8

Dynamics of a System of Particles

8.1. Introduction

The principles of mechanics for a particle are extended here to a system of n discrete material points. We begin with Newton's second law for a system of particles and formulate the momentum, impulse–momentum, moment of momentum, work–energy, conservation, and general energy principles for a system of particles. Several of the concepts introduced here are especially useful in the study (in Chapter 11) of Lagrange's general equations for arbitrary dynamical systems, and the development of the moment of momentum principle for a system of particles provides a foundation for the independent presentation (in Chapter 10) of parallel results for the moment of momentum of a rigid body.

8.2. Equation of Motion for the Center of Mass

The total force $\mathbf{F}_k = \mathbf{F}(P_k, t)$ that acts on the k^{th} particle of a system $\beta = \{P_i\}$ of n particles consists of a *total external force* $\mathbf{f}_k = \mathbf{f}(P_k, t)$ exerted by bodies outside of β and a *total internal force* $\mathbf{b}_k = \mathbf{b}(P_k, t)$ due to the mutual interaction between P_k and all other particles in β. Let $\mathbf{b}_{kj}$ denote the mutual internal force exerted on the particle P_k by the particle P_j. Then the total internal force on P_k is

$$\mathbf{b}_k = \sum_{\substack{j=1 \\ j\neq k}}^{n} \mathbf{b}_{kj}. \tag{8.1}$$

Thus, the total force $\mathbf{F}(\beta, t) = \sum_{k=1}^{n} \mathbf{F}_k = \sum_{k=1}^{n} (\mathbf{f}_k + \mathbf{b}_k)$ acting on the system is

$$\mathbf{F}(\beta, t) = \sum_{k=1}^{n} \mathbf{f}_k + \sum_{k=1}^{n} \sum_{\substack{j=1 \\ j \neq k}}^{n} \mathbf{b}_{kj}. \tag{8.2}$$

In accordance with the third law, the internal forces occur in equal, oppositely directed pairs so that

$$\mathbf{b}_{jk} = -\mathbf{b}_{kj}; \tag{8.3}$$

and hence the total internal force, the last sum in (8.2), vanishes. *Therefore, the total force that acts on a system of particles is equal to the total external force*:

$$\mathbf{F}(\beta, t) = \sum_{k=1}^{n} \mathbf{f}_k. \tag{8.4}$$

We recall from (5.7) that the total momentum of a system of particles is equal to the momentum of its center of mass, and use of this result in (5.40) leads to the familiar classical form (5.41) of Newton's second law of motion for a system of particles in which only external forces (8.4) arise.

Newton's principle of motion for a system of particles: *The total external force on a system of particles is equal to the time rate of change of the momentum of the center of mass relative to an inertial frame* Φ, *and is thus equal to the product of the total mass of the system and the acceleration of the center of mass in* Φ:

$$\mathbf{F}(\beta, t) = \dot{\mathbf{p}}^{*}(\beta, t) = m(\beta)\mathbf{a}^{*}(\beta, t). \tag{8.5}$$

This equation aids in determination of the motion of the center of mass of the system and the external forces that act on it. In applications, however, the auxiliary center of mass relations (5.5) through (5.8), as well as the separate equations of motion of the particles, often are needed in problem solutions. Of course, the motion of an individual particle is governed by (5.39), which depends on the action of *all* forces that act on the particle, including internal forces that do not appear in (8.5). Without these auxiliary equations, (8.5) alone may not be very helpful. Plainly, all of the principles of mechanics for a single particle apply directly to the unique center of mass particle of a system of particles subjected to only the total external force (8.4). The familiar principle of conservation of momentum (7.69), for example, may be read immediately from (8.5), as follows below. Afterwards, an important application of the principle illustrates the need for the aforementioned auxiliary equations for the center of mass and Newton's law for a particle.

The principle of conservation of momentum of a system of particles: *The total external force component in a fixed direction* $\mathbf{e}$ *vanishes for all time if and only if the corresponding component of the momentum of the center of mass is* *constant*:

$$\mathbf{F}(\beta, t) \cdot \mathbf{e} = 0 \Leftrightarrow \mathbf{p}^*(\beta, t) \cdot \mathbf{e} = \text{const.} \tag{8.6}$$

Thus, the momentum of the center of mass is a constant vector if and only if the total external force vanishes for all time, in which case the center of mass moves uniformly on a straight line, or, if at rest initially, it remains so.

8.3. The Two Body Problem

Our study in the last chapter of the central gravitational attraction of a body in its orbital motion about another body assumed *fixed* in an inertial reference frame led to proof of Kepler's empirical laws on the elliptical path and orbital period of the attracted body. Here we study the related classical problem of the *relative motions* of two celestial bodies due only to their mutual gravitational attraction. This so-called *two body problem* models the relative motions of two astronomical bodies like the Earth and the Moon, the Sun and a planet, or a double star, for example, all of whose mutual distances of separation are so great that a pair of these huge celestial objects may be treated as two particles remote from the gravitational influence of all other bodies.

Consider two bodies of mass m_1 and m_2 subject only to their mutual gravitational force. These are internal forces for the system. Therefore, the total external force on the system vanishes in (8.5). It follows that the center of mass must have a uniform motion or be at rest in the astronomical inertial frame. (Although this is useful information, it does not address the relative motion of the particles.) As a consequence, we may choose an inertial frame $\Psi = \{C; \mathbf{I}_k\}$ with its origin at the center of mass, and now introduce the relative position vectors ρ_k of the two particles in Ψ. Then by (5.6), relative to the center of mass, we have

$$m_1\rho_1 + m_2\rho_2 = \mathbf{0}. \tag{8.7}$$

This shows that the particles must be situated along a line through C. Consider the motion of m_2 relative to m_1 described by

$$\mathbf{r} \equiv \rho_2 - \rho_1. \tag{8.8}$$

The motion of each particle relative to the center of mass is then obtained from (8.7) and (8.8) in terms of the relative motion vector $\mathbf{r}$ by

$$\rho_1 = -\frac{m_2}{m_1 + m_2}\mathbf{r}, \qquad \rho_2 = \frac{m_1}{m_1 + m_2}\mathbf{r}. \tag{8.9}$$

The equations of motion of the separate bodies in the inertial frame Ψ are

$$m_1\ddot{\rho}_1 = \frac{Gm_1m_2}{r^2}\mathbf{e}, \qquad m_2\ddot{\rho}_2 = -\frac{Gm_1m_2}{r^2}\mathbf{e}, \tag{8.10}$$

wherein

$$\mathbf{e} \equiv \frac{\mathbf{r}}{r} = \frac{\rho_2 - \rho_1}{|\rho_2 - \rho_1|}, \tag{8.11}$$

is a unit vector directed from m_1 toward m_2. The equation for the motion (8.8) of m_2 relative to m_1 is now provided by $\ddot{\mathbf{r}} = \ddot{\rho}_2 - \ddot{\rho}_1$ and (8.10); namely,

$$m_r \ddot{\mathbf{r}} = -\frac{Gm_1 m_2}{r^2}\mathbf{e} = -\frac{G(m_1 + m_2)m_r}{r^2}\mathbf{e}. \tag{8.12}$$

This is the equation of motion of a single "particle" having the *reduced mass* m_r defined by

$$m_r \equiv \frac{m_1 m_2}{m_1 + m_2}. \tag{8.13}$$

The problem of the relative motion of two bodies is thus transformed to an equivalent single body problem for which we need determine only the motion of a fictitious particle of mass m_r under the same central force experienced by the two bodies separated a distance r.

We recall (6.4) in which $\mathbf{a} = \ddot{\mathbf{r}}$ and $\mathbf{e}_r = \mathbf{e}$ to obtain from (8.12) the scalar equations of motion in cylindrical coordinates:

$$\ddot{r} - r\dot{\phi}^2 = -\frac{G(m_1 + m_2)}{r^2}, \qquad r^2\dot{\phi} = \gamma, \text{ a constant.} \tag{8.14}$$

Use of the second equation in the first and integration in r leads to an energy equation of the same form as (7.96a) in which now $\mu = G(m_1 + m_2)$ and $m = m_r$. The orbit analysis is then similar to what is presented at the end of the last chapter. With the solution of (8.14) in hand, the individual motions of the two bodies relative to the center of mass in Ψ may be obtained from (8.9). The solution of (8.14) thus suffices to determine their relative orbital motion. We shall omit these details.

An immediate effect of our accounting for the relative motions of the two bodies is that Kepler's third law (7.98e) for the orbital periodic time τ is changed by the modified factor μ:

$$\tau = 2\pi\sqrt{\frac{a^3}{\mu}} = 2\pi\sqrt{\frac{a^3}{G(m_1 + m_2)}}. \tag{8.15}$$

Therefore, the orbital period for the two body problem with a moving attractor is smaller than the Kepler period (7.98e) *for the single body problem with a fixed attractor; so, the orbital period is **not** the same for all orbits having the same semi-major axis.* The period varies from planet to planet due to the presence in (8.15) of the mass m_2 of the orbiting body. Of course, the same thing follows symmetrically for the body of mass m_1 in its motion relative to m_2. If the mass of either of the bodies is much greater than the other, like the mass of the Sun compared with that of the Earth and other planets in our solar system, for example, the deviations of the periodic times of the different planets discovered in the two body problem are small, and only then does Kepler's third law prevail. On the other hand, because the mass of Jupiter is roughly $1/1000$ of the mass of the Sun, the two body departure from Kepler's approximate law for the period of Jupiter is observable.

8.4. The Impulse–Momentum Equation

Integration of (8.5) with respect to time yields

$$\mathbf{p}^*(\beta, t) - \mathbf{p}^*(\beta, t_0) = \int_{t_0}^{t} \mathbf{F}(\beta, t)dt \equiv \mathscr{I}(t; t_0), \tag{8.16}$$

in which $\mathscr{I}(t; t_0)$ is called the *impulse of the external force* $\mathbf{F}(\beta, t)$ on the system. Thus, (8.16) provides in the usual notation the following rule.

The impulse–momentum equation for a system of particles: *The impulse of the external force on a system of particles over the time interval* $[t_0, t]$ *is equal to the change in the momentum of the center of mass during that time*:

$$\Delta\mathbf{p}^* = \mathscr{I}(t; t_0). \tag{8.17}$$

The change of momentum is always in the direction of the impulse. Since (8.17) is similar to rule (7.2), rules similar to (7.7) and (7.8) characterizing an instantaneous impulse hold also for the center of mass. Of course, any finite external force will contribute nothing to the instantaneous impulse; but other impulsive external reaction forces that might act on the system must be accounted for. Now, if the instantaneous impulse is due to equal, oppositely directed and collinear internal impulsive forces only and all external forces are finite, then at the impulsive instant $\lim_{\Delta t \to 0} \mathscr{I}(t; t_0) = \mathbf{0}$ and (8.17) yields $\Delta\mathbf{p}^* = \mathbf{0}$. *Therefore, in this case, for a system of particles subject to finite external forces, the instantaneous momentum of the center of mass is constant during the internal impulsive interval.* This is shown differently in (7.12) for a system of two particles on which the mutual instantaneous impulsive forces are equal, oppositely directed internal forces.

Example 8.1. A gun of mass M fires a shell S of mass m with a muzzle velocity $\mathbf{v}_{SG} = \mathbf{v}_0$ relative to the gun barrel G, at an elevation angle α in the ground frame $\Phi = \{F; \mathbf{I}_k\}$ in Fig. 8.1. The gun carriage C is mounted on a greased horizontal track. (a) Find the instantaneous recoil velocity $\mathbf{v}_{GF}$ of the gun (i.e. the center of mass of the gun assembly). (b) Compare the magnitude v of the instantaneous, absolute muzzle velocity $\mathbf{v}_{SF}$ of the shell in Φ with its relative value v_0. (c) What is the instantaneous impulsive reaction exerted by the track on the gun carriage? □

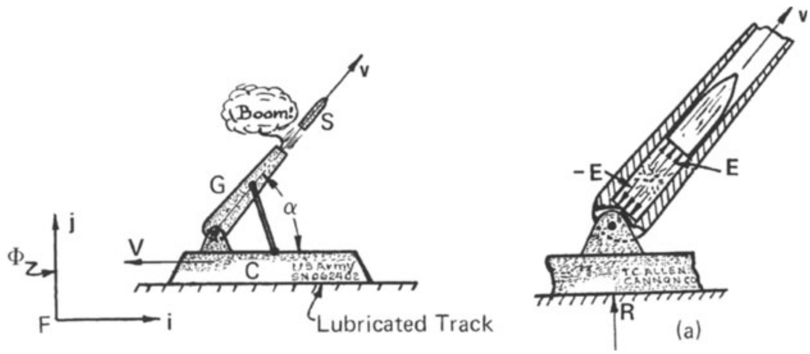


Figure 8.1. Momentum and impulse reaction in firing a gun.

Solution of (a). The gun and shell are modeled as center of mass objects—"a system of two particles." Then the instantaneous, absolute muzzle velocity of the shell S relative to the ground frame $\Phi = \{F; \mathbf{I}_k\}$ is $\mathbf{v}_{SF} = \mathbf{v}_{SG} + \mathbf{v}_{GF}$; that is,

$$\mathbf{v}_{SF} = \mathbf{v}_0 + \mathbf{v}_{GF} = (v_0 \cos\alpha - V)\mathbf{i} + v_0 \sin\alpha \mathbf{j}, \tag{8.18a}$$

wherein the instantaneous recoil velocity of the gun is $\mathbf{v}_{GF} = -V\mathbf{i}$. To find $\mathbf{v}_{GF}$, we observe that the only external forces that act on the system at the impulsive instant are vertical forces—the total weight of the system, the equipollent static track reaction force on the carriage, and the additional impulsive vertical reaction force $\mathbf{R}$ exerted on the gun carriage by the lubricated track. Therefore, in accordance with (8.6), the component $\mathbf{p}^* \cdot \mathbf{i}$ of the linear momentum of the center of mass, hence the system, is conserved. Initially, $\mathbf{p}^* = \mathbf{0}$; hence, after the impulse, the component $\mathbf{p}^* \cdot \mathbf{i}$ of the instantaneous momentum of the system must vanish in Φ:

$$\mathbf{p}^* \cdot \mathbf{i} = (M\mathbf{v}_{GF} + m\mathbf{v}_{SF}) \cdot \mathbf{i} = (-MV\mathbf{i} + m\mathbf{v}_{SF}) \cdot \mathbf{i} = 0. \tag{8.18b}$$

Equations (8.18a) and (8.18b) yield $-MV + m(v_0 \cos\alpha - V) = 0$; and hence the instantaneous recoil velocity of the gun is

$$\mathbf{v}_{GF} = -V\mathbf{i} = -\frac{m}{m+M} v_0 \cos\alpha \mathbf{i}. \tag{8.18c}$$

After the impulsive instant, additional forces exerted by a recoil spring and viscous damper, not shown in the diagram, retard the subsequent motion of the gun and restore it to its firing station. We shall not explore this motion.

Solution of (b). The instantaneous, absolute muzzle velocity $\mathbf{v}_{SF}$ of the shell follows from (8.18a):

$$\mathbf{v}_{SF} = v_0 \left(\frac{M}{m+M} \cos\alpha \mathbf{i} + \sin\alpha \mathbf{j} \right); \tag{8.18d}$$

and hence the absolute muzzle speed v is related to the relative speed v_0 by

$$v = v_0\left(1 - \frac{m(m+2M)}{(m+M)^2}\cos^2\alpha\right)^{1/2}. \tag{8.18e}$$

Inasmuch as $m \ll M$, to the first order in $\mu \equiv m/M$, $v = v_0(1-\mu\cos^2\alpha) < v_0$; the instantaneous, absolute muzzle speed of the shell in Φ is somewhat less than its muzzle speed v_0 relative to the gun.

Solution of (c). The instantaneous, external normal reaction impulse $\mathscr{T}^*_R$ exerted by the smooth track on the system is obtained from (8.17):

$$\mathscr{T}^*_R \equiv \lim_{t\to t_0}\int_{t_0}^{t}\mathbf{R}dt = \Delta\mathbf{p}^* = M\mathbf{v}_{GF} + m\mathbf{v}_{SF}. \tag{8.18f}$$

Since $\mathbf{R}\cdot\mathbf{i} = 0$, (8.18f) requires $\Delta\mathbf{p}^*\cdot\mathbf{i} = 0$, which is the same as (8.18b); and hence, by (8.18c) and (8.18d), the external impulsive reaction of the track on the gun is

$$\mathscr{T}^*_R = (\Delta\mathbf{p}^*\cdot\mathbf{j})\mathbf{j} = mv_0\sin\alpha\mathbf{j}. \tag{8.18g}$$

8.5. Moment of Momentum of a System of Particles

In preparation for the study of the moment of momentum principle for a system of particles, we next express the moment of momentum for the system in terms of the motion of its center of mass C. By (5.32), the moment about C of the momenta of the system relative to frame $\Phi = \{F; \mathbf{I}_k\}$ in Fig. 8.2 is given by

$$\mathbf{h}_C(\beta, t) = \sum_{k=1}^{n}\boldsymbol{\rho}_k \times m_k\dot{\mathbf{X}}_k, \tag{8.19}$$

wherein $\boldsymbol{\rho}_k$ is the position vector from C to the particle P_k and $m_k\dot{\mathbf{X}}_k$ is its momentum relative to the origin in Φ. It is useful to define the moment about C of the momenta of the system of particles relative to C in Φ, called *the moment of momentum relative to* C, in accordance with

$$\mathbf{h}_{rC}(\beta, t) \equiv \sum_{k=1}^{n}\boldsymbol{\rho}_k \times m_k\dot{\boldsymbol{\rho}}_k. \tag{8.20}$$

Here and in similar relations below, the subscript r is used for moments of momenta relative to points in Φ, but not with respect to the origin F in Φ, as emphasized in (8.19) and (8.20).

To relate $\mathbf{h}_{rC}$ to $\mathbf{h}_C$, introduce $\mathbf{X}_k = \mathbf{X}^* + \boldsymbol{\rho}_k$ in (8.19) to obtain

$$\mathbf{h}_C(\beta, t) = \sum_{k=1}^{n} m_k\boldsymbol{\rho}_k \times \dot{\mathbf{X}}^* + \sum_{k=1}^{n}\boldsymbol{\rho}_k \times m_k\dot{\boldsymbol{\rho}}_k. \tag{8.21}$$

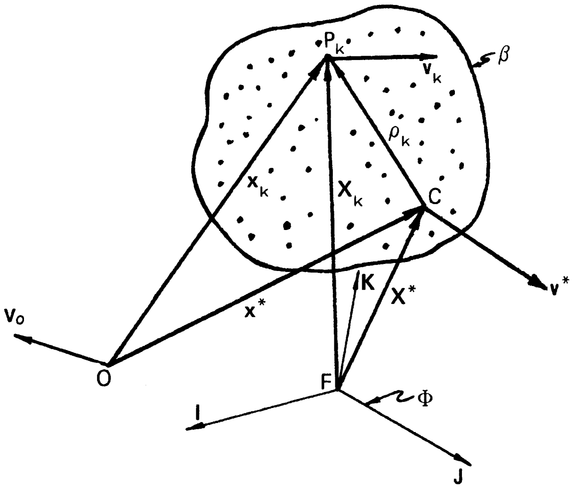


Figure 8.2. Schema for the moment of momentum of a system of particles.

By (5.6), however, the first product term vanishes, and we have with (8.20),

$$\mathbf{h}_C(\beta, t) = \sum_{k=1}^{n} \rho_k \times m_k \dot{\rho}_k = \mathbf{h}_{rC}(\beta, t). \tag{8.22}$$

In sum, *the moment about C of the momentum of the system relative to the origin in Φ is equal to the moment about C of the momentum of the system relative to C in Φ.*

The importance of this interesting property of the moment of momentum of a system of particles is revealed in the next section. The following exercises and subsequent example prepare the reader for development of principles presented there.

Exercise 8.1. The *moment of momentum relative to O in Φ* is defined by

$$\mathbf{h}_{rO}(\beta, t) \equiv \sum_{k=1}^{n} \mathbf{x}_k \times m_k \dot{\mathbf{x}}_k, \tag{8.23}$$

where $\mathbf{x}_k$ is the position vector of particle P_k from O in Fig. 8.2. This is the moment about a point O in Φ of the momenta of the system relative to O. Recall (5.32) for the moment of momentum $\mathbf{h}_O(\beta, t)$ of a system of particles about point O in Φ, introduce a velocity transformation, and show that

$$\mathbf{h}_O(\beta, t) = \mathbf{h}_{rO}(\beta, t) + m(\beta)\mathbf{x}_O^* \times \mathbf{v}_O, \tag{8.24}$$

in which $\mathbf{v}_O$ is the velocity of O in Φ and $\mathbf{x}_O^* = \mathbf{x}^*(\beta, t)$ is the position vector of C from O. Describe three cases for which $\mathbf{h}_O(\beta, t) = \mathbf{h}_{rO}(\beta, t)$. □

Exercise 8.2. Use the point transformation $\mathbf{x}_k = \mathbf{x}^* + \rho_k$ in Fig. 8.2 and show that the moment of momentum relative to a point O in Φ is related to the moment of momentum relative to the center of mass C in accordance with

$$\mathbf{h}_{rO}(\beta, t) = \mathbf{h}^*_{rO}(\beta, t) + \mathbf{h}_{rC}(\beta, t), \tag{8.25}$$

in which, by definition,

$$\mathbf{h}^*_{rO}(\beta, t) \equiv \mathbf{x}^*_O \times \mathbf{p}^*_O, \tag{8.26}$$

and $\mathbf{p}^*_O = m(\beta)\dot{\mathbf{x}}^*(\beta, t)$ is the momentum of the center of mass relative to O. The vector $\mathbf{h}^*_{rO}$, therefore, is the *moment of momentum of the center of mass relative to* O. Describe the content of (8.25) in words. □

Exercise 8.3. Introduce a velocity transformation in (5.32) to show that *the moment about point* O of *the momentum in* Φ *of a system of particles is equal to the moment about* O *of the momentum of the center of mass relative to the origin in* Φ *plus the moment of momentum relative to* C *in* Φ:

$$\mathbf{h}_O(\beta, t) = \mathbf{h}^*_O(\beta, t) + \mathbf{h}_{rC}(\beta, t), \tag{8.27}$$

where $\mathbf{h}^*_O \equiv \mathbf{x}^* \times m(\beta)\mathbf{v}^*$ with $\mathbf{v}^* = \dot{\mathbf{X}}^*$. Discuss the major difference between the moment of momentum vector $\mathbf{h}^*_O(\beta, t)$ and $\mathbf{h}^*_{rO}(\beta, t)$ in (8.26). Replacing O with C in (8.27) or (8.24), we recover (8.22). □

We conclude this extended review with an example demonstrating some calculations that include the application of (8.25).

Example 8.2. Two particles of mass $m_1 = m$ and $m_2 = 3m$ are moving with respective velocities $\mathbf{v}_1 = (4v, -7v, 0)$ and $\mathbf{v}_2 = (0, v, 4v)$ relative to frame $\Phi = \{F; \mathbf{i}_k\}$. (a) Find the velocity of the center of mass in Φ. (b) At a certain instant the respective particles are at $\mathbf{x}_1 = (0, -1, 3)$ and $\mathbf{x}_2 = (8, -1, 3)$ from a fixed point O in Φ. What is the moment about O of the momentum of the center of mass of the system in Φ? (c) What is the moment of momentum of C relative to O?

Solution of (a). To find $\mathbf{v}^*$, consider the momentum of the center of mass in $\Phi : \mathbf{p}^* = m(\beta)\mathbf{v}^*$, where $m(\beta) = m_1 + m_2 = 4m$. Then, by (5.7), the total momentum of the system is $\mathbf{p}^* = m_1\mathbf{v}_1 + m_2\mathbf{v}_2 = m(4v, -7v, 0) + 3m(0, v, 4v)$, i.e.

$$\mathbf{p}^* = 4m\mathbf{v}^* = 4mv(1, -1, 3); \tag{8.28a}$$

and hence the center of mass has velocity

$$\mathbf{v}^* = v(\mathbf{i} - \mathbf{j} + 3\mathbf{k}). \tag{8.28b}$$

Solution of (b). To determine $\mathbf{h}^*_O = \mathbf{x}^* \times \mathbf{p}^*$, the moment about point O of the momentum of the center of mass in Φ, we need to determine the position vector

$\mathbf{x}^*$ of the center of mass from O at the instant of interest. From (5.5), the reader will find that the center of mass is at the place from O given by

$$\mathbf{x}^* = 6\mathbf{i} - \mathbf{j} + 3\mathbf{k}; \tag{8.28c}$$

and with (8.28a) the moment about O of the momentum of C in Φ is

$$\mathbf{h}_O^* = \mathbf{x}^* \times \mathbf{p}^* = 4mv \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 6 & -1 & 3 \\ 1 & -1 & 3 \end{vmatrix} = -20mv(3\mathbf{j} + \mathbf{k}). \tag{8.28d}$$

Solution of (c). The moment about O of the momentum of C relative to the origin in Φ is given by $\mathbf{h}_O^* = \mathbf{x}^* \times m(\beta)\mathbf{v}^*$, in which $\mathbf{v}^* \equiv \dot{\mathbf{X}}^* = \mathbf{v}_O + \dot{\mathbf{x}}^*$ in Fig. 8.2. Because point O is fixed in Φ, $\mathbf{v}_O = \mathbf{0}$ and hence $\mathbf{p}^* = \mathbf{p}_O^*$. (In general, of course, $\mathbf{p}^* \neq \mathbf{p}_O^*$.) Then, by (8.26) and (8.28d), $\mathbf{h}_{rO}^* = \mathbf{h}_O^* = -20mv(3\mathbf{j} + \mathbf{k})$.

More generally, (8.24) holds for a system consisting of only a single particle, hence for the center of mass particle alone,

$$\mathbf{h}_O^* = \mathbf{h}_{rO}^* + m(\beta)\mathbf{x}_O^* \times \mathbf{v}_O. \tag{8.28e}$$

Therefore, when $\mathbf{v}_O = \mathbf{0}$, we have the special result $\mathbf{h}_O^* = \mathbf{h}_{rO}^*$ found in the example. Characterization of other situations for which this holds is left for the reader.

8.6. The Moment of Momentum Principle

We shall now prove that the moment of momentum principle (6.79) for a particle extends to a system of particles. First, differentiate (5.32) with respect to time in Φ to obtain

$$\frac{d\mathbf{h}_O(\beta, t)}{dt} = \sum_{k=1}^{n} (\dot{\mathbf{x}}_{Ok} \times \mathbf{p}_k + \mathbf{x}_{Ok} \times \dot{\mathbf{p}}_k), \tag{8.29}$$

in which* $\mathbf{p}_k = m_k\dot{\mathbf{X}}_k = m_k\mathbf{v}_k$ is the momentum relative to the origin in Φ of the particle P_k at the place $\mathbf{x}_{Ok} = \mathbf{x}_k$ from an arbitrary point O in Fig. 8.2. If O is fixed in Φ, then $\dot{\mathbf{x}}_{Ok} = \dot{\mathbf{X}}_k = \mathbf{v}_k$, and the first product term in (8.29) vanishes. The total force on P_k is $\mathbf{F}_k = \mathbf{f}_k + \mathbf{b}_k = \dot{\mathbf{p}}_k$; hence, by (8.29), *for O fixed in Φ*,

$$\frac{d\mathbf{h}_O(\beta, t)}{dt} = \sum_{k=1}^{n} \mathbf{x}_{Ok} \times \mathbf{F}_k. \tag{8.30}$$

The right-hand side of (8.30) is the total moment about a fixed point O of all forces that act on the system of particles. We accept that all internal forces (8.1)

* Unless explicitly stated otherwise, the summation convention on repeated indices is suspended throughout this chapter. Summation is explicitly indicated by a summation sign.

occur in equal, oppositely directed and collinear pairs; hence, by (8.3), the total moment of the internal forces about any point O whatsoever is zero:

$$\sum_{k=1}^{n} \mathbf{x}_{Ok} \times \mathbf{b}_k = \sum_{\substack{j=1 \\ j\neq k}}^{n} \sum_{k=1}^{n} \mathbf{x}_{Ok} \times \mathbf{b}_{kj} = \mathbf{0}. \tag{8.31}$$

The right-hand side of (8.30), therefore, reduces to the total moment about a fixed point O of only the external forces that act on the system, written as

$$\mathbf{M}_O(\beta, t) \equiv \sum_{k=1}^{n} \mathbf{x}_{Ok} \times \mathbf{f}_k. \tag{8.32}$$

From (8.30) and (8.32), we obtain the following important principle.

The moment of momentum principle for a system of particles: *The total moment of the external forces about a fixed point O in an inertial frame Φ is equal to the time rate of change of the moment of momentum of the system about O:*

$$\mathbf{M}_O(\beta, t) = \frac{d\mathbf{h}_O(\beta, t)}{dt}. \tag{8.33}$$

This yields the following easy supplementary rule.

The principle of conservation of moment of momentum of a system of particles: *The total torque of external forces about a fixed line with direction* $\mathbf{e}$ *through O may vanish in Φ if and only if the corresponding component of the moment of momentum of the system about O is constant*:

$$\mathbf{M}_O(\beta, t) \cdot \mathbf{e} = 0 \Leftrightarrow \mathbf{h}_O(\beta, t) \cdot \mathbf{e} = \text{const.} \tag{8.34}$$

Moreover, the moment about O of the external forces vanishes if and only if the moment of momentum of the system about O is a constant vector.

Integration of (8.33) with respect to time leads to the torque–impulse, moment of momentum relation for the moment about a fixed point O in the inertial frame Φ. The result is similar to (7.15), and its instantaneous form is similar to (7.17); so, these equations are not repeated here.

8.6.1. Moment of Momentum Principle for a Moving Reference Point

The moment of momentum law (8.33) holds only for an arbitrary *fixed point* O in an inertial frame Φ; but use of a moving reference point often is essential and more practicable. Therefore, we seek those circumstances for which the moment of momentum principle in the form (8.33) may hold for a moving reference point. First, consider the moment about a moving point Q of all forces that act on the system, and recall from (8.31) that the torque of the mutual internal forces about

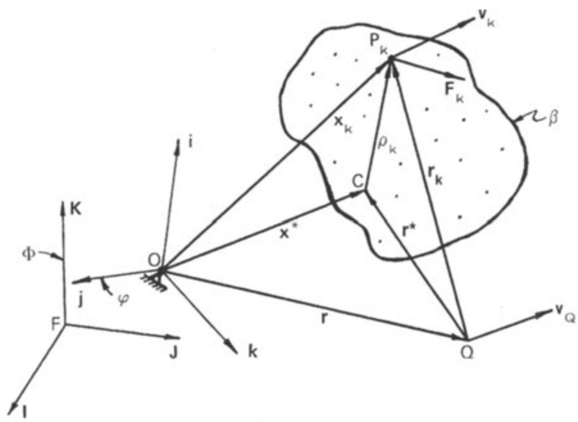


Figure 8.3. Schema for the moment of momentum about a moving reference point.

any point in Φ vanishes. Then using notation introduced in Fig. 8.3, we have

$$\mathbf{M}_Q = \sum_{k=1}^{n} \mathbf{r}_k \times \mathbf{F}_k = \sum_{k=1}^{n} \mathbf{r}_k \times \mathbf{f}_k = \sum_{k=1}^{n} \mathbf{r}_k \times \dot{\mathbf{p}}_k, \tag{8.35}$$

wherein† $\mathbf{F}_k = \mathbf{b}_k + \mathbf{f}_k = \dot{\mathbf{p}}_k$ and $\mathbf{r}_k$ is the vector of P_k from Q.

The moment about point Q of the momenta $\mathbf{p}_k = m_k\dot{\mathbf{x}}_k$ in Φ of all particles of the system is given by

$$\mathbf{h}_Q(\beta, t) = \sum_{k=1}^{n} \mathbf{r}_k \times \mathbf{p}_k, \tag{8.36}$$

and hence

$$\dot{\mathbf{h}}_Q = \sum_{k=1}^{n} \dot{\mathbf{r}}_k \times \mathbf{p}_k + \sum_{k=1}^{n} \mathbf{r}_k \times \dot{\mathbf{p}}_k. \tag{8.37}$$

Since $\mathbf{r}_k = \mathbf{x}_k - \mathbf{r}$ and $\dot{\mathbf{r}} = \mathbf{v}_Q$ in Fig. 8.3, the first product term yields

$$\sum_{k=1}^{n} \dot{\mathbf{r}}_k \times \mathbf{p}_k = -\mathbf{v}_Q \times \sum_{k=1}^{n} \mathbf{p}_k = -\mathbf{v}_Q \times \mathbf{p}^*.$$

Thus, with this relation in (8.37) and recalling (8.35), we obtain *the first form of the moment of momentum principle for a moving reference point Q*:

$$\mathbf{M}_Q(\beta, t) = \frac{d\mathbf{h}_Q(\beta, t)}{dt} + \mathbf{v}_Q \times \mathbf{p}^*. \tag{8.38}$$

Consequently, there exist moving points Q with respect to which $\mathbf{M}_Q(\beta, t) = \dot{\mathbf{h}}_Q(\beta, t)$ has the same form as (8.33) for a fixed point O, if and only if $\mathbf{v}_Q \times \mathbf{p}^* = \mathbf{0}$.

† Note that $\mathbf{X}_k = \mathbf{x}_k + \mathbf{R}$ is the vector of P_k in Φ and $\mathbf{R}$, not shown here, is the constant vector of the fixed point O from the origin F in Fig. 8.3. Hence, $\dot{\mathbf{X}}_k = \dot{\mathbf{x}}_k$, $\ddot{\mathbf{X}}_k = \ddot{\mathbf{x}}_k$ throughout these results. See also Fig. 8.2.

This holds when (i) trivially, either Q or the center of mass C is at rest in Φ, or (ii) when the velocity of Q is parallel to the velocity of the center of mass. In particular, this is so when Q is the center of mass. *Therefore, the moment about the center of mass of the external forces acting on a system of particles is equal to the time rate of change of the moment of momentum about the center of mass, which may be either at rest or moving arbitrarily in* Φ:

$$\mathbf{M}_C(\beta, t) = \frac{d\mathbf{h}_C(\beta, t)}{dt}. \tag{8.39}$$

This is the *first center of mass form of the moment of momentum principle.*

8.6.2. Second Form of the Moment of Momentum Principle for a Moving Point

Another formulation for the torque about a moving point Q, suggested by (8.22) and other results sketched in the previous section, is to express (8.38) in terms of *the moment of momentum relative to* Q, namely,

$$\mathbf{h}_{rQ}(\beta, t) = \sum_{k=1}^{n} \mathbf{r}_k \times m_k \dot{\mathbf{r}}_k, \tag{8.40}$$

in accordance with (8.23). Here $\mathbf{r}_k$ is the position vector of P_k from Q in Fig. 8.3. To relate (8.36) and (8.40), we recall (8.24) in which O is replaced by Q to obtain

$$\mathbf{h}_Q(\beta, t) = \mathbf{h}_{rQ}(\beta, t) + m(\beta)\mathbf{r}^* \times \mathbf{v}_Q, \tag{8.41}$$

wherein $\mathbf{r}^* = \mathbf{x}_Q^*$ is the position of the center of mass from Q.

Differentiation of (8.41) with respect to time in Φ gives

$$\dot{\mathbf{h}}_Q = \dot{\mathbf{h}}_{rQ} + m(\beta)\dot{\mathbf{r}}^* \times \mathbf{v}_Q + m(\beta)\mathbf{r}^* \times \mathbf{a}_Q,$$

in which $\mathbf{a}_Q = \dot{\mathbf{v}}_Q$ is the acceleration of Q in Φ. With $m(\beta)\dot{\mathbf{r}}^* = \mathbf{p}^* - m(\beta)\mathbf{v}_Q$ from Fig. 8.3, the last equation may be written as

$$\dot{\mathbf{h}}_Q + \mathbf{v}_Q \times \mathbf{p}^* = \dot{\mathbf{h}}_{rQ} + \mathbf{r}^* \times m(\beta)\mathbf{a}_Q. \tag{8.42}$$

Therefore, in place of (8.38), we find the *second form of the moment of momentum principle for a moving reference point* Q:

$$\mathbf{M}_Q(\beta, t) = \frac{d\mathbf{h}_{rQ}}{dt} + \mathbf{r}^* \times m(\beta)\mathbf{a}_Q. \tag{8.43}$$

Consequently, there exist points Q with respect to which $\mathbf{M}_Q(\beta, t) = \dot{\mathbf{h}}_{rQ}(\beta, t)$ has the same basic form (8.33) for a fixed point, if and only if $\mathbf{r}^* \times m(\beta)\mathbf{a}_Q = \mathbf{0}$. This occurs when (i) trivially, Q is either at rest or in uniform motion in Φ so that $\mathbf{a}_Q = \mathbf{0}$, in which case Q may be chosen as the origin of an inertial frame, (ii) the acceleration of Q is along a line passing through the center of mass so that $\mathbf{r}^*$ and $\mathbf{a}_Q$ are parallel vectors, or (iii) Q is the center of mass so that $\mathbf{r}^* = \mathbf{0}$, this being the most general of these situations for a moving reference point. *Therefore, the moment about the center of mass of the external* *forces acting on a system of particles is equal to the time rate of change of the moment of momentum relative to the center of mass, which may be either at ease, in uniform motion, or moving arbitrarily in* Φ:

$$\mathbf{M}_C(\beta, t) = \frac{d\mathbf{h}_{rC}(\beta, t)}{dt}. \tag{8.44}$$

This is the *second center of mass form of the moment of momentum principle.*

8.6.3. Summary: The Moment of Momentum Principle for a System of Particles

For an arbitrary moving reference point Q, the moment of momentum principle in (8.38) or (8.43) must be used. In view of (8.22), however, equations (8.39) and (8.44) are equivalent, and hence *the simplest formulation of the moment of momentum principle for a moving reference point is provided by the equation for the moving center of mass*:

$$\mathbf{M}_C(\beta, t) = \frac{d\mathbf{h}_C(\beta, t)}{dt} = \frac{d\mathbf{h}_{rC}(\beta, t)}{dt}. \tag{8.45}$$

Otherwise, for any point Q that either is fixed or has a uniform motion in an inertial frame Φ,

$$\mathbf{M}_Q(\beta, t) = d\mathbf{h}_{rQ}(\beta, t)/dt. \tag{8.46}$$

In the latter case, Q may be chosen as a fixed point at the origin of a new inertial frame Φ' with respect to which, trivially, $\dot{\mathbf{h}}_{rQ} = \dot{\mathbf{h}}_Q$.

The first law of motion (8.5) for a system of particles essentially determines the motion of the center of mass of the system, and the second law of motion (8.45) determines the motion of the system relative to the center of mass. In addition, however, we must bear in mind in applications that the moment of momentum about C may be referred to a moving frame φ having an angular velocity $\boldsymbol{\omega}_f$ relative to the inertial frame Φ. In this case (see (4.11) in Volume 1) $\mathbf{h}_C = \mathbf{h}_{rC}$ is a vector referred to a moving reference frame, and (8.45) is written as

$$\mathbf{M}_C(\beta, t) = \frac{\delta\mathbf{h}_C(\beta, t)}{\delta t} + \boldsymbol{\omega}_f \times \mathbf{h}_C(\beta, t). \tag{8.47}$$

Two examples that illustrate use of the results (8.22), (8.45), and (8.47) follow.

Example 8.3. A communications van has an antenna system modeled in Fig. 8.4 as two coils of equal mass m that move radially along a rigid control shaft that rotates with angular speed ω about the vertical antenna axis. At an instant of interest, each coil is at a distance d from the center C and is moving with center directed variable speed v relative to the shaft frame $\varphi = \{C; \mathbf{i}_k\}$. The van moves with speed v_O in the ground frame $\Phi = \{F; \mathbf{I}_k\}$. (i) What is the total momentum of the system? (ii) What is the moment of momentum of the system relative to the

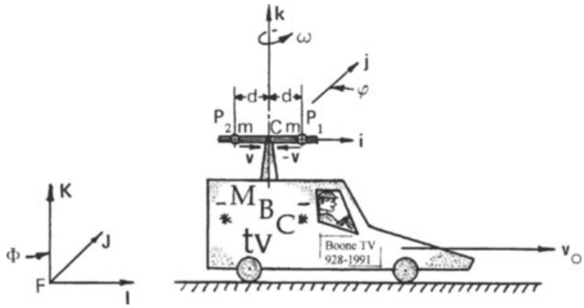


Figure 8.4. A two particle system model of moving antenna coils.

center of mass? (iii) Apply (8.19) to determine the moment of momentum about the center of mass. Ignore the mass of the shaft and model the coils as particles.

Solution of (i). The total momentum in Φ of the system of coils P_1 and P_2 is equal to the momentum of its center of mass whose velocity in Φ is $\mathbf{v}^* = \mathbf{v}_O = v_O\mathbf{I}$, the velocity of the van. Hence, $\mathbf{p}^* = m(\beta)\mathbf{v}^* = 2mv_O\mathbf{I}$.

Solution of (ii). The moment of momentum of the system relative to the center of mass is determined by (8.22). The respective relative position vectors of P_1 and P_2 from C are

$$\rho_1 = -\rho_2 = d\mathbf{i}. \tag{8.48a}$$

The angular velocity of the shaft frame φ is $\boldsymbol{\omega}_f = \omega\mathbf{k}$, and hence their velocity vectors relative to the center of mass at the instant of interest are

$$\dot{\rho}_1 = -\dot{\rho}_2 = -\mathbf{v} + \boldsymbol{\omega}_f \times \mathbf{d} = -v\mathbf{i} + \omega d\mathbf{j}. \tag{8.48b}$$

Use of (8.48a) and (8.48b) in (8.22) yields

$$\mathbf{h}_{rC} = \rho_1 \times m_1\dot{\rho}_1 + \rho_2 \times m_2\dot{\rho}_2 = 2m\rho_1 \times \dot{\rho}_1 = 2md\mathbf{i} \times (\omega d\mathbf{j} - v\mathbf{i}).$$

That is,

$$\mathbf{h}_{rC} = 2md^2\omega\mathbf{k}. \tag{8.48c}$$

Solution of (iii). To find the moment about C of the momentum relative to the origin in Φ, we shall need the total velocity of each particle in the inertial frame Φ, namely, $\dot{\mathbf{X}}_k = \mathbf{v}_O + \dot{\rho}_k$. Then, with (8.48a) and (8.48b), (8.19) gives

$$\mathbf{h}_C = \rho_1 \times m_1(\mathbf{v}_O + \dot{\rho}_1) + \rho_2 \times m_2(\mathbf{v}_O + \dot{\rho}_2) = 2\rho_1 \times m_1\dot{\rho}_1 = \mathbf{h}_{rC},$$

in agreement with the general rule (8.22): $\mathbf{h}_C = \mathbf{h}_{rC} = 2md^2\omega\mathbf{k}$. Notice in passing that the velocity of the center of mass $\mathbf{v}^* = \mathbf{v}_O$ does not affect the final result. In view of (5.6), the term $(m_1\rho_1 + m_2\rho_2) \times \mathbf{v}_O \equiv \mathbf{0}$. □

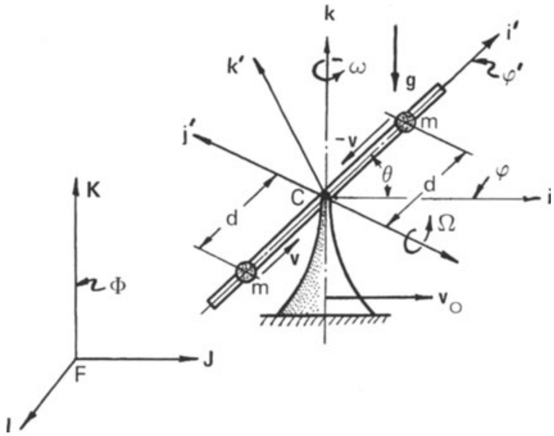


Figure 8.5. Moment of momentum of a system of antenna coils referred to a moving frame.

Example 8.4. The antenna coil system in the previous example has an additional angular velocity $\boldsymbol{\Omega}$ normal to the plane of $\mathbf{i}'$ and $\mathbf{k}$ in Fig. 8.5, and relative to its initially oriented shaft frame $1 = \varphi = \{C; \mathbf{i}_k\}$, which is turning with angular velocity $\boldsymbol{\omega}$ relative to the ground frame $0 = \Phi = \{F; \mathbf{I}_k\}$ at the instant shown. Find the applied torque about the center of mass required to sustain the motion of the system referred to the shaft frame $2 = \varphi' = \{C; \mathbf{i}'_k\}$. Ignore the mass of the control shaft.

Solution. The torque about C is given by (8.45), so we must first find $\mathbf{h}_C$ in (8.22) referred to the moving frame. The total angular velocity of the moving frame $2 = \varphi' = \left\{C; \mathbf{i}'_k\right\}$ fixed in the control shaft is $\boldsymbol{\omega}_f \equiv \boldsymbol{\omega}_{20} = \boldsymbol{\omega}_{21} + \boldsymbol{\omega}_{10} = \boldsymbol{\Omega} + \boldsymbol{\omega}$. Hence, with reference to Fig. 8.5, referred to φ',

$$\boldsymbol{\omega}_f = -\Omega\mathbf{j}' + \omega(\sin\theta\mathbf{i}' + \cos\theta\mathbf{k}'). \tag{8.49a}$$

The velocity of each coil relative to C is $\dot{\boldsymbol{\rho}}_k = (-1)^k\mathbf{v} + \boldsymbol{\omega}_f \times \boldsymbol{\rho}_k$, where we recall (8.48a) in which $\mathbf{i} \to \mathbf{i}'$. Specifically,

$$\dot{\boldsymbol{\rho}}_1 = -\dot{\boldsymbol{\rho}}_2 = -v\mathbf{i}' + \omega d\cos\theta\mathbf{j}' + \Omega d\mathbf{k}'. \tag{8.49b}$$

Then (8.22) yields

$$\mathbf{h}_C = 2\boldsymbol{\rho}_1 \times m\dot{\boldsymbol{\rho}}_1 = 2md^2(-\Omega\mathbf{j}' + \omega\cos\theta\mathbf{k}'). \tag{8.49c}$$

When $\theta = 0$ and $\Omega = 0$, we recover (8.48c).

The total torque about C required to sustain the motion of the system is determined by (8.45) for $\mathbf{h}_C$ given in (8.49c). But $\mathbf{h}_C$ is a vector referred to a moving reference frame, so we shall need to apply (8.47). With the aid of (8.49a), (8.49c), and noting that $\dot{d}(t) = -v(t)$ and $\dot{\theta} = \Omega$, we determine

$$\frac{\delta \mathbf{h}_C}{\delta t} = 2md[(2v\Omega - d\dot{\Omega})\mathbf{j}' + (d\dot{\omega}\cos\theta - d\omega\Omega\sin\theta - 2v\omega\cos\theta)\mathbf{k}'],$$

$$\boldsymbol{\omega}_f \times \mathbf{h}_C = 2md^2 \begin{vmatrix} \mathbf{i}' & \mathbf{j}' & \mathbf{k}' \\ \omega\sin\theta & -\Omega & \omega\cos\theta \\ 0 & -\Omega & \omega\cos\theta \end{vmatrix}$$
$$= 2md^2\,(-\omega^2 \sin\theta\cos\theta\mathbf{j}' - \Omega\omega\sin\theta\mathbf{k}').$$

Thus, by (8.47), the total moment about C of all external forces exerted on the coil system by the control shaft, by gravity, and by the drive mechanism is

$$\mathbf{M}_C = 2md[(2v\Omega - d\dot{\Omega} - d\omega^2\sin\theta\cos\theta)\mathbf{j}' + (d\dot{\omega}\cos\theta - 2d\Omega\omega\sin\theta - 2v\omega\cos\theta)\mathbf{k}']. \tag{8.49d}$$

When $\Omega = 0$ and $\theta = 0$, the applied torque required to sustain the motion of the system considered initially is $\mathbf{M}_C = 2md(d\dot{\omega} - 2v\omega)\mathbf{k}$, which also follows easily from (8.45) and (8.48c) wherein now $\mathbf{k}' = \mathbf{k}$ is fixed in Φ. □

8.7. Kinetic Energy of a System of Particles

The kinetic energy $K(\beta, t)$ of a system of particles $\beta = \{P_k\}$ in frame $\varphi = \{O; \mathbf{i}_k\}$ of Fig. 8.3 is defined as the sum of the kinetic energies $K_k(t) \equiv K(P_k, t)$ of particles P_k:

$$K(\beta, t) \equiv \sum_{k=1}^{n} K_k(t) = \sum_{k=1}^{n} \frac{1}{2} m_k \mathbf{v}_k \cdot \mathbf{v}_k, \tag{8.50}$$

where $\mathbf{v}_k = \dot{\mathbf{x}}_k$. With $m(\beta)$ defined by (5.3), *the kinetic energy $K^*(\beta, t)$ of the center of mass* is defined by

$$K^*(\beta, t) \equiv \frac{1}{2} m(\beta)\mathbf{v}^* \cdot \mathbf{v}^*, \tag{8.51}$$

wherein $\mathbf{v}^*(\beta, t) = \dot{\mathbf{x}}^*(\beta, t)$ is the velocity of the center of mass of the system in φ.

To relate (8.50) and (8.51), with reference to Fig. 8.3, substitute the relation $\mathbf{v}_k = \mathbf{v}^* + \dot{\boldsymbol{\rho}}_k$ in (8.50) and expand the result to obtain

$$K(\beta, t) = \frac{1}{2} m(\beta)\mathbf{v}^* \cdot \mathbf{v}^* + \mathbf{v}^* \cdot \sum_{k=1}^{n} m_k \dot{\boldsymbol{\rho}}_k + \sum_{k=1}^{n} \frac{1}{2} m_k \dot{\boldsymbol{\rho}}_k \cdot \dot{\boldsymbol{\rho}}_k.$$

Then, by (5.8), the second term vanishes and the last term is *the kinetic energy of the system relative to the center of mass C*, defined by

$$K_{rC}(\beta, t) \equiv \sum_{k=1}^{n} \frac{1}{2} m_k \dot{\boldsymbol{\rho}}_k \cdot \dot{\boldsymbol{\rho}}_k. \tag{8.52}$$

Hence, with (8.51), we obtain

$$K(\beta, t) = K^*(\beta, t) + K_{rC}(\beta, t). \tag{8.53}$$

That is, *the kinetic energy of a system of particles is equal to the kinetic energy of the center of mass plus the kinetic energy of the system relative to the center mass.*

Equations (8.51) and (8.52), therefore, are two independent kinetic energy relations for a system of particles, and (8.53) is the decomposition of the total kinetic energy (8.50) into these independent parts. In the two body problem, for example, the kinetic energy of the center of mass in the inertial frame Ψ at C is zero, and the reader will find from (8.52) that the kinetic energy relative to C is given by $K_{rC} = \frac{1}{2}m_r\dot{\mathbf{r}} \cdot \dot{\mathbf{r}} = \frac{1}{2}m_r(\dot{r}^2 + \gamma^2/r^2)$, which by (8.53) also is the total kinetic energy of the system in Ψ. This differs from the kinetic energy in (7.96a) for the one body problem. The results are approximately the same only when the mass of one body is much greater than that of the other, say $m_1 \gg m_2 = m$, so that by (8.13) $m_r \approx m$.

Example 8.5. Find the kinetic energy of the antenna system in Example 8.3, page 314.

Solution. The center of mass C of the two coil system has velocity $\mathbf{v}^* = \mathbf{v}_O$, and $m(\beta) = 2m$; so, by (8.51), the kinetic energy of the center of mass is

$$K^*(\beta, t) = mv_O^2. \tag{8.54a}$$

The velocity of each coil relative to C is given in (8.48b); therefore, by (8.52), the kinetic energy of the system relative to C is

$$K_{rC} = \frac{1}{2}m(\dot{\rho}_1 \cdot \dot{\rho}_1 + \dot{\rho}_2 \cdot \dot{\rho}_2) = m(v^2 + \omega^2 d^2). \tag{8.54b}$$

Finally, (8.53) yields the kinetic energy of the antenna coil system:

$$K(\beta, t) = m\big(v_O^2 + v^2 + \omega^2 d^2\big). \tag{8.54c}$$

The reader will find the same result on starting from (8.50). □

Exercise 8.4. What is the kinetic energy of the antenna coil system in Example 8.4? □

8.8. Work–Energy Equations for a System of Particles

The total external force acting on a system of particles may be considered to act on the center of mass particle whose motion is governed by (8.5) and from which a work–energy equation follows as a first integral. Let us think of the total external force $\mathbf{F}(\beta, t) = \mathbf{F}(\mathbf{x}^*)$ as varying only with the position of the center of

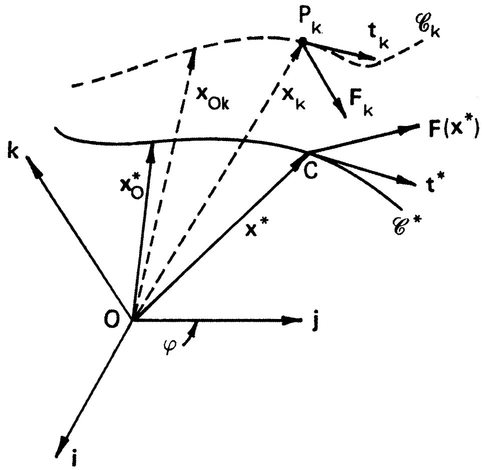


Figure 8.6. Schema for the work done by forces acting on a system of particles.

mass along its path $\mathscr{C}^*$ at time t in the inertial frame φ in Fig. 8.6. Then by (7.21) this force does work

$$\mathscr{W}^* \equiv \int_{\mathscr{C}^*} \mathbf{F}(\mathbf{x}^*) \cdot d\mathbf{x}^* = \int_{t_0}^{t} \mathbf{F}(\mathbf{x}^*) \cdot \mathbf{v}^* dt, \tag{8.55}$$

where t_0 and t are the instants when the center of mass is at its respective end states $\mathbf{x}_0^*$ and $\mathbf{x}^*$ on $\mathscr{C}^*$, $d\mathbf{x}^*$ is the elemental displacement vector tangent to $\mathscr{C}^*$, and $\mathbf{v}^* = \dot{\mathbf{x}}^*$. We recall (7.34) applied to the center of mass particle, use (8.5), and thus obtain from (8.55) the *work–energy equation for the center of mass*:

$$\mathscr{W}^* = \Delta K^*, \tag{8.56}$$

where $\Delta K^* = K^*(\beta, t) - K^*(\beta, t_0)$ is the change in the kinetic energy (8.51) of the center of mass during the time $[t_0, t]$. In sum, formally, *the work done by the total external force that acts at the center of mass of a system of particles is equal to the change in the kinetic energy of the center of mass.*

Moreover, similarly, by (7.37), (7.38), and (8.56), formally, *the mechanical power $\mathscr{P}^*$ expended by the total external force acting at the center of mass of a system of particles is equal to the time rate of change of the kinetic energy of the center of mass*:

$$\mathscr{P}^* \equiv \frac{d\mathscr{W}^*}{dt} = \frac{dK^*}{dt}. \tag{8.57}$$

The results (8.56) and (8.57) hinge on our writing the total external force as a function of the motion $\mathbf{x}^*$ of the center of mass particle in (8.55). Generally, however, this cannot be done. Nevertheless, in view of (8.5) and because these results are expressed in terms of the kinetic energy of the center of mass, they are meaningful—the work $\mathscr{W}^*$ and the power $\mathscr{P}^*$ are determined by the kinetic energy K^* of the unique center of mass particle. The work $\mathscr{W}^*$, however, is *not* the total work done on the system. Because the internal forces $\mathbf{b}_k$ act over paths $\mathscr{C}_k$ traversed by the individual particles P_k, these forces generally contribute to the work done by the total force $\mathbf{F}_k = \mathbf{f}_k + \mathbf{b}_k$ on P_k in Fig. 8.6. It is assumed that each of the forces depends on only the position $\mathbf{x}_k$ of the respective particle P_k. Hence, by (7.21), the work $\mathscr{W}_k$ done on the particle P_k in φ is

$$\mathscr{W}_k \equiv \int_{\mathscr{C}_k} \mathbf{F}_k \cdot d\mathbf{x}_k = \int_{t_0}^{t} (\mathbf{f}_k + \mathbf{b}_k) \cdot d\mathbf{x}_k = \Delta K_k, \tag{8.58}$$

wherein t_0 and t, respectively, are the instants when the particle P_k is at the end points $\mathbf{x}_{0k}$ and $\mathbf{x}_k$ on its path $\mathscr{C}_k$. Of course, all of the individual particle paths $\mathscr{C}_k$ and the path $\mathscr{C}^*$ in Fig. 8.6 generally are different; but the interval $[t_0, t]$ applies to the motion of every particle and to the center of mass of the system. Therefore, the total work $\mathscr{W} \equiv \Sigma_{k=1}^{n} \mathscr{W}_k$ done by all forces that act on the system is given by

$$\mathscr{W} \equiv \sum_{k=1}^{n} \int_{\mathscr{C}_k} \mathbf{F}_k \cdot d\mathbf{x}_k = \sum_{k=1}^{n} \int_{t_0}^{t} \mathbf{f}_k \cdot d\mathbf{x}_k + \sum_{k=1}^{n} \int_{t_0}^{t} \mathbf{b}_k \cdot d\mathbf{x}_k. \tag{8.59}$$

Recalling (8.50) for the total kinetic energy, with (8.58), we have *the work–energy equation for the system*:

$$\mathscr{W} = \Delta K, \tag{8.60}$$

where $\Delta K = \Delta \Sigma_{k=1}^{n} K_k$. *Hence, the total work done by all forces acting on a system of particles is equal to the change in the total kinetic energy of the system.*

Introduce $d\mathbf{x}_k = \mathbf{v}_k dt = (\mathbf{v}^* + \dot{\boldsymbol{\rho}}_k)dt$ in (8.59) to write the total work done as

$$\mathscr{W} = \int_{t_0}^{t} \sum_{k=1}^{n} \mathbf{F}_k \cdot d\mathbf{x}_k = \int_{t_0}^{t} \left(\sum_{k=1}^{n} \mathbf{F}_k \cdot \mathbf{v}^* + \sum_{k=1}^{n} \mathbf{F}_k \cdot \dot{\boldsymbol{\rho}}_k \right) dt.$$

In view of (8.4), the first term on the far right-hand side of this expression is equivalent to (8.55) for the total external force; therefore, with (8.56), (8.60) may be written as

$$\mathscr{W} - \mathscr{W}^* = \Delta(K - K^*) = \int_{t_0}^{t} \sum_{k=1}^{n} \mathbf{F}_k \cdot \dot{\boldsymbol{\rho}}_k dt. \tag{8.61}$$

Then, with (8.53) and K_{rC} in (8.52), *the work–energy equation relative to the center of mass* is

$$\mathscr{W}_{rC} \equiv \sum_{k=1}^{n} \int_{\mathscr{C}_k} \mathbf{F}_k \cdot d\boldsymbol{\rho}_k = \int_{t_0}^{t} \sum_{k=1}^{n} \mathbf{F}_k \cdot \dot{\boldsymbol{\rho}}_k dt = \Delta K_{rC}. \tag{8.62}$$

Therefore, the work $\mathscr{W}_{rC}$ done by all forces acting on a system of particles in motion relative to the center of mass is equal to the change in the total kinetic energy of the system relative to the center of mass.

To conclude, *the total work done by all forces acting on a system of particles is equal to the total of the work done by external forces acting at the center of mass* *and the work done by all forces in the motion relative to the center of mass*:

$$\mathscr{W} = \mathscr{W}^* + \mathscr{W}_{rC}. \tag{8.63}$$

Of course, $\mathscr{P}^*$ in (8.57) is not the total mechanical power expended; rather, the total power $\mathscr{P} = d\mathscr{W}/dt = dK/dt$ is easily seen to be $\mathscr{P} = \mathscr{P}^* + \mathscr{P}_{rC}$, in which $\mathscr{P}_{rC} = d\mathscr{W}_{rC}/dt = dK_{rC}/dt$.

Exercise 8.5. (a) Derive (8.56) as a formal first integral of (8.5). (b) Introduce the equation of motion for the k^{th} particle in the inertial frame φ, observe from Fig. 8.6 that $\mathbf{x}_k = \mathbf{x}^* + \boldsymbol{\rho}_k$, and thus confirm (8.62). □

Equations (8.56) and (8.62) are two independent work–energy equations for a system of particles; the first is related to the motion of the center of mass and is influenced by external forces only, whereas the second is related to the motion of the system relative to the center of mass and is influenced by both external and internal forces. The decomposition (8.63) of the total work (8.60) into these independent parts parallels the decomposition (8.53) of the total kinetic energy into corresponding independent parts; and, of course, the decomposition of the total mechanical power is similar.

A rigid system of particles is an important special case for which the distances between all pairs of particles are constant. Moreover, for a rigid system of particles, $\mathbf{v}_k = \mathbf{v}^* + \boldsymbol{\omega} \times \boldsymbol{\rho}_k$; and hence for mutual internal forces for which $\mathbf{b}_{ij} = -\mathbf{b}_{ji}$, use of (8.1) and (8.31), which holds for any point O and hence for C, yield

$$\sum_{k=1}^{n} \int_{\mathscr{C}_k} \mathbf{b}_k \cdot d\mathbf{x}_k = \int_{t_0}^{t} \left(\sum_{k=1}^{n} \mathbf{b}_k \cdot \mathbf{v}^* + \boldsymbol{\omega} \cdot \sum_{k=1}^{n} \boldsymbol{\rho}_k \times \mathbf{b}_k \right) dt = 0. \tag{8.64}$$

Therefore, mutual internal forces do no total work in any motion of a rigid system. *Consequently, for a rigid system of particles only the external forces contribute to the total work done.* Now, with (8.59), (8.60) may be written as

$$\mathscr{W} = \sum_{k=1}^{n} \int_{\mathscr{C}_k} \mathbf{f}_k \cdot d\mathbf{x}_k = \int_{t_0}^{t} \sum_{k=1}^{n} \mathbf{f}_k \cdot \dot{\mathbf{x}}_k dt = \Delta K. \tag{8.65}$$

Hence, the total work done by external forces acting on a rigid system of particles is equal to the change in the total kinetic energy of the system. Moreover, it follows from (8.62) that

$$\mathscr{W}_{rC} = \sum_{k=1}^{n} \int_{\mathscr{C}_k} \mathbf{f}_k \cdot d\boldsymbol{\rho}_k = \int_{t_0}^{t} \sum_{k=1}^{n} \mathbf{f}_k \cdot \dot{\boldsymbol{\rho}}_k dt = \Delta K_{rC}. \tag{8.66}$$

That is, *the work done by external forces acting on a rigid system of particles in motion relative to the center of mass is equal to the change in the total kinetic energy of the system relative to the center of mass.* Plainly, for a rigid system of

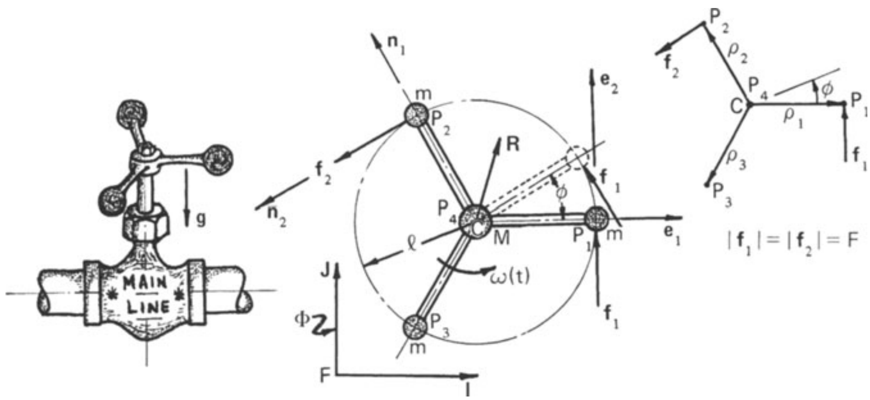


Figure 8.7. Forces acting on a rigid system of three particles.

particles, the total work done by external forces and the total kinetic energy in (8.65) may be decomposed in accordance with (8.63) and (8.53), respectively; and the total mechanical power expended, the rate of working of the external forces only, may be similarly decomposed.

Exercise 8.6. Show that (8.66) follows from (8.62). □

Example 8.6. A pipeline valve handle consists of three equally spaced handle grips of equal mass m attached to the valve body of mass M by thin rigid torque bars of equal length ℓ and negligible mass. The handle, initially at rest, is turned by forces $\mathbf{f}_1$ and $\mathbf{f}_2$ of equal constant magnitude F applied at two grips in the plane of, and perpendicular to the torque bars, as shown in Fig. 8.7. The handle turns freely without friction. (a) Apply the work–energy principle to find as a function of time the angular speed $\omega(t)$ of the handle. (b) Derive the same result by use of the moment of momentum principle.

Solution of (a). We model the handle as a rigid system of three particles of equal mass m attached by massless rigid rods to the valve body, which is a particle of mass M at C, the center of mass of the system. Since the handle turns freely without friction at C, the reaction force of the valve body is equipollent to a single force $\mathbf{R}$ at C that does no work in the motion; and, of course, the normal gravitational force also is workless. The work done by the remaining applied external forces, relative to the center of mass, is determined by the first equation in (8.66), and hence, with reference to Fig. 8.7,

$$\mathscr{W}_{rC} = \int_{\mathscr{C}_1} \mathbf{f}_1 \cdot d\boldsymbol{\rho}_1 + \int_{\mathscr{C}_2} \mathbf{f}_2 \cdot d\boldsymbol{\rho}_2 = \int_0^{\phi} F\mathbf{e}_2 \cdot \ell d\phi \mathbf{e}_2 + \int_0^{\phi} F\mathbf{n}_2 \cdot \ell d\phi \mathbf{n}_2.$$

Hence,

$$\mathscr{W}_{rC} = 2F\ell\phi. \tag{8.67a}$$

Notice that since C is fixed in Φ, $\mathscr{W}^* = 0$ in (8.55); hence, by (8.63), $\mathscr{W} = \mathscr{W}_{rC}$ also is the total work done on the rigid system.

Since each of the three handle particles has the same speed $|\dot{\rho}_1| = \ell\dot{\phi}$ relative to C, and because the system is at rest initially, the change in kinetic energy (8.52) relative to the center of mass is

$$\Delta K_{rC} = \frac{3}{2}m\ell^2\dot{\phi}^2. \tag{8.67b}$$

Use of (8.67a) and (8.67b) in the work–energy equation (8.66) thus yields

$$\omega(\phi) = \dot{\phi} = \sqrt{\frac{4F\phi}{3m\ell}}. \tag{8.67c}$$

To determine $\omega(t)$ as a function of time, we integrate this equation with $\phi(0) = 0$ initially to obtain the angular placement

$$\phi(t) = \alpha t^2, \qquad \text{with} \quad \alpha \equiv \frac{F}{3m\ell}. \tag{8.67d}$$

Now (8.67c) yields the desired result:

$$\omega(t) = 2\alpha t. \tag{8.67e}$$

Solution of (b). The same result may be obtained from the principle of moment of momentum relative to the center of mass in (8.45). First, we note that the applied torque about C is $\mathbf{M}_C = \rho_1 \times \mathbf{f}_1 + \rho_2 \times \mathbf{f}_2 = 2F\ell\mathbf{k}$. The moment about C of the momenta relative to C is $\mathbf{h}_{rC} = \rho_1 \times m_1\dot{\rho}_1 + \rho_2 \times m_2\dot{\rho}_2 + \rho_3 \times m_3\dot{\rho}_3 = 3m\ell^2\dot{\phi}\mathbf{k}$, and $d\mathbf{h}_{rC}/dt = 3m\ell^2\,\ddot{\phi}\,\mathbf{k}$. Hence, (8.45) yields $2F\ell\mathbf{k} = 3m\ell^2\,\ddot{\phi}\,\mathbf{k}$; that is,

$$\ddot{\phi} = \frac{2F}{3m\ell} = 2\alpha. \tag{8.67f}$$

Integration with respect to time, with $\dot{\phi}(0) = 0$ initially, returns (8.67e). □

Exercise 8.7. Determine the valve body reaction force $\mathbf{R}$ in the plane frame $\{\mathbf{P}_1; \mathbf{e}_k\}$. What is its magnitude? □

8.9. The Principle of Conservation of Energy

Suppose that for a system of particles the total external force (8.4) is conservative with potential energy $V^* = V^*(\mathbf{x}^*)$ depending only on the position $\mathbf{x}^*$ of the center of mass. Then the total external force is given by

$$\mathbf{F}(\beta, t) = -\boldsymbol{\nabla} V^*, \tag{8.68}$$

where $\boldsymbol{\nabla} \equiv \partial/\partial\mathbf{x}^*$, and hence (8.55) may be integrated to yield $\mathscr{W}^* = -\Delta V^*$. It then follows from (8.56) that

$$K^* + V^* = E^*, \text{ a constant.} \tag{8.69}$$

But this is a very weak and superficial principle of conservation of energy. In the first place, it applies only to a conservative **total** external force expressed as a function of $\mathbf{x}^*$, and it is unlikely that (8.4) will admit such a representation. Moreover, E^* is only the total energy of the center of mass, not the total energy of the system. Internal forces are not involved, and (8.69) suggests that the total energy of the center of mass may be conserved even in the presence of dissipative or other nonconservative internal forces. Of course, both workless external and internal forces also might be present. So, we discard (8.69) and seek a more substantial and meaningful principle of conservation of energy. If both the external and internal forces that act on every particle of a system are conservative, the system is called *conservative*; otherwise, it is called *nonconservative*. With this in mind, a useful general energy principle for a system of particles is developed.

8.9.1. External Potential Energy

Let $\phi_k(\mathbf{x}_k)$ be the *external potential energy* of the particle P_k due to the conservative external force $\mathbf{f}_k$. This potential function depends only on the position $\mathbf{x}_k$ of the particle P_k on which the force acts;

$$\mathbf{f}_k = -\boldsymbol{\nabla}_k \phi_k(\mathbf{x}_k), \qquad k = 1, 2, \ldots, n, \tag{8.70}$$

wherein $\boldsymbol{\nabla}_k \equiv \partial/\partial\mathbf{x}_k$. Thus, the total work done by a conservative system of external forces acting on all n particles of the system is

$$\sum_{k=1}^{n} \int_{t_0}^{t} \mathbf{f}_k \cdot d\mathbf{x}_k = -\sum_{k=1}^{n} \int_{t_0}^{t} \boldsymbol{\nabla}_k \phi_k \cdot d\mathbf{x}_k = -\Delta\Phi(\beta), \tag{8.71}$$

wherein each integrand is an exact differential $d\phi_k(\mathbf{x}_k)$, and by definition,

$$\Phi(\beta) \equiv \sum_{k=1}^{n} \phi_k(\mathbf{x}_k), \tag{8.72}$$

is the *total external potential energy of the system.* The function $\Phi(\beta) \equiv \Phi(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n)$ depends on the positions of all particles of the system on which external forces act along diverse particle paths having end points $\mathbf{x}_{Ok}$ and $\mathbf{x}_k$, the time interval $[t_0, t]$ being the same for all paths. Also, in (8.71), $\Delta\Phi(\beta) \equiv \sum_{k=1}^{n} \phi_k(\mathbf{x}_k) - \phi_k(\mathbf{x}_{Ok}) = \sum_{k=1}^{n} \Delta\phi_k$.

8.9.2. Internal Potential Energy

Let $\beta_{jk} = \psi(\mathbf{x}_j, \mathbf{x}_k)$ denote the *internal potential energy of a particle P_j due to a conservative mutual internal force $\mathbf{b}_{jk}$ exerted on P_j by P_k*. This function depends on the positions of both particles as emphasized by the subscripts, and $\psi(\mathbf{x}_j, \mathbf{x}_k)$ generally may be a different function for each pair of particles. In general, then, for a fixed k, the conservative internal force on P_j is given by

$$\mathbf{b}_{jk} = -\nabla_j \beta_{jk}, \tag{8.73}$$

where for each position $\mathbf{x}_j$, in Cartesian coordinates,

$$\nabla_j \equiv \partial/\partial \mathbf{x}_j = \sum_{p=1}^{3} \mathbf{i}_p \partial/\partial x_p^j. \tag{8.74}$$

It was proved in Section 5.18.3, page 81, that the mutual internal force $\mathbf{b}_{jk}$ between any pair of discrete material points P_j and P_k, which depends only on their distinct positions $\mathbf{x}_j$ and $\mathbf{x}_k$, has a magnitude that depends only on the distance $r = |\mathbf{r}_{jk}|$ between them and is directed along their common straight line. Therefore, for a conservative internal force, the mutual internal potential energy $\psi(\mathbf{x}_j, \mathbf{x}_k)$ for an arbitrary pair of interacting particles must be a function of only the magnitude of the vector $\mathbf{r}_{jk} \equiv \mathbf{x}_j - \mathbf{x}_k$ joining the two particles, namely, $\psi(\mathbf{x}_j, \mathbf{x}_k) = \psi(|\mathbf{r}_{jk}|)$. Bearing in mind that the function $\psi(|\mathbf{r}_{jk}|)$ generally may be a different function for each pair of particles, we have[‡]

$$\beta_{jk} = \psi(|\mathbf{r}_{jk}|). \tag{8.75}$$

Conversely, the reader will find that when the internal potential energy (8.75) of a pair of particles P_j and P_k is a function only of the distance $|\mathbf{r}_{jk}|$ between them, the mutual force $\mathbf{b}_{jk}$ must be parallel to $\mathbf{r}_{jk}$, hence directed along their mutual line. *Therefore, the internal potential energy for each pair of particles is a function of only the distance between the particles if and only if their mutual internal force is directed along their common line.* This kind of mutual action occurs in most interactions in nature, but not all. The general validity of this rule fails for molecular, atomic, electron, proton, or other elementary particle force interactions for which the internal potential energy does *not* depend on r alone.

Exercise 8.8. Show that $\nabla(\mathbf{x} \cdot \mathbf{x}) = 2\mathbf{x}$, and hence demonstrate that $\nabla(\sqrt{\mathbf{x} \cdot \mathbf{x}})$ is a unit vector parallel to $\mathbf{x}$. Apply this rule to prove the converse result stated above and confirm (8.77) and (8.78) below. Recall that $\nabla \equiv \partial/\partial \mathbf{x} = \sum_{p=1}^{3} \mathbf{i}_p \partial/\partial x_p$. □

It is evident that two particles P_j and P_k share the same internal potential energy, because the internal potential energy function is symmetric with respect

‡ An alternative but weaker proof of (8.75) that does not appeal to frame indifference is given in Appendix C.

to an interchange of the particles, namely, $\psi(|\mathbf{r}_{jk}|) = \psi(|\mathbf{r}_{kj}|)$, and hence

$$\beta_{jk} = \beta_{kj}. \tag{8.76}$$

By (8.73), $\mathbf{b}_{jk} = -(\partial\beta_{jk}/\partial r)(\mathbf{x}_j - \mathbf{x}_k)/r$, where $r = |\mathbf{r}_{jk}|$; and with (8.76) it follows that

$$\mathbf{b}_{jk} = -\frac{\partial\beta_{jk}}{\partial\mathbf{x}_j} = \frac{\partial\beta_{kj}}{\partial\mathbf{x}_k} = -\mathbf{b}_{kj}, \tag{8.77}$$

which is a statement of the third law in (8.3). It is also useful to observe that

$$\frac{\partial\beta_{jk}}{\partial\mathbf{x}_j} = \frac{\partial\beta_{jk}}{\partial\mathbf{r}_{jk}}. \tag{8.78}$$

The *total internal potential energy* $B(\beta)$ *of the system* is defined by

$$B(\beta) = \frac{1}{2}\sum_{\substack{j=1\\ j\neq k}}^{n}\sum_{k=1}^{n} \beta_{jk}(|\mathbf{r}_{jk}|), \tag{8.79}$$

in which $B(\beta) \equiv B(|\mathbf{r}_{12}|, |\mathbf{r}_{13}|, \ldots, |\mathbf{r}_{1n}|, |\mathbf{r}_{23}|, \ldots, |\mathbf{r}_{2n}|, |\mathbf{r}_{34}|, \ldots, |\mathbf{r}_{(n-1)n}|)$ is a function of the mutual distances between all pairs of particles, and the factor $1/2$ reflects the symmetry in (8.76). Consider, for example, a system of two particles. With (8.76), (8.79) yields the total internal potential energy $B(\beta) = \frac{1}{2}(\beta_{12} + \beta_{21}) = \beta_{12} \overset{\text{or}}{=} \beta_{21}$. And for a system of three particles, $B(\beta) = \beta_{12} + \beta_{13} + \beta_{23}$, each term of which is a function of the distance between the corresponding particles so that, for example, $\beta_{12} = \beta_{12}(|\mathbf{r}_{12}|)$. Therefore, (8.79) is the total of all of the internal potential energy functions.

We wish to relate the total internal potential energy to the work done by all of the conservative internal forces. By (8.1) and (8.73), the total conservative internal force due to the other $n-1$ particles acting on P_j is

$$\mathbf{b}_j = \sum_{\substack{k=1\\ k\neq j}}^{n} \mathbf{b}_{jk} = -\sum_{\substack{k=1\\ k\neq j}}^{n} \nabla_j\beta_{jk}(|\mathbf{r}_{jk}|). \tag{8.80}$$

Hence, with the aid of (8.77), the total work done by the conservative internal forces is determined by

$$\sum_{j=1}^{n}\int_{t_0}^{t} \mathbf{b}_j\cdot d\mathbf{x}_j = \sum_{\substack{j=1\\ j\neq k}}^{n}\sum_{k=1}^{n}\int_{t_0}^{t} \mathbf{b}_{jk}\cdot d\mathbf{x}_j = -\frac{1}{2}\sum_{\substack{j=1\\ j\neq k}}^{n}\sum_{k=1}^{n}\int_{t_0}^{t} \nabla_j\beta_{jk}\cdot d\mathbf{r}_{jk}. \tag{8.81}$$

To see this more clearly, it is best to write out several integrand terms of the sums in (8.81), and observe (8.1) and (8.77) to obtain

$$\begin{aligned}
\mathbf{b}_1\cdot d\mathbf{x}_1 + \mathbf{b}_2\cdot d\mathbf{x}_2 + \ldots &= (\mathbf{b}_{12} + \mathbf{b}_{13} + \ldots)\cdot d\mathbf{x}_1 + (\mathbf{b}_{21} + \mathbf{b}_{23} + \ldots)\cdot d\mathbf{x}_2 + \ldots \\
&= \mathbf{b}_{12}\cdot d(\mathbf{x}_1 - \mathbf{x}_2) + \mathbf{b}_{13}\cdot d(\mathbf{x}_1 - \mathbf{x}_3) \\
&\quad + \mathbf{b}_{23}\cdot d(\mathbf{x}_2 - \mathbf{x}_3) + \ldots \\
&= \mathbf{b}_{12}\cdot d\mathbf{r}_{12} + \mathbf{b}_{13}\cdot d\mathbf{r}_{13} + \mathbf{b}_{23}\cdot d\mathbf{r}_{23} + \ldots,
\end{aligned}$$

in which $\mathbf{r}_{jk} = \mathbf{x}_j - \mathbf{x}_k$. Introducing the internal potential energy from (8.73), we have

$$\sum_{j=1}^{n} \mathbf{b}_j \cdot d\mathbf{x}_j = \mathbf{b}_1 \cdot d\mathbf{x}_1 + \mathbf{b}_2 \cdot d\mathbf{x}_2 + \ldots$$
$$= -\boldsymbol{\nabla}_1 \beta_{12} \cdot d\mathbf{r}_{12} - \boldsymbol{\nabla}_1 \beta_{13} \cdot d\mathbf{r}_{13} - \boldsymbol{\nabla}_2 \beta_{23} \cdot d\mathbf{r}_{23} - \ldots.$$

Bearing in mind (8.77), it is seen that this is equivalent to the last sum in (8.81); indeed, consider, for example, the two terms $-\frac{1}{2}\boldsymbol{\nabla}_1 \beta_{12} \cdot d\mathbf{r}_{12} - \frac{1}{2}\boldsymbol{\nabla}_2 \beta_{21} \cdot d\mathbf{r}_{21} = -\boldsymbol{\nabla}_1 \beta_{12} \cdot d\mathbf{r}_{12}$. By (8.78), however, $\partial\beta_{12}/\partial\mathbf{x}_1 = \partial\beta_{12}/\partial\mathbf{r}_{12}$, $\partial\beta_{23}/\partial\mathbf{x}_2 = \partial\beta_{23}/\partial\mathbf{r}_{23}$, and so on. Since each $\beta_{jk} = \beta_{jk}(|\mathbf{r}_{jk}|)$ is a function of a single variable $\mathbf{r}_{jk}$, each term in the last sum above is an exact differential of the form $\nabla_1 \beta_{12} \cdot d\mathbf{r}_{12} = \partial\beta_{12}/\partial\mathbf{r}_{12} \cdot d\mathbf{r}_{12} = d\beta_{12}$, and so forth; therefore,

$$\sum_{j=1}^{n} \mathbf{b}_j \cdot d\mathbf{x}_j = -d\beta_{12} - d\beta_{13} - d\beta_{23} - \ldots = -d(\beta_{12} + \beta_{13} + \beta_{23} + \ldots).$$

The term in parentheses, however, is just the total of all of the internal potential energy functions given by (8.79). We thus obtain

$$\sum_{j=1}^{n} \int_{t_0}^{t} \mathbf{b}_j \cdot d\mathbf{x}_j = -\int_{t_0}^{t} d\left(\frac{1}{2}\sum_{j=1}^{n}\sum_{\substack{k=1\\k\neq j}}^{n} \beta_{jk}(|\mathbf{r}_{jk}|)\right) = -\int_{t_0}^{t} dB(\beta). \quad (8.82)$$

In sum, *the total work done by conservative internal forces is equal to the decrease in the total internal potential energy of the system*:

$$\sum_{k=1}^{n} \int_{t_0}^{t} \mathbf{b}_k \cdot d\mathbf{x}_k = -\,\Delta B(\beta). \quad (8.83)$$

The dummy summation index in (8.82) is here replaced by k for convenience below.

8.9.3. Total Energy of the System

Let $V(\beta) = V(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n)$ denote *the total potential energy of the system* defined by

$$V(\beta) \equiv \Phi(\beta) + B(\beta). \quad (8.84)$$

Then, with the aid of (8.71) and (8.83) in (8.59), *the total work done by all conservative external and internal forces* is given by

$$\mathscr{W} = -\Delta\Phi(\beta) - \Delta B(\beta) = -\Delta V(\beta). \quad (8.85)$$

Hence, the total work done on a conservative system of particles is equal to the decrease in the total potential energy of the system.

Now, the work energy equation (8.60) holds for all kinds of force fields $\mathbf{F}_k$, conservative or not. Therefore, with (8.85), we have our final result.

The principle of conservation of energy for a system of particles: *If the external and internal forces that act on a system of particles are conservative, or otherwise do no work in a given motion of the system, the sum of the total kinetic energy and the total potential energy of the system of particles, or briefly the total energy of the system, is constant:*

$$K + V = E, \text{ a constant.} \tag{8.86}$$

If the system is rigid, the particles are constrained in their relative positions; so, the internal forces are workless, as shown in (8.64). *Therefore, the total potential energy for a rigid system of particles is equal to the total external potential energy of the system*: $V(\beta) = \Phi(\beta)$.

Finally, let us confirm that the conservative forces derive from their corresponding potential energy functions. We can show that the total force $\mathbf{F}_k$ acting on the k^{th} particle may be deduced from (8.84) in accordance with

$$\mathbf{F}_k = -\nabla_k V(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n) = -\nabla_k \Phi - \nabla_k B. \tag{8.87}$$

From (8.72) and (8.70), we have

$$\nabla_k \Phi = \nabla_k(\phi_1(\mathbf{x}_1) + \ldots + \phi_k(\mathbf{x}_k) + \ldots + \phi_n(\mathbf{x}_n)) = \nabla_k \phi_k(\mathbf{x}_k) = -\mathbf{f}_k. \tag{8.88}$$

Similarly, recalling (8.76), we find with (8.79) and (8.80)

$$\nabla_k B = \sum_{\substack{j=1 \\ k \neq j}}^{n} \nabla_k \beta_{kj} = -\mathbf{b}_k. \tag{8.89}$$

Hence, (8.87) becomes $\mathbf{F}_k = \mathbf{f}_k + \mathbf{b}_k$, the total force acting on P_k. Furthermore, because the total internal force vanishes, $\Sigma_{k=1}^{n} \nabla_k B = -\Sigma_{k=1}^{n} \mathbf{b}_k = \mathbf{0}$; and hence, from (8.87) and in agreement with (8.4), the total force

$$\mathbf{F}(\beta, t) = \sum_{k=1}^{n} \mathbf{F}_k = -\sum_{k=1}^{n} \nabla_k V(\beta) = -\sum_{k=1}^{n} \nabla_k \Phi = \sum_{k=1}^{n} \mathbf{f}_k. \tag{8.90}$$

Finally, it follows from (8.87) and Newton's law for a particle that *the separate equations of motion of the particles of a conservative system may be expressed in terms of the total potential energy function* (8.84) *for the system*:

$$m_k \ddot{\mathbf{x}}_k = -\nabla_k V(\beta), \qquad k = 1, 2, \ldots, n. \tag{8.91}$$

8.9.4. The General Energy Principle for a System of Particles

Let $\mathbf{F}_{Nk} = \mathbf{f}_{Nk} + \mathbf{b}_{Nk}$ denote the total of the nonconservative external and internal forces acting on the k^{th} particle of a system. Then *the total work done by* *all nonconservative forces* is defined by

$$\mathcal{W}_N \equiv \sum_{k=1}^{n} \int_{\mathcal{C}_k} \mathbf{F}_{Nk} \cdot d\mathbf{x}_k. \tag{8.92}$$

Hence, the total work done by all conservative and nonconservative forces is given by $\mathcal{W} = -\Delta V(\beta) + \mathcal{W}_N$, with total potential energy (8.84). Now the general work–energy principle (8.60) may be written in the following alternative form.

General energy principle for a system of particles: *The change in the total energy $\mathcal{E} \equiv K + V$ of a system of particles is equal to the total work done by the nonconservative forces that act on the system:*

$$\Delta\mathcal{E} = \mathcal{W}_N. \tag{8.93}$$

Consequently, the total energy is conserved when and only when the nonconservative forces do no total work in the motion, or they are absent.

For a nonconservative system, the total force on the k^{th} particle may be written as $\mathbf{F}_k = -\nabla_k V(\beta) + \mathbf{F}_{Nk}$, in terms of its total conservative and total nonconservative parts. *Therefore, the separate equations of motion of the particles of a nonconservative system may be derived from*

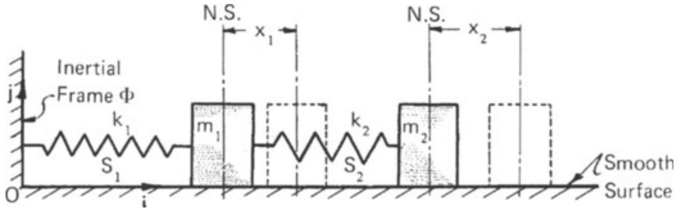
$$m_k\ddot{\mathbf{x}}_k = -\nabla_k V(\beta) + \mathbf{F}_{Nk}, \qquad k = 1, 2, \ldots, n. \tag{8.94}$$

Mutual nonconservative internal forces between a pair of particles must be equal but oppositely directed forces.

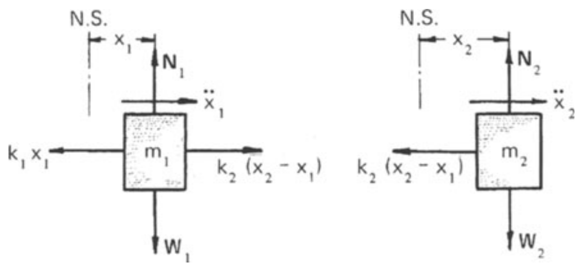
The following example illustrates some of the fine points and concepts encountered above. Afterwards, however, it will not be necessary to trace details of the foregoing construction leading to the general energy principle.

Example 8.7. A system shown in Fig. 8.8a consists of two blocks of mass m_1 and m_2 connected by a spring S_2 of stiffness k_2, while m_1 is fastened to a rigid wall by a spring S_1 of stiffness k_1. The system is displaced arbitrarily along its axis from its natural state and released to perform oscillations on a smooth horizontal surface. (i) Find the total energy of the system. (ii) Derive the differential equations of motion for the system. (iii) With reference to the free body diagram of each particle, derive each of the horizontal forces and their totals from their potential functions.

Solution of (i). We model the physical system in Fig. 8.8a as a system of two particles (center of mass objects) with mass m_1 and m_2, respectively. Their separate free body diagrams are shown in Fig. 8.8b. To find the total energy of the system $\beta = \{m_1, m_2\}$, we first note that the weights $\mathbf{W}_1$ and $\mathbf{W}_2$, and the normal, smooth surface reaction forces $\mathbf{N}_1$ and $\mathbf{N}_2$ are external forces that do no work in the motion, and the elastic spring forces are conservative. As usual, the infinitesimal



(a) The Physical System



(b) Free Body Diagrams

Figure 8.8. A spring-mass system modeled as a system of particles.

internal, mutual gravitational force between the particles is ignored. The system $\beta = \{m_1, m_2\}$, therefore, is conservative.

Let x_1 and x_2 denote the respective displacements of m_1 and m_2 from the natural state of the springs S_1 and S_2 in Fig. 8.8a. There are no constraints relating these variables, so the system has two degrees of freedom. The spring S_1 exerts an external force on m_1 with external potential energy ϕ_1; but no relevant external forces act on m_2, so $\phi_2 = 0$. The total external potential energy (8.72) is thus given by

$$\Phi = \phi_1 = \frac{1}{2}k_1 x_1^2. \tag{8.95a}$$

The mutual internal force on m_1 and m_2 is due to the spring S_2. These forces, shown in Fig. 8.8b, are equal but oppositely directed, so the total internal force is zero; but the total internal potential energy is not. The internal potential energy arising from the elastic force exerted on m_1 by m_2 is $\beta_{12} = \frac{1}{2}k_2(x_2 - x_1)^2$, and the internal potential energy arising from the equal but oppositely directed elastic force exerted on m_2 by m_1 is $\beta_{21} = \frac{1}{2}k_2(x_2 - x_1)^2$. Clearly, the mutual internal potential energy $\beta_{12} = \beta_{21}$ is a symmetric function of the change in distance, hence, also the current distance between the particles, as indicated in (8.75) and (8.76). The total internal potential energy (8.79) is thus determined by $B = \frac{1}{2}(\beta_{12} + \beta_{21})$, that is,

$$B = \beta_{12} = \frac{1}{2}k_2(x_2 - x_1)^2. \tag{8.95b}$$

Consequently, the total internal potential energy of the system in Fig. 8.8a is simply the elastic potential energy due to the spring S_2. Therefore, with (8.95a) and (8.95b), the total potential energy (8.84) of the system is

$$V(\beta) = \Phi + B = \frac{1}{2}k_1x_1^2 + \frac{1}{2}k_2(x_2 - x_1)^2. \tag{8.95c}$$

It is evident that the total potential energy function $V(\beta)$ may be written down immediately by inspection of the system in Fig. 8.8a. In fact, this is the usual procedure to follow.

The total kinetic energy (8.50) of the system in Fig. 8.8a is

$$K(\beta, t) = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2. \tag{8.95d}$$

The total energy of the conservative system $\beta = \{m_1, m_2\}$ now follows from (8.86):

$$K + V = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 + \frac{1}{2}k_1x_1^2 + \frac{1}{2}k_2(x_2 - x_1)^2 = E, \text{ a constant.} \tag{8.95e}$$

Solution of (ii). Application of the law of motion to each particle shown separately in the free body diagrams of Fig. 8.8b yields the following equations of motion for the system:

$$m_1\ddot{x}_1 = -k_1x_1 + k_2(x_2 - x_1), \qquad m_2\ddot{x}_2 = -k_2(x_2 - x_1). \tag{8.95f}$$

Notice that the sum of these equations is $m_1\ddot{x}_1 + m_2\ddot{x}_2 = -k_1x_1$, or $m(\beta)\ddot{x}^* = F(\beta)$, which is the equation of motion (8.5) of the system with total external force $F(\beta) = -k_1x_1$. This system equation is not useful. The motions x_1 and x_2 of the particles are coupled, but independent—the motion of one particle influences but does not determine the motion of the other. As a consequence, these equations cannot be separately integrated. The solution of coupled equations of this kind is considered in Chapter 11.

Solution of (iii). We next determine the horizontal forces and their totals from their potential functions. The total external force acting on each particle is determined by use of (8.95a) in (8.70):

$$\mathbf{f}_1 = -\boldsymbol{\nabla}_1\phi_1 = -\frac{\partial\phi_1}{\partial x_1}\mathbf{i} = -k_1x_1\mathbf{i}, \qquad \mathbf{f}_2 = -\boldsymbol{\nabla}_2\phi_2 = -\frac{\partial\phi_2}{\partial x_2}\mathbf{i} = \mathbf{0}. \tag{8.95g}$$

Hence, the total external force on the system, by (8.4), is $\mathbf{F}(\beta, t) = \mathbf{f}_1 + \mathbf{f}_2 = -k_1x_1\mathbf{i}$, noted earlier. All other forces that act must be the equal, oppositely directed external and internal contributions. Specifically, the equal and oppositely directed external forces are $\mathbf{N}_1 = -\mathbf{W}_1$, $\mathbf{N}_2 = -\mathbf{W}_2$; these have no influence on the motion. The total internal force acting on each particle is obtained by use of (8.95b) in (8.80):

$$\mathbf{b}_1 = -\nabla_1\beta_{12} = -\frac{\partial\beta_{12}}{\partial x_1}\mathbf{i} = k_2(x_2 - x_1)\mathbf{i},$$
$$\mathbf{b}_2 = -\nabla_2\beta_{21} = -\frac{\partial\beta_{21}}{\partial x_2}\mathbf{i} = -k_2(x_2 - x_1)\mathbf{i}. \tag{8.95h}$$

Of course, the total internal force acting on the system is $\mathbf{b}_1 + \mathbf{b}_2 = \mathbf{0}$. From (8.95g) and (8.95h), the total force $\mathbf{F}_k = \mathbf{f}_k + \mathbf{b}_k$ acting on each particle separately is

$$\mathbf{F}_1 = [-k_1x_1 + k_2(x_2 - x_1)]\mathbf{i}, \qquad \mathbf{F}_2 = -k_2(x_2 - x_1)\mathbf{i}. \tag{8.95i}$$

These are the totals of the forces on the right-hand side of the equations in (8.95f) and shown in the free body diagrams of Fig. 8.8b.

Finally, we apply equations (8.87), (8.89), and (8.90). Using (8.95c) in the first relation of (8.87), we have

$$\mathbf{F}_1 = -\nabla_1 V = [-k_1x_1 + k_2(x_2 - x_1)]\mathbf{i}, \qquad \mathbf{F}_2 = -\nabla_2 V = -k_2(x_2 - x_1)\mathbf{i}, \tag{8.95j}$$

in agreement with (8.95i). It is seen immediately that use of these relations in (8.91) returns the separate equations on motion (8.95f). With (8.95b) in (8.89), we find

$$\mathbf{b}_1 = -\nabla_1 B = k_2(x_2 - x_1)\mathbf{i}, \qquad \mathbf{b}_2 = -\nabla_2 B = -k_2(x_2 - x_1)\mathbf{i}, \tag{8.95k}$$

in accord with (8.95h). And, finally, substitution of (8.95j) into (8.90) yields

$$\mathbf{F}(\beta, t) = -\nabla_1 V - \nabla_2 V = -k_1x_1\mathbf{i}, \tag{8.95l}$$

the total external force on the system. □

This example illustrates the notation and several concepts introduced earlier in the development of the basic energy principles. Fortunately, in applications of the energy principles, it is not necessary to retrace these fine points. Consider the system in Fig. 8.8a and suppose, for example, that an additional mass m_3 is attached to the mass m_2 by a linear spring S_3 with stiffness k_3. We then have a new system of three collinear particles, and this new element introduces an additional internal potential energy function, which by inspection is given immediately by $\beta_{23} = \beta_{32} = \frac{1}{2}k_3(x_3 - x_2)^2$. This extra internal energy contribution is then added to (8.95c) to obtain the total potential energy for the system, and the total kinetic energy is increased by $\frac{1}{2}m_3\dot{x}_3^2$. Alternatively, suppose that the same spring is attached to m_2 at one end and to a rigid wall at the other. We now have a new system of two collinear particles with three spring elements. The new element, by inspection, introduces an additional external potential energy $\phi_3 = \frac{1}{2}k_3x_2^2$ in contribution

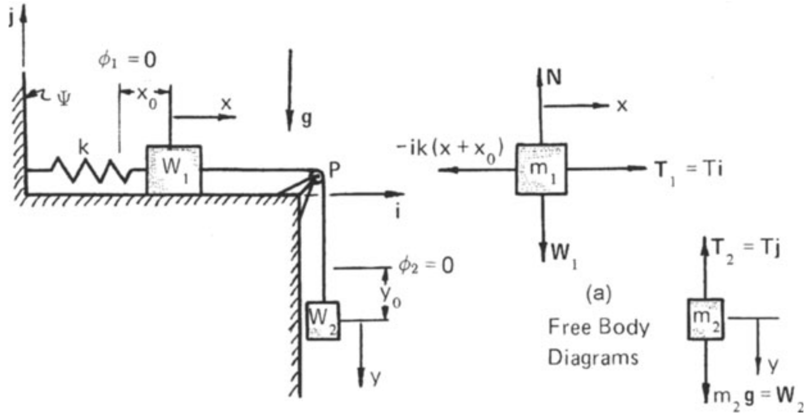


Figure 8.9. Gravity induced free vibrations of a system of two particles.

to (8.95c) for the total potential energy of our new system, and the total kinetic energy is given by (8.95d). Let the reader consider the forces that act on the free bodies involved in these additional models and write down the separate equations of motion for the particles of the respective systems. Try to derive the additional forces from their potential energy functions. Now let us turn to another example and apply the theory directly.

Example 8.8. Two small blocks of weight W_1 and W_2 are connected by a perfectly flexible and inextensible cable of negligible mass, as shown in Fig. 8.9. The weight W_1 rests on a smooth horizontal surface and is attached to a linear spring of stiffness k fastened to a rigid wall. The cable is free to slide over a smooth pulley at P and suspends the weight W_2. The system is at rest initially when W_2 is displaced vertically and released. (i) Find the total energy of the system. (ii) Derive the equation of motion and determine the frequency of the vibration. (iii) Describe alternative formulations of these issues.

Solution of (i). The physical system is modeled as a system of two particles of masses m_1 and m_2 for each of which the free body diagram is shown in Fig. 8.9a. The weight W_1 and the normal, smooth surface reactions at W_1 and at P do no work in the motion. The cable has negligible mass, so its motion around the smooth pulley may be ignored, and hence the oppositely directed, internal cable tensions $\mathbf{T}_1$ and $\mathbf{T}_2$ have equal magnitude T, say. The cable is inextensible and perfectly flexible, so the total internal potential energy of the system is zero: $B = 0$, and W_1 and W_2 share the same displacement so that $x = y$. In the equilibrium state, $x_0 = y_0$ is the static displacement of W_1 and W_2 so that $kx_0 = m_2 g$. The relevant external forces W_2 and the elastic spring force are conservative. The system, therefore, is conservative with total potential energy (8.84) equal to the total external potential energy $\Phi(\beta) = \phi_1 + \phi_2$ from (8.72), namely,

$$V(\beta) = \frac{1}{2}k(x + x_0)^2 - m_2 g(y + y_0), \tag{8.96a}$$

obtained directly by inspection of the system diagram. The total kinetic energy of the system, with the inextensibility constraint $x = y$ in mind, is

$$K(\beta, t) = K_1 + K_2 = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\dot{y}^2 = \frac{1}{2}m(\beta)\dot{x}^2, \tag{8.96b}$$

wherein $m(\beta) \equiv m_1 + m_2$. Hence, the principle of conservation of energy gives the constant total energy of the system:

$$E = K + V = \frac{1}{2}m(\beta)\dot{x}^2 + \frac{1}{2}k(x + x_0)^2 - m_2 g(x + x_0). \tag{8.96c}$$

Solution of (ii). Because the system has only one degree of freedom, the equation of motion may be found by differentiation of (8.96c) with respect to the path variable x (or the time t) to obtain

$$\ddot{x} + \frac{k}{m(\beta)}x = \frac{m_2 g - kx_0}{m(\beta)}. \tag{8.96d}$$

The system is in its static equilibrium state at $x = 0$ where $m_2 g - kx_0 = 0$, and hence the equation of motion for the system is

$$\ddot{x} + p^2 x = 0, \qquad p = \sqrt{\frac{k}{m_1 + m_2}}, \tag{8.96e}$$

in which p is the circular frequency.

Solution of (iii). Other methods will lead to the same results. Because the spring is linear, one alternative approach is to consider the motion from the static equilibrium position directly. The kinetic energy is unchanged in (8.96b), while the total potential energy may be written as

$$V(\beta) = \frac{1}{2}kx^2. \tag{8.96f}$$

This procedure, however, cannot be used when the spring is nonlinear, whereas the earlier method leading to (8.96a) can. The principle of conservation of energy (8.86) yields

$$\frac{1}{2}m(\beta)\dot{x}^2 + \frac{1}{2}kx^2 = E. \tag{8.96g}$$

Differentiation of this equation with respect to x (or t) returns (8.96e).

Another approach starts with the separate equations of motion for each particle. With reference to the free body diagrams in Fig. 8.9a, we find easily

$$m_1\ddot{x} = T - k(x + x_0), \qquad m_2\ddot{y} = m_2 g - T, \qquad W_1 = N. \tag{8.96h}$$

Eliminating T and introducing the inextensibility constraint $y = x$ and the equilibrium condition $m_2 g = kx_0$, we recover (8.96e). The equations of motion also may be formulated relative to the static state: $m_1\ddot{x} = T - kx$, $m_2\ddot{y} = -T$, which again lead to (8.96e).

8.10. An Application of the General Energy Principle

Suppose the cable in the previous example has elasticity characterized by a force-elongation equation $S = k_2\delta + k_3\delta^3$, in which k_2 and k_3 are constants and δ is the cable elongation measured from its natural unstretched state. To model this case, the vertical portion of the cable in Fig. 8.9 is replaced with a nonlinear spring characterized by S, the remaining part of the cable being inextensible and perfectly flexible. Of course, now $x \neq y$. In view of the nonlinear character of the cable spring, it is best to measure the respective particle displacements $X \equiv x + x_0$ and $Y \equiv y + y_0$ from the natural state of the springs, then the cable elongation $\delta = Y - X$. We wish to derive the energy equation of the system for $\delta \geq 0$. Afterwards, the equations of motion for the linear system with $k_3 = 0$ are described.

The total external potential energy is unchanged: $\Phi(\beta) = \frac{1}{2}kX^2 - m_2 gY$, and the total kinetic energy of the system is $K = \frac{1}{2}m_1\dot{X}^2 + \frac{1}{2}m_2\dot{Y}^2$. However, the total energy must include the internal elastic energy of the cable. We anticipate that the cable restoring force $\mathbf{F}_S = -S\mathbf{j}$ is conservative; but in the absence of its potential energy function, we may use the general energy principle (8.93) in which, with $\mathbf{F}_N = \mathbf{F}_S$ and $d\mathbf{x} = d\delta\mathbf{j}$,

$$\mathscr{W}_N = \int_{\mathscr{C}} \mathbf{F}_N \cdot d\mathbf{x} = \int_0^{Y-X} -(k_2\delta + k_3\delta^3)d\delta = -\frac{1}{2}k_2(Y-X)^2 - \frac{1}{4}k_3(Y-X)^4. \tag{8.97a}$$

Since the work done by the nonlinear spring force is path independent, this confirms that $\mathbf{F}_S$ is indeed conservative, and the negative of (8.97a) is the internal potential energy function of the nonlinear spring: $B(\beta) = \frac{1}{2}k_2(Y-X)^2 + \frac{1}{4}k_3(Y-X)^4$. With reference to the aforementioned modified model for which the system initially is at rest in its natural state, the general energy principle (8.93) yields

$$\frac{1}{2}m_1\dot{X}^2 + \frac{1}{2}m_2\dot{Y}^2 + \frac{1}{2}kX^2 + \frac{1}{2}k_2(Y-X)^2 + \frac{1}{4}k_3(Y-X)^4 - m_2 gY = 0. \tag{8.97b}$$

Because the system is conservative with total potential energy

$$V(\beta) = \frac{1}{2}kX^2 + \frac{1}{2}k_2(Y-X)^2 + \frac{1}{4}k_3(Y-X)^4 - m_2 gY, \tag{8.97c}$$

the separate equations of motion of m_1 and m_2 may be easily derived from (8.91). The reader will then find the static equilibrium relation $kx_0 = m_2 g = k_2\delta_0 + k_3\delta_0^3$,

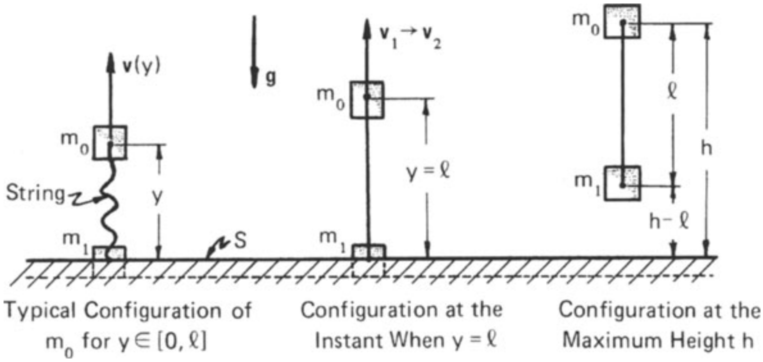


Figure 8.10. Application of the impulse–momentum and energy principles.

where $\delta_0 = y_0 - x_0$; but this offers no simplification of the equations by a coordinate shift to the static state. Now set $k_3 = 0$, recall that $X \equiv x + x_0$, $Y \equiv y + y_0$, and show that relative to the equilibrium state, the motion of the system is governed by the following coupled linear system of equations:

$$m_1\ddot{x} + (k + k_2)x - k_2 y = 0, \qquad m_2\ddot{y} + k_2 y - k_2 x = 0. \tag{8.97d}$$

8.11. An Application of the Energy and Impulse–Momentum Principles

Two blocks of masses m_0 and m_1 shown in Fig. 8.10 are connected by a flexible, inextensible string of length ℓ and negligible mass. The mass m_0 is projected vertically upward from the horizontal surface S with initial velocity $\mathbf{v}_0$. Apply the conservation of energy and instantaneous impulse–momentum principles to determine the maximum height attained by m_0 when its initial speed $v_0 > \sqrt{2g\ell}$.

Only the conservative gravitational force acts on m_0 prior to the impending impulse at $y = \ell$, while m_1 remains at ease at $y = 0$. Let v_1 denote the speed of m_0 at the instant just prior to the impulsive string reaction at $y = \ell$ (see Fig. 8.10); then the energy principle (8.86) gives

$$v_1^2 = v_0^2 - 2g\ell, \tag{8.98a}$$

where v_0 is the initial speed of m_0 at $y = 0$, the zero datum for the gravitational potential energy $V_g = m_0 g y$.

Henceforward, we suppose that $v_1 \neq 0$, i.e. $v_0 > \sqrt{2g\ell}$, and that the massless inextensible string does not break. The fully extended string at $y = \ell$ experiences an instantaneous internal impulse, so the instantaneous impulse of the total external force $\mathscr{I}^*(\beta) = \mathbf{0}$, finite forces contributing nothing. Therefore, by (8.17), $\Delta\mathbf{p}^* = \mathbf{0}$, i.e. the momentum of the system is constant during the impulsive instant: $m_0 v_1 \mathbf{i} = (m_0 + m_1) v_2 \mathbf{i}$. The impulse–momentum principle, with (8.98a), thus yields

$$v_2 = \frac{m_0}{m_0 + m_1} v_1 = \frac{m_0}{m_0 + m_1} \sqrt{v_0^2 - 2g\ell}. \tag{8.98b}$$

Now, after the impulsive instant, only the external conservative gravitational force acts on the system; therefore, the total energy of the system is conserved. Initially, at $y = \ell$, $E = \frac{1}{2}(m_0 + m_1)v_2^2 + m_0 g\ell$, and at the maximum height h where the system comes to rest, as shown in Fig. 8.10, $E = m_0 gh + m_1 g(h - \ell)$. Therefore, the principle of conservation of energy yields $h = \ell + v_2^2/2g$. Finally, use of (8.98b) leads to

$$h = \ell \left[1 - \left(\frac{m_0}{m_0 + m_1} \right)^2 \right] + \left(\frac{m_0}{m_0 + m_1} \right)^2 \frac{v_0^2}{2g}, \tag{8.98c}$$

for the greatest height attained by m_0, for $v_1 \neq 0$, i.e. $v_0 > \sqrt{2g\ell}$; otherwise, $h = \ell$.

8.12. Motion of a Chain on a Smooth Curve by the Energy Method

Let us consider the motion of a simple "deformable" body, a perfectly flexible and inextensible uniform chain, modeled as a contiguous system of particles. The chain has length $2l$ and mass ρ per unit length, and slides under gravity along a smooth, plane curved track $\mathscr{C}$ in the vertical plane, as shown in Fig. 8.11. The energy principle is applied to find the speed of the chain along the track. Then its motion along a cycloid is described.

Let s denote the arc length coordinate along $\mathscr{C}$ of the midpoint A of the chain from the origin O. Since $\mathscr{C}$ is smooth, only the gravitational force does work on the chain. The potential energy of the element of mass $dm = \rho d\sigma$ at the position $y(\sigma)$ in Fig. 8.11 is $dV = gy(\sigma)dm = \rho gy(\sigma)d\sigma$, wherein σ is the variable arc

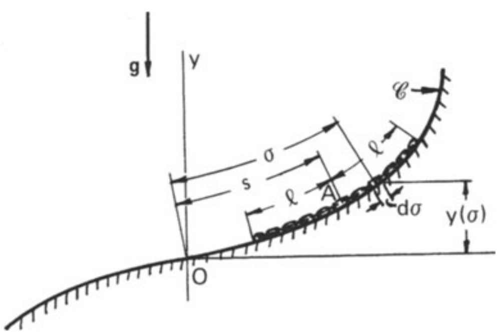


Figure 8.11. Motion of a chain on a smooth plane curve.

length parameter along the chain from O. Thus, the total potential energy of the contiguous system of chain elements is

$$V(s) = \rho g \int_{s-\ell}^{s+\ell} y(\sigma) d\sigma. \tag{8.99a}$$

The chain is inextensible, so all particles move with the same speed $\dot{s}$ along $\mathscr{C}$; hence, the total kinetic energy of the chain is

$$K = \int_{s-\ell}^{s+\ell} \frac{1}{2}\rho \dot{s}^2 d\sigma = \rho \ell \dot{s}^2. \tag{8.99b}$$

With (8.99a) and (8.99b), the energy principle (8.86) gives

$$\rho \ell \dot{s}^2 + \rho g \int_{s-\ell}^{s+\ell} y(\sigma) d\sigma = E. \tag{8.99c}$$

The constant E is determined from the assigned initial speed and position of A. The speed of the chain at any position s is thus determined by (8.99c) when the equation of the smooth track is specified.

The system's only degree of freedom is described by s. Hence, the equation of motion of the uniform chain on an arbitrary smooth curve in the vertical plane is obtained by differentiation of (8.99c) with respect to s. (See Problem 6.28, equation (P6.28c).) We thus find

$$\ddot{s} + \frac{g}{2\ell}(y(\sigma)|_{s+\ell} - y(\sigma)|_{s-\ell}) = 0. \tag{8.99d}$$

Now, suppose the track $\mathscr{C}$ is a cycloid defined by the parametric equations (7.94a), so that $y = 2a \sin^2(\beta/2)$, and recall that $\gamma = \beta/2$ in (7.94d). Then, in terms of the arc length coordinate σ of an arbitrary particle of the chain, we have

$$y(\sigma) = 2a \sin^2(\beta/2) = \frac{\sigma^2}{8a}. \tag{8.99e}$$

Hence, (8.99d) yields the equation of motion of a uniform chain on a smooth cycloid:

$$\ddot{s} + \frac{g}{4a} s = 0. \tag{8.99f}$$

Comparison of (8.99f) with (7.94e) reveals that the flexible, inextensible chain, regardless of its length and its mass, has the same motion as that of a single particle at its midpoint sliding under gravity along a smooth cycloid. The chain will perform simple harmonic oscillations with circular frequency $p = (g/4a)^{1/2}$ independent of the extent of its displacement from O. Consequently, the motion of the chain is governed by the motion of its midpoint, which from any initial position along the cycloid will reach the lowest point in the same quarter period time $t^* = \tau/4 = \pi\sqrt{a/g}$.

8.13. Law of Restitution

Consider the internal impulsive interaction of two bodies modeled as particles. We shall assume that all other forces remain finite during the impulsive instant and thus contribute nothing to the impulse. Then the total momentum of the center of mass, and hence of the system, is constant during the impulsive interval: $\Delta \mathbf{p}^* = \mathbf{0}$. Given their velocities $\mathbf{v}_k$ or momenta $\mathbf{p}_k$ immediately prior to the impulse, the problem of interest is to find the unknown velocities $\mathbf{v}'_k$ or momenta $\mathbf{p}'_k$ of each particle immediately after the impulse, altogether six unknowns. With this objective in mind, let $\mathbf{n}$ denote the direction of the instantaneous, mutual internal impulse, and let $\mathbf{i}_\alpha$ denote two suitably chosen orthogonal directions in the plane normal to $\mathbf{n}$. Then the momentum of the system in the direction of the impulse is conserved: $\Delta \mathbf{p}^* \cdot \mathbf{n} = (\mathbf{p}'^* - \mathbf{p}^*) \cdot \mathbf{n} = 0$; that is,

$$(\mathbf{p}'_1 + \mathbf{p}'_2) \cdot \mathbf{n} = (\mathbf{p}_1 + \mathbf{p}_2) \cdot \mathbf{n}, \tag{8.100}$$

and in the absence of all other forces during the impulsive instant, the components of the momentum of each particle and of the center of mass in the plane perpendicular to $\mathbf{n}$ must be conserved:

$$\mathbf{p}'_1 \cdot \mathbf{i}_\alpha = \mathbf{p}_1 \cdot \mathbf{i}_\alpha, \qquad \mathbf{p}'_2 \cdot \mathbf{i}_\alpha = \mathbf{p}_2 \cdot \mathbf{i}_\alpha, \qquad \alpha = 1, 2. \tag{8.101}$$

The addition of these components yields $(\mathbf{p}'_1 + \mathbf{p}'_2) \cdot \mathbf{i}_\alpha = (\mathbf{p}_1 + \mathbf{p}_2) \cdot \mathbf{i}_\alpha$ or $\mathbf{p}'^* \cdot \mathbf{i}_\alpha = \mathbf{p}^* \cdot \mathbf{i}_\alpha$, which together with (8.100) confirms that the momentum vector $\mathbf{p}^*$ of the center of mass is unchanged during an internal impulse, all other forces remaining finite. We now have a total of five equations for the six unknown momentum or velocity components. An additional equation relating the two unknown components $\mathbf{v}'_k \cdot \mathbf{n}$ to $\mathbf{v}_k \cdot \mathbf{n}$ is provided by the law of restitution introduced below. First, let us briefly consider some well-known effects of impact between real bodies.

When a ball strikes an essentially rigid wall, the ball actually becomes distorted during the impact, an effect that we have ignored for the sake of simplicity. If the ball is a highly elastic rubber ball, it will quickly recover its shape as it rebounds from the wall with almost no loss of energy. So this impact is perceived as a perfectly elastic collision. On the other hand, when two bodies collide, they generally suffer considerable distortion that is only partially recovered as the two bodies either separate or become so entangled that subsequent to the impact they continue in motion together. These are complex inelastic situations during which some portion of the distortional energy is expended in permanent deformation of the bodies, some is lost through heat and internal dissipation in creating their permanent distortion, and some through generation of substantial acoustic energy. Thus, during the impulsive instant the internal forces of deformation have done considerable work, some of which is recovered, most of which is not, at the expense of the total kinetic energy of the system. This loss of energy depends on the nature of the impact situation.

To model the impact of two colliding bodies, various kinds of impact situations, all ideal, are identified. Let P denote the point of contact of two colliding bodies. The line through the point P and perpendicular to the plane of contact tangent to both bodies at P is called the *line of action*. When the centers of mass of both bodies are situated on the line of action, the impact is called a *central impact*; otherwise, it is called an *eccentric impact*. In addition, if the initial velocities at the impulsive instant are parallel vectors, not necessarily collinear, the impact is called a *direct impact*; otherwise, the impact is called *oblique*. There are two kinds of direct impact. A direct impact for which the initial velocity vectors are directed along the line of action is called *collinear*; otherwise, it is called *noncollinear*. As a consequence, there are five general classes of impact: collinear and noncollinear direct central impact, an oblique central impact, a direct eccentric impact, and an oblique eccentric impact. Only the simplest models of direct central and oblique impact are studied here. Of course, in an impact situation one of the bodies may be at rest initially, and one may be so considerably more massive than the other that it suffers virtually no change in its state of rest.

The rapid deformation process in a real impact is just too complicated to describe in any analytical detail, and in any event this falls beyond the scope of our studies here. So, to avoid our getting into an awkward discussion of deformation of bodies and the time periods of their stress and recovery, we shall ignore all rotational, distortional, thermal, and acoustic phenomena that may occur in a real collision of bodies of finite size. For our purposes here, it is sufficient to suppose that the colliding bodies may be modeled as center of mass objects—two particles, pictured as small circular or spherical bodies, say, subject to an instantaneous internal impulse arising from a direct or oblique central impact only. To account for the energy lost in the impact, and to get around the impasse of complex deformation analysis, an empirical rule is introduced to model the restitution of the impacting bodies. With this in mind, let $\mathbf{v}_k$ and $\mathbf{v}'_k$ denote the respective instantaneous velocities of their centers of mass before and after the impact. Then $\mathbf{v}_2 - \mathbf{v}_1$ and $\mathbf{v}'_2 - \mathbf{v}'_1$ are the respective instantaneous relative velocities of approach and separation of their centers of mass before and after the impact. Let $\mathbf{n}$ be a unit vector along the line of action perpendicular to the plane of contact between the bodies. We shall assume that the impacting bodies are smooth, so that the impulsive force always is directed along $\mathbf{n}$. Now, to account for the energy loss in a direct or oblique central impact of two smooth bodies, we adopt the following empirical rule attributed to Newton. (See Cajori in the References.)

Law of restitution: *The normal component of the instantaneous relative velocity of separation of the centers of mass of two bodies after impact is proportional to the normal component of their instantaneous relative velocity of approach prior to impact; namely,*

$$(\mathbf{v}'_2 - \mathbf{v}'_1) \cdot \mathbf{n} = -e(\mathbf{v}_2 - \mathbf{v}_1) \cdot \mathbf{n} \tag{8.102}$$

in which the constant $e \in [0, 1]$ is called the coefficient of restitution.

A perfectly elastic impact for which the normal component of the relative velocity of separation is equal but oppositely directed to the normal component of the relative velocity of approach is described by $e = 1$. In the case when one body is fixed and the bodies are smooth, (8.101) and (8.102) show that the rebound velocity of the particle is the opposite of its approach velocity, so no kinetic energy is lost. A perfectly inelastic impact for which the relative velocity of separation vanishes because the bodies do not separate, and hence their normal velocities after impact are equal, is described by $e = 0$. Otherwise, the value of e, which must be the same for all observers, depends on the nature of the colliding bodies, i.e., their material, shape, size, surface roughness, and so on—it is a physical constant determined by experiment, a difficult task even for simple situations. More generally, however, it turns out that e may also depend on the impact velocities of the bodies, in which case the simple law of restitution (8.102) is no longer valid.

Consider a direct, collinear central impact (a head-on collision). Then the instantaneous velocities are along the line of action, so that $\mathbf{v}_k \cdot \mathbf{n} = v_k$, $\mathbf{v}'_k \cdot \mathbf{n} = v'_k$, and $\mathbf{v}_k \cdot \mathbf{i}_\alpha = \mathbf{v}'_k \cdot \mathbf{i}_\alpha = 0$. Hence, (8.101) is trivially satisfied, (8.100) becomes

$$m_1 v'_1 + m_2 v'_2 = m_1 v_1 + m_2 v_2 = (m_1 + m_2) v^*, \tag{8.103}$$

and by the law of restitution (8.102),

$$v'_2 - v'_1 = -e(v_2 - v_1). \tag{8.104}$$

In a perfectly inelastic head-on collision, $e = 0$ and hence the instantaneous velocities of the centers of mass following a direct, collinear central collision are the same: $v'_1 = v'_2$—i.e. the colliding bodies remain in contact following their impact. Moreover, (8.103) shows that their velocities are equal to the instantaneous velocity of the center of mass of the system: $v'_1 = v'_2 = v^* = (m_1 v_1 + m_2 v_2)/(m_1 + m_2)$. In a perfectly elastic impact, $e = 1$ and (8.104) becomes $v'_2 - v'_1 = -(v_2 - v_1)$; consequently, the instantaneous relative rebound velocity after the impact is equal but oppositely directed to the instantaneous relative velocity of approach.

More generally, solving (8.103) and (8.104) for the unknown instantaneous speeds v'_k in a direct, collinear central impact, we obtain

$$v'_1 = \frac{m_1 - e m_2}{m_1 + m_2} v_1 + (1 + e) \frac{m_2}{m_1 + m_2} v_2, \tag{8.105}$$

and

$$v'_2 = (1 + e) \frac{m_1}{m_1 + m_2} v_1 + \frac{m_2 - e m_1}{m_1 + m_2} v_2. \tag{8.106}$$

The direct, collinear central impact problem is now solved completely for all values of the coefficient of restitution. In particular, if the bodies have the same mass $m_1 = m_2 = m$, and the collision is perfectly elastic with $e = 1$, we obtain $v'_1 = v_2$ and $v'_2 = v_1$, which shows that in a perfectly elastic, direct collinear central impact, bodies having the same mass exchange their instantaneous impact velocities.

Exercise 8.9. Let K and K' denote the instantaneous total kinetic energy immediately before and after a direct collinear central impact of two center of mass objects. Consider the motion of the system relative to the center of mass so that (5.8) holds. Show that the respective kinetic energies relative to the center of mass are related by

$$K'_{rC} = e^2 K_{rC}. \tag{8.107}$$

Because $e^2 \leq 1$, $K'_{rC} \leq K_{rC}$, equality holding if and only if the impact is perfectly elastic with $e = 1$. In all other cases, kinetic energy is lost in the impact. The instantaneous loss of kinetic energy is given by

$$K - K' = (1 - e^2) K_{rC}. \tag{8.108}$$

There is no loss of kinetic energy when $e = 1$, and the greatest loss occurs when $e = 0$. The coefficient of restitution, therefore, is a measure of the loss of kinetic energy. □

References

1. Blanco, V. M., and Mccuskey, S. W., *Basic Physics of the Solar System,* Addison-Wesley, Reading Massachusetts, 1961. This is a concise treatment of the main physical and dynamical aspects of the solar system, including an introduction to the basic principles of celestial mechanics, written for scientists, engineers, and other nonspecialists with interests in space science. Celestial mechanics and the two body problem are introduced in Chapter 4. The three body problem and the general n-body problem are discussed in Chapter 5.
2. Cajori, F., *Newton's Principia.* English translation of *Mathematical Principles of Natural Philosophy* by Isaac Newton, 1687, University of California Press, Berkeley, 1947. The empirical principle of restitution is introduced in the Scholium (pp. 21–5) of Newton's laws of motion, as a consequence and in support of the third law. In primary work, Wallis, Wren, and Huygens, in the order of their priority according to Newton, "did severally determine the rules of the impact and reflection of hard bodies, and about the same time communicated their discoveries to the Royal Society, exactly agreeing among themselves as to those rules. But Sir Christopher Wren confirmed the truth of the thing before the Royal Society by the experiments on pendulums,... ." Newton addresses the effects of air resistance on impacting pendula, and further on states: "By the theory of Wren and Huygens, bodies absolutely hard return from one another with the same velocity with which they met. But this may be affirmed with more certainty of bodies perfectly elastic. In bodies imperfectly elastic the velocity of the return is to be diminished together with the elastic force; because that force is certain and determined, and makes the bodies to return one from the other with a relative velocity, which is in a given ratio to that relative velocity with which they met."
3. Housner, G. W., and Hudson, D. E., *Applied Mechanics,* Vol. II, *Dynamics,* 2nd Edition, Van-Nostrand, Princeton, New Jersey, 1959. Systems of particles are studied in Chapter 6 and the coefficient of restitution in Chapter 4. A few problems given below are modelled upon those provided in this text.
4. Marion, J. B., and Thornton, S. T., *Classical Dynamics of Particles and Systems,* 3rd Edition, Harcourt Brace Jovanovich, New York, 1988. Central force motion is investigated in Chapter 7. See also Chapter 10 of the earlier 2nd edition by Marion cited in the References to Chapter 6, page 197.

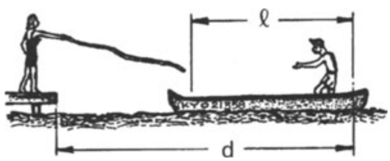
5. Meriam, J. L., and Kraige, L. G., *Engineering Mechanics*, Vol. 2 *Dynamics*, 3rd Edition, Wiley, New York, 1992. Direct and oblique central impact of smooth spheres are investigated in Chapter 3. Here the reader will find many additional examples and exercises for further study.
6. Shames, I. H., *Engineering Mechanics. Statics and Dynamics,* 4th Edition, Prentice-Hall, New Jersey, 1997. Additional examples and exercises on the central and oblique impact of particles are provided in Chapter 14, and problems on the eccentric impact of bodies by impulsive forces and torques (topics not treated herein) are discussed in Chapter 17.
7. Timoshenko, S., and Young, D. H., *Advanced Dynamics*, McGraw-Hill, New York, 1948. A classic, but non-vector treatment of applied topics in dynamics. A few problems provided in the present chapter are modelled upon examples found in Chapters 2 and 3 dealing with a system of particles.

Problems

8.1. Three particles with mass $m_1 = 5$ kg, $m_2 = 2$ kg, $m_3 = 4$ kg are initially located in $\Phi = \{F; \mathbf{i}_k\}$ at $\mathbf{X}_1 = 2\mathbf{i} + 3\mathbf{j} + 3\mathbf{k}$ m, $\mathbf{X}_2 = \mathbf{i} - 2\mathbf{k}$ m, $\mathbf{X}_3 = 2\mathbf{i} + \mathbf{k}$ m with initial velocities $\mathbf{v}_1 = 2\mathbf{i} + \mathbf{k}$ m/sec, $\mathbf{v}_2 = 3\mathbf{i} - \mathbf{j}$ m/sec, $\mathbf{v}_3 = -2\mathbf{j}$ m/sec. Determine for the initial instant (a) the location and velocity of the center of mass, (b) the momentum of the system, and (c) the total moment about F of the momenta of the system. (d) What is the total moment of momentum about a fixed point O at $\mathbf{X}_O = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ m in Φ? (e) Suppose the point O has velocity $\mathbf{v}_O = 2\mathbf{i} - 3\mathbf{j}$ m/sec. How does this affect the moment of momentum of the system about O?

8.2. The three particles described in the previous problem are acted upon by the respective external forces $\mathbf{F}_1 = 30\mathbf{i} - 6t\mathbf{j} + 45\mathbf{k}$ N, $\mathbf{F}_2 = 60t\mathbf{i} - 15\mathbf{k}$ N, $\mathbf{F}_3 = \mathbf{0}$. (a) Find the acceleration of the center of mass and determine its position in Φ after 2 sec. (b) What is the moment about F in Φ of the forces acting on the system at the initial instant?

8.3. A man of weight w is standing at the rear of a small boat of weight W. Initially, the man is adrift at a distance d from the pier, as shown. A friend tosses him a rope that is just long enough to reach the forward end of the boat, so the man moves forward a distance ℓ to improve his position. However, to their mutual surprise, the couple discovers that the rope does not reach the boat. What minimum additional length of rope is needed to reach the man at the forward end of the boat? In particular, what additional length is required when $W = w$? Neglect drag and current effects between the boat and the water.



Problem 8.3.

8.4. A bullet of mass m is fired with muzzle speed v_0 directed downward and inclined at an angle θ from a horizontal line perpendicular to the impact face of a large wooden block of mass M. The block is supported on smooth, rigid roller bearings at its base and by a series of recoil springs at the face opposite to the impact face. (a) Find the instantaneous impulsive force exerted on the bullet by the block. (b) What is the instantaneous impulsive reaction of the bearings on the block?

8.5. Two automobiles of masses m_A and m_B collide at an intersection in an oblique, direct central impact. The approach speed of car A was v_A directed at an angle θ north of east and that of car B was v_B directed south. After the collision, the entangled cars skidded together with unknown speed v directed at an angle ϕ north of east. (a) Determine the ratio v_B/v_A of their collision speeds. (b) It was found that $m_A = 5m_B/4$, $\theta = 30°$, and ϕ, though inaccurately measured, was smaller than θ. Which vehicle was traveling faster prior to the impact?

8.6. A system of four particles of equal mass m are situated at the ends of orthogonal cross-shaped rigid bars forming four spokes of length l and negligible mass. The system initially is rotating in the horizontal plane with constant angular velocity $\boldsymbol{\omega}_0 = \omega_0 \mathbf{k}$ when a constant torque $\mathbf{M} = M\mathbf{k}$ is suddenly applied. (a) Determine the new angular velocity of the system. (b) Suppose that $\mathbf{M}$ is reversed at $t = 0$. Determine the time required to bring the system to rest.

8.7. Two particles of equal mass m are symmetrically attached at a distance $L/2$ from the center O of a rigid rod of negligible mass and length $2L$. The rod is constrained by a bearing at O to rotate in the horizontal plane frame $\psi = \{O; \mathbf{i}, \mathbf{j}\}$ with a constant angular velocity $\boldsymbol{\omega} = \omega\mathbf{k}$. The particles are released simultaneously and move outward to the ends of the rod. (a) How is the angular speed of the system affected? (b) What will be the angular speed of the system if one of the particles fails to release? Find the bearing reaction torque required to sustain the plane motion.

8.8. A symmetrically balanced machine component consists of two particles of equal mass m attached to a massless rigid rod of length $l = 6r_0$ that rotates in the vertical plane about its central horizontal axle with constant counterclockwise angular velocity $\boldsymbol{\omega}$. The particles are released simultaneously at $r = r_0$ from the center of the rod at O, and a control mechanism then moves each particle toward its extreme end of the rod with the same speed $v(r)$ relative to the rod. (a) Determine the angular velocity and the angular acceleration of the system when each particle is at $r = 2r_0$ and $v(2r_0) = v_0$. (b) What is the angular speed when the particles reach their extreme positions at $3r_0$?

8.9. Two particles Q and S of equal mass m are attached to the ends of a rigid rod of negligible mass, and a third particle P of mass m is tied by an inextensible string to a point R of the rod at distances a and b from Q and S, respectively, with $a > b$. The system is at rest on a smooth horizontal surface when the particle P is projected in the horizontal plane with constant velocity $\mathbf{v}_0$ away from and perpendicular to the rod at R. Find the velocity of P immediately after the string becomes taut, and determine the angular speed of the rod. What are the results for the special case when $a = b$?

8.10. Two particles of mass m_1 and m_2 are connected by a massless rigid rod of length l initially situated on a smooth horizontal surface, along the **I**-axis of the plane frame $\Phi = \{O; \mathbf{I}, \mathbf{J}\}$ with m_1 at O. The system is given an initial angular velocity $\boldsymbol{\omega}_0 = \omega_0\mathbf{K}$ about O. (a) Find as functions of time the position $\mathbf{x}^*(t)$ of the center of mass C in Φ and the placement $\theta(t)$ of the rod. (b) Find as functions of time the moment of momentum about C and about O.

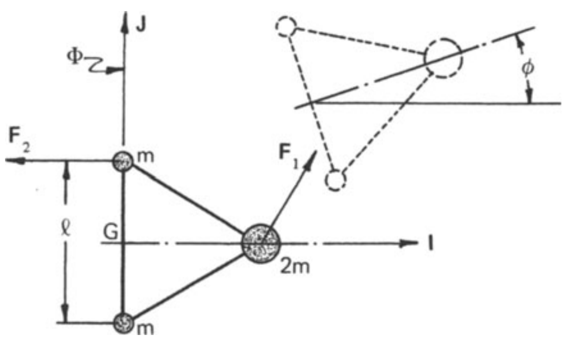
8.11. Two particles of mass m_1 and m_2 are connected by a massless rigid rod of length l initially situated along the **i**-axis of the vertical plane frame $\varphi = \{O; \mathbf{i}, \mathbf{j}\}$ with m_1 at O. The system is given an initial angular velocity $\boldsymbol{\omega}_0 = \omega_0\mathbf{k}$ about O. (a) Determine the position $\mathbf{x}^*(t)$ of the center of mass C in φ and the inclination $\theta(t)$ of the rod at time t. (b) Find as functions of time the moment of momentum about C and about O.

8.12. Three particles of mass m, $2m$, and $3m$ are moving with constant velocities $\mathbf{v}_1 = 2v\mathbf{i} + v\mathbf{j}$, $\mathbf{v}_2 = v\mathbf{k}$, and $\mathbf{v}_3 = v\mathbf{i} + 2v\mathbf{k}$, respectively, in $\Phi = \{O; \mathbf{i}_k\}$. Initially, the particles are at $\mathbf{X}_1 = \mathbf{j}+2\mathbf{k}$, $\mathbf{X}_2 = 3\mathbf{i} + \mathbf{k}$, and $\mathbf{X}_3 = 2\mathbf{i} + 3\mathbf{j}$ in Φ. (a) Find the motion of the center of mass, and determine its initial location in Φ. (b) Determine the momentum and the kinetic energy of the system. What is the kinetic energy relative to the center of mass? (c) Find the initial values of (i) the moment about O of the momentum of the center of mass, (ii) the moment about O

of the momentum of the system, and (iii) the moment of momentum relative to the center of mass.

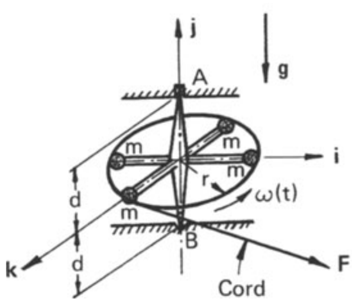
8.13. Two particles of equal mass m are connected by a rigid rod of length l and negligible mass, initially at rest along the X-axis on a smooth horizontal surface in frame $\Phi = \{O; \mathbf{i}, \mathbf{j}\}$. A constant force $\mathbf{F} = F\mathbf{j}$ acts on the particle at the right-hand end. Find the motion of the center of mass as a function of time; and find the angular speed of the rod as a function of its angular placement ϕ. Describe the motion of the rod.

8.14. A system of three particles of masses m, m, and $2m$ form the vertices of a massless rigid, equilateral triangular frame of side ℓ. The system, initially at rest in $\Phi = \{G; \mathbf{I}, \mathbf{J}\}$, moves in the horizontal plane under the action of follower forces $\mathbf{F}_1$ and $\mathbf{F}_2$ having equal constant magnitude F and directed always perpendicular to the sides of the triangle as shown in the figure. Find the angular speed $\dot{\phi}(t)$ of the triangular frame.



Problem 8.14.

8.15. A simple top consists of four particles of equal mass m spaced equally on a circular hoop of radius r. The rigid spokes, the axle, and the hoop have negligible mass, and the ball bearing at A and the ball-thrust bearing at B are frictionless. A cord of negligible mass is wound around the hoop as shown, and a constant tangential force $\mathbf{F}$ is applied for 2 sec. (a) Find the dynamic bearing reactions as functions of t. (b) Determine the angular velocity $\omega(t)$ of the system about the axle. (c) Find the angular speed when the cord is free.



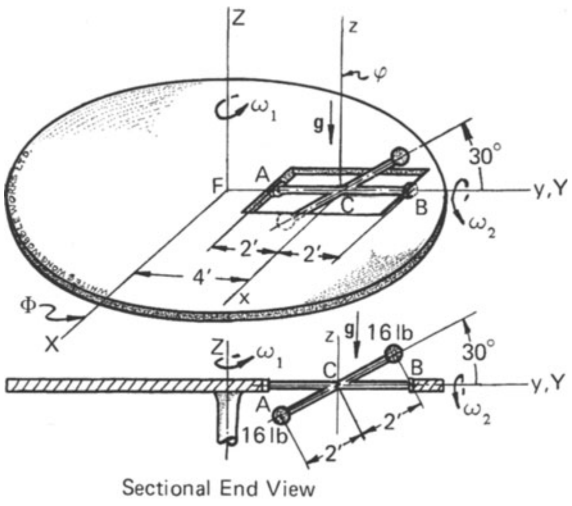
Problem 8.15.

8.16. Two particles of mass m_1 and m_2 are connected by an inextensible string of length l and negligible mass. The string passes through a small hole O in a smooth horizontal table. The mass m_1 moves on the table with cylindrical coordinates $(r(t), \phi(t))$, while m_2 hangs vertically below O. (a) Show that the differential equations of motion for the system are given by

$$\frac{d(m_1 r^2 \dot{\phi})}{dt} = 0, \qquad (m_1 + m_2)\ddot{r} - m_1 r \dot{\phi}^2 + m_2 g = 0. \tag{P8.16}$$

(b) Find the tension in the string as a function of r alone. (c) Equations (P8.16) show that a steady-state solution is possible for which m_2 is at rest while m_1 moves with a constant angular speed $\dot{\phi} = \omega$ on a circle of radius $r = a$, say. Find the steady-state values of a and ω. (d) Show that this dynamic equilibrium state is infinitesimally stable with respect to a small disturbance of m_1 for which $r = a + \eta$, where η and its derivatives are small quantities; in fact, establish that the system will execute small amplitude oscillations of frequency $f = (\omega/2\pi)[3m_1/(m_1 + m_2)]^{1/2}$.

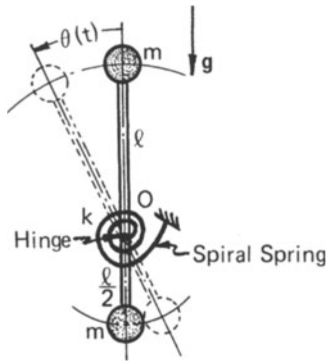
8.17. A wobble mechanism is modeled as a rigid system of two 16 lb balls attached symmetrically to the ends of a 4 ft rigid shaft welded at a 30° angle to a horizontal axle held in a roller bearing at A and restrained by an axial thrust bearing at B. The shaft–axle assembly rotates, as shown, with a constant angular speed $\omega_2 = 50$ rad/sec relative to a large circular control gear that has a constant angular speed $\omega_1 = 10$ rad/sec in a high elevation ground frame $\Phi = \{F; \mathbf{I}_k\}$ where $\mathbf{g} = -32\mathbf{k}$ ft/sec^2. At the instant of interest shown in the figure, the wobble shaft is in the vertical yz-plane. Find the dynamic bearing reactions at A and B at the moment of interest. Ignore the mass of the shaft–axle assembly.



Problem 8.17.

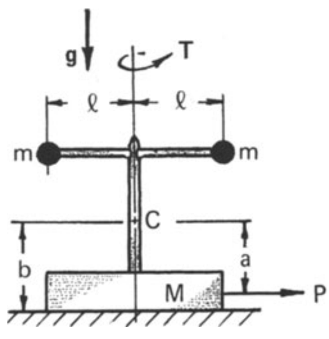
8.18. A pendulum rod of length l and negligible mass is hinged at O to a block of mass m_1 that slides freely on a smooth horizontal shaft parallel to the x-axis. The bob has mass m_2 and oscillates freely without friction in the vertical plane, suspended below m_1 with the placement $\theta(t)$. (a) Derive the coupled equations of motion for the system. (b) Show that when the variables x and θ together with their time derivatives are small quantities whose products are negligible, these equations reduce to two linear equations for x and θ, the latter being the equation of motion for a simple pendulum with period $\tau = 2\pi[m_1 l/g(m_1 + m_2)]^{1/2}$. (c) For small, but otherwise arbitrary initial data find the motions $x(t)$ and $\theta(t)$.

8.19. A pendulum shown in the figure consists of two bobs of equal mass m attached to the ends of a rigid rod of negligible mass. The rod is hinged at O to a torsion spring of stiffness k that supplies a restoring torque proportional to the angular placement θ from the vertical axis. (a) Derive the differential equation for the finite amplitude motion $\theta(t)$ of the system. (b) Determine the angular motion $\theta(t)$ for a small initial placement θ_0 of rest from the vertical position. (c) Discuss the infinitesimal stability of the equilibrium positions of the system.



Problem 8.19.

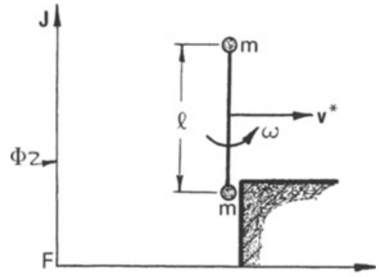
8.20. A block of mass M, subjected to a constant force $\mathbf{P} = P\mathbf{i}$ in the vertical plane at a distance a below the center of mass of the system at C, moves over a rough horizontal surface with coefficient of friction ν. Two small spheres of equal mass m are attached symmetrically to a rigid rod of length 2ℓ which is driven by a constant torque $\mathbf{T} = T\mathbf{j}$. The system shown in the figure is at rest initially when both $\mathbf{P}$ and $\mathbf{T}$ are applied simultaneously. The design geometry is such that the system does not tip. (a) Find the velocity of the block when the system has moved a distance d. (b) Determine the angular speed ω and the angular acceleration $\dot{\omega}$ of the rod after n revolutions. (c) Evaluate the results for the case when $M = 20$ slug, $m = 2$ slug, $\mathbf{P} = 832\mathbf{i}$ lb, $\mathbf{T} = 64\mathbf{k}$ ft · lb, $\nu = 1/3$, $\ell = 4$ ft, $d = 12$ ft, $n = 4$, and $g = 32$ ft/sec^2.



Problem 8.20.

8.21. A small object of mass M is in outer space where all gravitational forces are negligible. The object is initially at rest relative to an inertial frame $\Phi = \{E; \mathbf{i}_k\}$ when suddenly it explodes into two splinters having masses m_1 and m_2. Their subsequent relative velocity of separation is $\mathbf{v}$. Find their velocity vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ in Φ, and determine the kinetic energy of the system.

8.22. Two particles of equal mass m are attached to a rigid rod of negligible mass and length ℓ. The system is moving on a smooth horizontal surface with an angular velocity ω and center of mass velocity $\mathbf{v}^*$ when suddenly one end of the rod makes a normal impact with a wall, as illustrated. There is no loss of energy during the collision. (a) Find the instantaneous impulse of the force and the instantaneous torque impulse about the center of mass due to the impact. (b) Show that the impact results in an interchange of translational and rotational kinetic energies of the system. (c) Describe the subsequent motion of the system.

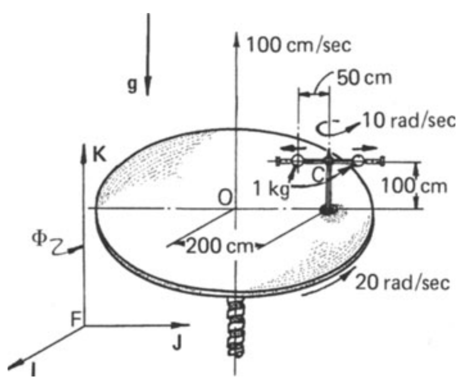


Problem 8.22.

8.23. The center of mass C of a rigid system of two particles of equal mass m separated a distance $2d$ is initially at rest in the vertical plane. The system is given a constant angular velocity $\boldsymbol{\omega} = \omega\mathbf{n}$ in a right-hand sense about an axis $\mathbf{n}$ at C, and the system is released to fall freely under gravity. The axis $\mathbf{n}$ is situated in the vertical plane at a fixed angle ϕ from the line joining the particles. What is the total kinetic energy of the system at time t?

8.24. In a general spatial motion of two particles of masses m_1 and m_2, the velocity of m_2 relative to m_1 is $\mathbf{v}$ and the center of mass has velocity $\mathbf{v}^*$. (a) What is the total kinetic energy of the system? (b) Let a be the perpendicular distance from one particle to the line through the other particle and parallel to $\mathbf{v}$. Show that the moment of momentum relative to the center of mass may be written as $\mathbf{h}_C = av[m_1m_2/(m_1 + m_2)]\mathbf{n}$, where $\mathbf{n}$ is a unit vector perpendicular to the plane containing the particles and the vector $\mathbf{v}$ and $v = |\mathbf{v}|$.

8.25. An antenna coil system consists of a rigid rod that rotates as shown with a constant angular speed of 10 rad/sec relative to a platform which is turning with an angular speed of 20 rad/sec while being raised vertically on a threaded shaft at a speed of 100 cm/sec in the ground frame Φ. At the instant of interest, each of two small coils of equal mass $m = 1$ kg are 50 cm from the center C and moving radially outward with a speed of 200 cm/sec relative to the rod. (a) Find the total momentum of the coil system in Φ. (b) Find the kinetic energy in Φ relative to the center of mass. (c) Determine the kinetic energy of the system in Φ. (d) What is the moment about point O in the platform of the momentum in Φ? (e) Determine the moment about O of the

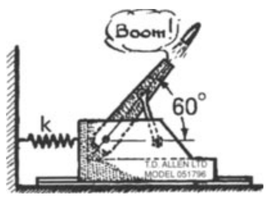


Problem 8.25.

momentum relative to O in Φ. (f) Find the moment about the center of mass C of the momentum in Φ. (g) What is the moment about C of the momentum relative to C in Φ.

8.26. Two particles of masses m and $2m$ are attached to the ends of a rigid rod of length l, negligible mass, and initially at rest along the Y-axis in the vertical plane frame $\Phi = \{O; \mathbf{I}, \mathbf{J}\}$. The mass m, initially at O, is subjected to a propulsive force $\mathbf{P}$ of constant magnitude and directed always perpendicular to the rod. Find the angular speed $\dot{\phi}(t)$ and the angular placement $\phi(t)$ of the system as functions of time. Formulate the problem in two ways: (i) write the work–energy equation with respect to the center of mass and (ii) write the equation for the moment of momentum relative to the center of mass. Notice that in neither case is it necessary to consider the motion of the center of mass.

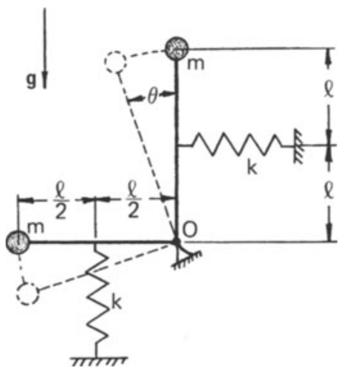
8.27. A 995 lb cannon with a recoil spring of stiffness $k = 193$ lb/ft is mounted on smooth horizontal rails. The gun fires a 5 lb shell with a muzzle velocity of 1500 ft/sec at a 60° angle, relative to the cannon. Determine the ultimate compression of the spring and the impulse reaction of the rails on the system.



Problem 8.27.

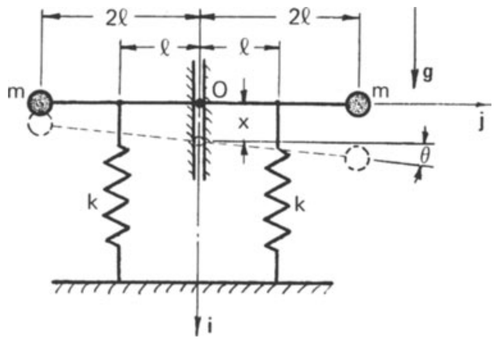
8.28. A shell explodes at the apex of its path into two pieces of equal mass m. One fragment is seen to fall vertically with initial speed $\dot{y}_0$. Find the path of the other splinter in a Cartesian reference frame with origin at the apex. Neglect frictional effects.

8.29. Two particles of equal mass m are attached to the ends of a rigid, right angle frame of negligible mass and supported by a smooth hinge at point O. Identical springs with stiffness k are attached at the midpoint of each rod, the horizontal spring being unstretched in the equilibrium configuration shown in the diagram. The system is given a small angular placement θ_0 and released from rest. (a) Discuss the infinitesimal stability of the equilibrium configuration of the system in terms of the static spring deflection θ_e by (i) use of the moment of momentum principle and (ii) use of the energy equation. (b) Find the motion $\theta(t)$ of the system.



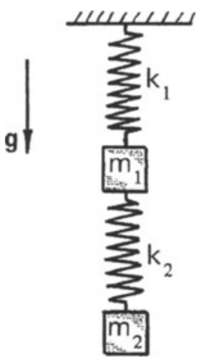
Problem 8.29.

8.30. Two particles of equal mass m are connected to the ends of a massless rigid rod of length 4ℓ and supported symmetrically by identical springs of stiffness k, as shown in the diagram. The center of the rod at O is constrained by smooth vertical rails to move only in the vertical plane. Initially, the rod is held horizontally so that the springs are unstretched, then turned clockwise about O through a small angle θ_0 and released to perform small oscillations in the vertical plane. (a) Apply momentum principles to derive the equations of motion of the system and solve them for the assigned initial conditions. What are the frequencies of the vertical and rotational oscillations? (b) Can the same results be obtained from the energy method? Explain.



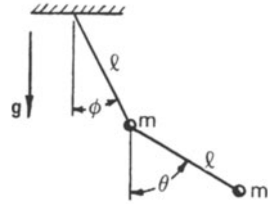
Problem 8.30.

8.31. A coupled system consists of a mass m_1 suspended vertically from a spring of stiffness k_1 and of another mass m_2 suspended from m_1 by a second spring of stiffness k_2. (a) Apply Newton's law to derive the equation of motion for each particle. (b) Write the energy equation for the system. Is it possible to derive from this equation the separate equations of motion for each particle? Explain.



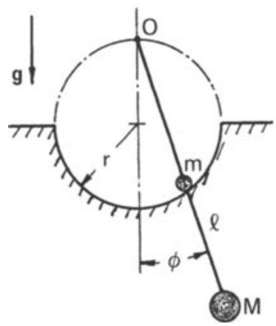
Problem 8.31.

8.32. A double pendulum consists of two bobs of equal mass m attached to the ends of two inextensible strings of equal length ℓ and negligible mass. The pendula are given small displacements shown in the figure and released to perform small plane oscillations. (a) Use Newton's law to derive the equations of motion for each particle. (b) Is it possible to derive these relations from the energy equation for the system? Explain.



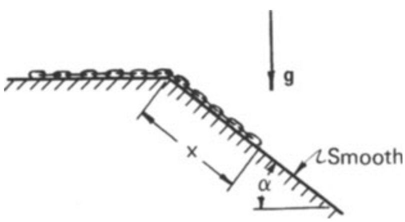
Problem 8.32.

8.33. A pendulum bob of mass M is attached to a smooth rigid rod of negligible mass and length ℓ supported at point O. A second mass m, which can slide freely along the pendulum rod, is constrained to move on a smooth circular surface of radius r as the system swings in the vertical plane, as illustrated. Derive the equation of motion for the system in two ways: (a) by use of the moment of momentum principle and (b) by use of the energy principle. (c) What is the first integral of the equation of motion when the system is released from rest at the placement ϕ_0? (d) What is the small amplitude oscillation frequency of the system?



Problem 8.33.

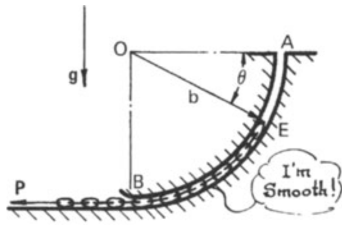
8.34. A uniform, inextensible heavy chain of length l, initially at rest on the horizontal section of the smooth surface (when $x = 0$), is given a small, ignorable disturbance causing it to slide down the inclined plane, shown in the figure. (a) Find the speed v of the chain as a function of its end distance x along the inclined surface. (b) Consider the case when $\alpha = \pi/2$ and the chain has an initial vertical overhang of length $x(0) = a$. Find the speed of the end of the chain as a function of x.



Problem 8.34.

8.35. A uniform, inextensible chain of length $l = \pi b/2$ is pulled by a constant force **P** from a smooth, quarter circle tube of radius b, situated in the vertical plane. The horizontal surface also is smooth. (a) If the chain initially is at rest when its end point E is at A, find its speed $v(\theta)$

as a function of its end placement θ shown in the figure. (b) What is the speed of E when it exits the tube at B? (c) Find $v(\theta)$ for the case when the chain merely slides from its initial state under gravity.



Problem 8.35.

8.36. Consider a body of mass M attached to a very long, uniform and inextensible coiled rope at rest in the horizontal plane. Suppose that the mass M is projected vertically upward from the plane with an initial speed v_0, so that the rope subsequently uncoils and follows vertically behind. This is an example of a variable mass system for which the principle of conservation of energy does not hold, even though the only force acting on this system is the conservative gravitational force. In this case, the first integral of the equation of motion $\mathbf{F} = d\mathbf{p}^*/dt$ does not lead to the work–energy principle. (a) Show that the first integral of this equation for a variable mass $m(t)$ is given by

$$\Delta \frac{1}{2}\mathbf{p}^* \cdot \mathbf{p}^* = \int_{t_0}^{t} \mathbf{F} \cdot \mathbf{p}^* dt. \tag{P8.36}$$

When the total mass is constant, this rule reduces to the familiar work–energy principle for the center of mass; otherwise, it does not. (b) Now, return to the coiled rope problem, let σ denote the mass per unit rope length and determine the maximum height h to which M will ascend. (c) What is the condition on v_0 in order that $h \geq l$ for a rope of length l? Assume that this condition holds for a sufficiently large initial velocity, and find the greatest height $h > l$ attained by M.

8.37. Two putty balls of masses m and $3m$ are moving toward one another with constant velocities $\mathbf{v}_1 = 2v\mathbf{i} + v\mathbf{j}$ and $\mathbf{v}_2 = v\mathbf{k}$, respectively, in $\Phi = \{O; \mathbf{i}_k\}$ when they collide in an oblique, direct impact and stick together. (a) Find the velocity of the single particle formed by the collision. (b) Determine the change in the kinetic energy of the system. Does it increase, decrease, or remain unchanged?

8.38. Two particles of equal mass m are connected by a rigid rod of length l and negligible mass, initially at rest along the X-axis on a smooth horizontal surface in frame $\Phi = \{O; \mathbf{I}, \mathbf{J}\}$. A third particle of mass m moving with velocity $\mathbf{v} = v\mathbf{J}$ in Φ collides with the particle at the right-hand end of the rod in a perfectly elastic, direct central impact. Find the subsequent motion of the center of mass of the original two particle system, and determine the angular speed of the rod.

8.39. A particle of mass m, attached by a light inextensible string to the center of a smooth horizontal table, is moving in a circle of radius r with speed v when it strikes an unconstrained particle of mass M at rest at a point r on the table. (a) Suppose the collision is perfectly inelastic. Find the angular speed after the collision, and show that the tension T in the string is reduced in the ratio $T/T_0 = m/(m + M)$, where T_0 is the initial tension. (b) Suppose the collision is perfectly elastic. Find the angular speed of m and the velocity of M after the impact, and determine the ratio in which the string tension is reduced.

8.40. Two pendulums of equal length l have bobs of masses m_1 and m_2, and both are suspended vertically from the same point O. The mass m_1 is displaced and released from rest at a height h (i.e. at an initial placement θ_0 from m_2). (a) Assume there is no energy loss in the impact, and find the velocities of m_1 and m_2 immediately afterward. (b) Apply the law of restitution to find these velocities. (c) Describe the results for three cases: $m_1 > m_2$, $m_1 < m_2$, and $m_1 = m_2$. (d) Find the common velocity of m_1 and m_2 following a perfectly inelastic collision, and determine the kinetic energy lost in the impact.