9

The Moment of Inertia Tensor

9.1. Introduction

We know that Euler's first law (5.43) relates the total external applied force on a rigid body to the motion of its center of mass, and in the next chapter we shall demonstrate that Euler's second law (5.44) relates the total external applied torque to the body's rotational motion through its moment of momentum vector. The latter involves introduction of the moment of inertia tensor studied here; and, of course, the first law involves the location of the center of mass of the body. We begin, therefore, with the concept of the center of mass of a complex structured body and illustrate its application to a materially nonhomogeneous body having a complex shape and cavities. Then the inertia tensor is introduced, and its components for some special homogeneous bodies are determined. Afterwards, some important physical properties of the moment of inertia tensor, properties actually characteristic of all kinds of symmetric tensors, are derived. Consequently, as an additional benefit, study of the inertia tensor provides tools useful, for example, in the study of the mechanics of deformable solid and fluid materials in which stress, strain, and deformation rate tensors play a major role.

9.2. The Center of Mass of a Complex Structured Rigid Body

The center of mass of a rigid body $\mathscr{B}$ is a unique point that moves with the body and whose position vector $\mathbf{x}^*(\mathscr{B})$ in an arbitrary spatial frame is defined by (5.12), namely,

$$m(\mathscr{B})\mathbf{x}^*(\mathscr{B}) = \int_{\mathscr{B}} \mathbf{x}(P)dm(P). \tag{9.1}$$

Although the center of mass point may or may not be situated at a material point, its motion and momentum are the same as those of a particle of mass $m(\mathscr{B})$, the body's mass. Determination of the center of mass of a homogeneous body having a simple geometrical shape is straightforward. In general, however, bodies are complex structures that often do not have conveniently simple geometrical shapes, they may not be materially homogeneous, and they may have cavities. A body composed of an assembly of materially different homogeneous bodies with holes is a typical example. In this case, the body may be treated as a composition of several simpler bodies each of whose mass and center of mass are readily determined. As a consequence, a complex structured body is called a *composite body*.

To derive the equation for the center of mass of a complex structured body, we consider an arbitrary rigid and possibly materially nonhomogeneous body $\mathscr{B}$ having a complex shape with cavities. Now divide $\mathscr{B}$ into n separate, geometrically or materially simple parts $\mathscr{B}_k$ so that $\mathscr{B} = \cup_{k=1}^{n} \mathscr{B}_k$. Then the total mass of $\mathscr{B}$ may be written as

$$m(\mathscr{B}) = \int_{\cup_{k=1}^{n}\mathscr{B}_k} dm(P) = \sum_{k=1}^{n} \int_{\mathscr{B}_k} dm(P) = \sum_{k=1}^{n} m_k, \tag{9.2}$$

in which $m_k \equiv m(\mathscr{B}_k)$ is the total mass of the k^{th} simple part. Each part may be materially different and nonhomogeneous. This natural result shows that *the total mass of a composite body $\mathscr{B}$ is equal to the sum of the masses of its simple parts $\mathscr{B}_k$.*

If $\mathscr{B}$ has p separate cavities $\mathscr{C}_k$, say, we may imagine that each cavity is filled with material having the same mass distribution as $\mathscr{B}$. In this case, we may consider an auxiliary solid body defined by $\mathscr{B}_S = \mathscr{B} \cup_{k=1}^{p} \mathscr{C}_k$ and apply (9.2) to obtain

$$m(\mathscr{B}_S) = m(\mathscr{B}) + \sum_{k=1}^{p} m(\mathscr{C}_k). \tag{9.3}$$

Therefore, the mass of the actual body is determined by

$$m(\mathscr{B}) = m(\mathscr{B}_S) - \sum_{k=1}^{p} m(\mathscr{C}_k), \tag{9.4}$$

whose interpretation is evident. The same relation follows from (9.2) applied directly to $\mathscr{B} = \mathscr{B}_S \backslash \cup_{k=1}^{p} \mathscr{C}_k$, the auxiliary solid body with all of its separate filled cavities removed. Clearly, (9.4) may be applied to any separate part $\mathscr{B}_j$ having a cavity $\mathscr{C}_j$.

In the same way, the integral in (9.1) for $\mathscr{B} = \cup_{k=1}^{n} \mathscr{B}_k$ may be written as

$$\int_{\mathscr{B}} \mathbf{x}(P) dm(P) = \sum_{k=1}^{n} \int_{\mathscr{B}_k} \mathbf{x}(P) dm(P) = \sum_{k=1}^{n} m(\mathscr{B}_k)\mathbf{x}^*(\mathscr{B}_k), \tag{9.5}$$

in which (9.1) has been applied to each separate part $\mathscr{B}_k$ in the first sum. Thus, by (9.1), *the center of mass $\mathbf{x}^*(\mathscr{B})$ of a composite body $\mathscr{B}$ is provided by*

$$m(\mathscr{B})\mathbf{x}^*(\mathscr{B}) = \sum_{k=1}^{n} m(\mathscr{B}_k)\mathbf{x}^*(\mathscr{B}_k). \tag{9.6}$$

Of course, each part may be materially different and nonhomogeneous. Notice that (9.6) has the same form as (5.5) for the center of mass of a system of particles, each "particle" being a center of mass object.

Similarly, if $\mathscr{B}$ contains p cavities $\mathscr{C}_k$, use of $\mathscr{B} = \mathscr{B}_S \backslash \cup_{k=1}^{p} \mathscr{C}_k$ in (9.1) delivers

$$m(\mathscr{B})\mathbf{x}^*(\mathscr{B}) = m(\mathscr{B}_S)\mathbf{x}^*(\mathscr{B}_S) - \sum_{k=1}^{p} m(\mathscr{C}_k)\mathbf{x}^*(\mathscr{C}_k), \tag{9.7}$$

in which $\mathbf{x}^*(\mathscr{B}_S)$ is the center of mass of the solid body $\mathscr{B}_S$ composed of n solid parts $\mathscr{B}_k^S$, and $\mathbf{x}^*(\mathscr{C}_k)$ is the center of mass of the k^{th} materially similar solid body that fills the hole $\mathscr{C}_k$ in $\mathscr{B}_k$. Clearly, $m(\mathscr{B}_S)$ may be found by use of (9.2) applied to $\mathscr{B}_S$. Then $m(\mathscr{B})$ is given by (9.4), and the first term on the right in (9.7) may be obtained by use of (9.6) applied to $\mathscr{B}_S$, that is, $m(\mathscr{B}_S)\mathbf{x}^*(\mathscr{B}_S) = \sum_{k=1}^{n} m(\mathscr{B}_{Sk})\mathbf{x}^*(\mathscr{B}_{Sk})$, where $\mathscr{B}_{Sk}$ is the k^{th} solid simple part of $\mathscr{B}_S$.

If we view a cavity $\mathscr{C}_k$ as a "body" of negative volume, hence negative mass, and materially similar to the simple body $\mathscr{B}_k$ containing $\mathscr{C}_k$, (9.7) may be rewritten in the same form as (9.6); and (9.4) can be cast in the form of (9.2). *Hence, the rule of composition for the center of mass* $\mathbf{x}^*(\mathscr{B})$ *of a complex structured body* $\mathscr{B}$ *of mass* $m(\mathscr{B})$ is *summarized by the familiar general formula*

$$m(\mathscr{B})\mathbf{x}^*(\mathscr{B}) = \sum_{k=1}^{n} m_k \mathbf{x}_k^*, \qquad \text{with} \quad m(\mathscr{B}) = \sum_{k=1}^{n} m_k. \tag{9.8}$$

Herein $m_k \equiv m(\mathscr{B}_k)$ and $\mathbf{x}_k^* \equiv \mathbf{x}^*(\mathscr{B}_k)$ denote the mass and the center of mass of the k^{th} "body" $\mathscr{B}_k$, respectively.

Example 9.1. For an easy illustration, consider the homogeneous cylinder in Fig. 5.3, page 13. The central cylindrical hole is identified as $\mathscr{C}_1$. The mass of the solid, homogeneous cylinder called $\mathscr{B}_S$ is $m(\mathscr{B}_S) = \rho\pi r_0^2 \ell$, and the mass of a materially similar solid that fills $\mathscr{C}_1$ is $m(\mathscr{C}_1) = \rho\pi r_i^2 \ell$. Thus, from (9.4) or the second equation in (9.8), the total mass of the tube $\mathscr{B} = \mathscr{B}_S \backslash \mathscr{C}_1$ is

$$m(\mathscr{B}) = m(\mathscr{B}_S) - m(\mathscr{C}_1) = \rho\pi\ell\left(r_0^2 - r_i^2\right). \tag{9.9a}$$

In addition, for a homogeneous solid circular cylinder, we have

$$m(\mathscr{B}_S)\mathbf{x}^*(\mathscr{B}_S) = \left(\rho\pi r_0^2 \ell\right)\frac{\ell}{2}\mathbf{k}, \qquad m(\mathscr{C}_1)\mathbf{x}_1^*(\mathscr{C}_1) = \left(\rho\pi r_i^2 \ell\right)\frac{\ell}{2}\mathbf{k}, \tag{9.9b}$$

in frame φ in Fig. 5.3. Hence, by (9.9a) and (9.7) or the first equation in (9.8),

$$m(\mathscr{B})\mathbf{x}^*(\mathscr{B}) = \rho\frac{\pi\ell^2}{2}\left(r_0^2 - r_i^2\right)\mathbf{k} = m(\mathscr{B})\frac{\ell}{2}\mathbf{k}. \tag{9.9c}$$

Thus, as we know, $\mathbf{x}^*(\mathscr{B}) = \frac{1}{2}\ell\mathbf{k}$ in frame φ.

9.3. The Moment of Inertia Tensor

Lex $\mathbf{x}(P)$ denote the position vector from a base point Q of a rigid body $\mathscr{B}$ to a material parcel of mass $dm(P)$ at P. The tensor $\mathbf{I}_Q(\mathscr{B})$ defined by

$$\mathbf{I}_Q(\mathscr{B}) = \int_{\mathscr{B}} [(\mathbf{x}\cdot\mathbf{x})\,\mathbf{1} - \mathbf{x}\otimes\mathbf{x}]\,dm, \tag{9.10}$$

is called the *moment of inertia tensor relative to* Q, sometimes, briefly, the *inertia tensor*. Herein we recall from (3.31) the identity tensor $\mathbf{1} = \delta_{ij}\mathbf{e}_{ij}$ and from (3.24) the tensor product of vectors $\mathbf{a}$ and $\mathbf{b}$ for which $\mathbf{a}\otimes\mathbf{b} = a_i b_j \mathbf{e}_{ij}$, where $\mathbf{e}_{ij} \equiv \mathbf{e}_i \otimes \mathbf{e}_j$ is the tensor product basis associated with the orthonormal vector basis $\mathbf{e}_k$. Also, we observe the summation rule for repeated indices. From (9.10), $[\mathbf{I}_Q] = [ML^2]$, typical measure units being slug $\cdot$ ft^2 or kg $\cdot$ m^2.

Referred to a frame $\varphi = \{Q; \mathbf{e}_k\}$, the inertia tensor has the representation

$$\mathbf{I}_Q = I_{ij}^Q \mathbf{e}_{ij}, \tag{9.11}$$

in terms of its scalar components $I_{ij}^Q = \mathbf{e}_i \cdot \mathbf{I}_Q \mathbf{e}_j$ referred to $\mathbf{e}_{ij}$, in accordance with (3.15). It is seen from (9.10) that the moment of inertia tensor is symmetric: $\mathbf{I}_Q = \mathbf{I}_Q^T$, that is, $I_{ij}^Q = I_{ji}^Q$. Hence, only six of its nine scalar components are independent. From here onward, to simplify the component notation, the superscript Q usually is written only when we wish to emphasize the reference point being used.

Observe in (9.10) and (9.11) that only the base point need be fixed relative to the body; the reference frame $\varphi = \{Q; \mathbf{e}_k\}$ need not be. If φ is not an imbedded reference frame, however, the moment of inertia tensor referred to φ generally will vary with time as the body turns relative to φ. But if φ is an imbedded frame, then $\mathbf{I}_Q$ is a constant tensor whose components at Q will depend only on the body's fixed orientation in φ. In general then, the components of $\mathbf{I}_Q$ depend on the choice of reference point and on the orientation of the basis directions in the body. We shall return to these aspects later; but first the rectangular Cartesian components of $\mathbf{I}_Q$ in a body reference frame are described and a few general examples are studied.

9.4. Rectangular Cartesian Components of the Inertia Tensor

Let $\mathbf{e}_k = \mathbf{i}_k$ be a rectangular Cartesian reference basis at Q. Then the position vector of P from Q in the body frame $\varphi = \{Q; \mathbf{e}_k\}$ is $\mathbf{x}(P) = x_k \mathbf{i}_k$, and $\mathbf{x}\otimes\mathbf{x} = x_j x_k \mathbf{i}_{jk}$. Use of these relations and $\mathbf{1} = \delta_{jk}\mathbf{i}_{jk}$ in (9.10) yields

$$\mathbf{I}_Q = \left[\int_{\mathscr{B}} (\mathbf{x}\cdot\mathbf{x}\delta_{jk} - x_j x_k)dm\right]\mathbf{i}_{jk}, \tag{9.12}$$

and hence t*he rectangular Cartesian components of the moment of inertia tensor* are given by

$$I_{jk} = \int_{\mathscr{B}} (\mathbf{x} \cdot \mathbf{x}\delta_{jk} - x_j x_k) dm. \tag{9.13}$$

Now let $(x_1, x_2, x_3) = (x, y, z)$ as usual, and note that $\mathbf{x} \cdot \mathbf{x} = x^2 + y^2 + z^2$. Then (9.13) yields the explicit component relations

$$I_{11} = \int_{\mathscr{B}} (y^2 + z^2) dm, \quad I_{22} = \int_{\mathscr{B}} (x^2 + z^2) dm, \quad I_{33} = \int_{\mathscr{B}} (x^2 + y^2) dm, \tag{9.14}$$

$$I_{12} = I_{21} = -\int_{\mathscr{B}} xy dm, \quad I_{13} = I_{31} = -\int_{\mathscr{B}} xz dm, \quad I_{23} = I_{32} = -\int_{\mathscr{B}} yz dm. \tag{9.15}$$

The three components (9.14) are called *normal components of inertia*, and the six symmetric components (9.15) are known as *products of inertia*. It is important to note that in many dynamics books the products of inertia are defined somewhat differently as follows:

$$I_{xy} \equiv -I_{12} = \int_{\mathscr{B}} xy dm, \quad I_{xz} \equiv -I_{13} = \int_{\mathscr{B}} xz dm, \quad I_{yz} \equiv -I_{23} = \int_{\mathscr{B}} yz dm. \tag{9.16}$$

Therefore, the reader must exercise caution when consulting other sources.

It follows from (9.14) that

$$I_{11} + I_{22} = I_{33} + 2\int_{\mathscr{B}} z^2 dm, \tag{9.17}$$

which occasionally is useful in calculations involving the normal components. Also,

$$\operatorname{tr}\mathbf{I}_Q = I_{11} + I_{22} + I_{33} = 2\int_{\mathscr{B}} r^2 dm, \tag{9.18}$$

where $r^2 = \mathbf{x} \cdot \mathbf{x}$ is the squared distance from Q to the mass element dm. This rule involves a principal invariant of $\mathbf{I}_Q$ whose value in every reference basis at Q is the same.

For a homogeneous body, the mass density $\rho = dm/dV$ is a constant which may be extracted from the inertia integrals (9.14) and (9.15). The volume integrals that remain define what are known as *volume moments of inertia.*

9.4.1. Moments of Inertia for a Lamina

A thin, flat body $\mathscr{B}$ of negligible thickness h and elemental plane material area $dA(P)$ may be conveniently modeled as a *plane body,* or *lamina* for which $h \to 0$ and $\eta(P) \equiv dm(P)/dA(P)$, the ratio of the element of mass at P to the element of area at P, is the mass density per unit area. Also, $A(\mathscr{B}) = \int_{\mathscr{B}} dA(P)$ defines the total plane material area of $\mathscr{B}$. Let the lamina plane be the xy-plane. Then when $z \to 0$ in (9.14) and (9.15), we obtain the *moment of inertia tensor components for a lamina*:

$$I_{11} = \int_{\mathscr{B}} y^2 dm, \qquad I_{22} = \int_{\mathscr{B}} x^2 dm, \qquad I_{33} = I_{11} + I_{22}, \tag{9.19}$$

$$I_{12} = -\int_{\mathscr{B}} xy dm, \qquad I_{13} = I_{23} = 0. \tag{9.20}$$

For a homogeneous lamina, the constant density η may be removed from these integrals. The area integrals that remain define what are known as *area moments of inertia*.

9.4.2. Moment of Inertia About an Arbitrary Axis and the Radius of Gyration

The normal components of inertia in (9.14) and (9.19) have the general form

$$I_{nn} = \int_{\mathscr{B}} \bar{r}^2 dm, \tag{9.21}$$

in which $\bar{r}$ is the perpendicular distance from the axis n (with unit vector $\mathbf{n}$) to the element of mass dm, as shown in Fig. 9.1. Thus, (9.21) is called the *moment of inertia about the axis n* through the point Q. The result (9.21), however, holds more generally for an arbitrary axis through Q. The general proof is left for the reader in Problem 9.1. Another viewpoint is described in Problem 9.2.

The *radius of gyration about the axis n* is a positive scalar R_n defined by

$$R_n \equiv \sqrt{\frac{I_{nn}}{m(\mathscr{B})}}. \tag{9.22}$$

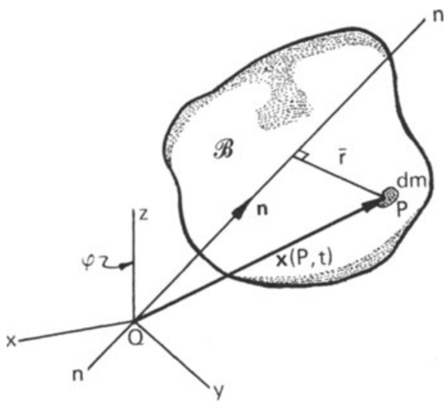


Figure 9.1. Schema for the moment of inertia about an axis n.

Since $m(\mathscr{B})R_n^2 = I_{nn}$, the squared radius of gyration R_n^2 is the average value of the integral (9.21). Further, it can be shown (see Problems 9.3 and 9.4.) that $I_{33} = mR^2$ is the moment of inertia about the central z-axis of a thin circular tube or ring of radius R. Therefore, by (9.22), the radius of gyration of any body $\mathscr{B}$ may be interpreted geometrically as the radius of an equivalent thin circular tube or ring having the same mass $m(\mathscr{B})$ and moment of inertia $I_{nn}(\mathscr{B})$ about the n-axis as those of the given body $\mathscr{B}$.

9.4.3. Moment of Inertia Properties of Symmetric Bodies

When n is the x-, y-, or z-axis, (9.21) coincides with the integrals in (9.14) and (9.19). Thus, $I_{11} = I_{xx}$, $I_{22} = I_{yy}$, $I_{33} = I_{zz}$, the normal Cartesian components, often are called *moments of inertia about the x-, y-, z-axes*, respectively. These integrals always are positive-valued, whereas the products of inertia (9.15) and (9.20) may be positive, negative, or zero. *For a homogeneous body having a plane of symmetry, if one of the coordinate planes contains the body plane of symmetry, the products of inertia involving the coordinate variable perpendicular to this plane will vanish.* Consider, for example, the homogeneous body shown in Fig. 9.2 for which the xz-plane is a body symmetry plane. Then the shape of the body surface to the right of this plane may be written as $y_s = f(x, z)$, and its surface to the left of this plane is described by $y_s = -f(x, z)$. Notice in (9.15) that both I_{12} and I_{23} contain y, the variable perpendicular to the xz-plane, the body symmetry plane. Thus, these products of inertia vanish upon integration with respect to the

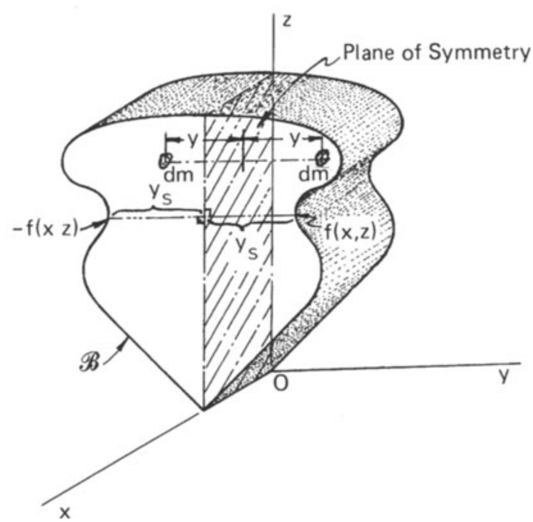


Figure 9.2. A homogeneous body having an xz-plane of symmetry.

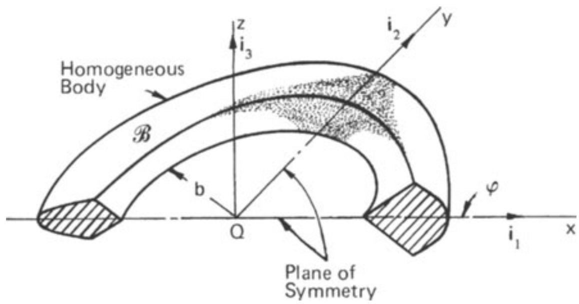


Figure 9.3. A homogeneous semicircular ring of variable cross section having an xy-plane of symmetry.

y variable. To see this, consider I_{12} in (9.15). With $dm = \rho dx dz dy \equiv d\mu dy$, we have

$$I_{12} = -\int_{(x,z)} \left[\int_{-f(x,z)}^{f(x,z)} y dy \right] x d\mu = 0. \tag{9.23}$$

Therefore, referred to coordinate planes that include a body plane of symmetry, at least two of the products of inertia for a homogeneous body will vanish. We shall return to this important property momentarily in some general remarks on axisymmetric bodies.

Example 9.2. What are the matrix and tensor forms of $\mathbf{I}_Q$ for the homogeneous body in Fig. 9.3 referred to the body frame $\varphi = \{Q; \mathbf{i}_k\}$?

Solution. Since the xy-plane is a body plane of symmetry for this homogeneous body, by (9.15), $I_{13} = I_{23} = 0$ in φ. Therefore, referred to the body frame φ in Fig. 9.3, we have the component matrix

$$I_Q = [\mathbf{I}_Q] = \begin{bmatrix} I_{11} & I_{12} & 0 \\ I_{12} & I_{22} & 0 \\ 0 & 0 & I_{33} \end{bmatrix}, \tag{9.24}$$

that is, in its tensor form, $\mathbf{I}_Q = I_{11}\mathbf{i}_{11} + I_{22}\mathbf{i}_{22} + I_{33}\mathbf{i}_{33} + I_{12}(\mathbf{i}_{12} + \mathbf{i}_{21})$. □

Now, consider a body having two orthogonal planes of symmetry. The line formed by the intersection of two orthogonal planes of symmetry of a body is called an *axis of symmetry*, and a body having an axis of symmetry is called *axisymmetric*. The plane geometrical figure formed by a cut through the body normal to an axis of symmetry is called a *cross section*. The cross section describes both the exterior and interior axisymmetric shapes of the body. Any line in the cross section through the axis of symmetry intersects the body at boundary points equidistant from the axis, and hence the point on the axis of symmetry in the cross section is called the *center of symmetry*. Now recall our previous result on a homogeneous body having a coordinate plane of symmetry. *In consequence, if the orthogonal planes of* *symmetry of an axisymmetric homogeneous body are chosen as coordinate planes of a body frame* $\varphi = \{Q; \mathbf{i}, \mathbf{j}, \mathbf{k}\}$, *then all products of inertia vanish relative to* Q *in* φ, *and hence the matrix* $[\mathbf{I}_Q]$ *for the moment of inertia referred to* φ *is diagonal*: $I_Q = \operatorname{diag}[I_{11}, I_{22}, I_{33}]$.

A body having a circular (or annular), but not necessarily axially uniform cross section perpendicular to its central axis is called a *body of revolution.* A body of revolution having a cavity may be characterized by an internal surface of revolution different from its exterior surface of revolution, in which case the cross section is a circular annulus whose width varies along the central axis of rotational symmetry. Clearly, a body of revolution is an axisymmetric body for which *every plane* through its axis of symmetry is an identical plane of symmetry; its exterior and interior boundaries in the cross section are circles. *Therefore, if the axis of symmetry of a homogeneous body of revolution is the* z*-axis of a body frame* $\varphi = \{Q; \mathbf{i}, \mathbf{j}, \mathbf{k}\}$, *say, then all products of inertia vanish and* $I_{11} = I_{22}$ *relative to every point* Q *on the axis of symmetry.* Hence, in every body frame at Q that includes the $\mathbf{i}_3$ coordinate direction, the moment of inertia tensor for a body of revolution, with or without a cavity of revolution, has the same diagonal form $\mathbf{I}_Q = I_{11}(\mathbf{i}_{11} + \mathbf{i}_{22}) + I_{33}\mathbf{i}_{33}$. A homogeneous solid circular cylinder or tube, an ellipsoid of revolution with a central spherical cavity, and a right circular cone with a central conical cavity are examples of bodies of revolution that share these properties.

Let the frame $\Phi = \{Q; \mathbf{I}_k\}$ be fixed in space, and $\varphi = \{Q; \mathbf{i}_k\}$ fixed in the body at point Q. Suppose that a homogeneous rigid body is turning about a fixed axis of rotational symmetry $\mathbf{I}_3 = \mathbf{i}_3$ through Q. Then at every instant of time the values of the moment of inertia components referred to Φ are indistinguishable from their corresponding values referred to φ, all constant. In fact, the moment of inertia components at every instant of time will have the same values with respect to any reference frame at Q that contains the fixed central axis of rotational symmetry. If the homogeneous body is axisymmetric but not a body of revolution, however, the moments of inertia about orthogonal axes perpendicular to the axis of symmetry and rotation will be independent of time only when these axes are fixed in the body. For illustration, picture a homogeneous right elliptical cylinder, an axisymmetric body, turning about the z-axis, the axis of symmetry through Q, and visualize the (ellipse) frame φ at different instants in its rotation relative to Φ. Clearly, the components I_{11}^Q and I_{22}^Q vary with the orientation of the body when determined in Φ, whereas both are constant when referred to φ. Therefore, we may rightly speculate that dynamical problems generally will be greatly simplified by use of a body reference frame.

9.5. Moments of Inertia for Some Special Homogeneous Bodies

The foregoing examples illustrate some general properties of all homogeneous bodies. In this section, the focus shifts to three specific axisymmetric homogeneous

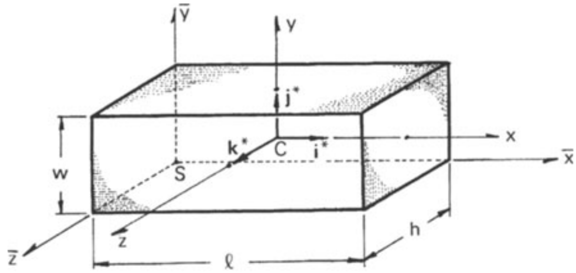


Figure 9.4. Geometry for a homogeneous rectangular block.

bodies—a rectangular body, a circular cylindrical body, and a sphere. The inertia tensors for these and some related slender bodies are derived.

9.5.1. Moment of Inertia Tensor for a Rectangular Body

The moment of inertia tensor $\mathbf{I}_C$ for a homogeneous rectangular block of length ℓ, width w, and height h is determined with respect to a body frame $\varphi = \{C; \mathbf{i}_k^*\}$ at its center of mass C and oriented as shown in Fig. 9.4. The results are then specialized to obtain the inertia tensor for a thin rectangular plate of thickness h and for a thin rod of length ℓ. The body frame at S in Fig. 9.4 is reserved for future use.

The homogeneous block has constant mass density ρ, and hence its center of mass C is at the centroid. The components I_{jk} of the inertia tensor $\mathbf{I}_C$ referred to φ are given by (9.14) and (9.15). Specifically, for the rectangular block,

$$I_{11} = \rho\ell \int_{-h/2}^{h/2} \left[\int_{-w/2}^{w/2} (y^2 + z^2) dy \right] dz = \rho \frac{\ell w h}{12} (w^2 + h^2).$$

That is, with $m \equiv m(\mathscr{B}) = \rho \ell w h$ for the mass of the block,

$$I_{11} = \frac{m(w^2 + h^2)}{12}. \tag{9.25a}$$

Noting the correspondence $(x, y, z) \sim (\ell, w, h)$ and permuting the symbols accordingly in the last two relations in (9.14) while bearing in mind (9.25a), we find

$$I_{22} = \frac{m(\ell^2 + h^2)}{12}, \qquad I_{33} = \frac{m(\ell^2 + w^2)}{12}. \tag{9.25b}$$

All products of inertia are zero. (Why?) Collecting these results in (9.11), we obtain the *moment of inertia tensor for a homogeneous rectangular parallelepiped*:

$$\mathbf{I}_C = \frac{m(w^2 + h^2)}{12} \mathbf{i}_{11}^* + \frac{m(\ell^2 + h^2)}{12} \mathbf{i}_{22}^* + \frac{m(\ell^2 + w^2)}{12} \mathbf{i}_{33}^*, \tag{9.26}$$

referred to the center of mass body frame $\varphi = \{C; \mathbf{i}_k^*\}$. Notice that the matrix of (9.26) is diagonal.

9.5.1.1. *Moment of Inertia of a Thin Rectangular Plate*

A thin plate of negligible thickness h is a lamina, a plane body in the xy-plane, with mass density $\eta = \rho h$ per unit area. Hence, $m = \eta \ell w$. Upon neglecting the h^2 terms in (9.26), we obtain *the moment of inertia tensor for a thin rectangular plate*:

$$\mathbf{I}_C = \frac{mw^2}{12}\mathbf{i}_{11}^* + \frac{m\ell^2}{12}\mathbf{i}_{22}^* + \frac{m(\ell^2 + w^2)}{12}\mathbf{i}_{33}^*, \tag{9.27}$$

referred to the center of mass body frame $\varphi = \{C; \mathbf{i}_k^*\}$. Notice that $I_{33} = I_{11} + I_{22}$ in accordance with the general rule in (9.19) valid for every plane body.

9.5.1.2. *Moment of Inertia of a Thin Rod*

Let $\sigma = \eta w$ denote the mass per unit length of the homogeneous body, so that $m = \sigma \ell$. Now, neglect terms of order w^2 in (9.27) to obtain the *moment of inertia tensor for a thin rod*:

$$\mathbf{I}_C = \frac{m\ell^2}{12}(\mathbf{i}_{22}^* + \mathbf{i}_{33}^*), \tag{9.28}$$

where the rod axis is the $\mathbf{i}_1^*$-axis in $\varphi = \{C; \mathbf{i}_k^*\}$.

9.5.2. Moment of Inertia of a Circular Cylindrical Body

The inertia tensor for a homogeneous, circular cylindrical tube of inside radius r_i, outside radius r_o, and length ℓ is derived relative to a center of mass body frame $\varphi = \{C; \mathbf{i}_k^*\}$ situated at $\ell/2$ from O in Fig. 5.3, page 13, with $\mathbf{i}_3^* = \mathbf{k}$. The result is then applied to find the inertia tensor for a solid cylinder, an annular lamina, a thin circular disk, and a thin rod.

We begin with the cylindrical tube. The moment of inertia about the z-axis is obtained from the last equation in (9.14). Introducing cylindrical coordinates with $x^2 + y^2 = r^2$ and noting that $dm = \rho 2\pi r \ell dr$, we have

$$I_{33} = \int_{\mathscr{B}} r^2 dm = 2\pi\rho\ell \int_{r_i}^{r_o} r^3 dr = \frac{\pi}{2}\rho\ell\left(r_o^4 - r_i^4\right).$$

With $m = \rho A \ell$ and the cross sectional area $A = \pi(r_o^2 - r_i^2)$, we obtain

$$I_{33} = \frac{m}{2}\left(r_o^2 + r_i^2\right). \tag{9.29a}$$

The z-axis is the central axis for this body of revolution; so, $I_{11} = I_{22}$, and hence by (9.17), $2I_{11} = I_{33} + 2\int_{\mathscr{B}} z^2 dm$, in which $dm = \rho A dz$. For a homogeneous material,

$$\int_{\mathscr{B}} z^2 dm = \rho A \int_{-\ell/2}^{\ell/2} z^2 dz = \frac{m\ell^2}{12}, \tag{9.29b}$$

and hence with (9.29a),

$$I_{11} = I_{22} = \frac{m}{4}\left(r_o^2 + r_i^2 + \frac{\ell^2}{3}\right). \tag{9.29c}$$

The products of inertia vanish. (Why?) Collecting (9.29a) and (9.29c), we reach the *moment of inertia tensor for a homogeneous circular cylindrical tube referred to the center of mass body frame* $\varphi = \{C; \mathbf{i}_k^*\}$:

$$\mathbf{I}_C = \frac{m}{4}\left(r_o^2 + r_i^2 + \frac{\ell^2}{3}\right)(\mathbf{i}_{11}^* + \mathbf{i}_{22}^*) + \frac{m}{2}(r_o^2 + r_i^2)\mathbf{i}_{33}^*. \tag{9.30}$$

9.5.2.1. *Moment of Inertia of a Solid Cylinder*

Now consider a solid cylinder for which $r_o = r$ and $r_i = 0$. Then (9.30) reduces to the *moment of inertia tensor for a solid circular cylinder referred to the center of mass body frame* $\varphi = \{C; \mathbf{i}_k^*\}$:

$$\mathbf{I}_C = \frac{m}{4}\left(r^2 + \frac{\ell^2}{3}\right)(\mathbf{i}_{11}^* + \mathbf{i}_{22}^*) + \frac{m}{2}r^2\mathbf{i}_{33}^*, \tag{9.31}$$

where $m = \rho\pi r^2\ell$. Upon neglecting the terms mr^2 in (9.31), we recover the inertia tensor (9.28) for a thin rod, except now the rod axis is $\mathbf{i}_3^*$. Therefore, the actual cross sectional geometry of a slender rod is unimportant.

9.5.2.2. *Moment of Inertia of an Annular Plate*

Relations (9.30) and (9.31) are used next to derive inertia tensors for similar plane bodies having mass density η per unit area. First, consider an annular plate, or flat washer, of negligible thickness ℓ and area $A = \pi(r_o^2 - r_i^2)$. Then with $m = \eta A$, (9.30) yields the *moment of inertia tensor for a homogeneous annular plate relative to the body frame* $\varphi = \{C; \mathbf{i}_k^*\}$:

$$\mathbf{I}_C = \frac{m}{4}(r_o^2 + r_i^2)(\mathbf{i}_{11}^* + \mathbf{i}_{22}^* + 2\mathbf{i}_{33}^*). \tag{9.32}$$

9.5.2.3. Moment of Inertia of a Thin Circular Disk

Finally, with $r_o = R$, $r_i = 0$, and $m = \eta\pi R^2$, (9.32) reduces to the *moment of inertia tensor for a homogeneous, thin circular disk relative to* $\varphi = \{C; \mathbf{i}_k^*\}$:

$$\mathbf{I}_C = \frac{m}{4}R^2(\mathbf{i}_{11}^* + \mathbf{i}_{22}^* + 2\mathbf{i}_{33}^*). \tag{9.33}$$

Notice that (9.33) also follows directly from (9.31) upon neglecting terms in ℓ^2. In agreement with the last rule in (9.19) for every plane body, it is seen in both (9.32) and (9.33) that $I_{33} = I_{11} + I_{22}$ holds.

9.5.3. Moment of Inertia Tensor for a Sphere

The moment of inertia tensor for a homogeneous sphere of radius R is derived relative to a body frame $\varphi = \{C; \mathbf{i}_k^*\}$ at its center. Clearly, every plane through C is a plane of symmetry, so all products of inertia vanish and $I_{11} = I_{22} = I_{33}$. Hence, by (9.18), $3I_{11} = 2\int_{\mathscr{B}} r^2 dm$. The surface area of a sphere of radius r is $4\pi r^2$, so the elemental mass of a spherical shell of thickness dr is $dm = \rho 4\pi r^2 dr$; and the foregoing relation yields

$$I_{11} = \frac{8}{3}\rho\pi\int_0^R r^4 dr = \frac{2}{5}mR^2,$$

where $m = \frac{4}{3}\rho\pi R^3$ is the total mass of the sphere. *Therefore, in every body reference frame at its center, the moment of inertia tensor for a homogeneous sphere is*

$$\mathbf{I}_C = \frac{2}{5}mR^2\mathbf{1}. \tag{9.34}$$

Exercise 9.1. Show that, relative to its center C, the moment of inertia tensor for a nonhomogeneous sphere of radius R whose mass density $\rho = \rho(r)$ varies with the radius $r \in [0, R]$ is spherical; i.e., $\mathbf{I}_C = I_C\mathbf{1}$, where $I_C = \frac{8\pi}{3}\int_0^R r^4\rho(r)dr$. For constant ρ, this returns (9.34). □

This concludes discussion of the moment of inertia tensor for a few homogeneous rigid bodies. Additional results are summarized in the table of properties in Appendix D, and some further examples are provided in the problems.

9.6. The Moment of Inertia of a Complex Structured Body

A complex structured body $\mathscr{B} = \cup_{k=1}^n \mathscr{B}_k$ may be regarded as a composition of several materially or geometrically simple bodies $\mathscr{B}_k$ whose moment of inertia tensors are known or may be readily determined. Hence, for the composite body $\mathscr{B}$, (9.10) may be written as

$$\mathbf{I}_Q(\mathscr{B}) = \sum_{k=1}^{n} \int_{\mathscr{B}_k} [(\mathbf{x}\cdot\mathbf{x})\mathbf{1} - \mathbf{x}\otimes\mathbf{x}]\, dm,$$

in terms of the simple parts $\mathscr{B}_k$ of $\mathscr{B}$, each of which may be materially different and nonhomogeneous. Then application of (9.10) to each body $\mathscr{B}_k$ yields the *composition rule for the moment of inertia tensor for a composite body, relative to Q*:

$$\mathbf{I}_Q(\mathscr{B}) = \sum_{k=1}^{n} \mathbf{I}_Q(\mathscr{B}_k). \tag{9.35}$$

Hence, with respect to an assigned body frame $\varphi = \{Q; \mathbf{e}_k\}$, the moment of inertia tensor for a composite body is equal to the sum of the inertia tensors for its simple parts referred to φ.

If $\mathscr{B}$ has p cavities $\mathscr{C}_k$, say, we may imagine that each cavity is filled with material having the same mass distribution as $\mathscr{B}$. The composition rule (9.35) applied to the auxiliary solid body $\mathscr{B}_S = \mathscr{B} \cup_{k=1}^{p} \mathscr{C}_k$ determines the moment of inertia tensor for $\mathscr{B} = \mathscr{B}_S \backslash \cup_{k=1}^{p} \mathscr{C}_k$ in accordance with

$$\mathbf{I}_Q(\mathscr{B}) = \mathbf{I}_Q(\mathscr{B}_S) - \sum_{k=1}^{p} \mathbf{I}_Q(\mathscr{C}_k). \tag{9.36}$$

On the other hand, if a cavity $\mathscr{C}_k$ is viewed as a "body" of negative mass and materially similar to the simple body $\mathscr{B}_k$ containing $\mathscr{C}_k$, (9.36) may be summarized in the form (9.35). Each inertia tensor in the sum (9.35), however, must be referred to the same body frame φ at Q. In consequence, we shall need to know how to transform the inertia tensor components for each of the separate parts, from a reference frame at a point P conveniently chosen for calculation of the components for a separate part, to another reference frame at another point Q appropriate for the tensor components for the composite body. Before we tackle this problem, however, let us consider two examples of homogeneous bodies that require only a single reference frame. The second example is noteworthy because it shows that neither symmetry nor material homogeneity of a body is necessary for the vanishing of products of inertia.

Example 9.3. A homogeneous rectangular parallelepiped of length ℓ and square cross section of side h has a circular hole of radius R drilled lengthwise through its center, as shown in Fig. 9.5. Determine the moment of inertia tensor components of the drilled block referred to the center of mass body frame $\varphi = \{C; \mathbf{i}_k^*\}$.

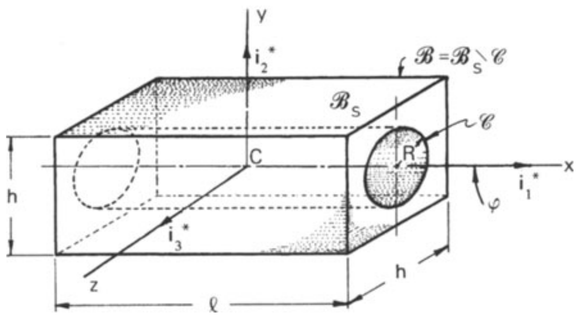


Figure 9.5. A homogeneous block $\mathscr{B}$ having a drilled hole.

Solution. The center of mass of the homogeneous drilled block $\mathscr{B} = \mathscr{B}_S \backslash \mathscr{C}$ is at the center of the hole. Here $\mathscr{B}_S$ denotes the solid rectangular block and $\mathscr{C}$ identifies a homogeneous circular cylinder of the same material which we imagine fills the hole. Thus, with respect to φ, (9.36) yields

$$\mathbf{I}_C(\mathscr{B}) = \mathbf{I}_C(\mathscr{B}_S) - \mathbf{I}_C(\mathscr{C}). \tag{9.37a}$$

Recalling (9.26) for a homogeneous solid parallelepiped with $w = h$ and (9.31) for a homogeneous solid cylinder of radius R, bearing in mind the arrangement of the coordinate axes in Fig. 9.5 and in Fig. 5.3, page 13, for the cylinder, we obtain from (9.37a), referred to the body frame $\varphi = \{C; \mathbf{i}_k^*\}$,

$$\begin{aligned}\mathbf{I}_C(\mathscr{B}) = {} & \frac{m_S h^2}{6}\mathbf{i}_{11}^* + \frac{m_S}{12}(h^2 + \ell^2)(\mathbf{i}_{22}^* + \mathbf{i}_{33}^*) \\ & - \left[\frac{m_C R^2}{2}\mathbf{i}_{11}^* + \frac{m_C}{4}\left(R^2 + \frac{\ell^2}{3}\right)(\mathbf{i}_{22}^* + \mathbf{i}_{33}^*)\right],\end{aligned} \tag{9.37b}$$

wherein the mass m_S of the solid block and m_C of the cavity body are given by

$$m_S \equiv m(\mathscr{B}_S) = \rho \ell h^2, \qquad m_C \equiv m(\mathscr{C}) = \rho \pi \ell R^2. \tag{9.37c}$$

Hence, by (9.4), the mass of the drilled block is

$$m(\mathscr{B}) = m_S - m_C = \rho \ell (h^2 - \pi R^2). \tag{9.37d}$$

Use of (9.37c) and (9.37d) in (9.37b) yields the moment of inertia tensor components for the homogeneous, drilled parallelepiped referred to the center of mass frame φ:

$$I_{11} = m(\mathscr{B})\left[\frac{h^4 - 3\pi R^4}{6(h^2 - \pi R^2)}\right], \qquad I_{22} = I_{33} = \frac{1}{2}I_{11} + \frac{m(\mathscr{B})\ell^2}{12}. \tag{9.37e}$$

□

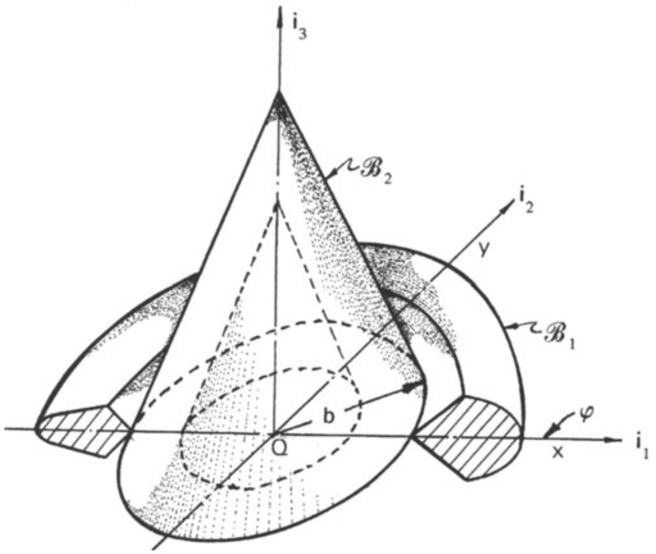


Figure 9.6. A complex structured rigid body composed of homogeneous, but materially different parts $\mathscr{B}_1$ and $\mathscr{B}_2$.

Example 9.4. The complex structured body $\mathscr{B} = \mathscr{B}_1 \cup \mathscr{B}_2$ in Fig. 9.6 consists of a semicircular ring $\mathscr{B}_1$ of variable cross section, shown in Fig. 9.3, and a right conical shell $\mathscr{B}_2$. The bodies are made of different homogeneous materials welded together along their common circular boundary in the xy-plane. Find the matrix of the moment of inertia tensor for the composite body $\mathscr{B}$ referred to $\varphi = \{Q; \mathbf{i}_k\}$.

Solution. Let $\mathbf{I}'_Q(\mathscr{B}_1)$ and $\mathbf{I}''_Q(\mathscr{B}_2)$ denote the moment of inertia tensors for $\mathscr{B}_1$ and $\mathscr{B}_2$ referred to the same body frame $\varphi = \{Q; \mathbf{i}_k\}$ in Fig. 9.6. For the composite body $\mathscr{B} = \mathscr{B}_1 \cup \mathscr{B}_2$, (9.35) yields $\mathbf{I}_Q(\mathscr{B}) = \mathbf{I}'_Q(\mathscr{B}_1) + \mathbf{I}''_Q(\mathscr{B}_2)$. The matrix of $\mathbf{I}'_Q$ referred to $\varphi = \{Q; \mathbf{i}_k\}$ has the form (9.24), and that of $\mathbf{I}''_Q$ for the conical shell is diagonal with $I''_{11} = I''_{22}$. (Why?) Thus, with $[\mathbf{I}''_Q] = \operatorname{diag}[I''_{11}, I''_{11}, I''_{33}]$ and introduction of the $'$ notation in (9.24) yields the moment of inertia tensor for the nonhomogeneous, composite body $\mathscr{B}$ referred to φ:

$$[\mathbf{I}_Q] = \begin{bmatrix} I'_{11} + I''_{11} & I'_{12} & 0 \\ I'_{12} & I'_{22} + I''_{11} & 0 \\ 0 & 0 & I'_{33} + I''_{33} \end{bmatrix}.$$

The xy-plane clearly is not a plane of geometrical symmetry for $\mathscr{B}$, not to mention the nonhomogeneous nature of the assembly, yet $I_{13} = I_{23} = 0$. Thus, while symmetry of a homogeneous body with respect to an $\alpha\beta$-plane normal to γ implies that the products of inertia $I_{\alpha\gamma} = I_{\beta\gamma} = 0$, body symmetry is not a necessary condition for the vanishing of products of inertia. The general nature of this noteworthy observation will be explored after the transformation laws for the inertia tensor under parallel translation and rotation of the reference frame are in hand. □

9.7. The Parallel Axis Theorem

The parallel axis theorem is a useful transformation rule that provides relations connecting the moment of inertia tensor components in parallel bases at two base points. To deduce this rule, let us consider a rigid body $\mathscr{B}$ of any shape, homogeneous or not, and suppose that $\mathbf{I}_Q(\mathscr{B})$ is known in the body frame $\varphi = \{Q; \mathbf{i}_k\}$ at a base point Q. The problem is to determine $\mathbf{I}_S(\mathscr{B})$ in a parallel body frame $\psi = \{S; \mathbf{i}_k\}$ at another base point S, as shown in Fig. 9.7. The position vectors of a material point P from Q and S are respectively denoted by $\mathbf{x}(P)$ and $\rho(P) = \mathbf{r}(Q) + \mathbf{x}(P)$, where $\mathbf{r}(Q)$ is the position vector of Q from S. Let $\circ$ denote either the $\cdot$ or $\otimes$ operation, recall (9.1), and consider

$$\begin{aligned}\int_{\mathscr{B}} \rho \circ \rho dm &= \mathbf{r} \circ \mathbf{r} \int_{\mathscr{B}} dm + \mathbf{r} \circ \int_{\mathscr{B}} \mathbf{x} dm + \int_{\mathscr{B}} \mathbf{x} dm \circ \mathbf{r} + \int_{\mathscr{B}} \mathbf{x} \circ \mathbf{x} dm \\ &= m(\mathscr{B})[\mathbf{r} \circ \mathbf{r} + \mathbf{r} \circ \mathbf{x}^* + \mathbf{x}^* \circ \mathbf{r}] + \int_{\mathscr{B}} \mathbf{x} \circ \mathbf{x} dm, \qquad (9.38)\end{aligned}$$

where $\mathbf{x}^*$ is the position vector of the center of mass C from Q in φ. Thus, use of (9.38) in (9.10) applied at S leads to the general point transformation rule for the moment of inertia tensor:

$$\mathbf{I}_S(\mathscr{B}) = \mathbf{I}_Q(\mathscr{B}) + m(\mathscr{B})[(\mathbf{r} \cdot \mathbf{r})\mathbf{1} - \mathbf{r} \otimes \mathbf{r} + (2\mathbf{r} \cdot \mathbf{x}^*)\mathbf{1} - \mathbf{r} \otimes \mathbf{x}^* - \mathbf{x}^* \otimes \mathbf{r}]. \qquad (9.39)$$

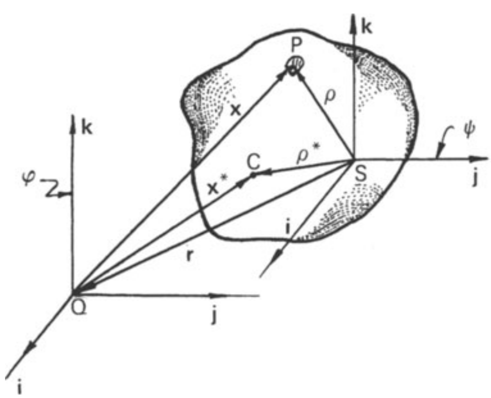


Figure 9.7. Schema for the parallel axis theorem.

This cumbersome expression is readily simplified by our choosing Q at the center of mass C located at $\boldsymbol{\rho}^*$ from S in Fig. 9.7; for then $\mathbf{x}^* = \boldsymbol{\rho}^* - \mathbf{r} = \mathbf{0}$, and (9.39) yields the following reduced point transformation rule:

$$\mathbf{I}_S(\mathscr{B}) = \mathbf{I}_C(\mathscr{B}) + \mathbf{I}_S^*(\mathscr{B}), \tag{9.40}$$

wherein, with $m = m(\mathscr{B})$,

$$\mathbf{I}_S^*(\mathscr{B}) \equiv m[(\boldsymbol{\rho}^* \cdot \boldsymbol{\rho}^*)\mathbf{1} - \rho^* \otimes \boldsymbol{\rho}^*], \tag{9.41}$$

is the *moment of inertia of the center of mass particle relative to S*. Accordingly, rule (9.40) states that *the moment of inertia relative to any point S is equal to the moment of inertia relative to the center of mass point C plus the moment of inertia of the center of mass particle relative to S.*

An important geometrical interpretation of (9.40) derives from its Cartesian component form in a body reference frame $\psi = \{S; \mathbf{i}_k\}$. Write $\boldsymbol{\rho}^* = x^*\mathbf{i} + y^*\mathbf{j} + z^*\mathbf{k}$ for the position vector of C from S in (9.41) to obtain the components of the inertia tensor of the center of mass particle relative to S:

$$[\mathbf{I}_S^*] = \begin{bmatrix} m(y^{*2} + z^{*2}) & -mx^*y^* & -mx^*z^* \\ -mx^*y^* & m(z^{*2} + x^{*2}) & -my^*z^* \\ -mx^*z^* & -my^*z^* & m(x^{*2} + y^{*2}) \end{bmatrix}. \tag{9.42}$$

So far, the frame $\psi = \{S; \mathbf{i}_k\}$ at S may have any orientation relative to $\varphi = \{C; \mathbf{i}_k^*\}$ at C. But we now consider the case when $\mathbf{i}_k = \mathbf{i}_k^*$, so that the coordinate directions at S are parallel to those at C. Then (9.40) yields the six component relations $I_{jk}^S = I_{jk}^C + I_{jk}^{S*}$, and use of (9.42) gives, as example,

$$I_{11}^S = I_{11}^C + md_1^2, \qquad I_{23}^S = I_{23}^C + I_{23}^{S*} = I_{23}^C - my^*z^*, \tag{9.43}$$

in which $d_1^2 = y^{*2} + z^{*2}$ is the square of the perpendicular distance between the parallel axes $\mathbf{i}_1$ at S and $\mathbf{i}_1^* = \mathbf{i}_1$ at C in Fig. 9.8. More generally, the six tensor component relations in a common Cartesian tensor basis $\mathbf{i}_{ab}$ are summarized by

$$I_{aa}^S = I_{aa}^C + md_a^2, \qquad I_{ab}^S = I_{ab}^C + I_{ab}^{S*}, \tag{9.44}$$

in which $a, b = 1, 2, 3$ (no sum) and $b \neq a$. Here d_a^2 is the square of the distance between the parallel $\mathbf{i}_a$-axis at S and the $\mathbf{i}_a^*$-axis at C, $I_{ab}^{S*} \equiv -ma^*b^*$, and a^*, b^* are the a, b coordinates of the center of mass particle from S. For $a = 2$, $b = 3$, for example, $a^* = y^*$, $b^* = z^*$, and hence $I_{23}^{S*} = -my^*z^*$ in (9.43) and $d_2^2 = x^{*2} + z^{*2}$. In view of (9.44), the point transformation rule (9.40) is characterized by the following useful theorem.

The parallel axis theorem: *The moment of inertia about an axis at S is equal to the moment of inertia about a parallel axis at the center of mass C plus the product of the mass and the square of the perpendicular distance between the parallel axes. Further, the product of inertia with respect to orthogonal axes at S is equal to the product of inertia with respect to the same parallel axes at C plus the corresponding product inertia of the center of mass particle relative to S.*

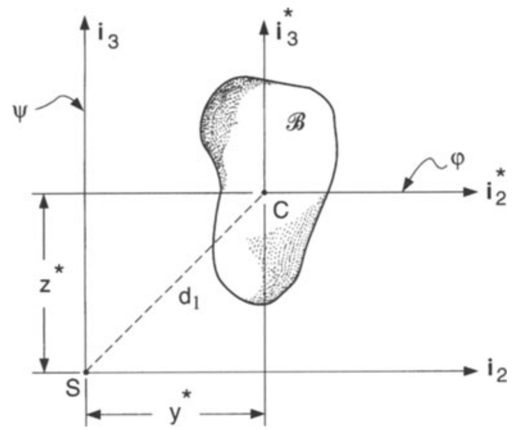


Figure 9.8. Geometrical interpretation of translation terms in the parallel axis rules (9.43).

Now consider a simple parallel shift of the axes along a coordinate direction, say, the z-axis. Then both base points S and C at $d = z^*$ from S are on the z-axis, and $x^* = y^* = 0$. Therefore, all products of inertia of the center of mass particle vanish in (9.42), and (9.44) simplifies to $I_{11}^S = I_{11}^C + md^2$, $I_{22}^S = I_{22}^C + md^2$, $I_{33}^S = I_{33}^C$, and $I_{ab}^S = I_{ab}^C$, $a \neq b = 1, 2, 3$, in accordance with the parallel axis theorem. If the body is homogeneous and the z-axis also is an axis of symmetry, then $I_{ab}^S = I_{ab}^C = 0$ as well. (What can be said if the mass density of the axisymmetric body varies with z?)

Example 9.5. The moment of inertia tensor $\mathbf{I}_C$ for a homogeneous rectangular block is given in (9.26). Find the inertia tensor components with respect to parallel axes at the corner point S in Fig. 9.4, page 364.

Solution. To determine $\mathbf{I}_S$, we apply the point transformation law (9.40). First, note that the center of mass C has coordinates $(x^*, y^*, z^*) = (\ell/2, w/2, h/2)$ in the parallel frame at S; therefore, (9.42) yields relative to S the moment of inertia component matrix for the center of mass particle:

$$[\mathbf{I}_S^*] = \begin{bmatrix} \frac{m}{4}(w^2 + h^2) & -\frac{m}{4}\ell w & -\frac{m}{4}\ell h \\ -\frac{m}{4}\ell w & \frac{m}{4}(h^2 + \ell^2) & -\frac{m}{4}wh \\ -\frac{m}{4}\ell h & -\frac{m}{4}wh & \frac{m}{4}(\ell^2 + w^2) \end{bmatrix}. \tag{9.45a}$$

The moment of inertia components of the block in a parallel frame at C are given in (9.26), and, with (9.45a), the point transformation law (9.40) delivers the inertia

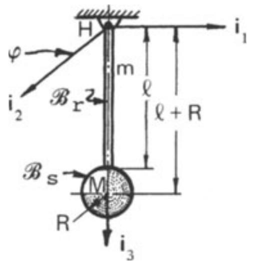


Figure 9.9. Application of the parallel axis theorem to a composite pendulum.

tensor components of the block in the parallel frame at S:

$$[\mathbf{I}_S] = \begin{bmatrix} \frac{m}{3}(w^2+h^2) & -\frac{m}{4}\ell w & -\frac{m}{4}\ell h \\ -\frac{m}{4}\ell w & \frac{m}{3}(h^2+\ell^2) & -\frac{m}{4}wh \\ -\frac{m}{4}\ell h & -\frac{m}{4}wh & \frac{m}{3}(\ell^2+w^2) \end{bmatrix}. \tag{9.45b}$$

Notice that no products of inertia occur in (9.26) for $\mathbf{I}_C$, whereas all products of inertia appear in (9.45b) for $\mathbf{I}_S$. Of course, the same result may be obtained from (9.44). This is left as an exercise for the reader. □

Example 9.6. A complex structured pendulum assembly in Fig. 9.9 consists of a homogeneous spherical body $\mathscr{B}_s$ of radius R and mass M fastened to a homogeneous, but materially different thin rod $\mathscr{B}_r$ of length ℓ, mass m, and supported by a small hinge pin at H. Find the moment of inertia tensor for the pendulum assembly referred to the body frame $\varphi = \{H; \mathbf{i}_k\}$.

Solution. The moment of inertia tensor relative to H for the composite pendulum assembly $\mathscr{B} = \mathscr{B}_s \cup \mathscr{B}_r$ is given by (9.35):

$$\mathbf{I}_H(\mathscr{B}) = \mathbf{I}_H(\mathscr{B}_s) + \mathbf{I}_H(\mathscr{B}_r). \tag{9.46a}$$

Therefore, we shall need to determine the inertia tensor for each simple part $\mathscr{B}_k$ in the body frame $\varphi = \{H; \mathbf{i}_k\}$ in Fig. 9.9. Since the separate homogeneous parts $\mathscr{B}_k$ are materially different, the composite body $\mathscr{B}$ is neither homogeneous nor materially uniform. Nevertheless, because each part is a homogeneous body of revolution that shares the same $\mathbf{i}_3$-axis of symmetry, all products of inertia for the separate bodies, and hence for the assembly, vanish with respect to all base points of parallel frames along the common $\mathbf{i}_3$-axis, specifically, relative to φ at H.

First, consider the homogeneous sphere of radius R and mass $M = \frac{4}{3}\pi\rho R^3$ for which $\mathbf{I}_C(\mathscr{B}_s)$ is given by (9.34) in every reference frame at its center C;

and note that $\ell + R$ is the distance between the parallel axes $\{\mathbf{i}_1^*, \mathbf{i}_2^*\}$ at C for the sphere and $\{\mathbf{i}_1, \mathbf{i}_2\}$ at H. Then, by the parallel axis theorem in (9.44), $I_{11}^H = I_{22}^H = \frac{2}{5}MR^2 + M(\ell + R)^2$, $I_{33}^H = \frac{2}{5}MR^2$, and hence the inertia tensor for the sphere relative to H in φ is

$$\mathbf{I}_H(\mathscr{B}_s) = \left[\frac{2}{5}MR^2 + M(\ell + R)^2\right](\mathbf{i}_{11} + \mathbf{i}_{22}) + \frac{2}{5}MR^2\mathbf{i}_{33}. \quad (9.46\text{b})$$

The same result follows easily from the parallel axis theorem in (9.40), from which $\mathbf{I}_H(\mathscr{B}_s) = \mathbf{I}_C(\mathscr{B}_s) + \mathbf{I}_H^*(\mathscr{B}_s)$, wherein $\mathbf{I}_H^*(\mathscr{B}_s) = M(\ell + R)^2(\mathbf{i}_{11} + \mathbf{i}_{22})$, the moment of inertia of the center of mass of the sphere relative to H. Thus, with (9.34), (9.46b) follows.

Now recall (9.28) for a thin rod of length ℓ, note in Fig. 9.9 that the rod axis is $\mathbf{i}_3$, and rewrite (9.28) to obtain, with $m \equiv m(\mathscr{B}_r) = \sigma\ell$,

$$\mathbf{I}_C(\mathscr{B}_r) = \frac{m\ell^2}{12}(\mathbf{i}_{11} + \mathbf{i}_{22}), \quad (9.46\text{c})$$

where now C is the center of mass of the rod which is at $\ell/2$ from H, i.e. the distance between the parallel axes $\{\mathbf{i}_1^*, \mathbf{i}_2^*\}$ at C for the rod and $\{\mathbf{i}_1, \mathbf{i}_2\}$ at H. Then, by the parallel axis theorem in (9.44), $I_{11}^H = I_{22}^H = m\ell^2/12 + m\ell^2/4$, $I_{33}^H = 0$; so, the inertia tensor for the homogeneous thin rod relative to H in φ is

$$\mathbf{I}_H(\mathscr{B}_r) = \frac{m\ell^2}{3}(\mathbf{i}_{11} + \mathbf{i}_{22}). \quad (9.46\text{d})$$

As before, the same result follows from the tensor form of the parallel axis theorem in (9.40), from which $\mathbf{I}_H(\mathscr{B}_r) = \mathbf{I}_C(\mathscr{B}_r) + \mathbf{I}_H^*(\mathscr{B}_r)$, wherein $\mathbf{I}_H^*(\mathscr{B}_r) = (m\ell^2/4)(\mathbf{i}_{11} + \mathbf{i}_{22})$, the moment of inertia of the center of mass of the rod relative to H.

We are now prepared to calculate $\mathbf{I}_H(\mathscr{B})$ for the pendulum assembly. Substitution of (9.46b) and (9.46d) into (9.46a) delivers the inertia tensor for the pendulum assembly relative to the hinge H in the body frame φ:

$$\mathbf{I}_H(\mathscr{B}) = \left[\frac{2}{5}MR^2 + M(\ell + R)^2 + \frac{m}{3}\ell^2\right](\mathbf{i}_{11} + \mathbf{i}_{22}) + \frac{2}{5}MR^2\mathbf{i}_{33}. \quad (9.46\text{e})$$

□

Let us return briefly to the first expression in (9.44) and notice that each of its terms is positive; hence, $I_{aa}^S > I_{aa}^C$ for every point S and for each choice of axis a. *Therefore, the moment of inertia about an axis has its smallest value at the center of mass.* A similar statement does not hold for the products of inertia in the second relation in (9.44), because their signs are indefinite; but their smallest absolute values at every point plainly are zero. Since we have infinitely many choices for center of mass axes, however, these observations prompt an interesting question: For what directions at the center of mass, or any other point, are the moments of inertia greatest and least? We shall return to this question later. First, we shall need to consider the transformation of tensor components induced by a rotation of the frame of reference.

9.8. Moment of Inertia Tensor Transformation Law

The inertia tensor $\mathbf{I}_Q$ referred to Cartesian frames $\varphi = \{Q; \mathbf{e}_k\}$ and $\varphi' = \{Q; \mathbf{e}'_k\}$ at Q has the same representation (9.11), and hence

$$\mathbf{I}_Q = I_{jk}\mathbf{e}_{jk} = I'_{lm}\mathbf{e}'_{lm}. \tag{9.47}$$

The change of vector basis defined by $\mathbf{e}'_j = A_{jk}\mathbf{e}_k$, or by its inverse $\mathbf{e}_k = A_{jk}\mathbf{e}'_j$, in which $A_{jk} \equiv \cos\langle \mathbf{e}'_j, \mathbf{e}_k\rangle$, induces a change of the corresponding tensor product bases in accordance with (3.101). Consequently, under a change of frame by a rotation of the bases at Q, in terms of the matrix notation in (3.108), we obtain *the Cartesian tensor transformation law for the moment of inertia components at Q:*

$$I'_Q = A I_Q A^T \quad \text{or} \quad I_Q = A^T I'_Q A, \tag{9.48}$$

where $A = [A_{jk}] = [\cos\langle \mathbf{e}'_j, \mathbf{e}_k\rangle]$. The reader may confirm this important rule by tracing its derivation starting with the change of basis in (9.47).

Example 9.7. Find the moment of inertia tensor for a homogeneous rectangular block of mass $m = 6$ slug, referred to body frames $\varphi = \{C; \mathbf{i}_k\}$ and $\varphi' = \{C; \mathbf{i}'_k\}$ defined in Fig. 9.10 at the center of mass C.

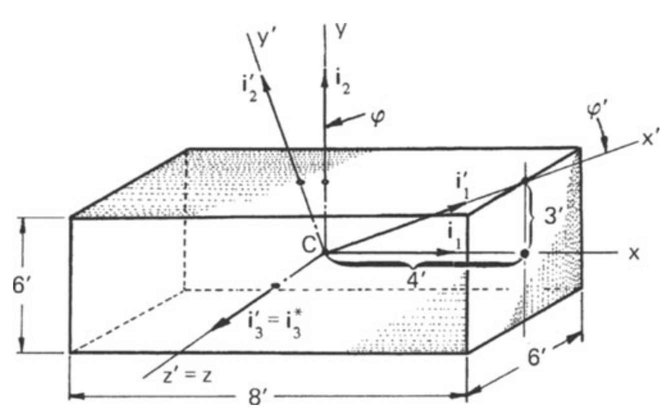


Figure 9.10. Application of the transformation law for the moment of inertia tensor referred to a rotated frame φ'.

Solution. The component matrix I_C of the moment of inertia tensor for the homogeneous block may be read from (9.26). With $m = 6$ slug, $\ell = 8$ ft, $w = h = 6$ ft, we find in $\varphi = \{C; \mathbf{i}_k\}$ the tensor component matrix

$$I_C = \begin{bmatrix} 36 & 0 & 0 \\ 0 & 50 & 0 \\ 0 & 0 & 50 \end{bmatrix}. \tag{9.49a}$$

The matrix I'_C in φ' is obtained from (9.48). It is seen from Fig. 9.10 that the transformation matrix $A = [\cos\langle \mathbf{i}'_p, \mathbf{i}_q\rangle]$ is given by

$$A = \begin{bmatrix} 4/5 & 3/5 & 0 \\ -3/5 & 4/5 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \tag{9.49b}$$

and its use in the first rule in (9.48) provides

$$I'_C = \begin{bmatrix} 4/5 & 3/5 & 0 \\ -3/5 & 4/5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 36 & 0 & 0 \\ 0 & 50 & 0 \\ 0 & 0 & 50 \end{bmatrix} \begin{bmatrix} 4/5 & -3/5 & 0 \\ 3/5 & 4/5 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{9.49c}$$

Hence, the matrix I'_C of the inertia tensor referred to $\varphi' = \{C; \mathbf{i}'_k\}$ in Fig. 9.10 is

$$I'_C = \begin{bmatrix} 1026/25 & 168/25 & 0 \\ 168/25 & 1124/25 & 0 \\ 0 & 0 & 50 \end{bmatrix} \text{slug} \cdot \text{ft}^2. \tag{9.49d}$$

Notice that all products of inertia vanish in φ but not in φ'.

It is useful to note the invariance of $\operatorname{tr}\mathbf{I}_Q$ as a check on the calculation. By (9.49d), $\operatorname{tr}\mathbf{I}'_C = 1026/25 + 1124/25 + 50 = 136$. This agrees with $\operatorname{tr}\mathbf{I}_C = 136$ obtained from (9.49a). □

Exercise 9.2. Because of the symmetry of a homogeneous sphere with respect to every plane through its center C, its inertia tensor (9.34) has the same components in every reference frame at C. (i) Consider any tensor $\mathbf{T}$ whose matrix T in a Cartesian basis $\mathbf{e}_k$ is a scalar multiple of the identity matrix: $T = \tau I$, say. Apply the tensor transformation law (3.108) to show that $\mathbf{T}$ has the same components in every basis $\mathbf{e}'_k$; indeed, $\mathbf{T} = \tau \mathbf{1}$, and hence $\mathbf{T}$ is called a *spherical tensor.* (ii) A homogeneous cube, of course, does not have global symmetry with respect to every plane through its center C; so it is surprising that its inertia tensor is spherical. Show that the *moment of inertia tensor for a homogeneous cube of side a with mass $m = \rho a^3$ is*

$$\mathbf{I}_C = \frac{1}{6} m a^2 \mathbf{1}. \tag{9.50}$$

Other striking examples are the hemispherical shell shown in Fig. D.11 and the hemisphere in Fig. D.13 of Appendix D. See Problems 9.10 and 9.11. □

9.9. Extremal Properties of the Moment of Inertia Tensor

We have seen that all of the products of inertia vanish for every homogeneous body having at least two orthogonal planes of symmetry with respect to a body reference frame. A homogeneous cube, however, is a particularly striking example in that (9.50) shows that the products of inertia for a cube vanish in every reference frame at its center, underscoring our earlier observation that geometrical symmetry of a body is not necessary for the vanishing of its products of inertia in a body reference frame. But the most remarkable part of the story is yet untold. We are going to show that however complex the body geometry and regardless of its material distribution, there always exists at each point of a body an imbedded reference frame, called a *principal frame*, with respect to which the products of inertia vanish. Moreover, two of these are the directions for which the normal components of the inertia tensor assume their greatest and least values at each point. To demonstrate this, however, it is convenient to first study the method of Lagrange multipliers, a neat analytical technique that enables one to determine the stationary values of a function of several variables related by some specified constraint conditions.

The principal problem is introduced next. Then the Lagrange method of undetermined multipliers is described, and the method is illustrated in an easily visualized mechanical control problem. Afterwards, the extremal properties of the normal components of the inertia tensor are determined by Lagrange's method, and these properties are then characterized geometrically by Cauchy's inertia ellipsoid. The procedure for finding the body axes relative to which all products of inertia vanish at a specified body point is illustrated.

9.9.1. Introduction to the Principal Values Problem

Let $\mathbf{n}$ be an arbitrary unit vector at Q in an assigned body frame $\varphi = \{Q; \mathbf{e}_k\}$. The moment of inertia about the axis $\mathbf{n}$ at Q is the normal component of the inertia tensor $\mathbf{I}_Q$ for the direction $\mathbf{n}$ defined by

$$I_{nn}^{Q} = \mathbf{n} \cdot \mathbf{I_Q n}. \tag{9.51}$$

Clearly, the value of I_{nn}^{Q} depends on the direction $\mathbf{n}$. *The main problem, therefore, is to find the directions* $\mathbf{n} = \nu_k \mathbf{e}_k$ *in the body frame* φ *at* Q *for which the normal components of the moment of inertia tensor are largest and least.*

The normal component I_{nn}^{Q} is a function of the three direction cosines ν_k of the unit vector $\mathbf{n}$, thus subject to the *constraint equation*

$$\mathbf{n} \cdot \mathbf{n} = \nu_k \nu_k = 1. \tag{9.52}$$

Therefore, the three variables ν_k are not independent. The constraint equation (9.52) can be used to express any one of the ν_k in terms of the others, the result substituted into (9.51), and the stationary values I_{nn}^{Q} then determined in the usual way. This procedure, though straightforward in principle, often proves tedious; and sometimes it does not give a correct solution (see Problem 9.46). A more convenient, systematic scheme, applicable to a function of p variables related by $q < p$ constraint equations, is provided by Lagrange's method of undetermined multipliers.

9.9.2. The Method of Lagrange Multipliers

Consider a scalar-valued function $D(\mathbf{u})$ of the Cartesian components u_k of the p-dimensional vector field variable $\mathbf{u} = u_k\mathbf{i}_k$. Let $\partial S(\mathbf{u})/\partial\mathbf{u} \equiv (\partial S/\partial u_k)\mathbf{i}_k$ define the gradient of a general scalar function $S(\mathbf{u})$ with respect to $\mathbf{u}$. With no constraints on $\mathbf{u}$, a necessary condition that $D(\mathbf{u})$ have a stationary value is that

$$dD(\mathbf{u}) = \frac{\partial D(\mathbf{u})}{\partial u_k} du_k = \frac{\partial D(\mathbf{u})}{\partial \mathbf{u}} \cdot d\mathbf{u} = 0, \tag{9.53}$$

hold for all values of the differentials du_k, that is, for all vectors $d\mathbf{u}$. Since the variables u_k are assumed independent, their differentials can be assigned arbitrarily. We are thus led by (9.53) to p equations $\partial D/\partial u_k = 0$ in the p scalar components u_k of $\mathbf{u}$, that is, $\partial D(\mathbf{u})/\partial\mathbf{u} = \mathbf{0}$. These are the usual necessary conditions for existence of extrema of $D(\mathbf{u})$.

Now suppose that the components u_k must satisfy a *constraint equation* $F(\mathbf{u}) = 0$. Then only $p - 1$ of the p components u_k are independent. In addition to (9.53), $\mathbf{u}$ also must satisfy

$$dF(\mathbf{u}) = \frac{\partial F(\mathbf{u})}{\partial \mathbf{u}} \cdot d\mathbf{u} = 0. \tag{9.54}$$

Since du_k cannot be varied arbitrarily in (9.53) and (9.54), the extreme values of $D(\mathbf{u})$ are no longer determined by the p equations $\partial D(\mathbf{u})/\partial u_k = 0$, nor equivalently by $\partial D/\partial\mathbf{u} = \mathbf{0}$. Also, in general, $\partial F(\mathbf{u})/\partial u_k \neq 0$. Observe, however, that (9.53) and (9.54) show that the vectors $\partial D/\partial\mathbf{u}$ and $\partial F/\partial\mathbf{u}$ are perpendicular to the same vector $d\mathbf{u}$ for which any $p - 1$ components du_k can be varied arbitrarily, the p^{th} component being fixed by the constraint equation $F(\mathbf{u}) = 0$. These conditions imply that $\partial D/\partial\mathbf{u}$ and $\partial F/\partial\mathbf{u}$ must be parallel vectors so that $\partial D/\partial\mathbf{u} = \lambda\partial F/\partial\mathbf{u}$ at the stationary point, where λ is an unspecified, essentially arbitrary scalar independent of $\mathbf{u}$, called a *Lagrange multiplier*, to be determined as needed.

To prove this, we introduce an auxiliary function

$$G(\mathbf{u}) \equiv D(\mathbf{u}) - \lambda F(\mathbf{u}), \tag{9.55}$$

where λ is an arbitrary scalar to be determined as needed. Then the extreme values of $D(\mathbf{u})$ subject to the constraint $F(\mathbf{u}) = 0$ are determined from the extrema of $G(\mathbf{u})$ upon setting $\partial G(\mathbf{u})/\partial\mathbf{u} = \mathbf{0}$. Indeed, since

$$dG(\mathbf{u}) = \left(\frac{\partial D}{\partial \mathbf{u}} - \lambda\frac{\partial F}{\partial \mathbf{u}}\right) \cdot d\mathbf{u} = \left(\frac{\partial D}{\partial u_k} - \lambda\frac{\partial F}{\partial u_k}\right) du_k = 0, \tag{9.56}$$

must hold for an arbitrary value of λ, we may choose λ so that the coefficient of any one of the p differentials du_k in (9.56) vanishes, assuming of course that *for this choice* $\partial F/\partial u_k \neq 0$. Then the components of $d\mathbf{u}$ that remain in (9.56) are independent and can be varied arbitrarily. In consequence, it follows that all coefficients of the differentials in (9.56) must vanish. Therefore, the necessary condition for an extremum of $G(\mathbf{u})$ is provided by

$$\frac{\partial G}{\partial \mathbf{u}} = \frac{\partial D}{\partial \mathbf{u}} - \lambda \frac{\partial F}{\partial \mathbf{u}} = \mathbf{0}. \tag{9.57}$$

Thus, (9.57) determines the stationary values of $D(\mathbf{u})$ subject to the constraint

$$F(\mathbf{u}) = 0. \tag{9.58}$$

Indeed, the system of $p + 1$ equations (9.57) and (9.58) determine the p components u_k and the scalar multiplier λ for which $G(\mathbf{u})$ has an extremum. Now, at an extremal point $\mathbf{u} = \mathbf{u}^*$, say, the constraint $F(\mathbf{u}^*) = 0$ must be satisfied, and hence (9.55) shows that $G(\mathbf{u}^*) = D(\mathbf{u}^*)$; that is, the stationary values of D are the same as those of G. This procedure is known as *the method of Lagrange multipliers*.

The method may be extended to $q < p$ constraints by introduction of q undetermined Lagrange multipliers λ_r. In this case, we introduce the auxiliary function $G(\mathbf{u}) \equiv D(\mathbf{u}) - \sum_{r=1}^{q} \lambda_r F_r(\mathbf{u})$, in which the q constraints to be satisfied at the stationary points are $F_r(\mathbf{u}) = 0$. Then the necessary conditions for an extremum of $D(\mathbf{u}) = 0$ subject to these constraints are given by $\partial G(\mathbf{u})/\partial \mathbf{u} = \partial D(\mathbf{u})/\partial \mathbf{u} - \sum_{r=1}^{q} \lambda_r \partial F_r(\mathbf{u})/\partial \mathbf{u} = \mathbf{0}$. By setting $\partial G(\mathbf{u})/\partial \lambda_r = -F_r(\mathbf{u}) = 0$, we may recover the q constraint equations.

An application of Lagrange's method to a mechanical control problem whose solution is easily visualized follows.

Example 9.8. A bell crank mechanism having a telescopic control arm OP is shown in Fig. 9.11. The control pin P is constrained to move in a straight slot defined by the equation $y = 1 - x$. To design the crank, the designer must know the shortest distance d from the origin to the line of motion of P, an easy geometry

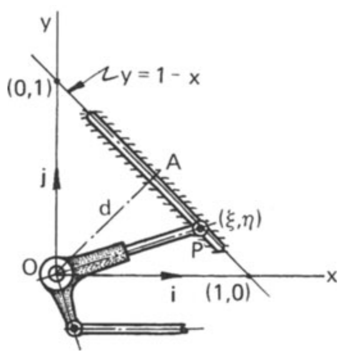


Figure 9.11. Application of the method of Lagrange multipliers to the design analysis of a bell crank mechanism.

problem. Find by geometry and then by the method of Lagrange multipliers the point on this line which is closest to the origin, and thus determine d.

Solution. The geometrical solution is evident in Fig. 9.11. The shortest line OA is the perpendicular bisector of the hypotenuse of the isosceles right triangle whose length is $\sqrt{2}$. Hence, $d = \sqrt{2}/2$ is the shortest distance from O to the line of motion of P, the nearest point to O being the midpoint A at $\mathbf{x} = \frac{1}{2}(\mathbf{i} + \mathbf{j})$.

Now let us see how the method of Lagrange multipliers is used to find the place $\mathbf{x} = \xi\mathbf{i} + \eta\mathbf{j}$ on the line $y = 1 - x$ which is nearest the origin in Fig. 9.11. The problem is to minimize the function $d(P) = (\mathbf{x} \cdot \mathbf{x})^{1/2}$, or more conveniently, the related squared distance function

$$D(\mathbf{x}) \equiv d^2(P) = \mathbf{x} \cdot \mathbf{x} = \xi^2 + \eta^2, \tag{9.59a}$$

subject to the constraint relation

$$F(\mathbf{x}) \equiv \xi + \eta - 1 = 0, \tag{9.59b}$$

specifying that the point (ξ, η) is constrained to the line $y + x = 1$. Notice that neither $\partial F(\mathbf{x})/\partial \xi$ nor $\partial F(\mathbf{x})/\partial \eta$ vanishes, as required below (9.56). Now use (9.59a) and (9.59b) to form the auxiliary function

$$G(\mathbf{x}) \equiv D(\mathbf{x}) - \lambda F(\mathbf{x}) = \xi^2 + \eta^2 - \lambda(\xi + \eta - 1), \tag{9.59c}$$

in accordance with (9.55). Then, by (9.57), the extremal points are determined by

$$\frac{\partial G(\mathbf{x})}{\partial \xi} = 2\xi - \lambda = 0, \qquad \frac{\partial G(\mathbf{x})}{\partial \eta} = 2\eta - \lambda = 0. \tag{9.59d}$$

Consequently, $\lambda = 2\xi = 2\eta$, and use of this result in the constraint equation (9.59b) yields the nearest point coordinates $\xi = \eta = \frac{1}{2}$, from which (9.59a) delivers the minimum value $D(\mathbf{x}) = d^2 = \frac{1}{2}$. Therefore, the nearest point on the line from O is at $\mathbf{x} = \frac{1}{2}(\mathbf{i} + \mathbf{j})$, at a distance $d = \sqrt{2}/2$ from O. □

Lagrange's systematic method of undetermined multipliers in this example is just about as easy as the elementary geometrical solution. Now consider another example where the conclusion is not so apparent.

Example 9.9. An atomic particle is confined to a rectangular box of sides a, b, c in which its ground state energy is $\mathscr{E} = k(1/a^2 + 1/b^2 + 1/c^2)$, where k is a constant. Find the dimensions of a box of constant volume for which the energy is least.

Solution. The rectangular box has volume $V(a, b, c) = abc = \gamma$, a constant. The problem is to find the smallest value of $D(a, b, c) \equiv \mathscr{E}(a, b, c)$ among all positive values of a, b, c for which the volume constraint $F(a, b, c) \equiv V(a, b, c) - \gamma = 0$ holds. To apply Lagrange's method, we form the auxiliary function

$$G(a,b,c) \equiv \mathscr{E}(a,b,c) - \lambda(V(a,b,c) - \gamma) = k\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right) - \lambda(abc - \gamma),$$

in accordance with (9.55). Notice that $\partial F/\partial a = bc \neq 0$, for example, and hence the assumption below (9.56) is satisfied. The extremal values of $G(a, b, c)$ are then determined by

$$\frac{\partial G}{\partial a} = -\frac{2k}{a^3} - \lambda bc = 0, \quad \frac{\partial G}{\partial b} = -\frac{2k}{b^3} - \lambda ac = 0, \quad \frac{\partial G}{\partial c} = -\frac{2k}{c^3} - \lambda ab = 0,$$

from which, with the aid of the constraint condition,

$$\frac{2k}{a^2} = \frac{2k}{b^2} = \frac{2k}{c^2} = -\lambda abc = -\lambda\gamma.$$

Hence, $a = b = c$; that is, the box for which the ground state energy is least is a cube of side a. Consequently, $\mathscr{E}_{\min} = 3k/a^2$ is the smallest value of the ground state energy consistent with the constant volume constraint. □

See Problems 9.23 through 9.30 and 9.45 for additional examples. We now return to the major problem posed earlier below (9.51).

9.9.3. Principal Values and Directions for the Inertia Tensor

Consider a rigid body of any sort, homogeneous or not. The principal problem is to find all directions $\mathbf{n}$ for which the normal components

$$D(\mathbf{n}) \equiv \mathbf{I}_Q\mathbf{n} \cdot \mathbf{n}, \tag{9.60}$$

of the inertia tensor at a point Q in a body reference frame are greatest and least, subject to the unit vector constraint condition

$$F(\mathbf{n}) \equiv \mathbf{n} \cdot \mathbf{n} - 1 = 0. \tag{9.61}$$

The problem is best solved by the method of Lagrange multipliers. We thus introduce the auxiliary function (9.55),

$$G(\mathbf{n}) \equiv \mathbf{I}_Q\mathbf{n} \cdot \mathbf{n} - \lambda(\mathbf{n} \cdot \mathbf{n} - 1), \tag{9.62}$$

in which λ is an undetermined scalar, independent of $\mathbf{n}$. The extreme values of (9.62) are then obtained by differentiation with respect to the three scalar components ν_k of $\mathbf{n} = \nu_k \mathbf{i}_k$ in accordance with (9.57). Bearing in mind the symmetry of $\mathbf{I}_Q$, we find in direct notation

$$\frac{\partial G(\mathbf{n})}{\partial \mathbf{n}} = 2\mathbf{I}_Q\mathbf{n} - 2\lambda\mathbf{n} = \mathbf{0}.$$

Hence, the extremal directions and values of the inertia tensor are determined by the vector equation

$$(\mathbf{I}_Q - \lambda \mathbf{1})\mathbf{n} = \mathbf{0}, \tag{9.63}$$

called the *principal vector equation*, together with the *constraint equation* (9.61). In index notation, bearing in mind the summation rule, this system of four algebraic equations for λ and ν_k is written as

$$\left(I_{kj}^Q - \lambda\delta_{kj}\right)\nu_j = 0, \qquad \nu_k\nu_k = 1, \tag{9.64}$$

or, explicitly, in expanded notation with the superscript Q suppressed,

$$\begin{aligned} (I_{11} - \lambda)\nu_1 + I_{12}\nu_2 + I_{13}\nu_3 &= 0, \\ I_{21}\nu_1 + (I_{22} - \lambda)\nu_2 + I_{23}\nu_3 &= 0, \\ I_{31}\nu_1 + I_{32}\nu_2 + (I_{33} - \lambda)\nu_3 &= 0, \\ \nu_1^2 + \nu_2^2 + \nu_3^2 &= 1. \end{aligned} \tag{9.65}$$

Nontrivial solutions of the homogeneous system (9.63), i.e. the first three equations of (9.65), exist if and only if

$$\det(\mathbf{I}_Q - \lambda\mathbf{1}) = \det \begin{bmatrix} I_{11} - \lambda & I_{12} & I_{13} \\ I_{12} & I_{22} - \lambda & I_{23} \\ I_{13} & I_{23} & I_{33} - \lambda \end{bmatrix} = 0. \tag{9.66}$$

This yields a cubic equation for λ, called the *characteristic equation*:

$$f(\lambda) \equiv -\lambda^3 + J_1\lambda^2 - J_2\lambda + J_3 = 0, \tag{9.67}$$

in which J_1, J_2, J_3 are defined by

$$J_1 \equiv \operatorname{tr}\mathbf{I}_Q, \qquad J_2 \equiv \frac{1}{2}\left(J_1^2 - \operatorname{tr}\mathbf{I}_Q^2\right), \qquad J_3 \equiv \det\mathbf{I}_Q. \tag{9.68}$$

These are the *principal invariants of the moment of inertia tensor* $\mathbf{I}_Q$. See (3.113).

The real cubic equation (9.67) has at least one real, positive root $\lambda = \lambda_1$, say, so there exists at least one real extremal direction $\mathbf{n} = \mathbf{n}_1$ determined by the system (9.65). In fact, it is proved later that because $\mathbf{I}_Q$ is a real-valued symmetric tensor, (9.67) always has three real roots λ_k, all positive. Therefore, there exist three corresponding mutually perpendicular directions $\mathbf{n}_k$ determined by (9.65) that define a special basis at Q, called the *principal basis* or *principal directions*. Two of these directions are the directions at Q with respect to which the normal components of the inertia tensor assume their maximum and minimum values, all determined by the roots λ_k of (9.67), called the *principal values* of $\mathbf{I}_Q$. To see this, let $\mathbf{n}$ be a principal direction for the characteristic root λ. Then from (9.63) and (9.61), $\mathbf{n} \cdot \mathbf{I}_Q\mathbf{n} = \lambda$ for each extremal direction $\mathbf{n}$. That is, the three roots λ_k of the characteristic equation (9.67) are the extreme values of the function (9.60) for which the constraint (9.61) is satisfied. Moreover, these are the moments of inertia about the axes defined by the corresponding principal directions $\mathbf{n}_k$. So, the principal values also are known as the *principal moments of inertia.*

For future notational convenience, henceforward, the principal basis is denoted by $\hat{\mathbf{e}}_k$, and $\hat{I}_{jk}$ denote the corresponding principal components of the inertia tensor at Q. Then, for the principal value $\lambda = \lambda_k$ and its corresponding principal direction $\mathbf{n} = \hat{\mathbf{e}}_k$, for a fixed value of $k = 1, 2, 3$, (9.63) becomes

$$\mathbf{I}_Q \hat{\mathbf{e}}_k = \lambda_k \hat{\mathbf{e}}_k \text{ (no sum on } k), \tag{9.69}$$

and the principal components $\hat{I}_{jk}$ of $\mathbf{I}_Q$ are given by

$$\hat{I}_{jk} = \hat{\mathbf{e}}_j \cdot \mathbf{I}_Q \hat{\mathbf{e}}_k = \lambda_k \hat{\mathbf{e}}_j \cdot \hat{\mathbf{e}}_k = \lambda_k \delta_{jk} \text{ (no sum on } k). \tag{9.70}$$

Consequently, in the principal reference basis $\hat{\mathbf{e}}_j$ at Q, we have

$$\begin{aligned} &\hat{I}_{11} = \lambda_1, \qquad \hat{I}_{22} = \lambda_2, \qquad \hat{I}_{33} = \lambda_3, \\ &\hat{I}_{12} = \hat{I}_{21} = \hat{I}_{13} = \hat{I}_{31} = \hat{I}_{23} = \hat{I}_{32} = 0. \end{aligned} \tag{9.71}$$

Therefore, all products of inertia vanish, and thus attain their smallest absolute values, in the principal reference frame. We shall have no need to determine their maximum absolute values. (See Problem 9.45.) It follows from the first three relations in (9.71) that the three principal values λ_k are the moments of inertia about the principal axes, and these values may be ordered so that

$$\hat{I}_{11} \geq \hat{I}_{22} \geq \hat{I}_{33}, \tag{9.72}$$

and hence, $\hat{I}_{11}$ and $\hat{I}_{33}$ are the largest and smallest values of I_{nn}^Q among all possible normal components of $\mathbf{I}_Q$ at Q. We thus have the following remarkable result.

The principal axes theorem: *At each point Q of an arbitrary rigid body, homogeneous or not, there exists an orthonormal basis $\hat{\mathbf{e}}_k$ with respect to which the products of inertia are zero, the moments of inertia about these axes assume their greatest and least values, and the inertia tensor at Q has the unique representation*

$$\mathbf{I}_Q(\mathscr{B}) = \hat{I}_{11}^Q \hat{\mathbf{e}}_{11} + \hat{I}_{22}^Q \hat{\mathbf{e}}_{22} + \hat{I}_{33}^Q \hat{\mathbf{e}}_{33}, \tag{9.73}$$

referred to the principal tensor basis $\hat{\mathbf{e}}_{jk} = \hat{\mathbf{e}}_j \otimes \hat{\mathbf{e}}_k$.

Recall that among all moments of inertia about parallel axes, regardless of the geometry and material distribution of the body, the smallest value occurs about an axis at the center of mass. Therefore, by the principal axes theorem, *the absolute minimum moment of inertia of a body about an axis is given by the smallest principal moment of inertia at its center of mass.* The principal axes of the inertia tensor for a homogeneous body having geometrical symmetry often are readily determined by inspection, as shown in earlier examples. For a nonhomogeneous or composite body, however, it is usually necessary to apply the principal axes analysis. The importance of the analysis rests on the reduction of the inertia tensor to its simplest diagonal form (9.73). The procedure is illustrated next in a numerical example.

Example 9.10. Find the principal values and directions for an inertia tensor $\mathbf{I}_Q$ whose component matrix at Q in frame $\varphi = \{Q; \mathbf{i}_k\}$ is

$$I_Q = \begin{bmatrix} 5/2 & -3/2 & 0 \\ -3/2 & 5/2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \text{kg} \cdot \text{m}^2. \tag{9.74a}$$

Solution. The principal values of $\mathbf{I}_Q$ at Q are determined by the characteristic equation (9.66) for the matrix (9.74a):

$$f(\lambda) \equiv \det \begin{bmatrix} \frac{5}{2} - \lambda & -\frac{3}{2} & 0 \\ -\frac{3}{2} & \frac{5}{2} - \lambda & 0 \\ 0 & 0 & 3 - \lambda \end{bmatrix} = (3 - \lambda)\left[\left(\frac{5}{2} - \lambda\right)^2 - \frac{9}{4}\right];$$

that is,

$$f(\lambda) = (3 - \lambda)(\lambda - 4)(\lambda - 1) = 0. \tag{9.74b}$$

Hence, the three principal values ordered so that $\lambda_1 \geq \lambda_2 \geq \lambda_3$ are

$$\lambda_1 = 4, \qquad \lambda_2 = 3, \qquad \lambda_3 = 1. \tag{9.74c}$$

It follows that the greatest and least normal components of the inertia tensor at Q are $\hat{I}_{11} = \lambda_1 = 4\ \text{kg} \cdot \text{m}^2$ and $\hat{I}_{33} = \lambda_3 = 1\ \text{kg} \cdot \text{m}^2$, respectively.

The principal directions at Q are determined by the system (9.65). With (9.74a), these take the form

$$\begin{gathered} \left(\frac{5}{2} - \lambda\right)\nu_1 - \frac{3}{2}\nu_2 = 0, \\ -\frac{3}{2}\nu_1 + \left(\frac{5}{2} - \lambda\right)\nu_2 = 0, \\ (3 - \lambda)\nu_3 = 0, \\ \nu_1^2 + \nu_2^2 + \nu_3^2 = 1. \end{gathered} \tag{9.74d}$$

This system of equations in ν_k is to be solved for each principal value in (9.74c). For $\lambda = \lambda_1 = 4$, (9.74d) yields $\nu_1 = -\nu_2 = \pm\sqrt{2}/2$, $\nu_3 = 0$, and hence the first principal direction $\hat{\mathbf{e}}_1 = \nu_k \mathbf{i}_k$ referred to φ is

$$\hat{\mathbf{e}}_1 = \pm\frac{\sqrt{2}}{2}(\mathbf{i}_1 - \mathbf{i}_2) \sim \lambda_1 = 4. \tag{9.74e}$$

Use of $\lambda = \lambda_2 = 3$ in (9.74d) delivers $\nu_1 = \nu_2 = 0,\ \nu_3 = \pm 1$. Thus, the second principal axis is

$$\hat{\mathbf{e}}_2 = \pm \mathbf{i}_3 \sim \lambda_2 = 3. \tag{9.74f}$$

Finally, the third principal direction orthogonal to $\hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_2$ is given directly by $\hat{\mathbf{e}}_3 = \hat{\mathbf{e}}_1 \times \hat{\mathbf{e}}_2$:

$$\hat{\mathbf{e}}_3 = \mp \frac{\sqrt{2}}{2}(\mathbf{i}_1 + \mathbf{i}_2) \sim \lambda_3 = 1. \tag{9.74g}$$

The triple of vectors $\hat{\mathbf{e}}_k$ define the principal directions of a frame $\hat{\varphi} = \{Q; \hat{\mathbf{e}}_k\}$ with respect to which the products of inertia vanish and the inertia tensor has diagonal form at Q. Notice that six unit vectors have been found. The difference in sign means only that either $\hat{\mathbf{e}}_k$ or its opposite may be chosen as a principal vector. It is customary, however, to select the principal basis to form a right-hand set, in which case the signs for $\hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_2$, say, may be chosen arbitrarily and $\hat{\mathbf{e}}_3$ is then determined in accordance with the right-hand rule. With this concluded, the results referred to the original Cartesian body frame at Q are

$$\begin{aligned}
\lambda_1 &= \hat{I}_{11} = 4\ \mathrm{kg \cdot m^2}, & \hat{\mathbf{e}}_1 &= \frac{\sqrt{2}}{2}(\mathbf{i}_1 - \mathbf{i}_2),\\
\lambda_2 &= \hat{I}_{22} = 3\ \mathrm{kg \cdot m^2}, & \hat{\mathbf{e}}_2 &= \mathbf{i}_3,\\
\lambda_3 &= \hat{I}_{33} = 1\ \mathrm{kg \cdot m^2}, & \hat{\mathbf{e}}_3 &= -\frac{\sqrt{2}}{2}(\mathbf{i}_1 + \mathbf{i}_2).
\end{aligned} \tag{9.74h}$$

The principal vectors $\hat{\mathbf{e}}_k$ are described relative to the original body frame $\varphi = \{Q; \mathbf{i}_k\}$. The orthogonal transformation matrix A: $\mathbf{i}_k \rightarrow \hat{\mathbf{e}}_k$, i.e. $A_{jk} = \cos\langle \hat{\mathbf{e}}_j, \mathbf{i}_k \rangle$, from the frame φ to the principal body frame $\hat{\varphi} = \{Q; \hat{\mathbf{e}}_k\}$ may be read from (9.74h):

$$A = \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 & 0 \\ 0 & 0 & 1 \\ -\sqrt{2}/2 & -\sqrt{2}/2 & 0 \end{bmatrix}. \tag{9.74i}$$

This matrix diagonalizes the original matrix I_Q in (9.74a), as the reader may confirm from the tensor transformation law (9.48) written as $\hat{I}_Q = A I_Q A^T$. Thus, in the principal basis,

$$\mathbf{I}_Q = 4\hat{\mathbf{e}}_{11} + 3\hat{\mathbf{e}}_{22} + \hat{\mathbf{e}}_{33}\ \mathrm{kg \cdot m^2}. \tag{9.74j}$$

□

The canonical form (9.73) of the inertia tensor in the principal reference system at Q is independent of the shape of the body and its distribution of material. In particular, however, the principal axes of inertia for a homogeneous rigid body with two orthogonal planes of symmetry are easily identified at a point Q on the axis of symmetry, the first principal axis; call it $\hat{\mathbf{e}}_1$. The other two principal axes

$\hat{\mathbf{e}}_2$ and $\hat{\mathbf{e}}_3$ lie in the orthogonal planes of symmetry, perpendicular to $\hat{\mathbf{e}}_1$ and to each other at Q, ordered so that $\hat{\mathbf{e}}_3 = \hat{\mathbf{e}}_1 \times \hat{\mathbf{e}}_2$. This identifies the principal basis at Q relative to which the products of inertia of a homogeneous axisymmetric body vanish and $\mathbf{I}_Q$ has the canonical form (9.73). When $\hat{\mathbf{e}}_3$ is the axis of a homogeneous body of revolution, any orthogonal pair of planes through the axis are identical orthogonal principal planes of symmetry, and hence every direction in the cross section is a principal body axis for which $\hat{I}_{11}^Q = \hat{I}_{22}^Q$.

More generally, however, the foregoing analysis for an arbitrary body has avoided the important special cases in which two or three of the principal values λ_k are equal. This topic is explored geometrically in the next section and analytically later. It is found that when two principal values are equal, say $\lambda_1 = \lambda_2 \neq \lambda_3$, every direction in the plane perpendicular to $\hat{\mathbf{e}}_3$ is a principal direction for $\mathbf{I}_Q$. This occurs, for example, in the special case described above, when $\hat{\mathbf{e}}_3$ is the axis of a homogeneous body of revolution, as illustrated in equations (9.30) through (9.33) for homogeneous circular cylinders and tubes. Moreover, if all three principal values of inertia of an arbitrary body are equal, then $\mathbf{I}_Q = \lambda\mathbf{1}$, and every direction at Q is a principal direction. The inertia tensors for a homogeneous sphere (9.34) and for a cube (9.50) have these spherical tensor properties. A thin hemispherical shell and a homogeneous hemisphere are especially striking additional examples for which the inertia tensor is spherical. (See Figs. D.11 and D.13 in Appendix D.)

An illuminating geometrical interpretation of the variation of the normal components of the inertia tensor with direction follows. The maximum and minimum normal components of the inertia tensor are related to the geometry of a quadric surface, and the principal directions corresponding to equal principal values are characterized.

9.9.4. Cauchy's Inertia Ellipsoid

Consider an arbitrary rigid body whose inertia tensor $\mathbf{I}_Q$ is known in frame $\varphi = \{Q; \mathbf{e}_k\}$, and introduce an arbitrary axis n through Q with direction $\mathbf{n}$ so that $\mathbf{x} \equiv R\mathbf{n}$ is the position vector on this radial line of a point P at $R = |\mathbf{x}|$ from Q, as shown in Fig. 9.12. Recall (9.51) and note that

$$\mathbf{x} \cdot \mathbf{I_Q x} = R^2 I_{nn}^Q, \tag{9.75}$$

is a quadratic form. Of course, the normal component I_{nn}^Q will vary with the direction $\mathbf{n}$, and R will change with the position of P. On each line n through Q, however, let us choose P so that its squared distance R^2 from Q is inversely proportional to I_{nn}^Q, the moment of inertia about that line; that is, let C be an arbitrary, positive constant and measure R along each axis n so that

$$R = \frac{C}{\sqrt{I_{nn}^Q}}. \tag{9.76}$$

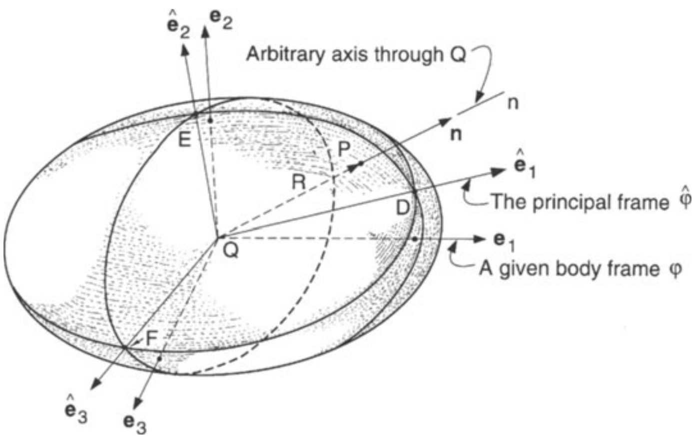


Figure 9.12. Cauchy's inertia ellipsoid at Q.

Then $R^2 I_{nn}^Q = C^2$, and (9.75), with $\mathbf{x} = x_k \mathbf{e}_k$ and $\mathbf{I_Q} = I_{jk}\mathbf{e}_{jk}$ in the body frame φ, describes the equation of a quadric surface, $\mathbf{x} \cdot \mathbf{I_Q x} = I_{jk} x_j x_k = C^2$ centered at Q. In expanded notation this becomes

$$I_{11}x_1^2 + I_{22}x_2^2 + I_{33}x_3^2 + 2I_{12}x_1x_2 + 2I_{13}x_1x_3 + 2I_{23}x_2x_3 = C^2. \quad (9.77)$$

In the principal reference frame $\hat{\varphi} = \{Q; \hat{\mathbf{e}}_k\}$, the products of inertia vanish and the position vector of P in $\hat{\varphi}$ is $\mathbf{x} = \hat{x}_k \hat{\mathbf{e}}_k$. Thus, (9.77) is transformed in the principal frame $\hat{\varphi}$ at Q to its simplest canonical form

$$\hat{I}_{11}\hat{x}_1^2 + \hat{I}_{22}\hat{x}_2^2 + \hat{I}_{33}\hat{x}_3^2 = C^2. \quad (9.78)$$

Equation (9.78), all of whose coefficients are positive, describes the closed quadratic surface shown in Fig. 9.12—an ellipsoid known as *Cauchy's inertia ellipsoid*, or *the momental ellipsoid.*

Equation (9.77) describes Cauchy's ellipsoid in the rotated position of the initially assigned body frame $\varphi = \{Q; \mathbf{e}_k\}$ in Fig. 9.12, with respect to which the inertia tensor $\mathbf{I}_Q = I_{jk}\mathbf{e}_{jk}$ is known. Equation (9.78) describes the same invariant ellipsoid in the principal body frame $\hat{\varphi} = \{Q; \hat{\mathbf{e}}_k\}$. For each choice of basis at the same point Q, equation (9.77) representing the ellipsoid in terms of the new component coefficients I_{jk} for a given rigid body $\mathscr{B}$ will be different. Nevertheless, each of these quadratic equations can be transformed to the unique canonical form (9.78); they all describe the same invariant ellipsoid centered at Q. Indeed, the momental ellipsoid has the invariant form $\mathbf{x} \cdot \mathbf{I}_Q \mathbf{x} = C^2$. However, if Q is changed to another point S, say, the inertia ellipsoid at Q will change to another invariant ellipsoid $\mathbf{X} \cdot \mathbf{I}_S \mathbf{X} = B^2$, say, centered at S; and at the new point S there exists another basis in which the ellipsoid is described by the canonical form (9.78) with principal values determined for $\mathbf{I}_S$.

We are now able to visualize the extremal properties of the inertia tensor in terms of the geometrical properties of its ellipsoid. Fix C in (9.76) to have any convenient constant value, say $C = 1$. Then $R = 1/\sqrt{I_{nn}^Q}$, and *the squared distance along a radial line from Q to a point P on the momental ellipsoid is numerically equal to the reciprocal of the moment of inertia of the body about that line.* In this case, with $\hat{R}_k \equiv 1/\sqrt{\hat{I}_{kk}},\ k = 1, 2, 3$, the inertia ellipsoid (9.78) in the principal reference frame at Q is described by

$$\left[\frac{\hat{x}_1}{\hat{R}_1}\right]^2 + \left[\frac{\hat{x}_2}{\hat{R}_2}\right]^2 + \left[\frac{\hat{x}_3}{\hat{R}_3}\right]^2 = 1, \tag{9.79}$$

where $\hat{R}_1 \geq \hat{R}_2 \geq \hat{R}_3$ are the ordered lengths of the three principal semidiameters of the inertia ellipsoid centered at Q, shown as $\overline{QD}$, $\overline{QE}$, and $\overline{QF}$ in Fig. 9.12. Among all possible lines from Q to any point P on this surface, none can be greater than $\hat{R}_1$ nor smaller than $\hat{R}_3$, the largest and least of the principal semidiameters. Accordingly, we have $\hat{I}_{33}^Q \geq \hat{I}_{22}^Q \geq \hat{I}_{11}^Q$; therefore, among all possible moments of inertia about axes through Q, none can be larger than $\hat{I}_{33}^Q$ nor smaller than $\hat{I}_{11}^Q$, the greatest and least of the principal moments of inertia. Moreover, if two of the principal components of the inertia tensor are equal, then two of the semidiameters of the inertia ellipsoid also are equal. Suppose, for example, that $\hat{R}_2 = \hat{R}_3 = \rho$. Then the surface is an ellipsoid of revolution about $\hat{\mathbf{e}}_1$ in $\hat{\varphi}$ and thus has a circular cross section for which no direction in its plane is distinguished. Consequently, the moment of inertia about every axis in this plane is numerically equal to $1/\rho^2$, and hence every axis in the plane at Q perpendicular to $\hat{\mathbf{e}}_1$ is a principal axis for the inertia tensor. If all three semidiameters of the ellipsoid, and therefore all three principal values of the inertia tensor, are equal, the ellipsoid is a sphere for which every axis is a principal axis of inertia at Q.

Exercise 9.3. Set $C = \sqrt{m(\mathscr{B})}$ in (9.76) and recall (9.22). Then $R = 1/R_n$, and hence in Fig. 9.12 *the distance from Q to the point P where the n axis intersects the inertia ellipsoid is numerically equal to the reciprocal of the radius of gyration of the body about that axis.* Review the properties of the momental ellipsoid in these terms. □

Not every ellipsoid centered at Q can be an inertia ellipsoid. *The class of inertia ellipsoids is restricted by the condition that the sum of any two normal Cartesian components of the inertia tensor is not less than the third.* It follows from (9.17), for example, that in any Cartesian frame $\varphi = \{Q; \mathbf{e}_j\}$ at Q, $I_{11} + I_{22} \geq I_{33}$ must hold, *the equality holding in φ if and only if the body is a plane body* for which $z = 0$ in φ, in accordance with (9.19); otherwise, the strict inequality holds. Thus, *if the body is not a plane body*, (9.14) yields the three constraints

$$I_{11} + I_{22} > I_{33}, \qquad I_{22} + I_{33} > I_{11}, \qquad I_{33} + I_{11} > I_{22}. \tag{9.80}$$

Plainly, the same relations hold for the principal components. *Consequently, if the body is not a plane body, the greatest principal moment of inertia must be smaller than the sum of the other two.*

Each invariant in (9.68), and specifically,

$$J_1 \equiv \mathrm{tr}\mathbf{I}_Q = I_{11} + I_{22} + I_{33}, \tag{9.81}$$

has the same value in every reference frame at Q. So, (9.80) and (9.81) are useful as a quick check on numerical computations.

In Example 9.10, page 385, for instance, we find from (9.74a) that $J_1 = 8$, and an easy check on the principal values in (9.74c) confirms the same sum. Further, (9.74a) shows that

$$I_{11} + I_{22} = 5 > I_{33} = 3, \qquad I_{22} + I_{33} = \frac{11}{2} > I_{11} = \frac{5}{2}, \qquad I_{11} = I_{22}, \tag{9.82a}$$

in the assigned frame φ. Similarly, for the principal values (9.74c), we obtain

$$\lambda_1 + \lambda_2 = 7 > \lambda_3 = 1, \qquad \lambda_2 + \lambda_3 = 4 = \lambda_1, \qquad \lambda_3 + \lambda_1 = 5 > \lambda_2 = 3, \tag{9.82b}$$

at the same point Q in the principal frame $\hat{\varphi} = \{Q; \hat{\mathbf{e}}_k\}$. The equality in the second principal axes relation in (9.82b) means that the body is a plane (thin) body in the principal 23-plane of $\hat{\varphi}$, a fact that is not evident from any relations in frame φ. In all cases, the constraints (9.80) are satisfied.

Similar geometrical interpretations of the normal components of symmetric tensors for stress and strain in terms of their ellipsoids arise in the study of the mechanics of deformable solids. This is a reflection of the analytical properties shared by all symmetric tensors, the principal aspects of which are sketched above. But to complete the picture some details beg further discussion and analytical clarification presented below. The reader who may wish to move on to the next chapter, however, will experience no serious loss of continuity.

9.10. Loose Ends and Generalities for Symmetric Tensors

The principal axes analysis developed for the inertia tensor is applicable to any symmetric tensor, and symmetric tensor quantities occur often in all areas of engineering, specifically in the study of mechanics of solids and fluids. In these and other areas, the tensor entities generally have real-valued components, and the tensor is said to be real-valued. When $\mathbf{T}$ is a real-valued symmetric tensor, its principal values and directions have important special properties—the principal values must be real, the principal vectors are mutually orthogonal regardless of any multiplicity of these principal values, and the component matrix referred to the principal basis is diagonal. Our objective is to explore these properties in general terms applicable to all symmetric tensor quantities and thus dispose of a few loose ends mentioned only briefly and previously described geometrically.

9.10.1. Summary of the Principal Values Problem

The principal values and directions for an arbitrary tensor $\mathbf{T}$ are determined by the principal vector equation

$$(\mathbf{T} - \lambda\mathbf{1})\mathbf{n} = \mathbf{0}, \tag{9.83}$$

subject to the unit vector constraint

$$\mathbf{n} \cdot \mathbf{n} = 1. \tag{9.84}$$

The homogeneous system of algebraic equations (9.83) for the components of $\mathbf{n}$ has a nontrivial solution if and only if

$$f(\lambda) \equiv \det(\mathbf{T} - \lambda\mathbf{1}) = 0. \tag{9.85}$$

This provides the characteristic equation for λ:

$$f(\lambda) = -\lambda^3 + J_1\lambda^2 - J_2\lambda + J_3 = 0, \tag{9.86}$$

where the principal invariants $J_1,\ J_2,\ J_3$ of the tensor $\mathbf{T}$ are defined by

$$J_1 = \operatorname{tr}\mathbf{T}, \qquad J_2 = \frac{1}{2}\left(J_1^2 - \operatorname{tr}\mathbf{T}^2\right), \qquad J_3 = \det(\mathbf{T}). \tag{9.87}$$

The real cubic equation (9.86) has at least one real root λ_1, say. Depending on the nature of $\mathbf{T}$, the other two roots $\lambda_2,\ \lambda_3$ are either real or they are complex conjugates. In any case, (9.86) may be expressed in terms of its factors λ_k to obtain

$$f(\lambda) = -(\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3) = 0. \tag{9.88}$$

Comparison of the coefficients in (9.86) and (9.88) shows that

$$J_1 = \lambda_1 + \lambda_2 + \lambda_3, \qquad J_2 = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1, \qquad J_3 = \lambda_1\lambda_2\lambda_3. \tag{9.89}$$

9.10.2. Reality of the Principal Values of a Symmetric Tensor

Recall that for real numbers a and b the conjugate of a complex number $z = a + ib$ is denoted by $\bar{z} = a - ib$. Then $\bar{z} = z$ is a real number if and only if $b = 0$. The magnitude of z (or $\bar{z}$) is defined by $|z|^2 = z \cdot \bar{z} = a^2 + b^2$. Similarly, let $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ be real vectors, i.e. vectors having real components. Then $\boldsymbol{\eta} = \boldsymbol{\alpha} + i\boldsymbol{\beta}$ and $\bar{\boldsymbol{\eta}} = \boldsymbol{\alpha} - i\boldsymbol{\beta}$ are complex conjugate vectors for which $\boldsymbol{\eta} = \bar{\boldsymbol{\eta}}$ is a real vector when and only when $\boldsymbol{\beta} = \mathbf{0}$. The magnitude of $\boldsymbol{\eta}$ (or $\bar{\boldsymbol{\eta}}$) is defined by $|\boldsymbol{\eta}|^2 = \boldsymbol{\eta} \cdot \bar{\boldsymbol{\eta}} = \boldsymbol{\alpha} \cdot \boldsymbol{\alpha} + \boldsymbol{\beta} \cdot \boldsymbol{\beta}$. Thus, if $\boldsymbol{\eta}$ (hence $\bar{\boldsymbol{\eta}}$) is a unit vector, then $\boldsymbol{\eta} \cdot \bar{\boldsymbol{\eta}} = 1$.

We consider only real symmetric tensors, i.e. those having real-valued components in every real basis. Equation (9.86), however, requires only that at least one characteristic value must be real, while the others might be complex conjugates. Therefore, it appears that three real principal values might not exist for a real symmetric tensor, in which case there would be no way to transform the tensor to a diagonal form whose components would be real. We now prove that *the principal values of a real symmetric tensor are real.*

Suppose that two of the principal values of a real symmetric tensor $\mathbf{T}$ are complex conjugates λ and $\bar{\lambda}$, and let $\mathbf{n} = \boldsymbol{\alpha} + i\boldsymbol{\beta}$ and $\bar{\mathbf{n}} = \boldsymbol{\alpha} - i\boldsymbol{\beta}$ denote the corresponding complex conjugate principal vectors for λ and $\bar{\lambda}$. Then, for any real tensor $\mathbf{T}$, by (9.83),

$$\mathbf{T}\mathbf{n} = \lambda\mathbf{n}, \qquad \mathbf{T}\bar{\mathbf{n}} = \bar{\lambda}\bar{\mathbf{n}}, \tag{9.90a}$$

where the unit principal vectors satisfy

$$\mathbf{n} \cdot \bar{\mathbf{n}} = 1. \tag{9.90b}$$

Now form the inner product of the first equation in (9.90a) by $\bar{\mathbf{n}}$, the second by $\mathbf{n}$, introduce (9.90b), and recall the transpose rule (3.42) to obtain

$$\mathbf{n} \cdot (\mathbf{T}^T - \mathbf{T})\bar{\mathbf{n}} = \lambda - \bar{\lambda}.$$

Thus, if $\mathbf{T} = \mathbf{T}^{\mathbf{T}}$, then $\lambda = \bar{\lambda}$, and hence λ is real. That is, the principal values of a real symmetric tensor are real. An alternative proof is provided as an exercise.

Exercise 9.4. Let $\lambda = a + ib$ and $\mathbf{n} = \boldsymbol{\alpha} + i\boldsymbol{\beta}$ be a principal pair for a real symmetric tensor $\mathbf{T}$. Use only the first equation in (9.90a), identify its real and imaginary parts, and prove from these relations that $b = 0$, and hence λ is real. □

9.10.3. Orthogonality of Principal Directions

Let λ_α and λ_β be any two *distinct* principal values of a symmetric tensor $\mathbf{T}$ with corresponding principal vectors $\mathbf{n}_\alpha$ and $\mathbf{n}_\beta$. Then, by (9.83),

$$\mathbf{T}\mathbf{n}_\alpha = \lambda_\alpha \mathbf{n}_\alpha, \qquad \mathbf{T}\mathbf{n}_\beta = \lambda_\beta \mathbf{n}_\beta \text{ (no sum on } \alpha \text{ and } \beta),$$

and by the previous argument, we reach

$$\mathbf{n}_\alpha \cdot (\mathbf{T}^T - \mathbf{T})\mathbf{n}_\beta = (\lambda_\alpha - \lambda_\beta)\mathbf{n}_\alpha \cdot \mathbf{n}_\beta \text{ (no sum on } \alpha \text{ and } \beta),$$

where $\lambda_\alpha \neq \lambda_\beta$. Hence, if $\mathbf{T}$ is symmetric, $\mathbf{n}_\alpha \cdot \mathbf{n}_\beta = 0$. *Consequently, the principal vectors corresponding to distinct principal values of a real symmetric tensor are mutually perpendicular.* The proof breaks down if any two principal values are equal.

9.10.4. Multiplicity of Principal Values

If a real tensor $\mathbf{T}$ has repeated principal values, then all must be real, whether $\mathbf{T}$ is symmetric or not; but it is not evident that the principal vectors must be orthogonal, nor in fact if they need be distinct. Suppose, for example, that all principal values of $\mathbf{T}$ are unity; then (9.83) provides only one system of equations $\mathbf{T}\mathbf{n} = \mathbf{n}$

and $\mathbf{n} \cdot \mathbf{n} = 1$ for the principal vectors corresponding to $\lambda = 1$. This raises the question of whether the number of principal vectors also reduces to a single vector. The answer is no. *If* $\mathbf{T}$ *is symmetric and has a principal value* λ_1 *of multiplicity* $m = 2$ *or* 3, *so that the characteristic equation* (9.88) *has a factor* $(\lambda - \lambda_1)^m$, *then there exist at least m orthogonal principal vectors corresponding to the same* λ_1. The proof follows.

We know that there exists at least one principal pair $\lambda = \lambda_3$ and $\mathbf{n} = \hat{\mathbf{e}}_3$, say, such that $\mathbf{T}\hat{\mathbf{e}}_3 = \lambda\hat{\mathbf{e}}_3$. Let $\mathbf{e}_k$ be an orthonormal basis for which $\mathbf{e}_3 = \hat{\mathbf{e}}_3$. Then $\mathbf{e}_1 \cdot \mathbf{T}\mathbf{e}_3 = T_{13} = T_{31} = 0,\ \mathbf{e}_2 \cdot \mathbf{T}\mathbf{e}_3 = T_{23} = T_{32} = 0,\ \mathbf{e}_3 \cdot \mathbf{T}\mathbf{e}_3 = T_{33} = \lambda_3$, and so the symmetric tensor $\mathbf{T}$ has the component matrix

$$T = \begin{bmatrix} T_{11} & T_{12} & 0 \\ T_{12} & T_{22} & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}; \tag{9.91a}$$

that is, referred to $\varphi = \{O; \mathbf{e}_1, \mathbf{e}_2, \hat{\mathbf{e}}_3\}$, $\mathbf{T}$ has the representation

$$\mathbf{T} = \mathbf{S} + \lambda_3\hat{\mathbf{e}}_{33}, \qquad \mathbf{S} \equiv T_{\alpha\beta}\mathbf{e}_{\alpha\beta},\ (\alpha, \beta = 1, 2), \tag{9.91b}$$

in which $\mathbf{S}$ is a two-dimensional symmetric tensor.

Regardless of the possible multiplicity of the principal values for $\mathbf{T}$, we now wish to determine if it is possible to find a nonzero vector $\mathbf{u} = u_\alpha\mathbf{e}_\alpha$, $(\alpha = 1, 2)$ in the plane P of $\mathbf{e}_1$ and $\mathbf{e}_2$ for which $\mathbf{T}\mathbf{u} = \lambda\mathbf{u}$ holds. From (9.91b), we note that $\mathbf{T}\mathbf{u} = \mathbf{S}\mathbf{u}$; therefore, we seek a vector $\mathbf{u} \neq \mathbf{0}$ in P such that

$$(\mathbf{S} - \lambda\mathbf{1})\mathbf{u} = \mathbf{0}; \text{ that is, } (T_{\alpha\beta} - \lambda\delta_{\alpha\beta})u_\beta = 0,\ (\alpha, \beta = 1, 2). \tag{9.91c}$$

The homogeneous system (9.91c) will have a nontrivial solution $\mathbf{u}$ provided that $\det(\mathbf{S} - \lambda\mathbf{1}) = 0$. Because $\mathbf{S}$ is real and symmetric, this real quadratic equation in λ has two real roots. In consequence, there exists at least one vector $\mathbf{u}$ in P for which (9.91c) holds. Therefore, $\lambda = \lambda_2$ and $\mathbf{u} \equiv \hat{\mathbf{e}}_2$, say, is a principal pair for $\mathbf{S}$, hence also a second principal pair for $\mathbf{T}$. Since the basis directions $\mathbf{e}_k$ in the plane P were arbitrary, we may now assign them so that $\mathbf{e}_2 = \hat{\mathbf{e}}_2$. Then referred to $\varphi = \{O; \mathbf{e}_k\}$, we have $\mathbf{S}\mathbf{e}_2 = \lambda_2\mathbf{e}_2$, and therefore $\mathbf{e}_2 \cdot \mathbf{S}\mathbf{e}_2 = T_{22} = \lambda_2,\ \mathbf{e}_1 \cdot \mathbf{S}\mathbf{e}_2 = T_{12} = T_{21} = 0$, wherein we recall the second equation in (9.91b). Thus, the matrix (9.91a) referred to $\varphi = \{O; \mathbf{e}_k\} = \{O; \mathbf{e}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3\}$ now has the diagonal form

$$T = \begin{bmatrix} T_{11} & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}. \tag{9.91d}$$

Finally, in view of (9.91d), we have $\mathbf{T}\mathbf{e}_1 = T_{k1}\mathbf{e}_k = T_{11}\mathbf{e}_1$; and hence $\mathbf{e}_1$ also is a principal direction for $\mathbf{T}$ and $T_{11} = \lambda_1$ is the corresponding principal value. Notice that we have nowhere assumed that the principal values of $\mathbf{T}$ must be distinct. *Consequently, regardless of the possible multiplicity of principal values for a symmetric tensor* $\mathbf{T}$, *there always exist at least three mutually orthogonal directions* $\mathbf{e}_k$ *that may be chosen as a principal basis* $\hat{\mathbf{e}}_k$ *with respect to which* $\mathbf{T}$

has the diagonal form

$$\mathbf{T} = \lambda_1 \hat{\mathbf{e}}_{11} + \lambda_2 \hat{\mathbf{e}}_{22} + \lambda_3 \hat{\mathbf{e}}_{33}. \tag{9.92}$$

This is called the *spectral representation* for $\mathbf{T}$.

Suppose, however, that $\mathbf{T}$ has a principal value of multiplicity $m = 2$, say, $\lambda_1 = \lambda_2 = \lambda$. Then the foregoing theorem assures existence of at least two orthogonal directions $\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2$ corresponding to the same λ so that $\mathbf{T}\hat{\mathbf{e}}_\alpha = \lambda \hat{\mathbf{e}}_\alpha$ $(\alpha = 1, 2)$. Now, an arbitrary unit vector $\mathbf{n}$ in the plane of $\hat{\mathbf{e}}_\alpha$ may be written as

$$\mathbf{n} = (\mathbf{n} \cdot \hat{\mathbf{e}}_\alpha)\hat{\mathbf{e}}_\alpha;$$

so, we have

$$\mathbf{Tn} = (\mathbf{n} \cdot \hat{\mathbf{e}}_\alpha)\mathbf{T}\hat{\mathbf{e}}_\alpha = (\mathbf{n} \cdot \hat{\mathbf{e}}_\alpha)\lambda\hat{\mathbf{e}}_\alpha = \lambda\mathbf{n}.$$

In consequence, every vector $\mathbf{n}$ *in the plane perpendicular to* $\hat{\mathbf{e}}_3$, *the direction corresponding to the distinct principal value for* $\mathbf{T}$, *is a principal direction for* $\mathbf{T}$ *corresponding to the repeated principal value* $\lambda_1 = \lambda_2 = \lambda$. *Similarly, if* $\mathbf{T}$ *has three equal principal values* $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$, *then* $\mathbf{T} = \lambda\mathbf{1}$, *and every spatial direction* $\mathbf{n}$ *is a principal direction for* $\mathbf{T}$.

The multiplicity properties of the symmetric tensor $\mathbf{T}$ are precisely those described geometrically by the Cauchy momental ellipsoid. A Cauchy ellipsoid with two equal principal radii is an ellipsoid of revolution, every direction in the cross section being a principal direction. When all three principal radii are equal, the ellipsoid is a sphere for which every direction is a principal direction. Some further topics on symmetric tensors are described in Problems 9.34, 9.37, 9.41, and 9.45. See also Problem 9.40 in which $\mathbf{T} \neq \mathbf{T}^T$. We conclude with two examples.

Example 9.11. A symmetric tensor $\mathbf{T}$ has scalar components

$$T = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{9.93a}$$

referred to $\varphi = \{O; \mathbf{e}_k\}$. Determine the principal values and directions for $\mathbf{T}$.

Solution. The principal values for $\mathbf{T}$ in (9.93a) are determined by (9.85):

$$\det(\mathbf{T} - \lambda\mathbf{1}) = \begin{vmatrix} 1-\lambda & 2 & 0 \\ 2 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = (1-\lambda)[(1-\lambda)^2 - 4] = 0,$$

which has three real roots

$$\lambda_1 = 3, \qquad \lambda_2 = -1, \qquad \lambda_3 = 1. \tag{9.93b}$$

Hence, $\lambda_2 = -1$ is the algebraically smallest normal component of $\mathbf{T}$ and $\lambda_1 = 3$ is the greatest. We note that $\mathrm{tr}T = 3$ from (9.93a) and confirm that the sum of the principal values (9.93b) is the same.

With $\mathbf{n} = \nu_k \mathbf{e}_k$ in φ and use of (9.93a), the principal vector equation (9.83) and the constraint (9.84) may be expanded as

$$\begin{aligned}(1-\lambda)\nu_1 + 2\nu_2 &= 0,\\ 2\nu_1 + (1-\lambda)\nu_2 &= 0,\\ (1-\lambda)\nu_3 &= 0,\\ \nu_1^2 + \nu_2^2 + \nu_3^2 &= 1.\end{aligned} \tag{9.93c}$$

With $\lambda = \lambda_3 = 1$, (9.93c) yields $\nu_1 = \nu_2 = 0$, $\nu_3 = \pm 1$; thus $\mathbf{n} \equiv \hat{\mathbf{e}}_3 = \pm\,\mathbf{e}_3$. Similarly, for $\lambda = \lambda_2 = -1$, $\nu_1 = -\nu_2 = \pm\sqrt{2}/2$, $\nu_3 = 0$, and hence $\mathbf{n} \equiv \hat{\mathbf{e}}_2 = \pm\,(\sqrt{2}/2)(\mathbf{e}_1 - \mathbf{e}_2)$. The third principal vector orthogonal to $\hat{\mathbf{e}}_2$ and $\hat{\mathbf{e}}_3$ is given by $\hat{\mathbf{e}}_1 = \hat{\mathbf{e}}_2 \times \hat{\mathbf{e}}_3 = \mp(\sqrt{2}/2)(\mathbf{e}_1 + \mathbf{e}_2)$. The signs are fixed as we please, but such that the triple $\hat{\mathbf{e}}_k$ forms a right-hand basis. We thus find the following principal values and directions for $\mathbf{T}$:

$$\begin{aligned}&\lambda_1 = 3 \sim \hat{\mathbf{e}}_1 = (\sqrt{2}/2)(\mathbf{e}_1 + \mathbf{e}_2),\\ &\lambda_2 = -1 \sim \hat{\mathbf{e}}_2 = (\sqrt{2}/2)(-\mathbf{e}_1 + \mathbf{e}_2),\\ &\lambda_3 = 1 \sim \hat{\mathbf{e}}_3 = \mathbf{e}_3.\end{aligned} \tag{9.93d}$$

Hence, by (9.92), in the principal basis $\hat{\mathbf{e}}_k$,

$$\mathbf{T} = 3\hat{\mathbf{e}}_{11} - \hat{\mathbf{e}}_{22} + \hat{\mathbf{e}}_{33}. \tag{9.93e}$$

Notice that $\mathbf{T}$ cannot be a moment of inertia tensor for a rigid body, because $\hat{T}_{22} < 0$. This is not evident from (9.93a) for which all of the diagonal components are positive and the inequalities (9.80) are satisfied, which is not true for (9.93e).

Finally, it is useful to note from (9.93d) the orthogonal transformation matrix $A : \mathbf{e}_k \to \hat{\mathbf{e}}_k$ for which $A_{ij} = \cos\langle \hat{\mathbf{e}}_i, \mathbf{e}_j \rangle$:

$$A = \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ -\sqrt{2}/2 & \sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{9.93f}$$

Hence, the transformation A that diagonalizes T in (9.93a) in accordance with the tensor transformation law $\hat{T} = ATA^T$ describes a 45° counterclockwise rotation of $\mathbf{e}_k \to \hat{\mathbf{e}}_k$ about their common axis $\mathbf{e}_3 = \hat{\mathbf{e}}_3$, to yield the matrix of the tensor $\mathbf{T}$ in (9.93e). □

Example 9.12. If the T_{33} component of T in (9.93a) is replaced by $T_{33} = -1$, the characteristic equation for the new tensor has a repeated root $\lambda_3 = \lambda_2 = -1$ and $\lambda_1 = 3$. Hence, every vector in the plane normal to $\hat{\mathbf{e}}_1$ is a principal direction corresponding to $\lambda_2 = -1$. In particular, the same two vectors $\hat{\mathbf{e}}_2$, $\hat{\mathbf{e}}_3$ given in (9.93d) are principal vectors for the new tensor.

References

1. Bowen, R. M., *Introduction to Continuum Mechanics for Engineers*, Plenum, New York, 1989. Appendix A presents a parallel development of the elements of tensor algebra in notation similar to that used here. The principal values and vectors for a tensor and the Cayley–Hamilton theorem also are discussed there.
2. Buck, R. C., *Advanced Calculus*, 2nd Edition, McGraw-Hill, New York, 1965. The method of Lagrange multipliers is described in Chapter 6.
3. Greenwood, D. T., *Principles of Dynamics*, Prentice-Hall, Englewood Cliffs, New Jersey, 1965. This intermediate level text is a good source for general collateral study. Some subtle aspects of the momental ellipsoid and its relation to the body are discussed in Chapter 7.
4. Kane, T. R., *Dynamics*, Holt, Reinhart and Winston, New York, 1968. Moments of inertia are nicely described in Chapter 3 as the components of both the second moment vector (See Problem 9.2.) and also as dyadic (tensor) components. Some further examples may be found here and in Kane's earlier work *Analytical Elements of Mechanics*, Vol. 1, *Dynamics*, Academic, New York, 1961.
5. Rosenberg, R. M., *Analytical Dynamics of Discrete Systems*, Plenum, New York, 1977. The moment of inertia tensor, its transformation properties, and description in terms of Cauchy's ellipsoid are presented in both index and expanded notation. Recommended for advanced readers.
6. Shames, I. H., *Engineering Mechanics*, Vol. 2, *Dynamics*, 2nd Edition, Prentice-Hall, Englewood Cliffs, New Jersey, 1966. An alternative formulation of the properties of the inertia tensor in expanded notation is presented in Chapter 16. See also Chapter 9 of the 3rd Edition, 1980.
7. Yeh, H., and Abrams, J. I., *Principles of Mechanics of Solids and Fluids*, Vol. 1, *Particle and Rigid Body Mechanics*, McGraw-Hill, New York, 1960. Chapter 11 deals with the inertia tensor mainly in expanded notation, though index notation also is used sparingly. Cauchy's momental ellipsoid is introduced to characterize the principal moments of inertia.

Problems

9.1. Let $\mathbf{n}$ be a unit vector along an arbitrary imbedded axis n through a base point Q in Fig. 9.1, page 360, and let $\mathbf{x}$ be the position vector from Q to the mass element dm. Begin with definition (9.10) and derive (9.21), wherein $I_{nn}^Q \equiv \mathbf{n} \cdot \mathbf{I}_Q \mathbf{n}$.

9.2. The *second moment vector* $\mathbf{I}_n^Q(\mathscr{B})$ relative to Q for a fixed direction $\mathbf{n}$ in Fig. 9.1 is defined by

$$\mathbf{I}_n^Q(\mathscr{B}) = \int_{\mathscr{B}} \mathbf{x} \times (\mathbf{n} \times \mathbf{x}) dm. \tag{P9.2a}$$

(i) Show by vector algebra that $\mathbf{n} \cdot \mathbf{I}_n^Q = I_{nn}^Q$, the integral in (9.21). (ii) More generally, expand the triple product to show in direct notation that

$$\mathbf{I}_n^Q(\mathscr{B}) = \mathbf{I}_Q(\mathscr{B})\mathbf{n}, \tag{P9.2b}$$

and thus prove that the component of the second moment vector in the direction $\mathbf{m}$ is given by $\mathbf{m} \cdot \mathbf{I}_n^Q = \mathbf{m} \cdot \mathbf{I}_Q \mathbf{n} = I_{mn}^Q$. Now show that $\mathbf{m} \cdot \mathbf{I}_n^Q = \mathbf{n} \cdot \mathbf{I}_m^Q$, and hence $I_{mn}^Q = I_{nm}^Q$. Notice that if $\mathbf{m} = \mathbf{n}$, this yields (9.21) for the moment of inertia about the axis $\mathbf{n}$; and if $\mathbf{m} \cdot \mathbf{n} = 0$, I_{mn}^Q is the product of inertia for the orthogonal directions $\mathbf{m}$ and $\mathbf{n}$.

9.3. Although the inertia tensor for a thin body may be derived as a limit case of a similar thick body, it is also straightforward to obtain results for thin bodies directly. Apply (9.14) to derive the inertia properties referred to $\varphi = \{C; \mathbf{i}_k^*\}$ at the center of mass for (a) the homogeneous thin tube in Fig. D.5 of Appendix D, (b) the homogeneous thin rod shown in Fig. D.7, and (c) the homogeneous thin spherical shell in Fig. D.10.

9.4. Find the mass, the center of mass, and the components of the moment of inertia tensor for the thin homogeneous rod forming the circular sector in Fig. D.8. Apply the results to determine these properties for a homogeneous semicircular wire and a thin circular ring.

9.5. Determine the mass, center of mass, and inertia tensor $\mathbf{I}_O$ for a homogeneous, circular cylindrical sector having a central angle 2θ, inner radius R_i, outer radius R_o, and length L. Refer all quantities to a body frame $\varphi = \{O; \mathbf{i}_k\}$ at the central point O with $\mathbf{i}_3$ being the cylinder axis and $\mathbf{i}_1$ bisecting both the central angle and the length. Derive as limit cases the properties for (a) a thin-walled circular sector, and (b) a thin circular rod described in Fig. D.8.

9.6. Find the mass, center of mass, and moment of inertia tensor for the homogeneous thin conical shell in Fig. D.4, referred to $\varphi = \{O; \mathbf{i}_k\}$.

9.7. Determine the mass, center of mass, and moment of inertia tensor for the homogeneous semicylinder of length ℓ and radius R shown in Fig. D.9, referred to the body frame $\varphi = \{O; \mathbf{i}_k\}$.

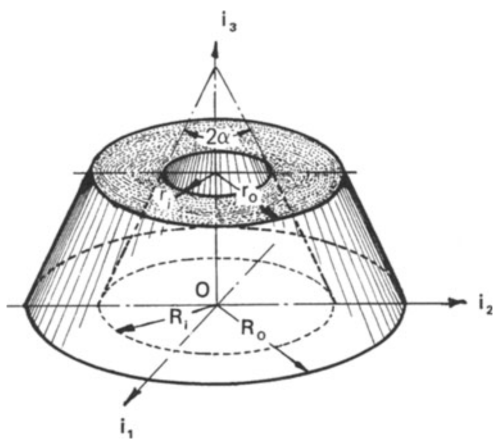
9.8. Derive the inertia tensor properties for the homogeneous right rectangular pyramid described in Fig. D.2.

9.9. Apply (9.29c) to derive the moment of inertia tensor for a homogeneous thin-walled circular tube of mean radius r, referred to $\varphi = \{C; \mathbf{i}_k^*\}$. Use the result to deduce the inertia tensor for a plane circular wire of radius R.

9.10. (a) Find the mass, center of mass, and moment of inertia tensor at O for the homogeneous thin hemispherical shell in Fig. D.11. (b) Derive from these results the same properties for the entire thin spherical shell in Fig. D.10.

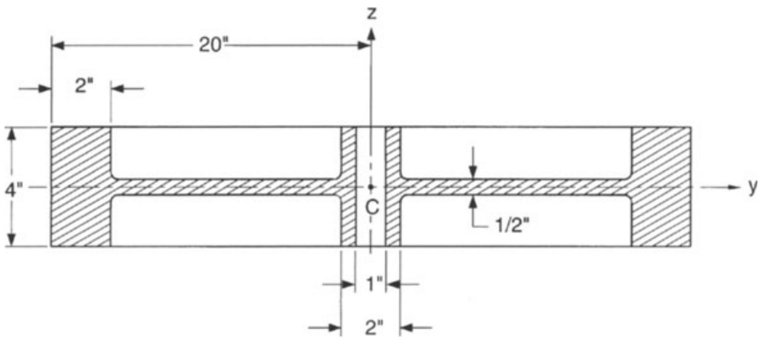
9.11. (a) Find the mass, center of mass, and moment of inertia tensor at O for the homogeneous hemisphere in Fig. D.13. (b) Derive from these results the same properties for a sphere of radius R. (c) Use the solution for a solid sphere to deduce the inertia tensor for the thick-walled, homogeneous spherical shell in Fig. D.10.

9.12. A portion of a thick-walled, homogeneous projectile casing whose inner and outer parallel surfaces are frustums of similar coaxial cones is shown in the figure. Apply the properties of a homogeneous right circular cone in Fig. D.3 to determine the mass of the casing and its moment of inertia about the z-axis, referred to the body frame $\varphi = \{O; \mathbf{i}_k\}$.



Problem 9.12.

9.13. The figure shows an arbitrary diametral cross section of a homogeneous flywheel made of a grade of steel of density $\rho = 15$ slug/ft^3. Determine its moment of inertia about the z-axis. What is the radius of gyration about the z-axis?



Problem 9.13.

9.14. The moment of inertia tensor for a sector of a homogeneous circular rod is given in Fig. D.8. Derive its inertia tensor in a parallel frame $\varphi = \{C; \mathbf{i}_k^*\}$ at its center of mass. What is the inertia tensor for a semicircular wire referred to φ?

9.15. Derive by integration the inertia tensor for the thin rod in Fig. D.7, referred to frame $\phi = \{O; \mathbf{i}_k\}$ at its end point O. Confirm the result by use of the parallel axis theorem applied to the tensor $\mathbf{I}_C$ given there.

9.16. A nonhomogeneous thin rigid rod of length l has a mass density $\rho(x)$ that varies with the distance x from one end O such that $d\rho(x)/dx = \rho_1$, a constant, and $\rho(0) = \rho_0$. (a) Find the mass of the rod and determine its moments of inertia relative to O. (b) Find the moments of inertia relative to the center of mass of the rod. (c) Derive from the results in (a) and (b) the corresponding properties for a homogeneous thin rod. (d) Consider a rod for which $\rho(l) = 2\rho_0$, and thus determine all of the properties found more generally in (a) and (b).

9.17. Apply the properties in Fig. D.9 for a homogeneous semicylinder to derive the moment of inertia tensor for (a) a solid cylinder referred to frame $\varphi = \{O; \mathbf{i}_k\}$, and (b) a semicylinder in a parallel frame $\phi^* = \{C; \mathbf{i}_k^*\}$ at its center of mass.

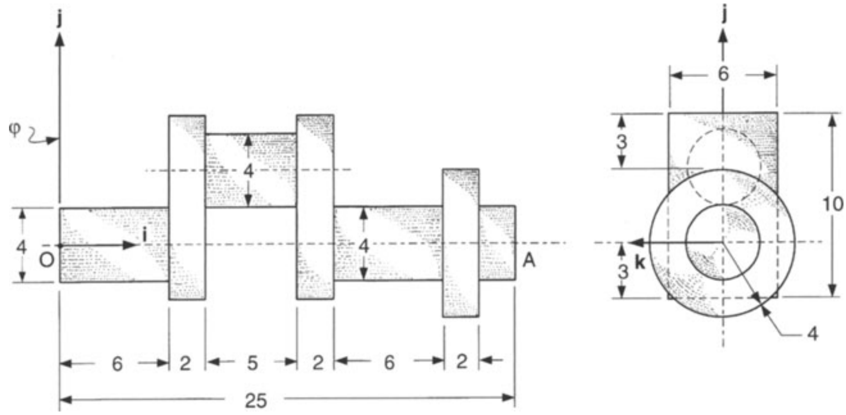
9.18. Derive the moment of inertia tensor for the homogeneous right circular cone in Fig. D.3, referred to frame $\varphi = \{O; \mathbf{i}_k\}$ in its base, and to a parallel frame at its center of mass.

9.19. Use the properties of the solid cone in Fig. D.3 to find its moment of inertia tensor $\mathbf{I}_Q$ referred to a parallel frame $\psi = \{Q; \mathbf{i}_k\}$ at its vertex Q. Let P be a point on the base circle at $\mathbf{r} = r\mathbf{i}_1$ from O, and determine at Q the moment of inertia tensor component about the edge line QP, referred to ψ.

9.20. A model of a crankshaft assembly for a one cylinder engine is shown in the figure. Use the table of properties in Appendix D to find the radius of gyration of the assembly about the axis OA, referred to $\varphi = \{O; \mathbf{i}_k\}$.

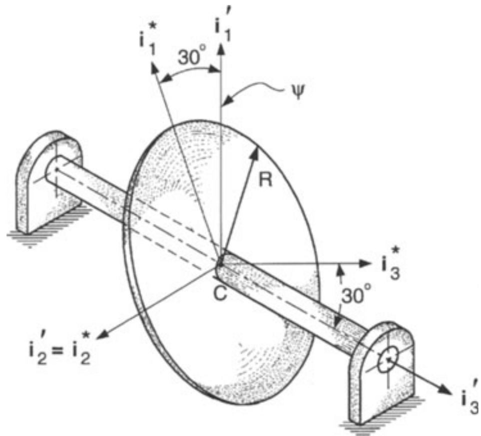
9.21. The plane of a homogeneous thin disk makes a 30° angle with the vertical plane, as shown in the figure. Determine the inertia tensor $\mathbf{I}_C$ for the disk in the body frame $\psi = \{C; \mathbf{i}'_k\}$.

9.22. A homogeneous, thin rectangular plate of mass $m = 2$ kg is welded to a horizontal shaft, as shown in the diagram. Find its inertia tensor $\mathbf{I}_C$ in the plate frame $\psi = \{C; \mathbf{e}_k\}$.

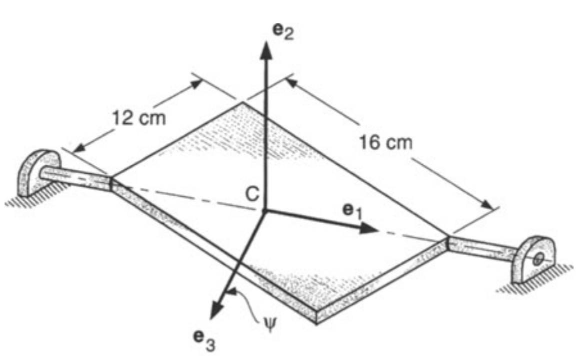


Note: All Dimensions are in Centimeters.

Problem 9.20.



Problem 9.21.



Problem 9.22.

9.23. Find by the method of Lagrange multipliers the point P in the plane $x_1 + x_2 + x_3 = 3$ nearest to the origin. Sketch the portion of this surface for $x_k \geq 0$ and provide a geometrical description of the point P in this region.

9.24. A particle moves on a curve defined by the intersection of the plane $2x + 4y = 5$ and the paraboloid $x^2 + z^2 = 2y$. What is the greatest elevation $z = h$ that the particle may reach in the motion? Note that there are two constraints here.

9.25. A particle P initially at the place $(0, 0, 12)$ is constrained by forces to move in the plane $2x + 3y + z = 12$. Find the equation of the straight path for which the motion of P passes the point where the potential energy function $V(\mathbf{x}) = 4x^2 + y^2 + z^2$ has a minimum.

9.26. What are the volume V and the moment of inertia tensor $\mathbf{I}_C$ for the largest homogeneous rectangular block having sides parallel to the coordinate planes and which can be inscribed in the ellipsoid $(x/a)^2 + (y/b)^2 + (z/c)^2 = 1$? The constants a, b, c are the principal semidiameters of the ellipsoid.

9.27. A particle P moves along the line through the points $(1, 0, 0)$ and $(0, 1, 0)$. Find the point on this line at which P is nearest to the line $x = y = z$. What is the shortest distance between these lines?

9.28. In continuum mechanics, a Bell material is a constrained elastic solid material for which the first principal invariant $J_1 = \mathrm{tr}\mathbf{V}$ of the symmetric Cauchy–Green deformation tensor $\mathbf{V}$ must satisfy the rule $J_1 = 3$ in every deformation from the undeformed state where $\mathbf{V} = \mathbf{1}$. The three principal values λ_k of $\mathbf{V}$, all positive, are called principal stretches. The principal invariants of $\mathbf{V}$ are defined by (9.89). Determine the extremal values of the second and third principal invariants of $\mathbf{V}$. Are these extrema their largest or smallest values?

9.29. Find the work done in moving a particle from a place at $\mathbf{x} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ to the place at which the potential energy function $V(x, y, z) = x - 2y + 2z$ has a maximum value among all points (x, y, z) located on a sphere of radius 3.

9.30. An electron moves in the plane $ax + by + cz + d = 0$. Find the point in this plane at which the electron can be closest to the origin in frame $\varphi = \{O; \mathbf{i}_k\}$, and determine its shortest distance from O.

9.31. Consider the symmetric tensor $\mathbf{T} = \mathbf{1} - 4(\mathbf{e}_{12} + \mathbf{e}_{21})$ referred to the Cartesian frame $\varphi = \{O; \mathbf{e}_k\}$. (i) Find the principal values and directions for $\mathbf{T}$. (ii) Identify the basis transformation matrix A: $\mathbf{e}_k \rightarrow \hat{\mathbf{e}}_k$, the principal basis for $\mathbf{T}$, and describe its geometrical character. (iii) Apply the tensor transformation law to demonstrate that A diagonalizes the matrix T in $\mathbf{e}_k$ to form $\hat{T}$ in $\hat{\mathbf{e}}_k$. Could $\mathbf{T}$ be the inertia tensor for some body in φ? Appropriate units are assumed.

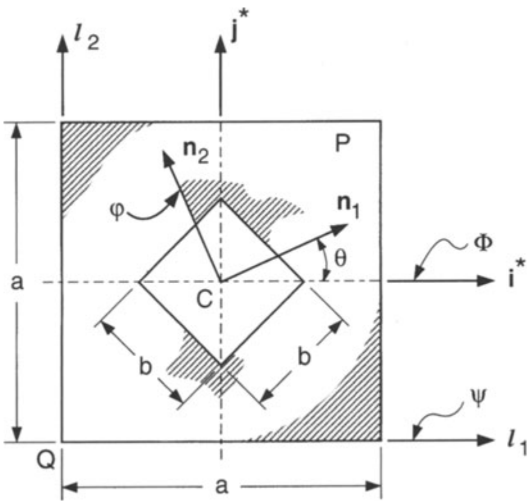
9.32. Consider two symmetric tensors: $\mathbf{T} = 2\mathbf{e}_{11} + 5\mathbf{e}_{22} - \mathbf{e}_{33} + 4(\mathbf{e}_{23} + \mathbf{e}_{32})$ in $\mathbf{e}_k$ and $\mathbf{U} = -\bar{\mathbf{e}}_{11} + 6\bar{\mathbf{e}}_{22} + \bar{\mathbf{e}}_{33} + 2(\bar{\mathbf{e}}_{23} + \bar{\mathbf{e}}_{32})$ in $\bar{\mathbf{e}}_k$. Determine the principal values and directions for $\mathbf{T}$ in $\mathbf{e}_k$. Is the tensor $\mathbf{U}$ the same as $\mathbf{T}$ but merely referred to another basis $\bar{\mathbf{e}}_k$?

9.33. A homogeneous, thin rectangular plate has sides $\hat{x} = a$ and $\hat{y} = 2a$. (a) Find by integration the moment of inertia tensor $\mathbf{I}_O$ referred to a Cartesian frame $\varphi = \{O; \hat{\mathbf{e}}_k\}$ at the corner point O and parallel to the plate edges $\hat{x}$ and $\hat{y}$. (b) Confirm the result by application of (9.27) and the parallel axis theorem. (c) Determine in φ the principal values and directions for the inertia tensor at O.

9.34. A certain symmetric tensor $\mathbf{T}$ is given by $\mathbf{T} = 15\mathbf{e}_{11} + 25\mathbf{e}_{22} + 30\mathbf{e}_{33} - 10(\mathbf{e}_{13} + \mathbf{e}_{31})$ in $\varphi = \{Q; \mathbf{e}_k\}$. (a) Write the equation for its ellipsoid in φ, and find its principal ellipsoid. (b) Could $\mathbf{T}$ be the inertia tensor at Q for some rigid body? Could it be the inertia tensor for a plane body?

9.35. Let T be the matrix in a Cartesian frame φ of a tensor $\mathbf{T}$, and consider another tensor $\mathbf{U}$ whose matrix in φ is $U = \alpha T$, where α is a scalar. (a) Prove that the same proportional relation holds in every Cartesian reference system. What can be said about the corresponding principal values and directions for **U** and **T**? (b) Let $\alpha = 1/5$ and consider the tensor **T** defined in Problem 9.34. Solve that problem for the tensor **U**.

9.36. A homogeneous, thin square plate of side a has a square hole of side b punched through its center, as illustrated. (a) First, consider the plate without the hole. Find by integration the inertia tensor for the solid plate referred to the frame $\Phi = \{C; \mathbf{i}_k^*\}$, and then read from this the result referred to the frame $\varphi = \{C; \mathbf{n}_k\}$ making an angle θ with Φ in the plane of the plate, as shown. What is the radius of gyration of the plate about the axis $\mathbf{n}_1$ and about any other axis in the plane? (b) Now consider the plate with the hole. Determine the inertia tensor for the punched plate (i) referred to Φ at C and (ii) referred to a parallel frame $\psi = \{Q; \mathbf{l}_k\}$ at the corner Q.



Problem 9.36.

9.37. The matrix in $\psi = \{Q; \mathbf{l}_k\}$ of the inertia tensor $\mathbf{I}_Q$ for the punched plate described in the previous problem has the general form

$$I_Q = \begin{bmatrix} A & -B & 0 \\ -B & A & 0 \\ 0 & 0 & 2A \end{bmatrix}, \tag{P9.37}$$

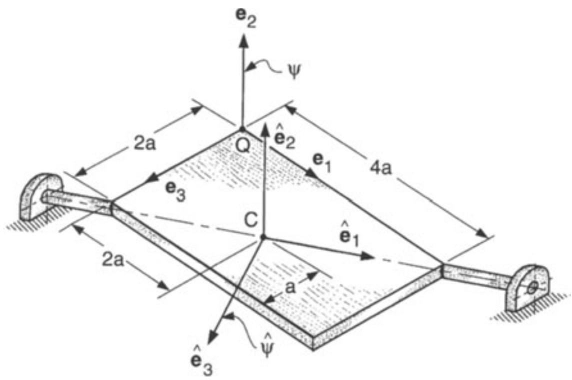
where $A > B > 0$. (a) Find the principal values and directions for $\mathbf{I}_Q$. (b) Interpret the geometry of the principal directions for $\mathbf{I}_Q$, and relate it to the geometry of the punched plate. Are these directions evident from the plate geometry? (c) Use the results of Problem 9.36 to find by the parallel axis theorem the principal values of the inertia tensor at Q.

9.38. The inertia tensor $\mathbf{I}_O$ in the frame $\psi = \{O; \mathbf{i}_k\}$ has the component matrix

$$I_O = \begin{bmatrix} \frac{1}{2}(\alpha+\beta) & \frac{1}{2}(\alpha-\beta) & 0 \\ \frac{1}{2}(\alpha-\beta) & \frac{1}{2}(\alpha+\beta) & 0 \\ 0 & 0 & \gamma \end{bmatrix}, \tag{P9.38}$$

where α, β, γ are positive constants. Find the principal values and directions for the inertia tensor. Describe the angular orientation of the principal frame relative to ψ.

9.39. (a) Find by integration the inertia tensor $\mathbf{I}_C$ for the homogeneous, rectangular plate of mass m in a plate frame $\psi = \{C; \mathbf{e}_k\}$, and thus determine $\mathbf{I}_C$ in the plate frame $\hat{\psi} = \{C; \hat{\mathbf{e}}_k\}$ shown in the figure. (b) Find the inertia tensor $\mathbf{I}_Q$ at the corner Q in the plate frame $\psi = \{Q; \mathbf{e}_k\}$. (c) Determine in ψ the principal values and directions for $\mathbf{I}_Q$ at Q. Here the $\hat{\mathbf{e}}_k$ are not principal vectors.



Problem 9.39.

9.40. Because a tensor and its transpose have the same principal invariants, they have the same characteristic equation and hence the same principal values. Their principal vectors, however, need not be the same. (a) Prove that $\mathbf{T}$ and $\mathbf{T}^T$ have the same principal invariants. (b) Now consider the tensor $\mathbf{T} = \mathbf{e}_{11} + \mathbf{e}_{22} + 3\mathbf{e}_{33} + \mathbf{e}_{12} + 2\mathbf{e}_{21}$ in the Cartesian frame $\varphi = \{O; \mathbf{e}_k\}$. Find the principal values and directions for $\mathbf{T}$ and for $\mathbf{T}^T$, and determine their principal invariants. (c) Determine the angles between the principal vectors for $\mathbf{T}$, and do the same for $\mathbf{T}^T$. Sketch the principal vectors for both tensors in φ, and describe the geometry. (d) Are the principal vectors of $\mathbf{T}$ mutually orthogonal? Are those of $\mathbf{T}^T$ mutually orthogonal?

9.41. Consider the symmetric tensor $\mathbf{T} = \frac{5}{2}(\mathbf{e}_{11} + \mathbf{e}_{22}) - \frac{3}{2}(\mathbf{e}_{12} + \mathbf{e}_{21}) + 3\mathbf{e}_{33}$ referred to $\mathbf{e}_k$. Show that the principal values of $\mathbf{T}$ are non-negative. Therefore, $\mathbf{T}$ has a unique, positive symmetric square root defined by $\mathbf{T}^{1/2} = \sqrt{\lambda_1}\hat{\mathbf{e}}_{11} + \sqrt{\lambda_2}\hat{\mathbf{e}}_{22} + \sqrt{\lambda_3}\hat{\mathbf{e}}_{33}$ in the principal basis $\hat{\mathbf{e}}_k$ of $\mathbf{T}$. Find $\mathbf{T}^{1/2}$ in $\mathbf{e}_k$ and check your solution by the matrix multiplication $T = T^{1/2}T^{1/2}$ in $\mathbf{e}_k$.

9.42. Determine in the principal basis $\hat{\mathbf{e}}_k$ the positive square root of the symmetric tensor $\mathbf{U}$ whose component matrix referred to $\varphi = \{O; \mathbf{e}_k\}$ is

$$U = \begin{bmatrix} 5 & 1 & -1 \\ 1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}. \tag{P9.42}$$

Let $\hat{U}^{1/2}$ denote the principal matrix of $\mathbf{U}^{1/2}$ in $\hat{\mathbf{e}}_k$. Identify the basis transformation matrix required to transform $\hat{U}^{1/2}$ into $U^{1/2}$ in φ. See Problem 9.41.

9.43. Since $\mathbf{I}_Q$ is a positive, symmetric tensor, in accordance with the theorem stated in Problem 9.41, it has a unique positive, symmetric square root $\mathbf{I}_Q^{1/2}$. Hence, we may define the unique *gyration tensor*

$$\mathbf{G}_Q \equiv \frac{1}{\sqrt{m(\mathscr{B})}}\mathbf{I}_Q^{1/2}, \tag{P9.43a}$$

so that $m\mathbf{G}_Q^2 = \mathbf{I}_Q$. The tensors $\mathbf{G}_Q$ and $\mathbf{I}_Q$ have the same principal directions, and the principal values of $\mathbf{G}_Q$ are the familiar radii of gyration (9.22) about the principal axes $\hat{\mathbf{e}}_k$, namely,

$$\hat{R}_n = \sqrt{\frac{\hat{I}_{nn}}{m}} = \hat{G}_{nn}. \tag{P9.43b}$$

The matrix (P9.37) shows that $R_1^Q = R_2^Q = \sqrt{A/m}$, $R_3^Q = \sqrt{2A/m}$ are the radii of gyration about the $\mathbf{l}_k$ axes at Q. (a) What are the principal components of $\mathbf{G}_Q$ for the tensor with matrix (P9.37)? (b) Identify the principal directions from Problem 9.37, and determine the components of $\mathbf{G}_Q$ referred to $\psi = \{Q; \mathbf{l}_k\}$ in terms of the principal radii of gyration (P9.43b). Of course, the normal components G_{nn}^Q of the gyration tensor $\mathbf{G}_Q$ in ψ generally are *not* the same as the radii of gyration R_n^Q given above.

9.44. A tensor $\mathbf{W}$ has the Cartesian component matrix

$$W = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \tag{P9.44}$$

referred to $\varphi = \{O; \mathbf{e}_k\}$. Find the positive square root of $\mathbf{W}$ referred to φ. See Problem 9.41.

9.45. Although the greatest values of the products of inertia of a rigid body are unimportant, the determination of the extremal values of the nondiagonal components of a symmetric tensor is important in the study of the properties of stress and strain tensors in continuum mechanics, for example. Let $\mathbf{T}$ be a symmetric tensor and $\mathbf{m}$ and $\mathbf{n}$ orthogonal unit vectors. The *orthogonal shear* or *product component* of $\mathbf{T}$ for the pair $(\mathbf{m}, \mathbf{n})$ is defined by $T_{mn} \equiv \mathbf{m} \cdot \mathbf{Tn}$. (a) Apply the method of Lagrange multipliers to derive a system of two vector equations that determine among all possible pairs $(\mathbf{m}, \mathbf{n})$ those orthogonal directions for which T_{mn} has its greatest absolute value $|T_{mn}|_{\max}$. (b) How are the Lagrange multipliers related to the components of $\mathbf{T}$ in the extremal directions and to $|T_{mn}|_{\max}$? (c) Derive from the vector equations in (a) two equations for the sum and difference of the extremal directions $\mathbf{m}$ and $\mathbf{n}$. Interpret these equations in terms of the principal values τ_a and principal directions $\hat{\mathbf{e}}_a$ for $\mathbf{T}$, and thus show that the product components of $\mathbf{T}$ have their maximum absolute value with respect to a basis with directions $\mathbf{m}$ and $\mathbf{n}$ that bisect the principal directions for $\mathbf{T}$. That is, show that*

$$|T_{mn}|_{\max} = \max\left\{\left|\frac{\tau_1 - \tau_2}{2}\right|, \left|\frac{\tau_2 - \tau_3}{2}\right|, \left|\frac{\tau_3 - \tau_1}{2}\right|\right\}, \tag{P9.45a}$$

where $\mathbf{m}$ and $\mathbf{n}$ are the orthogonal directions

$$\mathbf{m} = \frac{\sqrt{2}}{2}(\hat{\mathbf{e}}_a + \hat{\mathbf{e}}_b), \qquad \mathbf{n} = \frac{\sqrt{2}}{2}(\hat{\mathbf{e}}_a - \hat{\mathbf{e}}_b) \tag{P9.45b}$$

(or their opposites) for which $|T_{mn}|_{\max} = \frac{1}{2}|\tau_a - \tau_b|$ in (9P.45a). Of course, the least absolute value for the orthogonal shear or product components of $\mathbf{T}$ occurs for the principal basis where they all vanish.

* A very short elegant proof of the maximum orthogonal shear component of a symmetric tensor is given by Ph. Boulanger and M.A. Hayes, Shear, shear stress and shearing, *Journal of the Mechanics and Physics of Solids* **40**, 1449–1457 (1992). An earlier alternate proof that uses the geometrical properties of pairs of conjugate semi-diameters of ellipses is provided by M.A. Hayes, A note on maximum orthogonal shear stress and shear strain, *Journal of Elasticity* **21**, 117–120 (1989).

9.46. [†]Consider a particle P with potential energy $V(x, y) = 4x + y + y^2$. The motion of P is constrained so that the point (x, y) lies on the circle $x^2 + 2x + y^2 + y = 1$. Determine the extremal values of the potential energy. (a) First, apply the constraint equation to write $V(x, y) = \bar{V}(x)$ and thus show that the usual substitution procedure fails to deliver a real solution for any extrema of $V(x, y)$. (b) Apply the method of Lagrange multipliers and show that the potential energy has both maximum and minimum values at distinct points on the circle. Find these points and determine the energy extrema.

[†] I thank Professor Michael A. Hayes for suggesting this example and for recalling the aforementioned references on maximum orthogonal shear.