadratic functions of θ and $\dot{\theta}$, respectively. The Lagrangian $L(\dot{\theta}, \theta)$ for the pendulum problem is given exactly in (11.36a), where the kinetic energy $T(\dot{\theta}, \theta) = \frac{1}{2}M\dot{\theta}^2$ is independent of θ and already quadratic in $\dot{\theta}$, and $M \equiv m\ell^2$. The potential energy, however, with the power series expansion of $\cos\theta \cong 1 - \frac{1}{2}\theta^2$ to retain terms quadratic in θ in (11.36a), simplifies to $V(\theta) = \frac{1}{2}K\theta^2$, where $K \equiv mg\ell$. Hence, the power series expansion of the Lagrangian function to terms of the *second order* in $\dot{\theta}$ and θ is thus given by $L(\dot{\theta}, \theta) = \frac{1}{2}M\dot{\theta}^2 - \frac{1}{2}K\theta^2$; and, by (11.35), we find $M\ddot{\theta} + K\theta = 0$, the familiar linearized form of the exact nonlinear equation of motion (11.36c), for small oscillations of the pendulum about its equilibrium state at $\theta = 0$.

The same linearization process may be applied to any multidegree of freedom, conservative scleronomic system for which no generalized coordinates are ignorable and all remain small over time. For holonomic systems with rheonomic (time dependent) constraints and systems with ignorable coordinates, it proves best to linearize the final equations of motion, as before. For conservative scleronomic systems, however, the Lagrangian does not depend explicitly on time and the static equilibrium states in an inertial frame are readily determined by our taking all $\dot{q}_k = 0$, and hence $T = 0$, in Lagrange's equations (11.35), to obtain the static equilibrium equations $\partial V(q_r)/\partial q_k = 0$. These determine the values $q_k = q_k^*$ of the generalized coordinates in the static equilibrium state, some of which may not be stable. In the pendulum, Example 11.2, page 508, $\partial V(\theta)/\partial\theta = mg\ell\sin\theta$ vanishes at $q_1^* = \theta = 0, \pi$, the latter being an unstable equilibrium solution. Moreover, $\partial^2 V(\theta)/\partial\theta^2 = mg\ell\cos\theta$; and hence at the stable configuration $\theta = 0$, $\partial^2 V(\theta)/\partial\theta^2 = mg\ell > 0$, whereas $\partial^2 V(\theta)/\partial\theta^2 = -mg\ell < 0$ at the unstable state $\theta = \pi$. More generally, we have the following stability criterion.

Energy criterion for stability of static equilibrium: *The static equilibrium configuration $\mathscr{E}_s$ of a conservative scleronomic, holonomic system having n degrees of freedom with generalized coordinates $q_k = q_k^*$ in $\mathscr{E}_s$ is infinitesimally stable if the total potential energy of the linearized system is positive definite, that is, if*

$$\left.\frac{\partial^2 V}{\partial q_k \partial q_l}\right|_{q_k^*} u_k u_l > 0, \tag{11.80}$$

for all nonzero, n-dimensional vectors $\mathbf{u} = (u_k)$.

If the quadratic form (11.80) vanishes for some choice of u_k that are not all zero but is otherwise positive, the static equilibrium configuration at $q_k = q_k^*$ is called *neutrally stable*. If the quadratic form is negative for any choice of u_k, the equilibrium configuration is *unstable* at $q_k = q_k^*$. We shall return to this energy criterion momentarily in the presentation of the Lagrangian analysis of the theory of small vibrations about a stable equilibrium configuration of a nonlinear system.

11.13.2. The Theory of Small Vibrations

To formulate the general problem of small vibrations of a conservative and scleronomic holonomic system, let u_k denote small disturbances from a stable, static equilibrium state with corresponding specified coordinate values q_k^* so that the system has the perturbed generalized coordinates $q_k = q_k^* + u_k$, $k = 1, 2, \ldots, n$, the number of degrees of freedom. A Taylor series expansion of the total potential energy function about q_k^* to terms of the second order in u_k yields

$$V(q_r) = V^* + \frac{\partial V^*}{\partial q_k} u_k + \frac{1}{2} K_{kl} u_k u_l, \tag{11.81}$$

where repeated indices are summed over n, $V(q_r) \equiv V(q_1, q_2, \ldots, q_n)$ as usual, and quantities denoted by $*$ are evaluated in the static equilibrium state at $q_k = q_k^*$. For instance, $V^* = V(q_r^*)$, $\partial V^*/\partial q_k = \partial V(q_r)/\partial q_k|_{q_k = q_k^*}$, and, by definition, the constants

$$K_{kl} \equiv \frac{\partial^2 V^*}{\partial q_k \partial q_l} \tag{11.82}$$

are called *stiffness coefficients*. The series approximation (11.81) requires that *all* of the generalized coordinates are changed by a small perturbation, none are ignorable, and all remain small in time. Since only derivatives of V enter the equations of motion, we may omit the constant term V^* and note that in the equilibrium state $\partial V^*/\partial q_k = 0$. So far, the stiffness coefficients generally depend on the q_k^*s; however, no generality is lost in our assuming henceforward that all $q_k^* = 0$, and hence the generalized coordinates are measured from the equilibrium configuration. Then $u_r = q_r$ and the total potential energy (11.81) of the system may be written as a homogeneous quadratic function of the perturbed generalized coordinates q_k alone:

$$V(q_r) = \frac{1}{2} K_{kl} q_k q_l. \tag{11.83}$$

The matrix of stiffness coefficients (11.82) is square and symmetric: $K_{kl} = K_{lk}$, and the criterion (11.80) for stability of the equilibrium configuration requires, in matrix notation, that $K_{kl} q_k q_l \equiv K u \cdot u > 0$ for all nonzero vectors $u = (q_k)$. Therefore, for small displacements q_k from a stable, static equilibrium configuration, $V(q_r)$ is a positive definite, homogeneous quadratic form. Hence, alternatively, the stability criterion (11.80) holds if and only if $\det K$ and all of its principal minors are positive.

The total kinetic energy of a scleronomic system is a positive definite** quadratic function $T = \frac{1}{2} M_{kl}(q_r) \dot{q}_k \dot{q}_l$ of the perturbed generalized velocity components $\dot{u}_k = \dot{q}_k$. Because retention of terms linear and higher in the Taylor series expansion of $M_{kl}(q_r)$ about $q_k = q_k^* = 0$ introduces terms of order greater than the second in the total kinetic energy function, the coefficients $M_{kl}(q_r)$ may be replaced by their constant values in the equilibrium state: $M_{kl}(q_r) \cong M_{kl}^* \equiv M_{kl}$. The kinetic energy is then a homogeneous, positive definite quadratic function of the perturbed generalized velocities alone:

$$T(\dot{q}_r) = \frac{1}{2} M_{kl} \dot{q}_k \dot{q}_l. \tag{11.84}$$

The constants M_{kl} are called *inertia coefficients.* The matrix of inertia coefficients is square and symmetric: $M_{kl} = M_{lk}$.

Now, form from (11.83) and (11.84) the Lagrangian function

$$L(q_r, \dot{q}_r) = \frac{1}{2} M_{kl} \dot{q}_k \dot{q}_l - \frac{1}{2} K_{kl} q_k q_l. \tag{11.85}$$

Then Lagrange's equations (11.66), with the symmetry of K_{kl} and M_{kl} in mind, yield a system of n ordinary linear differential equations for small vibrations about

** It is easily seen that, in general, a quadratic form $F = Pu \cdot u$ that is positive definite in one reference system is positive definite in every reference system related to the first by an orthogonal transformation A. Consider the transformed quadratic form $F' = P'u' \cdot u'$, where $u' = Au$ denotes the transformed matrix vector. Then $F' = P'u' \cdot u' = P'Au \cdot Au = A^T(P'Au) \cdot u = Pu \cdot u = F$, in which the matrix $P = A^T P' A$, or $P' = APA^T$. Consequently, if the quadratic form F is positive definite in one system, it retains this property in every system related to first by an orthogonal transformation.

Now, the mass of every material object being positive, the kinetic energy, by its definition in (7.35) for a particle, (8.50) for a system of particles, and (10.90) for all bodies, is inherently positive definite. Like the potential energy function, it has the same positive definite value in every reference system. Note, however, that while the kinetic energy may be referred to any appropriate moving reference frame, it is always determined with respect to an inertial frame.

a stable, static equilibrium configuration of a conservative, scleronomic system:

$$M_{kl}\ddot{q}_l + K_{kl}q_l = 0, \qquad k, l = 1, 2, \ldots, n. \tag{11.86}$$

The stability of the equilibrium configuration must be confirmed in each application.

Example 11.11. Consider a system having $n = 2$ degrees of freedom. Then (11.86) is a coupled system of two linear equations:

$$\begin{aligned} M_{11}\ddot{q}_1 + M_{12}\ddot{q}_2 + K_{11}q_1 + K_{12}q_2 = 0, \\ M_{21}\ddot{q}_1 + M_{22}\ddot{q}_2 + K_{21}q_1 + K_{22}q_2 = 0, \end{aligned} \tag{11.87a}$$

in which $M_{12} = M_{21}$ and $K_{12} = K_{21}$.

In particular, recall the conservative scleronomic system of Fig. 11.1, page 514, for which the total kinetic energy (11.40a) is a homogeneous quadratic function of the generalized velocities $\dot{q}_k = \dot{x}_k$ and the total potential energy (11.40b) is a homogeneous quadratic function of all of the generalized coordinates $q_k = x_k$, none being absent. Because both functions, without any series approximations, are homogeneous quadratic functions of $\dot{q}_k$ and q_k, respectively, the equations of motion in (11.40f) corresponding to (11.87a) hold for large amplitude oscillations of the system. The exact energy relations (11.40a) and (11.40b) are to be compared with the respective homogeneous quadratic forms (11.83) and (11.84) of the linearized theory in which the inertia and stiffness coefficient matrices of the example are identified by

$$[M_{kl}] = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}, \qquad [K_{kl}] = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_1 + k_2 \end{bmatrix}, \tag{11.87b}$$

evident from (11.40f). Since $Ku \cdot u = k_1(u_1^2 + u_2^2) + k_2(u_1 - u_2)^2 > 0$ for all vectors $u = (u_1, u_2) \neq 0$, the equilibrium configuration at $x_1 = x_2 = 0$ is infinitesimally stable, which also is physically evident. Alternatively, we confirm from the stiffness matrix K that $\det K = k_1^2 + 2k_1k_2 > 0$ and both of its principal minors $k_1 + k_2 > 0$; hence, K is a positive definite matrix.

To obtain the general solution of the system (11.86), we consider trial solutions of the form $q_k^T = C_k \sin(pt + \phi)$, all having the same circular frequency p and initial phase ϕ, and where C_k are n constants. Substitution of q_k^T into (11.86) yields the following homogeneous system of n algebraic equations:

$$\left(K_{kl} - p^2 M_{kl}\right)C_l = 0. \tag{11.88}$$

For nontrivial amplitudes C_l, the $n \times n$ determinant of the coefficient matrix must vanish; that is,

$$\det\left|K_{kl} - p^2 M_{kl}\right| = 0. \tag{11.89}$$

This is a polynomial of degree n in the squared circular frequency p^2 called the *characteristic equation*, and its roots are called *characteristic frequencies, eigenfrequencies*, *normal mode* or *natural frequencies*. Because both matrices K_{kl} and M_{kl} are real-valued, symmetric matrices all of the squared eigenfrequencies are positive, and only positive solutions p_m are meaningful. For each p_m obtained from (11.89), there is a corresponding trial solution of the form $q_k^T = C_{km}\sin(p_m t + \phi_m)$ for each m (no sum) and for which ratios of the amplitudes C_{km} are determined from (11.88), now cast in the form

$$\left(K_{kl} - p_m^2 M_{kl}\right) C_{lm} = 0, \tag{11.90}$$

for $k, l, m = 1, 2, \ldots, n$, sum on l, no sum on m. Hence, the general solution of the system (11.86) for each q_k is the sum of all of the trial solutions corresponding to each p_m, ϕ_m pair:

$$q_k = \sum_{m=1}^{n} C_{km}\sin(p_m t + \phi_m), \qquad k = 1, 2, \ldots, n. \tag{11.91}$$

This systematic analysis assumes that all of the characteristic frequencies are distinct and nonzero; but this is not always the case. Dynamical systems for which some roots of the characteristic equation may be repeated or may be zero, called *degenerate systems*, are handled somewhat differently. These systems are not studied here. The analysis of degenerate systems may be found in several advanced texts cited in the references.

Example 11.12. A homogeneous thin body of mass M, shown in Fig. 11.8, is suspended in the vertical plane by a thin wire of mass $m \ll M$, length a, and with its ends hinged in smooth bearings. (i) Derive the equations for small vibrations of the body about its vertical equilibrium position $\phi = \theta = 0$. Identify the inertia

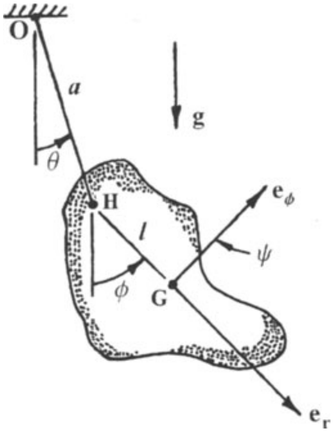


Figure 11.8. Small vibrations of a physical pendulum with two degrees of freedom.

and stiffness matrices. (ii) Sketch the general formulation of the solution. Then let $a = l$, and suppose that the body is a thin circular disk of radius $R = \sqrt{2}l$. Determine the normal mode frequencies p_m, the ratios of the amplitudes C_{lm}, and thus derive the solution of the coupled equations of motion. Identify the normal equations of motion and their normal mode solutions as functions of the generalized variables.

Solution of (i). The system is scleronomic with two degrees of freedom θ and ϕ defined in Fig. 11.8. Since the wire has negligible mass compared with M, its contribution to the total kinetic and potential energies is negligible. The total velocity of the center of mass G referred to the body frame $\psi = \{G; \mathbf{e}_r, \mathbf{e}_\phi\}$ is given by $\mathbf{v}^* = \mathbf{v}_H + \boldsymbol{\omega} \times \mathbf{l} = a\dot{\theta}\mathbf{t} + l\dot{\phi}\mathbf{e}_\phi$, where $\mathbf{t} = \sin(\phi - \theta)\mathbf{e}_r + \cos(\phi - \theta)\mathbf{e}_\phi$. The total kinetic energy of the system, by (10.101), is

$$T = \frac{1}{2}M[a^2\dot{\theta}^2\sin^2(\phi - \theta) + (a\dot{\theta}\cos(\phi - \theta) + l\dot{\phi})^2] + \frac{1}{2}I_G\dot{\phi}^2, \tag{11.92a}$$

and the total gravitational potential energy is given by

$$V = Mg[a(1 - \cos\theta) + l(1 - \cos\phi)]. \tag{11.92b}$$

We note that no coordinates are absent. For small vibrations, only terms of second order in all of the small quantities $\phi, \dot{\phi}, \theta, \dot{\theta}$ are retained in (11.92a) and (11.92b). Therefore, the first term in (11.92a) is of higher order and may be neglected; and with $\cos(\phi - \theta) \cong 1 - (\phi - \theta)^2/2$, to terms of second order the total kinetic and potential energies of the system for small vibrations about the vertical equilibrium configuration are thus given by

$$\begin{aligned} T &= \frac{1}{2}(I_G + Ml^2)\dot{\phi}^2 + \frac{1}{2}M(2la\dot{\phi}\dot{\theta} + a^2\dot{\theta}^2), \\ V &= \frac{1}{2}Mg(l\phi^2 + a\theta^2). \end{aligned} \tag{11.92c}$$

The potential energy is a positive definite, homogeneous quadratic function of the generalized coordinates θ and ϕ. Hence, clearly, the equilibrium configuration $\phi = \theta = 0$ is infinitesimally stable. Similarly, the total kinetic energy is a homogeneous quadratic function of the generalized velocities $\dot{\theta}$ and $\dot{\phi}$. With $I_H = I_G + Ml^2$ in accordance with the parallel axis theorem in (11.92c), the symmetric inertia and stiffness matrices in (11.84) and (11.83) are thus identified as

$$[M_{kl}] = \begin{bmatrix} I_H & Mla \\ Mla & Ma^2 \end{bmatrix}, \qquad [K_{kl}] = \begin{bmatrix} Mgl & 0 \\ 0 & Mga \end{bmatrix}. \tag{11.92d}$$

We note in passing that the stiffness matrix is diagonal and all of its nontrivial components are positive; so, it is quite evident that $Ku \cdot u > 0$ holds for all $u \neq 0$.

The hinge constraints are workless, so the system is conservative with a Lagrangian function

$$L = \frac{1}{2} I_H \dot{\phi}^2 + \frac{1}{2} M(2la\dot{\phi}\dot{\theta} + a^2\dot{\theta}^2) - \frac{1}{2} Mg(l\phi^2 + a\theta^2), \tag{11.92e}$$

and Lagrange's equations (11.66) now yield the equations of small vibration,

$$\begin{aligned} I_H\ddot{\phi} + Mal\ddot{\theta} + Mgl\phi &= 0, \\ Mla\ddot{\phi} + Ma^2\ddot{\theta} + Mga\theta &= 0. \end{aligned} \tag{11.92f}$$

Solution of (ii). The natural frequencies for the small vibrations are determined by (11.89). With (11.92d), we have

$$\det \begin{bmatrix} Mgl - p^2 I_H & -p^2 Mla \\ -p^2 Mla & Mga - p^2 Ma^2 \end{bmatrix} = 0.$$

This reduces to a quadratic equation in p^2:

$$\left(Mgl - p^2 I_H\right)\left(Mga - p^2 Ma^2\right) - p^4 M^2 l^2 a^2 = 0, \tag{11.92g}$$

which yields two eigenfrequencies p_m. Use of these in (11.90) provides two sets of equations for the ratios of the coefficients C_{lm}, $m = 1, 2$; namely,

$$\begin{bmatrix} Mgl - p_m^2 I_H & -p_m^2 Mla \\ -p_m^2 Mla & Mga - p_m^2 Ma^2 \end{bmatrix} \begin{bmatrix} C_{1m} \\ C_{2m} \end{bmatrix} = 0. \tag{11.92h}$$

Finally, use of these results in (11.91) yields the solutions for $q_1 = \phi$ and $q_2 = \theta$. We omit these general details and turn to a special case for illustration.

Consider a circular disk of radius $R = \sqrt{2}l$ and let $a = l$. Then $I_G = MR^2/2 = Ml^2$, and $I_H = 2Ml^2$. The characteristic equation (11.92g) thus simplifies to $p^4 - 3p_0^2 p^2 + p_0^4 = 0$ with positive roots

$$p_1 = p_0\sqrt{\frac{1}{2}(3 + \sqrt{5})}, \qquad p_2 = p_0\sqrt{\frac{1}{2}(3 - \sqrt{5})}, \tag{11.92i}$$

where $p_0 \equiv \sqrt{g/l}$, and (11.92h) becomes

$$\begin{bmatrix} p_0^2 - 2p_m^2 & -p_m^2 \\ -p_m^2 & p_0^2 - p_m^2 \end{bmatrix} \begin{bmatrix} C_{1m} \\ C_{2m} \end{bmatrix} = 0. \tag{11.92j}$$

For $m = 1, 2$ (no sum), we thus obtain the amplitude ratios

$$\frac{C_{21}}{C_{11}} = \frac{\left(p_0^2 - 2p_1^2\right)}{p_1^2}, \qquad \frac{C_{12}}{C_{22}} = \frac{p_2^2}{\left(p_0^2 - 2p_2^2\right)}. \tag{11.92k}$$

Substitution here of the characteristic roots (11.92i) yields

$$C_{21} = -\frac{1}{2}(1 + \sqrt{5})C_{11}, \qquad C_{12} = \frac{1}{2}(1 + \sqrt{5})C_{22}. \tag{11.92l}$$

Introducing these in (11.91) in which $q_1 = \phi$ and $q_2 = \theta$, we obtain the general solution

$$\begin{aligned}\phi &= C_{11}\sin(p_1 t + \phi_1) + \frac{1}{2}(1+\sqrt{5})C_{22}\sin(p_2 t + \phi_2),\\ \theta &= -\frac{1}{2}(1+\sqrt{5})C_{11}\sin(p_1 t + \phi_1) + C_{22}\sin(p_2 t + \phi_2).\end{aligned} \tag{11.92m}$$

The remaining constants $C_{11},\ C_{22},\ \phi_1$, and ϕ_2 are determined upon specification of the initial data.

The normal mode motions, readily identified from (11.92m), are defined by

$$\xi_1 = C_{11}\sin(p_1 t + \phi_1), \qquad \xi_2 = C_{22}\sin(p_2 t + \phi_2), \tag{11.92n}$$

and upon solving (11.92m) for the ξ_ks, we obtain the normal coordinates in terms of the original generalized coordinates:

$$\begin{aligned}\xi_1 &= -\frac{1}{10}\sqrt{5}\big(2\theta + \phi(1-\sqrt{5})\big),\\ \xi_2 &= -\frac{1}{20}(\sqrt{5}-5)\big(2\theta + \phi(1+\sqrt{5})\big).\end{aligned} \tag{11.92o}$$

These normal coordinates uncouple the original equations of motion (11.92f) applied to the circular disk so that the normal mode equations of motion are

$$\ddot{\xi}_k + p_k^2 \xi_k = 0, \qquad k = 1, 2 \text{ (no sum)}, \tag{11.92p}$$

in which the normal mode frequencies p_k are given in (11.92i). □

For linear systems having two degrees of freedom, the solution procedure illustrated above for free vibrations is straightforward. The theory of free vibrations of conservative, scleronomic systems studied above is the easiest class of problems within the theory of vibrations of systems having n degrees of freedom. For more general dynamical systems in the theory of small forced vibrations and having any number of degrees of freedom, the use of normal coordinates is especially advantageous because it eliminates the need to solve a coupled system of n simultaneous, linear nonhomogeneous differential equations of motion. Rather, in terms of normal coordinates ξ_k, a simpler system of n independent equations of the form $\ddot{\xi}_k + p_k^2 \xi_k = P_k(t)$, where $P_k(t)$ are certain generalized forcing functions corresponding to the normal coordinates, are to be solved. The methods of modal analysis and the general theory for the transformation from generalized coordinates to normal coordinates, and the analysis of degenerate systems, may be found in the works by Greenwood, Whittaker, and Yeh and Abrams, among others listed in the References.

11.14. Dissipative Dynamical Systems of the Stokes Type

In this section, Lagrange's equations are modified to account directly for the effects of linear viscous damping. We begin with a single particle and recall the Stokes drag force (6.29): $\mathbf{F}_D = -c\mathbf{v} = -c\dot{\mathbf{x}}$. Then for a scleronomic system $\mathbf{x} = \mathbf{x}(q_r)$, and the corresponding mechanical power $\mathscr{P}_D$ expended, i.e. the energy dissipated by the drag force alone, may be written as

$$\mathscr{P}_D = -c\mathbf{v}\cdot\mathbf{v} = -c\frac{\partial \mathbf{x}}{\partial q_i}\dot{q}_i \cdot \frac{\partial \mathbf{x}}{\partial q_j}\dot{q}_j = -c_{ij}\dot{q}_i\dot{q}_j < 0, \tag{11.93}$$

wherein we sum on i and j, the index range being the number of independent generalized coordinates of the particle, and, by definition,

$$c_{ij} \equiv c\frac{\partial \mathbf{x}}{\partial q_i}\cdot\frac{\partial \mathbf{x}}{\partial q_j}. \tag{11.94}$$

Of course, $c_{ij} = c_{ji}$, the *generalized damping coefficients*, are functions of the q_rs alone. For future convenience, we introduce a generalized dissipation function D defined by

$$D \equiv -\frac{\mathscr{P}_D}{2} = \frac{1}{2}c_{ij}(q_r)\dot{q}_i\dot{q}_j > 0, \tag{11.95}$$

a positive definite quadratic form in the $\dot{q}_r$s equal to the negative of half the rate at which energy is dissipated by the Stokes force. The function D is known as the *Rayleigh dissipation function.*

In the special case when the Stokes force is the only force that acts on the particle, $\mathscr{P}_D = \mathscr{P} = \dot{T}$, the total mechanical power; and then (11.93) may be written as $\dot{T} = -2\nu T = -2D$. Hence, $D = -\nu T$, where $\nu = c/m$ is a damping exponent. Let $T_0 = T(0)$ denote the initial kinetic energy of the particle. Then $T(t) = T_0 e^{-2\nu t}$, and the dissipation $D = \nu T$ decays to zero with the total kinetic energy T, as expected. In general, however, other forces that act on the particle contribute to the total power.

Now consider the virtual work done by the Stokes force in a general motion of a particle: $\delta\mathscr{W} = -c\mathbf{v}\cdot\delta\mathbf{x} = -c\mathbf{v}\cdot\partial\mathbf{x}/\partial q_r\delta q_r$; and, with the aid of (11.9), we find

$$\delta\mathscr{W} = -c\mathbf{v}\cdot\frac{\partial \mathbf{v}}{\partial \dot{q}_r}\delta q_r = \frac{\partial}{\partial \dot{q}_r}\left(-\frac{1}{2}c\mathbf{v}\cdot\mathbf{v}\right)\delta q_r = Q_r^D\delta q_r. \tag{11.96}$$

Hence, for a holonomic system, the *generalized Stokes force* Q_r^D is thus defined by $Q_r^D \equiv \partial(-\frac{1}{2}c\mathbf{v}\cdot\mathbf{v})/\partial\dot{q}_r$, and, in accordance with (11.93) for scleronomic systems, we obtain

$$Q_r^D = \frac{\partial(\mathscr{P}_D/2)}{\partial \dot{q}_r} = -\frac{\partial D}{\partial \dot{q}_r}. \tag{11.97}$$

With the total generalized force $Q_r = Q_r^D + \hat{Q}_r$, where $\hat{Q}_r$ denotes the total of all other generalized forces that are not of the Stokes type, Lagrange's equations (11.73) for scleronomic systems, accounting separately for dissipative forces of the Stokes type, may be written as

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_r}\right) - \frac{\partial T}{\partial q_r} + \frac{\partial D}{\partial \dot{q}_r} = \hat{Q}_r. \tag{11.98}$$

Further, with $\hat{Q}_r = Q_r^N - \partial V/\partial q_r$ in the presence of some conservative forces with total potential energy V and other nonconservative generalized forces Q_r^N, and writing $L = T - V$, as usual, we arrive at the alternative form of Lagrange's equations for holonomic systems of the scleronomic type under dissipative forces of the Stokes type:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_r}\right) - \frac{\partial L}{\partial q_r} + \frac{\partial D}{\partial \dot{q}_r} = Q_r^N. \tag{11.99}$$

Example 11.13. For an illustration recall our earlier Example 11.4, page 510, of a particle falling from rest in a Stokes medium. In this case, there is one degree of freedom with generalized coordinate $q_1 = y$. The Lagrangian function is $L = T - V = \frac{1}{2}m\dot{y}^2 + mgy$, the Rayleigh dissipation function (11.95) is given by $D = \frac{1}{2}c_{11}\dot{y}^2$, and all other nonconservative generalized forces $Q_r^N = 0$. From (11.99) and with $c \equiv c_{11}$, we thus obtain the equation for the motion of a particle falling in a Stokes medium: $m\ddot{y} - mg + c\dot{y} = 0$. This agrees with our earlier result.

In accordance with (11.73), it follows that (11.99) holds for any general scleronomic system of particles and rigid bodies so long as the damping is characterized by a Rayleigh dissipation function of the form (11.95). In particular, for a lineal rigid body $\mathscr{L}$ of length ℓ subject to a Stokes force over its entire length and on which all other forces are workless, the total power is $\mathscr{P} = \mathscr{P}_D$ and the Rayleigh dissipation function may be read from (10.144): $D = -\mathscr{P}/2 = \beta T(\ell, t)$, in which β is a damping exponent and $T(\ell, t) = \frac{1}{2}I\dot{q}^2$ is the total kinetic energy of the body about a fixed point or about its center of mass. This example is similar to the single particle problem discussed earlier, and clearly $T(t) = T_0 e^{-2\beta t}$, as before. The system has one degree of freedom with $q = \theta$ and $\dot{q} = \omega$, the angular spin of the body. All other forces being workless, (11.98) yields the universal equation of motion $\dot{\omega} + \beta\omega = 0$, as shown previously in (10.150). Now let us consider a system having two degrees of freedom.

Example 11.14. The damped free vibration of a two degree of freedom system moving in the xy-plane has total kinetic energy $T = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\dot{y}^2$, total elastic potential energy $V = \frac{1}{2}k_1x^2 + \frac{1}{2}k_2y^2$, and a total Stokes type dissipation described by the Rayleigh function $D = \frac{1}{2}(c_1\dot{x}^2 + 2c_{12}\dot{x}\dot{y} + c_2\dot{y}^2)$. Derive the equations of motion.

Solution. The Lagrangian is $L = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\dot{y}^2 - \frac{1}{2}k_1x^2 - \frac{1}{2}k_2y^2$. For the free vibrational motion $Q_r^N = 0$ and use of L and D in (11.99) yields the two equations of motion for the system:

$$\begin{aligned} m_1\ddot{x} + c_1\dot{x} + c_{12}\dot{y} + k_1x &= 0, \\ m_2\ddot{y} + c_{12}\dot{x} + c_2\dot{y} + k_2y &= 0. \end{aligned} \tag{11.100}$$

This is a coupled system of linear differential equations for which general solution methods are well known. See Whittaker in the References for further study of this topic. □

The theory of small vibrations about an equilibrium configuration of a system having n degrees of freedom is now easily extended to include dissipative forces of the Stokes type. In this case, the damping coefficients, to the lowest order, are constants: $c_{ij} = C_{ij}$, and the Rayleigh dissipation function $D = \frac{1}{2}C_{ij}\dot{q}_i\dot{q}_j$ is a homogeneous, positive definite quadratic function of only the generalized velocities. With the Lagrangian given by (11.85), the general equations (11.99) for small vibrations with Stokes damping become

$$M_{kl}\ddot{q}_l + C_{kl}\dot{q}_l + K_{kl}q_l = Q_k^N, \qquad k, l = 1, 2, \ldots, n. \tag{11.101}$$

When no additional driving forces act on the system, $Q_k^N = 0$ and these equations reduce to those for the free vibrations of a damped dynamical system having n degrees of freedom. See Problem 11.37.

Because the kinetic and potential energies and the Rayleigh dissipation function in the last example above are already quadratic functions of q_r and $\dot{q}_r$, the equations of motion (11.100) necessarily have the same form as the general equations of the theory of small vibrations in which $(q_1, q_2) = (x, y)$ and

$$[M_{kl}] = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}, \qquad [C_{kl}] = \begin{bmatrix} c_1 & c_{12} \\ c_{12} & c_2 \end{bmatrix}, \qquad [K_{kl}] = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}.$$

The difference, however, is that the motion in the example need not be small.

11.15. Closure

We have seen that Lagrange's analytical mechanics reduces the various principles of mechanics to an invariant system of differential equations that facilitates the formulation and solution of all kinds of dynamical problems. The development of Lagrange's equations from Hamilton's principle has demonstrated that these equations may be applied to general and complex conservative and nonconservative holonomic dynamical systems having any number of degrees of freedom. It is important to remember, however, that in the applications of Lagrange's equations, the energy is determined with respect to an inertial reference frame, a concept that was never mentioned in the derivation of these equations from Hamilton's principle. Moreover, the determination of the kinetic energy for a rigid body, in particular, requires knowledge of its moment of momentum, and this entails use of a body reference frame with origin at a special point—generally a fixed point in the inertial reference frame, one having a uniform motion in the inertial frame, or the center of mass. While none of these concepts is apparent in Lagrange's formulation, the influence of Euler's ideas is evident throughout Lagrange's work. Hence, while Lagrange's equations most certainly are convenient for the derivation of the equations of motion of complex systems, in its applications we must appeal to many classical concepts and methods due to others, notably Newton and Euler, whose profound classical ideas are developed throughout this book.

In its applications to systems of particles and rigid bodies the Newton-Euler theory often proves to be particularly tedious, because generally one must deal separately with each and every particle or body in the system and introduce all of the seprate internal and external forces, including all forces of constraint. On the other hand, in applications of the Lagrangian method, though in many respects simpler than the Newton-Euler formulation, it is necessary to bear in mind certain technical details that characterize the system, the nature of its constraints, and the corresponding special technical conditions for the applicability of the equations. We recall, for example, that in the Lagrangian formulation, the work-energy principle was derived specifically for only scleronomic systems; and the simple principles of conservation of momentum and moment of momentum in the Newton-Euler theory are imbedded in Lagrange's principle of conservation of generalized momentum for an ignorable coordinate for which the corresponding nonconservative part of the generalized force vanishes. The fact that a constraint is holonomic or nonholonomic, scleronomic or rheonomic, details essential to the structure of the Lagrangian formulation, is unimportant to the mathematical structure of the Newton-Euler laws of mechanics. In the latter instance, these important technical details are brought into the analysis in different ways that usually involve determination of forces of constraint. There are, however, countless situations in engineering practice where the intensity of constraining forces that act on the system must be determined for design considerations, and these forces are not provided by the Lagrangian theory. So, we really need the full body of theory and good models to successfully analyze the motion of complex dynamical systems.

In our studies here, we have not explored nonholonomic dynamical systems, we have not investigated Lagrange's unified approach to analytical mechanics essentially based on D'Alembert's principle, and we have not studied Hamilton's form of the equations of motion. These and other topics that are outside the scope of this Introduction may be found in advanced treatises on analytical mechanics. See, for example, the works by Lanczos, Pars, Rosenberg, and Whitttaker listed in the chapter References. So, this is not the end—there is much more to be learned about dynamics. It is hoped, however, that this introductory treatment may encourage the reader to continue study of dynamical systems at the advanced and more abstract levels of theoretical mechanics and the theory of equations.

References

1. Greenwood, D. T., *Principles of Dynamics*, Prentice-Hall, Englewood Cliffs, New Jersey, 1965. Cast at an intermediate level comparable to the present text, this book provides an excellent resource for collateral study. Lagrange's equations, including a thorough discussion of constraints and the method of virtual work, are discussed in Chapter 6. The theory of vibrations, including discussion of degenerate systems, is treated in Chapter 9.
2. Housner, G. W., and Hudson, D. E., *Applied Mechanics: Dynamics*, 2nd Edition, Van-Nostrand, Princeton, New Jersey, 1959. Lagrange's equations applied to the theory of small, free, and forced vibrations are studied in Chapter 9. The authors provide good preparation throughout for the student's subsequent study of advanced topics in dynamics. Some examples and problems in the current chapter are modeled after those presented in this book.
3. Lagrange, J. L., *Analytical Mechanics*, translated from the *Mécanique analytique novelle édition* of 1811, editors A. Boissonnade and V. V. Vagliente, Kluwer, Dordrecht, The Netherlands, 1997. The original of this enduring classical work often is unavailable for study, so this English translation is a most helpful substitute. The translators' "Introduction" provides an interesting sketch of the life and times of Lagrange. Lagrange's treatise, based on the method of virtual work, begins with principles of statics in Part I; the greater emphasis is on the various principles of dynamics in Part II.
4. Lanczos, C., *The Variational Principles of Mechanics*, University of Toronto Press, Toronto, Canada, 1949. A beautifully written text highly recommended for advanced study. The focus, however, is mainly on the theoretical aspects of Lagrange's differential equations of motion and the Hamilton–Jacobi canonical theory of equations. There are very few, though well chosen examples throughout. The method of undetermined multipliers is applied in the formulation of Lagrange's equations for systems with constraints. Our discussion of nonholonomic constraints mirrors that presented in this text.
5. Long, R. R., *Engineering Science Mechanics*, Prentice-Hall, Englewood Cliffs, New Jersey, 1963. Vector and Cartesian tensor methods are integrated in this treatment of topics on engineering mechanics. Hamilton's principle and Lagrange's equations are introduced in Chapter 3. The latter, however, are not uniformly applied throughout the text, which otherwise provides a good resource for collateral reading and for additional problems and examples.
6. Marion, J. B., *Classical Dynamics of Particles and Rigid Bodies*, Academic, New York, 1965. In Chapter 9, Lagrange's equations are derived from Hamilton's principle, and the method of undetermined multipliers to characterize constraints on a system is introduced. Multidegree of freedom systems are studied in Chapter 14. Applications of Euler's equations and the energy principle for rigid bodies, including the spinning top problem, are studied in Chapter 13. See also Marion, J. B., and Thornton, S. T., *Classical Dynamics of Particles and Rigid Bodies*, Harcourt Brace, New York, 1995.
7. Pars, L. A., *A Treatise on Analytical Dynamics*, Ox Bow Press, Woodbridge, Connecticut, 1965. This text provides many examples for advanced readers.
8. Rosenberg, R. M., *Analytical Dynamics of Discrete Systems*, Plenum, New York, 1977. This is a carefully written, thorough treatment of analytical dynamics based on the geometry of the configuration space of generalized coordinates, including a precise presentation of the geometry of constraints and virtual displacements, and their relation to constrained systems. Chapter 9 is a clear analysis and description of D'Alembert's principle that leads in subsequent chapters to Lagrange's equations for arbitrary systems. There are many worked examples throughout.

9. SYNGE, J. L., and GRIFFITH, B. A., *Principles of Mechanics*, 3rd Edition, McGraw-Hill, New York, 1959. Chapter 14 treats the general problem of the spinning top and the gyroscope by Euler's equations and the energy method, and Chapter 15 deals with Lagrange's equations with several examples, including the top problem.
10. TRUESDELL, C., *Essays in the History of Mechanics*, Springer-Verlag, Berlin, Heidelberg, New York, 1968. The author argues in Chapters 2 and 5 that Euler's laws are more general than Lagrange's equations, pointing specifically to the significance of Euler's principle of moment of momentum and noting that it is never mentioned or used by Lagrange.
11. WHITTAKER, E. T., *A Treatise on the Analytical Dynamics of Particles and Rigid Bodies*, 3rd Edition, Cambridge University Press, Cambridge, 1927. This classical volume, highly recommended to advanced readers, remains one of the best mathematical treatments of analytical dynamics. The treatise focuses entirely on the Lagrangian and Hamiltonian theories and includes many worked examples throughout. The reader may find the problems rather challenging, however. The theory of vibrations, including degenerate systems, is analyzed thoroughly in Chapter VII; nonholonomic systems are studied in Chapter VIII, and extension of Hamilton's principle to conservative and nonconservative, nonholonomic systems follows in Chapter IX.
12. YEH, H., and ABRAMS, J. I., *Principles of Mechanics of Solids and Fluids*, Vol. 1, *Particle and Rigid Body Mechanics* McGraw-Hill, New York, 1960. Lagrange's equations and some examples, including application to the vibrations of a structure, are discussed in Chapters 13.

Problems

11.1. Introduce $C_1 = -A/C$ and $C_2 = -B/C$, in which $A = A(q_1, q_2, q_3)$, $B = B(q_1, q_2, q_3)$, and $C = C(q_1, q_2, q_3)$, so that the differential constraint (11.5) becomes $A dq_1 + B dq_2 + C dq_3 = 0$. Show that the test condition (11.6) may be rewritten in the form

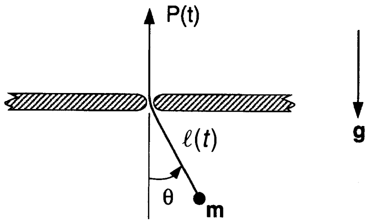
$$A\left(\frac{\partial B}{\partial q_3} - \frac{\partial C}{\partial q_2}\right) + B\left(\frac{\partial C}{\partial q_1} - \frac{\partial A}{\partial q_3}\right) + C\left(\frac{\partial A}{\partial q_2} - \frac{\partial B}{\partial q_1}\right) = 0. \qquad \text{(P11.1)}$$

Notice that this is satisfied identically when the terms in parentheses vanish. In this case, the constraint is integrable and hence holonomic. If these terms do not vanish, but (P11.1) vanishes identically, the constraint is holonomic, otherwise not. In either case, however, the integral of the differential constraint is not revealed (See Rosenberg, p. 46.), and it may be quite difficult to determine. If (P11.1) yields a relation $q_3 = q_3(q_1, q_2)$ that satisfies the conditions $\partial q_3/\partial q_1 = C_1$ and $\partial q_3/\partial q_2 = C_2$, then $q_3 = q_3(q_1, q_2)$ is the holonomic constraint corresponding to (P11.1). Apply this method to decide the nature of the differential constraint relation in Exercise 11.1, page 499.

11.2. A particle P moves on a space curve with path variable $s(t)$. Apply Lagrange's method to derive the intrinsic equation of motion of P.

11.3. Introduce spherical coordinates in Example 7.15, page 260, for the spherical pendulum. Apply Lagrange's equations to derive the equations of motion, and determine their first integrals.

11.4. A small mass m is attached to a weightless, inextensible string that passes through a tiny, smooth hole in a horizontal plate. The specified time variable force $P(t)$ shown in the figure controls the cord length $\ell(t)$ as a function of time so that the mass moves in the vertical plane. (a) How many degrees of freedom does this system have? (b) Apply Lagrange's equations to derive equations to determine $\theta(t)$ and $\ell(t)$. (c) What results follow from the moment of momentum principle? (d) Derive the same equations from Newton's law.



Problem 11.4.

11.5. A particle of mass m moves on the smooth inner surface of a thin paraboloidal shell of revolution defined by $r^2 = az$ in cylindrical coordinates (r, θ, z), where a is a constant. The particle, with weight $\mathbf{W} = -mg\mathbf{k}$, encounters air resistance described by a Stokes drag force $\mathbf{F}_d = -c\mathbf{v}$. Apply Lagrange's equations to derive the equations of motion. Find the generalized forces (a) by the method of virtual work, (b) by application of (11.14), and (c) by use of (11.20).

11.6. Consider the motion of the slider block S in Problem 6.54. Identify the rheonomic constraint. Let $\mathbf{R}$ denote the force exerted on the slider by the smooth rod. Show that $Q_1 = \mathbf{R} \cdot \partial\mathbf{x}/\partial r = 0$, where $\mathbf{x}$ is the position vector of S from F at the center of the table. Use Lagrange's equations to derive the equation of motion of S for the generalized coordinate $q_1 \equiv r(t)$. Find the motion of the slider when $r(0) = 0$ and $\dot{r}\ (0) = v_o$ initially.

11.7. The slider S described in Problem 6.55 is released from rest at O, relative to the table. Apply Lagrange's method to derive the equation of motion for S, and find its relative motion $r(t)$ for all constant values of the angular speed ω. Refer all quantities to the rod frame $\varphi = \{O; \mathbf{i}_k\}$ shown in Problem 6.54.

11.8. Identify any rheonomic constraints and apply Lagrange's method to derive the equation of motion of the slider block described in Problem 6.51.

11.9. Consider the system described in Problem 6.56. (a) Identify the rheonomic constraint and derive the equation of motion for the mass m by application of Lagrange's method. (b) Relax the constraint, obtain a second equation of motion involving the constraint reaction force R exerted by the rod on the slider, and thereby determine R in the case when $\omega = \omega_0$, a constant. (c) Suppose the slider is released from rest at $x = a$ to oscillate along the smooth rod. Find R as a function of x.

11.10. Determine the generalized forces and derive the Lagrange equations of motion for the pendulum bob described in Example 6.14, page 150. Show how the bob constraining force may be found.

11.11. A particle of mass m with cylindrical coordinates (r, θ, z) moves in a gravitational field $\mathbf{g} = -g\mathbf{k}$ on a smooth, concave upward surface of revolution defined by $r = r(z)$ with $r(0) = 0$. Use Lagrange's equations to derive the equation of motion for $z(t)$, and outline how the angular placement $\theta(t)$ may be found.

11.12. Apply Lagrange's equations (11.15) to find the applied forces required to control the uniform motion of the particle relative to the rotating frame in Example 5.9, page 71. Identify the physical nature of the pseudoforces described by $-\partial T/\partial q_k$.

11.13. Apply Lagrange's equations to derive the equations of motion for the system described in Problem 8.16.

11.14. Use Lagrange's equations to investigate Problem 8.29.

11.15. Derive Lagrange's equations for small amplitude oscillations of the system shown in Problem 10.14.

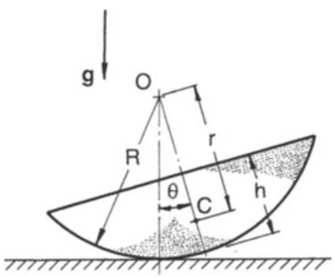
11.16. A uniform rod of mass m and length 2ℓ moves on a smooth horizontal plane with angular velocity $\boldsymbol{\omega} = \omega\mathbf{k}$. Its center C has a velocity $\mathbf{v}^* = u\mathbf{i} + v\mathbf{j}$ referred to a body frame $\varphi = \{C; \mathbf{i}_k\}$ with $\mathbf{i}$ directed along the rod. Apply Lagrange's method to find the impulsive force $\mathbf{P} = P_x\mathbf{i} + P_y\mathbf{j}$ applied at a point B distant b from C in order to bring point B instantaneously to rest. Express the result in terms of the assigned parameters.

11.17. Identify the generalized coordinates and the number of degrees of freedom of the log in Problem 10.58. Use Lagrange's method to deduce the equations of motion and thus determine the frequency of the vertical oscillations of the log.

11.18. The wire and bob assembly of the rotating simple pendulum shown in the diagram for Problem 6.47 is replaced by a thin rigid rod of length l and mass m. The rod is hinged in a smooth bearing at O and is free to slide on the smooth horizontal table. (a) Identify the rheonomic constraint and apply Lagrange's equations to derive the equation for finite amplitude oscillations of the rod relative to the table. (b) Relax the constraint, determine the generalized forces that act on the rod at its hinge bearing, and thus find the constraint reaction force as an exact function of the finite angular placement β for initial data $\beta(0) = \beta_0$ and $\dot{\beta}(0) = 0$.

11.19. Derive Lagrange's equation of motion for the rolling cylinder in Problem 10.43.

11.20. A homogeneous circular cylindrical segment of radius R, length L, height h, and mass m performs rocking oscillations without slipping on a rough horizontal surface. The center of mass is at r from the center O. The segment is released from rest at the placement $\theta(0) = \theta_0$. (a) Derive the differential equation for the finite angular motion $\theta(t)$ by (i) application of Lagrange's equations, and (ii) by use of the Newton–Euler equations. (b) Determine the first integral of the equation of motion. (c) Derive an equation for the period of the large amplitude oscillations. (d) Find the circular frequency for small oscillations.

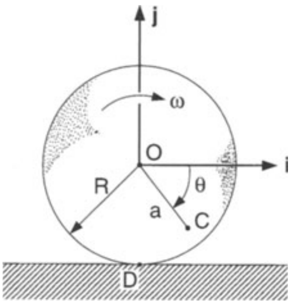


Problem 11.20.

11.21. A uniform, thin rigid rod shown in Problem 10.37 slides in the vertical plane with its ends on a smooth circle of radius r and subtending a central angle of 120°. (a) Derive the equation of motion by use of (i) Euler's laws, (ii) Lagrange's method, and (iii) the work–energy principle. (b) What is the first integral of the equation of motion? (c) Discuss briefly the exact solution for the motion $\theta(t)$ of the rod. (d) Find as functions of θ alone the contact forces acting on the rod. (e) What are the major differences among the three methods used in (a)? (f) What is the length ℓ, expressed in terms of r, of an equivalent simple pendulum having the same frequency?

11.22. Suppose the thin rod in the previous problem has its ends set in smooth bearings that slide along a circular hoop of radius r and negligible mass. The hoop rotates about its vertical central axis with a constant angular speed Ω. The rod is released from rest relative to the hoop at an angle $\theta_0 = \theta(0)$. (a) Use Lagrange's method to derive the equations of motion of the rod. Are there any surprising features of these results? (b) Derive the equations of motion by use of Euler's laws. (c) Find the bearing reaction forces exerted on the rod. (d) In what manner would the mass M of the hoop affect the results?

11.23. A nonhomogeneous circular cylinder has its center of mass C at a distance a from its geometrical center O, and its circular cross sectional plane through O is a plane of symmetry. The cylinder is released from rest when $\theta = 0$ and rolls without slipping on the horizontal surface. (a) Apply Lagrange's equations to determine the angular velocity $\boldsymbol{\omega}$ and angular acceleration $\dot{\boldsymbol{\omega}}$ of the cylinder as functions of θ. (b) Deduce the same results starting from the energy principle. (c) Find the surface reaction forces at D in terms of θ, ω, and $\dot{\omega}$. (d) Use Euler's equations to derive the equation of motion for the cylinder. (e) Discuss the principal difference between the methods of Euler and Lagrange.



Problem 11.23.

11.24. Use Lagrange's equations to formulate the equations of motion of the spring and pulley system described in Problem 7.49, about its static equilibrium state. The pulley has radius a, mass m, and rolls without slipping on its inextensible belt. How many degrees of freedom does this system have?

11.25. Use Lagrange's method to set up the equation for the finite motion of the system described in Problem 10.39 for a thin hoop whose mass m is the same as that of the thin rod. (a) Find the first integral of the equation of motion. (b) Derive an equation from which the exact period of the finite rocking oscillation is determined. (c) What is the circular frequency of small amplitude oscillations? (d) What is the length of a simple pendulum having the same small amplitude frequency as that of this system?

11.26. A smooth rigid rod shown in the figure for Problem 6.56 is attached to a table T that rotates in the horizontal plane about a smooth bearing at F. The table has mass M, radius of gyration K about F, and its variable angular speed due to an applied driving torque $\boldsymbol{\mu}_F(t) = \mu_F(t)\mathbf{k}$ about F is $\omega(t) = \dot{\theta}(t)$. The mass of the rod is negligible. A slider block of mass m, supported symmetrically by identical springs of stiffness k, is released from rest relative to the rod at a distance a from the unstretched state at O. (a) Derive the equations of motion for the system (i) by use of Lagrange's equations, and (ii) by use of the Newton–Euler laws. (b) Find the torque $\mu_F(t)$ required to sustain a stable motion of the system with a constant angular speed.

11.27. Apply Lagrange's method to derive the equations of motion for the system described in Problem 8.30. Solve these for the given initial conditions, and determine the small amplitude vertical and rotational frequencies of the motion.

11.28. Use Lagrange's equations to solve Problem 8.18.

11.29. Two uniform rigid rods, each of mass m and length 2ℓ, are connected end-to-end by a smooth hinge and placed in a straight line along the y-axis on a smooth horizontal table in the xy-plane. The end of one rod is struck suddenly by a force $\mathbf{P} = P\mathbf{i}$. Find the subsequent instantaneous generalized velocities of the system. What is the increase of the total energy of the system due to the impulse?

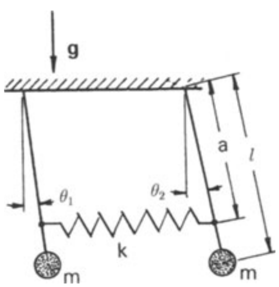
11.30. Consider the system described in Problem 6.57, but now suppose that the rigid rod is homogeneous with mass M. (i) Determine by integration the moment of inertia of the rod about the point O. (ii) Let θ denote the small angular placement from the horizontal equilibrium position. Derive the equations of motion and find the vibrational frequency of the system by use of (a) Euler's equations, (b) the energy method, and (c) Lagrange's equations. Which is the simplest, most direct method? (iii) Determine the dynamic part of the support reaction force as a function of θ for the case $b = 2a$. Does this depend on mass?

11.31. A pendulum device consists of a thin rod of mass m and length ℓ supported in the vertical plane by a smooth hinge H attached at the rim of a thin circular disk of radius R and mass M. The disk turns in the vertical plane with a steady angular speed ω about a smooth fixed axle at its center. Use Lagrange's method to derive the equations of motion.

11.32. Formulate Lagrange's equations for small vibrations of the system described in Problem 10.56.

11.33. Derive Lagrange's equations for small amplitude oscillations of the system in Problem 10.57.

11.34. Apply the theory of small vibrations to derive the equations of motion for the pendula shown in the figure, and solve these for the angular motions $\theta_1(t)$ and $\theta_2(t)$. Determine the eigenfrequencies, find the normal mode motions, and characterize these physically when the pendula are appropriately displaced and released from rest initially.



Problem 11.34.

11.35. (a) Derive Lagrange's equations of motion for the finite amplitude oscillations of the double pendulum described in Problem 8.32. Deduce from these results the equations for small amplitude oscillations. (b) Apply the theory of small vibrations to derive the latter equations of motion.

11.36. (a) Derive the equation for the finite amplitude motion of the system described in Problem 8.33. Then linearize the result to obtain the equation for small amplitude oscillations. (b) Apply the theory of small vibrations to derive the equation of motion.

11.37. Suppose that the system of pendula in Problem 11.34 moves in a Stokes medium, which might be the surrounding air for example. Find the Rayleigh dissipation function for the system of particles, and derive the equations for its small oscillations. Show that when $k = 0$, the coupled equations of motion reduce to those for the small damped oscillations of simple pendula.