7

Momentum, Work, and Energy

7.1. Introduction

Several methods of integration of the Newton–Euler vector equation of motion and its related scalar equations have been studied in a variety of applications in previous chapters. Although it is not possible to integrate these equations in general terms for all types of problems, certain kinds of problems do admit general first integrals that lead to several additional and useful basic principles of mechanics: the impulse–momentum principle, the torque–impulse principle, and the work–energy principle. Moreover, for certain kinds of forces, the work–energy principle may be reduced to a powerful fundamental law known as the principle of conservation of energy. The law of conservation of momentum and the law of conservation of moment of momentum are two more first integral principles that derive from the Newton–Euler law and the moment of momentum principle. This chapter concerns the development and application of these several additional principles.

7.2. The Impulse–Momentum Principle

The first integral of the equation of motion has been obtained in a variety of special problems where the force acting on a particle was a specified function of time. However, it is sometimes possible to obtain information about the motion even though a full specification of the force is not known. In particular, when a ball strikes a wall, the force exerted by the wall on the ball varies suddenly in time, and though we have no way of knowing the precise manner in which this impulsive force changes with time, we can still obtain useful information about the motion of the ball or the force exerted by the wall. To see how this may be done, we introduce the vector-valued integral function $\mathscr{I}(t; t_0)$, called the *impulse of the force* $\mathbf{F}(t)$,

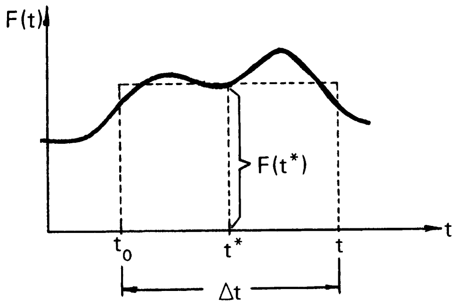


Figure 7.1. Graphical interpretation of the mean value theorem.

defined by

$$\mathscr{I}(t;t_0) \equiv \int_{t_0}^{t} \mathbf{F}(t)dt. \tag{7.1}$$

Then integrating the Newton–Euler equation of motion: $d\mathbf{p}/dt = \mathbf{F}(t)$ with respect to time on the interval $[t_0, t]$ in an inertial reference frame Φ and writing $\Delta\mathbf{p} \equiv \mathbf{p}(t) - \mathbf{p}(t_0)$ for the change in the momentum of the particle, we obtain the *impulse–momentum principle*:

$$\mathscr{I}(t;t_0) = \Delta\mathbf{p}. \tag{7.2}$$

In words, *the impulse of the force over the time interval* $[t_0, t]$ *is equal to the change in the linear momentum of the particle during that time*. We note that impulse has the physical dimensions $[\mathscr{I}] = [FT] = [MLT^{-1}]$.

The mean value of the force acting over the time interval $[t_0, t]$ is determined by the impulse. Consider first the one-dimensional graph shown in Fig. 7.1 for a force $F(t)$. According to the mean value theorem of integral calculus, there exists a value of t, say t^*, such that

$$\int_{t_0}^{t} F(t)dt = F(t^*)\Delta t, \tag{7.3}$$

wherein $t_0 \le t^* \le t$ and $\Delta t = t - t_0$. Geometrically, (7.3) shows that the area under the $F(t)$ curve on $[t_0, t]$ in Fig. 7.1 is equal to the area on $[t_0, t]$ of a rectangle of height $F(t^*)$. The value $F(t^*)$ is the *average value* of $F(t)$ on $[t_0, t]$. The same formula (7.3) may be applied to each force component. Therefore, more generally, the *average value* $\mathbf{F}^*$ of the force $\mathbf{F}(t)$ on the interval $[t_0, t]$ is defined by

$$\mathbf{F}^* \equiv \frac{1}{\Delta t}\int_{t_0}^{t} \mathbf{F}(t)dt = \frac{\mathscr{I}(t;t_0)}{\Delta t} = \frac{\Delta\mathbf{p}}{\Delta t}, \tag{7.4}$$

wherein (7.1) and (7.2) are introduced.

This result shows that although we may not know the actual impulsive force acting on the particle, its average value on the interval Δt is determined by the change in the linear momentum of the particle during that interval. Moreover, it is seen that in the limit as $t_0 \rightarrow t$, (7.4) returns the rule (5.34). The following example illustrates an average force calculation.

Example 7.1. A projectile S weighing 50 lb strikes a concrete bunker with a normal velocity of 1288 ft/sec (878 mph). The projectile imbeds itself in the wall and comes to rest in 10^{-2} sec. What is the average force exerted on the wall by the projectile during this time?

Solution. The change in the linear momentum of the projectile is

$$\Delta \mathbf{p} = -\frac{50}{32.2}(1288)\mathbf{n} = -2000\mathbf{n} \ \text{slug} \cdot \text{ft/sec}, \tag{7.5a}$$

in which $\mathbf{n}$ is the unit normal vector directed into the wall. The average force $\mathbf{F}_S^*$ acting on the projectile in the time $\Delta t = 10^{-2}$ sec is given by the last ratio in (7.4), and with (7.5a) we thus obtain

$$\mathbf{F}_S^* = -\frac{2000}{10^{-2}}\mathbf{n} = -2 \times 10^5 \mathbf{n} \ \text{lb} = -100\mathbf{n} \ \text{tons}. \tag{7.5b}$$

This estimates the total force exerted on the projectile by the concrete wall and gravity. Of course, the weight of the projectile compared with the total impulsive force (7.5b) is negligible, and hence the average force exerted on the wall by the projectile may be estimated by the equal and oppositely directed force $\mathbf{F}_W^* = 100\mathbf{n}$ tons. If the action time increment is smaller, the average force acting on the projectile or the penetration force acting on the wall grows larger. □

7.2.1. Instantaneous Impulse and Momentum

There are many physical situations in which a change in velocity induced by the exchange of deformation energy occurs so suddenly that it is very difficult to observe the transition from one state to another. When a cue strikes a billiard ball, for example, the ball experiences a finite change in velocity during an infinitesimally short interval of time. There is also no observable change in its position during the impact time. The same thing is true when a bullet strikes a block of wood and when an automobile impacts a pole. In these cases the impulse occurs virtually instantaneously. This physical idea of an instantaneous impulsive action is first characterized mathematically. Afterwards, the use of singularity functions to define the impulsive force and the instantaneous impulse are described.

In general, the average value $\mathbf{F}^*$ of the total force conveys no information about the nature or maximum intensity of the actual applied force $\mathbf{F}(t)$, rather, it provides only an estimate of $\mathbf{F}(t)$ that is independent of the duration of its application. A one-dimensional triangular loading, for example, has a mean value equal to one-half the height of the triangle regardless of the length of its time base. So, an average value estimate of $\mathbf{F}(t)$ might not be a very good one. On the other hand, when the impulse is almost instantaneous it is reasonable to imagine that the

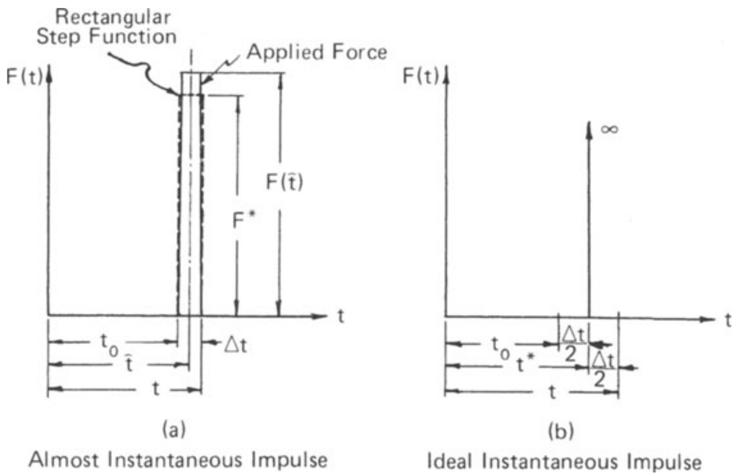


Figure 7.2. Graphical models of an almost instantaneous impulse and an ideal instantaneous impulse.

force–time graph, as shown in Fig. 7.2a for the one-dimensional case, is closely approximated by a rectangular step function. In this instance, the mean value $\mathbf{F}^*$ approximates very closely the extreme intensity $F(\hat{t})$ of the actual applied force. Of course, the impulse–momentum principle (7.2) holds for all time intervals Δt, large or small.

Furthermore, the particle's displacement $\Delta\mathbf{x} = \mathbf{x}(t) - \mathbf{x}(t_0)$ during any time interval $\Delta t = t - t_0$ is related to the *average value* $\mathbf{v}^*$ of its velocity $\mathbf{v}$ in accordance with the relation

$$\Delta\mathbf{x} = \int_{t_0}^{t} \mathbf{v}(t)dt = \mathbf{v}^*\Delta t. \tag{7.6}$$

Therefore, if a particle experiences a finite change in velocity in an infinitesimal time interval, it follows from (7.6) that the displacement during that interval must be infinitesimal, and as $\Delta t \to 0,\ \Delta\mathbf{x} \to \mathbf{0}$ also.

An instantaneous impulse, therefore, is characterized physically as a very large, suddenly applied force acting over a vanishing time interval, and resulting in an instantaneous but finite change in the particle's velocity with no change in its position. In accordance with (7.2), the *instantaneous impulse* $\mathscr{I}^*$ is defined by

$$\mathscr{I}^* \equiv \operatorname*{limit}_{t\to t_0} \int_{t_0}^{t} \mathbf{F}(t)dt = \Delta\mathbf{p}^*, \tag{7.7}$$

where $\Delta\mathbf{p}^*$ is the finite, but *instantaneous change in the linear momentum of the particle* (*or center of mass object*). By (7.6), since the average velocity must be finite, there is no instantaneous change $\Delta\mathbf{x}^*$ in the particle's position at the impulsive instant:

$$\Delta\mathbf{x}^* \equiv \underset{t\to t_0}{\text{limit}}\ \Delta\mathbf{x} = \mathbf{0}. \tag{7.8}$$

Further, in the limit $\Delta t \to 0$, all *finite* forces that contribute to the total force that acts on a particle will vanish from (7.7). In order for this limit to be nonzero, the total force must become very large as Δt is indefinitely diminished. Therefore, when calculating the effect of an instantaneous impulsive force, we may neglect the effect of all other finite forces, such as gravity, spring or friction forces, that may act on the particle at the time of the instantaneous impulse. Of course, the instantaneous change in momentum occurs always in the direction of the instantaneous impulse.

Exercise 7.1. A particle P of unit mass is acted upon by a constant force $\mathbf{F}$. Prove that the average velocity $\mathbf{v}^*$ of P in a time interval Δt is equal to $\mathbf{v}(P, \Delta t/2)$, its velocity at the midpoint of the interval. □

The delta function (1.120) introduced in Volume 1 may be used to describe the instantaneous impulse. Let us write $\mathbf{F}(t) = \mathscr{I}^*\delta(t)$, where $\mathscr{I}^*$ is the instantaneous impulse at the instant $t^* \in [t_0, t]$ and $\delta(t) = \langle t - t^*\rangle_{-1}$ is the *Dirac delta function.* Then the *ideal instantaneous impulsive force* is described by

$$\mathbf{F}(t) = \mathscr{I}^*\langle t - t^*\rangle_{-1} = \begin{cases} \mathbf{0} \text{ if } t \neq t^*, \\ \infty \text{ if } t = t^*. \end{cases} \tag{7.9}$$

This ideal force is illustrated in Fig. 7.2b. In accordance with (1.133), we also obtain from (7.9)

$$\int_{-\infty}^{t} \mathbf{F}(t)dt = \mathscr{I}^* \int_{-\infty}^{t} \langle t - t^*\rangle_{-1}dt = \mathscr{I}^*\langle t - t^*\rangle^0, \tag{7.10}$$

wherein we recall the unit step function $u(t) = \langle t - t^*\rangle^0$ in (1.117). Hence,

$$\mathscr{I}^*\langle t - t^*\rangle^0 = \begin{cases} \mathbf{0} \text{ if } t < t^*, \\ \mathscr{I}^* \text{ if } t > t^*, \\ \text{undefined at } t = t^*. \end{cases} \tag{7.11}$$

To relate this to the instantaneous change of momentum in (7.7), we integrate (7.9) over an infinitesimal time interval about t^*, say from $t_0 = t^* - \frac{1}{2}\Delta t$ to $t = t^* + \frac{1}{2}\Delta t$, and note that the impulsive force $\mathbf{F}(t)$ vanishes outside this interval. Then letting $\Delta t = t - t_0 \to 0$, as $t \to t^*$ from above, we obtain with (7.10) the instantaneous impulse defined in (7.7).

7.2.2. Linear Momentum in an Instantaneous Impulse

Suppose that the impulse of the force exerted on a particle P_1 is due to its interaction with another particle P_2. It is not necessary that the particles come into contact, but they may. Let $\mathscr{I}_{12}$ denote the impulse acting on P_1 by P_2; then the third law requires that the impulse $\mathscr{I}_{21}$ acting on P_2 by P_1 be equal, but oppositely directed. Therefore, if $\Delta \mathbf{p}_1$ and $\Delta \mathbf{p}_2$ denote the changes in linear momenta of P_1 and P_2, respectively, the impulse–momentum principle (7.2) and the law of mutual action together imply that for the same time interval $\Delta \mathbf{p}_1 = \mathscr{I}_{12} = -\mathscr{I}_{21} = -\Delta \mathbf{p}_2$, that is, $\Delta(\mathbf{p}_1 + \mathbf{p}_2) = \mathbf{0}$. This yields *the law of conservation of instantaneous momentum for a system of two particles—the total instantaneous linear momentum for a system of two particles is constant during an impulsive interval*:

$$\mathbf{p}_1 + \mathbf{p}_2 = \mathbf{c}, \quad \text{a constant.} \tag{7.12}$$

If any other forces acting on either particle are finite and the impulse is *instantaneous*, these external forces contribute nothing to the impulse of the force on that particle and may be ignored. Then, (7.12) holds even though other external, but nonimpulsive forces may act on each particle. Clearly, forces whose resultant is zero may be ignored, too. If other impulsive forces act on either particle, however, (7.12) does not hold, rather, the impulse–momentum principle (7.7) must be applied separately to each particle, accounting for the total impulsive force that acts on each.

Example 7.2. A *ballistic pendulum* is a device used to determine the muzzle speed of a gun. A bag of wet sand of mass M is suspended by a rope, and a bullet of mass m is fired into the sand with unknown muzzle speed β. The pendulum then swings through a small angle θ_0 from its vertical position of rest as shown in Fig. 7.3. Replace the sack by its center of mass object, and find the muzzle speed of the gun.

Solution. During the infinitesimally small time interval of impact of the bullet with the sack of sand, the only external forces that act on the pair are their weight

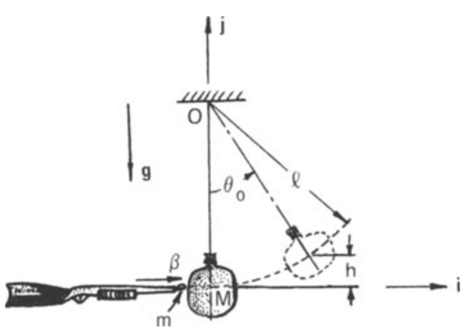


Figure 7.3. A ballistic pendulum model.

and the tension in the rope. These are finite external forces that contribute nothing to the instantaneous impulses and may be ignored. In fact, the resultant of the rope tension and the weight of the sand is zero during the impulsive instant. (See Exercise 7.3, page 229.) Hence, the total linear momentum at the instant of impact is constant. The bag being at rest, the linear momentum of the pair just prior to the instant t^* of impact is $m\beta\mathbf{i}$. Immediately afterward, when the bullet is lodged in the bag (captured by the center of mass object), which now has an instantaneous velocity $\mathbf{v}_0 = v_0\mathbf{i}$, the linear momentum is $(m + M)v_0\mathbf{i}$. Application of the momentum equation (7.12) yields $m\beta\mathbf{i} = (m + M)v_0\mathbf{i}$, and hence

$$\beta = \frac{m + M}{m} v_0. \tag{7.13a}$$

However, v_0 remains unknown. To find it, we consider a familiar problem.

After the impulse, the bag swings as a simple pendulum of length l and small amplitude θ_0 so that $h \ll \ell$ in Fig. 7.3. Therefore, the equation of motion for the bag carrying both the sand and the bullet is given by (6.67d), whose general solution for the initial conditions $\theta(0) = 0$ and $\ell\dot{\theta}(0) = v_0$ is provided by (6.67e). Hence, with $B = 0$ and $A = v_0/p\ell$, the solution is $\theta(t) = (v_0/p\ell)\sin pt$, from which the amplitude of the swing is $\theta_0 = v_0/p\ell$. With (6.67c), this yields $v_0 = \theta_0\sqrt{g\ell}$. Finally, use of this relation in (7.13a) delivers the muzzle speed of the gun:

$$\beta = \frac{m + M}{m} \theta_0\sqrt{g\ell}. \tag{7.13b}$$

Since $M \gg m$, the muzzle speed is closely estimated as $\beta = (M/m)\,\theta_0\sqrt{g\ell}$. □

7.3. The Torque–Impulse Principle

The moment of momentum principle (6.79) has the same analytical structure as the Newton–Euler law, so a parallel procedure is used to describe the impulse due to the moment of the force. The *impulse of the moment* $\mathbf{M}_O(t)$ *about the fixed point* O, called the *torque–impulse,* is the vector-valued integral function $\mathscr{M}_O(t; t_0)$ defined by

$$\mathscr{M}_O(t; t_0) \equiv \int_{t_0}^{t} \mathbf{M}_O(t)dt. \tag{7.14}$$

Let $\Delta\mathbf{h}_O \equiv \mathbf{h}_O(t) - \mathbf{h}_O(t_0)$ denote the change in the moment of momentum of the particle about O in the time interval $[t_0, t]$, and recall the moment of momentum principle: $d\mathbf{h}_O/dt = \mathbf{M}_O(t)$. Integrating this equation with respect to time, we obtain *the torque–impulse principle*:

$$\mathscr{M}_O(t; t_0) = \Delta\mathbf{h}_O. \tag{7.15}$$

Hence, the torque–impulse over the time interval $[t_0, t]$, *about a fixed point* O *in an inertial reference frame, is equal to the corresponding change in the particle's moment of momentum about* O. The torque–impulse and moment of momentum have the physical dimensions: $[\mathscr{M}_O] = [\mathbf{h}_O] = [FLT] = [ML^2T^{-1}]$.

The *average value* $\mathbf{M}_O^*$ of the torque $\mathbf{M}_O(t)$ on the interval $[t_0, t]$ is defined by

$$\mathbf{M}_O^* \equiv \frac{1}{\Delta t}\int_{t_0}^{t} \mathbf{M}_O(t)dt = \frac{\mathscr{M}_O(t;t_0)}{\Delta t} = \frac{\Delta \mathbf{h}_O}{\Delta t}, \tag{7.16}$$

wherein $\Delta t = t - t_0$ and we recall (7.14) and (7.15). In the limit as $\Delta t \to 0$, (7.16) returns the rule (6.79).

7.3.1. Instantaneous Torque–Impulse

When the torque is applied suddenly so that it results in a virtually instantaneous, finite change in the moment of momentum of the particle with no instantaneous change in its position, the torque–impulse is called an instantaneous torque–impulse. Symbolically, for a fixed point O in an inertial frame, the *instantaneous torque–impulse* $\mathscr{M}_O^*$ is defined by

$$\mathscr{M}_O^* \equiv \operatorname*{limit}_{t\to t_0} \int_{t_0}^{t} \mathbf{M}_O(t)dt = \Delta \mathbf{h}_O^*, \tag{7.17}$$

where $\Delta \mathbf{h}_O^*$ is the finite, but instantaneous change in the particle's moment of momentum about O.

The instantaneous torque–impulse about a fixed point O *is equal to the moment about* O *of the instantaneous impulse of the force acting on the particle, i.e.,*

$$\mathscr{M}_O^* = \mathbf{x}(t_0) \times \mathscr{I}^*. \tag{7.18}$$

To prove this, we recall that the particle's position vector from O does not change during an instantaneous impulse. Hence, with $\mathbf{x}_O = \mathbf{x}(t_0)$, (7.17) yields

$$\mathscr{M}_O^* = \operatorname*{limit}_{t\to t_0} \int_{t_0}^{t} \mathbf{x}_O(t) \times \mathbf{F}(t)dt = \mathbf{x}(t_0)\times \operatorname*{limit}_{t\to t_0} \int_{t_0}^{t} \mathbf{F}(t)dt,$$

from which (7.18) now follows by (7.7).

Exercise 7.2. Alternatively, form the difference $\Delta \mathbf{h}_O = \mathbf{h}_O(t) - \mathbf{h}_O(t_0)$, recall (5.31) and (7.2), and thus derive (7.18) differently. □

7.3.2. Moment of Momentum in an Instantaneous Torque–Impulse

Suppose that the impulse of the force exerted on a particle P_1 is due to its interaction with another particle P_2 so that $\mathscr{I}_{12}^* = -\mathscr{I}_{21}^*$. Let P_1 and P_2 have positions $\mathbf{x}_1$ and $\mathbf{x}_2$ from a fixed point O in an inertial frame Φ at the impulsive instant, and write $\mathbf{r} = \mathbf{x}_2 - \mathbf{x}_1$ for the vector of P_2 from P_1. Then (7.18) shows that for the system of two particles the total instantaneous torque–impulse about point O, defined by $\mathscr{M}_O^* = \mathscr{M}_{O1}^* + \mathscr{M}_{O2}^*$, is determined by

$$\mathscr{M}_O^* = \mathbf{x}_1 \times \mathscr{I}_{12}^* + \mathbf{x}_2 \times \mathscr{I}_{21}^* = \mathbf{r} \times \mathscr{I}_{21}^*. \tag{7.19}$$

We now assume that either (i) the particles collide, so that $\mathbf{r} = \mathbf{0}$, or (ii) they do not collide but their mutual interaction impulses are directed along the line joining the two particles, so that $\mathscr{I}_{21}^*$ is parallel to $\mathbf{r}$. In either case, the torque–impulse (7.19) vanishes; and, therefore, from (7.17), the change in the total moment of momentum of the system of two particles about the fixed point O, in accordance with (5.32), namely, $\Delta\mathbf{h}_O^* = \Delta(\mathbf{h}_{O1}^* + \mathbf{h}_{O2}^*)$, must vanish. This yields the *law of conservation of instantaneous moment of momentum for a system of two particles—at the instant of impulse, the total instantaneous moment of momentum about a fixed point O for a system of two particles is a constant vector*:

$$\mathbf{h}_{O1}^* + \mathbf{h}_{O2}^* = \mathbf{h}_O^*, \text{ a constant.} \tag{7.20}$$

Example 7.3. The rule (7.20) may be applied to find the muzzle speed of the bullet in the ballistic pendulum problem in Fig. 7.3. Just prior to impact, the moment of momentum of the bullet about point O is $\mathbf{h}_O^* = \ell m\beta\mathbf{k}$. Immediately thereafter, the moment of momentum of the system is $\mathbf{h}_O^* = \ell(m + M)v_0\mathbf{k}$. Hence, use of (7.20) yields (7.13a) giving the muzzle speed in terms of v_0. □

Exercise 7.3. Suppose the bullet enters the sack at a downward angle ψ from the horizontal axis in Fig. 7.3. The rope tension in this case will exert an additional instantaneous impulse on the sack. (a) Apply the impulse–momentum principle (7.7) to determine the muzzle speed β, and find the instantaneous impulse on the rope. (b) Apply the instantaneous torque–impulse principle (7.17) to find β. □

7.4. Work and Conservative Force

Consider a particle P in motion along a path $\mathscr{C}$ due to a force $\mathbf{F}(\mathbf{x})$ that varies only with the particle's position along $\mathscr{C}$, as shown in Fig. 7.4. The *work* $\mathscr{W}$ done by the force $\mathbf{F}(\mathbf{x})$ in moving P along $\mathscr{C}$ from the point $\mathbf{x}_1$ to the point $\mathbf{x}_2$ is defined by the *path*, or *line integral*

$$\mathscr{W} = \int_{\mathscr{C}} \mathbf{F}(\mathbf{x}) \cdot d\mathbf{x} = \int_{\mathbf{x}_1}^{\mathbf{x}_2} \mathbf{F}(\mathbf{x}) \cdot d\mathbf{x}. \tag{7.21}$$

The physical dimensions of work are $[\mathscr{W}] = [FL]$. Thus, if force is measured in Newtons or pounds and length in meters or feet, the measure units of work are expressed as N · m or lb · ft (or also as ft · lb), respectively.

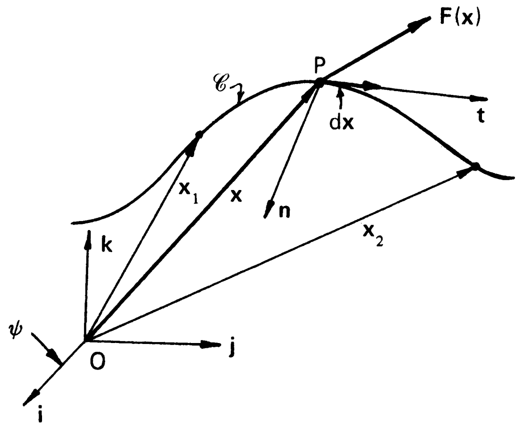


Figure 7.4. Schema for work done by a force **F**(**x**) acting on a particle over its path.

In accordance with (6.3), the total force referred to the intrinsic basis may be expressed as $\mathbf{F} = F_t\mathbf{t} + F_n\mathbf{n}$ and $d\mathbf{x} = ds\mathbf{t}$; therefore, in general, by (7.21), the work done by **F** is

$$\mathscr{W} = \int_{\mathscr{C}} F_t ds. \tag{7.22}$$

Consequently, only the component of force tangent to the path does work in moving P. Moreover, an increment of work $\Delta\mathscr{W} = F_t\Delta s$ is positive, negative, or zero according as the particle displacement Δs is in the same direction, the opposite direction, or perpendicular to the force acting on the particle, respectively. In particular, a propulsive force does positive work, whereas a drag force does negative work. The force (6.16) acting on a charged particle in a constant magnetic field does zero work; and the Coriolis force, $-2m\boldsymbol{\omega}\times\delta\mathbf{x}/\delta t$, being perpendicular to the relative velocity vector, does no work in the moving frame. Also, the property (7.8) of an instantaneous impulse shows that forces that act during an instantaneous impulse do no work.

The total work done by **F** depends not only on the end points, as emphasized by the second expression in (7.21), but generally also on the path traversed by P, as emphasized in the first expression. For certain forces, the work done is the same for every path joining the same end points, so the work done by these forces depends only on the values of the assigned end points. A force field **F**(**x**) that is independent of the path is called a *conservative force*. A force **F**(**x**) that is not conservative is called a *nonconservative force*; these are path dependent forces. Both propulsive and Coulomb friction forces, for example, always follow the motion along the specific path of the particle, so these are nonconservative forces that vary with the choice of path between fixed end states. Conservative forces are further characterized later on.

In rectangular Cartesian coordinates, we write $\mathbf{F}(\mathbf{x}) = F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k}$ and $d\mathbf{x} = dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k}$. Hence, for end states at $\mathbf{x}_j = (x_j, y_j, z_j)$, $j = 1, 2$, the second expression in (7.21) becomes

$$\mathscr{W} = \int_{(x_1, y_1, z_1)}^{(x_2, y_2, z_2)} (F_x dx + F_y dy + F_z dz). \tag{7.23}$$

This relation reveals the procedure for calculating the line integral. The force $\mathbf{F}(\mathbf{x})$ must be known as a function of position $\mathbf{x}$ along $\mathscr{C}$. In general, we must also know the equation of the path $\mathscr{C}$ so that the path variables x, y, z and their differentials can be related through this function; and, finally, the position coordinates (x_1, y_1, z_1) and (x_2, y_2, z_2) of the end points at $\mathbf{x}_1$ and $\mathbf{x}_2$ are required. This analysis is illustrated in the following problem.

Example 7.4. Find the work done by the force

$$\mathbf{F}(\mathbf{x}) = bxy\mathbf{i} + cy\mathbf{j}, \tag{7.24a}$$

in moving a particle from the origin (0, 0) to the point (1, a) along the paths defined by (i) a parabola $y = ax^2$ and (ii) the lines $x = 0$ and $y = a$. Here a, b, c are constants. (iii) What condition must be satisfied in order that the force (7.24a) may be conservative?

Solution of (i). The work done by the force (7.24a) in moving the particle from (0, 0) to $(1, a)$ on any path is given by (7.23):

$$\mathscr{W} = \int_{\mathscr{C}} \mathbf{F} \cdot d\mathbf{x} = \int_{(0,0)}^{(1,a)} bxydx + cydy. \tag{7.24b}$$

To integrate the first term in this equation, it is evident that we shall need to know how y is related to x. This means that the path must be specified, and hence the force (7.24a) is a nonconservative force.

For the parabolic path $y = ax^2$ shown in Fig. 7.5a, the path integral in (7.24b) becomes

$$\mathscr{W} = \int_{(0,0)}^{(1,a)} abx^3dx + cydy = \int_0^1 abx^3dx + \int_0^a cydy. \tag{7.24c}$$

Hence, the work done by the nonconservative force (7.24a) in moving the particle along the parabolic path $y = ax^2$ from (0, 0) to $(1, a)$ is

$$\mathscr{W} = \frac{ab}{4} + \frac{ca^2}{2}. \tag{7.24d}$$

Solution of (ii). Consider the work done by the same force (7.24a) acting over the path defined by the lines $x = 0$ and $y = a$ joining the same end points,

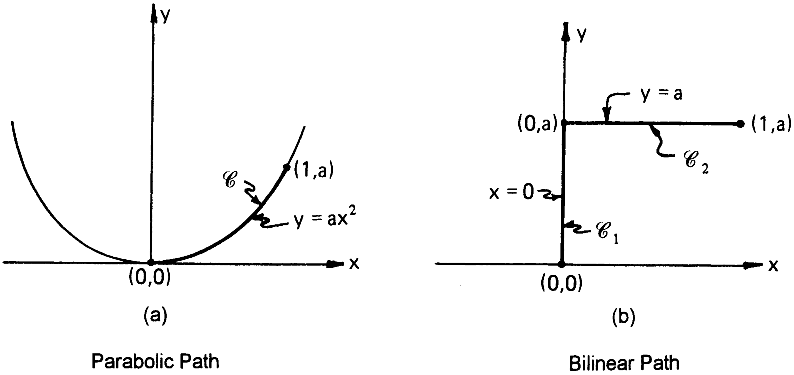


Figure 7.5. Distinct particle paths joining the same end points from (0, 0) to $(1, a)$.

as shown in Fig. 7.5b. The path $\mathscr{C}$ consists of two parts $\mathscr{C}_1$ and $\mathscr{C}_2$; hence, (7.21) is written as

$$\mathscr{W} = \int_{\mathscr{C}=\mathscr{C}_1\cup\mathscr{C}_2} \mathbf{F}\cdot d\mathbf{x} = \int_{\mathscr{C}_1} \mathbf{F}\cdot d\mathbf{x} + \int_{\mathscr{C}_2} \mathbf{F}\cdot d\mathbf{x}, \tag{7.25a}$$

wherein each integrand has the form of (7.24b). Now, $x = 0$ and y is variable on $\mathscr{C}_1$; hence, (7.24b) applied to $\mathscr{C}_1$ yields

$$\int_{\mathscr{C}_1} \mathbf{F}\cdot d\mathbf{x} = \int_{(0,0)}^{(0,a)} cydy = \int_0^a cydy = \frac{ca^2}{2}. \tag{7.25b}$$

Similarly, $y = a,\ dy = 0$ and x is variable on $\mathscr{C}_2$; hence, application of (7.24b) to $\mathscr{C}_2$ gives

$$\int_{\mathscr{C}_2} \mathbf{F}\cdot d\mathbf{x} = \int_{(0,0)}^{(1,a)} abxdx = \int_0^1 abxdx = \frac{ab}{2}. \tag{7.25c}$$

Therefore, the total work done by the force (7.24a) in moving the particle over the path $\mathscr{C} = \mathscr{C}_1 \cup \mathscr{C}_2$ between the same end states from (0, 0) to $(1, a)$, by (7.25a), is

$$\mathscr{W} = \frac{ab}{2} + \frac{ca^2}{2}. \tag{7.25d}$$

To conclude, let the reader consider the following additional example.

Exercise 7.4. Show that the work done by the force (7.24a) in moving the particle over the straight line path $y = ax$ joining (0, 0) to $(1, a)$ is given by

$$\mathscr{W} = \frac{ab}{3} + \frac{ca^2}{2}. \tag{7.25e}$$

Solution of (iii). Finally, we wish to determine the condition to be satisfied in order that (7.24a) may be conservative. The solutions (7.24d), (7.25d), and (7.25e) show that the work done by the same force will be the same for all paths considered above only if $b = 0$. This is the condition needed for the force (7.24a) to be conservative. Indeed, conversely, let us consider the force

$$\mathbf{F} = cy\mathbf{j}. \tag{7.26a}$$

Then the work done by $\mathbf{F}$ acting over a path $\mathscr{C}$ joining $(0, 0)$ to $(1, a)$ is given by

$$\mathscr{W} = \int_{\mathscr{C}} \mathbf{F} \cdot d\mathbf{x} = \int_{(0,0)}^{(1,a)} cydy = \int_0^a cydy = \frac{ca^2}{2}. \tag{7.26b}$$

Note that in this integration there is no need to mention a specific path $\mathscr{C}$. This result is independent of the path; it depends only on values at the end points. The force (7.26a) is a conservative force. We thus find that the work done by the force (7.24a) is independent of the particle's path, when and only when $b = 0$. □

There are many kinds of conservative and nonconservative forces. Gravitational and linear spring forces are conservative; the Coulomb frictional force is not. The work done by each of these forces is discussed next.

7.4.1. Work Done by a Constant Force

It is easy to show by (7.21) that *a constant force* $\mathbf{F}_c$ *is conservative.* The work done by $\mathbf{F}_c$ acting between the point $\mathbf{x}_0$ and any other point $\mathbf{x}$ is given by

$$\mathscr{W} = \mathbf{F}_c \cdot \int_{\mathbf{x}_0}^{\mathbf{x}} d\mathbf{x} = \mathbf{F}_c \cdot \Delta\mathbf{x}, \tag{7.27}$$

wherein $\Delta\mathbf{x} \equiv \mathbf{x} - \mathbf{x}_0$. Clearly, the work done by the constant force is independent of the path joining the end points, so $\mathbf{F}_c$ is conservative.

The apparent gravitational force on a particle of mass m near the surface of the Earth is a constant force $\mathbf{F}_c = -mg\mathbf{k}$. Therefore, by (7.27), *the apparent gravitational force near the surface of the Earth is a conservative force that does work*

$$\mathscr{W}_g = -mg\Delta z = -mgh, \tag{7.28}$$

in which $h \equiv \Delta z = \mathbf{k} \cdot \Delta\mathbf{x}$ is the vertical change in elevation through which m moves along its path. If $h > 0$, then the particle increases its height from the Earth and $\mathscr{W} < 0$. This means that the gravitational force acts oppositely to the particle's vertical displacement, and hence work is done against the force of gravity to increase the particle's elevation. On the other hand, $h < 0$ means that the elevation has decreased, i.e. the particle has moved in the direction of the force of gravity which now does positive work. Because the gravitational force is perpendicular to the horizontal contribution of the total displacement $\Delta\mathbf{x} = \Delta x\mathbf{i} + \Delta y\mathbf{j} + \Delta z\mathbf{k}$, it does no work in any horizontal displacement whatsoever.

7.4.2. Work Done by the Coulomb Frictional Force

A Coulomb frictional force of constant magnitude must not be confused as a conservative force. A constant force must have both a constant magnitude and a constant direction. Although a Coulomb drag force may have a constant magnitude in some cases, its direction always varies with the path.

The work done by the Coulomb frictional force $\mathbf{f}_d = -\nu N(s)\mathbf{t}$ over any path $\mathscr{C}$ of length d is given by

$$\mathscr{W}_f = \int_0^d \mathbf{f}_d \cdot d\mathbf{x} = -\nu \int_0^d N(s)ds. \tag{7.29}$$

In general, the Coulomb force need not have a constant magnitude, $N(s)$ in (7.29) may vary along the path. This happens, for example, when the particle slides down a rough curved path in the vertical plane. On the other hand, *in any motion for which N is constant, the work done by the Coulomb frictional force is*

$$\mathscr{W}_f = -\nu N d = -f_d d. \tag{7.30}$$

This formula is valid for all paths, but for different paths joining the same end states, the distance d along the path connecting these states will be different—the Coulomb frictional force $\mathbf{f}_d$ is not conservative.

7.4.3. Work Done by a Linear Force

Finally, consider the linear force $\mathbf{F}_L = \alpha\mathbf{x}$, where α is a constant. Recall (7.21) and note that the integrand may be written as $\mathbf{F}_L \cdot d\mathbf{x} = \alpha\mathbf{x} \cdot d\mathbf{x} = \alpha d(\frac{1}{2}\mathbf{x} \cdot \mathbf{x})$. Then the work done by $\mathbf{F}_L$ in moving a particle over an arbitrary path from a point $\mathbf{x}_0$ to any other point $\mathbf{x}$ is given by

$$\mathscr{W} = \int_{\mathbf{x}_0}^{\mathbf{x}} \mathbf{F}_L \cdot d\mathbf{x} = \frac{\alpha}{2}(\mathbf{x} \cdot \mathbf{x} - \mathbf{x}_0 \cdot \mathbf{x}_0), \tag{7.31}$$

which is independent of the path. *The linear force* $\mathbf{F}_L = \alpha\mathbf{x} = \alpha(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$ *is a conservative force*. Note that the force in (7.26a) is a special linear force of this type. **Caution**: Not every force linear in the variables $x,\ y,\ z$ is conservative; the force $\mathbf{F} = \alpha y\mathbf{i}$, for example, is not conservative.

Let x denote the change of length of a linear spring from its undeformed state and recall (6.64). Then the uniaxial force required to stretch or compress the spring is given by $\mathbf{F}_H = kx\mathbf{i}$, which is a linear force of the type $\mathbf{F}_L$. Therefore, from (7.31), the work done in stretching a linear spring from any initial state of stretch $\mathbf{x}_0$ is

$$\mathscr{W} = \int_{\mathbf{x}_0}^{\mathbf{x}} \mathbf{F}_H \cdot d\mathbf{x} = \frac{1}{2}k\left(x^2 - x_0^2\right). \tag{7.32}$$

If the initial state is the natural state, then $x_0 = 0$ and $\mathscr{W} = \frac{1}{2}kx^2$. *The force required to elongate or to compress a linear spring is a conservative force.*

Clearly, the equal but oppositely directed restoring force $\mathbf{F}_S = -kx\mathbf{i} = -\mathbf{F}_H$ exerted by the spring is a conservative force. By (7.32), *the work done by the elastic spring in its deformation from the natural state is given by*

$$\mathscr{W}_e = -\tfrac{1}{2}kx^2. \tag{7.33}$$

The work done by the spring force is negative, because the spring force opposes the displacement $d\mathbf{x}$.

7.5. The Work–Energy Principle

The concept of mechanical work is used to derive a general first integral of the equation of motion known as the work–energy principle. This principle is useful when the total force on a particle varies at most with its position $\mathbf{x}(P, t) = \mathbf{x}(s(t))$ along the path—the gravitational force of the Earth, the Coulomb frictional force, and the linear spring force being important examples. The notion of mechanical power is also introduced.

Let us consider the equation of motion when the total force is a function $\mathbf{F}(\mathbf{x})$ of the particle's position so that $m d\mathbf{v}/dt = \mathbf{F}(\mathbf{x})$. Now form its scalar product with $\mathbf{v}$, observe that $m\mathbf{v} \cdot d\mathbf{v}/dt = d(\frac{1}{2}m\mathbf{v} \cdot \mathbf{v})/dt$, and thereby obtain

$$\frac{dK(P,t)}{dt} = \mathbf{F}(\mathbf{x}) \cdot \mathbf{v}, \tag{7.34}$$

wherein, by definition, the new quantity

$$K(P,t) \equiv \tfrac{1}{2}m\mathbf{v} \cdot \mathbf{v} = \tfrac{1}{2}m\dot{s}^2, \tag{7.35}$$

is called the *kinetic energy* of the particle P. Then, recalling (7.21) and noting that $\mathbf{v}dt = d\mathbf{x}$, we integrate (7.34) over the path $\mathscr{C}$ traversed by the particle from $\mathbf{x}_0 = \mathbf{x}(t_0)$ to $\mathbf{x} = \mathbf{x}(t)$ to obtain the *work–energy principle*:

$$\mathscr{W} = \Delta K, \tag{7.36}$$

in which $\Delta K = K(P,t) - K(P,t_0)$ is the change in the kinetic energy of the particle that occurs in time $[t_0, t]$. *The work–energy equation states that the work done by the force* $\mathbf{F}(\mathbf{x})$ *acting on a particle over its path* $\mathscr{C}$ *from time* t_0 *to time* t *is equal to the change in the kinetic energy of the particle in that time.* It follows from (7.35) and (7.36) that $[K] = [MV^2] = [\mathscr{W}] = [FL]$.

The *mechanical power* $\mathscr{P}$ expended by the force is defined as the rate of working of the force:

$$\mathscr{P} = \frac{d\mathscr{W}}{dt} = \mathbf{F} \cdot \mathbf{v}. \tag{7.37}$$

With (7.36), this yields

$$\mathscr{P} = \frac{d\mathscr{W}}{dt} = \frac{dK}{dt}, \tag{7.38}$$

i.e. *the mechanical power expended by the force is equal to the time rate of change of the kinetic energy of the particle.* Power has the physical dimensions $[\mathscr{P}] = [FV] = [FLT^{-1}]$.

The work–energy equation is a single scalar equation, so it cannot replace equivalently the three scalar equations in the Newton–Euler vector equation of motion; rather, it often serves as a useful substitute for one of these equations integrated along the particle path. Since the work–energy equation was derived from the Newton–Euler law, we suspect that (7.36) may be applied conversely to derive the related single equivalent scalar equation of motion. In general, we see from (7.22) and (7.35) that in terms of intrinsic variables (7.36) may be written as

$$\frac{1}{2}m\dot{s}^2 - \frac{1}{2}mv_0^2 = \int_{s_0}^{s} F_t(s)ds, \tag{7.39}$$

where F_t is the tangential component of the total force $\mathbf{F}$ and $v_0 \equiv \dot{s}(t_0)$ is the particle's initial speed. Differentiation of this form of the work–energy equation with respect to the arc length parameter s and use of the relations $d(\frac{1}{2}m\dot{s}^2)/ds = m\ddot{s}$ and $d\mathscr{W}/ds = F_t$ show that

$$\Delta K = \mathscr{W} \iff m\ddot{s} = F_t. \tag{7.40}$$

(See (P6.28c) in Problem 6.28.) The reader may confirm that the same conclusion follows less directly by differentiation of (7.39) with respect to time.

The result (7.40) thus shows that the work–energy principle for a center of mass object is a convenient first integral of the tangential component of the Newton–Euler vector equation of motion, hence especially useful in single degree of freedom dynamical problems. On the other hand, if the value of $\mathscr{W}$ depends on the path, and we want to determine the particle's path, the work–energy rule might not be helpful. In other situations where the trajectory of the particle is known, or the force that acts on the particle does no work or is conservative so that its work is path independent, and especially when work is readily evaluated, the work–energy principle is most useful. The easy application of this rule is demonstrated in some examples that follow.

Example 7.5. Recall the ballistic pendulum problem in Fig. 7.3, page 226. Find the muzzle speed of the gun when the total angular placement may not be small enough to admit the approximate solution (7.13b) for which $h \ll \ell$.

Solution. The muzzle speed is still given by (7.13a), and v_0, the initial speed of the pendulum system, is the unknown of interest. The other end state condition and the path of the center of mass of this one-degree of freedom system are known. These facts strongly suggest that the work–energy principle will be helpful in this case. The total force that acts on this system is its total weight and the tension of the rope. The line tension is always normal to the circular path on which the center of mass moves, so it does no work as the system swings to its maximum placement θ_0. The work done by the constant force of gravity is determined by (7.28). Accordingly, in Fig. 7.3, the vertical height h through which the weight $(m + M)g$ is raised is $h = \ell(1 - \cos\theta_0)$, and hence $\mathscr{W} = -(m + M)g\ell(1 - \cos\theta_0)$. Since the system is at rest at θ_0 and has initial speed v_0, the change in the kinetic energy is $\Delta K = -\frac{1}{2}(m + M)v_0^2$. Thus, the work–energy principle (7.36) yields $-\frac{1}{2}(m + M)v_0^2 = -(m + M)g\ell(1 - \cos\theta_0)$. This gives the unknown v_0, and its use in (7.13a) provides the precise muzzle speed relation:

$$\beta = \frac{m + M}{m}\sqrt{2g\ell(1 - \cos\theta_0)}.$$

When $h/\ell = (1 - \cos\theta_0)$ is very small so that $1 - \cos\theta_0 = \frac{1}{2}\theta_0^2$, very nearly, the last equation reduces to our earlier approximate solution (7.13b). □

Example 7.6. A student is racing along in a sports car when suddenly, to avoid an impending collision, the driver slams on the brakes and skids along a straight line 200 ft to a stop in a 45 mph zone. Moments earlier, a policeman had checked the vehicle's speed on radar. Assume $\nu = 0.6$, ignore air resistance, and determine if the officer might give the student a ticket for exceeding the limit. Show for this example that the work–energy equation is the first integral of the equation of motion.

Solution. The assigned speed data in Fig. 7.6 show that the change in the kinetic energy of the car and its driver of total mass m is given by $\Delta K = -\frac{1}{2}mv^2$, and we want to find v. Because the path and the position varying nature of the forces acting on the system are known, we consider the work–energy method.

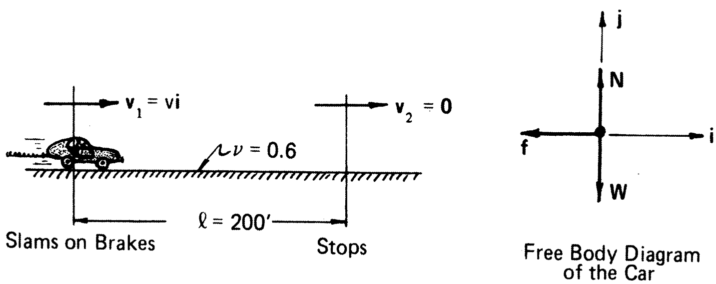


Figure 7.6. Motion of a braking vehicle over a rough road.

The forces that act on the car are shown in the free body diagram of Fig. 7.6. Both **N** and **W** do no work in the motion, and their magnitudes are equal: $N = W$. The work done by the nonconservative Coulomb frictional force $\mathbf{f}_d = -\nu N\mathbf{i}$ in the plane motion along a straight line is determined by (7.30) in which $d = \ell$, namely, $\mathscr{W}_f = -\nu W\ell$. The work–energy principle (7.36) thus yields the result $-\frac{1}{2}mv^2 = -\nu W\ell$, and with $W = mg$, the initial speed is determined by

$$v = \sqrt{2\nu g\ell}, \tag{7.41a}$$

which is *independent of the weight of the vehicle and its passenger.*

For the given data, $v = [2(.6)(32.2)(200)]^{1/2} = 87.91$ ft/sec, or very nearly 60 mph. In consequence, the student could receive a citation for speeding.

The first integral of the equation of motion $m\ddot{s} = d(\frac{1}{2}m\dot{s}^2)/ds = -\nu mg$ yields the general form of work–energy equation:

$$\tfrac{1}{2}m\dot{s}^2 - \tfrac{1}{2}mv^2 = -\nu mgs. \tag{7.41b}$$

When $s = \ell$, $\dot{s} = 0$, we obtain (7.41a). Conversely, differentiation of the work–energy equation (7.41b) with respect to either s or t yields the equivalent equation of motion. □

Example 7.7. A mass m is dropped from a height h onto a linear spring of constant stiffness k and negligible mass. Determine the maximum deflection δ of the spring, and compare this value with the static spring deflection δ_S produced by m. See Fig. 7.7. Assume that m maintains contact with the spring in its motion following the impact.

Solution. Since the velocity of m is zero at both its initial and terminal states at 1 and 3 in Fig. 7.7, the change in its kinetic energy on the path $\mathscr{C}$ is zero. Because the mass of the spring is negligible, its kinetic energy may be ignored. Moreover, all the forces that act on m are constant, workless, or vary only with the particle's position on $\mathscr{C}$. Therefore, the work–energy principle may be applied.

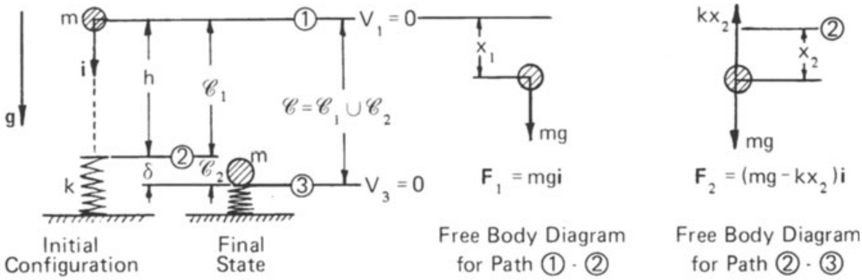


Figure 7.7. Spring deflection due to impact by a falling body.

The total work done by the forces acting on m from its initial position 1 to its final position 3 on $\mathscr{C}$ is determined by (7.21) in which $\mathscr{C} = \mathscr{C}_1 \cup \mathscr{C}_2$. We note that the instantaneous impulsive force of the spring does no work on m. The free body diagrams in Fig. 7.7 show that the force acting on m over $\mathscr{C}_1$ is $\mathbf{F}_1 = mg\mathbf{i}$ and over $\mathscr{C}_2$ is $\mathbf{F}_2 = (mg - kx_2)\mathbf{i}$. Hence,

$$\mathscr{W} = \int_{\mathscr{C}_1} \mathbf{F} \cdot d\mathbf{x} + \int_{\mathscr{C}_2} \mathbf{F} \cdot d\mathbf{x} = \int_0^h mg dx_1 + \int_0^\delta (mg - kx_2) dx_2. \tag{7.42a}$$

The work–energy equation (7.36) thus yields

$$\mathscr{W} = mgh + mg\delta - \tfrac{1}{2}k\delta^2 = \Delta K = 0, \tag{7.42b}$$

which determines the following deflection of the spring:

$$\delta = \frac{mg}{k} + \sqrt{\left(\frac{mg}{k}\right)^2 + \frac{2mgh}{k}}. \tag{7.42c}$$

The static deflection that would result from the weight alone is $\delta_S = mg/k$. Use of this relation in (7.42c) gives

$$\delta = \delta_S + \sqrt{\delta_S^2 + 2\delta_S h} \geq 2\delta_S. \tag{7.42d}$$

This formula shows that *the dynamic deflection δ is not less than twice the static deflection δ_S*. In particular, if m is released just at the top of the spring so that $h = 0$, then $\delta = 2\delta_S$.

The foregoing solution has illustrated the application of the work–energy equation when the work is calculated by use of the path integrals in (7.42a). Alternatively, we recognize that the work $\mathscr{W}_g$ done by the gravitational force acting over the entire path from 1 to 3 in the direction of the displacement is given by $\mathscr{W}_g = mg(h + \delta)$; the work $\mathscr{W}_e$ done by the elastic spring force acting on m over the path from 2 to 3 is $\mathscr{W}_e = -\frac{1}{2}k\delta^2$; and the impulsive force imposed on m at state 2 is workless. Hence, the total work done on m is $\mathscr{W} = \mathscr{W}_g + \mathscr{W}_e = mg(h + \delta) - \frac{1}{2}k\delta^2$, which is the same as (7.42b) derived above. □

Example 7.8. The work–energy equation for a relativistic particle may be derived from its intrinsic equation of motion (6.11). We form the scalar product of (6.11) with $\mathbf{v} = \dot{s}\mathbf{t}$ and recall equation (6.9) to obtain

$$\mathbf{F} \cdot \mathbf{v} = \frac{m}{1 - \beta^2}\ddot{s}\dot{s} = \frac{d}{dt}(mc^2). \tag{7.43a}$$

Integration with $d\mathbf{x} = \mathbf{v}dt$ leads to

$$\mathscr{W} = \int_{\mathscr{C}} \mathbf{F} \cdot d\mathbf{x} = mc^2 - \alpha, \tag{7.43b}$$

where α is an arbitrary constant. The *relativistic kinetic energy* is defined by

$$E \equiv mc^2. \tag{7.43c}$$

Therefore, (7.43b) yields the *relativistic work–energy equation*:

$$\mathscr{W} = \Delta E. \tag{7.43d}$$

The work done in moving a particle from its rest state where its mass is m_o and $\alpha = m_o c^2$ is determined by the following change in the relativistic kinetic energy:

$$\Delta E = \Delta mc^2. \tag{7.43e}$$

Here $\Delta m = m - m_o$ is the increase in the relativistic mass of the particle over its rest mass. Equation (7.43e) is the famous Einstein relation connecting mass and energy in the special theory of relativity.

When the speed v of the particle is much smaller than the speed of light c, so that $\beta = c/v \ll 1$, (6.9) may be written as $m = m_o(1 + \frac{1}{2}\beta^2)$, approximately. In this case, $\Delta mc^2 = (m - m_o)c^2 = \frac{1}{2}m_o\beta^2 c^2$, and hence the change in the kinetic energy (7.43e) from the rest state of the particle coincides with the change in the nonrelativistic kinetic energy (7.35): $\Delta E = \frac{1}{2}m_o v^2 = \Delta K$. The work–energy equation (7.43d) then reduces to (7.36). □

This concludes our introduction to the work–energy equation and some applications. We shall return to this principle following discussion of some related topics.

7.6. Potential Energy

Suppose we are given a general, continuous force function $\mathbf{F}(\mathbf{x})$ defined over a space region $\mathscr{R}$. For certain force functions, the work done is independent of the path in $\mathscr{R}$, while for others it is not. To determine the special property that a force function must have in order that its work may be path independent, the concept of a potential energy function is introduced. As a consequence, a simple criterion necessary and sufficient for existence of a potential energy function emerges. If the given force function satisfies this criterion everywhere in a so-called simply connected region* $\mathscr{R}$, it passes the test and the force is conservative; otherwise it is not.

* The region $\mathscr{R}$ where $\mathbf{F}(\mathbf{x})$ is defined is called *connected* if any two given points in $\mathscr{R}$ can be joined by an arc all of whose points are in $\mathscr{R}$. A region $\mathscr{R} = \mathscr{R}_1 \cup \mathscr{R}_2 \cup \mathscr{R}_3$, where $\mathscr{R}_1$ and $\mathscr{R}_3$ are tracts of land separated by a river spanned by a bridge $\mathscr{R}_2$, is connected; but if the bridge is washed away by a flood, the new region $\mathscr{R}^* = \mathscr{R}_1 \cup \mathscr{R}_3$ is not connected. A curve $\mathbf{x}(t)$ that does not cross itself at any point $t \in (a, b)$ is called *simple*; and it is said to be *closed* when joined at its end points, i.e. when $\mathbf{x}(a) = \mathbf{x}(b)$. A connected region $\mathscr{R}$ is thus called *simply connected* if every simple closed curve in $\mathscr{R}$ can be continuously shrunk to a point of $\mathscr{R}$. A connected plane region $\mathscr{R}$ containing a hole of any kind, for example, is not simply connected, because any curve that encircles the hole cannot be shrunk to a point of $\mathscr{R}$. The hole in $\mathscr{R}$ may be a single point which has been excluded from $\mathscr{R}$.

First, we show that if the scalar-valued integrand in (7.21) is an exact differential of a single-valued function† $V(\mathbf{x})$, the work done is independent of the path. Indeed, suppose that the integrand in (7.21) is an exact differential so that $\mathbf{F} \cdot d\mathbf{x} = -dV(\mathbf{x})$. (The negative sign is introduced for future convenience in Section 7.8.4.) Then, if $\mathscr{C}$ is any smooth curve joining two points at $\mathbf{x}_1$ and $\mathbf{x}_2$ in $\mathscr{R}$, we have

$$\int_{\mathscr{C}} \mathbf{F} \cdot d\mathbf{x} = \int_{\mathbf{x}_1}^{\mathbf{x}_2} -dV(\mathbf{x}) = -\Delta V, \tag{7.44}$$

where $\Delta V \equiv V(\mathbf{x}_2) - V(\mathbf{x}_1)$. The scalar-valued function $V(\mathbf{x})$ having this property is called the *potential energy*. Since $V(\mathbf{x})$ is single-valued, ΔV has a unique value determined only by the choice of end points. Then (7.21) shows that the work

$$\mathscr{W} = -\Delta V, \tag{7.45}$$

is independent of the path, and hence the force $\mathbf{F}(\mathbf{x})$ is conservative. *In consequence, the work done on a particle by a conservative force is equal to the decrease in the potential energy.*

7.6.1. Theorem on Conservative Force

But how are we to find this potential energy function? We show below that the components of a conservative force $\mathbf{F}$ are related to the partial derivatives of $V(\mathbf{x})$; and hence $V(\mathbf{x})$ may be found by integration of these equations. First, since $V(\mathbf{x}) = V(x, y, z)$, we have

$$dV(\mathbf{x}) = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz = \nabla V(\mathbf{x}) \cdot d\mathbf{x}, \tag{7.46}$$

wherein, by definition, the vector

$$\nabla V(\mathbf{x}) \equiv \frac{\partial V}{\partial x}\mathbf{i} + \frac{\partial V}{\partial y}\mathbf{j} + \frac{\partial V}{\partial z}\mathbf{k}. \tag{7.47}$$

This vector is called the *gradient* of $V(\mathbf{x})$, and sometimes it is written as $\operatorname{grad} V(\mathbf{x})$. The ∇ symbol for the gradient operation is defined by

$$\nabla \equiv \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}. \tag{7.48}$$

Consequently, if

$$\mathbf{F}(\mathbf{x}) = -\nabla V(\mathbf{x}), \tag{7.49}$$

† A function $y = f(x)$ is *single-valued* when $f(x)$ determines one and only one value y for each choice of x. The function $y = \sin x$, for example, is single-valued; but when its graph is turned through a right angle so that $y = \sin^{-1} x$, infinitely many values of y are determined for each choice of $x \in [-1, 1]$, and hence this function is *many-valued*. The parabola $y = x^2$ is a single-valued function for $x \in (-\infty, \infty)$, but the parabola $y = \pm x^{1/2}$ for $x > 0$, is not.

everywhere in a simply connected region $\mathscr{R}$, then $\mathbf{F}\cdot d\mathbf{x} = -dV$ follows from (7.46). In this case, the work done by $\mathbf{F}(\mathbf{x})$, as shown in (7.44), is independent of the path, and $\mathbf{F}(\mathbf{x})$ is conservative.

The relation (7.49) is a sufficient condition for $\mathbf{F}(\mathbf{x})$ to be conservative. Conversely, suppose that the work must be independent of the path in some region $\mathscr{R}$. We can then prove that there exists a scalar function $V(\mathbf{x})$, single-valued in $\mathscr{R}$, such that (7.49) holds everywhere in $\mathscr{R}$, except possibly at certain isolated points. Let $\mathbf{x}_1$ be an arbitrary fixed point and $\mathbf{x}$ a variable point in $\mathscr{R}$. Since $\mathscr{W}$ is independent of the path, the work integral over any curve $\mathscr{C}$ from $\mathbf{x}_1 = \mathbf{x}(s_1)$ to $\mathbf{x} = \mathbf{x}(s)$, where s denotes the distance along the path measured from any point on $\mathscr{C}$, say $\mathbf{x}_1$, is a single-valued function of the upper limit $\mathbf{x}$, and hence s alone. We thus write this work as

$$V(\mathbf{x}) = -\int_{\mathbf{x}_1}^{\mathbf{x}(s)} \mathbf{F}(\tilde{\mathbf{x}}) \cdot d\tilde{\mathbf{x}}, \tag{7.50}$$

where $\tilde{\mathbf{x}}$ is the dummy variable of integration introduced to avoid conflict with the variable limit. With the aid of Leibniz's rule (see Problem 6.28.), differentiation of the integral in (7.50) yields

$$\frac{dV(\mathbf{x})}{ds} = -\mathbf{F}(\mathbf{x}) \cdot \frac{d\mathbf{x}}{ds}. \tag{7.51}$$

An alternative derivation of (7.51) is left for the reader in the exercise below. It follows that $\mathbf{F}(\mathbf{x}) \cdot d\mathbf{x} = -dV(\mathbf{x}) = -\nabla V(\mathbf{x}) \cdot d\mathbf{x}$ is an exact differential that must be independent of the path. Therefore, $(\mathbf{F} + \nabla V) \cdot d\mathbf{x} = 0$ must hold for all $d\mathbf{x}(s)$; and hence (7.49) holds everywhere in $\mathscr{R}$. This completes the result summarized in the following theorem.

Theorem on conservative force: *A necessary and sufficient condition for* $\mathbf{F}(\mathbf{x})$ *to be conservative is that*

$$\mathbf{F}(\mathbf{x}) = -\nabla V(\mathbf{x}). \tag{7.52}$$

With (7.47), it is seen that (7.52) is equivalent to the following three partial differential equations for $V(\mathbf{x})$:

$$\frac{\partial V}{\partial x} = -F_x, \qquad \frac{\partial V}{\partial y} = -F_y, \qquad \frac{\partial V}{\partial z} = -F_z. \tag{7.53}$$

Exercise 7.5. Consider a neighboring point at $\mathbf{x} + \Delta\mathbf{x}$ and apply (7.50) to obtain the unique increment $\Delta V \equiv V(\mathbf{x} + \Delta\mathbf{x}) - V(\mathbf{x})$ in V. Let $\mathbf{x}^* \in [\mathbf{x}, \mathbf{x} + \Delta\mathbf{x}]$ be a point on $\mathscr{C}$ within the arc length element $\Delta s = |\Delta\mathbf{x}|$. Apply the mean value theorem of integral calculus to derive $\Delta V/\Delta s = -\mathbf{F}(\mathbf{x}^*) \cdot d\mathbf{x}^*/ds$ at $\mathbf{x}^*$. Then in the limit as $\Delta s = |\Delta\mathbf{x}| \to 0$ and hence $\mathbf{x}^* \to \mathbf{x}$, derive (7.51). □

Example 7.9. The relationship (7.52) between the potential energy function and a conservative force is illustrated for two situations. (i) Given $V(\mathbf{x})$, find the conservative force $\mathbf{F}(\mathbf{x})$; and, conversely, (ii) given a conservative force $\mathbf{F}(\mathbf{x})$, find the potential energy function $V(\mathbf{x})$. When $V(\mathbf{x})$ is known, the work done is readily determined by (7.45).

(i) If $V(\mathbf{x})$ is given, then the force $\mathbf{F}(\mathbf{x})$ derived from (7.52) is a conservative force. For example, consider

$$V(\mathbf{x}) = -\frac{cy^2}{2} + d, \tag{7.54a}$$

where c and d are constants. Then (7.47) gives $\nabla V = -cy\mathbf{j}$, and (7.52) shows that the force $\mathbf{F} = -\nabla V = cy\mathbf{j}$ is conservative.

(ii) Conversely, suppose we know that a force $\mathbf{F}(\mathbf{x})$ is conservative. Then (7.52) holds and the potential energy function $V(\mathbf{x})$ may be found by integration of (7.53). For example, we know that $\mathbf{F}(\mathbf{x}) = cy\mathbf{j}$ is a conservative force, hence (7.53) become

$$\frac{\partial V}{\partial x} = 0, \qquad \frac{\partial V}{\partial y} = -cy, \qquad \frac{\partial V}{\partial z} = 0. \tag{7.54b}$$

The first and last of these equations show that V is independent of x and z. Hence, $V(\mathbf{x}) = V(y)$ is at most a function of y; and the second equation in (7.54b) becomes $dV/dy = -cy$. The solution of this equation is given by (7.54a) in which d is an arbitrary integration constant that may be chosen to meet any convenient purpose. For example, we may wish to define $V(0) = 0$; then $d = 0$ for this choice.

We recall that $\mathbf{F} = cy\mathbf{j}$ is the same force encountered earlier in (7.26a) and for which the work done in (7.26b) is independent of the path. An easier calculation for the work done by a conservative force is now provided by the rule (7.45). Let the potential energy be given by (7.54a), for example. Then, by (7.45), the work done on any path between the end points $\mathbf{x}_1 = (0, 0)$ and $\mathbf{x}_2 = (1, a)$ is

$$\mathscr{W} = -\Delta V = -[V(\mathbf{x}_2) - V(\mathbf{x}_1)] = -[-\frac{ca^2}{2} + d - d] = \frac{ca^2}{2}, \tag{7.54c}$$

precisely the result derived differently in (7.26b). Clearly, the physical dimensions of potential energy are those of work: $[V] = [\mathscr{W}] = [FL]$. □

Notice in this example that the constant potential energy $V_0 = d$ in (7.54a) is of no consequence whatsoever in evaluating either the force or the work done. This is typical—only differences in potential energy are relevant. A constant force $\mathbf{F}_c$, for example, is a conservative force that does work given by (7.27); so, by (7.45), the corresponding potential energy function may be written as $V(\mathbf{x}) = V_0 - \mathbf{F}_c \cdot \Delta\mathbf{x}$, where $\Delta\mathbf{x} = \mathbf{x} - \mathbf{x}_0$ is the particle displacement vector. Similarly, by (7.45), the potential energy function for the conservative linear force $\mathbf{F}_L = \alpha\mathbf{x}$ may be read from (7.31): $V(\mathbf{x}) = V_0 - \frac{1}{2}\alpha(x^2 - x_0^2)$. In these and any other potential energy functions, the arbitrary constant potential energy V_0 is unimportant, it affects neither the force nor the work done. Therefore, sometimes the constant V_0 is suppressed in expressions for the potential energy, and sometimes its value is fixed for convenience. No generality is lost, if V_0 is discarded. For a constant force, this is equivalent to our assigning the value $V(\mathbf{x}_0) = V_0 = 0$ at $\mathbf{x} = \mathbf{x}_0$, say. In this case, $\mathbf{x} = \mathbf{x}_0$ becomes the datum point of zero potential energy with respect to which $V(\mathbf{x}) = -\mathbf{F}_c \cdot \Delta\mathbf{x}$. Another possibility is to absorb the constant part of the work terms in V_0 and write $V(\mathbf{x}) = \hat{V}_0 - \mathbf{F}_c \cdot \mathbf{x}$, or to choose $\hat{V}_0 \equiv V_0 + \mathbf{F}_c \cdot \mathbf{x}_0 = 0$ so that $V(\mathbf{x}) = -\mathbf{F}_c \cdot \mathbf{x}$ now vanishes at $\mathbf{x} = \mathbf{0}$. Then $\mathbf{x} = \mathbf{0}$ serves as the datum point of zero potential energy. A similar thing may be done for any other potential energy function.

7.6.2. The Needle in the Haystack

One very important question remains—How can we know if a given force $\mathbf{F}(\mathbf{x})$ admits a potential energy function or not, without our actually having to find it? Otherwise, the problem we face resembles the search for a needle in a haystack with no knowledge that there is a needle to be found. Therefore, before we begin looking for the needle, it would be a good idea to first locate some sort of detection device. Then, if we detect it, we might continue and try to find it; otherwise, we discontinue the search. In the same spirit, before we start searching for a potential energy function, it is best to find an easy test to which we may subject any suitably continuous force $\mathbf{F}(\mathbf{x})$ to determine first, if a corresponding potential energy function exists. We are going to show that the relation (7.52) can hold when and only when the force $\mathbf{F}(\mathbf{x})$ satisfies the condition

$$\boldsymbol{\nabla} \times \mathbf{F} = \mathbf{0}, \tag{7.55}$$

everywhere in a simply connected region $\mathscr{R}$. (The importance of $\mathscr{R}$ being simply connected is shown in Problem 7.22.) The operation $\boldsymbol{\nabla} \times \mathbf{F}$ is called the *curl of* $\mathbf{F}$, and sometimes (7.55) is written as curl$\mathbf{F} = \mathbf{0}$. The curl operation is defined with the aid of (7.48): curl$\mathbf{F} = (\mathbf{i}\partial/\partial x + \mathbf{j}\partial/\partial y + \mathbf{k}\partial/\partial z) \times (F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k})$, which is more conveniently represented by the familiar determinant-like, vector product representation in its expansion across the top row:

$$\boldsymbol{\nabla} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_x & F_y & F_z \end{vmatrix}. \tag{7.56}$$

The rule (7.55) is a neat device for detecting the possible existence of our needle in the haystack—it provides the test for existence of a potential energy function based on the following useful theorem.

Criterion for existence of a potential energy function: *A force* $\mathbf{F}(\mathbf{x})$ *is conservative in a simply connected region* $\mathscr{R}$ *if and only if* curl $\mathbf{F}(\mathbf{x}) = \mathbf{0}$ *everywhere in* $\mathscr{R}$. *In consequence, there exists a potential energy function such that* $\mathbf{F}(\mathbf{x}) = -\boldsymbol{\nabla} V(\mathbf{x})$ *holds everywhere in* $\mathscr{R}$.

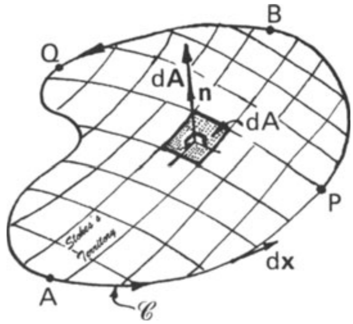


Figure 7.8. A simple closed curve bounding a region of area $\mathscr{A}$ in Stokes's theorem.

That (7.55) is necessary follows upon substitution of (7.53) into (7.56), which shows that (7.55) holds everywhere provided the continuous function $\mathbf{F}(\mathbf{x})$ is single-valued with continuous partial derivatives so that the order of the mixed partial derivatives of $V(\mathbf{x})$ may be reversed. Then, for example, $\partial F_y/\partial x - \partial F_x/\partial y = \partial^2 V/\partial x \partial y - \partial^2 V/\partial y \partial x = 0$.

Conversely, if (7.56) vanishes everywhere, the line integral of $\mathbf{F}(\mathbf{x})$ is independent of the path; hence, $\mathbf{F}$ has the form (7.52). The proof follows from Stokes's theorem of vector integral calculus, namely,

$$\mathscr{W} = \oint_{\mathscr{C}} \mathbf{F} \cdot d\mathbf{x} = \int_{\mathscr{A}} \operatorname{curl}\mathbf{F} \cdot d\mathbf{A}, \tag{7.57}$$

in which $d\mathbf{A}$ is the elemental vector area of the region $\mathscr{A}$ bounded by a simple closed curve $\mathscr{C}$, shown in Fig. 7.8. The symbol $\oint$ means that the integral is around the closed path $\mathscr{C}$, counterclockwise, with the region $\mathscr{A}$ on the left-hand side, as suggested in Fig. 7.8. The proof of (7.57) may be found in standard works on vector analysis. See, for example, the reference by Lass.

Stokes's theorem is applied to prove that $\operatorname{curl}\mathbf{F} = \mathbf{0} \implies \mathbf{F} = -\nabla V$. We thus require that (7.55) hold everywhere in a region containing $\mathscr{A}$, then (7.57) implies that

$$\oint_{\mathscr{C}} \mathbf{F} \cdot d\mathbf{x} = 0, \tag{7.58}$$

for every simple *closed* path in $\mathscr{R}$. Now, since only the unit tangent vector is reversed when a path is traversed in the opposite sense, we see in Fig. 7.8 that $\int_{BQA} \mathbf{F} \cdot d\mathbf{x} = -\int_{AQB} \mathbf{F} \cdot d\mathbf{x}$, where A and B are any two points on $\mathscr{C}$. Thus, by (7.58),

$$\int_{APB} \mathbf{F} \cdot d\mathbf{x} + \int_{BQA} \mathbf{F} \cdot d\mathbf{x} = \int_{APB} \mathbf{F} \cdot d\mathbf{x} - \int_{AQB} \mathbf{F} \cdot d\mathbf{x} = 0,$$

for any two points A and B on $\mathscr{C}$, that is,

$$\int_{APB} \mathbf{F} \cdot d\mathbf{x} = \int_{AQB} \mathbf{F} \cdot d\mathbf{x}.$$

Consequently, the condition (7.58) states, equivalently, that the work done by $\mathbf{F}$ is independent of the path joining A to B. However, this means that there exists a continuous single-valued, differentiable function $V(\mathbf{x})$ such that (7.52) holds everywhere in $\mathscr{R}$. This completes the proof. The thrust of the theorem is summarized as follows:

The work done by a force $\mathbf{F}(\mathbf{x})$ *is independent of the path, hence* $\mathbf{F}(\mathbf{x})$ *is conservative, if and only if the* curl$\mathbf{F}(\mathbf{x})$ *vanishes everywhere in the simply connected region* $\mathscr{R}$ *bounded by the closed path* $\mathscr{C}$, *and therefore if and only if* $\mathbf{F}(\mathbf{x})$ *is the gradient of a single-valued twice continuously differentiable potential energy function* $V(\mathbf{x})$. *That is, symbolically,*

$$\oint_{\mathscr{C}} \mathbf{F}\cdot d\mathbf{x} = 0 \iff \nabla\times\mathbf{F} = \mathbf{0} \iff \mathbf{F} = -\nabla V. \tag{7.59}$$

To illustrate our criterion for existence of a potential energy function, let us return to Example 7.4, page 231, and determine if $\mathbf{F}(\mathbf{x})$ in (7.24a) is conservative or not. We compute curl$\mathbf{F}$ using (7.56) and find

$$\nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ bxy & cy & 0 \end{vmatrix} = \mathbf{k}(-bx) \neq \mathbf{0} \text{ everywhere.}$$

Therefore, no potential energy function exists; and hence $\mathbf{F}$ is not conservative. Indeed, $\mathbf{F}$ is conservative if and only if $b = 0$, as shown earlier. Only then does the force (7.24a) admit the potential energy function (7.54a).

7.7. Some Basic Potential Energy Functions

In anticipation of future applications, the potential energy function for the apparent gravitational force near the surface of the Earth, its variation with distance from the Earth, and the elastic potential energy for a linear spring are described next.

7.7.1. Gravitational Potential Energy

The work done by the constant gravitational force is given by (7.28); and by (7.45), we write $\Delta V_g = -\mathscr{W}_g$. *Therefore, the gravitational potential energy of a particle or center of mass object on or near the surface of the Earth is*

$$V_g = mgh, \tag{7.60}$$

in which $h = z - z_0$ *is measured vertically from* $z = z_0$, *the datum point of zero gravitational potential energy.* The gravitational potential energy increases with h above the datum level z_0 and decreases, becoming negative with h, below the z_0

reference level. The question of how close to the surface the particle must be for (7.60) to hold is discussed below.

The elementary rule (7.60) ignores the variation in the gravitational field strength with the distance from the Earth. To account for the elevation effect on the gravitational potential energy, let us consider a particle P of mass m that is free to move in the gravitational field of the Earth whose mass is M. In terms of its spherical coordinates in a Cartesian reference frame fixed at the center of the Earth at O, the gravitational force (5.58) exerted on P has the familiar form

$$\mathbf{F}(P;\mathbf{x}) = m\mathbf{g}(\mathbf{x}) = -\frac{GmM}{r^2}\mathbf{e}_r, \tag{7.61}$$

where $\mathbf{x} = r\mathbf{e}_r \neq \mathbf{0}$ is the radius vector of P from O. The reader may confirm through the following exercise that the gravitational force (7.61) is conservative.

Exercise 7.6. With $\mathbf{e}_r = \mathbf{x}/r$, express (7.61) in terms of rectangular Cartesian coordinates (x, y, z), and show that $\operatorname{curl}\mathbf{F} = \mathbf{0}$ everywhere. □

In accordance with (7.61), the conservative gravitational force at every point on the surface of a sphere of radius r is the same for every direction $\mathbf{e}_r = \mathbf{e}_r(\theta, \phi)$; so, by (7.61) and (7.52), the potential energy is at most a function of r. Therefore, with $\nabla V = \partial V(r)/\partial r \mathbf{e}_r$ and (7.61), equation (7.52) yields the relation $dV/dr = GmM/r^2$ whose integration delivers *the gravitational potential energy as a function of the distance r from the center of the Earth*:

$$V(r) = V_0 - \frac{GmM}{r}. \tag{7.62}$$

The reader may confirm the result (7.62) by an alternative derivation described in the following exercise based on an alternative proof that (7.61) is conservative.

Exercise 7.7. Show that the work done by the gravitational force (7.61) acting on a particle is independent of the path, and hence this force is conservative. Then deduce (7.62). □

The constant V_0 may be chosen so that $V(A) = 0$ at $r = A$, the surface of the Earth. Then $V_0 = GmM/A$ and (7.62) becomes

$$V(r) = GmM\left(\frac{1}{A} - \frac{1}{r}\right). \tag{7.63}$$

Recalling (5.61) for the acceleration of gravity, namely, $g = GM/A^2$, introducing $h(r) \equiv r - A$ into (7.63) and writing $V(r) = \hat{V}(h(r))$, we obtain the *gravitational potential energy as a function of the elevation h from the surface of the Earth*:

$$\hat{V}(h) = mgh\left(\frac{1}{1 + \dfrac{h}{A}}\right). \tag{7.64}$$

The elementary formula (7.60) thus follows from (7.64) when the particle is sufficiently close to the surface so that the term h/A may be considered negligible compared to unity. Otherwise, to account for the variation of gravity with elevation, the more precise relation (7.62) or (7.64) must be used. For example, at six miles (about 10 km) above the surface of the Earth, the approximate cruising altitude of a commercial jet airliner, the potential energy variance of (7.60) from (7.64) is only 0.16% (larger); and for a satellite at 250 miles (about 420 km) above the Earth, the variance is roughly 6.3% (larger). Hence, in most applications (7.60) is a very good approximation for motion on or near the surface of the Earth.

7.7.2. Elastic Potential Energy of a Spring

The elastic work done by a linear spring relative to its undeformed state is given by (7.33). Therefore, with (7.45), we write $\Delta V_e = -\mathscr{W}_e$ to obtain *the elastic potential energy stored by the spring*:

$$V_e = \tfrac{1}{2}kx^2, \tag{7.65}$$

wherein x is the change of length of the spring measured from its undeformed state, the reference state of vanishing potential energy. For a linear spring, the elastic potential energy function is the same in both compression $x < 0$ and tension $x > 0$.

7.8. General Conservation Principles

We now derive from the Newton–Euler law of motion and the work–energy principle for a center of mass particle the conservation principles of linear momentum, moment of momentum, and energy. We start with a general conservation theorem from which the two momentum conservation laws follow. The work–energy equation leads to the important principle of conservation of energy for a particle acted upon by conservative forces; and, finally, the general form of the work–energy principle in which the total work is split into its conservative and nonconservative parts is presented. Afterwards, these fundamental laws are illustrated in several further applications of physical interest.

7.8.1. A General Conservation Theorem

Let $\mathbf{A}(t)$ be a vector-valued function of time that is equal to the time derivative of another vector-valued function $\mathbf{u}(t)$:

$$\mathbf{A}(t) = \frac{d\mathbf{u}(t)}{dt}. \tag{7.66}$$

Let **e** be a constant unit vector and form the scalar product

$$\mathbf{A} \cdot \mathbf{e} = \frac{d}{dt}(\mathbf{u} \cdot \mathbf{e}). \tag{7.67}$$

We note that $\mathbf{A} \cdot \mathbf{e}$ and $\mathbf{u} \cdot \mathbf{e}$ are the respective components of **A** and **u** in the direction **e**. Thus, $\mathbf{A} \cdot \mathbf{e} = 0$, and hence **A** is perpendicular to **e**, if and only if $\mathbf{u} \cdot \mathbf{e} = constant$, in which case the quantity **u** is said to be *conserved in the direction* **e**. In summary, (7.67) reveals a useful theorem with application to the principles of mechanics.

General conservation theorem: *Consider the vector differential equation* $\mathbf{A(t)} = \dot{\mathbf{u}}(\mathbf{t})$ *and a fixed direction* **e**. *Then the component of* **A** *in the direction* **e** *is zero if and only if the component of* **u** *in the direction* **e** *is constant, that is, when and only when the quantity* **u** *is conserved in the direction* **e**.

7.8.2. The Principle of Conservation of Linear Momentum

Equation (5.34) has the form (7.66), so for any constant vector **e**,

$$\mathbf{F} \cdot \mathbf{e} = \frac{d}{dt}(\mathbf{p} \cdot \mathbf{e}), \tag{7.68}$$

which thus yields the following conservation law.

The principle of conservation of linear momentum: *The component of the force acting on a center of mass object in a fixed direction* **e** *vanishes for all time if and only if its momentum in the direction* **e** *is constant*:

$$\mathbf{F} \cdot \mathbf{e} = 0 \iff \mathbf{p} \cdot \mathbf{e} = \text{const.} \tag{7.69}$$

Further, the Newton–Euler law (5.34) shows that *the linear momentum is a constant vector when and only when the total force on the particle is zero*. The same result follows from (7.69) for all directions **e**. The principle (7.69), valid for *all time* in the motion, differs from our earlier conservation rule (7.12) for a system of *two* particles whose momentum is constant only during the impulsive instant. An easy application of the rule (7.69) follows.

Example 7.10. A particle of mass m is released from rest at A and slides down a smooth circular track of radius R shown in Fig. 7.9. At the lowest point B, the particle is projected horizontally and continues its motion until it strikes the ground at C. Determine the horizontal component of the particle's velocity at C.

Solution. The free body diagram in Fig. 7.9 on the path from B to C shows that no horizontal forces act on the particle. Therefore, the linear momentum in

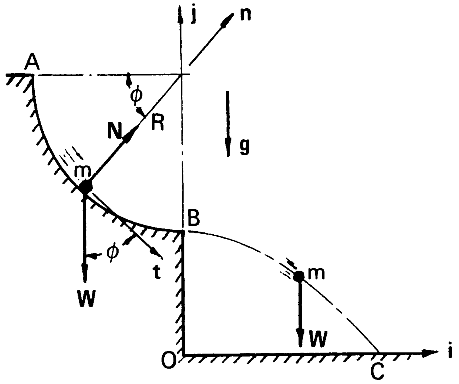


Figure 7.9. Application of the principles of conservation of momentum and work–energy.

the fixed horizontal direction **i** is conserved: $\mathbf{p} \cdot \mathbf{i} = m\dot{x} = \gamma$, a constant. Consequently, on the entire path BC, specifically at C, the horizontal component of the particle's velocity $\mathbf{v} \cdot \mathbf{i} = \dot{x}$ is a constant whose value is determined by its speed at point B. This value may be found by application of the work–energy principle.

The free body diagram of the particle on the circular path AB is shown in Fig. 7.9. The normal force **N** is workless on AB, while gravity does work $\mathscr{W}_g = mgR$ in reaching B. Hence, the total work done by the forces acting on m is $\mathscr{W} = mgR$. The increase in the kinetic energy as the particle slides from rest at A to the end state at B is $\Delta K = \frac{1}{2}mv_B^2$. The work-energy principle $\mathscr{W} = \Delta K$ determines the speed at B, and hence the horizontal component of the particle's velocity at C is given by

$$\dot{x} = v_B = \sqrt{2gR}. \qquad \square$$

Exercise 7.8. What is the normal force exerted on m by the surface at B? □

7.8.3. The Principle of Conservation of Moment of Momentum

The moment of momentum principle (6.79) also has the form (7.66). Therefore, for a fixed direction **e**,

$$\mathbf{M}_O \cdot \mathbf{e} = \frac{d}{dt}(\mathbf{h}_O \cdot \mathbf{e}), \tag{7.70}$$

where $\mathbf{M}_O \cdot \mathbf{e}$, the component of $\mathbf{M}_O$ in the direction **e**, characterizes the turning effect of the force about a line $\mathscr{L}$ through O having the direction **e**, as shown in Fig. 7.10. Thus, $\mathbf{M}_O \cdot \mathbf{e}$ is *the moment of the force about the axis* **e** *through* O. Similarly, $\mathbf{h}_O \cdot \mathbf{e}$ is *the moment of momentum about the axis* **e** *through* O. In these terms, the following conservation theorem is evident from (7.70).

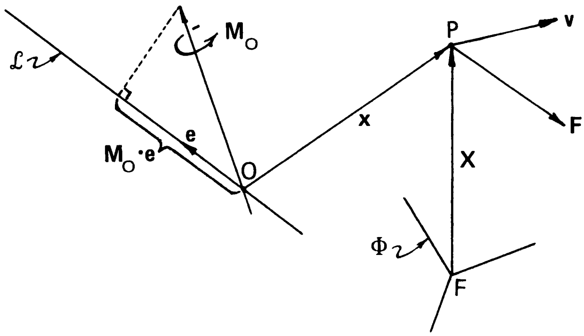


Figure 7.10. Schema for the torque about a line through the moment point O.

The principle of conservation of moment of momentum: *The moment of the force about an axis* **e** *through a fixed point O in an inertial frame* Φ *vanishes for all time when and only when the corresponding moment of momentum about that axis is constant*:

$$\mathbf{M}_O \cdot \mathbf{e} = 0 \Longleftrightarrow \mathbf{h}_O \cdot \mathbf{e} = \text{const.} \tag{7.71}$$

Moreover, the moment of momentum principle (6.79) shows that *the moment* $\mathbf{M}_O$ *about a fixed point vanishes for all time when and only when the moment of momentum is a constant vector.* The principle (7.71), valid for *all time* in the motion, differs from our earlier conservation rule (7.20) for a system of *two* particles whose moment of momentum is constant only during the impulsive instant. A classical application of our conservation law follows.

Example 7.11. *Central Force Motion and Kepler's Second Law.* A force directed invariably along a line through a fixed point is called a *central force.* A familiar example of a central force is the tension in a pendulum string; another is the gravitational force exerted by the Earth on a satellite shown in Fig. 7.11. Derive **Kepler's second law:** *A particle in motion under a central force alone must move in a plane; and if its path is not a straight line through the fixed central point O, its position vector from the fixed point sweeps out equal areas in equal intervals of time.*

Solution. Consider a central force **F** directed through the fixed origin O of an inertial frame $\varphi = \{O; \mathbf{e}_k\}$ in Fig. 7.11. Then the moment of **F** about O is zero: $\mathbf{M}_O = \mathbf{x} \times \mathbf{F} = 0$, wherein $\mathbf{x} = r\mathbf{e}_r$; and the moment of momentum principle (6.79) shows that the moment of momentum about O is a constant vector:

$$\mathbf{h}_O = \mathbf{x} \times m\mathbf{v}, \quad \text{a constant.} \tag{7.72a}$$

It follows that $\mathbf{h}_O = \mathbf{0}$ if and only if the position vector **x** is parallel to the velocity: $\mathbf{v} = k\mathbf{x}$, where k is a constant. In this instance the motion $\mathbf{x}(t) = \mathbf{x}_0 e^{kt}$, with $\mathbf{x}_0 = \mathbf{x}(0)$ initially, is along a straight line through O. Otherwise, when

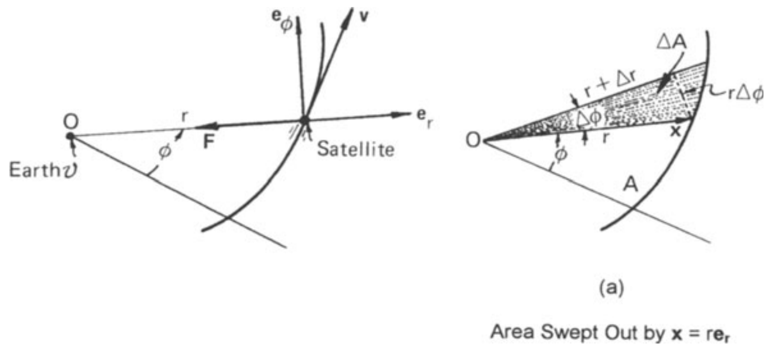


Figure 7.11. Satellite motion under a central force, and the orbital area swept out by the radius vector in an infinitesimal time.

$\mathbf{h}_O \neq \mathbf{0}$, both $\mathbf{x}$ and $\mathbf{v}$ are always perpendicular to the constant vector $\mathbf{h}_O$, and hence the particle must move in a plane whose vector equation is (7.72a). *Consequently, all central force motions are plane motions.*

To establish Kepler's equal area rule, we introduce polar coordinates and write $\mathbf{v} = \dot{r}\mathbf{e}_r + r\dot{\phi}\mathbf{e}_\phi$ and $\mathbf{x} = r\mathbf{e}_r$. Then $\mathbf{h}_O = h\mathbf{e}_z$ is normal to the plane of motion, and (7.72a) yields $mr^2\dot{\phi}\mathbf{e}_z = h\mathbf{e}_z$, that is,

$$r^2\dot{\phi} = \frac{h}{m} = \gamma, \quad \text{a constant.} \tag{7.72b}$$

Now, it may be seen in Fig. 7.11a that the element ΔA of the plane area swept out by the position vector is $\Delta A = \frac{1}{2}(r + \Delta r)r\Delta\phi$. Therefore, in the $\text{limit}_{\Delta t \to 0}\Delta A/\Delta t$, we have

$$\dot{A} = \tfrac{1}{2}r^2\dot{\phi}. \tag{7.72c}$$

This gives the rate of change of the area swept out by the position vector. Thus, with (7.72b), $\dot{A} = \gamma/2$; hence $A(t) = \frac{1}{2}\gamma t + A_0$, where $A_0 = A(0)$ is a constant, usually taken as zero. This is **Kepler's Second Law:** *The radius vector sweeps out equal areas in equal intervals of time.* □

Exercise 7.9. Apply the moment of momentum principle (6.79) to find $\mathbf{v}_B$ in Example 7.10, page 249. □

7.8.4. The Energy Principle

The important and useful principle of conservation of energy is derived next, and its equivalence with one of the Newton–Euler scalar equations of motion is demonstrated. Afterwards, the work–energy equation is recast in a form that separates the total work into its conservative and nonconservative parts. This procedure leads to the general energy principle, which includes the conservation law as a special case and shows clearly the roles of both nonconservative forces and workless normal forces in the work–energy equation.

7.8.4.1. The Principle of Conservation of Energy

The work–energy equation, $\mathscr{W} = \Delta K$, is valid for every conservative or nonconservative total force $\mathbf{F}(\mathbf{x})$. In addition, every conservative force is characterized by a scalar potential energy function $V(\mathbf{x})$ such that $\mathscr{W} = -\Delta V$. Therefore, when the total force acting on a particle (or center of mass object) is conservative, we have $\Delta K + \Delta V = \Delta(K + V) = 0$, from which the following conservation law is evident.

Principle of conservation of energy: *The sum of the kinetic and the potential energies of a particle acted upon by purely conservative forces is constant throughout the motion*:

$$K + V = E, \text{ a constant.} \tag{7.73}$$

Use of the negative sign for the potential energy in (7.44) was motivated by our desire at this point to assign a simple additive property in the conservation law (7.73). In general, the energy constant E is fixed by specified conditions at any instant in the motion. In accordance with (7.73), if the kinetic energy increases by some amount, the potential energy must decrease by the same amount, and vice versa. Indeed, when the kinetic energy in any motion of a conservative system attains its maximum value, the potential energy at that place must be least, and vice versa. Thus, for every conservative system

$$K_{\max} + V_{\min} = K_{\min} + V_{\max} = E, \text{ a constant.} \tag{7.74}$$

In a mechanical vibrations problem, for example, $K_{\min} = 0$ at the extreme position of instantaneous rest in the oscillation of the load. The same thing holds in any problem where the motion of a particle begins from rest. In either case, at the corresponding instant in the motion, the energy constant in (7.74) is $E = V_{\max}$. Specific examples are provided later on.

Since (7.73) was derived from the Newton–Euler law for a purely conservative total force $\mathbf{F}(\mathbf{x}) = -\nabla V(\mathbf{x})$, this equation may be applied conversely to derive a single equivalent scalar equation of motion, as shown in (7.40) for the general work–energy equation. It is instructive to review this important result for a conservative system. In terms of intrinsic variables, we have $K = \frac{1}{2}m\dot{s}^2$ and $V = V(\mathbf{x}(s))$. Thus, by (7.73),

$$\tfrac{1}{2}m\dot{s}^2 + V(\mathbf{x}(s)) = E, \text{ a constant.} \tag{7.75}$$

This is to be compared with the general equation (7.39). Differentiation of (7.75) with respect to the arc length parameter s and use of the relation $dV/ds =$

$\partial V/\partial \mathbf{x} \cdot d\mathbf{x}/ds = -\mathbf{F} \cdot \mathbf{t} = -F_t$, where F_t is the tangential component of the total conservative force $\mathbf{F}$, yields $m\ddot{s} - F_t = 0$; hence,

$$K + V = E \iff m\ddot{s} = F_t. \tag{7.76}$$

The same result also may be obtained by differentiation of (7.75) with respect to time. Of course, this is the same as (7.40) applied to a conservative system. Thus, the principle of conservation of energy for a center of mass object is the first integral of the intrinsic, tangential component of the Newton–Euler law. In view of the equivalence relation (7.76), the principle of conservation of energy is especially useful in single degree of freedom dynamical problems. Notice that $V_{\min}$ occurs at places in the motion for which $dV/ds = 0$ holds, that is, at places for which $F_t = 0$. For conservative systems, these are static equilibrium positions of the particle.

Clearly, since forces of constraint perpendicular to the path do no work in the motion, these normal forces contribute nothing to the energy of an otherwise conservative system of forces. Accordingly, the energy principle can provide no information about such forces of constraint. The forgoing conclusions and remarks are illustrated in an example.

Example 7.12. (i) Apply the principle of conservation of energy to find the velocity $\mathbf{v}_B$ at which the particle in Example 7.10, page 249, is projected from point B shown in Fig. 7.9, and show that an arbitrary constant reference potential energy does not alter the conclusion. (ii) Derive from the energy equation the equivalent Newton–Euler scalar equation of motion for the mass on the circular path AB.

Solution of (i). First, we need to confirm that the energy principle (7.73) may be applied. The forces that act on the mass m on the circular path AB are shown in Fig. 7.9. Since the weight $\mathbf{W}$ is a conservative force and the normal surface reaction force does no work on AB, the total energy is conserved.

The point A is clearly a convenient datum for zero gravitational potential energy. However, we recall that only *differences* in the potential energy are relevant. Moreover, an arbitrary reference potential energy V_0 does not alter the energy balance in (7.73), for the same constant potential energy will appear in both sides of the equation. To demonstrate this, let us choose an arbitrary value V_0 for the reference potential energy at A. Since the particle is at rest at A, the kinetic energy at A is zero. Thus, initially the total energy is $E = K_A + V_A = V_0$. With A as the reference state, the potential energy of the mass m at point B is $V_B = V_0 - mgR$, and its kinetic energy is $K_B = \frac{1}{2}mv_B^2$, where $\mathbf{v}_B = v_B\mathbf{i}$ is the velocity of m when it projects from B. The energy principle (7.73) now yields

$$K_B + V_B = \frac{1}{2}mv_B^2 + V_0 - mgR = E = V_0, \tag{7.77a}$$

that is,

$$v_B = \sqrt{2gR}, \tag{7.77b}$$

so $\mathbf{v}_B = \sqrt{2gR}\mathbf{i}$, the same result found in Example 7.10 by application of the work–energy principle. Notice that the arbitrary reference potential energy V_0 cancels from (7.77a). Hence, the particular value assigned to the datum potential energy V_0 has no effect whatsoever on the solution.

Solution of (ii). The scalar equation of motion equivalent to the energy principle is readily derived from the energy equation. At an arbitrary point on the path AB, $K = \frac{1}{2}mR^2\dot{\phi}^2$ and $V = -mgR\sin\phi$, where we now fix $V_0 = 0$ at A. Since $E = 0$ at A, (7.73) yields the energy equation on the path AB,

$$\frac{1}{2}mR^2\dot{\phi}^2 - mgR\sin\phi = 0. \tag{7.77c}$$

Differentiation of (7.77c) with respect to the path variable ϕ (or with respect to t) yields the equivalent tangential component of the Newton–Euler equation of motion, namely,

$$mR\ddot{\phi} - W\cos\phi = 0. \tag{7.77d}$$

Notice, in agreement with (7.76), that $R\ddot{\phi} = \ddot{s}$ is the tangential component of the acceleration, and $W\cos\phi = F_t$ is the conservative tangential component of the total force $\mathbf{F} = \mathbf{W} + \mathbf{N}$ acting on m in Fig. 7.9, whose workless normal component is $F_n = N - W\sin\phi$. □

Let the reader consider the following example.

Exercise 7.10. Apply the principle of conservation of energy to solve Example 7.7, page 238. Derive the equation for the maximum spring deflection resulting from the impact by a mass m falling through a height h shown in Fig. 7.7. □

7.8.4.2. Remarks on Time Varying Potential Functions

The centripetal acceleration of a particle in a moving frame $\varphi = \{O; \mathbf{e}_k\}$ gives rise to a central directed, apparent centrifugal force $\mathbf{P}(\mathbf{x}, t) \equiv -m\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{x})$. So, consider a radial motion with $\mathbf{x} = r\mathbf{e}_r$ in φ and $\boldsymbol{\omega}(t) = \omega(t)\mathbf{e}_z = \alpha t\mathbf{e}_z$, say, then $\mathbf{P}(\mathbf{x}, t) = mr\omega^2\mathbf{e}_r$ in a cylindrical reference basis. Notice that this force has a potential function $\mathscr{V}(\mathbf{x}, t) = -\frac{1}{2}mr^2\omega^2$, such that $\mathbf{P} = -\nabla\mathscr{V}(\mathbf{x}, t) = -\partial\mathscr{V}/\partial r\mathbf{e}_r = mr\omega^2\mathbf{e}_r$. But this is not a conservative force, because the potential function $\mathscr{V}(\mathbf{x}, t)$ varies with both position and time, indeed, with $\omega = \alpha t$, $\partial\mathscr{V}/\partial t = -mr^2\alpha^2 t$. Moreover, with $d\mathbf{x} = dr\mathbf{e}_r$, the work done by this force, defined by $\mathscr{W}(\mathbf{x}, t) = \int \mathbf{P}(\mathbf{x}, t)\cdot d\mathbf{x} = m\omega^2\int r\,dr = -\mathscr{V}(\mathbf{x}, t)$, to within an arbitrary constant, also varies with time. In fact, it is possible to consider more general kinds of forces $\mathbf{F}(\mathbf{x}, \dot{\mathbf{x}}, t)$ for which the work done is defined by $\mathscr{W}(\mathbf{x}, \dot{\mathbf{x}}, t) = \int \mathbf{P}(\mathbf{x}, \dot{\mathbf{x}}, t)\cdot d\mathbf{x}$. We shall not have an occasion to encounter these here.

Although it is possible to have a scalar-valued, time dependent function $V(\mathbf{x}, t)$ for which $\mathbf{F}(\mathbf{x}, t) = -\nabla V(\mathbf{x}, t)$, it is important to bear in mind that *the principle of conservation of energy holds only for conservative forces with potential functions that are independent of time.* To understand the reason for this, let us suppose that the potential energy function $V(\mathbf{x}, t)$ is time dependent. First, note that (7.34) holds for a force $\mathbf{F} = \mathbf{F}(\mathbf{x}, t)$. Consequently, in the present case,

$$\frac{dK(\mathbf{x}, t)}{dt} = \mathbf{F}(\mathbf{x}, t) \cdot \mathbf{v}(t) = -\nabla V(\mathbf{x}, t) \cdot \mathbf{v}(t). \tag{7.78}$$

However, since $V(\mathbf{x}, t)$ now depends explicitly on both the position vector $\mathbf{x}(t) = (x(t), y(t), z(t))$ and the time t, the total time rate of change of $V(\mathbf{x}, t)$ is

$$\frac{dV(\mathbf{x}, t)}{dt} = \frac{\partial V(\mathbf{x}, t)}{\partial \mathbf{x}} \cdot \frac{d\mathbf{x}(t)}{dt} + \frac{\partial V(\mathbf{x}, t)}{\partial t} = \nabla V(\mathbf{x}, t) \cdot \mathbf{v}(t) + \frac{\partial V(\mathbf{x}, t)}{\partial t}.$$

Adding this result to (7.78), we obtain

$$\frac{d}{dt}(K + V) = \frac{\partial V}{\partial t}. \tag{7.79}$$

Therefore, the time rate of change of the sum of the kinetic and potential energies is not zero, and hence the principle of conservation of energy does not hold when $V = V(\mathbf{x}, t)$. In fact, (7.79) shows that *the energy conservation law holds if and only if the potential energy is independent of time, i.e. when and only when $V = V(\mathbf{x})$*. Consequently, only those forces derivable from a potential energy $V = V(\mathbf{x})$, which is a function of position alone, are conservative forces; all other forces, even though they might have a potential function, are not conservative.

7.8.4.3. The General Energy Principle

It is useful to recast the work–energy equation in terms of the conservative and nonconservative parts of the total work done. We thus separate the total force $\mathbf{F}$ acting on a particle into its conservative part $\mathbf{F}_C = -\nabla V(\mathbf{x})$ and its nonconservative part $\mathbf{F}_N$. Clearly, any force perpendicular to the path, whatever its nature, necessarily is workless, and hence contributes nothing to the total work done. Therefore, with the aid of (7.21) and the work–energy equation (7.36), the total work done by $\mathbf{F} = \mathbf{F}_C + \mathbf{F}_N$ is related to the energy in accordance with

$$\Delta K = \int_{\mathscr{C}} \mathbf{F}_C \cdot d\mathbf{x} + \int_{\mathscr{C}} \mathbf{F}_N \cdot d\mathbf{x} = -\Delta V + \mathscr{W}_N,$$

in which $\mathscr{W}_N \equiv \int_{\mathscr{C}} \mathbf{F}_N \cdot d\mathbf{x}$ is the work done by the nonconservative force. Plainly, this is merely another form of (7.36). Let $\mathscr{E} \equiv K + V$ denote the *total energy*, and note that $\Delta K + \Delta V = \Delta(K + V) = \Delta\mathscr{E}$ is the change in the total energy. Consequently, the work–energy principle (7.36) yields the following equivalent law.

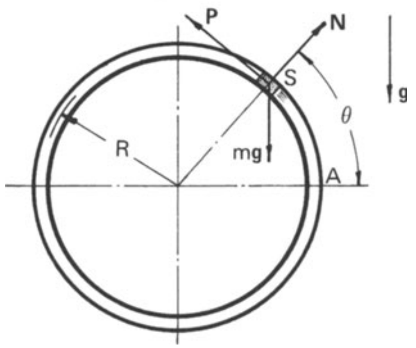


Figure 7.12. Propulsive motion of a slider block.

The general energy principle: *The change in the total energy is equal to the work done by the nonconservative part of the total force acting on the particle*:

$$\Delta\mathscr{E} = \mathscr{W}_N. \tag{7.80}$$

Hence, the total energy is constant if and only if the nonconservative part of the force does no work in the motion or, trivially, when nonconservative forces are absent.

It is useful to distinguish conservative and nonconservative forces, if possible; but if the nature of a force is uncertain, the ambiguous force is considered nonconservative until proven otherwise. The following example illustrates the straightforward application of the general energy principle (7.80).

Example 7.13. A propulsive force **P** of constant magnitude moves a slider S of mass m in a smooth circular track in the vertical plane, as shown in Fig. 7.12. The slider starts from rest at the horizontal position A. Determine the speed of S as a function of θ. What is its angular speed after n complete turns?

Solution. The total force that acts on S in the Fig. 7.12 consists of the workless normal reaction force **N** exerted by the smooth tube, the conservative gravitational force $\mathbf{F}_C = m\mathbf{g}$, and the nonconservative propulsive force $\mathbf{F}_N = \mathbf{P} = P\mathbf{t}$ which always is tangent to the path of S. The change in the potential energy of S is $\Delta V = mgR\sin\theta$, the datum being at A, and the change in the kinetic energy from the initial rest position at A is $\Delta K = \frac{1}{2}mv^2$. Therefore, with $\Delta\mathscr{E} = \Delta K + \Delta V$ and $\mathbf{F}_N \cdot d\mathbf{x} = P\,ds$, the general energy principle (7.80) yields

$$\frac{1}{2}mv^2 + mgR\sin\theta = P\int_0^{R\theta} ds = PR\theta, \tag{7.81a}$$

and hence the speed of S as a function of θ is given by

$$v(\theta) = \sqrt{\frac{2R}{m}(P\theta - mg\sin\theta)}. \tag{7.81b}$$

The angular speed of S is determined by $v = R\omega$. Thus, after n complete revolutions, $\theta = 2\pi n$ and (7.81b) provides the angular speed

$$\omega(n) = \sqrt{\frac{4n\pi P}{mR}}. \tag{7.81c}$$

□

7.9. Some Further Applications of the Fundamental Principles

Every problem of particle dynamics can be formulated entirely by use of the Newton–Euler law (5.34). Together with appropriate initial conditions, this law provides a complete system of three scalar equations for at most three unknown quantities. But the several auxiliary, first integral and moment of momentum principles derived from this law and discussed earlier in this chapter often deliver easily and more directly pieces of information that simplify the problem solution and often provide further physical insight as well. The remainder of this chapter is devoted to some further applications that demonstrate these attributes.

We begin with a familiar example that illustrates the joint application of the principles of conservation of momentum and energy in the elementary problem of projectile motion. The next example is an application of the principles of conservation of moment of momentum and energy in the formulation of the spherical pendulum problem. Finally, the phase plane curves for a simple harmonic oscillator and the motion of a spring-mass system are studied by the energy method. Some advanced topics are then presented in the sections that follow.

Example 7.14. *Application to projectile motion.* A projectile P is fired from a gun at O with muzzle speed v_0 at an elevation angle α from the horizontal ground plane in frame $\varphi = \{O; \mathbf{i}, \mathbf{j}\}$. Find the speed of the projectile as a function of its altitude; determine the maximum height h reached by P; and find its speed when it returns to the ground plane. Neglect air resistance.

Solution. A simple free body diagram of the projectile will show that the only force acting on P is the conservative gravitational force $m\mathbf{g} = -mg\mathbf{j}$ in frame φ. Consequently, the linear momentum in the horizontal direction $\mathbf{i}$ in φ, namely, $\mathbf{p} \cdot \mathbf{i} = m\dot{x}$, is constant. Initially, $\mathbf{p} \cdot \mathbf{i} = mv_0 \cos\alpha$; hence, for all time,

$$\dot{x} = v_0 \cos\alpha. \tag{7.82a}$$

This easy result provides auxiliary information for later use.

Clearly, the system is conservative, and with $y = 0$ as the zero reference for the potential energy, the total energy initially is $E = \frac{1}{2}mv_0^2$. At any subsequent position, the potential energy is $V(y) = mgy$ and the kinetic energy is $K(y) = \frac{1}{2}mv^2$. The energy principle (7.73) requires

$$\tfrac{1}{2}mv^2 + mgy = \tfrac{1}{2}mv_0^2, \tag{7.82b}$$

which determines the projectile's speed as a function of its altitude y:

$$v(y) = \sqrt{v_0^2 - 2gy}. \tag{7.82c}$$

The projectile's speed is independent of the gun's angle of elevation and the mass of the projectile.

To find the greatest height attained, we recall that $v^2 = \dot{x}^2 + \dot{y}^2$. Clearly, the projectile attains its maximum altitude h when $\dot{y} = 0$. With (7.82a), the speed $v(h) = \dot{x} = v_0 \cos\alpha$, and hence (7.82b) or (7.82c) yields the maximum altitude reached by P:

$$h = \frac{v_0^2}{2g} \sin^2 \alpha. \tag{7.82d}$$

Consequently, the greatest height attained depends on the angle of elevation, but not the mass of the projectile.

Finally, when P returns to the ground at Q, $y = 0$ and (7.82c) shows that the shell lands with speed equal to its muzzle speed v_0. □

The following exercises are left for the reader.

Exercise 7.11. Show that the projectile's horizontal range is given by

$$r = \frac{v_0^2}{g} \sin 2\alpha. \tag{7.82e}$$

□

Exercise 7.12. Show that (7.82a) and (7.82b) are equivalent to two scalar equations of motion provided by the Newton–Euler law (5.34). Integrate these equations with respect to time to obtain $(x(t), y(t))$ in φ. (a) Introduce (7.82d) and (7.82e), and show that the projectile's trajectory, absent any environmental effects, is a parabola defined by

$$Y = -\frac{g}{2v_0^2 \cos^2 \alpha} X^2, \tag{7.82f}$$

where $Y \equiv y - h$ and $X \equiv x - r/2$ in the frame $\Phi = \{Q; \mathbf{I}_k\}$. Identify frame Φ. (b) Determine, as a function of 2α, the projectile's coordinates (x_m, y_m) at its greatest height in φ. Show that the loci of all points of maximum height is an ellipse

$$\frac{x_m^2}{a^2} + \frac{(y_m - b)^2}{b^2} = 1, \tag{7.82g}$$

where $a = v_0^2/2g$ and $b = a/2$. □

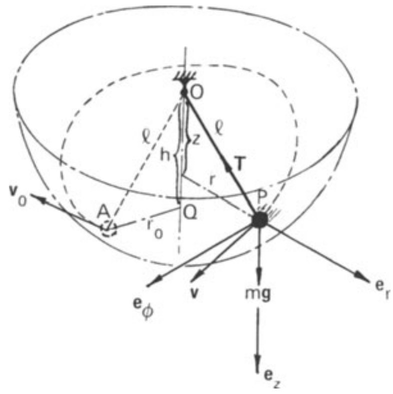


Figure 7.13. Spherical pendulum motion.

Example 7.15. *The spherical pendulum.* One end of a thin, rigid rod of length ℓ and negligible mass is fastened to a bob P of mass m, and its other end is attached to a smooth ball joint at O. In view of the constraint, the bob moves on a spherical surface of radius ℓ, so this device is called a *spherical pendulum*. The bob is given an arbitrary initial velocity $\mathbf{v}_0$ at a point A located in the horizontal plane at the distance h below O in Fig. 7.13. Find three equations that determine the velocity of P as a function of the vertical distance z below O, and describe how the motion $\mathbf{x}(P, t)$ may be found from the results.

Solution. To find $\mathbf{v}(z)$, it proves convenient to introduce cylindrical coordinates (r, ϕ, z) with origin at the ball joint O and basis directed as shown in Fig. 7.13, with $\mathbf{e}_z$ downward. The velocity of P is given by (see (4.59) in Volume 1)

$$\mathbf{v}(P, t) = \dot{r}\mathbf{e}_r + r\dot{\phi}\mathbf{e}_\phi + \dot{z}\mathbf{e}_z. \tag{7.83a}$$

We wish to determine $\dot{r}$, $r\dot{\phi}$, and $\dot{z}$ as functions of z. Three equations are needed.

The first equation is obtained from the energy principle. The forces that act on P are its weight $m\mathbf{g}$ and the workless force $\mathbf{T}$ exerted by the rod. Therefore, the principle of conservation of energy (7.73), with $V_0 = 0$ at O, yields

$$\tfrac{1}{2}mv^2 - mgz = mE_0, \tag{7.83b}$$

where $E_0 \equiv E/m$ is the total energy per unit mass. The speed v of P is thus determined by (7.83a) and (7.83b):

$$v^2 = \dot{r}^2 + r^2\dot{\phi}^2 + \dot{z}^2 = 2(E_0 + gz). \tag{7.83c}$$

This provides one relation among the three unknown functions. The constant E_0 is fixed by the initial conditions at A, $E_0 = \frac{1}{2}v_0^2 - gh$ at $z = h$ in (7.83b).

Another equation may be obtained from the principle of conservation of moment of momentum. The rod tension $\mathbf{T}$ has no moment about O, and $m\mathbf{g}$ exerts no moment about the vertical OQ axis. Hence, by (7.71), the moment of momentum about this line is conserved: $\mathbf{h}_O \cdot \mathbf{e}_z = \eta$, a constant. Recalling (7.83a), we see that only the component $mr\dot{\phi}$ of the linear momentum $m\mathbf{v}$ has a moment about the line OQ, namely, $r(mr\dot{\phi})\mathbf{e}_z$. Therefore, $\mathbf{h}_O \cdot \mathbf{e}_z = mr^2\dot{\phi} = \eta$, and with $\gamma \equiv \eta/m$, we have

$$r^2\dot{\phi} = \gamma. \tag{7.83d}$$

This provides another equation relating the unknown functions. The constant γ is determined from the initial conditions. Let $\hat{\mathbf{e}}_\phi$ denote $\mathbf{e}_\phi$ at A. Then $m\mathbf{v}_0 \cdot \hat{\mathbf{e}}_\phi$ is the only component of the initial linear momentum having a moment about the line OQ, and hence $\gamma = r_0\mathbf{v}_0 \cdot \hat{\mathbf{e}}_\phi = r_0 v_0 \cos\langle \mathbf{v}_0, \hat{\mathbf{e}}_\phi\rangle$, wherein $r_0 = (\ell^2 - h^2)^{1/2}$ in Fig. 7.13.

The final equation is derived from the suspension constraint: $\ell^2 = r^2 + z^2$. This gives $r = (\ell^2 - z^2)^{1/2}$, and hence

$$\dot{r} = -\frac{z\dot{z}}{\sqrt{\ell^2 - z^2}}. \tag{7.83e}$$

A few moments reflection reveals that $\dot{r}$, $r\dot{\phi}$, and $\dot{z}$ are now known as functions of z. Indeed, upon substituting (7.83d) and (7.83e) into (7.83c), we reach

$$\dot{z}^2 = \frac{2g}{\ell^2}[(\ell^2 - z^2)(z + \frac{E_0}{g}) - \frac{\gamma^2}{2g}], \tag{7.83f}$$

which determines $\dot{z}(z)$. And with $r(z) = (\ell^2 - z^2)^{1/2}$, $\dot{r}(z)$ given by (7.83e), and $r\dot{\phi} = \gamma/r(z)$ from (7.83d), it is now a straightforward matter to write the velocity (7.83a) as a function of z alone. We omit these details.

Finally, we need to say how the motion $\mathbf{x}(P, t) = r\mathbf{e}_r + z\mathbf{e}_z$ may be read from the results. In principle, integration of (7.83f) determines $z(t)$, hence $r(t)$, and (7.83d) provides

$$\phi(t) = \int_0^t \frac{\gamma}{r^2} dt, \tag{7.83g}$$

to fix $\mathbf{e}_r(\phi)$, which thus determines the motion. The exact solution for $z(t)$ may be obtained from (7.83f) in terms of Jacobian elliptic functions introduced later; however, we shall not pursue this problem further. (See Synge and Griffith.) □

Exercise 7.13. Apply the Newton–Euler law to formulate the spherical pendulum problem. Hint: Show that $z\ddot{r} = -\frac{1}{2}r d\dot{r}^2/dz$. □

Example 7.16. *Constant energy curves in the phase plane.* Use the energy principle to derive the differential equation for the smooth, horizontal motion of the linear spring-mass system in Fig. 6.13, page 134. Show that the phase plane trajectories, the curves in the xv-plane, are curves of constant total energy.

Solution. The free body diagram is shown in Fig. 6.13a. The weight of the oscillator and the normal surface reaction do no work in any rectilinear motion along the smooth horizontal surface. The elastic potential energy of the linear spring force acting on m is given by (7.65): $V(x) = \frac{1}{2}kx^2$, wherein $V(0) = 0$ in the natural state $x = 0$. The system is conservative with kinetic energy $K = \frac{1}{2}m\dot{x}^2$, so the energy principle (7.73) yields

$$\tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}kx^2 = E. \tag{7.84a}$$

The equation of motion is obtained by differentiation of (7.84a) with respect to the path variable x or with respect to time; we find $m\ddot{x} + kx = 0$. This agrees with (6.65a) in which $p = \sqrt{k/m}$.

Now let us examine the curves in the xv-plane, called the *phase plane*. Because $k > 0$, the total energy E in (7.84a) is a positive constant determined from assigned initial data. Suppose that $x(0) = x_0$ and $\dot{x}(0) = v_0$ at $t = 0$, then $E = \frac{1}{2}mv_0^2 + \frac{1}{2}kx_0^2$. We introduce

$$\varepsilon^2 \equiv \frac{2E}{m}, \tag{7.84b}$$

and write $v = \dot{x}$ to cast (7.84a) in the form

$$\left(\frac{x}{\varepsilon/p}\right)^2 + \left(\frac{v}{\varepsilon}\right)^2 = 1. \tag{7.84c}$$

For any given spring-mass pair (m, k), the phase plane curve described by (7.84c) is an ellipse whose axes are determined by the constant ε. For each choice of initial data, ε has a different value; and hence (7.84c) describes a family of concentric ellipses each of which is traversed in the same time $\tau = 2\pi/p$, the period of the oscillation, and on each of which ε is a constant fixed by the total energy E. In consequence, the phase plane curves for a conservative dynamical system are called *energy curves*. In physical terms, (7.84c) shows that ε is equal to the maximum speed in the periodic motion, which occurs at the natural state $x = 0$, and $x_A \equiv \varepsilon/p$ is the symmetric amplitude of the oscillation, the maximum displacement from the natural state—it marks the extreme states in the motion at which $v = 0$. □

Example 7.17. *Motion and the energy of a spring-mass system.* An unstretched linear spring shown in Fig. 7.14 is attached to a mass m that rests on a hinged board supported by a string. Find the motion of the load when the string is cut and the board falls clear from under it. Describe the energy curve for the motion.

Solution. The free body diagram in Fig. 7.14a shows the gravitational and elastic forces that act on m when the string is cut. These are conservative forces with total potential energy $V(x) = -mgx + \frac{1}{2}kx^2$, wherein $V(0) = 0$. The kinetic

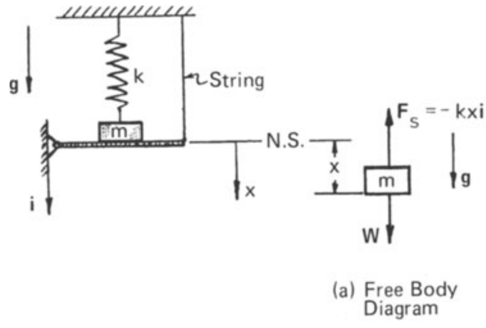


Figure 7.14. Gravity induced vibration of a simple harmonic oscillator.

energy is $K = \frac{1}{2}m\dot{x}^2$. Since the total energy initially is zero, the energy principle (7.73) gives

$$\tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}kx^2 - mgx = 0. \tag{7.85a}$$

Because this is the first integral of the equation of motion for which the path variable is x, differentiation of (7.85a) with respect to x (or t) yields the familiar equation of motion:

$$m\ddot{x} + kx = mg. \tag{7.85b}$$

The motion is a gravity induced, free vibration of a simple harmonic oscillator. The general solution of this equation is

$$x = \frac{mg}{k} + A\sin pt + B\cos pt, \tag{7.85c}$$

in which $p = \sqrt{k/m}$. The initial circumstances $x(0) = 0$ and $\dot{x}(0) = 0$ require $A = 0$ and $B = -mg/k$ in (7.85c), so the motion of the mass is described by

$$x(t) = \frac{mg}{k}(1 - \cos pt). \tag{7.85d}$$

It is seen from (7.85b) that $x_S \equiv mg/k$ is the static equilibrium displacement. Hence, (7.85d) shows that the load oscillates about the equilibrium state with circular frequency $p = \sqrt{k/m}$ and amplitude equal to x_S. The reader may readily confirm that $dV/dx = 0$ at x_S, and hence $V_{\min} = V(x_S) = -\frac{1}{2}kx_S^2$. As a consequence, $K_{\max} + V_{\min} = E = 0$ yields the maximum speed $v_{\max} = px_S$. It is simpler, however, to note from (7.85d) that $\dot{x}(t) = px_S \sin pt$, hence $v_{\max} = px_S$.

Now let us consider the energy curve. Due to the static displacement, the curve described by the energy equation (7.85a) in terms of x and $\dot{x}$ is an ellipse whose center is shifted a distance x_S along the x-axis. To see this, introduce the coordinate transformation $z = x - x_S$, which describes the motion of the load relative to its equilibrium position. Use of this relation in (7.85a) and (7.85b)

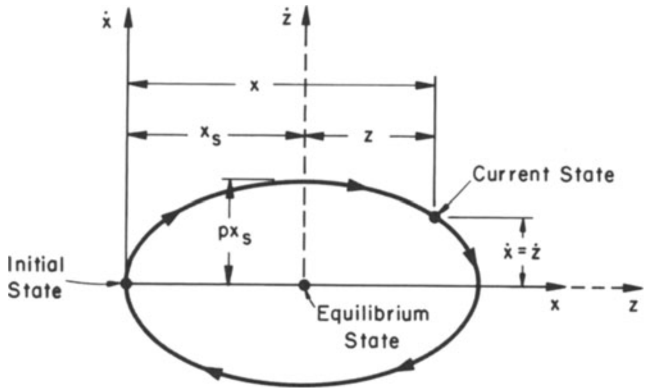


Figure 7.15. Phase plane diagram for the free vibration of a load on a linear spring.

yields the corresponding transformed equations:

$$\left(\frac{\dot{z}}{px_S}\right)^2 + \left(\frac{z}{x_S}\right)^2 = 1, \tag{7.85e}$$

$$\ddot{z} + p^2 z = 0. \tag{7.85f}$$

Clearly, the energy equation (7.85e) is an ellipse centered at the origin in the phase plane of z and $\dot{z}$; hence, the motion is periodic with circular frequency p and symmetric amplitude $z_{\max} = x_S$. The energy curve is the graph of (7.85e) shown in Fig. 7.15. Both the original and transformed variables are indicated. The geometry characterizes the motion of a simple harmonic oscillator described by (7.85f) whose solution is just the transformation of (7.85d) given by $z = -x_S \cos pt$. □

We now turn to some advanced applications. These include the finite amplitude oscillations of a simple pendulum, the plane motion of a particle on an arbitrary concave path, Huygens's isochronous pendulum, orbital motion, and Kepler's first and third laws. Elliptic functions and integrals are introduced along the way. In a first reading, however, these topics may be omitted without significant loss of continuity, if the reader may prefer to move forward to the next chapter.

7.10. The Simple Pendulum Revisited: The Exact Solution

Our earlier study of the simple pendulum focused on its small amplitude solution for which the motion is simple harmonic and hence isochronal, that is, the period is a constant, independent of the amplitude. Here we explore the exact solution for the large amplitude motion, which is neither simple harmonic nor isochronal. As a consequence, the accuracy of the approximation in the small amplitude solution is determined. The nonoscillatory, periodic motion of the revolving pendulum is also described. The importance of this classical problem lies in the parallel application of the method of analysis to a great variety of other physical systems described by a similar differential equation.

7.10.1. The Energy Equation and the Rod Tension

The energy equation will be applied to reformulate the finite motion problem for which the exact equation of motion is already given by the first equation in (6.67b). The energy equation, we recall, is the first integral of this equation of motion. An easy, exact result for the rod tension as a function of the angular motion $\theta(t)$ then follows immediately.

The problem geometry and the free body diagram of the bob are shown in Fig. 6.15, page 138. Recall that the supporting rod of length ℓ has negligible mass compared with the bob's mass m. Clearly, the variable rod tension $\mathbf{T}$ does no work in the motion, and the gravitational potential energy (7.60) is given by $V_g(\theta) = mg\ell(1 - \cos\theta)$. The system is conservative with kinetic energy $K = \frac{1}{2}m\ell^2\dot{\theta}^2$. Hence, the energy principle (7.73) yields

$$\tfrac{1}{2}m\ell^2\dot{\theta}^2 + mg\ell(1 - \cos\theta) = E, \quad \text{const.} \tag{7.86a}$$

The energy constant E may be evaluated from known conditions at any point in the motion, regardless of what the initial data might be. Here we consider an oscillatory motion with finite amplitude α so that $\dot{\theta} = 0$ when $\theta = \pm\alpha$. Then $E = mg\ell(1 - \cos\alpha)$, and (7.86a) becomes

$$\dot{\theta}^2 = 2p^2(\cos\theta - \cos\alpha), \tag{7.86b}$$

where $p = \sqrt{g/\ell}$. This is the exact integral of the first of the equations of motion in (6.67b). Therefore, substitution of (7.86b) into the second equation in (6.67b) yields the rod tension as a function of $\theta \in [-\alpha, \alpha]$, namely,

$$T(\theta) = W(3\cos\theta - 2\cos\alpha). \tag{7.86c}$$

7.10.2. The Finite Pendulum Motion and Its Period

We now turn to the exact analysis of the finite amplitude motion. The finite oscillatory motion of the pendulum and its period are essentially determined upon integration of (7.86b) to obtain

$$pt = \int_0^{\theta} \frac{d\theta}{\sqrt{2(\cos\theta - \cos\alpha)}}, \tag{7.87a}$$

wherein the value $\theta(0) = 0$ at $t = 0$ has been assigned for convenience, that is, time is measured from the instant when the bob is at its lowest vertical position.

To be consistent with the initial data, the positive root has been chosen in (7.86b). Equation (7.87a) thus provides the exact, but implicit solution for the periodic angular motion $\theta(t)$. The motion is periodic but not simple harmonic. The precise period of the finite oscillation, denoted by τ^*, follows from (7.87a). Let us write the integral in (7.87a) as $f(\theta)$ and note that $t = \tau^*/4$ is the time to reach the state $\theta = \alpha$ at the end of the pendulum's primary swing. Then the periodic time $\tau^* = 4f(\alpha)/p$. It is seen that the period varies, in fact increases, with the amplitude α, and hence the motion is not isochronous. This phenomenon is typical of nonlinear oscillation problems.

The integral in (7.87a) cannot be evaluated in terms of elementary functions, but it can be expressed in the form of an elliptic integral or a corresponding Jacobian elliptic function whose numerical value may be found from mathematical tables or computed directly. To cast the integral in its standard form, we introduce a new variable ϕ defined by the transformation

$$\sin\frac{\theta}{2} = k\sin\phi, \qquad k = \sin\frac{\alpha}{2}, \tag{7.87b}$$

so that $0 < k < 1$. Then $\cos\alpha = 1 - 2k^2$ and $\cos\theta = 1 - 2k^2\sin^2\phi$ follow from the familiar double angle trigonometric identities, and use of these relations in (7.87a) yields the standard formula

$$pt = \int_0^{\phi} \frac{d\phi}{\sqrt{1 - k^2\sin^2\phi}}. \tag{7.87c}$$

The integral in (7.87c) is called the *elliptic integral of the first kind*, usually denoted by

$$F(\phi; k) \equiv \int_0^{\phi} \frac{d\vartheta}{\sqrt{1 - k^2\sin^2\vartheta}}, \tag{7.87d}$$

for $0 < k < 1$. The variable limit ϕ is the *argument* of the integral whose dummy variable ϑ replaces ϕ in (7.87c). The constant k, defined by the second relation in (7.87b), is called the *modulus*. The two equations in (7.87b) determine the argument and the modulus in terms of the pendulum variable θ and its amplitude α. As the physical angular placement θ grows from 0 to α, the argument ϕ increases from 0 to $\pi/2$. The special integral obtained from (7.87d) at $\phi = \pi/2$, written as

$$K(k) \equiv F(\frac{\pi}{2}; k) = \int_0^{\pi/2} \frac{d\vartheta}{\sqrt{1 - k^2\sin^2\vartheta}}, \tag{7.87e}$$

is called the *complete elliptic integral of the first kind*. The use of K for this integral is conventional and is not to be confused with the kinetic energy function, and, of course, F is not a force. The values of $F(\phi; k)$ and $K(k)$ are tabulated‡ in

‡ See, for example, P. F. Byrd and M. D. Friedman, *Handbook of Elliptic Integrals for Engineers and Scientists*, 2nd Edition, revised, Springer-Verlag, New York, 1971. This is an especially valuable

handbooks of mathematical tables, and nowadays may be routinely calculated by computer. The elliptic integral and the complete elliptic integral of the second kind are introduced later in Exercise 7.19, page 283. An elliptic integral and a complete elliptic integral of the third kind also arise in nonlinear dynamical problems, but we shall not encounter these here.

The period τ^* of the oscillation is now readily determined. With $t = \tau^*/4$ at $\theta = \alpha$, (7.87c) and (7.87e) yield the exact periodic time of the pendulum motion,

$$\tau^* = \frac{4}{p}K(k) = \frac{2\tau}{\pi}K(k). \tag{7.87f}$$

Here $\tau = 1/f = 2\pi/p = 2\pi\sqrt{\ell/g}$ denotes the constant small amplitude period of the pendulum defined in (6.67g). Recall that k in the second of (7.87b) depends on the amplitude α. The result (7.87f), therefore, describes the precise manner in which the period τ^*, and hence the frequency $f^* \equiv 1/\tau^*$, varies with the amplitude, and it renders explicit the relation of the exact period τ^* (frequency f^*) to the elementary period τ (frequency f). We can now determine which is greater.

Because $0 < k < 1$, (7.87e) shows that $K(k) > \int_0^{\pi/2} d\vartheta = \frac{\pi}{2}$, that is, $2K(k)/\pi > 1$. By (7.87f), therefore, $\tau^* > \tau$, so $f^* < f$: *The exact period (frequency) of the finite amplitude oscillations of a simple pendulum is always greater (smaller) than the period (frequency) of its small amplitude, simple harmonic motion.*

Further, notice in (7.87e) that $K(k)$ increases monotonically with $k \in (0, 1)$, while the second relation in (7.87b) shows that k increases with the amplitude angle α. Therefore, $K(k)$ increases with α. As perceived earlier, (7.87f) shows that *the period (frequency) of the finite pendulum motion increases (decreases) when the amplitude is increased.*

7.10.3. Introduction to Jacobian Elliptic Functions

The solution (7.87c) provides the travel time in terms of the argument $\phi(\theta)$ in accordance with $pt = F(\phi; k) = f(\theta)$. By the introduction of a Jacobian elliptic function, however, this integral relation may be inverted to obtain the solution in the closed form $\theta = \theta(t) \equiv f^{-1}(pt)$. To motivate the idea of the Jacobian elliptic functions, let us recall first the elementary integral

$$u \equiv \int_0^y \frac{d\vartheta}{\sqrt{1-\vartheta^2}} = \sin^{-1} y, \tag{7.88a}$$

where ϑ is a dummy variable in all of these standard integrals. Then the inverse of the integral u whose argument is y is the familiar circular function $y = \sin u$.

resource for formulas, identities, descriptions of properties and graphics for elliptic functions, and it provides transformations of general elliptic integrals of all sorts to their standard forms. It contains adequate explanatory material on elliptic functions and integrals for those unfamiliar with the subject.

Similarly, the inverse of the elementary integral

$$u \equiv \int_0^y \frac{d\vartheta}{1-\vartheta^2} = \tanh^{-1} y, \tag{7.88b}$$

is the hyperbolic function $y = \tanh u \equiv \sinh u / \cosh u$.

The same idea may be applied to invert the elliptic integral (7.87c). We first introduce a new argument

$$y = \sin\phi, \tag{7.88c}$$

into (7.87c) and replace the integrand variable y with the dummy variable ϑ to obtain the following alternate standard formula for the elliptic integral of the first kind:

$$F(\phi(y);k) \equiv u(y) = \int_0^y \frac{d\vartheta}{\sqrt{(1-\vartheta^2)(1-k^2\vartheta^2)}}, \tag{7.88d}$$

for $0 < k < 1$. Notice that this integral reduces to (7.88a) when $k = 0$ and to (7.88b) when $k = 1$. With $\phi = \pi/2$, we have $y = 1$; hence, in accordance with (7.87e), the alternate standard form of the complete elliptic integral of the first kind is

$$K(k) = u(1) = \int_0^1 \frac{d\vartheta}{\sqrt{(1-\vartheta^2)(1-k^2\vartheta^2)}}. \tag{7.88e}$$

Now, the *Jacobian elliptic sine function* sn$u$, read as "*ess* − *en* − *u*", is similarly defined by

$$u(y) = \int_0^y \frac{d\vartheta}{\sqrt{(1-\vartheta^2)(1-k^2\vartheta^2)}} \equiv \mathrm{sn}^{-1} y. \tag{7.88f}$$

Therefore, the inverse of the elliptic integral u whose argument is y is the Jacobian elliptic sine function $y = \mathrm{sn}u$. With (7.88c),

$$y = \sin\phi = \mathrm{sn}u, \tag{7.88g}$$

yields the desired *inverse solution* $\phi = \sin^{-1}(\mathrm{sn}u)$.

We next establish the properties of the Jacobian elliptic sine function. In accordance with (7.88g), $\phi = 0$ implies $y = 0$, and (7.88f) gives $u = 0$; therefore, $\mathrm{sn}0 = 0$. Similarly, $\phi = \pi/2$ yields the corresponding amplitude $y = 1 = \mathrm{sn}u(1)$, and hence by (7.88e), $\mathrm{sn}K(k) = 1$ is the amplitude of the graph sn$u$ at $u(1) = K(k)$. Because $\sin\phi$ is an odd periodic function with quarter period $\pi/2$, the elliptic sine function in (7.88g) also is an odd periodic function: $\mathrm{sn}(-u) = -\mathrm{sn}u$, but with quarter period $K(k)$ that varies with k. Hence, sn$u$ has amplitude 1 and period $4K$. The graph of sn$u$, therefore, is similar to the map of the familiar sine function to which it reduces when $k = 0$. The reader may find it helpful to sketch the graph of the periodic function $y = \text{sn}u$ to illustrate the properties of the Jacobian elliptic sine function, namely,

$$\text{sn}(qK) = \begin{cases} 0 \text{ if } q = 0, 2, 4, \\ 1 \text{ if } q = 1, \\ -1 \text{ if } q = 3. \end{cases} \tag{7.88h}$$

$$\text{sn}(u + 4K) = \text{sn}u, \qquad \text{sn}(-u) = -\text{sn}u. \tag{7.88i}$$

Two additional Jacobian elliptic functions are defined in terms of the elliptic sine function. Study of these functions and their properties is left for the student in Problems 7.54 and 7.55. We shall find no need for these additional functions as we continue discussion of the pendulum problem.

7.10.4. The Pendulum Motion in Terms of an Elliptic Function

With the aid of (7.87d) and (7.88d), the simple pendulum solution (7.87c) may be written as $pt = F(\phi; k) = u(y)$. Consequently, the inverse of the elliptic integral for the pendulum problem is provided by (7.88g):

$$y = \sin\phi = \text{sn}(pt). \tag{7.89a}$$

Finally, use of this result in the first equation in (7.87b) delivers, explicitly, *the oscillatory motion $\theta(t)$ of a simple pendulum with amplitude α*:

$$\theta(t) = 2\sin^{-1}[k\text{sn}(pt)], \tag{7.89b}$$

in which $p = \sqrt{g/\ell}$ and $k = \sin(\alpha/2)$.

The periodicity of $\text{sn}u$ in the first of (7.88i) shows that the physical period τ^* of the pendulum motion is given by $p\tau^* = 4K$, which agrees with (7.87f). And at the quarter period $t = \tau^*/4 = K/p$, we confirm that (7.89b) yields the amplitude $\theta(\tau^*/4) = \alpha$ in (7.87b). This concludes the introduction to elliptic functions and their application to the simple pendulum problem.

7.10.5. The Small Amplitude Motion

We now know precisely the manner in which the periodic time of a simple pendulum increases with the amplitude. The elementary small amplitude solution, on the other hand, predicts a smaller constant period $\tau = 2\pi\sqrt{\ell/g}$. With the exact solution in hand, we can now assess the accuracy of the simple harmonic approximation. For small amplitudes α and hence small modulus k, we may approximate the elliptic integral (7.87c) and its modulus in (7.87b) by the first few terms of their power series expansions. We begin with the elliptic integral.

Since $k^2 \sin^2 \phi < 1$, the binomial expansion of the elliptic integral (7.87c) yields

$$pt = F(\phi; k) = \int_0^\phi [1 + \frac{1}{2}k^2 \sin^2 \phi + \frac{1 \cdot 3}{2 \cdot 4}k^4 \sin^4 \phi + \cdots]d\phi,$$

and term by term integration provides the series solution

$$pt = \phi + \frac{k^2}{2}\left(\frac{\phi}{2} - \frac{\sin 2\phi}{4}\right) + O(k^4), \tag{7.90a}$$

accurate to terms of the order k^4. With $t = \tau^*/4$ at $\phi = \pi/2$, (7.90a) gives the period as an approximate function of the modulus k—*the period increases monotonically with k^2*:

$$\tau^* = \tau\left(1 + \frac{k^2}{4}\right) + O(k^4). \tag{7.90b}$$

The series expansion for $k(\alpha)$ in (7.87b) yields

$$k = \frac{\alpha}{2} - \frac{1}{6}\left(\frac{\alpha}{2}\right)^3 + O\left(\alpha^5\right), \tag{7.90c}$$

and hence (7.90b) delivers the estimate of the period as a function of the amplitude:

$$\tau^* = \tau\left(1 + \frac{\alpha^2}{16}\right) + O(\alpha^4), \tag{7.90d}$$

accurate to terms of the order α^4. *Consequently, for small amplitudes, the period increases with the square of the amplitude.*

When terms of order greater than α and hence terms of order larger than k may be considered negligible, (7.90a), (7.90d), and the first equation in (7.87b) yield the isochronal simple harmonic solution as the lowest order approximation:

$$\theta = \alpha \sin pt, \qquad \tau^* = \tau = \frac{2\pi}{p}. \tag{7.90e}$$

Finally, we now assess the error that results from use of the small amplitude period as compared with the second order approximation in (7.90d), rewritten as

$$\frac{\tau^* - \tau}{\tau} = \left(\frac{\alpha}{4}\right)^2. \tag{7.90f}$$

Suppose the estimated error is not to exceed 1%. Then with $\tau^*/\tau \leq 1.01$, by (7.90f), the amplitude ought not to exceed $\alpha = 0.4$ rad, or 23° very nearly. Therefore, the small amplitude approximation of the oscillatory pendulum motion as a simple harmonic motion with constant period $\tau = 2\pi/p$ is very good for amplitudes smaller than 23°. Indeed, the solution computed for the complete elliptic integral with $\sin^{-1} k = \alpha/2 = 11.5°$ in the exact equation (7.87f) is $p\tau^*/4 = 1.5868$, that is, $\tau^*/\tau = 1.0101$. Therefore, the error involved in the second order approximation (7.90d) in comparison with the exact solution computed for an amplitude of 23° is insignificant. This analysis emphasizes that the "smallness role" of some quantities considered in approximations is not always so infinitesimally small as is sometimes imagined.

7.10.6. Nonoscillatory Motion of the Pendulum

If the initial angular speed $\dot{\theta}(0) \equiv \omega_0$ at $\theta(0) = 0$ was sufficiently great, the bob eventually could reach its highest point at $\alpha = \pi$ or swing past it. The energy equation (7.86a) shows that the angular speed $\omega_0 = \omega_z$ required for the bob to just reach its zenith is given by

$$\omega_z = 2p. \tag{7.91a}$$

Now consider starting the system at its lowest point $\theta(0) = 0$ with angular speed $\dot{\theta}(0) = \omega_0$. The corresponding amplitude function determined by (7.86b) and (7.91a) is $\cos\alpha = 1 - 2\omega_0^2/\omega_z^2$, and hence (7.86b) may be written as

$$\dot{\theta}^2 = \omega_0^2[1 - \kappa^2 \sin^2(\theta/2)], \qquad \kappa \equiv \frac{\omega_z}{\omega_0}. \tag{7.91b}$$

Integration of this equation provides the travel time as a function of $\theta \in [0, \pi]$:

$$t = \frac{1}{\omega_0} \int_0^\theta \frac{d\theta}{\sqrt{1 - \kappa^2 \sin^2(\theta/2)}}. \tag{7.91c}$$

The physical nature of the motion depends on whether $\kappa < 1$, $= 1$, or > 1.

First, consider the case $\kappa = 1$. Equation (7.91c) shows that $t \to \infty$ as $\theta \to \pi$. *Therefore, when* $\omega_0 = \omega_z$, *the pendulum approaches its zenith without reaching it in finite time, and hence the period is infinite.*

Now suppose that $\kappa > 1$ so that the initial angular speed $\omega_0 < \omega_z$. In this case, the bob cannot reach the zenith, so the motion is oscillatory with amplitude

$$\alpha = \cos^{-1}\left[1 - 2\left(\frac{\omega_0}{\omega_z}\right)^2\right]. \tag{7.91d}$$

With the aid of the second relation in (7.87b), we find $k = \kappa^{-1} = \omega_0/\omega_z < 1$ and $p = \omega_z/2$. The motion with amplitude (7.91d) and the corresponding period are respectively determined by (7.89b) and (7.87f).

Exercise 7.14. Show that for $\kappa > 1$ equation (7.91c) may be cast in the form (7.87c) for the oscillatory motion. □

Finally, suppose that $\kappa < 1$. Then $\omega_0 > \omega_z$ and now the bob turns past the zenith, because the initial kinetic energy $\frac{1}{2}ml^2\omega_0^2$ exceeds the gravitational potential energy $2mgl$ attained as the bob rises to its highest point. In this case, from (7.91b), $\dot{\theta}$ is positive for all θ, however large, and hence the angular speed $\omega \equiv |\dot{\theta}|$ varies from its greatest value ω_0 at $\theta = 0$ to its least value $\omega_0(1-\kappa^2)^{1/2}$ at $\theta = \pi$:

$$\omega_0\sqrt{1-\kappa^2} \le \omega \le \omega_0. \tag{7.91e}$$

We set $\phi = \theta/2$ in (7.91c) to obtain the travel time in the revolving pendulum motion:

$$t = \frac{2}{\omega_0}\int_0^{\frac{\theta}{2}} \frac{d\phi}{\sqrt{1-\kappa^2\sin^2\phi}} = \frac{2}{\omega_0}F(\frac{\theta}{2};\kappa), \tag{7.91f}$$

in which $F(\theta/2;\kappa)$ is the elliptic integral of the first kind with modulus κ. Hence, the bob reaches its zenith at $\theta = \pi$ in the finite time t_z given by

$$t_z = \frac{2}{\omega_0}K(\kappa), \tag{7.91g}$$

in which $K(\kappa)$ is the complete elliptic integral of the first kind.

Thus, if $\omega_0 > \omega_z$, the pendulum, in the absence of any frictional and aerodynamic effects, spins forever in the same direction about its support. The angular speed varies periodically between the extremes (7.91e); the motion is periodic, but not oscillatory. Indeed, with the aid of (7.91g), *the periodic time $\tau_o = 2t_z$, the time for the whirling pendulum to complete one orbit about its support, is given by $\tau_o = 4K(\kappa)/\omega_0$.*

Exercise 7.15. Show that the orbital motion $\theta(t)$ of the revolving pendulum is described by $\theta(t) = 2\sin^{-1}(\mathrm{sn}\frac{1}{2}\omega_0 t)$. □

7.11. The Isochronous Pendulum

The finite amplitude motion of a simple pendulum is not isochronous; its period varies with the amplitude. Here we explore the existence of a pendulum whose finite amplitude oscillation is isochronal. The gravity induced, oscillatory motion of a particle on a smooth symmetric curve, concave upward in the vertical plane is studied. The equation for the finite amplitude motion is obtained from the energy equation, and the frequency for small amplitude, simple harmonic oscillations on an arbitrary concave curve is derived. To study the finite amplitude motion, however, the curve geometry must be specified. The finite amplitude motion on a cycloidal curve is investigated, and it is shown that the cycloidal oscillator is exactly simple harmonic, hence isochronous. Moreover, the cycloid is the only plane curve having this property.

7.11.1. Equation of Motion on an Arbitrary Concave Path

Consider a particle P of mass m free to slide on a smooth and concave upward, but otherwise arbitrary curve $\mathscr{C}$ in the vertical plane. The free body diagram of P is shown in Fig. 7.16a. The normal, surface reaction force **N** is workless, and the gravitational force has potential energy $V(y) = mgy$. The system is conservative with kinetic energy $K(P, t) = \frac{1}{2}m\dot{s}^2$, where $s(t)$ is the arc length along $\mathscr{C}$ measured from point O at $y = 0$, say. The energy principle (7.73) requires

$$\tfrac{1}{2}m\dot{s}^2 + mgy = E. \tag{7.92a}$$

Differentiation of (7.92a) with respect to s yields $\ddot{s} + g\,dy/ds = 0$. Noting in Fig. 7.16a that

$$\sin\gamma(s) = \frac{dy}{ds}, \qquad \text{where} \quad \gamma = \tan^{-1}\frac{dy}{dx}, \tag{7.92b}$$

we obtain the equation of motion of P on $\mathscr{C}$:

$$\ddot{s} + g\sin\gamma(s) = 0. \tag{7.92c}$$

This is the tangential component of the intrinsic equation of motion of P.

Let $v(s) \equiv \dot{s}(t)$ and $v_0 \equiv v(s_0)$, $s_0 \equiv s(0)$ at $y = y_0$ initially. Introducing these initial data in (7.92a), integrating the first relation in (7.92b), and noting that $ds/v(s) = dt$, we obtain the general solution in terms of $s(t)$:

$$v^2(s) = v_0^2 - 2g\int_{s_0}^{s}\sin\gamma(s)ds, \qquad t = \int_{s_0}^{s}\frac{ds}{v(s)}, \tag{7.92d}$$

To do more, we shall need to know the shape function $\gamma(s)$.

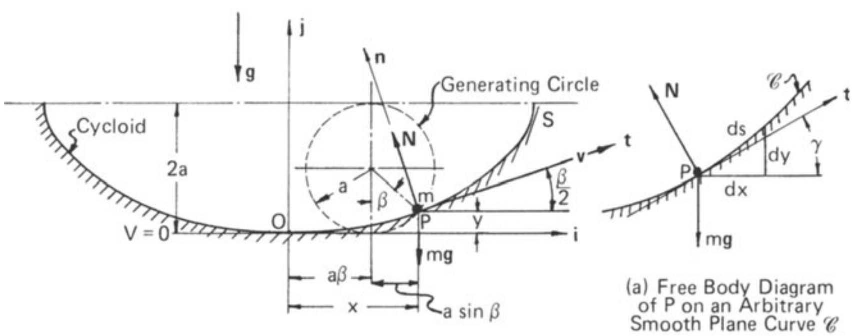


Figure 7.16. Motion on a smooth cycloid.

7.11.2. Small Oscillations on a Shallow, Concave Curve

Consider a shallow symmetric curve with a horizontal tangent at O where $s_0 = 0$, so that $\gamma(s)$ is a small inclination. Then $\sin\gamma = \gamma$, approximately. By (7.92c), the point O is the static equilibrium state at which $\gamma(s_0) = 0$. For a small amplitude oscillation of P about O, the power series expansion of the shape function $\gamma(s)$ about $s = s_0 = 0$ gives

$$\gamma(s) = \frac{d\gamma(0)}{ds}s + O(s^2). \tag{7.93a}$$

The path has the curvature $\kappa(s) = d\gamma(s)/ds$. Hence, to terms of the first order in s, (7.93a) yields a general, though approximate relation for $\gamma(s)$:

$$\gamma(s) = \kappa_0 s = \frac{1}{R_0}s, \tag{7.93b}$$

where $R_0 \equiv 1/\kappa_0 \equiv 1/\kappa(0)$ is the radius of curvature of $\mathscr{C}$ at the origin O.

Consequently, for small amplitude oscillations of a particle on a smooth, shallow and symmetric concave curve in the vertical plane, the equation of motion (7.92c) *reduces to the equation for the simple harmonic oscillator*:

$$\ddot{s} + p^2 s = 0, \qquad p = \sqrt{\frac{g}{R_0}} = \sqrt{g\kappa_0}. \tag{7.93c}$$

The small amplitude frequency $f = p/2\pi$ and period $\tau = 1/f$ are determined by the radius of curvature of the path at the equilibrium point O. For a circular arc of radius $R_0 = \ell$, (7.93c) describes the small amplitude oscillations of a simple pendulum of length ℓ, for example.

7.11.3. Finite Amplitude Oscillations on a Cycloid

Now consider the finite amplitude oscillations of a particle on a smooth cycloid generated by a point P, starting at O, on a circle of radius a, as shown in Fig. 7.16. As the circle rolls toward the right, without slipping on the horizontal line at $y = 2a$, the radial line turns counterclockwise through the angle $\beta \in [0, \pi]$ measured from its initial vertical direction at O. Hence, the parametric equations of the cycloid are described by the Cartesian coordinates of P, namely,

$$x = a(\beta + \sin\beta), \qquad y = a(1 - \cos\beta). \tag{7.94a}$$

Clearly, $y \in [0, 2a]$, and for symmetric oscillations, $\beta \in [-\pi, \pi]$ and $x \in [-\pi a, \pi a]$. (See Example 2.5, page 109, in Volume 1.)

The tangent angle $\gamma(s)$ in (7.92b) and the curvature $\kappa(s)$ of the cycloid are readily determined from (7.94a). With the aid of the double angle trigonometric identities, we first obtain

$$dx = 4a\cos^2\frac{\beta}{2}d\frac{\beta}{2}, \qquad dy = 4a\sin\frac{\beta}{2}\cos\frac{\beta}{2}d\frac{\beta}{2}.$$

These yield

$$\frac{dy}{dx} = \tan\frac{\beta}{2}, \qquad R = \frac{1}{\kappa} = \frac{ds}{d(\beta/2)} = 4a\cos\frac{\beta}{2}. \tag{7.94b}$$

Therefore, from the second relation in (7.92b), the tangent angle $\gamma(s)$ in Fig. 7.16 and the radius of curvature $R(\gamma)$ of the cycloid are given by

$$\gamma = \frac{\beta}{2}, \qquad R(\gamma) = \frac{ds}{d\gamma} = 4a\cos\gamma. \tag{7.94c}$$

Hence, $\gamma \in [-\pi/2, \pi/2]$ and $R(\gamma)$ decreases from $R(0) = 4a$ to $R(\pm\pi/2) = 0$. The greatest amplitude is restricted by the curve geometry shown in Fig. 7.16 for $\gamma \in [-\pi/2, \pi/2]$.

Integration of the last equation in (7.94c) determines the function $\gamma(s)$:

$$s = \int_0^{\gamma} 4a\cos\gamma\, d\gamma = 4a\sin\gamma(s). \tag{7.94d}$$

Use of this relation in (7.92c) yields *the exact equation of motion of a particle free to slide on a smooth cycloid in the vertical plane*:

$$\ddot{s} + p^2 s = 0, \qquad p = \sqrt{g/4a}. \tag{7.94e}$$

We thus find a most interesting result: *The finite amplitude, cycloidal motion is exactly simple harmonic and hence isochronous.* The period of the cycloidal pendulum for all amplitudes is a constant given by

$$\tau = 4\pi\sqrt{\frac{a}{g}}. \tag{7.94f}$$

The result (7.94f) is truly astonishing: *If a particle of arbitrary mass slides from a position of rest at any point whatsoever on a smooth cycloid, it reaches the bottom always in the same time $t^* = \tau/4 = \pi\sqrt{a/g}$.*

We notice from (7.94c) that $R_0 \equiv R(0) = 4a$ at the equilibrium position $\gamma = 0$. Hence, the small amplitude formulas (7.93c) are the same, of course, as the exact relations (7.94e) for arbitrary amplitudes.

Exercise 7.16. The analysis reveals some additional geometrical properties of the cycloid. Consider the cycloidal curve from O to its orthogonal intersection with the line $y = 2a$ at S in Fig. 7.16 and derive the following properties. (a) The length σ of the cycloid from O to S is equal to its radius of curvature at O: $\sigma = 4a = R_0$. (b) The slope of the cycloid at a point P situated at a distance s from O is equal to the product of the curvature $\kappa(s) = 1/R(s)$ and the arc length

s at P: $\tan\gamma = \kappa s = s/R$. (c) At a point P on a cycloid, the sum of squares of its radius of curvature and its arc length from O is a constant equal to the square of the radius of curvature $R_0 = 4a$ at its lowest point: $R^2 + s^2 = 16a^2$, and hence s and $R(s)$ at every point on a cycloid describe the same circle of radius R_0 in the Rs-plane. □

It is also known that the cycloid is the unique curve of quickest descent between two points in the vertical plane. This is the classical brachistochrone problem of the calculus of variations, a topic beyond the scope of our current studies. See Problem 7.68 for an example.

7.11.4. Uniqueness of the Isochronal, Cycloidal Pendulum

Glancing back to (7.92b) and noting the proportionality in (7.94d), we ask: Are there any curves besides the cycloid for which

$$\sin\gamma = \frac{dy}{ds} = cs, \qquad \cos\gamma = \frac{dx}{ds} = \sqrt{1 - c^2s^2}, \qquad \gamma = \tan^{-1}\frac{dy}{dx}, \tag{7.95a}$$

where c is a constant? If so, (7.92c) becomes $\ddot{s} + cgs = 0$, and hence the motion on any such smooth curve is simple harmonic, hence isochronous. To address the question, we need to find the parametric equations of all plane curves characterized by (7.95a).

We fix the Cartesian origin at the equilibrium point defined by $\gamma = 0$, and integrate the first two equations in (7.95a) to obtain

$$y = \frac{cs^2}{2}, \qquad x = \frac{1}{2c}[cs\sqrt{1 - c^2s^2} + \sin^{-1}(cs)].$$

Then we use (7.95a) to write these expressions in terms of the tangent angle γ, and afterwards introduce the double angle trigonometric identities to obtain

$$x = \frac{1}{4c}[2\gamma + \sin 2\gamma], \qquad y = \frac{1}{4c}(1 - \cos 2\gamma). \tag{7.95b}$$

The parametric equations (7.95b) describe a cycloid whose generating circle is described by

$$\left(x - \frac{2\gamma}{4c}\right)^2 + \left(y - \frac{1}{4c}\right)^2 = \left(\frac{1}{4c}\right)^2. \tag{7.95c}$$

The radius is $a \equiv 1/4c$. As the circle turns counterclockwise, rolling on the line $y = 2a$, its center thus moves horizontally toward the right a distance $2\gamma/4c = 2\gamma a$, and hence the circle turns through an angle $\beta \equiv 2\gamma$. Therefore, we find exactly our original parametric equations (7.94a). The relation (7.94d) between the arc length s and the tangent angle $\gamma(s)$ is uniquely characteristic of the cycloid.

In summary, *the unique plane curve on which the motion of a particle is simple harmonic for all amplitudes is the cycloid. Hence, the cycloidal pendulum is the only exact isochronal pendulum.*

The reader may consider the following similar problem.

Exercise 7.17. Apply (7.92b) to prove that the unique plane curve whose tangent angle is proportional to its arc length is a circle. Show in this case that (7.92c), valid for the finite amplitude oscillations of a particle on any smooth, concave curve in the vertical plane, yields the equation for the finite motion of a simple pendulum. □

7.11.5. Huygens's Isochronous Clock

The isochronous, cycloidal pendulum was invented in 1673 by the Dutch scientist and ingenious clockmaker, Christian Huygens (1629–1695). The idea was used in construction of a pendulum clock to assure that its period would not change with variations in the amplitude of its swing. Huygens was able to produce a cycloidal motion of the bob by applying the property that the evolute of a cycloid is another cycloid of the same kind as the generating curve. The evolute of the cycloid is the path traced by the center of curvature of the generating cycloid. In Fig. 7.17, the evolute of the cycloid arc OS is the similar cycloid arc QS, both are generated by a circle of radius a. As P moves from S toward O, the center of curvature T of the arc OS traces the arc from S to Q. In other words, if a string of length $4a$ is tied to a fixed point Q that forms the cusp of an inverted cycloidal curve in Fig. 7.17, and the string is pulled over the contour arc QS to point S where the bob is released from rest, the bob will describe the same cycloidal path OS as our sliding particle in Fig. 7.16. On the basis of these unique

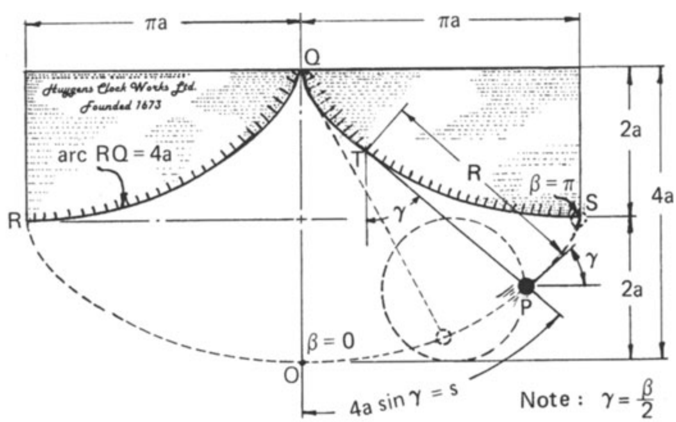


Figure 7.17. Huygens's isochronal pendulum.

properties of the cycloid, Huygens's isochronal pendulum may be constructed with shortened cycloidal surfaces at the cusp support Q so that the bob P moves on a shorter cycloidal path of some practical design dimensions. Subsequent inventors introduced certain drive control devices to adjust for energy losses due to frictional effects that would otherwise lead to variations in the amplitude.

7.12. Orbital Motion and Kepler's Laws

Consider a body P of mass m moving relative to an inertial frame $\varphi = \{O; \mathbf{e}_k\}$ under a central directed gravitational force (7.61) due to a body S of mass M with its center of mass fixed at O. For example, S might be the Sun and P a planet, or S the Earth and P the Moon or a satellite. It is natural to model the two bodies as a system of two center of mass particles, interactions with all other bodies being ignored. Then P moves in a plane such that, by (7.72b), $r^2\dot{\phi} = \gamma$, a constant. This has the geometrical interpretation that the radius vector of P sweeps out the same area in equal time intervals, the second of three laws deduced empirically by the German astronomer and mathematician Johannes Kepler (1571–1630) based on precise astronomical observations of the positions of stars and planets by the Danish astronomer Tycho Brahe (1546–1601), Kepler's mentor. More than half a century later, Kepler's laws were deduced by Newton (1642–1727) from his mathematical theory of planetary motion. Here we determine the motion of an orbital body, characterize its path, and derive Kepler's first and third laws of planetary motion.

7.12.1. Equation of the Path

We introduce cylindrical coordinates identified in Fig. 7.11, page 252, with origin at O in the inertial frame $\varphi = \{O; \mathbf{e}_r, \mathbf{e}_\phi\}$ and with respect to which the polar coordinate equation of the path of a particle P is described by $r = r(\phi)$. The only force acting on P is the conservative gravitational force with potential energy given in (7.62). The constant $V_0 = 0$ may be chosen so that $V \to 0$ when $r \to \infty$, and hence $V = -\mu m/r$, where the constant $\mu \equiv GM$. The kinetic energy of P is given by $K = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2)$. The energy principle together with (7.72b) yields

$$\dot{r}^2 + \frac{\gamma^2}{r^2} - \frac{2\mu}{r} = \frac{2E}{m}, \tag{7.96a}$$

in which γ is the constant moment of momentum per unit mass of P and E is the constant total energy.

To find $r(\phi)$, it is convenient to introduce a change of variable

$$u(\phi) = \frac{1}{r(\phi)}. \tag{7.96b}$$

Then, by (7.72b), $\dot{\phi} = \gamma u^2(\phi)$, and from (7.96b), $\dot{r} = -\gamma du/d\phi$. Now (7.96a) may be recast as

$$\left(\frac{du}{d\phi}\right)^2 + u^2 - \frac{2\mu}{\gamma^2}u = \frac{2E}{m\gamma^2}. \tag{7.96c}$$

Although this form of the energy equation may be readily integrated for $u(\phi)$, it is easier to first differentiate (7.96c) with respect to ϕ to obtain the equation of motion:

$$\frac{d^2u}{d\phi^2} + u = \frac{\mu}{\gamma^2}, \tag{7.96d}$$

whose easy general solution is

$$u(\phi) = \frac{1}{r(\phi)} = \frac{\mu}{\gamma^2} + C\cos(\phi - \phi_0), \tag{7.96e}$$

in which C and ϕ_0 are integration constants. The base line for ϕ may be chosen so that $\phi_0 = 0$, and C may be expressed in terms of γ and E by substitution of (7.96e) into (7.96c). We find

$$C^2 = \frac{\mu^2}{\gamma^4}\left(1 + \frac{2E\gamma^2}{m\mu^2}\right). \tag{7.96f}$$

Hence, (7.96e) yields the path equation

$$r(\phi) = \frac{d}{1 + e\cos\phi}, \tag{7.96g}$$

in which, by definition,

$$d \equiv \frac{\gamma^2}{\mu}, \qquad e \equiv \sqrt{1 + \frac{2E\gamma^2}{m\mu^2}}. \tag{7.96h}$$

Thus, the plane motion of P for all time is given by

$$\mathbf{x}(P,t) = r(\phi)\mathbf{e}_r(\phi) = \frac{d}{1 + e\cos\phi}\mathbf{e}_r(\phi). \tag{7.96i}$$

7.12.2. Geometry of the Orbit and Kepler's First Law

The total energy E in the second relation of (7.96h) may be positive, negative, or zero. Thus, in particular, when $E = -m\mu/2d < 0$, $e = 0$ and, by (7.96g), the orbit is a circle of radius $r = d$. Otherwise, (7.96g) describes the polar equation of a conic section in Fig. 7.18—defined as the locus of a point P that moves in a plane in such a way that the ratio of its distance $|\overline{OP}|$ from a fixed point O in the plane to its distance $|\overline{DP}|$ from a fixed line is constant. The fixed point is called the *focus*. The fixed line is known as the *directrix*, and the constant ratio

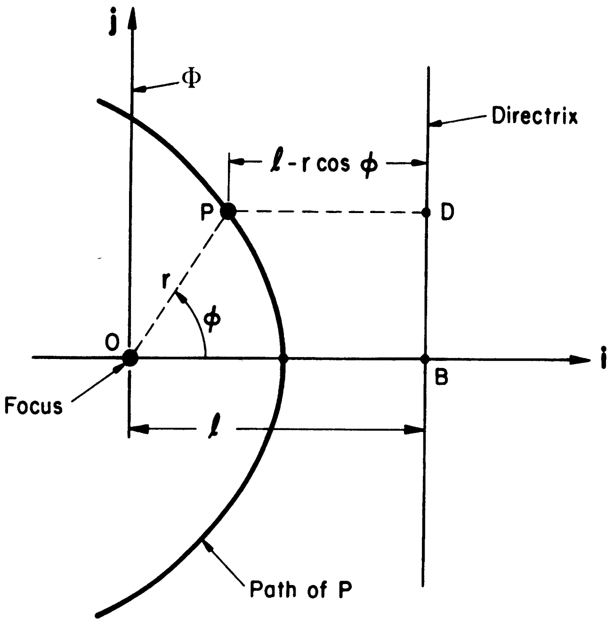


Figure 7.18. Geometry of a conic section.

of the two distances is called the *eccentricity*. With the focus at the origin O of frame $\Phi = \{O; \mathbf{i}, \mathbf{j}\}$ in Fig. 7.18, the directrix is a straight line BD parallel to $\mathbf{j}$ at a distance ℓ from O along $\mathbf{i}$. The eccentricity is defined by

$$e = \frac{|\overline{OP}|}{|\overline{DP}|} = \frac{r}{\ell - r\cos\phi} > 0. \tag{7.97a}$$

Solving this relation for $r(\phi)$, we obtain the general equation (7.96g) in which

$$d \equiv \ell e. \tag{7.97b}$$

Equation (7.96g) shows that $r(-\phi) = r(\phi)$, so the path is symmetric about the line $\phi = 0$, the $\mathbf{i}$-axis. The chord along the $\mathbf{j}$-axis (parallel to the directrix) through the focus O is called the *latus rectum*. When $\phi = \pi/2$, (7.96g) shows that $r(\pi/2) = d$, and hence $2d$ is the length of the latus rectum.

If $e > 1$, the conic described by (7.96g) is a hyperbola; if $e = 1$, the conic is a parabola; and if $e < 1$, the conic is an ellipse. The circle is a degenerate ellipse for which $e = 0$. It is amazing that the type of conic trajectory is uniquely characterized in terms of the total energy E in accordance with (7.96h), namely,

ellipse	$e < 1$ if $E < 0$
circle	$e = 0$ if $E = -m\mu/(2d) < 0$
parabola	$e = 1$ if $E = 0$
hyperbola	$e > 1$ if $E > 0$.

(7.97c)

The circle is a degenerate ellipse for which $E < 0$, and whose eccentricity vanishes when the total energy, $E = -m\mu/(2d) = \frac{1}{2}V(\pi/2)$, is one-half the potential energy at the semi-latus rectum.

A planet or satellite having a parabolic or hyperbolic path ultimately would leave the solar system forever. Astronomical observations, however, dictate that the planets have closed orbits around the Sun, and Newton's theory proves that these orbits are elliptical with the Sun situated at one focus O. This is **Kepler's first law**: *The planets travel on elliptical paths with the sun at one focus.*

7.12.3. Kepler's Third Law

We now turn to Kepler's third law on the orbital period. The orbital geometry is described in Fig. 7.19. To determine the *periodic time* τ in which P describes its elliptical orbit, we recall Kepler's second law relating the area swept out to the time, namely,

$$A = \tfrac{1}{2}\gamma t. \tag{7.98a}$$

The area $A = \pi ab$ enclosed by the elliptical path is thus covered in the time

$$\tau = \frac{2\pi ab}{\gamma}, \tag{7.98b}$$

where a and b are the respective semi-major and semi-minor axes. We prefer, however, to express this result in terms of the geometrical and gravitational constants.

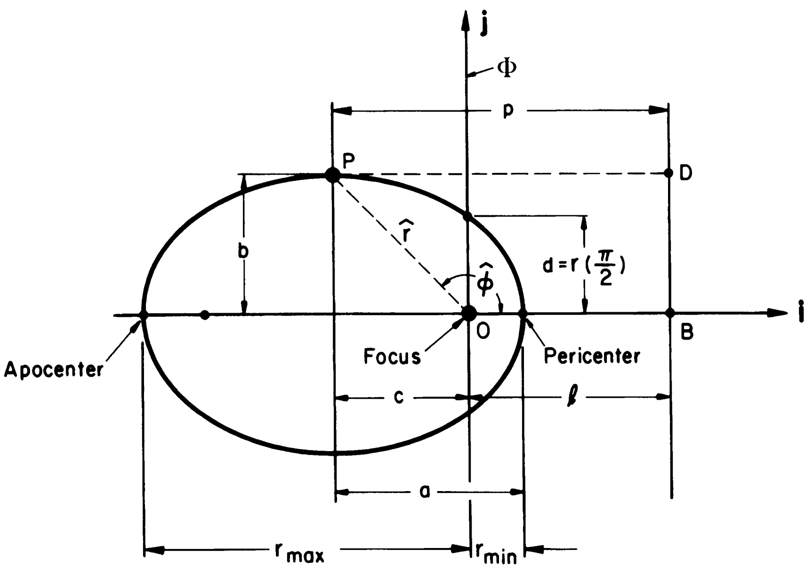


Figure 7.19. Geometry of an elliptical orbit.

The constant γ is related to the gravitational constant through (7.96h): $\gamma = \sqrt{\mu d}$, and hence the next step is to relate d, a, and b.

The ratio d/a may be found by use of (7.96g). The focus O is on the major axis, and the nearest location of P to O, called the *pericenter*, is at $\phi = 0$ in Fig. 7.19. Thus, by (7.96g), $r(0) = r_{\min} = d/(1+e)$. Similarly, at $\phi = \pi$, the greatest distance of P from O, named[§] the *apocenter*, is $r(\pi) = r_{\max} = d/(1-e)$. The length of the major axis, therefore, is $2a = r_{\min} + r_{\max}$, and hence

$$\frac{d}{a} = 1 - e^2. \tag{7.98c}$$

We next seek a relation for b/a. Equation (7.97a) applied to the point on the minor axis in Fig. 7.19 gives $e = \hat{r}/(c + \ell)$, wherein $c \equiv a - r_{\min} = a - d/(1 + e)$. Hence, by (7.98c), $c = ae$; and, with the aid of (7.97b), we have $\hat{r} = e(c + \ell) = a$. Now observe from Fig. 7.19 that $\hat{r}^2 = b^2 + c^2$, introduce $\hat{r}$ and c, and thus derive the ratio

$$\frac{b}{a} = \sqrt{1 - e^2}. \tag{7.98d}$$

Finally, use of (7.98c) in (7.98d) gives $b = \sqrt{ad}$. We now return to (7.98b), recall the first equation in (7.96h) to obtain $b/\gamma = \sqrt{a/\mu}$, and thus derive the periodic time for the elliptical orbit:

$$\tau = 2\pi\sqrt{\frac{a^3}{\mu}}. \tag{7.98e}$$

It is remarkable that the orbital period involves only one geometrical constant a and the physical constant $\mu = GM$. Therefore, the ratio τ^2/a^3 is the same for all planets in motion about the Sun. This is **Kepler's third law:** *The square of the periodic time of a planet is proportional to the cube of the semi-major axis of its orbit.*

Exercise 7.18. (a) It is remarkable also that the third law may be cast in terms of only one dynamical constant, the total energy E. Show that

$$\tau = \frac{2\pi\mu}{\sqrt{(-2E/m)^3}}, \tag{7.98f}$$

in which $E < 0$ for an elliptical orbit. (b) Show that the squared speed of P is given by

$$v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right), \tag{7.98g}$$

which is independent of its mass. Hint: Use (7.96h) and (7.96a). □

§ These general terms are used when no specific focal body is identified. However, when the body at O is the Earth and the orbital body P is the Moon or a satellite, the point nearest the Earth in the orbit of P is called the *perigee*, and the point farthest from the Earth is termed the *apogee*. The point nearest the Sun in the orbit of a planet or another body is known as the *perihelion*, and the most remote point in its orbit is called the *aphelion*.

Exercise 7.19. *Distance on an elliptical orbit.* Let $\psi = \{C; \mathbf{i}_k\}$ be a Cartesian reference frame at the center of an ellipse whose parametric equations are given by $x = a\sin\phi$, $y = b\cos\phi$, where ϕ denotes the central, clockwise angle from the y-axis and a and $b < a$ are the corresponding semi-axes of the ellipse

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1. \tag{7.99a}$$

(a) Show that the area A enclosed by this ellipse is $A = \pi ab$.

(b) The name elliptic integral derives from the following problem of determining the length of an elliptic arc. Starting at $(x, y) = (0, b)$, show that the distance s traveled on an elliptical orbit is given by

$$s(\phi) = aE(k;\phi), \tag{7.99b}$$

in which $E(k;\phi)$, not to be confused with the constant total energy, is standard notation for the *elliptic integral of the second kind,* defined by

$$E(k;\phi) \equiv \int_0^{\phi} \sqrt{1 - k^2 \sin^2 \vartheta}\, d\vartheta, \tag{7.99c}$$

in which $k^2 \equiv (a^2 - b^2)/a^2 = e^2$, so that $0 < k < 1$. By (7.99b), the circumference Γ of the elliptical orbit is thus determined by

$$\Gamma \equiv 4s(\pi/2) = 4aE(k;\pi/2), \tag{7.99d}$$

wherein $E(k;\pi/2)$ is the *complete elliptic integral of the second kind.* In particular, for a circle of radius a, $k = e = 0$; hence, (7.99b) yields the circular arc length $s(\phi) = aE(0;\phi) = a\phi$ and (7.99d) gives the circumference $\Gamma = 4as(\pi/2) = 2\pi a$. In general, the value of E for a given modulus $k \in (0, 1)$ and a specified angle $\phi \in [0, \pi/2]$ may be found from tables of elliptic integrals or by computation.

(c) Consider an elliptical orbit for which $a = 2b$. Find in terms of a the distance traveled when $\phi = \pi/4, \pi/2$, and 2π. □

The foregoing theory requires that the focal body S be fixed while the moving body P is attracted only by S. Of course, Newton's law of gravitation holds for every pair of bodies in the world, and disturbances induced by the mutual attractions with other bodies have been ignored. An accurate dynamical treatment of the solar system, the major problem of celestial mechanics, entails far greater complexities than those embodied in the simple model studied here. If the ratio of m/M of the masses of P and S is small, and their mutual distance and their separation from all other bodies is great, the elementary model gives a very close estimate of the facts. On the other hand, it is natural to question what may be said about the motion of a system of two or more bodies free to move under their mutual Newtonian attraction. The two body interaction problem and the effect of their relative motion on Kepler's law for the orbital period is studied in the next chapter.

References

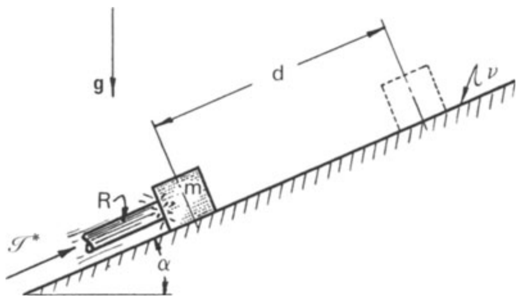
1. Blanco, V. M., and McCuskey, S. W., *Basic Physics of the Solar System*, Addison-Wesley, Reading, Massachusetts, 1961. The principles of celestial and orbital mechanics are presented mainly for engineers, scientists, and other nonspecialists interested in space dynamics, the physical aspects of the solar system, and the motion of artificial satellites.
2. Cajori, F., *Newton's Principia*, English translation of *Mathematical Principles of Natural Philosophy* by Isaac Newton, 1687, University of California Press, Berkeley, 1947. The motion of Huygens's cycloidal pendulum is discussed in Book 1: Motion of Bodies, pp. 153–60.
3. Fernandez-Chapou, J. L., Salas-Brito, A. L., and Vargas, C. A., An elliptic property of parabolic trajectories, *American Journal of Physics* **72**, 1109 (2004). The authors show that the loci of points of maximum height of a projectile is an ellipse, an easy projectile problem described in Exercise 7.12.
4. Lamb, H., *Dynamics*, Cambridge University Press, Cambridge, Massachusetts 1929. An excellent old text in which many straightforward and interesting additional problems and examples may be found. The conservation laws are presented in Chapter 4. The circular and cycloidal pendulums are discussed in Chapter 5, orbital motions under central forces in Chapter 11.
5. Lass, H., *Vector and Tensor Analysis*, McGraw-Hill, New York, 1950. This text deals mostly with vector analysis. The line integral and potential function are treated in Chapter 4. Additional examples for collateral study may be found there.
6. Meriam, J. L., and Kraige, L. G., *Engineering Mechanics.* Vol. 2, *Dynamics*, 3rd Edition, Wiley, New York, 1992. Work, energy, impulse–momentum, and central force motion are discussed in Chapter 3. The energy method is applied in vibrations problems in Chapter 8.
7. Pollard, H., *Celestial Mechanics*, No. 18 of the Carus Mathematical Monographs, Mathematical Association of America, Washington, D.C., 1976. This small book presents a brief but thorough mathematical introduction to celestial mechanics. Details concerning the central force problem, including analysis of the two body problem, are provided in Chapter 1. Advanced topics in two further chapters include analysis of the n-body problem and use of the Hamilton–Jacobi theory of analytical dynamics.
8. Routh, E. J., *Dynamics of a Particle*, Dover, New York, 1960; originally published by the Cambridge University Press in 1898. The cycloidal pendulum is discussed in Articles 204–9, the simple pendulum in Articles 213–16.
9. Shames, I. H., *Engineering Mechanics.* Vol. 2, *Dynamics*, 2nd Edition, Prentice-Hall, Englewood Cliffs, New Jersey, 1966. This is an often cited useful resource for collateral study and additional problems. Central force motion is covered in Chapter 12, work and energy in Chapter 13, and momentum methods follow in Chapter 14.
10. Sommerfeld, A., *Mechanics. Lectures on Theoretical Physics*, Vol. 1. Academic, New York, 1952. See Chapter III for discussion of the cycloidal pendulum.
11. Synge, J. L., and Griffith, B. A., *Principles of Mechanics*, 3rd Edition, McGraw-Hill, New York, 1959. Orbital motion, stability, and Kepler's laws are discussed in Chapter 6. The motion of a spherical pendulum, its solution in terms of Jacobian-elliptic functions, and its connection with potential experimental error in measurements of the Foucault effect are described in Chapter 13. For further discussion of the practical difficulties and errors encountered in measurements of the Foucault effect, see the excellent Foucault biography by Tobin cited in the previous chapter.
12. Timoshenko, S., *Vibration Problems in Engineering*, 3rd Edition, Van Nostrand, Princeton, New Jersey, 1955. The energy method is treated briefly in Chapter 1. This is a classical resource for additional mechanical vibrations problems in engineering, including topics on vibrations of continua.

Problems

7.1. A small ball of mass m strikes a horizontal surface with speed v_1 at an angle θ from the surface, and it bounces off with speed v_2 at an angle ϕ from the surface. Show that the magnitude $|\mathscr{I}^*|$ and direction ψ of the impulse exerted on the ball by the wall are determined by

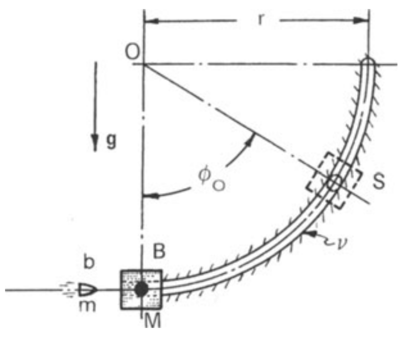
$$|\mathscr{I}^*| = m\left[v_1^2 + v_2^2 - 2v_1 v_2 \cos(\theta + \phi)\right]^{1/2}, \qquad \tan\psi = \frac{v_2 \sin\phi + v_1 \sin\theta}{v_2 \cos\phi - v_1 \cos\theta}.$$

7.2. A rigid rod R strikes a small block of mass m initially at rest on a rough plane inclined at an angle α shown in the figure. The block moves up the incline a distance d where it comes to rest. The dynamic coefficient of friction is ν. Find the initial impulse of the force exerted by the rod on the block.



Problem 7.2.

7.3. A ballistic pendulum consists of a heavy block B of mass M initially at rest but free to slide in a rough, circular guide slot of radius r. A bullet b of mass m is fired into the block which then swings from its initial vertical position shown in the figure through an angle ϕ_0 where the system $S = \{b, B\}$ comes to rest. The dynamic coefficient of friction is ν. (a) Derive the Newton–Euler differential equation giving the squared speed of S as a function of its angular placement, and thus find $v^2(\phi)$ exactly. (b) Determine the initial velocity $\mathbf{v}_0$ of S. (c) What relations will determine exactly the impulse of the force $\mathscr{I}^*$ exerted by the bullet on the block and the bullet's impact speed β? (d) Find $\mathscr{I}^*$ and β when ϕ_0 is a small angle.



Problem 7.3.

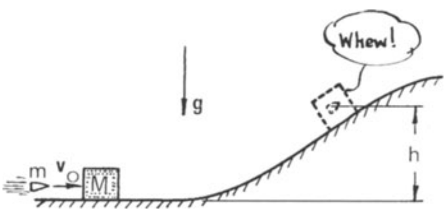
7.4. A force $\mathbf{F} = (3x^2 + 4y - 6z^3)\mathbf{i} + (x - 2y^2 - 3xz)\mathbf{j} + xy\mathbf{k}$ moves a particle P along the path $y = 4x - 2x^2$ from the origin $(0, 0, 0)$ to the point $(2, 0, 0)$. Find the work done by $\mathbf{F}$. Compare this with the work done by $\mathbf{F}$ in moving P along a straight line joining these points.

7.5. (a) What work is done by the force $\mathbf{F}(\mathbf{x}) = xy\mathbf{i} + y^2\mathbf{j}$ in moving a particle from the point $(0, 0)$ to the point $(1, 2)$ along (i) the parabola $y = ax^2$, (ii) the orthogonal paths $y = 0$ and $x = 1$, and (iii) the straight line $y = kx$? Is the force $\mathbf{F}(\mathbf{x})$ conservative? (b) The force $\mathbf{F}(\mathbf{x}) = y^3\mathbf{i} + x\mathbf{j}$ moves a particle on a plane path defined by the time-parametric equations $x = at^2$, $y = bt^3$. Find the work done during the period $t = 0$ to $t = 1$.

7.6. A hockey puck of mass m is driven over a frozen lake with an initial speed of 6 m/sec. Its speed 3 seconds later is 5 m/sec. Apply the work–energy principle to find the dynamic coefficient of friction, and determine the distance traveled by the puck during this time.

7.7. An electron E of mass m initially at rest at $(0, 0)$ is acted upon by a plane propulsive force $\mathbf{P}$ of constant magnitude. (a) What work is done by $\mathbf{P}$ in moving E along an arbitrary simple path from $(0, 0)$ to $(2, 2)$? (b) Find the work done when E moves between the same end points on (i) a circle centered at (0,2) and (ii) on a straight line. (c) If all other forces acting on E are workless, determine its speed at the point $(2, 2)$ on each of the three paths. (d) What can be said about the motion, if $\mathbf{P}$ were the only force acting on the electron?

7.8. A bullet of mass m is fired with muzzle velocity $\mathbf{v}_0$ into a block of mass M initially at rest on a smooth horizontal surface, as shown in the figure. After the impact, the block and imbedded bullet move on a smooth curve in the vertical plane. The system ultimately comes to rest at the distance h above the plane. Find the muzzle speed.



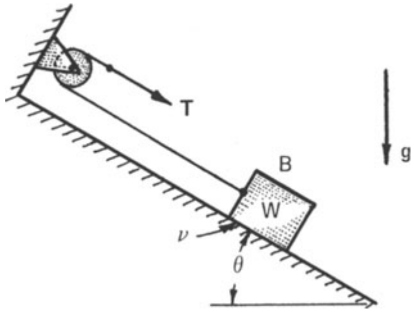
Problem 7.8.

7.9. A particle P of unit mass is acted upon by a force equal to twice its velocity. The initial velocity is $\mathbf{v}_0 = \mathbf{u}$ at the place $\mathbf{x}_0 = \frac{1}{2}\mathbf{u}$, where $\mathbf{u}$ is a unit vector. (a) Determine the change of the kinetic energy of P and the mechanical power expended during a time t. (b) Describe the trajectory of P.

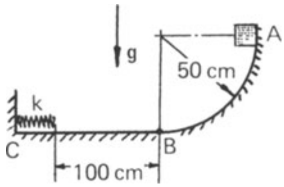
7.10. A cable under constant tension T passes over a smooth pulley and is attached to a block B of weight W initially at rest on a rough inclined plane shown in the figure. Suppose that $T = W$ and the dynamic coefficient of friction is ν. Neglect the mass of the cable and pulley. (a) What is the speed v_B of B after being dragged a distance d? (b) At the moment when B reaches d_0, the cable snaps. Find the additional distance $\hat{d}$ traveled by B. (c) Find equations for v_B and $\hat{d}$ for the motion of B on a horizontal surface.

7.11. A 40 N weight is released from rest at A on a smooth circular surface shown in the diagram. At B, it continues to move on a rough horizontal surface BC with dynamic coefficient of friction $\nu = 0.4$, and it subsequently strikes a spring with stiffness $k = 10$ N/cm at C. Find the deflection of the spring.

7.12. Apply the work–energy equation to determine the angular speed $\dot{\theta}(t)$ following impact of the ballistic pendulum described in Fig. 7.3, page 226. Then (i) show that the result is equivalent to one of the two scalar equations of motion for the load, and (ii) find the rope tension as a function of $\theta(t)$.



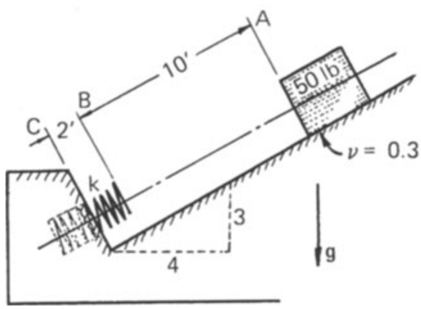
Problem 7.10.



Problem 7.11.

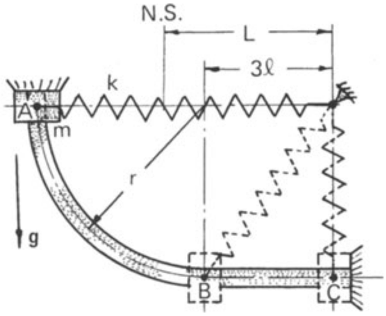
7.13. The motion of a particle P falling from rest with air resistance given by Stokes's law is described in Example 6.11, page 120. (a) Find the total mechanical power of the forces acting on P as a function of its speed v. Then show that the terminal speed v_∞ is the particle speed at which the power generated by gravity is balanced by the power dissipated by air resistance, and hence v_∞ is the speed at which the total power vanishes. (b) Determine as a function of time the total work done on P, and thus show that as $t \to \infty$, $\mathscr{W} \to \frac{1}{2}mv_\infty^2$, the kinetic energy of P at its terminal speed.

7.14. A 50 lb crate is released from rest at A on an inclined surface shown in the figure. It ultimately strikes and fully compresses a spring of modulus k before coming to rest at C. The dynamic coefficient of friction is 0.30. Find the modulus k.



Problem 7.14.

7.15. A linear spring of modulus k and unstretched length $L = 4\ell$ is attached to a slider block of mass m shown in the figure. The block experiences a negligible disturbance from rest at A and slides in the vertical plane on a smooth circular rod of radius $r = L$. Find in terms of L the speed of the block at B and C.



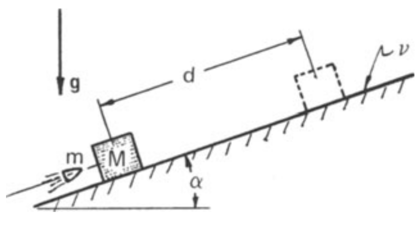
Problem 7.15.

7.16. It appears from (7.34) that the work–energy principle (7.36) can be used only when the particle mass is constant. (a) Show from (5.34) that if $m = m(v)$ is a function of the particle speed $v(t)$, the work done by a force $\mathbf{F}(\mathbf{x})$ acting over the particle path from time t_0 to time t is determined by

$$\mathscr{W} = m(v)v^2 - m(v_0)v_0^2 - \frac{1}{2}\int_{v_0}^{v} m(v)dv^2, \qquad \text{(P7.16)}$$

wherein $v_0 = v(t_0)$. Let $m(v) = m$, a constant, and *derive* the work–energy principle (7.36). (b) Now let $m(v)$ be the relativistic mass (6.9), and suppose that the work–energy relation (7.36) is *postulated* as a fundamental principle of mechanics. Use (P7.16) to show that the change in the relativistic kinetic energy is $\Delta K = [m(v) - m(v_0)]c^2$. With $m_o \equiv m(0)$ and $K \equiv E$, it is seen that this reduces to (7.43e), obtained somewhat differently in the text. (c) Verify that when $v/c \ll 1$, $\Delta K = \frac{1}{2}m_o(v^2 - v_0^2)$.

7.17. A bullet of mass m is fired into a block of mass M initially at rest on a rough inclined plane, as shown. After the impact, the system moves up the plane a distance d where it comes to rest. Apply the energy principle to find the impulse of the force exerted by the bullet on the block, and determine its impact velocity.



Problem 7.17.

7.18. Show that the force $\mathbf{F}(\mathbf{x}) = 2xy\mathbf{i} + (x^2 + ay)\mathbf{j}$ is conservative, and determine the work done on an arbitrary path from the origin to the point (0, 2, 0). What is the work done by $\mathbf{F}$ along a straight line through the points (2, 2, 0) and (0, 2, 0) when $a = 1$?

7.19. (a) Find the potential energy function for the conservative force

$$\mathbf{F}(\mathbf{x}) = (2z^2 \cosh x - y^2)\mathbf{i} + 2y(z - x)\mathbf{j} + (4z \sinh x + y^2)\mathbf{k}.$$

What is the work done by $\mathbf{F}$ in moving a particle from the origin along the path $y = \sin(\pi x/2)$ to the point (2, 0, 0)? (b) Is the following force conservative?

$$\mathbf{F}(\mathbf{x}) = (2z^2 + 5\cos y)\mathbf{i} + (z - 5x\sin y)\mathbf{j} + (4zx - y)\mathbf{k}.$$

7.20. (a) Establish that the force $\mathbf{F}(\mathbf{x}) = -xy^2\mathbf{i} - yx^2\mathbf{j}$ is conservative, and find the potential energy function. (b) Write down the scalar equations for the plane motion of a particle of unit mass moving under this force alone, and find a first integral of this system of equations.

7.21. Show that the force $\mathbf{F}(\mathbf{x}) = (y\mathbf{i} - x\mathbf{j})/r^2$, with $r^2 = x^2 + y^2 \neq 0$, is conservative. Determine the potential energy function.

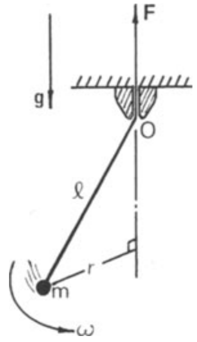
7.22. This problem illustrates the necessity for the conservative force to be defined over a simply connected region. The force $\mathbf{F}(\mathbf{x}) = \mathbf{k} \times \mathbf{x}/r^2$, where $\mathbf{x} = x\mathbf{i} + y\mathbf{j}$ and $r = |\mathbf{x}|$, has the curious property that curl $\mathbf{F}(\mathbf{x}) = \mathbf{0}$ while the work done by $\mathbf{F}$ in moving a particle around a circle $\mathscr{C}$ about the origin O in the plane region $\mathscr{R}$ does not vanish. (a) Does $\nabla \times \mathbf{F}(\mathbf{x}) = \mathbf{0}$ hold everywhere in $\mathscr{R}$? Is $\mathbf{F}(\mathbf{x})$ defined at all points of $\mathscr{R}$, specifically at $\mathbf{x} = \mathbf{0}$? (b) Show that on a circle $\mathscr{C}$ of radius R centered at O in the plane of the force, $\mathscr{W} = \oint_{\mathscr{C}} \mathbf{F} \cdot d\mathbf{x} = 2\pi$. Hence, by Stokes's theorem (7.57), $\int_{\mathscr{A}}$ curl $\mathbf{F}(\mathbf{x}) \cdot d\mathbf{A}$ over the area bounded by $\mathscr{C}$ equals 2π. This plainly implies that $\nabla \times \mathbf{F} \neq \mathbf{0}$ everywhere. (c) To construct a simply connected region that excludes point O, first draw a small circle c of radius ε around O. Next, remove tiny slices from c and $\mathscr{C}$ at their intersections with the x-axis, and join corresponding points on c and $\mathscr{C}$ by lines drawn parallel to the x-axis. This produces a single closed region $\mathscr{R}$ without O that looks like a split washer. Now, compute the work done by $\mathbf{F}$ in moving a particle around this new closed path and determine its value as $\varepsilon \to 0$.

7.23. Prove that the force

$$\mathbf{F}(\mathbf{x}) = (y + 5z\sin x)\mathbf{i} + (x + 4yz)\mathbf{j} + (2y^2 - 5\cos x)\mathbf{k},$$

is conservative, and derive the potential energy function. What is the work done by $\mathbf{F}$ in moving a particle from the origin to the point (4, 3, 0) along (i) a circular path centered on the x-axis and joining the end points, and (ii) a straight line between the same end points?

7.24. A particle of mass m is suspended vertically by a light inextensible string of length ℓ and twirled with angular speed ω in a circular path of radius r, as indicated. The restraining force $\mathbf{F}$ is slowly increased so that the particle moves in a circle of radius $r/2$. Use the moment of momentum principle to find its new angular speed Ω.



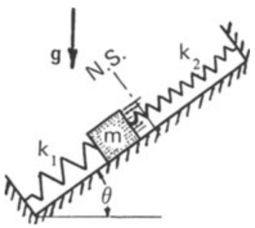
Problem 7.24.

7.25. Two projectiles of masses m_1 and m_2 are fired consecutively with the same initial speed v_0, but at different angles of elevation θ_1 and θ_2, respectively. The second shell is fired t_1 seconds after the first. The shells subsequently collide at the time t_2. Neglect air resistance, observe the principle of conservation of momentum, and find the angle θ_2 at which the second shell was fired. Is θ_2 smaller or larger than θ_1?

7.26. A particle of mass m, supported by a smooth horizontal surface, is fastened to a string of length 50 cm and twirled anticlockwise at 25 rad/sec about a fixed point O in the surface. If the string is pulled through a small hole at O with a constant speed of 150 cm/sec as the particle moves around the fixed surface, what is the absolute speed of the particle at 25 cm from O?

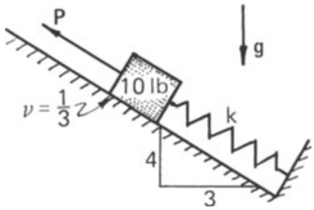
7.27. A particle of mass m, supported by a smooth horizontal surface, is attached to a string of length L and twirled about a fixed point O in the surface with constant counterclockwise angular velocity ω. The string strikes a nail inserted suddenly through the surface at a distance R from O. (a) Apply the energy principle to find the new angular velocity $\boldsymbol{\Omega}$ of m. (b) Apply the principle of conservation of moment of momentum to find $\boldsymbol{\Omega}$.

7.28. Two springs having moduli k_1 and k_2 are fastened to a mass m, as shown. The load is released from rest at the natural unstretched state of both springs to oscillate on the smooth, inclined plane. Determine by the energy method the maximum displacement of m, and find the period and the amplitude of the oscillations.



Problem 7.28.

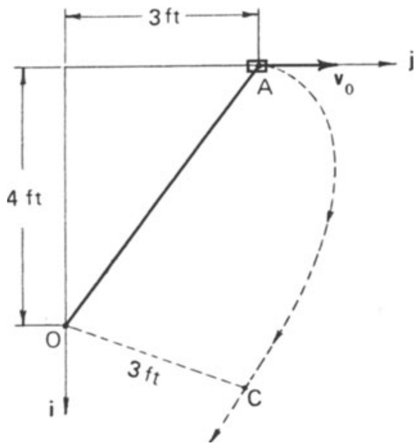
7.29. During an interval of interest, a constant force $\mathbf{P} = 20\mathbf{i}$ lb is applied to a 10 lb block attached to a spring of stiffness $k = 20$ lb/in. At the initial instant, the system is compressed an amount $\delta = 6$ in. from its natural state and released from rest on a rough surface inclined as shown in the figure. Assume that $g = 32$ ft/sec^2 and $\nu = \frac{1}{3}$. Find the speed of the block when it has moved 9 in.



Problem 7.29.

7.30. One end of a linearly elastic rubber string, having an unstretched length of 3 ft and a modulus $k = 9$ lb/ft, is fixed at O on the $\mathbf{i}$-axis. Its other end is fastened to a small block of mass 0.03 slug. The string is stretched to the point A shown in the figure, and the block is given a velocity $\mathbf{v}_0 = 20\mathbf{j}$ ft/sec at A. The block then slides on a smooth horizontal supporting surface. (a) Find the speed of the block at the instant the string becomes slack at the place C, and determine its subsequent closest approach to O. Describe the path of the block before the string loses its slack in the motion beyond C. (b) Determine the block's greatest distance from O on the path AC, and find its speed there.

7.31. The initial velocity $\mathbf{v}_0 = v_0\mathbf{e}_r$ that will enable a rocket to just escape from the Earth's gravitation is called the *escape velocity*. Consider a simple model of a rocket for space which has no propulsion system of its own after it has been projected vertically from the ground with an initial speed v_0. Account for the variation in the gravitational attraction with the distance from the center of the Earth, whose radius is 4000 miles, neglect air resistance and the Earth's rotation,



Problem 7.30

and assume that the speed vanishes at great distance from the Earth. Find the escape velocity in mph for this model.

7.32. A small block of mass m is attached to a linear spring of length l and stiffness k. The system is at rest in its natural state on a smooth horizontal plane when a suddenly applied horizontal force imparts to the block a velocity $\mathbf{v}_0 = v_0\mathbf{j}$ perpendicular to the spring axis $\mathbf{i}$. (a) Apply the moment of momentum and energy principles to determine the speed of the block as a function of the spring's extension δ. (b) Find the extensional rate $\dot{\delta}$ of the spring, and thus formulate an integral expression for the travel time t as a function of δ.

7.33. A slider block B of unknown mass is moving with a constant speed v_0 inside a smooth, horizontal straight tube when suddenly it strikes a linear spring of unknown stiffness. Its measured speed subsequently is reduced to v_1 when the observed spring compression is δ_1. Determine the deceleration of B at the maximum spring deflection δ_m. What is δ_m?

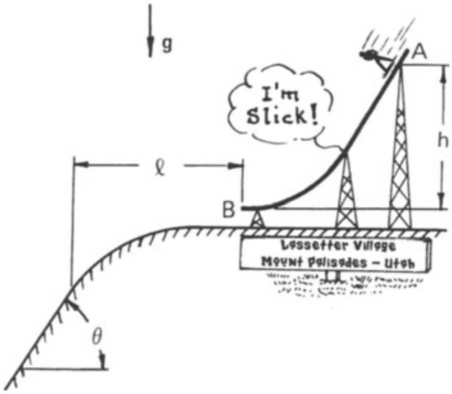
7.34. Solve Problem 6.57 by the energy method.

7.35. A rectangular steel plate AB weighing 100 lb is suspended by four identical springs of elasticity $k = 20$ lb/in. attached to its corner points. A 300 lb block is then placed centrally on the plate and released. (a) Apply the energy method to find the subsequent maximum displacement d of the system. (b) Derive the equation of motion, and compute the vibrational period of the total load.

7.36. A skier starts from rest at A, slides down a slick ski ramp of height h, and after leaving the ramp at B eventually lands smoothly on a steep slope inclined at the angle θ shown in the figure. The horizontal landing distance from B is $\ell = 2h$. Find the angle θ.

7.37. A particle of mass m slides in a smooth parabolic tube $y = kx^2$ in the vertical plane frame $\varphi = \{O; \mathbf{i}, \mathbf{j}\}$. In addition to the usual forces, a conservative force with potential $V^*(y)$ acts on P so that the horizontal component of its oscillatory motion within the tube is $x = A\cos\omega t$, A and ω being constants. (a) Find the potential energy $V^*(y)$. (b) Determine as a function of y the magnitude of the normal force exerted on P by the tube.

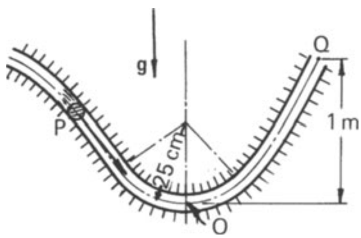
7.38. The force $\mathbf{F} = F_0[\sin(\pi y/a)\mathbf{i} + (\pi x/a)(1 + \cos(\pi y/a))\mathbf{j}]$, where F_0 is a constant, acts on a particle of mass m initially at rest at O in $\Phi = \{O; \mathbf{i}_k\}$. The particle moves in the vertical plane on a circle of radius a and center at $(a, 0)$. (a) Apply Stokes's theorem (7.57) to



Problem 7.36.

find the speed of the particle after one revolution. (b) Use the line integral (7.21) to calculate the work done by **F** in one revolution.

7.39. The figure shows a particle P of mass 2 kg moving in a smooth, curved tube in the vertical plane on a planet where the apparent acceleration of gravity is $g = 8.0$ m/sec^2. The particle has a speed of 5.0 m/sec when it passes the point O on the circular arc of radius 25 cm. (a) What force does the tube exert on the particle at O? (b) Determine the speed of P at the exit point Q. (c) If P started from rest initially, what was its initial location at h above O?

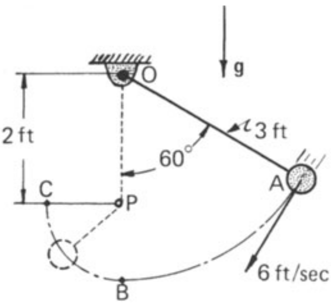


Problem 7.39.

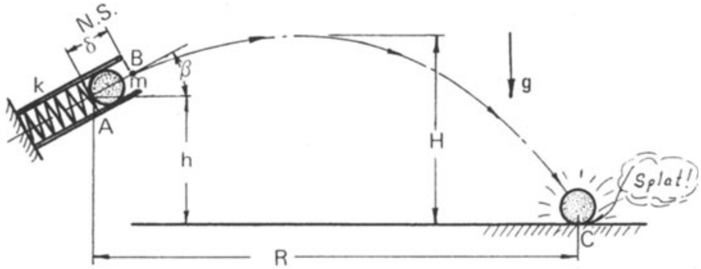
7.40. Apply the energy method to solve Problem 6.46. What is the static displacement of the load?

7.41. A pendulum bob is released from the position A with a speed of 6 ft/sec, as shown in the figure. In its vertical position OB, the cord strikes a fixed pin P, and the bob continues to swing on a smaller circular arc BC. (a) Apply the general energy principle to find the speed of the bob at its horizontal position at C. (b) What is the ratio of the angular speeds immediately after and before the string strikes P? (c) Derive equations for the angular speed $\dot{\theta}$ as a function of the angular placement θ of the bob measured from the vertical line OB, so that $\theta < 0$ on AB and $\theta > 0$ on BC. Plot $\dot{\theta}$ versus $\theta \in [-60°, 180°]$. Identify all important points of the plot. What is the jump in the angular speed at B?

7.42. A spring of stiffness k is compressed an amount δ from its natural state at B. When released, it projects a small mass m which lands at the point C located in the figure. Neglect friction and air resistance. (a) Find the speed of m when it strikes the ground at C. (b) What is the greatest height H attained in the motion?

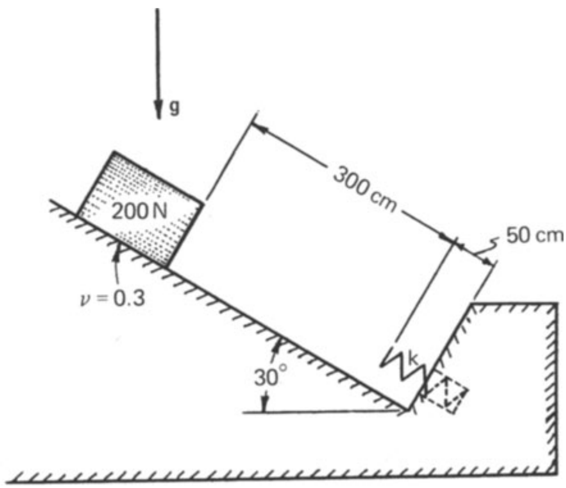


Problem 7.41.



Problem 7.42.

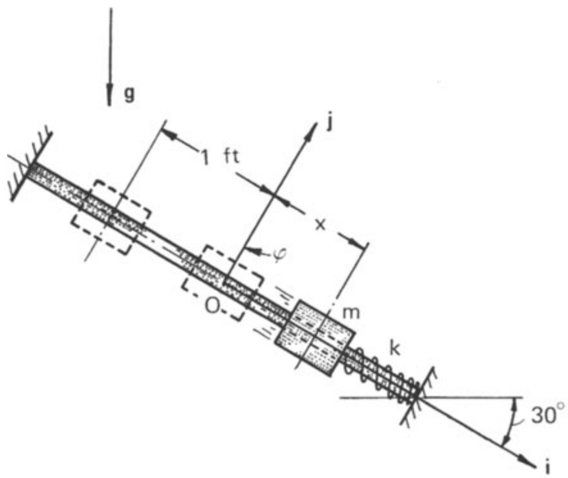
7.43. A block weighing 200 N is released from rest at the position shown on an inclined plane surface for which $\nu = 0.3$, and it ultimately contacts a spring of modulus $k = 20$ N/cm. Apply the general energy principle to find the maximum compression of the spring.



Problem 7.43.

7.44. The pilot of a cargo carrier is making an airdrop to a remote, tornado stricken area. At an appropriate time, an airman pushes a bundle of blankets from the rear of the aircraft with a constant speed v_B relative to the plane which has a ground speed v_P at an altitude h. (a) Apply conservation principles to determine the distance d from the recovery target area at which the bundle should be released, and find its trajectory. Find the drop time t^* to the target. Model the bundle as a particle and ignore environmental effects. (b) Evaluate the results for the case when $v_B = 5$ mph, $v_P = 185$ mph, and $h = 1610$ ft.

7.45. A slider block of mass $m = 0.5$ slug is attached to a spring of stiffness $k = 32$ lb/ft at a place where $g = 32$ ft/sec^2. The block is displaced, as shown, 1 ft from the natural state at O and released to move on a smooth, inclined supporting rod. (a) Apply the energy method to derive the equation of motion for m. (b) Find the equilibrium position of m, and determine the period and circular frequency of the oscillations. (c) Find the motion as a function of time, determine its amplitude, and sketch its graph for one period.



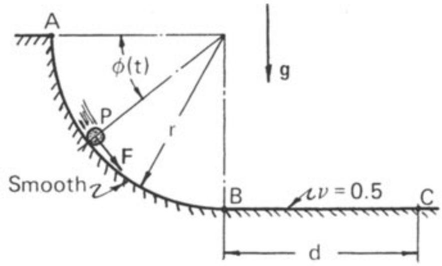
Problem 7.45.

7.46. The slider block described in Problem 6.53 is given an initial displacement $x(0) = x_0$ and released from rest relative to the table. Apply only the general energy and moment of momentum principles to derive its equation of motion. Interpret the energy equation for a stable equilibrium state at the origin, if initially $x_0 = 0$.

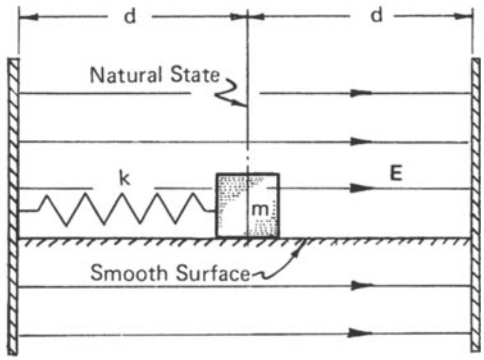
7.47. A particle P of weight W starts from rest at A, shown in the figure, and is driven along a smooth circular track of radius r by a tangential propulsive force of variable intensity $F(t) = 2W \cos\phi(t)$ for $\phi \in [0, \pi/2]$. At B, it transfers to a rough horizontal surface BC on which $\nu = 0.5$. (a) What is the horizontal distance d traveled by P before coming to rest at C? (b) Determine the surface force exerted on P at the instant it reaches B on the circular arc and immediately afterward.

7.48. A body of mass m, initially at rest on a smooth, electrically insulated, horizontal surface, is attached to an insulated linear spring. The assembly, in its natural state, is placed in a constant electric field of strength $\mathbf{E}$, directed as shown. The body is suddenly charged an amount q and oscillates within the field. What is the work done by the electrical force? Apply the work–energy principle to find the motion of m and describe its physical characteristics. Sketch the solution function.

7.49. The spring and pulley suspension system shown in the figure for Problem 6.46 is modified to introduce a spring of modulus k connected at the end of the rigid supporting rod OA and attached to the load M. Neglect friction, ignore the mass of the pulley and the support system, and suppose that the pulley belt is inextensible. (a) Find the static displacement of the load M and



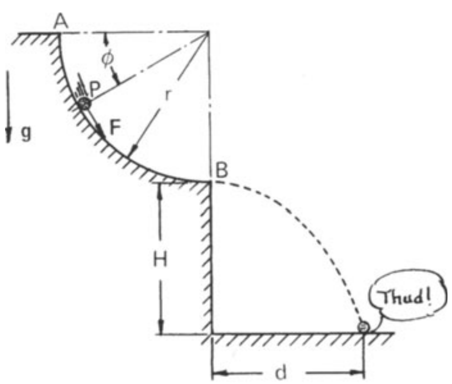
Problem 7.47.



Problem 7.48.

derive its equation of motion about the static equilibrium state. (b) What is the stiffness of an equivalent simple spring-mass system having the same frequency for the same load? (c) Describe the major vibrational characteristics of the motion. (d) How many degrees of freedom does this system have?

7.50. A particle P of mass m is moved along a smooth circular track by a tangential propulsive force whose magnitude $F = F_0 \cos \phi(t)$ varies with the position angle $\phi(t) \in [0, \pi/2]$ shown in the figure. The particle starts from rest at A and projects from the track at point B. Find the subsequent trajectory of P, and determine the horizontal distance d at which P strikes the horizontal plane at the vertical distance H below B.



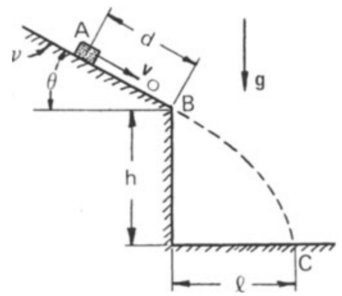
Problem 7.50.

7.51. The rod tension exerted on a simple pendulum bob is given by $\mathbf{T}(\mathbf{x}) = T(\theta)\mathbf{n}(\theta)$, where $T(\theta)$ is defined by (7.86c) and $\mathbf{n}(\theta)$ is the unit normal vector shown in Fig. 6.15, page 138. (a) Write $\mathbf{T}(\mathbf{x})$ as a function of the x- and y-position coordinates of the bob, and show that $\mathbf{T}(\mathbf{x})$ is not conservative. (b) The curl of a vector field $\mathbf{u} = u_r\mathbf{e}_r + u_\theta\mathbf{e}_\theta + u_z\mathbf{e}_z$ in cylindrical coordinates is defined by

$$\nabla \times \mathbf{u} = \left(\frac{1}{r}\frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z}\right)\mathbf{e}_r + \left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right)\mathbf{e}_\theta + \frac{1}{r}\left(\frac{\partial (r u_\theta)}{\partial r} - \frac{\partial u_r}{\partial \theta}\right)\mathbf{e}_z. \quad \text{(P7.51)}$$

Write $\mathbf{T}(\mathbf{x})$ in cylindrical coordinates and apply (P7.51) to show that $\mathbf{T}(\mathbf{x})$ is not conservative. Fortunately, $\mathbf{T}$ does no work in the motion.

7.52. A sled of mass m is driven at a constant velocity $\mathbf{v}_0$ down a rough plane with $\nu = 0.5$ and inclined at an angle $\theta = 45°$. The power is suddenly lost at point A, shown in the figure, but the sled continues to slide down the plane to point B where it leaves the surface and later impacts a horizontal plane at the point C, a distance $h = 2\ell$ below B. Find in terms of $v_0 = |\mathbf{v}_0|$ and ℓ the distance d from B at which the power was lost.



Problem 7.52.

7.53. (a) A bullet of mass m is fired with velocity β and passes through a block of mass M initially at ease on a smooth horizontal surface S. The block travels the distance d shown in the figure, projects from S at A, and ultimately lands at C at a distance D from B on a horizontal plane at H below A. Determine the exit speed α of the bullet. Ignore frictional effects and energy and small mass losses from permanent deformation and tearing of the block and the bullet. (b) Now suppose that the surface S is rough with coefficient of friction ν. Find the exit speed of the bullet. What is the largest initial velocity that M may have and remain on S?

7.54. Two other Jacobian elliptic functions related to the elliptic sine function $\text{sn}u$ in (7.88f) are defined by

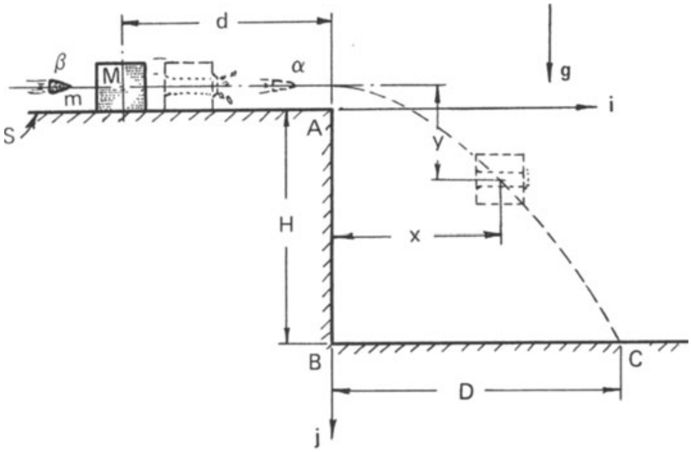
$$\text{cn}u = \cos\phi, \qquad \text{dn}u = \sqrt{1 - k^2\sin^2\phi}, \quad \text{(P7.54a)}$$

in which ϕ is the argument of the elliptic integral in (7.87d), or equivalently (7.88d): $u = F(\phi; k)$. (a) Show that

$$\text{sn}^2u + \text{cn}^2u = 1, \qquad k^2\text{sn}^2u + \text{dn}^2u = 1. \quad \text{(P7.54b)}$$

(b) Prove that the *elliptic cosine function* $\text{cn}u$, read as *"see—en—u"*, is an even periodic function of period $4K(k)$ and $-1 \le \text{cn}u \le 1$. (c) Show that when $k = 0$,

$$\text{sn}u = \sin u, \qquad \text{cn}u = \cos u, \qquad \text{dn}u = 1, \qquad K(0) = \frac{\pi}{2}, \quad \text{(P7.54c)}$$



Problem 7.53.

and hence the Jacobian elliptic functions reduce to the trigonometric functions. In all, there are twelve Jacobian elliptic functions; the others are defined in terms of the three basic functions described above. For details, see the Byrd and Friedman Handbook cited in footnote ‡, page 266.

7.55. (a) Recall (7.88f), (7.88g), and note that $u = F(\phi; k)$ in (7.87d). Show that these relations and those of the previous problem yield the following derivatives of the Jacobian elliptic functions:

$$\frac{d}{du}(\mathrm{sn}u) = \mathrm{cn}u\ \mathrm{dn}u, \tag{P7.55a}$$

$$\frac{d}{du}(\mathrm{cn}u) = -\mathrm{sn}u\ \mathrm{dn}u, \tag{P7.55b}$$

$$\frac{d}{du}(\mathrm{dn}u) = -k^2\mathrm{sn}u\ \mathrm{cn}u. \tag{P7.55c}$$

Verify that when $k = 0$, these reduce to the familiar trigonometric rules. (b) Recall (7.88h) and (7.88i). On the same plot, sketch graphs of the three basic Jacobian elliptic functions, and thus show that the third elliptic function dn$u$, read as *"dee—en—u"*, is a positive-valued, even periodic function of period $2K(k)$ with values in the interval $k' \le \mathrm{dn}u \le 1$, where *the complementary modulus* $k' \equiv (1 - k^2)^{1/2}$.

7.56. The exact relation for the period of a simple pendulum is given by (7.87f). (a) Use the ratio τ^*/τ to determine the percentage error in the period that occurs when the small amplitude solution τ is used for large amplitudes $\alpha = 30°,\ 60°$, and $90°$. (b) Compute the percentage error based on the second order approximate solution in (7.90d) in comparison with (7.87f). How does the error in this case compare with that in part (a) for the same amplitudes?

7.57. A bead B of mass m is constrained to slide in the vertical plane on a smooth, fixed circular wire of radius a and vertical diameter AC. The bead is projected counterclockwise from the lowest point A with initial speed $v_0 = \alpha\nu$, where ν is the smallest initial speed that will drive B to its highest point at C and α is a constant. (a) Determine the speed ν. (b) Let ϕ denote the angle at C between the diameter CA and the chord CB at time t. Find the time required for the bead to describe the arc AB subtended by ϕ. Then analyze three cases: (i) $\alpha = 1$, (ii) $\alpha > 1$, and (iii) $\alpha < 1$, and interpret their physical nature.

7.58. The pendulum for the data described in Problem 6.47 is a scaled model for a certain low speed vibration control device. The preliminary design requires that the period of the finite amplitude oscillations of the pendulum must be two seconds. The bob design mass $m = 0.01$ kg. Derive the exact equations of motion for the pendulum, and find as functions of the amplitude angle β_0 exact relations for the angular speed of the table and the maximum tension in the string. The project leader wants you to present the results to the technical management team who may not recall the mathematics used to express the solution. The results for all values of $\beta_0 \leq 60^\circ$ must be discussed, but it is anticipated that questions concerning the effects of variations in the period and the potential influence of larger amplitudes may arise. Provide your supervisor with a brief preliminary report that will convey clearly all of the desired information.

7.59. (a) The small amplitude period of a simple pendulum is 2 sec. What is its period for an amplitude $\alpha = \pm\pi/2$ rad? (b) Suppose the same pendulum has just adequate initial velocity to complete a full revolution. Find the time required for the bob to advance 90° from its lowest position.

7.60. A spring-mass system similar to that in Fig. 6.13, page 134, consists of a mass m attached to two concentric springs. The inner spring has linear response with stiffness k_1. The other is a nonlinear conical spring with stiffness k_2, whose spring force is proportional to the cube of its extension x from the natural state. The mass is given an initial displacement x_0 from the natural state and released to oscillate on the smooth horizontal surface. (a) Derive the equation of motion for m, and solve it to obtain an integral for the travel time $t = t(x)$. (b) Introduce the change of variable $x = x_0 \cos\phi$ to obtain $t = t(\phi)$, and derive exactly the motion $x(m, t)$ in terms of a Jacobian elliptic function. (c) What is the period of the finite amplitude oscillations of m?

7.61. A bead of mass m slides on a smooth wire in the vertical plane. Find its small amplitude frequency when the wire is (a) a parabola $y = ax^2$ and (b) a catenary $y = a\cosh x$, where a is the same positive constant. On which curve is the frequency greater?

7.62. Solve the last problem for (a) an ellipse $x^2/a^2 + y^2/b^2 = 1$ and (b) a hyperbola $y^2/b^2 - x^2/a^2 = 1$, where a and b are the usual constants. Confirm the solution for the ellipse in its special application to a circular wire. What is the particle's small amplitude frequency on an equilateral hyperbola? What is interesting about these results?

7.63. A particle is given a small displacement from a stable equilibrium state on a smooth Archimedean spiral $r = a\phi$ in the vertical plane. Here a is a positive constant. Find the frequency of the oscillation about the first stable equilibrium state. Is the frequency about other stable equilibrium states larger or smaller? Explain this and support your answer with an example. See Problem 4.51 in Volume 1.

7.64. A particle of mass m, initially at rest, slides in the vertical plane on a smooth cycloid shown in Fig. 7.16, page 273, and described by (7.94a). Let $\gamma(s)$ denote the slope angle of the curve at s, and γ_0 its value at the particle's initial position. (a) Find as a function of γ and γ_0 the time to reach a lower point on the curve. Do this two ways: (i) apply the energy integral (7.92a) and (ii) use the general solution of (7.94e). (b) Hence, show that regardless of its initial position, the particle will always reach the minimum point on the cycloid in the time $t = \tau/4$.

7.65. A bead of mass m slides on a smooth wire in the vertical plane. If the distance it travels in time t is $s(t) = a\sinh(nt)$, where a and n are constants, determine the shape of the wire and the initial conditions.

7.66. A particle of mass m moves on a smooth convex curve in the vertical plane. If its speed is proportional to the distance traveled from the highest point on the curve, determine the path.

7.67. A particle of mass m is at rest at the vertex of a smooth, inverted cycloid in the vertical plane. When slightly disturbed, it slides down the cycloidal surface. (a) Find the vertical distance below the vertex at which the particle leaves the surface. (b) Determine the distance traveled and the speed at that instant.

7.68. Consider an arbitrary point P on a smooth cycloid (7.94a) in the vertical plane frame $\Phi = \{O; \mathbf{i}, \mathbf{j}\}$ in Fig. 7.16, page 273. A particle of mass m is released from rest to slide on a smooth, straight wire from the point P to the origin O. (a) Derive an equation for the normalized time of descent $T_l \equiv t_l/\sqrt{a/g}$ along the wire, as a function of the angle β at the initial point P on the cycloid. (b) Show that for all values of $\beta \in [0, \pi]$ the normalized time of descent T_l exceeds the normalized time of descent $T_c \equiv t_c/\sqrt{a/g}$ of a particle sliding on the cycloid from the same point P. In particular, show that for $\beta \in [0, \pi]$, $T_l \in [4, (\pi^2 + 4)^{1/2}]$, which is everywhere greater than T_c. Hence, no matter where the motion starts, a particle that slides on the cycloid from point P always is the first to reach O; moreover, the result is independent of the particle's mass.

7.69. The orbit of a boomerang B thrown from point O is a petal of a lemniscate described by the polar coordinate equation $r^2 = a^2 \cos 2\theta$, where a is its greatest distance from O. Find the total distance L traveled by B in its return to O.