6

Dynamics of a Particle

6.1. Introduction

We have seen that in an inertial reference frame, Euler's first law (5.43) for the motion of the center of mass "particle" of a rigid body $\mathscr{B}$, a fictitious material point of mass $m(\mathscr{B})$ that moves with the body, has the same form as Newton's second law (5.39) for the motion of a particle P of mass $m(P)$. Hence, the motion of any such "material point" or "particle" is governed by the Newton–Euler law of motion, here written in its various forms as

$$\mathbf{F} = \dot{\mathbf{p}} = m\mathbf{a} = m\dot{\mathbf{v}} = m\ddot{\mathbf{x}}, \tag{6.1}$$

in which m is the mass of the "particle," $\mathbf{p} = m\mathbf{v}$, and $\mathbf{x}$, $\mathbf{v}$, and $\mathbf{a}$ are its respective current position, velocity, and acceleration in an inertial reference frame.

Our objective now is to study a variety of physical applications and solutions of the Newton–Euler equation of motion of a particle for various kinds of forces and motions and thus demonstrate its predictive value. In some examples, the principal body of interest may be small in some sense. An electron, a grain of sand, and a fluid droplet are typical examples of infinitesimal or small bodies commonly modeled as particles. Larger bodies like a ball, a pendulum bob, a crate, a person, and an automobile are modeled as center of mass objects of rigid bodies. So long as the rigid body has no rotation itself, there is no intrinsic difference between these two models. In fact, in many such problems in which the body is replaced by its center of mass "particle," precise identification of the center of mass point is not necessary; the mass distribution and the specific body geometry play no major roles; and the actual points of application of the resultant forces that act on the body are unimportant—they act on the particle. All of these virtually inconsequential matters, however, have great importance later when rotational effects of a rigid body are introduced. We recall, for example, the simple problem of a block sliding down an inclined plane without tipping over. In this case, the body's physical and geometrical properties, the location of the points of application of forces that act on it, and their moments were all very important to the description and analysis of the block's motion. These sorts of underlying potential complications are avoided when rotational effects are absent and a rigid body is modeled as a particle.

The study of particle dynamics thus deals with the analysis of the vector differential equation (6.1) for the motion of a particle and the forces that produce it. When the motion, the velocity, or the acceleration is known either as a function of time or as a function of a time dependent parameter, such as arc length along a path, the force required to produce the motion is readily determined by (6.1). The converse problem, to determine the motion of a particle under various kinds of assigned forces, however, is more difficult, because it involves the integration of (6.1). Moreover, the specification of some forces together with some components of acceleration, velocity, or position leads to a mixed variety of problem types. Some easy methods of integration useful in the analysis of (6.1) were studied in earlier chapters. Additional methods and several new concepts will be introduced as our study unfolds.

6.2. Component Forms of the Newton–Euler Law

We recall that the motion of a particle may be described in terms of different coordinate systems that offer special advantages in applications; and, clearly, in applications of (6.1), the force vector and the motion eventually must be represented in the same reference basis. For handy reference, the vector representations of the Newton–Euler law in four familiar kinds of reference bases are provided below.

Rectangular Cartesian reference frame $\Phi = \{O; \mathbf{i}, \mathbf{j}, \mathbf{k}\}$: The acceleration is given by (1.12) and (6.1) may be written as

$$\mathbf{F} \equiv F_x \mathbf{i} + F_y \mathbf{j} + F_z \mathbf{k} = m(\ddot{x}\mathbf{i} + \ddot{y}\mathbf{j} + \ddot{z}\mathbf{k}). \tag{6.2}$$

Intrinsic reference frame $\psi = \{P; \mathbf{t}, \mathbf{n}, \mathbf{b}\}$: Equation (1.71) provides the acceleration and (6.1) becomes

$$\mathbf{F} \equiv F_t \mathbf{t} + F_n \mathbf{n} = m(\ddot{s}\mathbf{t} + \kappa \dot{s}^2 \mathbf{n}). \tag{6.3}$$

Notice that there can be no intrinsic force component F_b normal to the osculating plane. Hence, if the motion is constrained to a plane, the total force component perpendicular to the plane must vanish. This is a property of every plane motion.

Cylindrical reference frame $\varphi = \{O; \mathbf{e}_r, \mathbf{e}_\phi, \mathbf{e}_z\}$: The Newton–Euler law (6.1) and the acceleration vector in (4.60) yield the representation

$$\mathbf{F} \equiv F_r \mathbf{e}_r + F_\phi \mathbf{e}_\phi + F_z \mathbf{e}_z = m\left[(\ddot{r} - r\dot{\phi}^2)\mathbf{e}_r + \frac{1}{r}\frac{d}{dt}(r^2\dot{\phi})\mathbf{e}_\phi + \ddot{z}\mathbf{e}_z\right]. \tag{6.4}$$

Spherical reference frame $\varphi = \{O; \mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi\}$: The acceleration components are defined in (4.71). Hence, (6.1) becomes

$$\mathbf{F} \equiv F_r\mathbf{e}_r + F_\theta\mathbf{e}_\theta + F_\phi\mathbf{e}_\phi = m\left[(\ddot{r} - r\dot{\theta}^2 - r\dot{\phi}^2\sin^2\theta)\mathbf{e}_r \right.$$
$$\left. + \left(\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) - r\dot{\phi}^2\sin\theta\cos\theta\right)\mathbf{e}_\theta + \left(\frac{1}{r}\frac{d}{dt}(r^2\dot{\phi}\sin\theta) + r\dot{\phi}\dot{\theta}\cos\theta\right)\mathbf{e}_\phi\right]. \tag{6.5}$$

The left-hand expressions in (6.2) through (6.5) define the respective component forms of the total force. The force components are then related to the acceleration components by equating their corresponding scalar components in these expressions. The intrinsic force components F_t and F_n in (6.3), for example, are thus related to the intrinsic acceleration components by $F_t = m\ddot{s}$, $F_n = m\kappa\dot{s}^2$. The procedure is the same for the others. The component equations are called the *scalar equations of motion*. In general, however, we first formulate each problem in its vector form, and afterwards identify the corresponding scalar equations of motion.

It is not always necessary to introduce a specific component form of (6.1). Sometimes it is possible to solve a problem in direct vector form without mention of any components, but more often than not this approach proves tedious and impractical; therefore, the component forms find wider use in applications.

6.3. Some Introductory Examples and Additional Concepts

We shall begin with several introductory examples that employ the foregoing representations in some problems where the motion is essentially known and certain force conditions are to be determined. Some earlier concepts are reviewed, and some new concepts are introduced as the examples progress. The importance of the Newton–Euler law in its generic form (5.34) is underscored in characterizing the motion of a relativistic particle.

6.3.1. Some Applications in a Rectangular Cartesian Reference Frame

Three problems that use a rectangular Cartesian reference frame are solved. The first is an easy application of (6.1) in which the acceleration is known and a certain force is to be found. The example demonstrates the importance of our distinguishing the inertial reference frame in applications of the Newton–Euler law. The second exercise illustrates an application in which the acceleration of one body is known, and a Coulomb condition for relative sliding of another contacting body is to be determined. The results will be used in the third example to illustrate

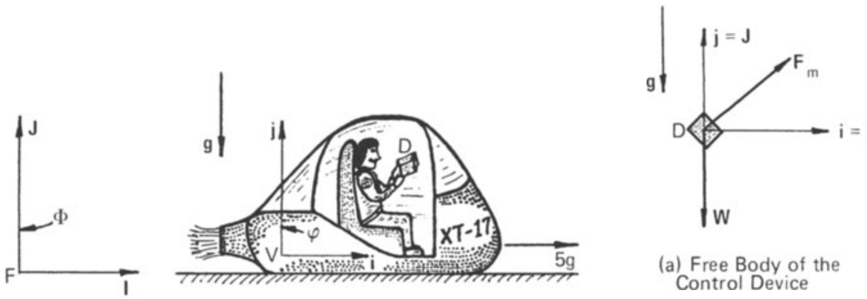

Figure 6.1. Motion in an accelerating reference frame.

the converse problem in which the forces are known and information about the motion is to be obtained. The form of the law in (6.2) is evident in the applications.

Example 6.1. A rocket propelled test vehicle V in Fig. 6.1 is used to study man's ability to function at high rates of acceleration and deceleration.* (a) Suppose the vehicle is accelerating at $5g$ along a straight track in the inertial frame $\Phi = \{F; \mathbf{I}_k\}$. What force does the operator need to apply to a 2 lb control device D to impart to its center of mass a relative acceleration $\mathbf{a}_{DV} = 16\mathbf{i} + 80\mathbf{j}$ ft/sec^2 in the vehicle frame $\varphi = \{V; \mathbf{i}_k\}$? (b) Compare the result with the force required to perform the same task when the vehicle has a uniform motion in Φ. Assume that the local acceleration of gravity is 32 ft/sec^2.

* The example brings to mind the daring exploits of U.S. Air Force Colonel John P. Stapp, MD, Ph.D., the biomedical engineering pioneer, who in December 1947, at Edwards (then Muroc) Air Force Base, California, became the first human to ride a rocket propelled test sled to study human tolerance to severe decelerations of the sort sustained in the crash of an automobile or aircraft. Based on Stapp's research studies, appropriate safety harnesses, helmets, restraints, and other essential equipment could be developed. Stapp demonstrated firsthand that a properly harnessed and protected driver, pilot, or astronaut could indeed survive an incredible impact, the wind blast, and deceleration of ejection from an aircraft traveling at supersonic speeds at great altitudes, or the large acceleration of a rocket lift-off, himself having withstood test sled decelerations of 25 to more than 40 times the acceleration of gravity. With new facilities at the Holloman Air Force Base, New Mexico, where subsequently he set up and directed his biomedical engineering and crash research programs, in 1954 Stapp rode the rocket vehicle "Sonic Wind" from 632 mph to a dead stop in 1.4 sec, suffering only minor injuries in a deceleration of more than 40 gs! A 2200 lb (1000 kg) automobile smashing into a brick wall at 50 mph ($\approx$80 kmph) would subject its driver to roughly the same impulsive shock. Other human volunteers in his program tested the security of safety belts in decelerations that exceeded 25 gs. See *Time*, *The Weekly Newsmagazine*, Volume **66**, No. 11, September 12, (1955), 80–2, 85–6, 88. Stapp's adventures, his sense of humor, and his generosity to others are portrayed here. Dr. Stapp, then dubbed "the fastest man on earth," died at his New Mexico home on November 13, 1999, at age 89. I thank Professor O. W. Dillon, who during the early 1950s was stationed at Holloman when Stapp was directing these research programs, and upon reading the manuscript reminded me of Stapp's heroic feats.

Solution of (a). We begin with the problem kinematics. The absolute acceleration of the vehicle in the inertial frame Φ is given as $\mathbf{a}_{VF} = 5g\mathbf{I}$, where $g = 32$ ft/sec^2. Thus, recalling the simple relative acceleration rule (4.50) and the assigned center of mass acceleration $\mathbf{a}_{DV} = 16\mathbf{i} + 80\mathbf{j}$ ft/sec^2 of D in the vehicle frame φ in which $\mathbf{i}_k = \mathbf{I}_k$, we determine the absolute acceleration of D in frame Φ:

$$\mathbf{a}_{DF} = \mathbf{a}_{DV} + \mathbf{a}_{VF} = 176\mathbf{I} + 80\mathbf{J} \text{ ft/sec}^2. \tag{6.6a}$$

This completes the kinematical analysis.

We now turn to the force analysis. The free body diagram of D is shown in Fig. 6.1a. As usual, we shall assume that the contact force due to the surrounding air is self-equilibrated to zero. Then the total force $\mathbf{F}(D, t)$ acting on D is the sum of its weight $\mathbf{W}$ and the force $\mathbf{F}_m$ exerted by the operator. Hence, the Newton–Euler law (6.1) applied to D in the inertial frame Φ yields

$$\mathbf{F}(D, t) = \mathbf{W} + \mathbf{F}_m = m(D)\mathbf{a}_{DF}, \tag{6.6b}$$

in which $\mathbf{W} = -mg\mathbf{J} = -2\mathbf{J}$ lb and $m(D) = 1/16$ slug. The kinematics in (6.6a) is now coupled with the force analysis in (6.6b) to yield the solution

$$\mathbf{F}_m = 11\mathbf{I} + 7\mathbf{J} \text{ lb}. \tag{6.6c}$$

Solution of (b). We note from (6.6c) that $|\mathbf{F}_m| = \sqrt{170} \approx 13.04$ lb. We wish to compare this result with the force needed to perform the same task when the vehicle has a uniform motion in Φ. To impart the same acceleration to the device when the vehicle has a constant velocity or may be at rest in Φ so that now $\mathbf{a}_{VF} = \mathbf{0}$ and $\mathbf{a}_{DF} = \mathbf{a}_{DV}$, we find from (6.6b) that the operator must apply a force $\mathbf{F}_m = m\mathbf{a}_{DV} - \mathbf{W} = \mathbf{I} + 7\mathbf{J}$ lb. Hence, $|\mathbf{F}_m| = 5\sqrt{2} \approx 7.07$ lb. Therefore, if the Newton–Euler law were applied in the accelerating reference frame, the operator would conclude incorrectly that a force of about 7 lb is needed, while the task actually requires nearly twice that. We thus learn that *when the operator works in the accelerating vehicle, nearly twice the effort must be expended to perform the assigned task.* □

This example demonstrates the important role of the inertial reference frame in applications of the Newton–Euler law. The next problem concerns the prediction of relative sliding of a body in contact with an accelerating surface.

Example 6.2. A truck carrying a crated load W is moving down a 15° grade in Fig. 6.2. The driver suddenly applies the brakes and the truck decelerates at the steady rate of 4 ft/sec^2 along its straight path. The coefficient of static friction between the crate and the trailer bed is $\mu = 0.3$. Determine for the given values of the parameters whether the crate will slide or remain stationary relative to the trailer.

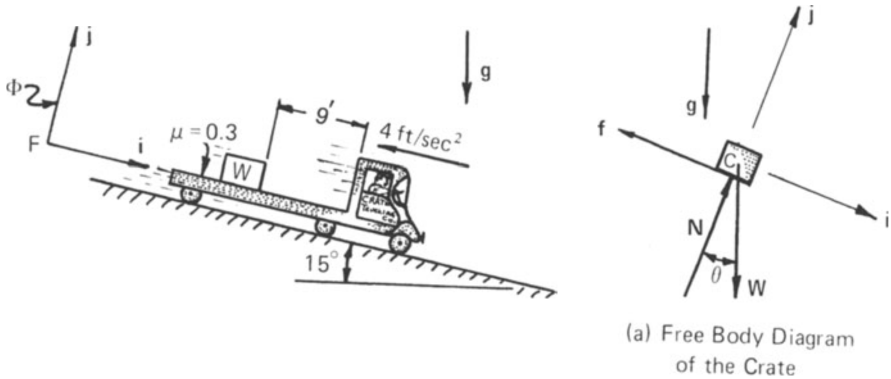

Figure 6.2. Relative motion of a crate on an accelerating truck.

Solution. We shall assume initially that the crate does not slide relative to the truck and seek a Coulomb condition sufficient to assure this. If this condition fails for the assigned data, we then know that the crate will slide. This strategy will enable us to decide the issue.

To investigate the motion of the crate C, we first draw its free body diagram in Fig. 6.2a. To simplify matters, all contact forces due to the Earth's atmosphere, including air flow effects due to the truck's motion and other wind effects, are neglected. Then the total force $\mathbf{F}(C, t)$ acting on C is approximated by its weight $\mathbf{W}$ and the resultant normal and tangential contact forces $\mathbf{N}$ and $\mathbf{f}$ exerted by the trailer bed. The equation of motion (6.1) for C becomes

$$\mathbf{F}(C, t) = \mathbf{W} + \mathbf{f} + \mathbf{N} = m\mathbf{a}_{CF}, \tag{6.7a}$$

wherein $m = m(C)$ is the total mass of C and $\mathbf{a}_{CF}$ is its total rectilinear acceleration in the inertial ground frame $\Phi = \{F; \mathbf{i}, \mathbf{j}, \mathbf{k}\}$. The vectors in (6.7a) are given by

$$\mathbf{W} = W(\sin\theta\mathbf{i} - \cos\theta\mathbf{j}), \qquad \mathbf{f} = -f\mathbf{i}, \qquad \mathbf{N} = N\mathbf{j}, \qquad \mathbf{a}_{CF} = a_C\mathbf{i}, \tag{6.7b}$$

and hence

$$(W\sin\theta - f)\mathbf{i} + (N - W\cos\theta)\mathbf{j} = ma_C\mathbf{i}. \tag{6.7c}$$

Therefore, the scalar equations of motion for the crate are

$$ma_C = W\sin\theta - f, \qquad N - W\cos\theta = 0. \tag{6.7d}$$

When a_C is known, equations (6.7d) determine the unknown forces N and f. Thus, with $W = mg$,

$$N = W\cos\theta, \qquad f = W(\sin\theta - a_C/g). \tag{6.7e}$$

Recalling the strategy proposed earlier, we note that the crate will not slip if the frictional force f is smaller than its critical static Coulomb value (5.70), that is, provided that $f < f_c = \mu N$. (See also (5.72).) In this case, because the crate is assumed not to slip, its acceleration is the same as that of the truck, namely, $\mathbf{a}_{TF} = a_T\mathbf{i}$. Thus, with the aid of (6.7e) and $a_C = a_T$, the Coulomb no slip criterion is

$$\sin\theta - \frac{a_T}{g} < \mu\cos\theta. \tag{6.7f}$$

This conclusion is independent of the weight, the size, and the shape of the crate. Actually, however, we have tacitly assumed in (6.7f) that the crate geometry is consistent with the no tip condition, which imposes limitations on the crate geometry. The reader may confirm, for example, that for a rectangular box of height $2h$ and a square cross section of side $2b$, the crate will not topple before slip occurs, if it occurs at all, provided that $b/h > \mu$.

The crate will not slide if (6.7f) holds for the assigned data; otherwise, it will. We now test (6.7f) for the assigned values $a_T = -4$ ft/sec^2, $g = 32.2$ ft/sec^2, $\mu = 0.3$, and $\theta = 15°$. The terms on the left side of (6.7f) yield the value $l \equiv 0.383$ while those on right give $r \equiv 0.290$. Since $l > r$, (6.7f) does *not* hold, and the crate will slide. For an alternative approach, the reader may show that the critical acceleration $\hat{a}_T$ of the truck for which sliding of the crate is imminent is given by $\hat{a}_T = g(\sin\theta - \mu\cos\theta) = -1$ ft/sec^2, the condition for equality in (6.7f). Since $|a_T| = 4$ ft/sec$^2 > |\hat{a}_T|$, the crate will slide, as concluded previously. □

The simple relative motion of the crate on the truck bed is examined next in illustration of the converse problem in which the forces are known and the velocity and the motion of the crate are to be found.

Example 6.3. The coefficient of dynamic friction between the crate and the trailer bed is $\nu = 0.25$. What is the rectilinear acceleration of the crate relative to the trailer? Determine the distance on the bed traveled by the crate after 1 sec and after 2 sec.

Solution. The crate C has a rectilinear acceleration $\mathbf{a}_{CT}$ relative to the truck T given by

$$\mathbf{a}_{CT} = \mathbf{a}_{CF} - \mathbf{a}_{TF}, \tag{6.8a}$$

wherein $\mathbf{a}_{TF} = a_T\mathbf{i}$ is the known absolute acceleration of the truck in the inertial frame Φ. We need to find $\mathbf{a}_{CF}$, the total acceleration of the crate in Φ.

The vector equation for the sliding motion of the crate is the same as (6.7a), and hence the scalar equations of motion for the crate in Φ are given in (6.7d). But this time, because the crate is sliding on the trailer bed, the Coulomb frictional force is given by (5.71). (See also (5.73).) Thus, with the last of (6.7d), we have $f = f_d = \nu N = \nu W\cos\theta$, and use of this relation in the first equation in (6.7d) yields a_C. That is, $\mathbf{a}_{CF} = a_C\mathbf{i} = g(\sin\theta - \nu\cos\theta)\mathbf{i}$. Hence, (6.8a) delivers the first of the desired results:

$$\mathbf{a}_{CT} = a_{CT}\mathbf{i} = [g(\sin\theta - \nu\cos\theta) - a_T]\mathbf{i}. \tag{6.8b}$$

Therefore, the rectilinear acceleration of the crate relative to the truck is independent of the weight, the size, and shape of the crate, consistent with the no tip condition.

The relative acceleration (6.8b) is a constant vector. With $a_T = -4$ ft/sec^2, $g = 32.2$ ft/sec^2, $\nu = 0.25$, and $\theta = 15°$, we find $\mathbf{a}_{CT} = 4.56\mathbf{i}$ ft/sec^2. To determine the distance traveled by the crate on the bed, we first integrate the differential equation $\delta\mathbf{v}_{CT}/\delta t = \mathbf{a}_{CT}$ with the initial condition $\mathbf{v}_{CT}(0) = \mathbf{0}$ to obtain $\mathbf{v}_{CT} = \mathbf{a}_{CT}t = 4.56t\mathbf{i}$. Hence, the relative speed of C is $\dot{s}(t) = 4.56t$; and with $s(0) = 0$, the distance traveled by the crate is $s(t) = 2.28t^2$. Therefore, after 1 sec the crate has moved a distance $s(1) = 2.28$ ft. After 2 secs, $s(2) = 9.12$ ft, and the crate, regardless of its physical features, slams into the cab, initially only 9 ft away in Fig. 6.2. □

6.3.2. Intrinsic Equation of Motion for a Relativistic Particle

In this section, the intrinsic equation of motion for a relativistic particle whose "effective" mass varies with its speed is derived, and the result is applied to examine the nature of a purely normal force that acts on the particle in its motion along a smooth curved path. The Newton–Euler law in the form (6.1), however, cannot be used in problems where the mass of the particle is variable; so we return to the basic law (5.34).

In relativistic mechanics, the *relativistic mass* m of a particle P in a frame Φ varies with its speed $\dot{s}$ relative to Φ in accordance with the rule

$$m = \gamma m_0 = \frac{m_0}{\sqrt{1-\beta^2}} \qquad \text{with} \quad \beta \equiv \frac{\dot{s}}{c}. \tag{6.9}$$

The constant m_0, the invariant mass of the particle, is called the *rest mass* of P in Φ and the constant c is the speed of light in a vacuum. The relativistic mass m is not the intrinsic mass of P. Rather, the concept of mass is retained as an invariant, intrinsic property of an object, and hence m_0 is identified as the invariant mass of the object, the same for all observers and for all times. The principle of conservation of mass applies to m_0, not to m. Although nowadays it is unfashionable to refer to m as the relativistic mass, it is convenient in this text to retain the symbolic relation $m \equiv \gamma m_0$ defined by (6.9) and continue to call it the relativistic mass.

These semantics aside, the *relativistic momentum* of P is defined by $\mathbf{p} \equiv m\mathbf{v} = \gamma m_0\mathbf{v}$, where $\mathbf{v} = d\mathbf{x}/dt$ is the usual time derivative; and the rule governing the motion of P is retained in the general Newtonian form $\mathbf{F} = d\mathbf{p}/dt$ stated in (5.34). Although m changes with $\dot{s}$, it is easy to show that $\mathbf{F} = \mathbf{0}$ holds if and only if the motion is uniform in Φ. This conforms with the condition set by the first law, i.e. $\mathbf{F} = \mathbf{0} \Longleftrightarrow \mathbf{a} = \mathbf{0}$. Otherwise, in view of (6.9), the second law becomes

$$\mathbf{F} = m\mathbf{a} + \frac{dm}{dt}\mathbf{v}. \tag{6.10}$$

Now, with the aid of (6.9) and $\mathbf{v} = \dot{s}\mathbf{t} = c\beta\mathbf{t}$, we find

$$\frac{dm}{dt}\mathbf{v} = \frac{m_0 \beta \dot{\beta} \mathbf{v}}{(1-\beta^2)^{3/2}} = \frac{m\beta^2}{1-\beta^2}\ddot{s}\mathbf{t}.$$

Therefore, use of this result and (1.71) for the intrinsic acceleration in (6.10) leads to the *intrinsic equation of motion for a relativistic particle*:

$$\mathbf{F} = m\left(\frac{\ddot{s}}{1-\beta^2}\mathbf{t} + \kappa\dot{s}^2\mathbf{n}\right). \tag{6.11}$$

When $\dot{s} \ll c$ so that $\beta \ll 1$, (6.9) reduces approximately to $m = m_0$ and we recover from (6.11) the classical, nonrelativistic intrinsic equation in (6.3). It follows from (6.11) that *the total force* $\mathbf{F}$ *acting on a particle may be normal to its path, hence perpendicular to its velocity vector* $\mathbf{v} = \dot{s}\mathbf{t}$, *if and only if its speed is constant.* (See Problem 1.5, Volume 1.) This is illustrated below.

Example 6.4. A particle P, free from gravitational force, experiences a relativistic motion in a smooth, spatially curved tube. Find the force exerted on the particle by the tube and characterize the tube geometry in order that the force may have a constant magnitude.

Solution. The reader's free body diagram of P will show that the total force on P is simply the normal reaction force exerted by the smooth tube. Hence, use of $\mathbf{F} = \mathbf{N} = N\mathbf{n}$ in (6.11) yields the desired information:

$$N = m\kappa\dot{s}^2 \quad \text{and} \quad \ddot{s} = 0. \tag{6.12}$$

Indeed, the second of these equations shows that the particle speed must be constant; and hence the relativistic mass in (6.9) must be constant too. Therefore, the first relation in (6.12) shows that *in a smooth motion with constant speed, the normal reaction force intensity at each point along the path is proportional to the curvature and is directed toward the center of curvature.* Clearly, $N = 0$ if and only if the motion is uniform, in which case the tube must be straight. *In general, N may be constant if and only if the tube has a constant curvature.* A cylindrical helix is a familiar example of a space curve having a constant curvature. (See Example 1.14.) If the motion is a plane motion, the tube must be circular. The following further example is left for the reader. □

Exercise 6.1. A particle P moves on a smooth surface S so that the only force on P is the normal surface reaction force $\mathbf{R}$. Prove that the principal normal vector **n** must be perpendicular to S at each point of the trajectory of P and hence the path is a geodesic on S. (See Example 1.16 in Volume 1.) □

The results for the motion of a relativistic particle in a smooth tube hold independently of relativistic considerations when $\beta \ll 1$. It is shown later that the same behavior occurs when an electrically charged particle, relativistic or not, moves in a uniform magnetic field.

6.3.3. Electric and Magnetic Forces on a Charged Particle

Two basic laws that describe electric and magnetic body forces are introduced. Afterwards, the trajectory of an electrically charged particle moving in a steady and uniform magnetic field is described.

First, consider the mutual force of attraction or repulsion between two particles with electric charges q_1 and q_2 respectively situated at $\mathbf{X}_1$ and $\mathbf{X}_2$ in an arbitrarily assigned reference frame so that the distance between them is $r = |\mathbf{X}_2 - \mathbf{X}_1|$. Let $\mathbf{F}_{12}$ denote the force exerted on q_1 by q_2, and write $\mathbf{e}$ for the unit vector directed from q_2, the source of the action, toward q_1. The force exerted on q_2 by q_1 is equal and oppositely directed so that $\mathbf{F}_{21} = -\mathbf{F}_{12}$. Experiments support the following principle governing the mutual interaction of electrically charged particles.

Coulomb's law of electrostatics: *Between any two charged particles in the world, there exists a mutual electrostatic force which is directly proportional to the product of the charges, inversely proportional to the square of the distance between them, and directed along their common line in the sense of mutual repulsion or attraction according as the charges are of the same or opposite kind, respectively; that is,*

$$\mathbf{F}_{12} = \frac{kq_1q_2}{r^2}\mathbf{e}. \tag{6.13}$$

The value of the positive constant k depends on the nature of the medium in which the charges are placed. The physical dimensions of k are fixed by (6.13): $[k] = [FL^2Q^{-2}]$, where $[Q] = [q]$ denotes the physical dimension of electric charge. The metric measure unit of q is named the *coulomb*. Experiments on charges in vacuum show that $k = 9 \times 10^9\ \mathrm{N} \cdot \mathrm{m}^2/\mathrm{coulomb}^2$. Notice that only the relative position vector $\mathbf{r} = r\mathbf{e}$ of q_1 from q_2 is important.

The rule (6.13) is a particular example of Noll's general rule (5.115) governing the internal force between any pair of particles, in this case charged particles; and the formal similarity of (6.13) with Newton's law of gravitation (5.46) is evident. We thus introduce the parallel idea of an electric field $\mathcal{E}$ that arises from the existence of a charged particle situated in space. And when a particle of charge q is placed in this space, it experiences a force of attraction or repulsion determined by (6.13). An *electric field* $\mathcal{E}$ is said to exist throughout space due to a particle of positive charge q_0, called the *source* of the electric field, whenever a force is felt by another charged "test" particle placed anywhere in $\mathscr{E}$. Thus, the *electric field strength* **E** at the place **X** due to q_0 is defined by

$$\mathbf{E}(\mathbf{X}) = \frac{kq_0}{r^2}\mathbf{e}(\mathbf{X}), \tag{6.14}$$

where **e** is the unit vector directed from q_0 toward the field point **X** at r from q_0. Hence, the *electric force* $\mathbf{F}_e$ that acts on a particle P of charge q at the place **X** is a body force given by

$$\mathbf{F}_e(P; \mathbf{X}) = q(P)\mathbf{E}(\mathbf{X}). \tag{6.15}$$

The same rule holds when the charged particle moves in the electrostatic field $\mathscr{E}$.

The electric body force is in the direction of **E** (repulsive) when q is positive and opposite to **E** (attractive) when q is negative. Hence, the action of this force alone will move a charged particle in a straight line in the direction of **E** if $q > 0$, oppositely if $q < 0$. The *principle of conservation of electric charge* asserts that the total charge Q for a closed system of n charges q_k is a constant equal to their algebraic sum: $Q = \sum_{k=1}^{n} q_k$. Thus, in a manner parallel to that demonstrated for a gravitational field, the resultant electric force on a particle of charge q placed in the electric field of a system of charged particles or, similarly, in the field of a charged continuum is given by the fundamental law (6.15). In general, then, the electric force acting on a particle of charge q having a motion $\mathbf{X}(q, t)$ in an electrostatic field of strength $\mathbf{E}(\mathbf{X})$ is given by (6.15).

A *magnetic field of strength* **B** arises in a similar way from the existence in space of some kind of magnetic object. When a charged particle moves with a velocity **v** in a time independent magnetic field **B**, it experiences a body force $\mathbf{F}_m$, the *magnetic force*, given by

$$\mathbf{F}_m = q\mathbf{v} \times \mathbf{B}. \tag{6.16}$$

This equation shows that the magnetic body force $\mathbf{F}_m$ on a charged particle is always perpendicular to **v**, and hence to the particle's path. Under the action of this force alone the particle, from (6.12), must move with a constant speed v_0, say; so, the magnitude of its momentum $|\mathbf{p}| = mv_0$ is constant.

Example 6.5. Consider a relativistic charged particle of rest mass m_0 moving in a constant magnetic field of strength **B**. (a) Prove that the charge moves in a circular helix, a curve of constant curvature, and hence $\mathbf{F}_m$ has a constant magnitude. (b) Derive the equation of the path for a plane motion perpendicular to a constant magnetic field $\mathbf{B} = B\mathbf{k}$.

Solution of (a). To determine the trajectory of a particle of charge q moving in a magnetic field of constant strength **B**, we recall Newton's law in (5.34) and consider the relation

$$\frac{d}{dt}(\mathbf{p} \cdot \mathbf{B}) = \frac{d\mathbf{p}}{dt} \cdot \mathbf{B} = \mathbf{F}_m \cdot \mathbf{B} = 0, \tag{6.17a}$$

wherein (6.16) is the total force on q. Therefore, the component of the momentum in the direction of $\mathbf{B}$ is constant:

$$\mathbf{p} \cdot \mathbf{B} = m\mathbf{v} \cdot \mathbf{B} = C, \text{ a constant.} \tag{6.17b}$$

Since the magnitudes of $\mathbf{p}$ and $\mathbf{B}$ are constant, (6.17b) implies that the angle between the fixed axis of $\mathbf{B}$ and the tangent to the space curve along which q moves is constant everywhere along the path. Consequently, as described in Example 1.14, the path is a circular helix, a space curve of constant curvature; therefore, $|\mathbf{F}_m| = qv_0 B \sin\langle \mathbf{v}, \mathbf{B}\rangle$ is constant. Conversely, it follows from (6.16) that if $\mathbf{F}_m$ has a constant magnitude, $\sin\langle \mathbf{v}, \mathbf{B}\rangle$ is constant and hence the path is a circular helix.

The initial velocity $\mathbf{v}_0$ may be considered arbitrary. If the velocity is initially perpendicular to $\mathbf{B}$, then, by (6.17b), $\mathbf{p} \cdot \mathbf{B} = 0$ always, and the path is a circle in the plane perpendicular to $\mathbf{B}$. If the initial velocity $\mathbf{v}_0$ is parallel to $\mathbf{B}$, the constant force $\mathbf{F}_m = \mathbf{0}$; the motion is uniform and the path is a straight line along the axis of $\mathbf{B}$. The circle and the line are degenerate kinds of helices. In summary, *the trajectory of a charged particle which is given an arbitrary initial velocity in a constant magnetic field is a circular helix.*

Solution of (b). The path of a charge q in a plane motion perpendicular to the constant vector $\mathbf{B}$ is a circle. To describe this circle, we apply Newton's law in (6.16) to write $d\mathbf{p}/dt = d(q\mathbf{x} \times \mathbf{B})/dt$. Integration yields $\mathbf{p} - \mathbf{qx} \times \mathbf{B} = \mathbf{A}$, a constant vector. Let $\mathbf{B} = B\mathbf{k}$ and consider a plane motion perpendicular to $\mathbf{B}$, so that $\mathbf{x} = x\mathbf{i} + y\mathbf{j}$. Then $\mathbf{p} = (A_1 + qBy)\mathbf{i} + (A_2 - qBx)\mathbf{j}$, and with $\mathbf{p} \cdot \mathbf{p} = |\mathbf{p}|^2 = m^2 v_0^2$, a constant, this yields the equation of a circular orbit of radius $R \equiv mv_0/qB$:

$$\left(x - \frac{A_2}{qB}\right)^2 + \left(y + \frac{A_1}{qB}\right)^2 = R^2. \tag{6.17c}$$

We thus find with (6.9) that a charged relativistic particle in a uniform magnetic field moves on a circular orbit with angular speed $\omega = v_0/R = qB/m = (qB/m_0)\sqrt{1-\beta^2}$. This is known as the *circular cyclotron frequency.* □

When both fields (6.15) and (6.16) are present, the total *electromagnetic force,* known as the *Lorentz force,* is

$$\mathbf{F}_e + \mathbf{F}_m = q\mathbf{E} + q\mathbf{v} \times \mathbf{B}. \tag{6.18}$$

Many interesting effects may be produced by an electromagnetic field. In some cases of physical interest an electromagnetic force is used to accelerate atomic

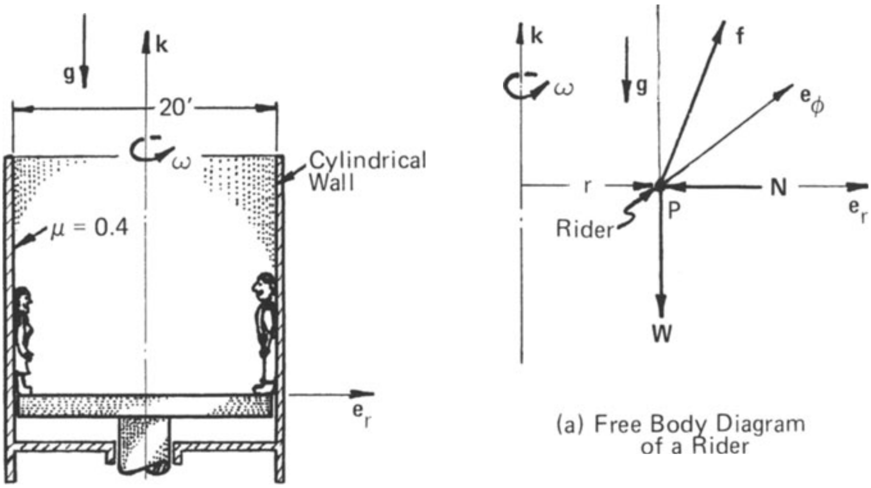

Figure 6.3. Relative equilibrium of passengers in an amusement park centrifuge.

particles in a cyclotron to speeds nearly as great as the speed of light. In these applications the electromagnetic force on the particle is considerably greater than the usual gravitational force, which is ignored. In further applications presented below, unless explicitly stated otherwise, it will be assumed that the speed of the particle is small compared with the speed of light so that the classical, Newton–Euler form (6.1) of the equation of motion for a particle or center of mass object is appropriate.

6.3.4. Fun at the Amusement Park

Our final illustration in this section concerns a design analysis of an amusement park ride to assess the safety of its occupants during its rotational motion. The cylindrical coordinate representation (6.4) for the equation of motion is illustrated.

Example 6.6. An amusement park ride shown in Fig. 6.3 consists of a 20 ft diameter cylindrical room that turns about its axis. People stand against the rough cylindrical wall. After the room has reached a certain angular speed, the floor drops from under the riders. What must be the angular speed of the room to assure that a person will not slide on the wall? The design coefficient of static friction is $\mu = 0.4$.

Solution. To assess the safe angular speed design, we seek a no-slip Coulomb condition sufficient to assure that a rider does not slide on the wall of the rotating room. The free body diagram of a rider represented as a center of mass object P is shown in Fig 6.3a. The rider's weight is $\mathbf{W} = -W\mathbf{k}$, and $\mathbf{N} = -N\mathbf{e}_r$ and $\mathbf{f} = f_\phi \mathbf{e}_\phi + f_z \mathbf{k}$ are the normal and the tangential frictional forces exerted by the wall. Thus, the total force **F** on a rider in a cylindrical frame that turns with the room is

$$\mathbf{F}(P,t) = \mathbf{N} + \mathbf{f} + \mathbf{W} = -N\mathbf{e}_r + f_\phi \mathbf{e}_\phi + (f_z - W)\mathbf{k}. \qquad (6.19a)$$

For the safety of a rider, we require that the rider remain at rest relative to the wall. Then by (6.4) in which $\dot{\phi} = \omega$, or by (4.48) in which $\boldsymbol{\omega}_f = \omega\mathbf{k}$, it follows that $m\mathbf{a}_P = -mr\omega^2\mathbf{e}_r$. Equating this to the force in (6.19a), we obtain the scalar equations of motion

$$N = mr\omega^2, \qquad W = f_z, \qquad f_\phi = 0. \qquad (6.19b)$$

In the steady rotation of the room, no circumferential component f_ϕ of the frictional force is exerted on the rider by the wall; and the second of these relations shows that the rider will not slide down the wall if the Coulomb condition $W = f_z \le f_c = \mu N$ holds. Therefore, with the first equation in (6.19b), the design criterion for safety of the riders is given by $\mu m r\omega^2 \ge W$. That is,

$$\omega \ge \sqrt{\frac{g}{r\mu}}, \qquad (6.19c)$$

equality holding when slip is imminent; the smallest value $\omega^* = \sqrt{g/r\mu}$ being the critical angular speed of the room. *The result is independent of the weight of the rider; so all persons, fat or thin, will stay on the wall, provided that their coefficient of friction with the wall is not less than the design value chosen for* μ.

For the given conditions $r = 10$ ft and $\mu = 0.4$, the critical angular speed is $\omega^* = 2.84$ rad/sec, which is about 27 rpm. Thus, to secure the safety of the riders, the room must spin at a rate greater than 27 rpm. □

6.3.5. Formulation of the Particle Dynamics Problem

The foregoing examples show that when information about the motion is known, various questions involving the nature of the applied forces may be addressed. Some unanticipated physical conclusions are also pointed out, and the predictive value of the classical principles of mechanics is demonstrated. A review of the methods used in these examples reveals a fairly orderly arrangement of steps followed in the formulation and in the solution procedure applied to the particle dynamics problem; namely,

1. To begin, identify and express the data and the unknown quantities in mathematical form, and ask the key question: what relations connect the given data to the information to be found? Write these down and decide upon an initial problem attack strategy; but be prepared to modify your strategy as the attack advances and additional data is revealed.
2. To continue, construct a free body diagram that shows all of the properly directed contact forces and body forces that act on the free body in an appropriate reference frame.

3. Write down the total, $\mathbf{F}$, of all forces identified in the free body diagram and express these various forces by their vector component representations in the chosen reference basis.
4. Determine the absolute acceleration $\mathbf{a}$ of the particle in the inertial frame but referred to the reference basis used above.
5. Assemble the results of steps 3 and 4 into the vector differential equation of motion: $\mathbf{F} = m\mathbf{a}$.
6. Equate the corresponding scalar components to obtain the scalar equations of motion, and proceed to solve these equations subject to the assigned data. Other laws appropriate to the problem, such as Newton's third law or Coulomb's laws, should be recalled and included here.

This basic procedural model is encountered repeatedly throughout our work. The outlined program, however, is not rigid. The examples suggest that sometimes it is useful, or simply a matter of personal preference, to begin with the kinematics in step 4 and then advance to the formulation of the force relations described in steps 2 and 3. Sometimes the vector equation in step 5, as shown in Example 6.5, page 105, may be solved directly without decomposing the vectors into their scalar components, eliminating steps 3 and 6. The student must be prepared to modify this schedule as other methods are introduced below. But the primary organizational step 1 always should be considered first and revisited as the solution unfolds.

With these ideas in mind, we shall begin the study of a variety of situations in which certain forces are prescribed functions and information concerning the motion and other forces is to be determined. This will require integration of the vector equation of motion (6.1). Some new forces of nature will be introduced along the way. We begin with some familiar examples.

6.4. Analysis of Motion for Time Dependent and Constant Forces

Problems of the motion of a particle under time varying and constant forces are readily solved by the method of separation of variables, a familiar approach used often in earlier examples. The formal solution of problems in this class, first presented as kinematical problems in Chapter 1, is reviewed next. The results are then applied in some elementary examples.

6.4.1. Motion under a Time Varying Force

Let us consider a total force $\mathbf{F} = \mathbf{F}(P, t)$ acting on a particle P in an inertial frame, given as a specified function of time. Then (6.1) yields $\mathbf{a}(P, t) = \mathbf{F}(P, t)/m(P)$, a known function of time. Hence, with $d\mathbf{v} = \mathbf{a}dt$, this differential equation is readily integrated in direct vector form to obtain the velocity of P:

$$\mathbf{v}(P,t) = \frac{1}{m}\int \mathbf{F}(P,t)dt + \mathbf{c}_1, \tag{6.20}$$

in which $\mathbf{c}_1$ is a constant vector of integration.

A second integration with $d\mathbf{x} = \mathbf{v}dt$ gives the motion of P:

$$\mathbf{x}(P,t) = \int \mathbf{v}(P,t)dt + \mathbf{c}_2, \tag{6.21}$$

wherein $\mathbf{c}_2$ is another constant vector of integration. The constants $\mathbf{c}_1$ and $\mathbf{c}_2$ are fixed by the assigned initial data. The reader will notice that (6.20) and (6.21) are respectively equivalent to the kinematical equations (1.24) and (1.23). A typical example follows. (See also Example 1.7 in Volume 1.)

Example 6.7. A particle P in an inertial reference frame has an initial velocity $\mathbf{v}_0$ at the place $\mathbf{x}_0$, and subsequently moves under the influence of a force that is proportional to the time and acts in a fixed direction $\mathbf{e}$. Find the position and velocity of P at time t.

Solution. The force on P is given by $\mathbf{F}(P,t) = kt\mathbf{e}$, where k is a constant and $\mathbf{e}$ is a constant unit vector. Use of this relation in (6.1) and integration of the result as shown in (6.20) with the initial value $\mathbf{v}(P,0) = \mathbf{v}_0$ yields the velocity $\mathbf{v}(P,t) = kt^2/2m\mathbf{e} + \mathbf{v}_0$. With the initial value $\mathbf{x}(P,0) = \mathbf{x}_0$, a second integration described by (6.21) yields the motion $\mathbf{x}(P,t) = kt^3/6m\mathbf{e} + \mathbf{v}_0 t + \mathbf{x}_0$. Let the reader show that if P starts at the origin with velocity $\mathbf{v}_0 = v_0\mathbf{j}$ and the force acts in the direction $\mathbf{e} = \mathbf{i}$, the path of P is a cubic parabola $x = cy^3$. Identify the constant c. □

6.4.2. Motion under a Constant Force

In the special case when $\mathbf{F}(P,t) = \mathbf{F}_0$ is a constant force, the acceleration $\mathbf{a}(P,t) = \mathbf{F}_0/m$ is also a constant vector. Hence, (6.20) reduces to

$$\mathbf{v}(P,t) = \frac{\mathbf{F}_0}{m}t + \mathbf{v}_0 \tag{6.22}$$

with $\mathbf{c}_1 = \mathbf{v}(P,0) \equiv \mathbf{v}_0$. Integration of (6.22) in accordance with (6.21) and use of $\mathbf{c}_2 = \mathbf{x}(P,0) \equiv \mathbf{x}_0$ delivers the motion

$$\mathbf{x}(P,t) = \frac{\mathbf{F}_0}{2m}t^2 + \mathbf{v}_0 t + \mathbf{x}_0. \tag{6.23}$$

These elementary formulas are applied below to study projectile motion and the motion of a particle that falls from rest relative to the Earth. To simplify matters, the spin of the Earth and aerodynamic and atmospheric drag effects are neglected. Then the two problems are similar because they occur under the same constant gravitational force $\mathbf{F}_0 = \mathbf{W} = m\mathbf{g}$, while only the initial conditions are different. Any motion under gravity alone is called *free fall.*

6.4.2.1. *Galileo's Principle for Free Fall of a Particle*

The initial conditions in the free fall problem of a particle P released from rest at the origin are $\mathbf{v}_0 = \mathbf{0}$, $\mathbf{x}_0 = \mathbf{0}$, and (6.22) and (6.23) thus yield the familiar elementary *equations for the free fall motion, velocity, and acceleration of the particle*:

$$\mathbf{x}(P, t) = \frac{1}{2}\mathbf{g}t^2, \qquad \mathbf{v}(P, t) = \mathbf{g}t, \qquad \mathbf{a}(P, t) = \mathbf{g}. \tag{6.24}$$

The results (6.24) *are independent of the mass or any other property of the object, and hence, for the same circumstances, we learn that all bodies fall with the same speed along the plumb line of* **g**. This is known as *Galileo's principle.* Accordingly, if two balls, one made of cast iron and the other of wood, were simultaneously released from the summit of the Leaning Tower of Pisa, an experiment *alleged*† to have been done in 1590 by the famous Italian scientist, Galileo Galilei (1564–1642), then together they would fall, and together they would strike the ground. Of course, common experience with feathers and stones contradicts this principle. But this happens because the physical attributes of the feather are not accurately modeled by the assumptions—specifically, the primary assumption of negligible air resistance which is plainly essential to our physical interpretation of the theoretical results. On the contrary, experiments conducted on bodies falling in a vacuum, including feathers and stones, lend support to Galileo's principle, which otherwise is especially altered by air resistance and to a lesser extent by the rotation of the Earth, effects that are investigated later.

6.4.2.2. *Motion of a Relativistic Particle under Constant Force*

Many elementary but interesting problems concern the motion of a particle when the total force is either a constant vector or an elementary function of time. It is not intended, however, that any of the foregoing formulas should be memorized. On the contrary, the examples serve to review procedures used often in Volume 1 to obtain solutions to similar problems by the easy method of separation of variables. While the same basic procedure may be applied to investigate the motion of a relativistic particle, for example, the formulas derived above cannot be used at all. This is illustrated next. Afterwards, the results are compared with those in (6.22) and (6.23) when $\mathbf{x}_0 = \mathbf{0}$, $\mathbf{v}_0 = \mathbf{0}$.

† See Cooper's study described in the References.

Example 6.8. A relativistic particle P, initially at rest at the origin in frame ψ, is moving along a straight line under a constant force $\mathbf{F}_0$. Determine the relativistic speed and the distance traveled by P as functions of time.

Solution. The equation of motion for the relativistic particle is given by (5.34) in which $\mathbf{F}(P, t) = \mathbf{F}_0$ is a constant force and (6.9) is to be used. Hence, separation of the variables and integration of $\mathbf{F}_0 dt = d(m\mathbf{v}) = d(\gamma m_0 \mathbf{v})$, with the initial values $\mathbf{v}(P, 0) = \mathbf{0}$ and $\gamma = 1$, yields $m\mathbf{v} = \mathbf{F}_0 t$. Thus, recalling (6.9) and noting that $\mathbf{v} = v\mathbf{t}$ and $\mathbf{F}_0 = F_0\mathbf{t}$ are parallel vectors, we have only one nontrivial component equation: $m_0 v/(1 - v^2/c^2)^{1/2} = F_0 t$. This scalar equation yields the rectilinear, relativistic speed

$$v(P, t) = \frac{ckt}{\sqrt{1 + (kt)^2}} \qquad \text{with} \quad k \equiv \frac{F_0}{m_0 c}. \tag{6.25a}$$

Introducing $v = \dot{s}$ into (6.25a), separating the variables, and integrating $ds = v dt$ with the initial value $s(0) = 0$, we obtain the rectilinear distance traveled by P:

$$s(P, t) = \frac{c}{k}\left(\sqrt{1 + (kt)^2} - 1\right). \tag{6.25b}$$

Notice in (6.25a) that $v/c < 1$ for all t, and $v/c \to 1$ as $t \to \infty$; that is, *under a constant force, the relativistic particle speed cannot exceed the speed of light c.* This result is quite different from the corresponding speed $v = F_0 t/m_0$ described by (6.22) for a Newtonian particle of mass $m = m_0$ initially at rest and subject to a constant force F_0; in this case $v \to \infty$ with t. If $m_0 c$ is large compared with $F_0 t$ so that $kt \ll 1$, then (6.25a) and (6.25b) reduce approximately to

$$v(P, t) = ckt = \frac{F_0}{m_0} t, \qquad s(P, t) = \frac{1}{2} ckt^2 = \frac{F_0}{2m_0} t^2. \tag{6.25c}$$

These are the Newtonian formulas described by (6.22) and (6.23) for the corresponding rectilinear motion of a particle of mass m_0 initially at rest at the origin and acted upon by a constant force F_0. In the present relativistic approximation, however, these results are valid for only a *sufficiently small* time for which $v/c = kt \ll 1$. □

6.4.2.3. Elements of Projectile Motion

Equations (6.22) and (6.23) are applied next in two examples involving projectile motion and the simultaneous rectilinear or free fall motion of another target body. Afterwards, a fascinating technological application of a controlled projectile motion is studied. In addition to earlier assumptions, frictional effects are ignored.

Example 6.9. Percy Panther is snoozing in an open-top artillery truck when he senses the presence of the mischievous Arnold Aardvark lurking beneath. He

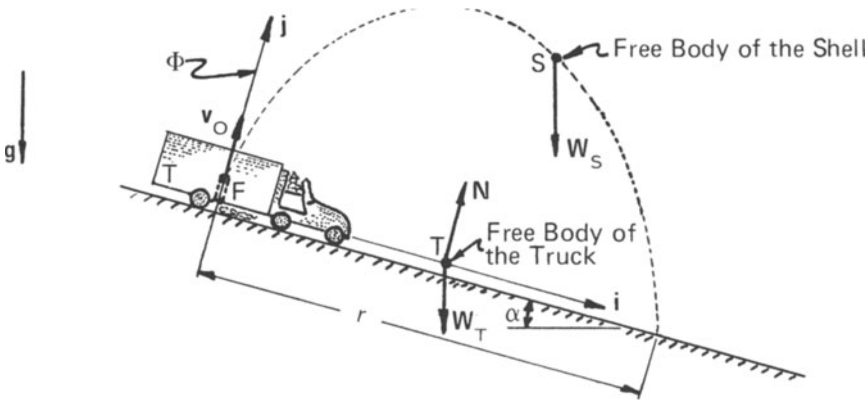

Figure 6.4. Projectile motion in an inertial reference frame without friction.

quietly releases the handbrake to escape down the hill inclined at an angle α. Arnold Aardvark having quietly rigged a remote trigger, immediately fires the gun, launching a shell of mass m straight up from the truck, as shown in Fig. 6.4. The gun has a muzzle velocity $\mathbf{v}_0$, and the total mass of the truck and its strange driver is M. Determine the time and the location at which the shell impacts the ground, and find the location of Percy Panther at that time.

Solution. First, we determine the motion of the shell S, whose free body diagram is shown in Fig. 6.4. The total force acting on S is its weight $\mathbf{W}_S = m\mathbf{g}$. Thus, in the inertial frame $\Phi = \{F; \mathbf{i}_k\}$ fixed in the ground, the constant force in (6.22) and (6.23) is $\mathbf{F}_0 = \mathbf{W}_S = mg(\sin\alpha\mathbf{i} - \cos\alpha\mathbf{j})$; and with $\mathbf{v}_0 = v_0\mathbf{j}$ and $\mathbf{x}_0 = \mathbf{0}$ initially, we obtain, in evident notation,

$$\mathbf{v}_S(t) = v_0\mathbf{j} + gt(\sin\alpha\mathbf{i} - \cos\alpha\mathbf{j}), \tag{6.26a}$$

$$\mathbf{x}_S(t) = \frac{1}{2}gt^2\sin\alpha\mathbf{i} + \left(v_0 t - \frac{1}{2}gt^2\cos\alpha\right)\mathbf{j}. \tag{6.26b}$$

Let the reader derive these results starting from (6.1), determine the maximum height reached by S, and show that its trajectory is a parabola.

The shell returns to the ground after a time t^* when $\mathbf{x}_S(t^*) = r\mathbf{i}$ in Fig. 6.4, and hence by (6.26b),

$$r = \frac{1}{2}gt^{*2}\sin\alpha, \qquad t^* = \frac{2v_0}{g\cos\alpha}. \tag{6.26c}$$

The results are independent of the mass or any other property of the shell. Elimination of t^* from the first of (6.26c) yields the impact range r in terms of the muzzle speed v_0 and the angle α that the gun makes with the vertical axis of $\mathbf{g}$:

$$r = \frac{2v_0^2 \tan\alpha}{g\cos\alpha}. \tag{6.26d}$$

Now consider the free body diagram of the truck in Fig. 6.4. The total force $\mathbf{F}_T$ acting on the truck is its total weight $\mathbf{W}_T$ and the normal surface reaction force $\mathbf{N}$. Without frictional effects, (6.1) becomes

$$\mathbf{F}_T = \mathbf{N} + \mathbf{W}_T = N\mathbf{j} + Mg(\sin\alpha\mathbf{i} - \cos\alpha\mathbf{j}) = M\mathbf{a}_T. \tag{6.26e}$$

Since the truck accelerates along the $\mathbf{i}$ direction, $N = Mg\cos\alpha$ and $\mathbf{a}_T = g\sin\alpha\mathbf{i}$. Hence, two easy integrations with $\mathbf{v}_0 = \mathbf{0}$ and $\mathbf{x}_0 = \mathbf{0}$ yield

$$\mathbf{v}_T(t) = gt\sin\alpha\mathbf{i}, \tag{6.26f}$$

$$\mathbf{x}_T(t) = \frac{1}{2}gt^2\sin\alpha\mathbf{i}. \tag{6.26g}$$

Comparison of the $\mathbf{i}$ components in (6.26b) and (6.26g) parallel to the truck's motion reveals that the shell at each instant is directly above the truck, now coasting toward the ultimate *surprise*! But a few tenths of a second before the impending catastrophe, Percy Panther spots the converging shell and slams on the brakes. The shell explodes violently in front of the truck, destroying it. Through the smoky haze, Arnold Aardvark spies the black, whisker-singed and disheveled driver crawling safely away to seek revenge another day. □

Example 6.10. Arnold Aardvark is sunbathing on a lookout platform at $\mathbf{x}_0 = a\mathbf{i} + b\mathbf{j}$ in the frame $\Phi = \{O; \mathbf{i}_k\}$ when he spots Percy Panther at O preparing to fire an artillery gun pointed directly toward the platform, as shown in Fig. 6.5. The gun has a muzzle velocity $\mathbf{v}_0$ and the tower is well within its range r. At the moment the gun is fired, Arnold Aardvark, sensing impending danger, grabs his umbrella, steps through a hole in the platform, and falls freely in pursuit of safety toward the ground. Determine the distance d that separates Arnold Aardvark and the shell at the instant t^* when it crosses his line of fall.

Solution. The free body diagrams of the shell S and Arnold Aardvark B are shown in Fig. 6.5, in which $\mathbf{W}_S = m_S\mathbf{g}$ and $\mathbf{W}_B = m_B\mathbf{g}$ denote their respective weights. Their free fall equations of motion, in evident notation, are

$$\mathbf{F}_B = m_B\mathbf{g} = m_B\mathbf{a}_B, \qquad \mathbf{F}_S = m_S\mathbf{g} = m_S\mathbf{a}_S. \tag{6.27a}$$

Therefore, B and S have the same constant, free fall acceleration, $\mathbf{a}_B = \mathbf{a}_S = \mathbf{g}$, but their respective initial conditions differ. Integration of this equation, i.e. $d\mathbf{v}_B = d\mathbf{v}_S$, with $\mathbf{v}_B(0) = \mathbf{0}$ and $\mathbf{v}_S(0) = \mathbf{v}_0$, the muzzle velocity of the gun, gives

$$\mathbf{v}_B = \mathbf{v}_S - \mathbf{v}_0 \qquad \text{with} \quad \mathbf{v}_0 = v_0(\cos\beta\mathbf{i} + \sin\beta\mathbf{j}). \tag{6.27b}$$

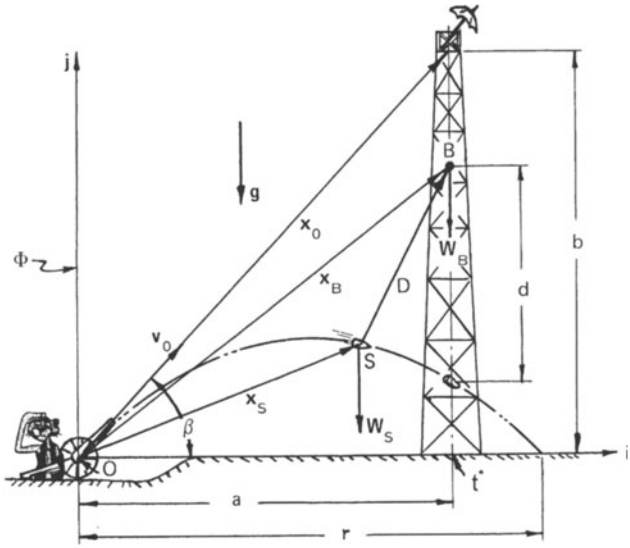

Figure 6.5. An unusual lesson on projectile motion.

A second integration with $\mathbf{x}_B(0) = \mathbf{x}_0$ and $\mathbf{x}_S(0) = \mathbf{0}$ yields the relative position vector $\mathbf{D} \equiv \mathbf{x}_B - \mathbf{x}_S$ of B from S at any time t:

$$\mathbf{D} = \mathbf{x}_0 - \mathbf{v}_0 t \qquad \text{with} \quad \mathbf{x}_0 = a\mathbf{i} + b\mathbf{j}. \tag{6.27c}$$

At the instant t^* when the shell crosses Arnold Aardvark's line of escape $x = a < r$, $\mathbf{D} = d\mathbf{j}$. Thus, with $\mathbf{v}_0$ given in (6.27b), (6.27c) yields $d\mathbf{j} = (a - v_0 t^* \cos\beta)\mathbf{i} + (b - v_0 t^* \sin\beta)\mathbf{j}$. The $\mathbf{i}$ component determines t^*, and the $\mathbf{j}$ component yields

$$d = b - a\tan\beta \tag{6.27d}$$

for the distance separating Arnold Aardvark and the unyielding shell at t^*. But Percy Panther had directed the gun on the line toward the platform with $\tan\beta = b/a$; so, Arnold Aardvark is headed straight toward an unpleasant *surprise* at the instant t^*! But a few moments before disaster strikes, he spies the approaching shell and quickly fixes the crook-handled umbrella to a tower beam, instantly arresting his fall. The shell explodes violently beneath him, destroying the tower. Arnold Aardvark, his snout scorched and twisted, escapes the assault with renewed mischief in mind.

So long as the tower is within the gun's range, the result is independent of the muzzle speed and of the masses of the objects involved; it depends only on the initial coordinates of B and the angle of elevation of the gun. Explain why Arnold Aardvark, living in a world where this solution is meaningful, was wise not to have used the umbrella as a parachute. □

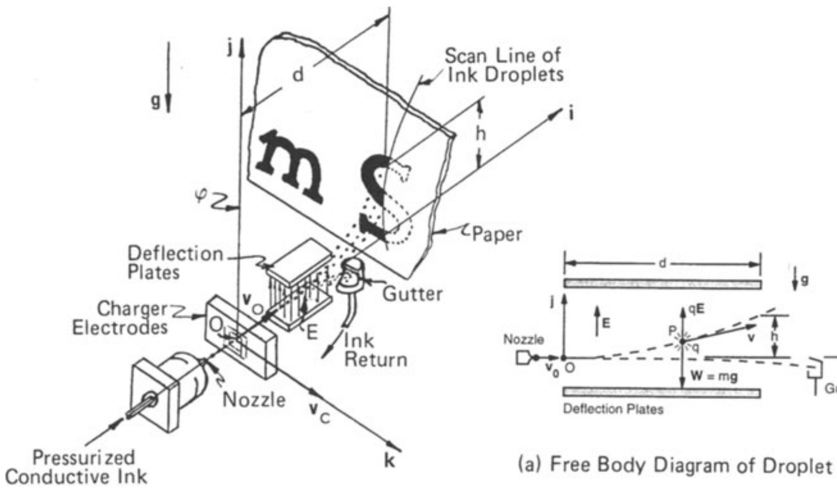

Figure 6.6. Schema of the IBM ink jet printing process. Copyright 1977 by International Business Machines Corporation; reprinted by permission.

6.4.2.4. Ink Jet Printing Technology

The same projectile ideas together with the basic law (6.15) for the electric force on a charged particle have a fascinating application in ink jet printing technology. An ink jet printer, illustrated schematically in Fig. 6.6, produces an image from tiny, charged spherical droplets of electrically conductive ink fired from a drop generating nozzle, approximately 1/1000 in. diameter, at the rate of 117,000 drops per second. The conductive droplets pass between charging electrodes where they are selectively charged electrostatically by command from programmed electronic control circuits that describe the image characters in terms of charge–no charge language. Moving at roughly 40 mph initially, the charged droplets pass through a constant electric field that directs them onto the paper. As vertical scanning occurs, an electromechanical control mechanism moves the printer carriage parallel to the paper at a constant speed of 7.7 in./sec. In this way, the ink jet printer quietly composes characters of high quality at a rate of about 80 characters per second, a full line of type across a standard page in about 1 sec. Of course, these operating rates will vary with printer design and evolving technology.

To understand its fundamental working principle, we shall determine the relative motion of a droplet P of mass m and charge q having an initial velocity $\mathbf{v}_0$ relative to the printer carriage. Since the carriage has a uniform velocity $\mathbf{v}_C$, as indicated in Fig. 6.6, the reference frame $\varphi = \{O; \mathbf{i}_k\}$ fixed in the charger at O is an inertial frame in which Newton's law may be applied. For simplicity, aerodynamic drag and wake effects, and the influence of electric repulsive forces between the charged droplets are neglected. Then, as shown in the free body diagram in Fig. 6.6a, the total force $\mathbf{F}(P, t) = \mathbf{F}_e + \mathbf{W}$ acting on a drop P is due to its weight $\mathbf{W} = -mg\mathbf{j}$ and the constant applied electric force $\mathbf{F}_e = q\mathbf{E} = qE\mathbf{j}$. Hence, $\mathbf{F}(P, t) = (qE - mg)\mathbf{j}$ is a constant force. From (6.1) and the initial condition $\mathbf{v}_0 = v_0\mathbf{i}$, we obtain the velocity of the drop relative to the printer carriage whose constant velocity is $\mathbf{v}_C = v_C\mathbf{k}$:

$$\mathbf{v}(P, t) = v_0\mathbf{i} + (cE - g)t\mathbf{j} \qquad \text{with} \quad c \equiv q/m. \tag{6.28a}$$

With $\mathbf{x}_0 = \mathbf{0}$ initially, integration of (6.28a) yields the motion of a droplet relative to the printer carriage:

$$\mathbf{x}(P, t) = v_0 t\mathbf{i} + \frac{1}{2}(cE - g)t^2\mathbf{j}. \tag{6.28b}$$

Hence, the path of the droplet relative to the carriage is a parabola

$$y(x) = \frac{1}{2v_0^2}(cE - g)x^2. \tag{6.28c}$$

Let us imagine for simplicity that the deflection plates of length d extend from the origin at the charger to the paper surface, as suggested in Fig. 6.6. Then (6.28c) holds for $0 \le x \le d$. (See Problem 6.22.) Therefore, at $x = d$, the droplet deflection or scan height $h \equiv y(d)$ at the paper surface is determined by

$$h = \frac{d^2}{2v_0^2}(cE - g). \tag{6.28d}$$

The result (6.28d) shows that when an electrostatically charged drop enters the uniform electric field, the electric force alters its free fall trajectory and deflects it vertically by an amount proportional to its charge. An uncharged drop is collected in a gutter that returns the unused ink to its reservoir as shown in Fig. 6.6. A charged drop impacts the paper. Alphabetic or any other characters, shown schematically in Fig. 6.6, are formed by directing the ink dots onto the paper in patterns determined by the printer electronics. The decision to charge or not to charge is made automatically 117,000 times each second. The formula (6.28d) shows that the character height is inversely proportional to the square of the stream speed v_0 which is controlled by the pump pressure. The printer controls the character height automatically by its pump control circuit. In this way, the ink jet printer is able to rapidly generate various characters of high quality. Some interesting style effects may be produced by varying the carriage rate.

A remarkable stroboscopic microphotograph of droplets of ink emerging from an ink jet printer is reproduced[‡] in Fig. 6.7. A jet of ink that originated in the drop generator to the right has dissociated into spherical droplets. The lower line of drops

[‡] This extraordinary photograph by Mr. Carl Lindberg was adapted from the color photograph on the cover of the Number 1 issue of the 1977 IBM Journal of Research and Development. Copyright 1977 by International Business Machines Corporation; reprinted by permission. In Fig. 6.7, however, the intensity of the droplets has been enhanced for greater clarity.

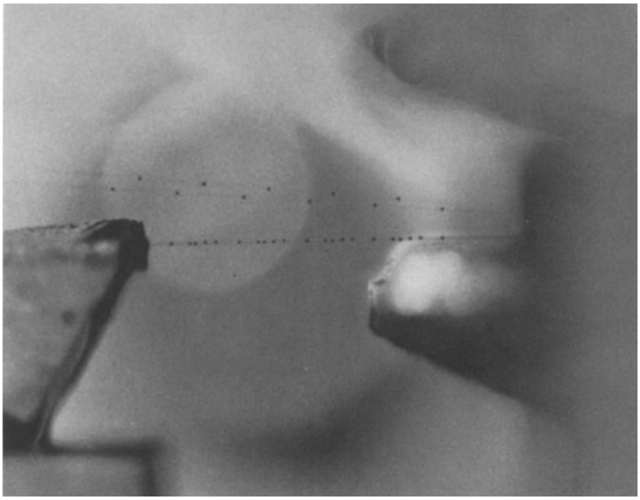

Figure 6.7. Stroboscopic microphotograph of ink drops in a jet printer. Copyright 1977 by International Business Machines Corporation; reprinted by permission.

were not charged, so these are moving toward the ink gutter to the left. The larger gaps between these uncharged drops are the vacated positions formerly occupied by the field deflected, charged drops that are traveling on the trajectories above.

The same ink jet technique was first applied in a similar way in the construction of a strip chart recorder, a high speed device for recording rapidly changing electrical signals on a moving paper chart. The disintegrating fluid jet concept has found other applications that include the sorting of cells in blood samples and the atomization of fuels for combustion. The deflection of a charged particle by an electric field also is used to control the motion of an electron stream in an oscilloscope and to produce images on a television screen or a computer monitor. Technological advances in electronic imaging, however, have led to the replacement of cathode ray tube devices by liquid crystal and high resolution plasma display systems whose basic operating principles are altogether different, and far more complex. The practical use of liquid crystal technology, for example, is evident in its increasingly diverse applications to computer and television screens, computer games, digital cameras, calculators, cellular phones, digital clocks and watches, microwave ovens, and a great host of other consumer and military electronic products.

6.5. Motion under Velocity Dependent Force

So far, complications due to air resistance have been ignored. Realistically, however, a projectile experiences atmospheric drag forces that slow it down and alter its trajectory. The same is true of an aircraft, a sky diver, and a raindrop; and water behaves similarly to retard the motion of swimmers, water skiers, and ships. Experience in such situations shows that the retarding force varies with the speed of the body.

For objects moving slowly through the air, the resistance is roughly proportional to the speed; but this simple rule breaks down at speeds typical of low velocity projectiles for which the air resistance varies roughly with the square of the speed. For an aircraft or a rocket whose velocity may approach the speed of sound, the drag force increases in proportion to some higher power of the speed, and so on. The retarding force is also a function of the density of air and hence varies with the altitude. Of course, aerodynamic design plays an important role too. These complications aside, we may gain physical insight into the nature of air and water resistance by study of special, ideal models.

6.5.1. Stokes's Law of Resistance

The simplest model used to study the nature of phenomena arising from drag effects of air and water on an object moving at low speeds is described by *Stokes's law*: *The drag force* $\mathbf{F}_D$ *on a particle is oppositely directed and proportional to its velocity* $\mathbf{v}$:

$$\mathbf{F}_D = -c\mathbf{v}. \tag{6.29}$$

The constant $c > 0$ is called the *drag* or *damping coefficient*. This model is applied later to investigate the motion of a projectile and of a particle falling with air resistance. First, however, we formulate the problem for a more general model for which the drag force is an unspecified function of the speed.

6.5.2. Formulation of the Resistance Problem

Figure 6.8 shows a particle P moving in the vertical plane of frame $\varphi = \{O; \mathbf{i}_k\}$, under a total force $\mathbf{F}(P, t) = \mathbf{W} + \mathbf{F}_D$ consisting of its weight $\mathbf{W} = m\mathbf{g}$

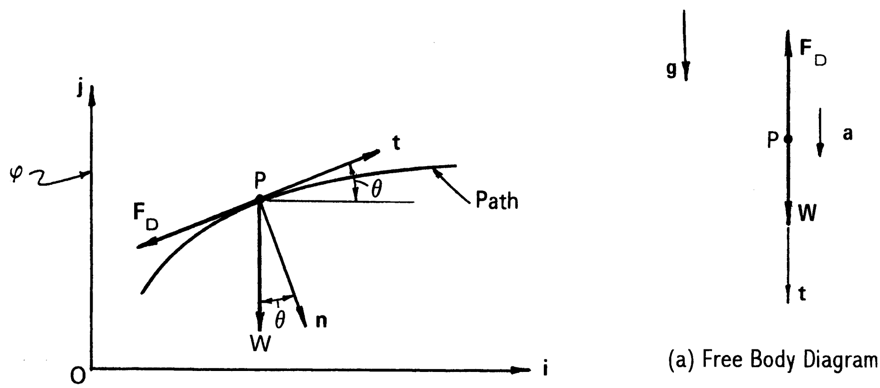

Figure 6.8. Motion of a particle under a drag force $\mathbf{F}_D$.

and the drag force $\mathbf{F}_D \equiv -R(v)\mathbf{t}$, where $R(v)$ is an unspecified, positive-valued function of the particle speed v. The equation of motion, by (6.1), is

$$m\mathbf{g} - R(v)\mathbf{t} = m\mathbf{a}(P, t). \tag{6.30}$$

Two cases are considered—rectilinear motion and plane motion.

6.5.2.1. Rectilinear Motion with Resistance

Let us consider a vertical rectilinear motion in the direction of $\mathbf{g} = g\mathbf{t}$ in Fig. 6.8a. Then with $\mathbf{a}(P, t) = \dot{v}\mathbf{t}$, (6.30) becomes

$$\dot{v} = g - \frac{R(v)}{m} \equiv F(v). \tag{6.31}$$

Integration of (6.31) yields the travel time as a function of the particle velocity in the resisting medium,

$$t = \int \frac{dv}{F(v)} + c_0, \tag{6.32}$$

where c_0 is a constant. Theoretically, this equation will yield $v(t) = ds/dt$ which may be solved to find the distance $s(t)$ traveled in time t. Alternatively, using $\dot{v} = v dv/ds$ in (6.31), we find the distance traveled as a function of the speed,

$$s = \int \frac{v dv}{F(v)} + c_1, \tag{6.33}$$

in which c_1 is another constant of integration. In principle, the integrals in (6.32) and (6.33) can be computed when the resistance function $R(v)$ is specified in (6.31). The following example illustrates these ideas for Stokes's linear rule (6.29).

Example 6.11. *Falling body with air resistance.* A particle of mass m, a raindrop for example, falls from rest through the atmosphere. Neglect the Earth's motion, wind effects, and the buoyant force of air, and adopt Stokes's law to model the air resistance. Determine as functions of time the rectilinear speed and the distance traveled by the particle.

Solution. The solution may be read from the foregoing results in which the drag force is modeled by Stokes's law (6.29) so that $\mathbf{F}_D = -R(v)\mathbf{t} = -cv\mathbf{t}$. Hence, use of $R(v) = cv$ in (6.31) gives

$$\frac{dv}{dt} = g - \nu v \equiv F(v) \qquad \text{with} \quad \nu \equiv \frac{c}{m}. \tag{6.34a}$$

For the initial condition $v(0) = 0$, we find by (6.32)

$$t = \int_0^v \frac{dv}{g - \nu v} = -\frac{1}{\nu}\log\left(1 - \frac{\nu v}{g}\right);$$

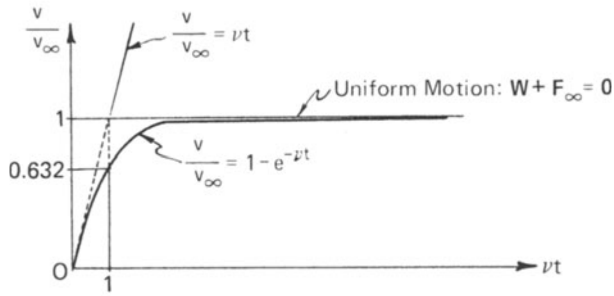

Figure 6.9. Graph of the normalized speed versus the normalized time for the vertical motion of a particle falling with resistance proportional to its speed.

so, the rectilinear speed of the particle in its fall from rest is

$$v(t) = v_\infty(1 - e^{-\nu t}) \qquad \text{with} \quad v_\infty \equiv \frac{g}{\nu}. \tag{6.34b}$$

In consequence, as $t \to \infty$, the particle speed v approaches a constant value $v_\infty \equiv g/\nu = W/c$, named *the terminal speed.* When the particle achieves its terminal speed, its weight is balanced by the drag force so that $cv_\infty = W$, and the particle continues to fall without further acceleration.

These facts are illustrated in Fig. 6.9. Equation (6.34b) shows that the rate at which $v(t)$ changes is governed by the coefficient of *dynamic viscosity* $\nu = c/m$, which has the physical dimensions $[\nu] = [F/MV] = [T^{-1}]$. Thus, at the instant $t = \nu^{-1}$, by (6.34b), $v(\nu^{-1}) = v_\infty(1 - e^{-1}) \approx 0.632 v_\infty$. Therefore, the speed reaches 63.2% of the terminal speed in the time $t = \nu^{-1}$, called the *retardation time*. The straight line of slope 1 in Fig. 6.9 shows that this also is the time at which the speed would reach the terminal value if it had continued to change at its initial constant rate $a(0) = g$, without air resistance. As the particle's speed approaches the terminal speed of its ultimate uniform motion shown by the horizontal asymptote, the weight $\mathbf{W} = W\mathbf{t}$ is balanced by the drag force $\mathbf{F}_D \to \mathbf{F}_\infty = -cv_\infty\mathbf{t}$.

Finally, with $v = ds/dt$ and the initial condition $s(0) = 0$, (6.34b) yields the distance through which the particle falls in time t:

$$s(t) = v_\infty \int_0^t (1 - e^{-\nu t})dt = v_\infty t - \frac{v_\infty}{\nu}(1 - e^{-\nu t}). \tag{6.34c}$$

Hence, the distance traveled in the retardation time interval is $s(1/\nu) = v_\infty/(\nu e) \approx 0.368 v_\infty/\nu$. The result (6.34c) also may be read from (6.33).

The reader may verify that in the absence of air resistance when $\nu \to 0$ the limit solutions of (6.34b) and (6.34c) are the elementary solutions (6.24). Now consider the case when the viscosity ν is small. First, recall the power series expansion of e^z about $z = 0$:

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots. \tag{6.34d}$$

Then use of (6.34d) in (6.34b) and (6.34c) yields, to the first order in ν, an approximate solution for the case of small air resistance:

$$v(t) = gt\left(1 - \frac{\nu}{2}t\right), \qquad s(t) = \frac{g}{2}t^2\left(1 - \frac{\nu}{3}t\right). \tag{6.34e}$$

When $\nu \to 0$, we again recover (6.24) for which air resistance is absent. □

6.5.2.2. Plane Motion with Resistance

Now, let us consider the plane motion of a particle in frame $\varphi = \{O; \mathbf{i}, \mathbf{j}\}$, as shown in Fig. 6.8. With $\mathbf{t} = \mathbf{v}/v = \dot{x}/v\mathbf{i} + \dot{y}/v\mathbf{j}$ and $\mathbf{g} = -g\mathbf{j}$ in (6.30), the component equation (6.2) yields

$$\ddot{x} = -\frac{R(v)}{mv}\dot{x}, \qquad \ddot{y} = -g - \frac{R(v)}{mv}\dot{y}. \tag{6.35}$$

These equations are difficult to handle in this general form. For resistance governed by Stokes's law (6.29), however, the ratio $R(v)/mv = c/m$ is constant; and (6.35) simplifies to

$$\ddot{x} = -\nu\dot{x}, \qquad \ddot{y} = -g - \nu\dot{y} \qquad \text{with} \quad \nu \equiv \frac{c}{m}. \tag{6.36}$$

Example 6.12. *Projectile motion with air resistance.* A projectile S of mass m is fired from a gun with muzzle speed v_0 at an angle β with the horizontal plane. Neglect the Earth's motion and wind effects and assume that air resistance is governed by Stokes's law. Determine the projectile's motion as a function of time.

Solution. The equations of motion with air resistance governed by Stokes's law are given in (6.36). To find the motion $\mathbf{x}(S, t)$, we first integrate the system (6.36) to obtain $\mathbf{v}(S, t)$. Use of the initial condition $\mathbf{v}_0 = v_0(\cos\beta\mathbf{i} + \sin\beta\mathbf{j})$ yields

$$\int_{v_0\cos\beta}^{\dot{x}} \frac{d\dot{x}}{\dot{x}} = -\nu t, \qquad \int_{v_0\sin\beta}^{\dot{y}} \frac{d\dot{y}}{g + \nu\dot{y}} = -t.$$

These deliver the projectile's velocity components as functions of time:

$$\dot{x} = (v_0\cos\beta)e^{-\nu t}, \qquad \dot{y} = -\frac{g}{\nu} + \left(v_0\sin\beta + \frac{g}{\nu}\right)e^{-\nu t}. \tag{6.37a}$$

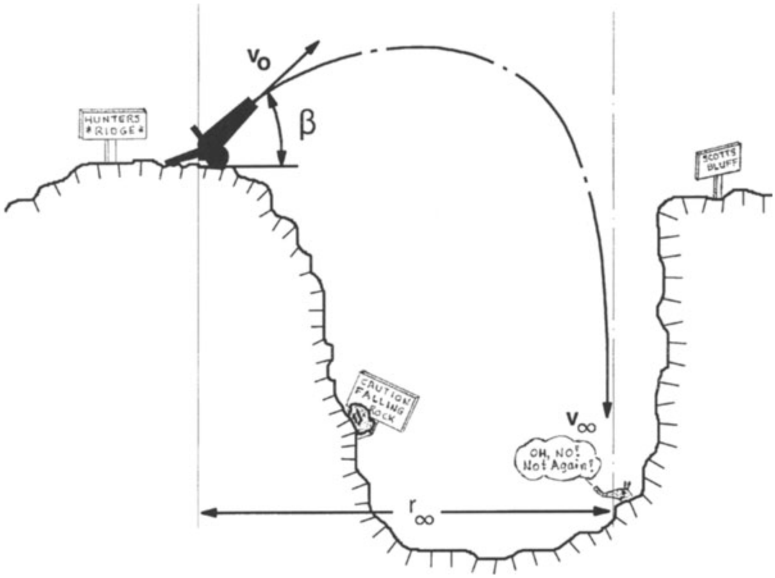

Figure 6.10. Projectile motion with air resistance.

Then integration of (6.37a) with the initial condition $\mathbf{x}_0 = \mathbf{0}$ yields the motion of the projectile as a function of time:

$$x(t) = \frac{v_0 \cos\beta}{\nu}(1 - e^{-\nu t}), \tag{6.37b}$$

$$y(t) = -\frac{g}{\nu}t + \frac{1}{\nu}\left(v_0 \sin\beta + \frac{g}{\nu}\right)(1 - e^{-\nu t}). \tag{6.37c}$$

Let us imagine that the projectile is fired from a hilltop into a wide ravine, as shown in Fig. 6.10. Then, as $t \to \infty$, in the absence of impact, (6.37a) gives $\dot{x} \to 0$ and $\dot{y} \to -g/\nu$. Hence, the projectile attains the terminal speed $v_\infty = g/\nu$ at which its weight is balanced by air resistance; and (6.37b) and (6.37c) show that the projectile approaches asymptotically, the vertical range line at $r_\infty \equiv \lim_{t\to\infty} x(t) = (v_0 \cos\beta)/\nu$ in Fig. 6.10. In the absence of air resistance, the range for the same situation would grow indefinitely with the width of the ravine. The simple Stokes model thus provides a more realistic picture of projectile motion with air resistance that limits its range. □

6.5.2.3. *The Millikan Oil Drop Experiment*

When oil is sprayed in fine droplets from an atomizer, the droplets become electrostatically charged, presumably due to frictional effects. The charge is usually negative, which means that the drops have acquired one or more excess electrons.

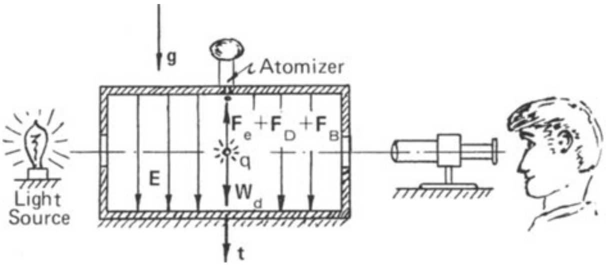

Figure 6.11. Schematic of the Millikan oil drop experiment.

This fact was exploited in 1909 by the famous American physicist Robert A. Millikan in a classic experiment designed to measure accurately the charge of an individual electron. Millikan's experimental method, its relation to our study of air resistance, and his remarkable result§ are discussed next.

A schematic of the oil drop test is shown in Fig. 6.11. Charged oil droplets, about a thousandth of a millimeter in diameter, are ejected from an atomizer at the top of the apparatus. A few drops escape through a small hole into an illuminated electric field **E** directed as shown. A lighted drop is seen in a telescope as a tiny, bright particle of mass m and negative electric charge $-q$ falling slowly under the influence of its weight $\mathbf{W}_d$, the electric force $\mathbf{F}_e$, the drag force $\mathbf{F}_D$, and the buoyant force $\mathbf{F}_B$ of the air, as shown in the free body drawing in Fig. 6.11; so, the total force on the droplet is $\mathbf{F}(P, t) = \mathbf{F}_e + \mathbf{F}_D + \mathbf{F}_B + \mathbf{W}_d$. The use of oil eliminates effects due to fluid evaporation, so only the drag force varies with time. Independent tests confirmed that the charge on the drops does not affect the air resistance to its motion, and because the particle's rate of fall is small, Stokes's law of resistance is applicable.

The intensity of the electric field, hence the electric force $\mathbf{F}_e = -q\mathbf{E}$ on a negatively charged drop, is adjusted until the droplet becomes stationary, spatially suspended in equilibrium between the field plates. In this case, $\mathbf{F}_D = \mathbf{0}$ and the equilibrium equation yields

$$\mathbf{F}_e + \mathbf{W}_d + \mathbf{F}_B = -q\mathbf{E} + \mathbf{W} = \mathbf{0}. \tag{6.38a}$$

The effective weight $\mathbf{W} \equiv \mathbf{W}_d + \mathbf{F}_B$ of a droplet in air depends on the mass density

§ In 1923, Robert A. Millikan was awarded the Nobel Prize for physics, principally for his work identifying precisely the unit of electric charge. Nearly 25 years subsequent to his death in 1953, however, he was strongly criticized for his treatment of students and others, and for his mishandling of the data. See the balanced account by D. Goodstein (among the references under Millikan) for the rest of the story. The importance that Millikan placed on his amended form of Stokes's law is underscored in this article. Also, it should be mentioned that besides frictional effects that induce negative charges on the droplets, the electric arc lamp is a source of ionization radiation of the space between the horizontal capacitor plates that also induces positive charges on an atom of an oil droplet, so the droplets are sometimes referred to as ions.

of the oil and of the sealed air as well as on the size of the drop. The mass densities are known, but the diameters of the drops are too small to be accurately measured directly with the telescope. Millikan devised an ingenious, indirect method for finding the size of the drops.

When the electric field is turned off, the drop accelerates in its fall until its terminal speed is reached. This happens when the drag force given by Stokes's rule balances the effective weight of the drop so that the particle is in a state of relative equilibrium at its constant terminal speed. Thus, the equation of the uniform motion yields

$$\mathbf{F}_\infty + \mathbf{W} = \mathbf{0}, \tag{6.38b}$$

in which $\mathbf{F}_\infty = -cv_\infty\mathbf{t}$ is the air resistance at the terminal speed v_∞. By timing the distance traveled at the constant slow rate of fall of the drop, Millikan measured the terminal speed and applied the result (6.38b) to compute the droplet size. Then the drag coefficient c, which depends on the size of the drop and the known viscosity of air, could be evaluated by a separate formula derived by Stokes from hydrodynamic theory. But Millikan found that Stokes's formula, due to the small size of the drops compared with the mean free path of a gas molecule, was inaccurate, and he provided an empirical correction to account for the discrepancy. With this adjustment in mind, c and v_∞ may be considered known. Thus, in effect, the charge on the drop is determined by eliminating $\mathbf{W}$ between (6.38a) and (6.38b) to obtain $-q\mathbf{E} = \mathbf{F}_\infty$. Clearly, the error in Stokes's formula for the calculation of c does not affect the basic linear nature of the rule (6.29), and hence the droplet charge is determined by

$$q = \frac{cv_\infty}{E}. \tag{6.38c}$$

Millikan and his co-workers found in many measurements the remarkable result that *every droplet had a charge q equal to an integral multiple of a number* $e = 1.6019 \times 10^{-19}$ coulomb, *the basic amount of negative charge of one electron. Thus, Millikan's conclusive experimental result that*

$$q = ne, \qquad n = 1, 2, 3, \ldots, \tag{6.38d}$$

showed that electric charge exists in nature only in integral units of magnitude e.

The procedure to obtain the data on one particular droplet sometimes took hours. At times, when interrupted while working on a drop, Millikan would put it into balance with the field and leave it. On one occasion he went home to dinner and returned after more than an hour to find the droplet only slightly displaced from where he had left it. At another time, Millikan realized he would not finish his experiment in time to attend dinner at home with invited guests, so he phoned Mrs. Millikan to explain that "I have watched an ion for an hour and a half and have to finish the job," but insisted that she and their guests go ahead with dinner. He learned later that Mrs. Millikan advised their guests that Robert would be delayed because he had "washed and ironed for an hour and a half and had to finish the job."

Measurement of e had been done earlier, but never with the accuracy achieved by Millikan's suspended oil drop test. He was studying the fundamental building block out of which, it is now believed, all electrical charges in the universe are composed, always in integral multiples of the basic unit electron charge e. The entire basis for the measurement of its magnitude rested on application of Stokes's law to the terminal speed of spherical droplets of oil in air. The apparatus was a device for catching and essentially seeing an individual electron riding on a drop of oil. Millikan recalled later in his autobiography this exciting observational experience: "He who has seen that experiment has in effect seen the electron."

Additional examples of particle motion with air resistance are provided in Problems 6.23 through 6.27. We continue with a new topic.

6.6. An Important Differential Equation

Many physical systems are governed by the second order differential equation

$$\ddot{u}(t) + r^2 u(t) = h(t), \tag{6.39}$$

for a scalar function $u(t)$. Herein r is a real or complex constant and $h(t)$ is a specified function of the independent variable t. We are going to encounter lots of applications in which one or more of the scalar equations of motion are of the type (6.39); so it is most helpful to understand the physical nature of its solution in general terms.

The solution of (6.39) when $r = 0$ describes a motion under a time varying force. This case was studied in Section 6.4.1, page 109; therefore, we shall assume that $r \neq 0$. In the general case, we recall from the theory of differential equations that the solution of (6.39) is given by the sum

$$u(t) = u_H(t) + u_P(t), \tag{6.40}$$

in which $u_H(t)$, called the *homogeneous solution*, is the general solution of the *related homogeneous equation*

$$\ddot{u}_H + r^2 u_H = 0, \tag{6.41}$$

and $u_P(t)$ is a *particular solution* that satisfies (6.39):

$$\ddot{u}_P + r^2 u_P = h(t). \tag{6.42}$$

6.6.1. General Solution of the Homogeneous Equation

The general solution of the homogeneous equation is obtained by consideration of a trial function $u_T = Ce^{\lambda t}$ in which λ and C are constants. This function satisfies (6.41) for each root of the *characteristic equation* $\lambda^2 + r^2 = 0$, namely, $\lambda = \pm ir$ in which $i = \sqrt{-1}$, so both $u_T(t) = C_1 e^{irt}$ and $u_T(t) = C_2 e^{-irt}$ are solutions of (6.41). Hence, the general solution of the homogeneous equation that contains two arbitrary integration constants C_1 and C_2 is given by the sum of these independent solutions:

$$u_H(t) = C_1 e^{irt} + C_2 e^{-irt}. \tag{6.43}$$

The homogeneous solution (6.43) is also known as the *complementary function*.

6.6.2. Particular Solution of the General Equation

The hardest part of our problem is to find a particular solution of (6.42) for a given function $h(t)$. Standard methods are available that may be applied to find one. The *method of variation of parameters*, for example, is a powerful procedure applicable to equations with variable or constant coefficients, but the complementary function must be known in advance. This presents no difficulty in the present problem for which it can be shown that this method leads to the following general relation for a particular solution of (6.42):

$$u_P(t) = \int^t \frac{h(\tau)}{\lambda_1 - \lambda_2} \left[e^{\lambda_1(t-\tau)} - e^{\lambda_2(t-\tau)} \right] d\tau, \tag{6.44}$$

wherein $\lambda = (\lambda_1, \lambda_2)$ are distinct roots of the characteristic equation. In the present case $\lambda = \pm ir$ yields $\lambda_2 = -\lambda_1 = -ir$. In evaluation of the indefinite integral in (6.44) arbitrary constants are omitted; they have no importance in the particular solution. The solution (6.44) also may be verified by its substitution into (6.42). (See Problems 6.28 and 6.29.) In many problems of physical interest, use of the formal relation (6.44) to compute the particular solution may be avoided. For the kinds of problems we shall encounter ahead, it is much easier to generate a particular solution on an ad hoc basis.

Example 6.13. Let us consider a particular solution for the case when $h(t)$ is a linear function of t, namely,

$$h(t) = c + bt, \tag{6.45a}$$

for constants b and c. Then because $\ddot{h}(t) = 0$, we see that a particular solution that satisfies (6.42) is

$$u_P(t) = \frac{h(t)}{r^2} = r^{-2}(c + bt). \tag{6.45b}$$

In this instance $\ddot{u}_P = 0$. Indeed, a particular solution of (6.42) has the property $\ddot{u}_P(t) = 0$ if and only if $u_P(t)$ is a linear function like (6.45b), and hence when and only when $h(t)$ is the linear function (6.45a). Therefore, in accordance with (6.40), the general solution of (6.39) for this case is given by the sum of

(6.43) and (6.45b):

$$u(t) = C_1 e^{irt} + C_2 e^{-irt} + r^{-2}(c + bt). \tag{6.45c}$$

The solution for other special functions $h(t)$ will be considered as the need arises.

6.6.3. Summary of the General Solution

In summary, we find with (6.43), (6.44), and (6.40) that the general solution of the differential equation (6.39) may be written as

$$u(t) = C_1 e^{irt} + C_2 e^{-irt} + u_P(t), \tag{6.46}$$

where the particular solution is defined formally by

$$u_P(t) = \int^t \frac{h(\tau)}{2ir}\left[e^{ir(t-\tau)} - e^{-ir(t-\tau)}\right] d\tau. \tag{6.47}$$

This is a convenient means of representing a particular solution of (6.39) for an arbitrary smooth function $h(t)$. Remember, however, that in many cases of practical interest, depending on the nature of $h(t)$, a particular solution of (6.39) may be obtained by simpler ad hoc methods.

6.6.4. Physical Character of the Solution

Now let us consider two important cases of physical interest. In the first instance we suppose that $r = p$ is a real constant so that $r^2 = p^2 > 0$. This leads to a trigonometric type solution. In the second case, we take $r = iq$, a pure complex constant, so that $r^2 = -q^2 < 0$. This leads to an exponential type solution which is then expressed in terms of hyperbolic functions. As a consequence, the physical nature of these two classes of solutions of (6.39) is quite different. (See Problem 6.33.)

6.6.4.1. Trigonometric Solution: $r = p$, a real constant

Equation (6.39) for this case becomes

$$\ddot{u}(t) + p^2 u(t) = h(t),\ p \text{ real}. \tag{6.48}$$

Of course, the general solution of this equation has precisely the form (6.46) with r replaced by p. But the complex exponential solution, convenient in some problems, suffers the undesirable disadvantage that the constants C_1 and C_2 are complex quantities. It proves more convenient, therefore, to transform this solution to its trigonometric form by use of *Euler's identity*

$$e^{\pm ipt} = \cos pt \pm i \sin pt. \tag{6.49}$$

Then, with $r = p$, the general solution (6.43) of the homogeneous equation (6.41), namely,

$$\ddot{u}_H + p^2 u_H = 0, \ p \text{ real}, \tag{6.50}$$

may be written as

$$u_H(t) = A \sin pt + B \cos pt, \tag{6.51}$$

wherein A and B are two real constants of integration. Thus, the complementary function has an oscillatory character typical of trigonometric functions.

Substitution of (6.49) into (6.47), with $r = p$, leads to the following expression for the particular solution of (6.48):

$$u_P(t) = \int^t \frac{h(\tau)}{p} \sin p(t - \tau) d\tau. \tag{6.52}$$

The general solution of (6.48) is the sum of (6.51) and (6.52). Formally,

$$u(t) = A \sin pt + B \cos pt + u_P(t). \tag{6.53}$$

The trigonometric functions in (6.51) and (6.53) have well-known periodic behavior whose physical relevance is discussed further in applications ahead.

Exercise 6.2. Let $C_1 = a_1 + ib_1$, $C_2 = a_2 + ib_2$, and set $r = p$ in (6.43). Use Euler's identity and show that the homogeneous solution (6.43) is real-valued when and only when $b_1 + b_2 = 0$ and $a_1 - a_2 = 0$. Determine in these terms the real constants in (6.51). □

6.6.4.2. *Hyperbolic Solution*: $r = iq$, *a complex constant*

The equation (6.39) for this case becomes

$$\ddot{u}(t) - q^2 u(t) = h(t), \ q \text{ real}. \tag{6.54}$$

It is important to recognize that *the principal difference between* (6.54) *and* (6.48) *is merely the sign of the second term.* This results in significantly different kinds of solutions. The general solution of (6.54) is given by (6.46) with r replaced by iq. We thus obtain

$$u(t) = C_1 e^{-qt} + C_2 e^{qt} + \int^t \frac{h(\tau)}{2q} \left(e^{q(t-\tau)} - e^{-q(t-\tau)}\right) d\tau. \tag{6.55}$$

Notice that the homogeneous solution, the first two terms in (6.55), has an exponential character. Unlike the oscillatory solution (6.51), this exponential solution grows increasingly large with t. Hence, plainly, equations (6.48) and (6.54) will describe totally distinct kinds of physical effects.

It is useful to observe that hyperbolic functions may be introduced to express the solution by formulas analogous to those used in the trigonometric case. For comparison, the results are presented in order parallel to the trigonometric formulas (6.49)–(6.53).

Representation in terms of hyperbolic functions. The hyperbolic sine and cosine functions are defined by

$$\sinh z = \frac{1}{2}(e^z - e^{-z}), \qquad \cosh z = \frac{1}{2}(e^z + e^{-z}). \tag{6.56}$$

These equations may be solved to obtain the exponential functions e^z and e^{-z}:

$$e^{\pm z} = \cosh z \pm \sinh z. \tag{6.57}$$

This is similar to (6.49). Then, with $r = iq$, the general solution (6.43) of the homogeneous equation associated with (6.54), namely,

$$\ddot{u}_H - q^2 u_H = 0, q \text{ real}, \tag{6.58}$$

may be written as

$$u_H(t) = A \sinh qt + B \cosh qt, \tag{6.59}$$

wherein A and B are two real constants of integration. Use of the first of (6.56) in (6.47) when $r = iq$ yields the following formula for a particular solution of (6.54):

$$u_P(t) = \int^t \frac{h(\tau)}{q} \sinh q(t - \tau)\, d\tau. \tag{6.60}$$

The general solution of (6.54), given by (6.55), is the sum of (6.59) and (6.60). Formally,

$$u(t) = A \sinh qt + B \cosh qt + u_P(t). \tag{6.61}$$

This completes the parallel representation of results (6.58)–(6.61) which are to be compared with the corresponding equations (6.50)–(6.53) for the trigonometric solution. Although the forms of solutions (6.53) and (6.61) are similar, it is evident that their physical nature is quite different. The trigonometric functions in (6.53) are periodic, they recur over and over again. But, as seen by (6.56), the hyperbolic functions in (6.61) grow indefinitely with the variable t. The graphs and some additional basic properties of the hyperbolic functions follow.

Further properties of the hyperbolic functions. Graphs of the functions (6.56) and some basic properties of the hyperbolic functions provide a helpful picture of their growth behavior. To start with, differentiation of (6.56) shows that

$$\frac{d}{dz}(\sinh z) = \cosh z, \qquad \frac{d}{dz}(\cosh z) = \sinh z. \tag{6.62}$$

We thus see an important difference in the derivatives of the hyperbolic functions

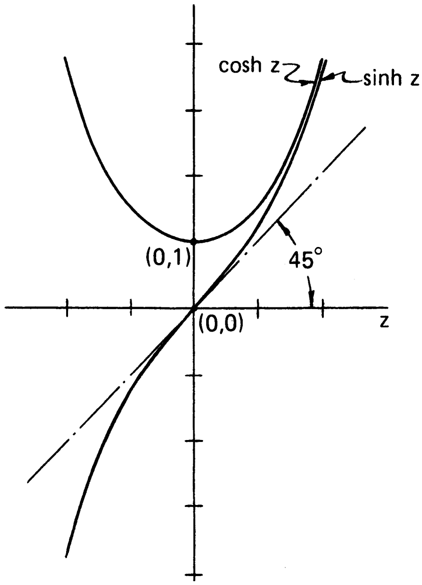

Figure 6.12. Plots of the hyperbolic functions sinh z and cosh z.

compared with their trigonometric counterparts. It also follows easily from (6.56) or (6.57) that

$$\cosh^2 z - \sinh^2 z = 1. \tag{6.63}$$

This identity reveals a simple geometrical property that accounts for the name of these functions. Indeed, with $x = \cosh z$ and $y = \sinh z$, (6.63) yields $x^2 - y^2 = 1$, the equation of equilateral hyperbolas with asymptotes along the bisectors of the coordinate lines. Hence, the functions (6.56) are named *hyperbolic functions*. The trigonometric functions $x = \cos z$ and $y = \sin z$, on the other hand, yield $x^2 + y^2 = 1$, the equation of a unit circle. And we recall that the trigonometric functions are also known as *circular functions*.

The identity (6.63) shows that, unlike their trigonometric cousins, $\cosh z > \sinh z$ for all values of z. This means that their graphs never intersect; the graph of $\cosh z$ lies always above the graph of $\sinh z$. Moreover, (6.56) and (6.62) show that $\sinh z$ vanishes at $z = 0$ where its slope, $\cosh z$, has value 1. Since $d^2(\sinh z)/dz^2 = 0$, the graph of sinh z has an inflection at $z = 0$. Equation (6.56) shows that $\sinh(-z) = -\sinh z$ is an odd function of z. The graph of $\sinh z$ thus has the form shown in Fig. 6.12. The graph of $\cosh z$, also shown there, has a minimum at $z = 0$ where its value is 1, and, by (6.56), $\cosh(-z) = \cosh z$ shows that $\cosh z$ is an even function of z. Clearly, as z grows indefinitely large, (6.56) indicates that both functions grow indefinitely, as shown in Fig. 6.12. It can be proved from statics that the graph of the hyperbolic cosine function, also called the *catenary*, is the shape assumed by a uniform, heavy cord supported at its ends and hanging under its own weight, an easy experiment for the reader.

6.7. The Simple Harmonic Oscillator

The differential equations (6.48) and (6.54) occur in a wide variety of dynamical problems, the simplest kind being those for which $h(t) = 0$. These equations reduce in this case to the respective homogeneous equations (6.50) and (6.58). In particular, the oscillations of a mass attached to an ideal linear spring and the small amplitude oscillations of a pendulum are motions of physical systems that are governed by the same homogeneous equation (6.50)—the equation of a so-called *simple harmonic oscillator.* An example in which (6.58) occurs will follow shortly. We begin with the linear spring/mass system.

6.7.1. Hooke's Law of Linear Elasticity

We usually think of a spring as a helically wound wire device. But all solid bodies, like a solid rubber block or cord, behave in the same springy way, except that the deformation of most solid bodies is usually very small. So all solid bodies whatsoever, whether metal, wood, glass, or stone; hair, silk, tissue, or bone; and so on, are springs too. In general, to characterize the uniform, uniaxial elastic behavior of a deformable solid body under tensile or compressive end loads, we adopt an ideal spring model described by *Hooke's law*: *The uniaxial force F_H required to stretch or to compress an ideal spring is proportional to the uniaxial change of length δ of the spring from its natural, undeformed state*:

$$F_H = k\delta. \tag{6.64}$$

The constant k is called the *spring constant.* Sometimes the terms *elasticity, modulus*, or *stiffness* are also used. Clearly, $[k] = [F/L]$. An ideal spring for which (6.64) holds is known as a *linear spring.*

The linear force–deformation law (6.64) was proposed by Robert Hooke in 1675. To protect his discovery from use by others while he exploited its applications, he claimed priority for the law and published its substance in a Latin anagram, "*ceiiinosssttuu.*" Three years later, and 18 years since his first knowledge of it, Hooke unscrambled the puzzle to read:[¶] "*ut tensio sic vis*;" that is, the

[¶] Notice that the anagram has a double "u," contrary to its Latin decipherment by Hooke. See R. Hooke, *De Potentia Restitutiva or of Spring*, 1678; reproduced in R. T. Gunther, *Early Science in Oxford*, Volume VIII, The Cutler Lectures of Robert Hooke, pp. 331–56, Oxford University Press, Oxford, 1931. This is not an error. In early Latin manuscripts v often appears in print as u. In fact, M. Espinasse in *Robert Hooke,* University of California Press, Berkeley, 1962, p. 78, writes literally, "*ut tensio sic uis.*" Hooke's own decipherment, however, is commonly adopted in books on elasticity, its history, and Hooke's life. See Volume 1, p. 5, of I. Todhunter and K. Pearson, *A History of the Theory of Elasticity and of the Strength of Materials*, Dover, New York, 1960; L. Jardin, *Ingenious Pursuits: Building the Scientific Revolution*, pp. 322–3, Doubleday, New York, 1999; and the remarkable treatise by J. F. Bell, "The Experimental Foundations of Solid Mechanics," *Flügge's Handbuch der Physik*, Volume **VIa/1**, pp. 156–60, Springer-Verlag, New York, 1973.

extension of any spring increases in proportion to the tension. Hooke demonstrated by experiments that in addition to solid bodies, the rule also holds for helical wire springs for which the deformation δ may be large. Because Hooke's law is linear, it follows that the extra force F^* required to stretch (or compress) the spring an additional amount η, say, is proportional to η; i.e., $F^* = k\eta$. This simple superposition rule does not apply to any nonlinear spring. Any potential confusion about the effect of initial deformation of a spring in the formulation of a problem may be avoided by use of a deformation variable defined with respect to the natural state.

Hooke's rule is not a fully accurate characterization of a springy body for all cases of practical interest. It ignores, for example, the possibly large twisting effect induced by uniaxial loading of a helical spring, whose torsional stiffness and mass usually are neglected in applications of (6.64). And it does not hold for large deformations possible in nonlinear rubberlike materials or biological tissues. On the other hand, Hooke's law provides a mathematically simple and useful description of the physical nature of phenomena in a great variety of practical cases where the elastic response of a solid may be reasonably modeled by a linear spring.

6.7.2. The Linear Spring-Mass System

Let us consider a linear spring fixed at one end and having a mass m (sometimes called the load) attached to its other end, and either suspended vertically or supported by a smooth plane surface. The mass of the spring is generally considered negligible in comparison with the mass m; so, henceforward, its mass is ignored. When m is displaced a distance δ from the natural, unstretched spring configuration, it exerts on the spring a uniaxial force F_H given by (6.64). In response, the spring exerts an equal but oppositely directed restoring force $F_S = -F_H = -k\delta$, called *the spring force,* that acts always to return the mass toward the natural state of the spring. Hence, if released, the mass will vibrate under the alternating extension and compression reactions of the spring itself. Let us first study the oscillations of the mass on a smooth horizontal surface, as shown in Fig. 6.13.

6.7.2.1. Horizontal Vibrations of a Mass on a Linear Spring

To characterize the horizontal oscillatory motion of the mass, we suppose that m is given an initial uniaxial velocity $\mathbf{v}_0 = v_0\mathbf{i}$ from its natural equilibrium configuration in $\Phi = \{F; \mathbf{i}, \mathbf{j}\}$ shown in Fig. 6.13. The free body diagram of m is shown in Fig. 6.13a. The weight $\mathbf{W}$ is balanced by the normal reaction $\mathbf{N}$ of the smooth surface, so the only force that affects the horizontal, uniaxial motion of m is the spring force $\mathbf{F}_S = -kx\mathbf{i}$, in which $x = \delta$ denotes the displacement of m, the change of length of the spring from its natural state. Therefore, the equation

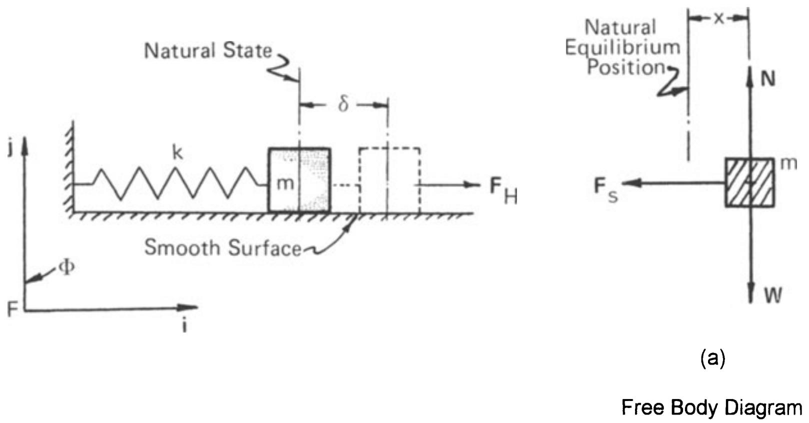

Figure 6.13. An ideal spring-mass system.

of motion of m, namely, $\mathbf{F}_S = m\ddot{x}\mathbf{i}$, becomes

$$\ddot{x} + p^2 x = 0 \qquad \text{with} \quad p = \sqrt{\frac{k}{m}}. \tag{6.65a}$$

This equation has the form (6.50) whose general solution is given by (6.51):

$$x(t) = A \sin pt + B \cos pt. \tag{6.65b}$$

An oscillatory motion described by (6.65b) is called a *simple harmonic motion.*

The integration constants A and B are determined by specified initial conditions; presently, $x(0) = 0$, $\dot{x}(0) = v_0$. Since $x(0) = B = 0$, (6.65b) reduces to $x(t) = A \sin pt$, and with $\dot{x}(0) = Ap = v_0$, we have the general solution

$$x(t) = \frac{v_0}{p} \sin pt. \tag{6.65c}$$

The maximum displacement of m from its equilibrium state is called the *amplitude* of the oscillation. The amplitude of the motion (6.65c) is given by $x_A \equiv v_0/p$. The graph of the motion (6.65c) and the corresponding velocity $\dot{x} = v_0 \cos pt$ are shown in Fig. 6.14. The motion of m varies from x_A to $-x_A$ over and over again. Also, the displacement $x(t)$, and similarly the velocity $v(t)$, has the same value at times $t + 2n\pi/p = t + n\tau$ for $n = 0, 1, 2, \ldots$; that is, $\sin(pt + 2n\pi) = \sin p(t + n\tau) = \sin pt$. Hence, the motion (6.65c) is said to be *periodic*, and the least nonzero time $\tau = 2\pi/p$ for which $x(t) = x(t + \tau)$ is called the *period* of the motion—it is the time required to complete one oscillation. (See Fig. 6.14.) The number of periods that occur in a unit of time is the number of oscillations of the mass per unit time. This number, denoted by

$$f = \frac{1}{\tau} = \frac{p}{2\pi}, \tag{6.65d}$$

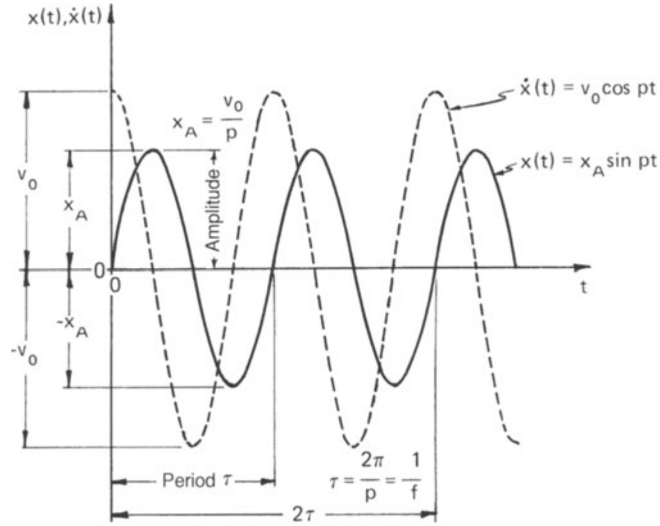

Figure 6.14. Graphs illustrating the periodic nature of the simple harmonic motion and the simultaneous velocity of the load in a linear spring-mass system.

is called the *frequency* of the oscillations. The measure units of f are expressed as cycles per unit time. When the time is in seconds, the measure of f commonly is stated in cycles per second or Hertz, abbreviated 1 cps$\equiv$ 1 Hz. Since there are 2π radians in one cycle, $p = 2\pi f$ is called the *circular frequency*; its measure units are radians per unit time. Clearly, $[p] = [f] = [T^{-1}]$ and $[\tau] = [T]$.

The relation for the circular frequency of a simple harmonic motion may be read immediately from the coefficient of the differential equation of motion (6.65a). Consequently, the period and the frequency of the motion of the mass of a linear spring-mass system may be obtained at once from (6.65d). We thus find

$$\tau = 2\pi\sqrt{\frac{m}{k}}, \qquad f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}. \tag{6.65e}$$

The graph of the uniaxial velocity $\dot{x} = v = v_0 \cos pt$ versus the uniaxial position $x = x_A \sin pt$, called a *phase plane diagram*, is an ellipse centered at the origin and having semi-axes determined by x_A and v_0:

$$\left[\frac{x}{x_A}\right]^2 + \left[\frac{\dot{x}}{v_0}\right]^2 = 1. \tag{6.65f}$$

We may suppose that p in (6.65a) is known. Then for each choice of initial velocity v_0, the pair $(x_A = v_0/p, v_0)$ determines a different ellipse, and hence (6.65f) describes a family of concentric ellipses. Moreover, the normalized plot of $\dot{x}/v_0$ versus x/x_A reduces every member of the family (6.65f) to a single unit circle.

The periodic nature of the motion is exhibited by these closed phase plane paths, all of which are traversed in the same time $\tau = 2\pi/p$, the period of the motion. Equation (6.65f) has exactly two solutions $x = \pm x_A$ for which $\dot{x} = 0$, and hence *the amplitudes* $\pm x_A$ *are the extreme points in the motion at which the mass comes momentarily to rest.* The $\pm$ sign reflects the symmetry of the motion about $x = 0$, the equilibrium state of rest—the unique time independent solution of (6.65a). And, by (6.65f), *the greatest velocity* $\dot{x} = v_0$ *also occurs at* $x = 0$. These results are evident in Fig. 6.14. We shall find in Chapter 7 that the phase plane curves are related to the energy of the system.

For other initial conditions, the form of the solution (6.65c), hence also the relations describing the amplitude and the phase plane trajectory, will be somewhat different. General formulas for the amplitude and the phase plane graph for arbitrary initial data assigned in any simple harmonic motion (6.65b) are presented later.

6.7.2.2. *Vertical Vibrations of a Mass on a Linear Spring*

Now let us consider the effect of gravity on the oscillatory motion of a load m supported vertically by a linear spring. The weight produces a static deflection δ_E of the spring from its natural state, and the mass is then set into vertical oscillatory motion about this equilibrium state. We shall see that the motion of m relative to the unstretched state is described by an equation that may be transformed to another having the same form as (6.65a) relative to the static equilibrium state.

Let us fix the origin at the natural state of the spring so that $\mathbf{i}$ is in the downward direction of $\mathbf{g} = g\mathbf{i}$. Construction of the free body diagram of m is left for the reader. The weight $\mathbf{W} = mg\mathbf{i}$ produces a static deflection δ_E such that the spring force exerted on m is $\mathbf{F}_S = -k\delta_E\mathbf{i}$; hence, the static equilibrium equation $\mathbf{W} + \mathbf{F}_S = (mg - k\delta_E)\mathbf{i} = \mathbf{0}$ yields

$$k\delta_E = mg. \tag{6.66a}$$

When the mass is set into vertical motion, the weight $\mathbf{W}$ is unchanged but the spring force becomes $\mathbf{F}_S = -kx\mathbf{i}$, where x denotes the stretch of the spring from its natural state. Hence, the equation of motion $\mathbf{W} + \mathbf{F}_S = (mg - kx)\mathbf{i} \doteq m\ddot{x}\mathbf{i}$ yields

$$\ddot{x} + p^2 x = g, \qquad \text{with} \quad p^2 = \frac{k}{m} = \frac{g}{\delta_E}, \tag{6.66b}$$

wherein (6.66a) is introduced. This equation has the form of (6.48) in which $h(t) \equiv g$ is constant. Therefore, recalling the method leading to (6.45b) and (6.53), we see that the general solution of (6.66b) is

$$x(t) = C\cos pt + D\sin pt + \frac{g}{p^2}, \tag{6.66c}$$

in which C and D are integration constants to be fixed by the initial data.

This shows that the motion of m is simple harmonic, but the center of the oscillations is displaced to the position at $x_E = g/p^2 = mg/k = \delta_E$, the static equilibrium position of m. Hence, introducing the new variable $z \equiv x - \delta_E$ to describe the displacement of m from its static equilibrium position, the equation of motion in (6.66b) transforms to

$$\ddot{z} + p^2 z = 0. \tag{6.66d}$$

This has the same form as our earlier equation (6.65a) for the horizontal motion; so, the solution (6.66c) may be cast in the form

$$z(t) = x(t) - \delta_E = C \cos pt + D \sin pt. \tag{6.66e}$$

Hence, both linear spring-mass systems are governed by the same kind of equation. When the displacement is measured from the vertical static equilibrium position of a *linear* spring-mass system, in effect, the static deflection and the weight of the load may be ignored in view of the balance equation (6.66a). Therefore, the static equilibrium state is a convenient reference state from which to study the motion of a linear system, because we need only consider the additional spring force $F^* = -kz = m\ddot{z}$ for the displacement from that state. This superposition procedure, however, cannot be used to study the motion of a load on a nonlinear spring; in this case, the undeformed reference state must be used.

The frequency $f = p/2\pi = (\sqrt{k/m})/2\pi$ of the vibration of m is independent of the initial data. In view of (6.66a), this may be rewritten in terms of the static deflection alone, namely,

$$f = \frac{1}{2\pi}\sqrt{\frac{g}{\delta_E}}. \tag{6.66f}$$

Thus, regardless of the spring stiffness and independently of the amplitude, any vertical loading that produces the same static deflection in different linear springs will oscillate with the same frequency. Of course, for springs of different moduli, the loads needed to produce the same static deflection differ; nevertheless, the measured frequency of their oscillations is identical for all amplitudes, and hence, in this sense, formula (6.66f) is *universal.*

We now study the small amplitude oscillations of a pendulum. Although this physical system is quite different from the spring-mass system, both are governed by the same basic equation of motion characteristic of a simple harmonic oscillator.

6.7.3. The Simple Pendulum

A *simple pendulum,* shown in Fig. 6.15, consists of a small heavy body of mass m, called the *bob*, attached to one end of a thin rigid rod or string of length ℓ, negligible mass, and suspended from a smooth pin or hinge at the point O. The pendulum is displaced to swing about its vertical equilibrium position. Air resistance and the mass of the rod are ignored. We wish to determine the

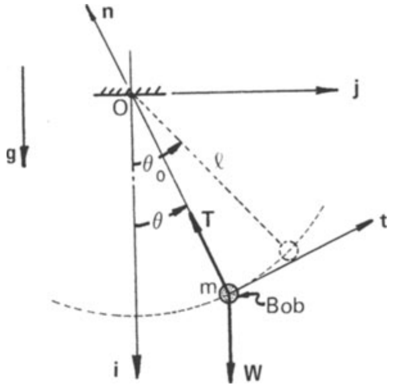

Figure 6.15. A simple pendulum and its free body diagram.

frequency and period of its small amplitude oscillations in the vertical plane, when the pendulum is released from rest at a small angle θ_0. The problem, however, is first formulated exactly for large amplitude oscillations.

The free body diagram in Fig. 6.15 shows the gravitational force $\mathbf{W} = m\mathbf{g}$ and the rod tension $\mathbf{T}$ acting on the bob. The equation of motion for the angular placement $\theta(t)$ of the bob from its vertical equilibrium position is readily described in terms of intrinsic variables. In accordance with (6.3), $\mathbf{F} = \mathbf{W} + \mathbf{T} = m(\ddot{s}\mathbf{t} + \kappa\dot{s}^2\mathbf{n})$, in which $\dot{s} = \ell\dot{\theta}$, $\ddot{s} = \ell\ddot{\theta}$, and $\kappa = 1/\ell$. Therefore,

$$m(\ell\ddot{\theta}\mathbf{t} + \ell\dot{\theta}^2\mathbf{n}) = T\mathbf{n} - W(\sin\theta\mathbf{t} + \cos\theta\mathbf{n}). \tag{6.67a}$$

This yields the two scalar equations of motion:

$$\ddot{\theta} + p^2\sin\theta = 0, \qquad T = m\ell(\dot{\theta}^2 + p^2\cos\theta), \tag{6.67b}$$

where

$$p \equiv \sqrt{\frac{g}{\ell}}. \tag{6.67c}$$

Let the reader confirm these equations by use of (6.4).

The first equation in (6.67b) is an ordinary nonlinear differential equation for the angular motion $\theta(t)$ and the second gives the rod tension $T(\theta)$ as a function of θ. The exact solution of these equations for finite amplitude oscillations of a pendulum will be studied in Chapter 7. Presently, however, we consider only small values of θ so that all squared and higher order terms in θ and its derivative $\dot{\theta}$ may be neglected. Then use of the series functions (2.17) in (6.67b) leads to

$$\ddot{\theta} + p^2\theta = 0, \qquad T = mg = W. \tag{6.67d}$$

Certainly, for sufficiently small placements $\theta(t)$, it is expected that the rod tension does not vary significantly from its static value, the weight of the bob, as shown in (6.67d). The first equation in (6.67d) has the same form as (6.65a); so, the small amplitude pendulum motion is simple harmonic with circular frequency p defined by (6.67c). Therefore,

$$\theta(t) = A \sin pt + B \cos pt. \tag{6.67e}$$

The constants A and B are determined by the given initial conditions that the pendulum is released from rest at a small angle θ_0, so that $\theta(0) = \theta_0$ and $\dot{\theta}(0) = 0$. From (6.67e), the angular speed is $\dot{\theta} = Ap \cos pt - Bp \sin pt$. We thus find $A = 0$ and $B = \theta_0$, and (6.67e) yields the solution

$$\theta(t) = \theta_0 \cos pt. \tag{6.67f}$$

The angle θ_0 is the amplitude, the maximum angular placement from the vertical equilibrium position in the motion (6.67f). Recalling the relations (6.65d), we find the small amplitude frequency and period of a simple pendulum with circular frequency (6.67c):

$$f = \frac{1}{2\pi}\sqrt{\frac{g}{\ell}}, \qquad \tau = 2\pi\sqrt{\frac{\ell}{g}}. \tag{6.67g}$$

The period is the time required for the pendulum to swing from θ_0 to $-\theta_0$ and back again to θ_0. The period of the small amplitude, simple harmonic motion of a pendulum is independent of this amplitude. The finite amplitude motion of a pendulum described by the first equation in (6.67b), though still periodic, is not simple harmonic. It is shown in Chapter 7 that the periodic time in the finite motion varies with the amplitude.

6.7.4. The Common Mathematical Model in Review

The linear spring-mass system and the simple pendulum (for small amplitude oscillations) are merely two examples of a great many physical systems that are characterized by the same mathematical model. Their common model, called the *simple harmonic oscillator*, is described by the homogeneous differential equation

$$\ddot{u} + p^2 u = 0, \tag{6.68}$$

whose solution

$$u = A \sin pt + B \cos pt, \tag{6.69}$$

is simple harmonic. The constant circular frequency p and period $\tau = 2\pi/p$, or the frequency $f = 1/\tau$, may be read immediately from the positive coefficient in (6.68).

The amplitude of the motion of a harmonic oscillator may be obtained by introduction of two other constants U and α defined by

$$A = U \cos\alpha, \qquad B = U \sin\alpha. \tag{6.70}$$

Therefore, the new constants are related to the former by

$$U = \sqrt{A^2 + B^2}, \qquad \tan\alpha = \frac{B}{A}. \tag{6.71}$$

We lose no generality in taking U positive, and α may be either positive or negative valued.

Use of (6.70) in (6.69) yields the following alternative form for the motion $u(t)$ of the simple harmonic oscillator:

$$u(t) = U\sin(pt + \alpha). \tag{6.72}$$

The reader may show that with $A = U\sin\beta,\ B = U\cos\beta$, where $\beta = \pi/2 - \alpha$, an alternative form of the solution (6.72) is given by $u(t) = U\cos(pt - \beta)$. Either solution shows that U, the maximum value of $u(t)$, is the *amplitude* of the motion. The angle $pt + \alpha$ (or $pt - \beta$) is called the *phase angle* or simply the *phase* of the motion; it characterizes the state of the oscillation at a specific time. The *phase constant* α (or β) defines the *initial phase* of the motion. From (6.72), the velocity may be written as

$$\dot{u}(t) = Up\cos(pt + \alpha). \tag{6.73}$$

Thus, if initially we are given $u(0) \equiv u_0$ and $\dot{u}(0) \equiv v_0$, then (6.72) and (6.73) yield $u_0 = U\sin\alpha,\ v_0 = Up\cos\alpha$. In consequence, by (6.71), the amplitude and initial phase may be expressed in terms of the initial data:

$$U = \sqrt{u_0^2 + \left(\frac{v_0}{p}\right)^2}, \qquad \alpha = \tan^{-1}\left(\frac{u_0 p}{v_0}\right).$$

A graphical description of the motion is obtained from (6.72) and (6.73). *For arbitrary initial data and for each fixed frequency p, the graph of $\dot{u}(t)$ versus $u(t)$ for the simple harmonic oscillator motion is a family of concentric ellipses having semi-axes determined by U and Up*:

$$\left(\frac{u}{U}\right)^2 + \left(\frac{\dot{u}}{Up}\right)^2 = 1. \tag{6.74}$$

In general, the plane graph of $\dot{u}$ versus u for any single degree of freedom system is called the *phase plane graph.* Thus, for each choice of initial data, the phase plane graph for the simple harmonic oscillator is an ellipse defined by (6.74). However, it is seen further that the normalized plot of $\dot{u}/Up$ versus u/U reduces every member of the family (6.74) to a single unit circle. The periodic nature of the simple harmonic motion is exhibited by these closed curves. Since p is fixed and the period does not depend on the initial data, all trajectories in the phase plane are traversed in the same time τ. Equation (6.74) has exactly two solutions $u = \pm U$ for which $\dot{u} = 0$ and two solutions $\dot{u} = \pm Up$ at the equilibrium position $u = 0$, the unique time independent solution of (6.68). *Hence, the amplitudes $\pm U$ mark the extreme positions in a simple harmonic motion at which the velocity* *momentarily vanishes; and the greatest velocity* $\pm Up$ *occurs at the equilibrium state.*

This concludes our study of the simple harmonic oscillator. The effect of viscous damping on mechanical vibrations, and some effects of inertial forces induced by rotating bodies, including effects of the Earth's rotation, and other kinds of forces are explored later in this chapter. It is important to recognize that not every vibration need be periodic, and not every periodic motion need be vibratory. Random vibrations, for example, are not periodic, and steady orbital motions are periodic but not vibratory. The following example of a particle moving in an electromagnetic field exhibits a motion that is periodic but not oscillatory. The solution procedure, however, is the same.

6.8. Motion of a Charged Particle in an Electromagnetic Field

A particle of charge q and mass m is ejected from an electronic device, with initial velocity $\mathbf{v}_0 = v\mathbf{j}$ at the place $\mathbf{x}_0 = R\mathbf{i}$ in an inertial frame $\Phi = \{F; \mathbf{i}_k\}$. The charge moves under the influence of constant and oppositely directed electric and magnetic fields that are parallel to the axis of the gravitational field, as shown in Fig. 6.16. The total body force acting on q is $\mathbf{F} = \mathbf{F}_e + \mathbf{F}_m + \mathbf{W}$; hence, with (6.18), the equation of motion may be written as

$$\ddot{\mathbf{x}} - c\dot{\mathbf{x}} \times \mathbf{B} = \frac{d}{dt}(\dot{\mathbf{x}} - c\mathbf{x} \times \mathbf{B}) = c\mathbf{E} + \mathbf{g}, \tag{6.75a}$$

where in $c \equiv q/m$. This vector equation is readily integrated to obtain the velocity

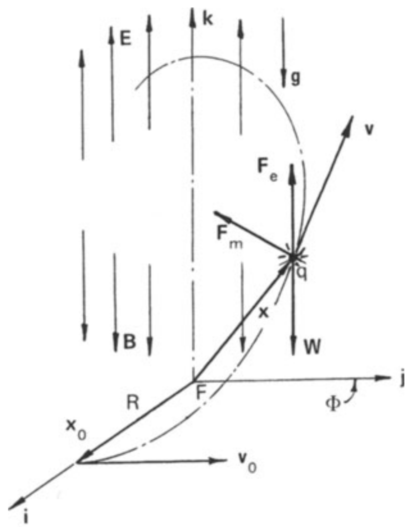

Figure 6.16. Motion of a charged particle in uniform and oppositely directed electric and magnetic fields.

$\dot{\mathbf{x}}$ as a function of $\mathbf{x}$ and t; thus,

$$\dot{\mathbf{x}} = c\mathbf{x} \times \mathbf{B} + (c\mathbf{E} + \mathbf{g})t + \mathbf{C}_0. \tag{6.75b}$$

The constant vector of integration is fixed by the initial data $\mathbf{x}(0) = \mathbf{x}_0$, $\dot{\mathbf{x}}(0) = \mathbf{v}_0$, so that

$$\mathbf{C}_0 = \mathbf{v}_0 - c\mathbf{x}_0 \times \mathbf{B}. \tag{6.75c}$$

Although (6.75b) cannot be integrated further, its use in (6.75a) leads to another integrable result. Bearing in mind that the vectors **B**, **E**, and **g** are parallel, we obtain

$$\ddot{\mathbf{x}} - c^2(\mathbf{x} \times \mathbf{B}) \times \mathbf{B} = c\mathbf{C}_0 \times \mathbf{B} + c\mathbf{E} + \mathbf{g}. \tag{6.75d}$$

Finally, substitution of (6.75c) into (6.75d), expansion of the triple products in the result, use of the orthogonality condition $\mathbf{B} \cdot \mathbf{x}_0 = 0$, and $\omega \equiv cB$ yields the vector differential equation

$$\ddot{\mathbf{x}} + \omega^2\mathbf{x} - c^2(\mathbf{x} \cdot \mathbf{B})\mathbf{B} = \omega^2\mathbf{x}_0 + c\mathbf{v}_0 \times \mathbf{B} + c\mathbf{E} + \mathbf{g}. \tag{6.75e}$$

Exercise 6.3. Show that (6.75e) may be written as $\ddot{\mathbf{x}} + \mathbf{P}^2\mathbf{x} = \gamma$, in which $\mathbf{P}^2 \equiv \omega^2(\mathbf{i} \otimes \mathbf{i} + \mathbf{j} \otimes \mathbf{j})$ and γ is a constant vector. □

We now introduce $\mathbf{B} = -B\mathbf{k}$, $\mathbf{E} = E\mathbf{k}$, $\mathbf{g} = -g\mathbf{k}$, $\mathbf{x}_0 = R\mathbf{i}$, $\mathbf{v}_0 = \nu\mathbf{j}$, and $\mathbf{x} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ into (6.75e) and equate the corresponding vector components to obtain the following three scalar equations of motion:

$$\ddot{x} + \omega^2 x = \omega^2\left(R - \frac{\nu}{\omega}\right), \qquad \ddot{y} + \omega^2 y = 0, \qquad \ddot{z} = 2A, \tag{6.75f}$$

in which $2A \equiv cE - g$. The first pair of these equations shows that both x and y are simple harmonic functions, and hence the general solution of the system (6.75f) is given by

$$x(t) = a + K\cos\omega t + L\sin\omega t, \qquad \text{with} \quad a \equiv R - \frac{\nu}{\omega}, \tag{6.75g}$$

$$y(t) = M\cos\omega t + N\sin\omega t \quad \text{and} \quad z(t) = At^2 + Pt + Q. \tag{6.75h}$$

The initial data $\mathbf{x}(0) = \mathbf{x}_0 = R\mathbf{i}$ and $\dot{\mathbf{x}}(0) = \mathbf{v}_0 = \nu\mathbf{j}$ determine the integration constants $L = M = P = Q = 0$ and $K = N = \nu/\omega$. Hence, the foregoing system has the solution

$$x(t) = a + \frac{\nu}{\omega}\cos\omega t, \qquad y(t) = \frac{\nu}{\omega}\sin\omega t, \qquad z(t) = At^2. \tag{6.75i}$$

It is seen that $(x - a)^2 + y^2 = (\nu/\omega)^2$ is the equation of a circle centered at $(a, 0)$, hence (6.75i) suggests that the trajectory of q looks a bit like a cylindrical helix of radius $\rho \equiv \nu/\omega$. By taking $R = \rho$ in (6.75g), we have $a = 0$, and the cylinder axis is shifted to the origin of Φ. The first two equations in (6.75f) have essentially the same form as equation (6.68) for the harmonic oscillator, but the motion of q is not oscillatory. On the other hand, the motion in the xy-plane is periodic; the circular frequency $\omega = cB$, evident from (6.75f), describes the constant rate of rotation of q about the cylinder axis, and the periodic time $\tau = 2\pi/\omega$ is the time required for the particle to make one full swing around that axis as it advances along z. Notice that the tangent to the path does not make a fixed angle with the cylinder axis; rather, $\mathbf{t} \cdot \mathbf{k} = 2At(\nu^2 + 4A^2t^2)^{-1/2}$. The pitch increases with the square of the number of turns: $p_n \equiv z(n\tau) = n^2 z(\tau)$, and $z(\tau) = A/4\pi\omega^2 = p_1$. So, the path is not a true cylindrical helix.

6.9. Motion of a Slider Block in a Rotating Reference Frame

We now turn to a different class of problems whose solutions involve the hyperbolic functions. Two problems concerning the motion of a slider block in a slot milled in a rotating table are studied. The first concerns the free sliding motion of the block due to inertial forces induced by the table's rotation. The second problem is similar, but more interesting. An additional controlling spring is introduced, and depending on the nature of two physical parameters, one due to the rotation and the other due to the spring, the governing equation of motion may have a solution of either trigonometric or hyperbolic type, or neither.

6.9.1. Uncontrolled Motion of a Slider Block

A block S of mass m shown in Fig. 6.17 is constrained initially by a cord fastened at the end point A of a smooth slot milled in a table that turns in the horizontal plane with a constant angular speed ω. When the string is cut suddenly, the block slides freely in the slot. We wish to determine the motion $\mathbf{x}(S, t)$ of the slider block relative to the spinning table, and the behavior of the force that acts on the block as a function of its position in the slot and as a function of time.

The free body diagram of the sliding block is shown in Fig. 6.17a. Of course, the string force $\mathbf{F}_S = \mathbf{0}$, and the weight of the block is $\mathbf{W} = -W\mathbf{k}$. Because the slot is smooth, it exerts on S only the normal contact forces $\mathbf{N} = -N\mathbf{j}$ in the plane of the table and $\mathbf{R} = R\mathbf{k}$ perpendicular to it. The total force acting on S is $\mathbf{F} = \mathbf{N} + \mathbf{R} + \mathbf{W}$, and hence the equation of motion for S in the inertial frame $\Phi = \{F; \mathbf{I}_k\}$ fixed in the laboratory is given by

$$\mathbf{F} = -N\mathbf{j} + (R - W)\mathbf{k} = m\mathbf{a}_S. \tag{6.76a}$$

The absolute acceleration $\mathbf{a}_S$ of S in Φ may be obtained from (4.48). With $\mathbf{a}_O = \mathbf{0}$, $\boldsymbol{\omega}_f = \omega\mathbf{k}$, $\dot{\boldsymbol{\omega}}_f = \mathbf{0}$, and $\mathbf{x}(S, t) = x\mathbf{i} + a\mathbf{j}$ in the reference frame $\varphi = \{O; \mathbf{i}_k\}$ fixed in the table, as shown in Fig. 6.17, the total acceleration of S referred

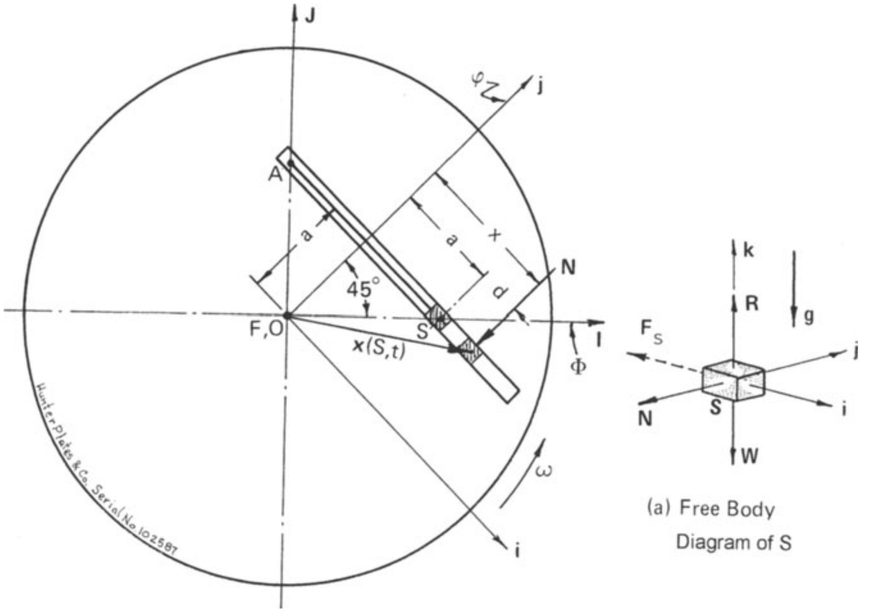

Figure 6.17. Relative motion of a slider block on a rotating table.

to φ is

$$\mathbf{a}_S = (\ddot{x} - \omega^2 x)\mathbf{i} + (2\omega\dot{x} - a\omega^2)\mathbf{j}. \tag{6.76b}$$

Substitution of (6.76b) into (6.76a) yields the scalar equations

$$\ddot{x} - \omega^2 x = 0, \qquad N = m(a\omega^2 - 2\omega\dot{x}), \qquad R = W. \tag{6.76c}$$

The first of these equations determines the motion $x(t)$ of S relative to the table, and the next one determines the normal contact force N either as a function of x or of t. The last relation confirms that since there is no motion of S normal to the table, the slot reaction force $\mathbf{R}$ balances the weight $\mathbf{W}$, so that $\mathbf{R} + \mathbf{W} = \mathbf{0}$ in (6.76a). Therefore, in future problems where the motion is constrained to a smooth horizontal plane, for simplicity, the trivial normal equilibrated forces may be ignored.

The first equation in (6.76c) has the same form as the homogeneous equation (6.58) whose solution is given by (6.59). Therefore, the slider's motion is given by

$$x(t) = A \sinh \omega t + B \cosh \omega t. \tag{6.76d}$$

The slider is initially at rest at $x(0) = a$ in frame φ, as shown in Fig. 6.17, and hence $\dot{x}(0) = 0$. Thus, with $\dot{x}(t) = A\omega \cosh \omega t + B\omega \sinh \omega t$, it follows that

$A = 0$, $B = a$; hence

$$x(t) = a \cosh \omega t, \qquad \dot{x}(t) = a\omega \sinh \omega t. \tag{6.76e}$$

Therefore, the motion of S relative to the table frame may be written as

$$\mathbf{x}(S, t) = a(\cosh \omega t \mathbf{i} + \mathbf{j}). \tag{6.76f}$$

Use of (6.76e) in the second equation in (6.76c) gives the slot reaction force $\mathbf{N} = -N\mathbf{j}$ as a function of time;

$$\mathbf{N} = \tilde{\mathbf{N}}(t) = -ma\omega^2(1 - 2 \sinh \omega t)\mathbf{j}. \tag{6.76g}$$

Alternatively, use of the identity (6.63) yields the slot reaction force as a function of the slider's position along the slot:

$$\mathbf{N} = \hat{\mathbf{N}}(x) = -m\omega^2(a - 2\sqrt{x^2 - a^2})\mathbf{j}. \tag{6.76h}$$

Let the reader show that the same result may be derived directly by integration of the first equation in (6.76c) to find $\dot{x}(x)$.

These results show from (6.76g) that initially $\tilde{\mathbf{N}}(0) = -ma\omega^2\mathbf{j}$, and, as time advances, the normal force $\tilde{\mathbf{N}}(t)$ decreases to zero in the time t^* for which $\omega t^* = \sinh^{-1}(1/2) \approx 0.481$, the instant when the slider is at the place $x^* = a\sqrt{5}/2 \approx 1.118a$ in the slot. Afterwards, the normal, slot reaction force reverses its sense of application to the opposite side of the block and grows again, indefinitely for as long as the block is able to move outward. Suppose, for example, that $\mathbf{x}(S, 0) = a(\mathbf{i} + \mathbf{j}) = 25\mathbf{I}$ cm and $\omega = 20\pi/3$ rad/sec (200 rpm). Then $a = 25/\sqrt{2}$ cm, and the previous formulas show that $\mathbf{N}$ vanishes, and then reverses its sense of application, after $t^* \approx 0.023$ sec when S has moved a distance $d^* = x^* - a \approx 2.086$ cm from its initial position.

When the string was cut, the motion of the block along the slot was no longer controlled, and the inertial effect of the table's rotation drove the slider increasingly farther from its rest state toward the end of the slot. The controlling effect of an additional spring force is illustrated next.

6.9.2. Controlled Motion and Instability of a Slider Block

Suppose that the string shown in Fig. 6.17 is replaced by a linear spring of stiffness k fastened at A and to the block S, initially at rest at the natural state of the spring at $x = a$ but otherwise free to slide in the smooth slot. We wish to investigate the motion $\mathbf{x}(S, t)$ of the block relative to the rotating table.

The free body diagram of the sliding block is shown in Fig. 6.17a in the table frame φ. The forces are the same as before with the addition of the spring force $\mathbf{F}_S = -k(x - a)\mathbf{i}$. Since there is no motion normal to the horizontal plane, $\mathbf{R} + \mathbf{W} = \mathbf{0}$, as noted before. Therefore, the equation of motion for S in the inertial frame Φ, but referred to the table frame φ, is given by

$$\mathbf{F} = \mathbf{N} + \mathbf{F}_S = -N\mathbf{j} - k(x-a)\mathbf{i} = m\mathbf{a}_S. \tag{6.77a}$$

Here we recall (6.76b) to obtain the scalar equations

$$\ddot{x} + p^2(1-\eta^2)x = ap^2, \tag{6.77b}$$

$$N = m(a\omega^2 - 2\omega\dot{x}), \tag{6.77c}$$

wherein, by definition,

$$p \equiv \sqrt{\frac{k}{m}}, \qquad \eta \equiv \frac{\omega}{p}. \tag{6.77d}$$

The physical nature of the motion determined by (6.77b) depends on the coefficient $p^2(1-\eta^2)$. There are three cases to explore: (i) $\eta < 1$, (ii) $\eta = 1$, and (iii) $\eta > 1$. Each case is studied in turn for the assigned initial data

$$x(0) = a, \qquad \dot{x}(0) = 0. \tag{6.77e}$$

Case (i): $\eta < 1$; i.e. the angular speed $\omega < p$. Then the equation of motion in (6.77b) has the form of (6.48) in which p^2 is replaced by $\Omega^2 \equiv p^2(1-\eta^2)$ and $h(t) \equiv ap^2$ is constant. Recalling (6.45b) and (6.53), we see that the general solution of (6.77b) is

$$x(t) = A\sin\Omega t + B\cos\Omega t + \frac{a}{1-\eta^2}, \qquad \text{with} \quad \Omega = p\sqrt{1-\eta^2}. \tag{6.77f}$$

The relative motion of S is simple harmonic, but the center of the oscillation is displaced to the relative equilibrium position at

$$x_E = \frac{a}{1-\eta^2}, \tag{6.77g}$$

defined by the unique time independent solution of the equation of motion (6.77b). Notice that $x_E > a$. Hence, introducing the new variable $z \equiv x - x_E$ to describe the displacement of S from its relative equilibrium position, we may write

$$z(t) \equiv x(t) - \frac{a}{1-\eta^2} = A\sin\Omega t + B\cos\Omega t. \tag{6.77h}$$

Consequently, the equation of motion in (6.77b) transforms to the familiar equation

$$\ddot{z} + \Omega^2 z = 0, \tag{6.77i}$$

the differential equation for the simple harmonic oscillator.

The initial values (6.77e) yield $B = a - x_E = -a\eta^2/(1-\eta^2)$ and $A = 0$; so, $z(t) = B\cos\Omega t$. The oscillations occur symmetrically about the relative equilibrium position x_E with the amplitude $z_{\max} = |B| = a\eta^2/(1-\eta^2)$ and circular frequency Ω given in (6.77f). We thus find that the motion of the slider in the case when $\omega < p$ is given by $\mathbf{x}(S, t) = x(t)\mathbf{i} + a\mathbf{j} \overset{\text{or}}{=} (z(t) + x_E)\mathbf{i} + a\mathbf{j}$, in which

$$x(t) = \frac{a}{1-\eta^2}(1 - \eta^2 \cos \Omega t), \qquad z(t) = -\frac{a\eta^2}{1-\eta^2} \cos \Omega t. \tag{6.77j}$$

Case (ii): $\eta = 1$; i.e. the angular speed $\omega = p$. The general solution of the differential equation of motion (6.77b), for the initial data (6.77e), is given by

$$x(t) = \frac{1}{2} a p^2 t^2 + a. \tag{6.77k}$$

This result suggests that $\omega = p$ is the critical angular speed of the table at which the motion of S about its relative equilibrium position x_E ceases to be oscillatory and now tends to grow indefinitely with time. The previously stable relative equilibrium position (6.77g) of the slider block about which it oscillates when $\omega < p$, no longer exists, a fact evident from (6.77b) for which no time independent solution exists when $\eta = 1$. And hence, the relative equilibrium position x_E of the slider block is said to be unstable at $\omega = p$. In our study of infinitesimal stability defined later, it is proved that the relative equilibrium state is stable if and only if $\omega < p$. Investigation of the physical nature of the slot reaction force (6.77c), both here and below, is left for the reader.

Case (iii): $\eta > 1$; i.e. the angular speed $\omega > p$. The equation of motion (6.77b), in which the coefficient is now negative, has the form of (6.54) in which $q^2 \equiv p^2(\eta^2 - 1)$ and $h(t) \equiv ap^2$. Therefore, with (6.45b) and (6.61) in mind, the general solution of (6.77b) is given by

$$x(t) = A \sinh qt + B \cosh qt - \frac{a}{\eta^2 - 1}, \tag{6.77l}$$

where $q \equiv p(\eta^2 - 1)^{1/2}$. Alternatively, the change of variable $\xi(t) \equiv x(t) + a/(\eta^2 - 1)$ transforms the equation of motion (6.77b) to $\ddot{\xi} - q^2\xi = 0$, an equation of the type (6.58) whose solution is given by (6.59).

The initial data (6.77e) yields $B = a\eta^2/(\eta^2 - 1)$ and $A = 0$. We thus obtain from (6.77l) the relative motion $\mathbf{x}(S, t) = x(t)\mathbf{i} + a\mathbf{j} \overset{\text{or}}{=} (\xi(t) - a/(\eta^2 - 1))\mathbf{i} + a\mathbf{j}$ in which

$$x(t) = \frac{a}{\eta^2 - 1}(\eta^2 \cosh qt - 1), \qquad \xi(t) = \frac{a\eta^2}{\eta^2 - 1} \cosh qt. \tag{6.77m}$$

The motion $x(t)$ relative to the table frame when $\omega > p$ and the slider block is released from rest at $x(0) = a$ thus tends to grow increasingly large with time. At some point, of course, Hooke's law fails, the limiting extensibility of the spring restricts the extent of the motion, and (6.77m) is no longer valid. Notice that the time independent solution of (6.77b) in this case is not a physically meaningful relative equilibrium state.

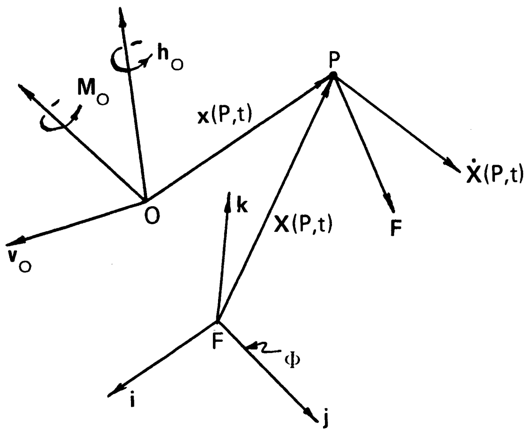

Figure 6.18. Schema for the moment of momentum principle.

6.10. The Moment of Momentum Principle

In this section the Newton–Euler law is applied to derive an additional principle of motion that relates torque and the moment of momentum of a particle. First, however, we recall the definition (5.20) to write *the moment* $\mathbf{M}_O$ *of a force* $\mathbf{F}$ *about a point* O*, either fixed or in motion relative to an assigned frame* Φ:

$$\mathbf{M}_O = \mathbf{x} \times \mathbf{F}, \tag{6.78}$$

in which $\mathbf{x} \equiv \mathbf{x}_O$ is the position vector from O to the particle P on which the total force $\mathbf{F}$ acts, as shown in Fig. 6.18. Let O be a *fixed point* in the inertial frame $\Phi = \{F; \mathbf{i}_k\}$ in Fig. 6.18, so that $\dot{\mathbf{x}} = \dot{\mathbf{X}} = \mathbf{v}$, the velocity of P in Φ. Now recall the definition (5.31) of the moment of momentum of a particle P, differentiate it with respect to time, and use (5.34) to obtain

$$\frac{d\mathbf{h}_O}{dt} = \mathbf{x} \times \frac{d\mathbf{p}}{dt} + \mathbf{v} \times m\mathbf{v} = \mathbf{x} \times \mathbf{F}.$$

In view of (6.78), this yields our additional principle of motion.

The moment of momentum principle: *The moment about a fixed point* O *of the total force acting on a particle* P *in an inertial frame* Φ *is equal to the time rate of change of the moment about* O *of the momentum of* P *in* Φ:

$$\mathbf{M}_O = \frac{d\mathbf{h}_O}{dt}. \tag{6.79}$$

6.10.1. Application to the Simple Pendulum Problem

The moment of momentum principle (6.79) provides an alternative and often simpler means to derive the appropriate equation of motion for a particle without our having to address details concerning certain forces of constraint; otherwise, it delivers no more information on the motion than may be obtained from the Newton–Euler law. This is demonstrated in our review of the equation of motion for a simple pendulum.

The forces that act on the bob are shown in Fig. 6.15, page 138. To apply (6.79), we first determine the moment of these forces about the fixed point O. The central directed string tension has no moment about O, while the weight exerts a torque about O given by

$$\mathbf{M}_O = \mathbf{x} \times \mathbf{W} = -mg\ell \sin\theta \mathbf{b},$$

where $\mathbf{b} = \mathbf{t} \times \mathbf{n}$ is a constant unit vector perpendicular to the plane of motion. The moment of momentum of the bob about the fixed point O in Fig. 6.15 is given by $\mathbf{h}_O = \mathbf{x} \times m\mathbf{v} = -\ell\mathbf{n} \times m\ell\dot{\theta}\mathbf{t} = m\ell^2\dot{\theta}\mathbf{b}$, and hence

$$d\mathbf{h}_O/dt = m\ell^2\ddot{\theta}\mathbf{b}.$$

Collecting this data in (6.79), equating the components, and writing $p^2 = g/\ell$, we obtain the equation $\ddot{\theta} + p^2 \sin\theta = 0$ for the angular motion $\theta(t)$ of the pendulum bob, which is the same as the first equation in (6.67b). Because the cord tension has no moment about O, the moment of momentum principle eliminates the need to consider it further in the discussion of the motion of the bob.

6.10.2. The Moment of Momentum Principle for a Moving Point

The moment of momentum principle (6.79) holds only for an arbitrary point O fixed in the inertial frame Φ. We now determine the form of this principle when O is an arbitrary moving point in Φ.

The moment about O of the momentum $\mathbf{p}(P, t) = m(P)\dot{\mathbf{X}}(P, t)$ in the inertial frame Φ is defined by (5.31) in which point O may be either a fixed or a moving moment center. Hence, when O has an arbitrary velocity $\mathbf{v}_O$ in Φ, the derivative of (5.31) with respect to time in Φ is given by

$$\dot{\mathbf{h}}_O = \dot{\mathbf{x}} \times \mathbf{p} + \mathbf{x} \times \mathbf{F},$$

wherein $\dot{\mathbf{x}} = \dot{\mathbf{X}} - \mathbf{v}_O$. Hence, use of (6.78) now yields the *moment of momentum principle for an arbitrary moving reference point* O:

$$\mathbf{M}_O = \dot{\mathbf{h}}_O + \mathbf{v}_O \times \mathbf{p}. \tag{6.80}$$

Therefore, the moment of momentum principle (6.79) *may hold with respect to a moving point O if and only if $\mathbf{v}_O \times \mathbf{p} = \mathbf{0}$, i.e. when and only when the velocity of O is parallel to the velocity of the particle P; otherwise, O must be a fixed point.*

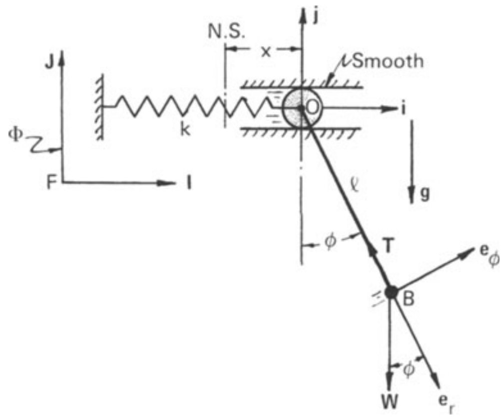

Figure 6.19. Motion of a pendulum having a moving support.

In general, then, the modified principle (6.80) must be used when O is a moving reference point. An application of this rule follows.

Example 6.14. A pendulum bob B attached to a rigid rod of negligible mass and length ℓ is suspended from a smooth movable support at O that oscillates about the natural undeformed state of the spring so that $x(t) = x_O \sin \Omega t$ in Fig. 6.19. Apply equation (6.80) to derive the equation of motion for the bob.

Solution. The forces that act on the pendulum bob B are shown in the free body diagram in Fig. 6.19. Notice that the tension $\mathbf{T}$ in the rod at B is directed through the moving point O. Moreover, the spring force and normal reaction force of the smooth supporting surface also are directed through O; but these forces do not act on B, so they hold no direct importance in its equation of motion. Consequently, the moment about the point O of the forces that act on B at $\mathbf{x}_B = \ell \mathbf{e}_r$ in the cylindrical system shown in Fig. 6.19 is given by

$$\mathbf{M}_O = \mathbf{x}_B \times \mathbf{W} = -\ell W \sin\phi \mathbf{k}. \tag{6.81a}$$

The absolute velocity of B is determined by $\mathbf{v}_B = \mathbf{v}_O + \boldsymbol{\omega} \times \mathbf{x}_B$, in which $\omega = \dot{\phi}\mathbf{k}$ and $\mathbf{v}_O = \dot{x}\mathbf{i} = x_O \Omega \cos \Omega t \mathbf{i} = v_O \mathbf{i}$. Thus,

$$\mathbf{v}_B = v_O \mathbf{i} + \ell\dot{\phi}\mathbf{e}_\phi, \qquad \text{with} \quad v_O = x_O \Omega \cos \Omega t. \tag{6.81b}$$

With the linear momentum $\mathbf{p} = m\mathbf{v}_B$ and use of (6.81b), we find

$$\mathbf{v}_O \times \mathbf{p} = v_O \mathbf{i} \times m\ell\dot{\phi}\mathbf{e}_\phi = m v_O \dot{\phi} \ell \sin\phi \mathbf{k}. \tag{6.81c}$$

The moment of momentum about O is given by $\mathbf{h}_O = \mathbf{x}_B \times \mathbf{p} = m\ell(v_O \cos\phi + \ell\dot{\phi})\mathbf{k}$, and its time rate of change is

$$\dot{\mathbf{h}}_O = m\ell(a_O \cos\phi - v_O \dot{\phi} \sin\phi + \ell\ddot{\phi})\mathbf{k}, \tag{6.81d}$$

in which $a_O = \dot{v}_O = -x_O\Omega^2 \sin \Omega t$. Substituting (6.81a), (6.81c), and (6.81d) into (6.80), we find $-\ell W \sin\phi \mathbf{k} = m\ell(-x_O\Omega^2 \sin\Omega t \cos\phi + \ell\ddot{\phi})\mathbf{k}$. Hence, with $W = mg$, the equation of motion for the bob may be written as

$$\ddot{\phi} + p^2 \sin\phi = \frac{x_O\Omega^2}{\ell} \cos\phi \sin\Omega t, \tag{6.81e}$$

where $p^2 = g/\ell$. The solution of (6.81e) for small $\phi(t)$ is discussed later in our study of mechanical vibrations. (See Example 6.15, page 161.) □

6.11. Free Vibrations with Viscous Damping

The simple harmonic oscillator is the fundamental model of the theory of mechanical vibrations. Its motion is a perpetual sinusoidal oscillation; once set into motion, the oscillation continues indefinitely. In real situations, however, there usually is a dissipative or viscous drag force, called a *damping force*, that causes the vibration eventually to die out. If the damping force is very small, the simple harmonic oscillator often is a useful model. On the other hand, when friction devices or shock absorbers are used in mechanical systems, it is the intent of the design that their damping effect be considerable. The suspension system of an automobile, for example, is designed to dampen smoothly and quickly the vibrations induced by the irregular motion of the vehicle over a rough road. The viscous damper used to ease the automatic closing of a door and prevent its slamming is another example of the useful effects of damping. Other cases where damping effects are sometimes desirable and sometimes not arise in instrument design. Damping of the potentially violent needle motion of a galvanometer can prevent damage to the instrument when the current is measured, whereas dissipative effects in a gravitometer may seriously affect the accuracy of gravity measurements.

The analysis of induced motion, damped or not, is also important. The motion of a structure induced by an earthquake or by aerodynamic effects of wind, the sudden wing vibration of an aircraft exposed to high winds or turbulence, and the vibration of a vehicle induced by a bumpy road obviously are undesirable but unavoidable environmental effects. On the other hand, magnification of induced motions is essential in the design of seismographs and certain flight instruments.

The analysis of the kinds of problems described above generally is quite complex, especially when vibrational effects are nonlinear; however, a great variety of problems that involve damping and induced motions can be adequately modeled by a simplified damped spring-mass system that consists of a load of mass m, a linear spring of constant stiffness k, and a linear *viscous damper* or *dashpot*. A typical model of a damped spring-mass system is shown in Fig. 6.20.

A dashpot consists of a piston that moves within a cylinder containing a fluid, usually oil. When the piston is moved by the load, it exerts a viscous retarding force on the load. For simplicity, we model this viscous force by Stokes's law (6.29) and

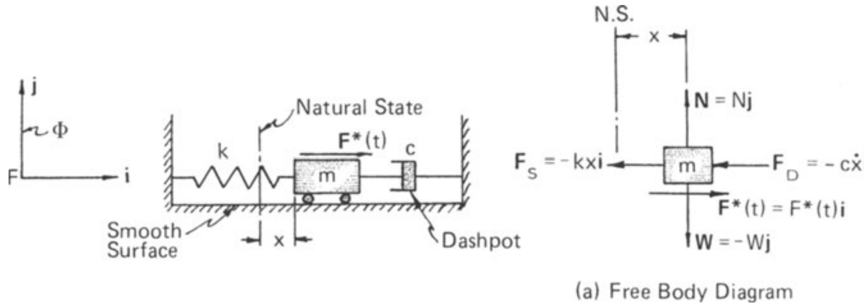

Figure 6.20. Model of a damped spring-mass system.

write $\mathbf{F}_D = -c\dot{x}\mathbf{i}$, in which c is a constant damping coefficient. The spring force is a restoring force given by $\mathbf{F}_S = -kx\mathbf{i}$, where $x(t)$ denotes the displacement of the load from the natural state of the system. The other applied forces in Fig. 6.20 include a disturbing force $\mathbf{F}^*(t) = F^*(t)\mathbf{i}$, attributed to certain environmental effects of the sort mentioned above. The free body diagram in Fig. 6.20a shows that the weight $\mathbf{W}$ is balanced by the normal reaction force $\mathbf{N}$ of the smooth surface, and hence the motion $x(t)$ is determined by the differential equation

$$m\ddot{x} + c\dot{x} + kx = F^*(t). \tag{6.82}$$

If the disturbing force $F^*(t) = F_0$ is constant, the motion is called a *free vibration*; otherwise, it is called a *forced vibration*. When c is zero or may be considered negligible, the motion is said to be *undamped*. The undamped, free vibrational motion is just the simple harmonic motion (6.65a) studied earlier. We next consider the problem of damped, free vibrations of the load.

6.11.1. The Equation of Motion for Damped, Free Vibrations

In a free vibration, the only effect of a constant disturbing force $F^* = F_0$, such as gravity, is to shift the origin to the new position $z \equiv x - x_E$, where $x_E = F_0/k$ is the unique time independent, relative equilibrium solution of (6.82). Therefore, by this simple transformation, all damped, free vibrations of the system in Fig. 6.20 are characterized by *the differential equation for the damped, free vibrational motion of the load m about its relative equilibrium position*:

$$\ddot{z} + 2\nu\dot{z} + p^2 z = 0, \tag{6.83}$$

wherein the coefficients are constants defined by

$$2\nu \equiv \frac{c}{m}, \qquad p \equiv \sqrt{\frac{k}{m}}, \tag{6.84}$$

in which p is the circular frequency of the familiar undamped spring-mass system. The coefficient ν is named the *damping exponent*. The damping coefficient has the physical dimensions $[c] = [FV^{-1}] = [MT^{-1}]$, and hence $[\nu] = [p] = [T^{-1}]$. The dimensionless ratio

$$\zeta \equiv \frac{\nu}{p} = \frac{c}{2mp}, \tag{6.85}$$

is known as the *viscous damping ratio.*

6.11.2. Analysis of the Damped, Free Vibrational Motion

The general solution of (6.83) may be obtained by several methods. One familiar approach is described at the end of this section in an exercise for the reader. Another useful method that simplifies the presentation and emphasizes the physical nature of the damping adopts a trial solution of the form

$$z(t) = e^{-\beta t} u(t). \tag{6.86a}$$

The constant β and the function $u(t)$ are then chosen to eliminate the damping term from the transformed equation for $u(t)$. Substitution of (6.86a) into (6.83) yields

$$\ddot{u} + 2(\nu - \beta)\dot{u} + (\beta^2 - 2\nu\beta + p^2)u = 0.$$

We thus choose $\beta = \nu$ to remove the damping term; then $u(t)$ is given by the general solution of the homogeneous equation

$$\ddot{u} + r^2 u = 0, \tag{6.86b}$$

wherein, with the aid of (6.85),

$$r^2 \equiv p^2 - \nu^2 = p^2(1 - \zeta^2). \tag{6.86c}$$

Equation (6.86b) has the structure of equation (6.41) whose general solution for $r \neq 0$ is given in (6.43) in which r may be either real or complex. We use this result in (6.86a) to obtain the solution of (6.83) in the general form

$$z(t) = e^{-\nu t}(C_1 e^{irt} + C_2 e^{-irt}), \tag{6.86d}$$

in which $C_1,\ C_2$ are arbitrary constants. The role of the damping exponent ν is now clear. From (6.86c), there are three physical cases to consider: $\nu < p,\ \nu > p,\ \nu = p$. In the latter case, $r \equiv 0$ and we need only solve the equation $\ddot{u}(t) = 0$. We shall begin with the case for which $\nu < p$.

Case (i): Lightly damped motion. If $\zeta = \nu/p < 1$, then $r^2 > 0$ in (6.86c); hence (6.86b), with $r \equiv \omega > 0$, has the general solution $u(t) = A\cos\omega t + B\sin\omega t$, wherein

$$\omega \equiv p\sqrt{1 - \zeta^2} < p. \tag{6.86e}$$

Therefore, the general solution of (6.83) provided by (6.86a) is

$$z(t) = e^{-\nu t}(A \cos \omega t + B \sin \omega t), \tag{6.86f}$$

wherein A and B are real constants determined by the initial data.

The solution (6.86f) is oscillatory but not periodic. Because of the *damping factor* $e^{-\nu t}$, the oscillations decay in time so that $z \to 0$ as $t \to \infty$; but in its oscillatory motion the load returns again and again to the relative equilibrium state at $z = 0$. In fact, by (6.86f), if the mass passes through $z = 0$ in a given direction at time t_o, then at time $t = t_o + 2\pi/\omega$ it will pass $z = 0$ again in the same direction. The time $\tau = 2\pi/\omega$, therefore, is called the *period of the lightly damped motion*, and the constant ω defined in (6.86e) is named the *damped circular frequency*. Hence,

$$f_d \equiv \frac{1}{\tau} = \frac{\omega}{2\pi} \tag{6.86g}$$

defines the *frequency of the damped, free vibration*. Notice, however, that the motion itself in (6.86f) is not periodic, because $z(t + \tau) \neq z(t)$.

The lightly damped motion (6.86f) may also be visualized from its equivalent form

$$z(t) = z_0 e^{-\nu t} \cos(\omega t + \lambda) \overset{\text{or}}{=} z_0 e^{-\nu t} \sin(\omega t + \psi), \tag{6.86h}$$

in which z_0 and λ (or ψ) are integration constants. The graph of the first equation is illustrated in Fig. 6.21. The initial displacement is $z_0 \cos \lambda$. The initial phase

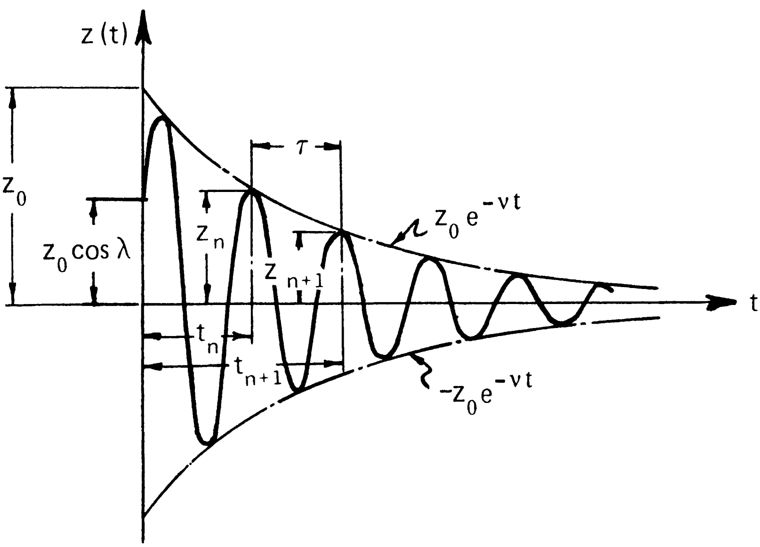

Figure 6.21. Graph of the motion of a lightly damped harmonic oscillator.

λ, however, may be chosen to adjust the time origin so that z_0 is the initial displacement. The damping factor $e^{-\nu t}$ reduces in time the amplitudes of successive oscillations; these occur in time τ. We see from (6.86e) that the *damped circular frequency ω is smaller than the circular frequency p for the undamped, simple harmonic case. Therefore, the effect of damping is to decrease the frequency of the oscillations compared with those of the undamped case.* However, if $\nu \ll p$, so that the damping is very slight, the term $e^{-\nu t}$ stays close to unity for large values of t, and (6.86f) models more precisely the actual physical behavior of the ideal simple harmonic oscillator.

An oscillographic recording of the motion in Fig. 6.21 may be obtained by experiment, and this graph can be used to determine the damping parameters from measurements of any two successive amplitudes at times t_n and $t_{n+1} = t_n + \tau$. Although the peak values of $z(t)$ do not quite touch the exponential envelope lines, they often are sufficiently close for practical experimental purposes. With (6.86h) and $z_n = z(t_n)$, we find $z_n/z_{n+1} = e^{\nu\tau}$. Thus, the natural logarithm of this ratio, called the *logarithmic decrement* Δ, determines ν and hence c in terms of measurable quantities:

$$\Delta \equiv \log \frac{z_n}{z_{n+1}} = \nu\tau. \tag{6.86i}$$

Therefore, with (6.86g), the damping exponent is determined by $\nu = f_d\Delta$, and (6.84) yields the damping coefficient $c = 2mf_d\Delta = 2m\Delta/\tau$. Alternatively, with the aid of (6.85) and (6.86e) in (6.86i), Δ may be written in terms of the viscous damping ratio ζ; we find $\Delta = 2\pi\nu/\omega = 2\pi\zeta/(1-\zeta^2)^{1/2}$. Then ζ may be expressed in terms of the frequency ratio $\omega/p = f_d/f$ or the logarithmic decrement Δ, which are measurable quantities, to obtain $\zeta = (1-(f_d/f)^2)^{1/2} = \Delta/(4\pi^2+\Delta^2)^{1/2}$.

It is useful to observe for the experimental situation that the damping parameters can be evaluated by use of data for any number of complete cycles in the oscillograph record in Fig. 6.21. Let z_1 and z_{n+1} denote the measured amplitudes at times t_1 and $t_1 + n\tau$, for integers $n = 1, 2, \ldots$. Then, in view of (6.86i) applied in turn to each n in the set just indicated,

$$\log\left(\frac{z_1}{z_{n+1}}\right) = \log\left(\frac{z_1}{z_2}\cdot\frac{z_2}{z_3}\cdot\frac{z_3}{z_4}\cdots\frac{z_n}{z_{n+1}}\right) = n\log\left(\frac{z_n}{z_{n+1}}\right) = n\Delta.$$

Therefore,

$$\Delta = \frac{1}{n}\log\left(\frac{z_1}{z_{n+1}}\right), \tag{6.86j}$$

which may be used to determine the damping parameters ν, c, and ζ, as shown above. This rule is particularly helpful in reducing experimental measurement error when recorded successive amplitudes are so close together that even small measurement errors in the amplitude and period will generate significant errors in data used to compute the damping parameters.

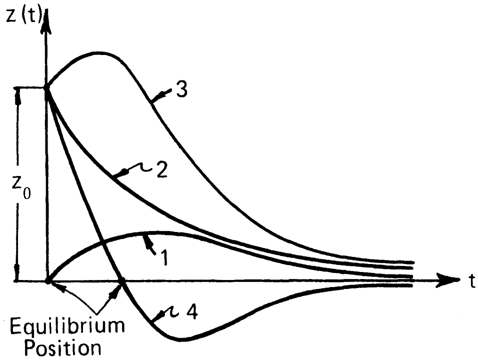

Figure 6.22. Graph of four typical motions of a heavily damped system.

Case (ii): Heavily damped motion. If $\zeta = \nu/p > 1$, then $r^2 = -q^2 < 0$ in (6.86c), where

$$q \equiv \sqrt{\nu^2 - p^2} = p\sqrt{\zeta^2 - 1} < \nu. \tag{6.86k}$$

Hence, with $r = \pm iq$ in (6.86d), the general solution of (6.83) in this case is

$$z(t) = e^{-\nu t}(Ae^{qt} + Be^{-qt}). \tag{6.86l}$$

The constants A and B are determined by the initial data. Equation (6.86l) may also be expressed in terms of hyperbolic functions.

This motion is not oscillatory. Since $q < \nu$, the damping factor $e^{-\nu t}$ is dominant; so, whatever initial conditions may be assigned, once the particle passes through its relative equilibrium position, if at all, it will never do so again. The unique null solution of (6.86l) is obtained in the time

$$t_o = \frac{\log(-B/A)}{2q}. \tag{6.86m}$$

The viscosity in a heavily damped system is so great that the load cannot vibrate about its relative equilibrium position; rather, it must creep slowly back to it as $t \to \infty$.

Some typical cases are shown in Fig. 6.22. Curve 1 occurs for the initial conditions $z(0) = 0$, $\dot{z}(0) = v_0$, from which $-B/A = 1$ and hence (6.86m) has only the trivial solution $t_o = 0$. This motion begins with a push away from the equilibrium position and the mass can never cross it again; for, $z \to 0$ again only as $t \to \infty$. Curve 2 in Fig. 6.22 illustrates the case $z(0) = z_0$, $\dot{z}(0) = 0$. For the general case $z(0) = z_0$, $\dot{z}(0) = v_0$, the motion may resemble either curve 2, 3, or 4. In the last instance, the load passes through its equilibrium position only once and then creeps gradually back to it from below. See Problem 6.62.

Case (iii): Critically damped motion. If $\zeta = \nu/p = 1$, the general solution of (6.86b) for which $r^2 = 0$ is $u = A + Bt$, where A and B are integration constants.

Thus, by (6.86a), the general solution of (6.83) for the critically damped motion is

$$z(t) = (A + Bt)e^{-\nu t}. \tag{6.86n}$$

As $t \to \infty$, the motion $z(t) \to 0$. The critically damped motion, therefore, is similar to that for the heavily damped model illustrated in Fig. 6.22. Discussion of the motion graphs is left for the reader in Problem 6.63.

From (6.85), the damping coefficient for this case has the value

$$c^* = 2mp = 2\sqrt{mk}, \tag{6.86o}$$

which is named the *critical damping coefficient*. This is the value of the damping coefficient at which the motion loses its oscillatory, lightly damped character in transition to a nonvibratory, heavily damped decaying motion. In view of (6.86o), the damping ratio (6.85) in the general case is the ratio of the damping coefficient to its critical value:

$$\zeta = \frac{c}{c^*} = \frac{\nu}{p}. \tag{6.86p}$$

In both the oscillatory lightly damped case $\zeta < 1$ and the nonoscillatory heavily damped case $\zeta > 1$, the load takes a longer time to come to rest than it does in the critically damped case $\zeta = 1$. This effect is illustrated by the familiar automatic storm-door closer. If the closer mechanism is adjusted to have light damping, the door will want to swing through its closed equilibrium position in an effort to oscillate, so the door will slam. If the closer is adjusted to have too much damping, the heavily damped door will close too slowly, perhaps not at all. The optimum case is when the closer is critically adjusted so that the door will close as quickly as possible, without slamming. Thus, the critical damping case $\zeta = 1$ describes the most efficient damping condition, because the motion is damped in the least time.

6.11.3. Summary of Solutions for the Damped, Free Vibrational Motion

For the damped, free vibrational motions, $z(t) \to 0$ as $t \to \infty$, so all of these motions eventually die out. To summarize, equation (6.83) for the damped, free vibrational motion of the load about its relative equilibrium position is characterized by three physical situations depending on the value of the viscous damping ratio $\zeta = \nu/p = c/2mp$:

- Lightly damped motion, $\zeta < 1$:

$$z(t) = e^{-\nu t}(A\cos\omega t + B\sin\omega t), \qquad \omega = p\sqrt{1-\zeta^2}. \qquad \text{(cf. 6.86f)}$$

- Heavily damped motion, $\zeta > 1$:

$$z(t) = e^{-\nu t}(Ae^{qt} + Be^{-qt}), \qquad q = p\sqrt{\zeta^2 - 1}. \qquad \text{(cf. 6.86l)}$$

- Critically damped motion, $\zeta = 1$:

$$z(t) = e^{-\nu t}(A + Bt). \qquad \text{(cf. 6.86n)}$$

The reader may explore the following additional elements.

Exercise 6.4. The usual solution method for linear equations with constant coefficients adopts a trial solution $z_T = Ae^{\lambda t}$. Find the characteristic equation for λ in order that (6.83) may be satisfied. Determine its roots, and thus show that the solution of (6.83) is given by (6.86d). □

Exercise 6.5. The method based on (6.86a) may be applied more generally in problems where the coefficients ν and p^2 in (6.83) are functions of time. Let $z(t) = u(t)e^{h(t)}$ and find $h(t)$ and $r^2(t)$ in order that (6.83) may be transformed to an equation of the form (6.86b) for the function $u(t)$. The solution $u(t)$ will now depend on the nature of the function $r(t)$; so, in general, $u(t)$ need not be a periodic function. □

6.12. Steady, Forced Vibrations with and without Damping

The oscillatory motion of a mechanical system subjected to a time varying external disturbing force is called a *forced vibration*. In this section, we investigate the forced vibration of the system in Fig. 6.20 due to a steady, sinusoidally varying disturbing force

$$F^*(t) = F_0 \sin \Omega t. \qquad (6.87)$$

The constant F_0 is the *force amplitude* and the constant circular frequency Ω is called the *forcing* or *driving frequency*.

The motion of a load induced by a time varying driving force of the kind (6.87) is known as a *steady, forced vibration;* otherwise, the response is called *unsteady* or *transient*. In general, a vibratory motion consists of identifiable steady and transient parts. The transient part of the motion eventually dies out, and the subsequent remaining part of the motion is called the *steady-state vibration*. A disturbing force that changes suddenly by a constant value, called a step function, and an impulsive exciting force which is suddenly applied for only a very short time, are examples of forces for which the response is transient. Some other examples are described in the problems. In the text, however, we shall explore only the steady, forced vibration problem for which the equation of motion (6.82) has the form

$$\ddot{x} + 2\nu\dot{x} + p^2x = Q \sin \Omega t, \qquad (6.88)$$

in which ν and p are defined in (6.84) and

$$Q \equiv \frac{F_0}{m}. \tag{6.89}$$

We recall that p is the free vibrational circular frequency of the undamped oscillator; it is the intrinsic frequency of the system. Therefore, for future clarity and brevity, p is called the *natural (circular) frequency*.

The general solution x_H of the homogeneous equation associated with (6.88) when $Q = 0$ is given by (6.65b) when $\nu = 0$, and by (6.86f), (6.86l), or (6.86n), according as $0 < \zeta = \nu/p < 1, \zeta > 1$, or $\zeta = 1$, respectively, as summarized earlier (page 157). Consequently, the general solution of (6.88) is obtained by adding to the appropriate homogeneous solution $x_H = e^{-\nu t}u(t)$ a particular solution x_P of (6.88) that gives the effect of the external force.

A particular solution of (6.88) may be obtained by the method of undetermined coefficients. Accordingly, we take $x_P = C_1 \sin \Omega t + C_2 \cos \Omega t$, where $\Omega \neq p$ is the forcing frequency and the constants C_1, C_2 are chosen to satisfy (6.88) identically. Substitution of x_P into (6.88) yields

$$\left[(p^2 - \Omega^2)C_1 - 2\nu\Omega C_2 - Q\right] \sin \Omega t + \left[2\nu\Omega C_1 + (p^2 - \Omega^2)C_2\right] \cos \Omega t = 0,$$

which holds identically for all t if and only if the coefficients vanish. This provides two equations for the constants C_1 and C_2, which yield

$$C_1 = \frac{X_S(1 - \xi^2)}{(1 - \xi^2)^2 + (2\xi\zeta)^2}, \qquad C_2 = \frac{-2X_S\xi\zeta}{(1 - \xi^2)^2 + (2\xi\zeta)^2}, \tag{6.90a}$$

wherein, by definition,

$$X_S \equiv \frac{Q}{p^2} = \frac{F_0}{k}, \qquad \xi \equiv \frac{\Omega}{p}, \tag{6.90b}$$

and ζ is the viscous damping ratio defined in (6.86p). Notice that X_S is the static deflection of the spring due to F_0, and ξ is the ratio of the forcing frequency to the natural frequency.

The general solution of (6.88) is the sum $x(t) = x_H + x_P$. This gives the forced vibrational motion

$$x(t) = e^{-\nu t}u(t) + C_1 \sin \Omega t + C_2 \cos \Omega t, \tag{6.90c}$$

provided that $\Omega \neq p$. The first term in (6.90c) is the transient part of the motion. It describes the damped, free vibrational part of the motion for which $u(t)$ is identified in (6.86f) for the lightly damped problem, in (6.86l) for the heavily damped case and in (6.86n) for the critically damped problem. In any event, the transient, damped part of the motion (6.90c) vanishes as $t \to \infty$, and the motion attains the *steady-state* simple harmonic form described by the last two terms. When $\nu = 0$, however, $u(t)$ is the simple harmonic solution of (6.68); and this part of the undamped, forced vibrational motion (6.90c) does not die out, it is not a transient motion. Nevertheless, the part of the undamped, forced vibrational motion described by the last two terms in (6.90c) is still named the steady-state part. Thus, in every case the effect of the sinusoidal driving force is to superimpose on the free, damped or undamped vibrational motion a simple harmonic motion whose frequency Ω equals that of the driving force (6.87) and whose steady-state amplitude, in accordance with (6.90a), is defined by

$$H \equiv \sqrt{C_1^2 + C_2^2} = \frac{X_S}{\sqrt{(1-\xi^2)^2 + (2\xi\zeta)^2}}. \tag{6.90d}$$

See (6.71). The steady-state amplitude is constant for a fixed value of Ω, hence ξ; but it grows larger as $\xi \to 1$, that is, as the forcing frequency Ω approaches the natural frequency p.

The foregoing results are used to study the ideal undamped, and the lightly, heavily, and critically damped vibration problems. We begin with the undamped case.

6.12.1. Undamped Forced Vibrational Motion

The equation for the undamped forced vibrational motion of the load is obtained from (6.88) with $\nu = p\zeta = 0$:

$$\ddot{x} + p^2 x = Q \sin \Omega t. \tag{6.91}$$

We recall (6.90a) and (6.90d) to obtain $(C_1, C_2) = (H, 0)$; then (6.90c), in which $u(t)$ is the simple harmonic solution of (6.68), yields the general solution of (6.91):

$$x(t) = A \cos pt + B \sin pt + H \sin \Omega t, \tag{6.92a}$$

where, with (6.90b),

$$H = \frac{Q}{p^2(1-\xi^2)} = \frac{F_0/k}{1-\xi^2} = \frac{X_S}{1-\xi^2}, \qquad \xi = \frac{\Omega}{p} \neq 1. \tag{6.92b}$$

The motion (6.92a) is the superposition of two distinct simple harmonic motions. The first two terms, which contain the two integration constants, represent an undamped, free vibration of circular frequency p. The third term is the steady-state, forced vibrational contribution; it depends on the driving force amplitude in (6.92b) but is independent of the initial data and has the same circular frequency Ω as the disturbing force. In general, the two motions have different amplitudes, frequencies, and phase. Therefore, their composition, and hence the motion, is not periodic unless the ratio $\xi = \Omega/p$ is a rational number, or unless A and B are zero. Thus, the undamped, forced vibrational motion (6.92a) usually is a complicated aperiodic motion.

Suppose, for example, that the system is given an initial displacement $x(0) = x_0$ and velocity $\dot{x}(0) = v_0$. Then (6.92a) yields $A = x_0$ and $B = (v_0 - H\Omega)/p$, and the undamped, forced vibrational motion is described by

$$x(t) = x_0 \cos pt + \frac{v_0}{p} \sin pt + H(\sin \Omega t - \xi \sin pt). \tag{6.92c}$$

Even if the system were started from its natural rest state so that $x_0 = v_0 = 0$, the solution $x(t) = H(\sin \Omega t - \xi \sin pt)$ still contains both free and forced vibration terms. This motion generally is not periodic. Suppose, however, that the initial data may be chosen so that $x_0 = 0$ and $v_0 = H\Omega$ for a fixed forcing frequency. Then $A = B = 0$ and the motion (6.92a) reduces to the steady-state, periodic motion $x(t) = H \sin \Omega t$.

The effects of damping and the critical case when $\xi = 1$ will be discussed momentarily. First, we consider an example that illustrates the application of these results to a mechanical system.

Example 6.15. The equation for the undamped, forced vibration of the pendulum device described in Fig. 6.19, page 150, is given in (6.81e). Solve this equation for the case when both the motion of the hinge support and the angular motion of the pendulum are small. Assume that the pendulum is released from rest at a small angle ϕ_0.

Solution. The differential equation (6.81e) describes a complicated nonlinear, undamped, forced vibrational motion of the pendulum. To simplify matters, we consider the case when the angular placement is sufficiently small that terms greater than first order in ϕ may be ignored. Then (6.81e) simplifies to

$$\ddot{\phi} + p^2 \phi = \frac{x_O \Omega^2}{\ell} \sin \Omega t, \tag{6.93a}$$

where $p^2 = g/\ell$. This equation has the same form as (6.91); it describes the small, undamped, steady forced vibrational motion of the pendulum. For consistency with the small motion assumption, however, we consider only the case for which the motion of the hinge support O also is small, so that $x_O/\ell \ll 1$. Because the amplitude of the disturbing force in (6.93a) varies with its frequency, for small motions $\phi(t)$, the range of operating frequencies also is limited.

The general solution of (6.93a), with $Q \equiv x_O \Omega^2/\ell$, may be read from (6.92a):

$$\phi(t) = A \cos pt + B \sin pt + H \sin \Omega t, \tag{6.93b}$$

in which A and B are constants and the steady-state amplitude, by (6.92b), is

$$H = \frac{x_O \xi^2}{\ell(1 - \xi^2)}, \qquad \xi \equiv \frac{\Omega}{p} \neq 1. \tag{6.93c}$$

The assigned initial data determine the constants in (6.93b),

$$\phi(0) = A = \phi_0, \qquad \dot{\phi}(0) = Bp + H\Omega = 0, \tag{6.93d}$$

which then yields the solution for the small angular motion of the pendulum:

$$\phi(t) = \phi_0 \cos pt + \frac{x_O \xi^2}{\ell(1-\xi^2)} (\sin \Omega t - \xi \sin pt)\,. \tag{6.93e}$$

It is evident that this small motion solution is meaningful only for sufficiently small values of the driving frequency ratio ξ; otherwise, the smallness of $\phi(t)$ is violated. □

6.12.1.1. *The Resonance Phenomenon*

As $\xi = \Omega/p \to 1$, the motion (6.92c) in response to the driving force grows increasingly larger, and at $\xi = 1$, its amplitude (6.92b) is infinite. The condition $\xi = \Omega/p = 1$ when the forcing frequency is tuned to the natural frequency of the system is known as *resonance*. It is useful to examine the solution for the undamped motion at the resonant frequency.

Let $x^*(t)$ denote the motion at the resonant frequency $\Omega = p$, and recall (6.92b). Then from (6.92c), we evaluate $x^*(t) = \lim_{\Omega \to p} x(t)$ to obtain

$$x^*(t) = (x_0 - Kt)\cos pt + \frac{1}{p}(v_0 + K)\sin pt, \qquad K \equiv \frac{p}{2} X_S = \frac{F_0}{2mp}.$$

This is not a steady-state motion; its amplitude increases continuously with time, so the vibrations grow increasingly larger. Although the condition of resonance does not occur instantaneously, the motion of the load may grow excessively and exceedingly large in a short time.

6.12.1.2. *Steady-State Amplitude Factors*

Two kinds of dimensionless *amplitude factors* arise often in forced vibration problems, both characterize the steady-state response of the system in terms of the frequency ratio. One of these amplitude factors, defined by

$$\alpha_0 \equiv \frac{1}{1-\xi^2}, \tag{6.94a}$$

called the *magnification factor*, appears in the steady-state amplitude relation (6.92b). The magnification factor is the ratio of the steady-state dynamic response amplitude H to the static amplitude X_s of the system, hence $\alpha_0 = H/X_s$ is a measure of the dynamic displacement compared to the static displacement of the load.

A different dimensionless amplitude factor, defined by

$$\alpha_1 \equiv \frac{\xi^2}{1-\xi^2}, \tag{6.94b}$$

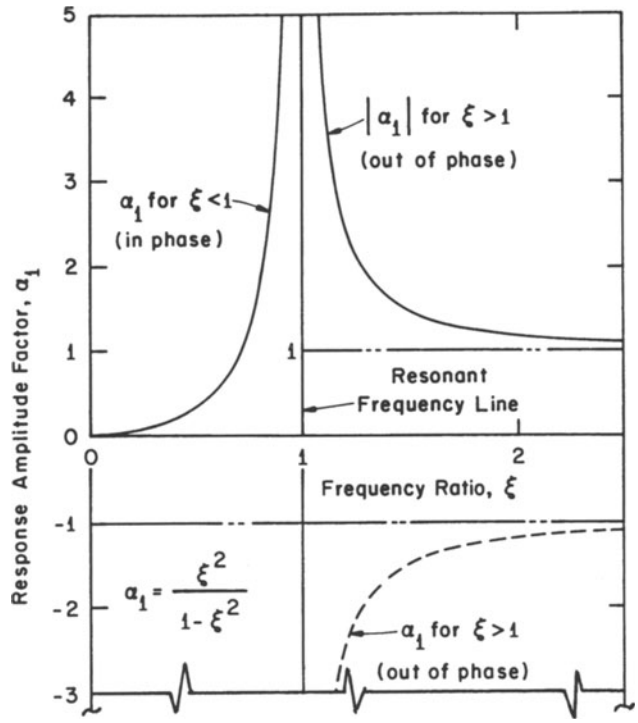

Figure 6.23. Response amplitude factor $\alpha_1(\xi)$ for steady-state forced vibrations without damping as a function of the system frequency ratio $\xi = \Omega/p$.

appears in the steady-state amplitude relation (6.93c) for the forced vibration of the pendulum. Since $\phi\ell$ describes the small horizontal motion of the pendulum bob, we see that $H\ell$ is its maximum value in the steady-state motion $\phi_\sigma \equiv H \sin \Omega t$. Thus, the amplitude factor in this case, according to (6.93c), is the ratio of the dynamical amplitude $H\ell$ of the bob to the amplitude x_O of the support; hence α_1 is a measure of the dynamical response of the system.

Graphs of the amplitude factors (6.94b) and (6.94a) are shown in Figs. 6.23 and 6.24, respectively. These *response graphs* are independent of the particular physical problems in which these amplitude factors may arise. The general physical relevance of (6.94b), however, is readily illustrated in connection with the driven pendulum example.

The map of (6.94b) is shown in Fig. 6.23. Accordingly, at small operating frequencies ξ, the amplitude factor α_1 also is small, both near zero. Thus, the influence of the vibrating support on the small amplitude oscillations of the pendulum is insignificant, and the motion in (6.93b) is essentially a simple harmonic motion of natural frequency p. Moreover, for $\xi < 1, \alpha_1 > 0$ and $H = x_O\alpha_1/\ell > 0$.

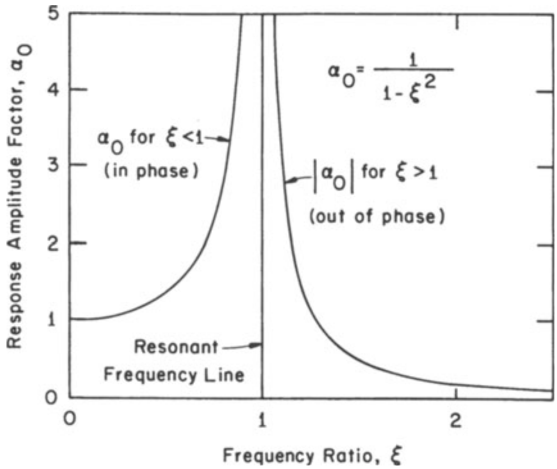

Figure 6.24. Response amplitude factor $\alpha_0(\xi)$ for steady-state forced vibrations without damping as a function of the system frequency ratio $\xi = \Omega/p$.

Therefore, the steady-state motion $\phi_\sigma \equiv H \sin \Omega t$ of the bob is in phase with the driving force (6.87); that is, the bob's motion is in the direction in which the support is moving. This is characterized by the solid left-hand curve in Fig. 6.23. At resonance, the forcing frequency is tuned to the natural frequency at $\xi = 1$, and therefore the amplitude factor (6.94b), and hence the amplitude of the pendulum motion, becomes infinite, as indicated by the vertical line in the response graph. But this is not an instantaneous effect, rather it indicates a growth in the amplitude in time, growth which eventually violates the small amplitude motion assumption used in the solution. When $\xi > 1$, the amplitude factor $\alpha_1 < 0$, and hence $H = x_O \alpha_1 / \ell < 0$ also. Thus, the steady-state response of the pendulum, the part $\phi_\sigma = H \sin \Omega t = |H| \sin(\Omega t \pm \pi)$, is simple harmonic and 180° out of phase with the driving force (6.87); that is, the bob's motion is opposite to the direction in which the support is moving. This case is represented by the dotted response curve in Fig. 6.23. At high operating frequencies for which $\xi \gg 1, \alpha_1 \to -1$; that is, $H \to -x_O/\ell$. Because $x_O \ll \ell$, the high frequency, steady-state dynamical amplitude of the pendulum swing will be small, and the steady-state pendulum motion (6.93b) is a high frequency, simple harmonic vibration, but 180° out of phase with the motion of the support. For graphical convenience, it is customary to plot the absolute value of the amplitude factor. When this is done for α_1, the dotted curve in Fig. 6.23 is transformed into its mirror reflection shown as the solid right-hand curve above it.

Interpretation of the general physical relevance of the magnification factor (6.94a) in its relation to the response graph shown in Fig. 6.24 is a bit different. In accordance with (6.92b), for a small operating frequency the magnification factor

$\alpha_0 \approx 1$, as shown in Fig. 6.24. This means that the steady-state motion of the mass shown in Fig. 6.20 has an amplitude equal to the static displacement of the spring due to a force F_0. The motion is in phase with the driving force, so the mass moves in the direction of this force. As $\xi \to 1$ at resonance, the amplitude grows indefinitely great, as described earlier. Beyond resonance $\xi > 1$; so, the steady-state motion in (6.92a) is out of phase with the driving force, and hence the mass in Fig. 6.20 moves in a direction opposite to the disturbing force. Under a high frequency driving force for which $\xi \to \infty$ in Fig. 6.24, the steady-state amplitude response $\alpha_0(\xi) \to 0$, and hence the steady-state amplitude in (6.92b) approaches zero. Therefore, the high frequency vibration of the supporting structure has virtually no effect on the motion of the system, and the mass in Fig. 6.20 remains essentially stationary.

Of course, some sort of damping or friction is always present in real mechanical systems. Damping effects in the forced vibration of a load are studied next.

6.12.2. Steady-State Vibrational Response of a Damped System

When damping is present, the free vibrational part of the motion, the first term in (6.90c) called the *transient state*, eventually dies out, and the vibrational motion thus converges toward a harmonic motion having the same frequency as the disturbing force, the steady-state heartbeat of the system. In consequence, only the steady-state part of the motion (6.90c) of a damped system need be considered.

Let x_σ denote the steady-state solution. Then by (6.90c)

$$x_\sigma = H \sin(\Omega t - \lambda), \tag{6.95a}$$

where H is defined in (6.90d) and, from (6.90a), the initial phase λ is given by

$$\tan \lambda = -\frac{C_2}{C_1} = \frac{2\xi\zeta}{1-\xi^2}. \tag{6.95b}$$

Clearly, for $\xi = 1$, $\lambda = 90°$ at resonance; and in this case, when $\Omega t = \pi/2$, $F^* = F_0$ in (6.87) and $x_\sigma = 0$ in (6.95a). Hence, at resonance, the vibrating body in Fig. 6.20 is moving through its mid position in its steady-state motion at the same instant when the driving force is at its greatest value. Notice that the *response amplitude H* in (6.90d) *does not depend on any initial data.* Thus, regardless of how the system may be set into motion initially, after a time, it settles down to the steady-state motion (6.95a) whose amplitude (6.90d) and phase (6.95b) depend upon the damping and frequency ratios.

The *amplitude factor* defined by

$$\alpha \equiv \frac{1}{\sqrt{(1-\xi^2)^2 + (2\xi\zeta)^2}} = \frac{H}{X_s} \tag{6.95c}$$

is a measure of the dynamic response; it is the ratio of the dynamic amplitude H to the static spring deflection X_s of the load due to the maximum disturbing force

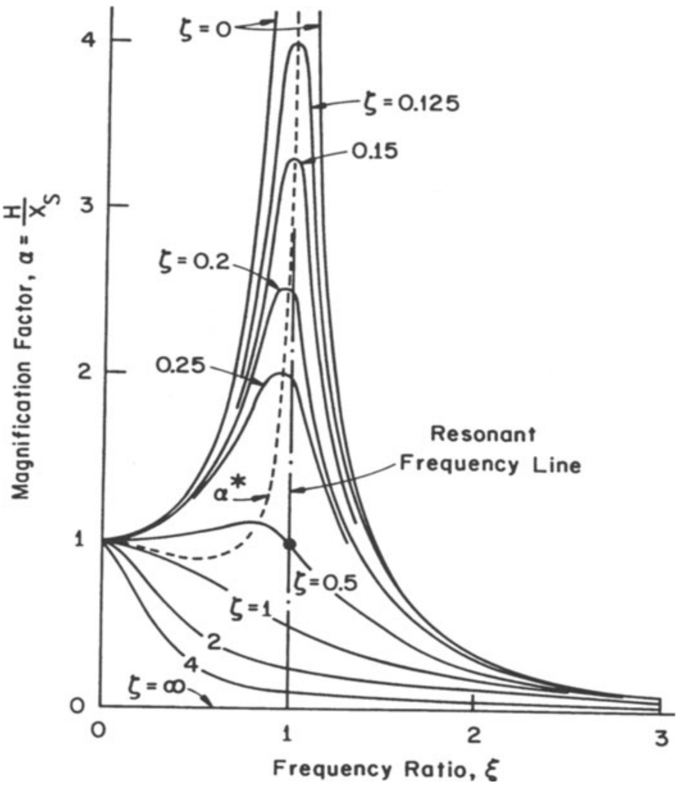

Figure 6.25. Magnification factor as a function of the frequency ratio ξ for various values of the damping parameter ζ in a forced vibration of a system.

F_0. Notice that when $\zeta = 0$, $\alpha = |\alpha_0|$ in (6.94a), and hence α is also known as the *magnification factor*. The *response curves* corresponding to (6.95c) for various values of the damping ratio are shown in Fig. 6.25. The curve for $\zeta = 0$ is the same as the plot of $|\alpha_0|$ in Fig. 6.24.

At low frequencies, $\xi = \Omega/p$ is very close to zero, and (6.95c) shows that α is very nearly equal to 1 in Fig. 6.25. In this case, the disturbing force has such a low frequency Ω in comparison with the undamped natural frequency p that it behaves very nearly as a static dead load; hence H is nearly the same as the static response to the disturbing force: $H = X_s = F_0/k$, very nearly. Notice by (6.95c) that for $\xi = 1$, the curve for $\zeta = \frac{1}{2}$ yields $\alpha = 1$. This is the emphasized point on the resonance line $\xi = 1$ in Fig. 6.25.

At high frequencies, $\xi \gg 1$, and (6.95c) shows that the dynamic response amplitude H becomes very small with α and approaches zero as $\xi \to \infty$ in Fig. 6.25. The frequency of the disturbing force in this instance changes so rapidly that the mass cannot respond but slightly, though at the same frequency, in accordance with (6.95a). Figure 6.25 thus shows that for very small or very large values of ξ, the effect of any sort of damping is insignificant.

At the resonant frequency, the forcing frequency Ω is tuned to the natural frequency p so that $\xi = 1$. Then (6.95c) gives $H = X_s/2\zeta = F_0/2\zeta k$, (6.95b) yields $\lambda = \pi/2$ for the angle by which the disturbing force F^* in (6.87) leads the steady-state motion x_σ in (6.95a), which becomes

$$x_\sigma = -\frac{F_0}{2\zeta k} \cos pt. \tag{6.95d}$$

Hence, if the damping ratio is small, the amplitude of the steady motion may become seriously large when Ω is close to p. Resonance in the undamped system corresponds to $\zeta = 0$ in Fig. 6.25. *The effect of damping is to reduce the response amplitude, and at the resonant frequency ratio $\xi = 1$ the reduction may be especially significant.* Thus, the intensity of the resonant motion may be substantially reduced by the introduction of damping in the system.

The peak magnification in the damped motion, however, does not occur at $\xi = 1$. For fixed values of ζ and p, the maximum magnification occurs when ξ has the value

$$\xi^* \equiv \sqrt{1 - 2\zeta^2}. \tag{6.95e}$$

This is known as the *damped resonant frequency ratio* and $\Omega^* = p\xi^*$ is called the *damped resonant forcing frequency*. From (6.95e), *the peak frequency Ω^* occurs at a ratio ξ^* which is somewhat smaller than* 1, *depending upon the degree of damping.*

At the damped resonant frequency ratio ξ^*, the maximum dynamic amplitude is H^* and the magnification factor (6.95c) has the maximum value

$$\alpha^* = \frac{1}{2\zeta\sqrt{1 - \zeta^2}} \equiv \frac{H^*}{X_S}, \tag{6.95f}$$

which depends on the damping ratio. The locus of these maxima, indicated by the dotted curve in Fig. 6.25, shows that the peak value α^* increases as the damping ratio ζ decreases. Since ζ usually is much less than 1, (6.95e) shows that $\xi^* = 1$; that is, the value of the lightly damped resonant forcing frequency Ω^* differs very little from the undamped, free vibrational frequency p of the system. In this case, from (6.95f), the maximum dynamic amplitude at the damped resonant frequency is $H^* \equiv X_s\alpha^* = X_s/2\zeta$, very nearly. For small damping the amplitude is greatest near the resonant frequency ratio $\xi = 1$. As ζ increases, α^* decreases and shifts toward the left until it reaches $\alpha^* = 1$ at $\xi^* = 0$ for $\zeta = \sqrt{2}/2$. Afterwards, the peak $\alpha^* = 1$ is a relative maximum value for all $\zeta > \sqrt{2}/2$, and (6.95f) is no longer applicable.

6.12.3. Force Transmissibility in a Damped System

The vibrating load in its steady-state obviously transmits force to the supporting structure of the system. Therefore, it is important to have a measure of the intensity of this force. In this section, a certain transmissibility factor is introduced, and effects due to variation in the damping and in the operating frequency are discussed.

In the steady-state motion (6.95a), the spring and damping forces for the mechanical system in Fig. 6.20 are given by

$$F_S = kx_\sigma = kH\sin(\Omega t - \lambda), \qquad F_D = c\dot{x}_\sigma = c\Omega H\cos(\Omega t - \lambda), \quad (6.96a)$$

whose amplitudes are $\hat{F}_S = kH$ and $\hat{F}_D = cH\Omega$. Each force in (6.96a) contributes to the *total force transmitted to the support*: $F_S + F_D = F_T \sin(\Omega t - \lambda + \psi)$ where $\tan\psi \equiv c\Omega/k$ and the maximum impressed force, denoted by F_T, is defined by

$$F_T \equiv \sqrt{\hat{F}_S^2 + \hat{F}_D^2} = H\sqrt{k^2 + c^2\Omega^2}. \quad (6.96b)$$

Then the ratio of the total impressed force to the maximum value of the disturbing force $F_0 = kX_s$ defines the *transmission ratio* T_R, also known as the *transmission factor* or the *transmissibility*. Thus, with (6.95c), we find the *transmission ratio*

$$T_R = \frac{F_T}{F_0} = \sqrt{\frac{1 + (2\xi\zeta)^2}{(1 - \xi^2)^2 + (2\xi\zeta)^2}}. \quad (6.96c)$$

The graph of the transmission ratio as a function of the frequency ratio $\xi = \Omega/p$ for various values of the damping ratio $\zeta = c/2mp$ is shown in Fig. 6.26. The greatest transmission to the supporting structure for small damping occurs at resonance, and the effect of increased damping is to decrease the amplitude of the transmission and shift it toward the left of the resonant frequency line $\xi = 1$. Notice, however, that a transmission factor $T_R = 1$ occurs at a universal frequency ratio $\xi = \sqrt{2}$ (shown as the small circle in Fig. 6.26), regardless of the amount of damping. For $\xi > \sqrt{2}$, the transmission ratio $T_R < 1$, and hence the transmitted force is smaller than the applied disturbing force. Moreover, the transmission ratio actually is made smaller by decreasing the amount of damping at high operating frequencies. Therefore, less vibrational force is transmitted to the supporting structure. As a result, smoother operation may be expected. At very low operating frequencies, the transmissibility is again close to 1 for all values of the damping. Otherwise, Fig. 6.26 shows that increasing the amount of damping ζ when $0 < \xi < \sqrt{2}$ decreases the maximum transmitted force. In summary, *if* $\xi < \sqrt{2}$, $T_R > 1$ *and greater damping is recommended for smoother operation of the system; however, when* $\xi > \sqrt{2}$, $T_R < 1$ *and decreased damping will result in smoother operation, that is, the effect of the transmitted force intensity is reduced.*

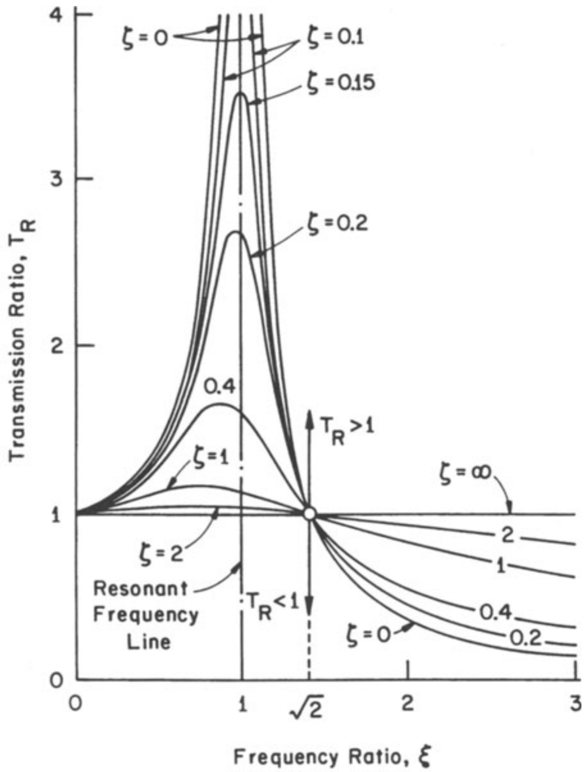

Figure 6.26. Transmissibility as a function of the frequency ratio ξ for various values of the damping ratio ζ in the forced vibration of a system.

For $\xi = \sqrt{2}$, $T_R = 1$ *for every damped (linear) mechanical system.* Mechanical design with these ideas in mind is known as *vibration isolation.*

6.13. Motion under a General Nonlinear Force $f(x, \dot{x})$

So far, we have considered free and forced vibrations of damped and undamped systems subjected to forces that are linear in x and $\dot{x}$. Here we study the motion $x(t)$ of a particle under a general nonlinear force $f = f(x, \dot{x})$ per unit mass. This total force may include inertial forces as well as other sorts of linear and nonlinear contact and body forces. The equation of motion is

$$\ddot{x} = f(x, \dot{x}). \tag{6.97}$$

Although exact solutions of such equations can be obtained, this is not always possible, and the analysis of (6.97) often is difficult. Some readily integrable situations arise when $f(x, \dot{x})$ has special properties. The reader will see easily, for example, that for a nonlinear force of the form $f(x, \dot{x}) = g(x)h(\dot{x})$ for smooth functions $g(x)$ and $h(\dot{x})$, the equation of motion (6.97) has the first integral $\int h^{-1}(\dot{x})d\dot{x}^2 = 2\int g(x)dx + C$, where C is a constant. Another example follows.

6.13.1. Special Class of Nonlinear Equations of Motion

A variety of dynamical systems are characterized by an integrable nonlinear equation of motion (6.97) of the form

$$q(x)\ddot{x} + \frac{1}{2}\dot{x}^2\frac{dq(x)}{dx} = g(x), \tag{6.98a}$$

for any smooth functions $q(x)$ and $g(x)$. This equation may be written as

$$\frac{d}{dx}\left[\frac{1}{2}\dot{x}^2 q(x)\right] = g(x), \tag{6.98b}$$

which is twice integrable. We first derive

$$\dot{x}^2 q(x) = 2\int g(x)dx + C \equiv p(x), \tag{6.98c}$$

where C is a constant, and thus obtain the velocity function

$$v(x) = \dot{x}(x) = \pm\sqrt{\frac{p(x)}{q(x)}}. \tag{6.98d}$$

A second integration yields the travel time in the motion:

$$t = \pm\int\sqrt{\frac{q(x)}{p(x)}}dx + t_0, \tag{6.98e}$$

t_0 denoting the initial instant. In principle, this determines the nonlinear motion $x(t)$; then $v(t)$ can be found from (6.98d). The inversion of (6.98e), however, may require numerical integration. Two explicit examples are provided in the following exercises.

Exercise 6.6. The motion of a particle free to slide on a smooth parabolic wire $y = \frac{1}{2}kx^2$ that rotates about its vertical y-axis with a constant angular speed is described by the nonlinear equation

$$(1 + k^2x^2)\ddot{x} + \Omega x + k^2 x\dot{x}^2 = 0,$$

where k and Ω are constants. Derive a first integral for $\dot{x}(x)$. □

Exercise 6.7. The motion of a dynamical system is governed by the equation

$$(h^2 + r^2\theta^2)\ddot{\theta} + r^2\theta\dot{\theta} + kr\theta\cos\theta = 0,$$

where h, k, and r are constants. The system is initially at rest at $\theta = 0$. Derive an integral giving the travel time in the motion. □

6.13.2. Radial Oscillations of an Incompressible Rubber Tube

Nonlinear equations of the type (6.98a) arise often in physical problems. An important example in nonlinear elasticity theory, discovered by J. K. Knowles in 1960, concerns the finite amplitude, free radial oscillations of a very long cylindrical tube made of an incompressible, rubberlike material. The tube has an inner radius r_1 and outer radius r_2 in its undeformed state and is initially inflated uniformly by an internal pressure. A purely radial motion of the tube is induced by its sudden deflation, so that the radial motion of any concentric cylindrical material surface of radius R in the deformed state at time t is described by $R = R(r, t)$, where r is the radius of the corresponding undeformed cylindrical material surface. Let R_1, R_2 respectively denote the inner and outer radii of the deformed tube surfaces at time t. Because of the incompressibility of the material, these radii are related by $R^2 - R_1^2 = r^2 - r_1^2$. Hence, the motion is determined completely if $R_1(t)$ is known. It proves convenient to introduce the dimensionless ratios

$$x(t) \equiv \frac{R_1(t)}{r_1}, \qquad \mu \equiv \frac{r_2^2}{r_1^2} - 1. \tag{6.99a}$$

Knowles found for arbitrary rubberlike materials that the free radial motion of the tube is described by the nonlinear differential equation

$$x \log\left(1 + \frac{\mu}{x^2}\right)\ddot{x} + \left(\log\left(1 + \frac{\mu}{x^2}\right) - \frac{\mu}{\mu + x^2}\right)\dot{x}^2 + h(x, \mu) = 0, \tag{6.99b}$$

where $h(x, \mu)$ is a known function that depends on the constitutive character of the rubberlike material. Notice that while this problem concerns the motion of a highly deformable body, the equation of motion actually involves only the motion of a particle on the inner surface of the tube. All other particles on the inner surface have the same radial motion.

At first glance, equation (6.99b) certainly appears formidable. Upon multiplication by x, however, it is seen that (6.99b) assumes the form (6.98a) and may be written as

$$\frac{d}{dx}\left(\frac{1}{2}\dot{x}^2 x^2 \log\left(1 + \frac{\mu}{x^2}\right)\right) + xh(x, \mu) = 0. \tag{6.99c}$$

This yields the first integral

$$\dot{x}^2 x^2 \log\left(1 + \frac{\mu}{x^2}\right) = -2\int xh(x, \mu)dx + C \equiv p(x). \tag{6.99d}$$

The integration constant C depends on the specified initial data $x(0) = x_0, \dot{x}(0) = v_0$. Equation (6.99d) thus determines the radial "velocity" function

$$v(x) = \dot{x}(x) = \pm \sqrt{\frac{p(x)}{x^2 \log\left(1 + \frac{\mu}{x^2}\right)}}, \tag{6.99e}$$

in which $\dot{x} = \dot{R}_1/r_1$, hence $[v(x)] = [T^{-1}]$. The analytical properties of the function $p(x)$ show that the phase plane curves described by (6.99e) are closed and that (6.99e) yields exactly two values $x = a$, $x = b > a$ for which $v(a) = v(b) = 0$; so the motion is periodic. See the referenced paper by Knowles for details.

Integration of (6.99e), with the appropriate sign chosen to render $t > 0$, yields the travel time

$$t = \int_{x_0}^{x} \frac{dx}{v(x)}. \tag{6.99f}$$

The finite periodic time τ of the purely radial oscillations of the tube, the time required for the tube to pulsate from $x = a$ to $x = b$ and back again, is thus determined by the formula

$$\tau = 2\int_{a}^{b} \frac{dx}{v(x)}. \tag{6.99g}$$

It turns out that the exact solution of (6.99g) may be obtained for special kinds of rubberlike materials. Without getting into these matters, however, we see that these general results are useful because they provide physical insight into what is otherwise a very difficult dynamical problem. Some additional simpler examples may be found in the problems at the end of this and subsequent chapters. (See Problems 6.68 and 6.69.) Similar ideas are applied in Chapter 7 to determine exactly the motion and period of the finite amplitude oscillations of a pendulum.

6.14. Infinitesimal Stability of the Relative Equilibrium States of a System

In other problems for which the exact solution of (6.97) is not possible, a variety of analytical and graphical methods described in other works may be used to construct an approximate solution or to study various physical aspects of the motion of the dynamical system. An important physical attribute of particular interest is the infinitesimal stability of the relative equilibrium states of a dynamical system governed by (6.97).

Relative equilibrium solutions of (6.97), if any exist, are the time independent solutions x_E of the equation

$$f(x_E, 0) = 0. \tag{6.100}$$

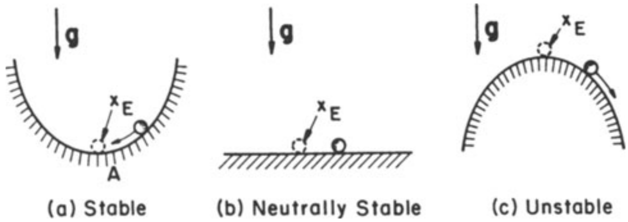

Figure 6.27. Schematic illustrating the concepts of (a) infinitesimal stability, (b) neutral stability, and (c) instability.

This provides the positions x_E at which the mass is at relative rest. In the special case when f is linear in x, there is only one equilibrium solution of (6.100), but for nonlinear systems there may be many relative equilibrium positions. In particular, if $f(x_E, 0)$ is a polynomial in x_E, there are as many equilibrium positions as there are real roots of (6.100); but some of these may not be stable.

The question of how the system behaves if disturbed only slightly from a relative equilibrium position is of special interest. If the body either returns eventually to the relative equilibrium position x_E, or oscillates about x_E so that its motion always remains in a small neighborhood of x_E, the relative equilibrium position is said to be *infinitesimally stable*, or briefly, *stable*. For greater clarity, the term *asymptotically stable* is also used to characterize the relative equilibrium position in the case when the body returns eventually to this state. If the body, following its arbitrary small disturbance from an equilibrium position, remains at a fixed small distance from the relative equilibrium position, the equilibrium state is called *neutrally stable*. On the other hand, if the body moves away indefinitely from x_E, the relative equilibrium state is called *unstable*. These three situations are illustrated in Fig. 6.27 for the small disturbance of a heavy particle from its equilibrium position x_E. The particle will perform small oscillations indefinitely about the equilibrium state at the lowest point of the bowl in Fig. 6.27a, and hence this state is infinitesimally stable. Now suppose the bowl contains water, then the oscillations eventually will die out as the heavy particle settles down to x_E; in this instance x_E is asymptotically stable. If the particle is given a small displacement from x_E on the horizontal plane surface in Fig. 6.27b and released from rest, it will remain there; therefore, the equilibrium state x_E is neutrally stable. Finally, in Fig. 6.27c, if the particle is disturbed only very slightly from its equilibrium position at the vertex of the inverted bowl, it will move away indefinitely from x_E, so this position is unstable.

To investigate the motion in the neighborhood of a relative equilibrium position x_E, we write

$$x(t) = x_E + \chi(t), \tag{6.101}$$

where $\chi(t)$ is a small disturbance from x_E, compatible with any constraints on x, so that $\dot{x} = \dot{\chi}$ also is a small quantity of the same order. The function $f(x, \dot{x})$ is then expanded in a Taylor series about x_E to obtain to the second order in χ and $\dot{\chi}$,

$$f(x, \dot{x}) = f(x_E, 0) + \left.\frac{\partial f(x, \dot{x})}{\partial x}\right|_{x_E} \chi + \left.\frac{\partial f(x, \dot{x})}{\partial \dot{x}}\right|_{x_E} \dot{\chi} + O(\chi^2, \dot{\chi}^2).$$

Thus, recalling (6.100) and (6.101), introducing

$$\alpha \equiv -\left.\frac{\partial f(x, \dot{x})}{\partial \dot{x}}\right|_{x_E}, \qquad \beta \equiv -\left.\frac{\partial f(x, \dot{x})}{\partial x}\right|_{x_E}, \tag{6.102}$$

and neglecting all terms of order greater than the first in χ and $\dot{\chi}$, we thus obtain from (6.97) the *linearized differential equation of motion of the body about the relative equilibrium position* x_E:

$$\ddot{\chi} + \alpha\dot{\chi} + \beta\chi = 0. \tag{6.103}$$

The relative equilibrium position will be stable if and only if the solution $\chi(t)$ of this equation remains bounded for all time t or approaches zero as $t \to \infty$. Otherwise, the initial infinitesimal displacement grows with time and eventually violates the smallness assumptions leading to (6.103); so, the position x_E is unstable.

We recognize that (6.103) is similar to (6.83) for the damped, free vibrations of a body about its relative equilibrium state. Here, however, the constant coefficients obtained from (6.102) are arbitrary; they may be positive, negative, or zero, so all possible solutions of (6.103) must be examined. The usual trial solution $\chi_T = Ae^{\lambda t}$ of (6.103) yields the characteristic equation

$$\lambda^2 + \alpha\lambda + \beta = 0, \tag{6.104a}$$

which has the two solutions

$$\lambda_1 = -\frac{\alpha}{2} + \sqrt{\left(\frac{\alpha}{2}\right)^2 - \beta}, \qquad \lambda_2 = -\frac{\alpha}{2} - \sqrt{\left(\frac{\alpha}{2}\right)^2 - \beta}. \tag{6.104b}$$

Therefore, the general solution of (6.103) is

$$\chi(t) = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t}, \tag{6.104c}$$

in which A_1, A_2 are arbitrary constants. The physical nature of the solution, and hence the stability of the relative equilibrium positions, is characterized by the signs of α and β, which determine the roots λ_1 and λ_2. There are several cases to explore.

1. Roots λ_1, λ_2 are real and negative. Then (6.104c) shows that $\chi(t) \to 0$ as $t \to \infty$. Therefore, the equilibrium position is asymptotically stable. For real roots (6.104b), $(\alpha/2)^2 > \beta$ must hold. Moreover, $\alpha > 0$ is necessary for a negative root λ_1. If $\beta = 0$ or $\beta < 0$, λ_1 will be non-negative, contrary to the initial requirement. Consequently, it is necessary and sufficient

that $\alpha > 0$, $(\alpha/2)^2 > \beta > 0$ hold. Hence, $\alpha > 0$, $\beta > 0$ in (6.104a) imply asymptotic stability.

2. Roots λ_1, λ_2 are real and positive. Then $\chi(t) \to \infty$ as $t \to \infty$ in (6.104c), and hence the equilibrium position is unstable. Real roots require $(\alpha/2)^2 > \beta$. For $\lambda_2 > 0$, $\alpha < 0$ is necessary, and hence $\beta = 0$ or $\beta < 0$ cannot hold. Therefore, it is necessary and sufficient that $\alpha < 0$, $(\alpha/2)^2 > \beta > 0$ hold. Thus, $\alpha < 0$, $\beta > 0$ in (6.104a) imply instability.
3. Roots $\lambda_1 > 0, \lambda_2 < 0$, or conversely. The second term in (6.104c) $\to 0$ and the first $\to \infty$, or conversely; so the relative equilibrium position is unstable. Real roots require $(\alpha/2)^2 > \beta$; and $\beta \neq 0$, otherwise $\lambda_1 = 0$. Case 1 and Case 2 show that $\beta > 0$, $\alpha > 0$ and $\beta > 0$, $\alpha < 0$ cannot satisfy the assigned conditions. Therefore, $\beta < 0$ must hold, and the conditions on λ_1, λ_2 are then satisfied for all real α. So, $\beta < 0$, α arbitrary imply instability.
4. Roots λ_1, λ_2 are complex conjugates. Now $\beta > (\alpha/2)^2 > 0$ must hold and (6.104c) may be written as

$$\chi(t) = e^{-\alpha t/2}\left(A_1 e^{irt} + A_2 e^{-irt}\right), \tag{6.104d}$$

where $r = (\beta - (\alpha/2)^2)^{1/2}$ is real and positive. If $\alpha > 0$, we have Case 1: $\alpha > 0$, $\beta > 0$, and hence the equilibrium position is asymptotically stable. Notice that $\chi(t) \to 0$ as $t \to \infty$. If $\alpha = 0$, the motion (6.104d) is simple harmonic, and hence the relative equilibrium position is infinitesimally stable. Finally, when $\alpha < 0$, we have Case 2: $\alpha < 0$, $\beta > 0$, and $\chi(t) \to \infty$ with t. The equilibrium state is unstable.
5. For $\beta = 0$, (6.104b) yields $\lambda_1 = 0$, $\lambda_2 = -\alpha$, and hence the motion is given by

$$\chi(t) = A_1 + A_2 e^{-\alpha t}. \tag{6.104e}$$

When $\alpha > 0$, $\chi \to A_1$ as $t \to \infty$; the equilibrium position is neutrally stable. When $\alpha < 0$, $\chi(t) \to \infty$ with t, and the equilibrium position is unstable. The degenerate case when $\alpha = 0$ also yields $\ddot{\chi} = 0$ in (6.103); so $\chi(t) = A_1 + A_2 t$. The equilibrium state is again unstable.

In summary, for all real or complex characteristic roots (6.104b), the infinitesimal stability of the relative equilibrium states is characterized by the following four circumstances expressed in terms of the *infinitesimal stability parameters* α and β, the coefficients (6.102) of the linearized equation of motion (6.103). *The relative equilibrium position is*

$$\left.\begin{array}{l}\text{(a) } \textit{infinitesimally stable when } \alpha = 0,\ \beta > 0,\\ \text{(b) } \textit{asymptotically stable for } \alpha > 0,\ \beta > 0,\\ \text{(c) } \textit{neutrally stable for } \alpha > 0,\ \beta = 0,\\ \text{(d) } \textit{unstable for all remaining cases.}\end{array}\right\} \tag{6.105}$$

These results also may be conveniently arranged in a matrix shown below.

$\alpha \downarrow \mid \beta \rightarrow$	>0	$=0$	<0
>0	A	N	U
$=0$	S	U	U
<0	U	U	U

Notice that the system is always unstable when either α or β is negative.

6.14.1. Stability of the Equilibrium Positions of a Pendulum

We now investigate the infinitesimal stability of the (relative) equilibrium positions of a simple pendulum whose finite angular motion is described by (6.67b):

$$\ddot{\theta} = -p^2 \sin\theta. \tag{6.106a}$$

This has the form of (6.97) in which $f(\theta, \dot{\theta}) = -p^2 \sin\theta$ is independent of $\dot{\theta}$. Hence, by (6.102), $\alpha \equiv 0$ and $\beta = p^2 \cos\theta_E$ at an equilibrium position θ_E. Thus, from (a) in (6.105), θ_E is a stable equilibrium position if and only if $\beta > 0$.

The (relative) equilibrium states, by (6.100), are given by

$$f(\theta_E) = -p^2 \sin\theta_E = 0, \tag{6.106b}$$

also evident from (6.106a). This yields infinitely many equilibrium positions $\theta_E = \pm n\pi$, $n = 0, 1, 2, \ldots$. But only two, $\theta_E = 0, \pi$, are physically distinct positions. For $\theta_E = 0$, $\beta = p^2 > 0$, and for $\theta_E = \pi$, $\beta = -p^2 < 0$. Hence, $\theta_E = 0$ is an infinitesimally stable equilibrium position, whereas $\theta_E = \pi$ is unstable.

To see this somewhat differently, recall (6.101), write $\theta(t) = \theta_E + \chi(t)$, and then linearize equation (6.106a) to obtain

$$\ddot{\chi} + (p^2 \cos\theta_E)\chi = 0. \tag{6.106c}$$

This corresponds to the linearized equation (6.103). Specifically, then

$$\ddot{\chi} + p^2\chi = 0 \text{ for } \theta_E = 0, \qquad \ddot{\chi} - p^2\chi = 0 \text{ for } \theta_E = \pi. \tag{6.106d}$$

We know that the first of (6.106d) yields a stable simple harmonic solution for any given initial data, whereas the second yields a solution that grows exponentially with time. Hence, we again conclude that $\theta_E = 0$ is an infinitesimally stable equilibrium position, while $\theta_E = \pi$ is unstable.

The physical nature of the results is evident. Any small disturbance of the pendulum bob from its lowest point at $\theta_E = 0$ results in a small oscillation about this equilibrium position. Any infinitesimally small disturbance from its extreme vertical position $\theta_E = \pi$, on the other hand, grows increasingly larger and quickly violates the smallness assumption leading to (6.106c).

6.14.2. Application to Linear Oscillators

The foregoing discussion has focused on infinitesimal stability for nonlinear problems in the class defined by (6.97), but the same infinitesimal perturbation procedure can be applied to all sorts of dynamical systems, including problems in which $f(x, \dot{x})$ is linear in either one or both variables x and $\dot{x}$. For illustration, let us reexamine the stability of the equilibrium positions of the rotating spring-mass system studied in Section 6.9.2, page 145. Equation (6.77b) gives the equation of motion in the form (6.97): $\ddot{x} = f(x, \dot{x}) = ap^2 - p^2(1 - \eta^2)x$, independent of $\dot{x}$ and linear in x. Equation (6.100) yields the evident equilibrium position $x_E = a/(1 - \eta^2)$, the same as (6.77g). The infinitesimal stability parameters in (6.102) are $\alpha \equiv 0$, $\beta = -\, df(x)/dx|_{x_E} = p^2(1 - \eta^2)$. Therefore, the equilibrium position x_E is infinitesimally stable if and only if $\beta > 0$, that is, when and only when $\eta \equiv \omega/p < 1$. This is precisely the result derived earlier for arbitrary amplitude oscillations consistent with obvious constraints but based on the familiar nature of equation (6.77i).

The free vibrational motion of the general linear damped oscillator is described in (6.83), and this equation is not restricted to infinitesimal motions $z(t)$ from the equilibrium position $z_E = 0$. In view of the physical nature of the damping and spring coefficients, the infinitesimal stability parameters in (6.103) are positive; we identify $\alpha \equiv 2\nu > 0$ and $\beta \equiv p^2 > 0$. Therefore, we know from infinitesimal stability analysis that the equilibrium position $z_E = 0$ is asymptotically stable. In fact, it is physically clear that the system, when disturbed by any amount consistent with design constraints, will return eventually to its equilibrium position. If $\alpha \equiv 0$, the motion about the equilibrium state will be stable for $\beta \equiv p^2 > 0$, as learned earlier.

It is not necessary to recall the details of the formal infinitesimal stability analysis of the equilibrium states of a dynamical system. In special problems, it is straightforward to simply determine the equilibrium states from the equation of motion, introduce a disturbance like (6.101) for an infinitesimal perturbation from these states, and then carry out a linearized analysis of the equation of motion. This process leads to an incremental equation of motion similar to (6.103) from which the stability may be determined in accordance with (6.105). For further study of vibration problems and stability analysis see the referenced text by Meirovitch.

6.15. Equations of Motion Relative to the Earth

To investigate effects of the Earth's rotation on the motion of a particle, we recall the *equation of motion of a particle relative to the Earth*:

$$m\mathbf{a}_\varphi = \mathbf{F} - 2m\boldsymbol{\Omega} \times \mathbf{v}_\varphi. \qquad \text{(cf. 5.102)}$$

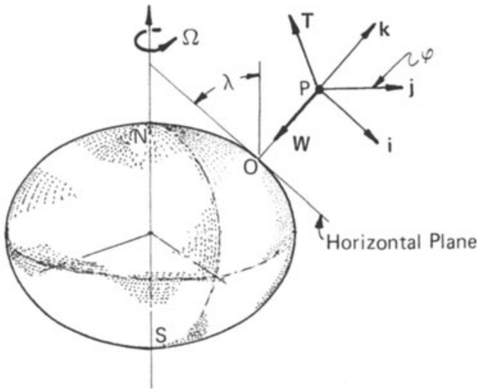

Figure 6.28. Motion of a particle relative to the Earth.

Let the Earth frame $\varphi = \{O; \mathbf{i}_k\}$ be oriented so that **i** is directed southward and **j** is eastward in the horizontal plane tangent to the Earth's surface at the latitude λ, the angle of elevation of the Earth's axis above the horizontal plane, as shown in Fig. 6.28. Then **k** is normal to the surface, directed skyward. Referred to φ, the angular velocity of the Earth frame is

$$\boldsymbol{\Omega} = \Omega(-\cos\lambda\mathbf{i} + \sin\lambda\mathbf{k}). \tag{6.107}$$

Hence, the Coriolis acceleration is given by

$$2\boldsymbol{\Omega} \times \mathbf{v}_\varphi = -2\Omega\dot{y}\sin\lambda\mathbf{i} + 2\Omega(\dot{x}\sin\lambda + \dot{z}\cos\lambda)\mathbf{j} - 2\Omega\dot{y}\cos\lambda\mathbf{k}, \tag{6.108}$$

wherein $\mathbf{v}_\varphi \equiv \delta\mathbf{x}/\delta t$ and $\mathbf{x}(P,t) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ is the relative position vector. Finally, the total force acting on the particle P is $\mathbf{F} = \mathbf{T} + \mathbf{W}$, where $\mathbf{W} = -mg\mathbf{k}$ is its apparent weight and $\mathbf{T} = Q\mathbf{i} + R\mathbf{j} + S\mathbf{k}$ is the total of all other contact and body forces that act on P. Then, use of (6.108) in (5.102) yields *the scalar equations for the particle's motion relative to the Earth*:

$$m\ddot{x} = Q + 2m\Omega\dot{y}\sin\lambda, \tag{6.109}$$

$$m\ddot{y} = R - 2m\Omega(\dot{x}\sin\lambda + \dot{z}\cos\lambda), \tag{6.110}$$

$$m\ddot{z} = S - mg + 2m\Omega\dot{y}\cos\lambda. \tag{6.111}$$

Some interesting Coriolis effects of the Earth's rotation may be read from these equations, or more directly from (6.108). When a particle is traveling eastward so that $\mathbf{v}_\varphi = \dot{y}\mathbf{j}$, for example, the Coriolis force $-2m\boldsymbol{\Omega} \times \mathbf{v}_\varphi = 2m\Omega\dot{y}(\sin\lambda\mathbf{i} + \cos\lambda\mathbf{k})$ for $\lambda > 0$ in the northern hemisphere drives the particle toward the right, southward and upward; and at the same latitude in the southern hemisphere for which $\lambda < 0$, it drives the particle toward the left, northward and upward. Therefore, in the moving Earth frame over a period of time, a ship or plane in its eastward directed motion must make a small course correction northward in the northern hemisphere and southward in the southern hemisphere, to counter the Coriolis force effect in (5.102). At the equator $\lambda = 0$, only the vertical component is active: $-2m\boldsymbol{\Omega} \times \mathbf{v}_\varphi = 2m\Omega\dot{y}\mathbf{k}$, so no course adjustment is needed. Other subtle

Coriolis effects on the motion of a particle relative to the Earth are demonstrated in some applications that follow.

6.16. Free Fall Relative to the Earth—An Exact Solution

The elementary result (6.24) for the motion of a particle that falls from rest relative to the Earth shows that the particle falls on a straight line—the plumb line, a result that ignores the Coriolis effect of the Earth's spin. Due to the Earth's rotation, however, the particle in its free fall from rest is deflected horizontally from the vertical plumb line. This Coriolis deflection effect is determined, and afterwards the theoretical result is compared with experimental data. For simplicity, however, effects due to air resistance, wind, and buoyancy are ignored.

The free fall problem is the simplest example for which an exact solution of the equations of motion of a particle relative to the Earth may be obtained. In this case, with $(Q, R, S) = 0$ and $\mathbf{v}_\varphi(P, 0) = \mathbf{0}$ initially, (6.109)–(6.111) may be readily integrated to obtain

$$\dot{x} = 2\Omega y \sin\lambda, \tag{6.112a}$$

$$\dot{y} = -2\Omega(x \sin\lambda + z\cos\lambda), \tag{6.112b}$$

$$\dot{z} = -gt + 2\Omega y \cos\lambda. \tag{6.112c}$$

The next step is less evident. We first substitute (6.112a) and (6.112c) into (6.110) and set $R = 0$ to obtain

$$\ddot{y} + 4\Omega^2 y = 2\Omega g t \cos\lambda. \tag{6.112d}$$

The general solution of (6.112d) is given by

$$y = \frac{gt\cos\lambda}{2\Omega} + A\cos 2\Omega t + B \sin 2\Omega t. \tag{6.112e}$$

Without loss of generality, the origin may be chosen so that $\mathbf{x}(P, 0) = \mathbf{0}$. Thus, with $y = 0$ and $\dot{y} = 0$ at $t = 0$, we find $A = 0$, $B = -(g\cos\lambda)/4\Omega^2$, and hence

$$y = \frac{g\cos\lambda}{4\Omega^2}(2\Omega t - \sin 2\Omega t). \tag{6.112f}$$

Now substitute this relation into (6.112a) and (6.112c), recall the initial data, and integrate the results to derive *the exact time-parametric equations for the particle path in its free fall relative to the rotating Earth frame*:

$$x = \frac{g\sin 2\lambda}{8\Omega^2}(2\Omega^2 t^2 - 1 + \cos 2\Omega t), \tag{6.112g}$$

$$y = \frac{g\cos\lambda}{4\Omega^2}(2\Omega t - \sin 2\Omega t), \tag{6.112h}$$

$$z = -\frac{gt^2}{2} + \frac{g\cos^2\lambda}{4\Omega^2}(2\Omega^2 t^2 - 1 + \cos 2\Omega t). \tag{6.112i}$$

Notice that both horizontal and vertical components of the motion are affected by the Earth's rotation, and that the results are independent of the particle's mass. When the Earth's rotational rate $\Omega \to 0$, these equations show that $x \to 0$, $y \to 0, z \to -\frac{1}{2}gt^2$. That is, $\mathbf{x}(P,t) = z\mathbf{k} = \frac{1}{2}\mathbf{g}t^2$, the elementary solution (6.24) for which the Earth's rotation is neglected.

6.16.1. Free Fall Deflection Analysis

Since Ω is small, but not zero, and the time of fall near the Earth's surface is of short duration, the path equations (6.112g)–(6.112i) may be simplified by series expansion of the trigonometric functions to retain only terms of $O(\Omega t)^2$. This yields

$$\mathbf{x}(P,t) = \frac{gt^2}{12}((\Omega t)^2 \sin 2\lambda \mathbf{i} + 4\Omega t \cos\lambda \mathbf{j} - 2(3 - 2(\Omega t)^2 \cos^2\lambda)\mathbf{k}). \qquad (6.112\text{j})$$

We thus find an eastward ($\mathbf{j}$-directed) deflection of the first order and a north–south ($\mathbf{i}$-directed) essentially negligible deflection of the second order in Ωt. To terms of the first order in Ωt, therefore, the motion is described by

$$\mathbf{x}(P,t) = \frac{1}{3}g\Omega t^3 \cos\lambda \mathbf{j} - \frac{1}{2}gt^2\mathbf{k}. \qquad (6.112\text{k})$$

The first term describes the Coriolis deflection, and the second is the elementary solution (6.24). *Therefore, a particle P in its free fall relative to the Earth experiences in either hemisphere an eastward directed deflection from the vertical axis.* The trajectory of P, shown in Fig. 6.29, to the first order in Ωt is a semicubical parabola in the east–west vertical plane:

$$y^2 = -\frac{8\Omega^2 \cos^2\lambda}{9g} z^3. \qquad (6.112\text{l})$$

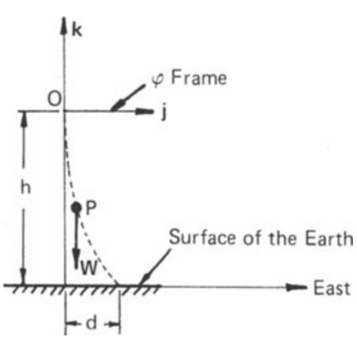

Figure 6.29. Free fall deflection of a particle relative to the Earth.

The deflection $y = d$ for fall through a height $z = -h$ is

$$d = \frac{2}{3}h\Omega\cos\lambda\sqrt{\frac{2h}{g}}. \tag{6.112m}$$

The deflection is greatest at the equator ($\lambda = 0$) and vanishes at the poles ($\lambda = \pm\pi/2$). For example, the greatest deflection of a raindrop falling freely through 10,000 ft (3049 m), without air resistance, wind, and buoyancy effects, according to (6.112m), is $d_{\max} = 12.1$ ft (3.69 m). Though only 0.12% of the height, the deflection in the ideal free fall case would be clearly observable. In fact, some experimental results on falling solid pellets have been reported.

6.16.2. Reich's Experiment

The free fall of pellets down a deep mine shaft at Freiberg, Germany was studied by F. Reich in 1831 and published a few years before Coriolis reported his formula for relative rotational effects in 1835. The depth of the mine was 158.5 m, and Reich observed an average deflection of 28.3 mm in 106 trials. The corresponding value estimated by (6.112m) for the data $\Omega = 7.29 \times 10^{-5}$ rad/sec, $g = 9.82$ m/sec^2, and $\lambda = 51°N$ is 27.5 mm. Our theoretical estimate thus demonstrates excellent agreement with Reich's experimental result on the eastward deflection. It is known, however, that the eastward deflection is slightly reduced by air resistance.

Long before the expression for the Coriolis acceleration was discovered, the eastward deflection due to the Earth's rotation was argued intuitively by natural philosophers, though usually incorrectly, and Reich knew about this. In addition, however, Reich found a small *southerly* deflection at Freiberg. This north–south deviation is determined exactly by (6.112g) and to terms of the order $(\Omega t)^2$ by (6.112j). If the time of fall τ from the height h is estimated by their omission in (6.112j) so that $\tau^2 = 2h/g$, the north–south deflection is approximated by $\delta = x(\tau) = (h^2\Omega^2/3g)\sin 2\lambda$. Hence, the southerly deflection predicted for Reich's experimental data is roughly 0.004 mm. Within the error of experiment, this would be zero and in fact negligible; so, it seems unlikely that such a minute free fall effect could be accurately measured. The fact that Reich and others have observed and reported the effect at all is surprising.

6.17. Foucault's Pendulum

In 1851, J. B. Léon Foucault** (1819–1868) discovered by experiment that the effect of the Earth's rotation on the motion of a carefully constructed pendulum

** The story of Léon Foucault's life, his pendulum experiments, his invention of the gyroscope, his numerous other accomplishments, and the illustrious period of French history during which he is to produce relative to the Earth an apparent rotation of its plane of oscillation at an angular rate $\omega = \Omega \sin \lambda$, clockwise in the northern hemisphere ($\lambda > 0$) and anticlockwise in the southern hemisphere ($\lambda < 0$). Foucault's first pendulum consisted of a 5 kg brass bob attached to a 2 m long steel wire suspended from the ceiling in the cellar of his house, its end held in a device that enabled the pendulum's unhindered rotation. To avoid disturbing extraneous vibrations from the thunderous clatter of passing carriages and other neighborhood noise, echoes of busy Paris streets that followed him to his cellar laboratory, he worked during the wee small hours of the night. His first test, 1–2 AM, Friday, January 3, 1851, ended quickly in failure when suddenly the wire broke. Several days later, modifications concluded, at two o'clock in the morning of Wednesday, January 8, 1851, he recorded in his journal the slow steady rotation of the plane of the pendulum's swing. Secluded from the rest of the world in the cellar of his house, without reference to heavenly bodies, he thus witnessed for the first time in history direct proof of the rotation of the Earth about its axis! (Incidentally, to relate the time of Foucault's pendulum experiments in France to American history, we may recall that Millard Fillmore was 13^{th} President of the United States (1850–1853).)

Needless to say, Foucault was most anxious to demonstrate his important discovery to French scientists, but he needed a prominent public place to display his pendulum. Moreover, the effect could be enhanced by the use of a longer pendulum wire—remember, the period of oscillation for a simple pendulum is increased with its length; so, with a longer wire the pendulum swings more slowly, and the turning of the Earth is more easily seen.

Having no scientific credentials himself, he was generally not well-regarded by the members of the French Academy of Sciences. On the other hand, François Arago, a man of scientific prominence and a member of the Academy, the renowned and distinguished Director of the Paris Observatory, a large building with a high dome, was a somewhat friendly, admiring associate, who was certain to appreciate his discovery. Foucault convinced Arago to permit the presentation of his pendulum discovery in the Meridian Hall, the largest, longest, and highest room in the Observatory and, though unimportant to the experiment, perfectly aligned lengthwise with the Paris Meridian. (This is the very Meridian a certain specified length of which was proposed to define the length of the standard meter, but because of errors of its measurement, which is another story, actually it does

struggled for recognition by his colleagues in the Academy of Sciences, is told in the books by Aczel and by Tobin cited in the References and from which this summary narrative is adapted. There are, however, some ambiguities and discrepancies in their reports. For instance, it is not clear from their separate presentations that Foucault's pendulum demonstration and his paper presented by Arago at the Academy announcing the discovery occurred on the same day. Also, Tobin, page 141, sets the time for Foucault's Meridian Hall invitation at 2–3 PM, while Aczel, page 93, reports 3–5 PM; and they express a difference of opinion on other historical matters, including the date of Foucault's first successful test! Consequently, when I perceived a conflict, unable to check the original sources myself, though the difference might seem insignificant, I generally leaned toward Tobin's view.

not.) The high ceiling of Meridian Hall would allow use of a pendulum of 11 m length.

Foucault prepared invitations and sent them to all members of the Academy and some others—"You are invited to come to see the Earth turn, in the Meridian Hall of the Paris Observatory, tomorrow, from two to three."—an invitation clearly designed to stimulate curiosity and to drive attendance. On February 3, 1851, Foucault (see the References) announced his pendulum discovery in a paper presented to the Academy by then supportive Arago. Later that day, many of France's most famous scientists and mathematicians assembled in Meridian Hall to see the Earth turn. Word of Foucault's pendulum experiment success instantly excited the interest of science-minded Louis-Napoléon, President of the French Republic, who decreed straightaway that the experiment be repeated in the Panthéon, a grand temple and mausoleum for great Frenchmen, the highest domed building in all of Paris. A new pendulum 67 m long and weighing 28 kg, the then longest and heaviest in the world, was fabricated. At the end of March 1851, Foucault's dream was realized—the Panthéon pendulum exhibition was open for all visitors to witness. Later that year, a report of a pendulum experiment in Brazil confirmed the counterclockwise, southern hemisphere ($\lambda < 0$) rotation of the pendulum in agreement with Foucault's empirical sine relation $\omega = \Omega \sin \lambda$.

The dynamical equations of motion of a particle relative to a moving reference frame were widely known long before 1851. The earliest derivation appears to have come from A. Clairaut in 1742 (see Dugas in the References). The result, however, is commonly attributed to G. G. de Coriolis (1792–1843), a student of Siméon Denis Poisson (1781–1840), who presented the correct equations in a paper read to the Academy of Sciences in 1831 and published a year later. Moreover, it is known that probably around 1837, Poisson had analyzed the Coriolis effect on the motion of a pendulum; but failing to appreciate its cumulative effect, he rejected the result as too small to be noticeable and apparently never published it. Foucault's demonstration sparked new interest among mathematicians and scientists to explain by analysis Foucault's empirical sine rule. At a meeting of the Academy a few days after Arago's presentation of Foucault's memoir, J. P. M. Binet, an obscure professor of mechanics and astronomy, wrote down the equations of motion from the principles of dynamics and following some approximations and a lengthy analysis, there, for the first time, derived Foucault's equation for the rate of rotation of the pendulum. (See the References.)

Though widely acclaimed around the world for his work in science and engineering, the ultimate honor that Foucault desperately desired, his election as a member of the Academy of Sciences, was continually denied to him. A seat in the Academy opened only upon the death of a member and then, of course, the number of candidates seeking election was many, to say least about vote-rigging politics that sometimes raised its ugly head. Foucault had narrowly missed election several times. Finally, on January 23, 1865, 14 years after his famous demonstration of the Earth's rotation and 3 years before his death, his quest was finally realized when he was elected to the Academy of Sciences. Foucault described the long awaited approbation of his peers, his election to Academy membership, as one of the great joys of his life. (See Tobin in the References.)

Nowadays, one may find a Foucault pendulum in just about every major city around the world. In Lexington, Kentucky, for example, a Foucault pendulum swings in the Public Library on Main Street. Surprisingly, the pendulum has been exhibited at the Panthéon only since 1995. In St. Petersburg (formerly Leningrad), Russia, during the Soviet years from 1931, interrupted by the war of 1941–1945, and thereafter continuing until the late 1980s, the world's longest Foucault pendulum, nearly 100 m in length, was suspended from the dome of St. Isaac's Cathedral, one of the tallest churches in the world, built in 1818–1858. Students were taken regularly by their professors to see this remarkable display proving the rotation of the Earth. Soon after Soviet President Mikhail Gorbachev's initiation of perestroika and his rise to power in 1988, St. Isaac's was returned to the Church, and the phenomenal Foucault pendulum, the incongruous centerpiece of St. Isaac's swinging from its cupola, was promptly removed. Today, the image of a white dove in flight adorns the pinnacle of the incredibly beautiful and spectacular ceiling within the golden dome of this magnificent church. Though still principally a museum as decreed by the Soviet government in 1931, from time to time St. Isaac's nowadays holds religious services on special occasions, and a Foucault pendulum may be seen at the St. Petersburg Planetarium. Everyone who has observed the swing of a Foucault pendulum has, in effect, seen the rotation of the Earth!

6.17.1. General Formulation of the Problem

We now turn to the analysis of Foucault's pendulum phenomenon. Let us consider a pendulum bob of mass m attached by a long wire of length ℓ to a fixed point $(0, 0, \ell)$ along the vertical plumb line in the Earth frame $\varphi = \{O; \mathbf{i}_k\}$ in Fig. 6.30. The relative position vector of m in φ is $\mathbf{x}(m, t) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. The total force $\mathbf{F}$ on the bob is its apparent weight $m\mathbf{g}$ and the wire tension $\mathbf{T} = T\mathbf{n} = Q\mathbf{i} + R\mathbf{j} + S\mathbf{k}$, where $\mathbf{n} = -x/\ell\mathbf{i} - y/\ell\mathbf{j} + (1 - z/\ell)\mathbf{k}$. Hence, the general equations (6.109)–(6.111) yield the following relations for the motion of the pendulum bob relative to the Earth:

$$m\ddot{x} = -\frac{Tx}{\ell} + 2m\Omega\dot{y}\sin\lambda, \tag{6.113a}$$

$$m\ddot{y} = -\frac{Ty}{\ell} - 2m\Omega(\dot{x}\sin\lambda + \dot{z}\cos\lambda), \tag{6.113b}$$

$$m\ddot{z} = \frac{T(\ell - z)}{\ell} - mg + 2m\Omega\dot{y}\cos\lambda. \tag{6.113c}$$

These equations cannot be integrated exactly for large amplitude oscillations. The manner in which the wire tension varies with the motion is unknown, and its elimination from these equations serves only to further complicate matters. It is possible, however, to derive an approximate solution for small amplitude oscillations.

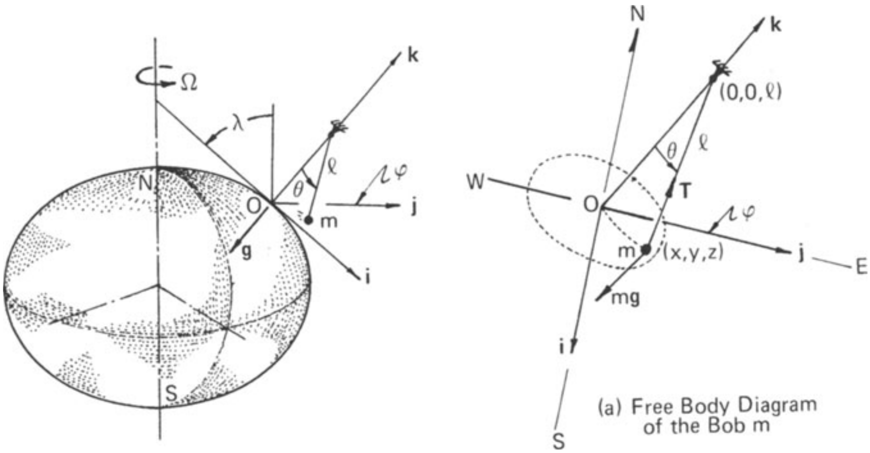

Figure 6.30. Foucault's pendulum and its motion relative to the Earth.

6.17.2. Equations for Small Amplitude Oscillations

Let us assume that the wire is long compared with the displacement so that x/ℓ, y/ℓ, and all of their time derivatives are small terms. Since $\ell - z = \ell[1 - (x^2 + y^2)/\ell^2]^{1/2}$, our smallness assumption shows that $z/\ell = (x^2 + y^2)/2\ell^2$, approximately. Hence, z/ℓ and its time derivatives are small quantities of the second order and may be discarded from (6.113a)–(6.113c). In particular, (6.113c) then yields an equation for the wire tension,

$$T = m(g - 2\Omega \dot{y} \cos\lambda). \tag{6.113d}$$

Since Ω is very small, (6.113d) shows that the tension, as expected, is very nearly equal to the apparent weight of the bob.

Using (6.113d) in (6.113a) and (6.113b) and neglecting terms of second order, we obtain the simpler, but coupled system of linear equations

$$\ddot{x} - 2\omega\dot{y} + p^2 x = 0, \qquad \ddot{y} + 2\omega\dot{x} + p^2 y = 0, \tag{6.113e}$$

in which

$$p \equiv \sqrt{\frac{g}{\ell}}, \qquad \omega \equiv \Omega \sin\lambda. \tag{6.113f}$$

The constant p is the familiar small amplitude, circular frequency of the simple pendulum when the Earth's rotation is ignored. It is evident from (6.113e), however, that *the motion of Foucault's pendulum is not simple harmonic.*

6.17.3. Solution of the Small Amplitude Equations

The solution of the coupled system (6.113e) may be obtained following an unusual change of variable. We multiply the second of (6.113e) by $i = \sqrt{-1}$, add the result to the first equation in (6.113e), and introduce the new complex variable

$$\xi(t) = x(t) + iy(t), \tag{6.113g}$$

to obtain the single complex equation

$$\ddot{\xi} + 2i\omega\dot{\xi} + p^2\xi = 0. \tag{6.113h}$$

The general solution of (6.113h) is

$$\xi(t) = A_1 e^{i\alpha_1 t} + A_2 e^{i\alpha_2 t}, \tag{6.113i}$$

in which A_1 and A_2 are integration constants, possibly complex, and the characteristic exponents are given by

$$\alpha_1 = -\omega - \omega^*, \qquad \alpha_2 = -\omega + \omega^*, \qquad \text{with} \quad \omega^* \equiv \sqrt{\omega^2 + p^2}. \tag{6.113j}$$

To determine the constants A_1 and A_2, let us suppose that the pendulum is released from rest at $x(0) = x_0,\ y(0) = 0$ at time $t = 0$. Then, by (6.113g), the initial values of the complex variable are $\xi(0) = x_0,\ \dot{\xi}(0) = 0$, and hence (6.113i) delivers

$$A_1 = \frac{x_0\alpha_2}{\alpha_2 - \alpha_1}, \qquad A_2 = -\frac{x_0\alpha_1}{\alpha_2 - \alpha_1}. \tag{6.113k}$$

Finally, use of (6.113j) in (6.113k) yields the real-valued constants

$$A_k = \frac{x_0}{2}\left(1 + (-1)^k \frac{\omega}{\omega^*}\right), \qquad k = 1, 2. \tag{6.113l}$$

We recall Euler's identity (6.49) to cast (6.113i) in the form

$$\xi(t) = (A_1 \cos\alpha_1 t + A_2 \cos\alpha_2 t) + i(A_1 \sin\alpha_1 t + A_2 \sin\alpha_2 t). \tag{6.113m}$$

It now follows with (6.113g) that the solution of the coupled equations in (6.113e) for the small amplitude motion of Foucault's pendulum is

$$\left.\begin{aligned} x(t) &= A_1 \cos\alpha_1 t + A_2 \cos\alpha_2 t, \\ y(t) &= A_1 \sin\alpha_1 t + A_2 \sin\alpha_2 t, \end{aligned}\right\} \tag{6.113n}$$

where the constants α_k and A_k are given by (6.113j) and (6.113l). Let the reader consider the following alternative procedure.

Exercise 6.8. Notice that (6.113h) is similar to the damped oscillator equation (6.83), and hence the solution method starting from (6.86a) is applicable. Begin with $\xi(t) = e^{\beta t}u(t)$, recall (6.113f) and (6.113j), and show that the general solution of (6.113h) for the assigned initial data for $\xi(t)$ yields the motion

$$\left.\begin{aligned} x(t) &= x_0\left(\cos\omega^* t\cos\omega t + \frac{\omega}{\omega^*}\sin\omega^* t\sin\omega t\right), \\ y(t) &= x_0\left(-\cos\omega^* t\sin\omega t + \frac{\omega}{\omega^*}\sin\omega^* t\cos\omega t\right). \end{aligned}\right\} \tag{6.113o}$$

Show that the same results follow from (6.113n). □

6.17.4. Physical Interpretation of the Solution

The motion (6.113n) is harmonic in time, but not simple, and it is not periodic unless α_1/α_2 is a rational number. Nevertheless, a period characteristic of the oscillation may be defined that will facilitate our physical understanding of the Foucault phenomenon.

The half-period $\tau/2$ is defined as the time required for the pendulum to complete its outward swing from its initial position. To find the period τ, we first determine all times $\hat{T} \neq 0$ for which $\dot{\mathbf{x}}(\hat{T}) = \mathbf{0}$. Differentiation of (6.113n) and use of (6.113k) shows that $\hat{T}$ must satisfy $\sin(\alpha_1\hat{T}) = \sin(\alpha_2\hat{T})$ and $\cos(\alpha_1\hat{T}) = \cos(\alpha_2\hat{T})$. Hence, $\hat{T}\alpha_2 = \hat{T}\alpha_1 \pm 2n\pi$ for all integers n. Use of (6.113j) in this expression yields the (positive) rest times

$$\hat{T}(n) = \frac{n\pi}{\omega^*} = \frac{n\pi}{\sqrt{\omega^2 + p^2}}, \qquad n = 1, 2, \ldots. \tag{6.113p}$$

At each time $\hat{T}(n)$, the bob attains a position of instantaneous rest. Thus, for the first outward swing, $\hat{T}(1) = \tau/2$, and hence the period of the oscillations is

$$\tau = \frac{2\pi}{\omega^*} = \frac{2\pi}{\sqrt{\omega^2 + p^2}}. \tag{6.113q}$$

Thus, ω^* defines the circular frequency of the oscillations, and the frequency is given by

$$f = \frac{1}{\tau} = \frac{\omega^*}{2\pi} = \frac{1}{2\pi}\sqrt{\omega^2 + p^2}. \tag{6.113r}$$

When the Earth's rotation is neglected so that $\omega = 0$, (6.113q) and (6.113r) reduce to the period and frequency for the simple pendulum. Otherwise, *the Earth's rotational effect on the oscillations of a pendulum is to increase its frequency (decrease its period) very slightly compared with that of the simple pendulum.* Moreover, in view of (6.113f), *the frequency is greatest (the period least) at the poles and least (greatest) at the equator where the effect vanishes to yield the simple pendulum, small amplitude value.* That is, the frequency varies from $p/2\pi$ at the equator to $(p^2 + \Omega^2)^{1/2}/2\pi$ at the poles.

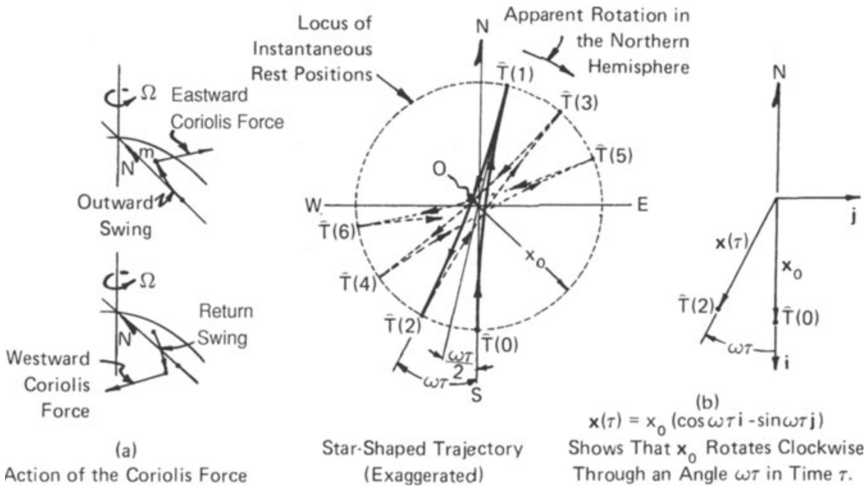

Figure 6.31. The Coriolis effect on the trajectory relative to the Earth of Foucault's pendulum viewed from its point of support at a place in the northern hemisphere where its apparent rotation is clockwise.

The rest positions of the bob at each half-period $\hat{T}(n) = n\tau/2 = n\pi/\omega^*$ may be obtained from (6.113o), which yields

$$x\left(\frac{n}{2}\tau\right) = (-1)^n x_0 \cos\left(\frac{n}{2}\omega\tau\right), \qquad y\left(\frac{n}{2}\tau\right) = (-1)^{n+1} x_0 \sin\left(\frac{n}{2}\omega\tau\right). \tag{6.113s}$$

In particular, the position of the bob after one full swing out and back is, for $n = 2$,

$$x(\tau) = x_0 \cos(\omega\tau), \qquad y(\tau) = -x_0 \sin(\omega\tau). \tag{6.113t}$$

Since $x(n\tau/2)^2 + y(n\tau/2)^2 = x_0^2$, it is seen that the locus of rest positions (6.113s) is a circle of radius x_0. Hence, (6.113t) shows that the initial position vector $\mathbf{x}_0 = x_0\mathbf{i}$, viewed from the point of support, has been rotated through an angle $\omega\tau$, which is clockwise when $\omega > 0$ and counterclockwise when $\omega < 0$. The second relation in (6.113f) shows that $\omega > 0$ in the northern hemisphere, $\omega < 0$ in the southern hemisphere, and $\omega = 0$ at the equator where the motion is always simple harmonic. Therefore, as first demonstrated by Foucault, *relative to the Earth, the plane of oscillation of a pendulum has an apparent clockwise rotation in the northern hemisphere, a counterclockwise rotation in the southern hemisphere, and no rotation at the equator.*

The motion is illustrated in Fig. 6.31 for the northern hemisphere. The pendulum starts from a southward displaced position of rest at a small distance x_0 from the plumb line. As the bob moves on its outward swing, it experiences a Coriolis force directed eastward; but on its return swing, the Coriolis force is directed westward. The deflection always is toward the right of the direction of the swing in the northern hemisphere. This is shown in Fig. 6.31a. Hence, the bob, after one period, has undergone a net displacement westward to the position $\mathbf{x}(\tau) = x_0(\cos \omega\tau \mathbf{i} - \sin \omega\tau \mathbf{j})$, the same distance from the origin, but rotated clockwise through a small angle $\omega\tau$ from $\mathbf{x}_0$, as shown in Fig. 6.31b. At each time $\hat{T}(n) = n\tau/2$, the same thing is repeated over and over, so the bob traces the star shaped trajectory described by (6.113o) and illustrated in Fig. 6.31. The apparent motion in the southern hemisphere for which $\lambda < 0$ is counterclockwise. The vertical plane of the pendulum's oscillations thus rotates relative to the Earth with Foucault's angular speed $\omega = \Omega \sin \lambda$, as indicated in (6.113t). The number of days $\tau_d(\lambda)$ required to complete one full revolution of the plane of oscillation of the pendulum is thus given by $\tau_d(\lambda) = 1/\sin \lambda$. *Consequently, Foucault's pendulum takes* 1 *day to complete its apparent turn at the poles where* $\lambda = \pm\pi/2$, *and this cyclic time increases as the latitude* λ *decreases toward the equator where the effect disappears.* Specifically, at $\lambda = \pi/6$, $\tau_d(\pi/6) = 2$ days/revolution, and at the equator $\tau_d(0) = \infty$ days/revolution, that is, the Foucault effect vanishes.

6.18. Relative Motion under a Constant Force

The scalar equations (6.109)–(6.111) for the motion of a particle relative to the Earth may be integrated exactly for any constant force components (Q, R, S). However, the general description of motion of a particle P relative to the Earth under a constant force $\mathbf{f} = \mathbf{F}/m$ per unit mass also may be derived as an easy approximate solution of the vector equation of motion (5.102), namely,

$$\frac{\delta \mathbf{v}}{\delta t} = \mathbf{f} - 2\boldsymbol{\Omega} \times \mathbf{v}. \tag{6.114a}$$

This is a first order, vector differential equation for the relative velocity $\mathbf{v} \equiv \mathbf{v}_\varphi(P, t) = \delta\mathbf{x}/\delta t$. Since $\boldsymbol{\Omega}$ is a constant vector, (6.114a) may be readily integrated to obtain

$$\mathbf{v}(P, t) = \frac{\delta \mathbf{x}}{\delta t} = \mathbf{f}t - 2\boldsymbol{\Omega} \times (\mathbf{x} - \mathbf{x}_0) + \mathbf{v}_0, \tag{6.114b}$$

in which $\mathbf{x}_0 \equiv \mathbf{x}(P, 0)$, $\mathbf{v}_0 \equiv \mathbf{v}(P, 0)$ are assigned initial values. For example, when gravity is the only force on a particle at rest initially at the origin, $\mathbf{f} = -g\mathbf{k}$, $\mathbf{x}_0 = \mathbf{0}$, $\mathbf{v}_0 = \mathbf{0}$, and (6.114b) is then equivalent to the system of scalar equations (6.112a)–(6.112c) for the motion of a particle in free fall relative to the Earth.

The equation for the motion $\mathbf{x}(P, t)$ of P relative to the Earth under the general constant force $\mathbf{f}$ follows by use of (6.114b) in (6.114a); we find

$$\frac{\delta^2 \mathbf{x}}{\delta t^2} - 4\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{x}) = \mathbf{f} - 2\boldsymbol{\Omega} \times (\mathbf{f}t + \mathbf{v}_0 + 2\boldsymbol{\Omega} \times \mathbf{x}_0). \tag{6.114c}$$

Upon discarding terms of order Ω^2, we obtain the easily integrable vector differential equation

$$\frac{\delta \mathbf{v}}{\delta t} = \frac{\delta^2 \mathbf{x}}{\delta t^2} = \mathbf{f} - 2\boldsymbol{\Omega} \times (\mathbf{f}t + \mathbf{v}_0). \tag{6.114d}$$

With the initial values $\mathbf{x}_0$ and $\mathbf{v}_0$ in mind, the first integral is

$$\mathbf{v} = \frac{\delta \mathbf{x}}{\delta t} = \mathbf{v}_0 + \mathbf{f}t - 2\boldsymbol{\Omega} \times \left(\frac{1}{2}\mathbf{f}t^2 + \mathbf{v}_0 t\right), \tag{6.114e}$$

and hence the approximate motion of P relative to the Earth is given by

$$\mathbf{x} = \mathbf{x}_0 + \mathbf{v}_0 t + \frac{1}{2}\mathbf{f}t^2 - \boldsymbol{\Omega} \times \left(\frac{1}{3}\mathbf{f}t^3 + \mathbf{v}_0 t^2\right). \tag{6.114f}$$

To check the result, let us consider the motion of a particle in free fall from rest at the origin. Then $\mathbf{f} = \mathbf{g} = -g\mathbf{k}$ and (6.114f) simplifies to

$$\mathbf{x}(P, t) = \frac{1}{2}\mathbf{g}t^2 - \boldsymbol{\Omega} \times \frac{1}{3}\mathbf{g}t^3. \tag{6.114g}$$

Use of (6.107) yields (6.112k) derived earlier for the free fall case in which terms of order Ω^2 were neglected.

6.18.1. First Order Vector Solution for Projectile Motion

The approximate solution (6.114f) for the motion of a particle under a constant force is applied to investigate the Coriolis effect on the motion of a projectile P fired at $\mathbf{x}_0 = \mathbf{0}$ with a relative muzzle velocity

$$\mathbf{v}_0 = V(\cos\alpha \mathbf{i} + \cos\beta \mathbf{j} + \cos\gamma \mathbf{k}). \tag{6.115a}$$

Here α, β, γ are the direction angles of the gun in the frame $\varphi = \{O; \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ defined in Fig. 6.28. The usual extraneous effects are neglected. Then only the body force $\mathbf{f} = \mathbf{g} = -g\mathbf{k}$ per unit mass acts on P. We thus recall (6.107) and (6.115a) to derive from (6.114f) the following estimate for the projectile's motion relative to the Earth:

$$\begin{aligned}\mathbf{x}(P, t) = {} & Vt(\cos\alpha + \Omega t \cos\beta \sin\lambda)\mathbf{i} \\ & + Vt\left[\cos\beta - \Omega t(\cos\gamma \cos\lambda + \cos\alpha \sin\lambda) + \frac{\Omega g t^2}{3V}\cos\lambda\right]\mathbf{j} \\ & + Vt\left(\cos\gamma + \Omega t \cos\beta \cos\lambda - \frac{gt}{2V}\right)\mathbf{k}. \end{aligned} \tag{6.115b}$$

Example 6.16. Determine the Coriolis deflection of a projectile fired eastward at latitude λ. Derive the classical relations for the motion and the range when the Earth's rotation is neglected.

Solution. Since the projectile is fired due east (the **j** direction in Fig. 6.28), the angle of elevation is β. Then $\alpha = \pi/2$, $\gamma = \frac{\pi}{2} - \beta$, and (6.115b) becomes

$$\begin{aligned}\mathbf{x}(P,t) &= \Omega V t^2 \cos\beta \sin\lambda \mathbf{i} \\ &+ Vt\left(\cos\beta - \Omega t \sin\beta\cos\lambda + \frac{\Omega g t^2}{3V}\cos\lambda\right)\mathbf{j} \\ &+ Vt\left(\sin\beta + \Omega t\cos\beta\cos\lambda - \frac{gt}{2V}\right)\mathbf{k}.\end{aligned} \quad (6.116a)$$

First consider the case when the Earth's rotation is neglected. With $\Omega = 0$, (6.116a) reduces to the classical elementary solution for projectile motion:

$$\mathbf{x}(P,t) = Vt\cos\beta\mathbf{j} + Vt\left(\sin\beta - \frac{gt}{2V}\right)\mathbf{k}. \quad (6.116b)$$

The time of flight $t^* = (2V\sin\beta)/g$ for which $z(t^*) = 0$ is then used to find the projectile's range $r \equiv y(t^*)$, namely,

$$r = \frac{V^2}{g}\sin 2\beta. \quad (6.116c)$$

Now consider the Earth's rotational effect. Equation (6.116a) indicates a lateral (**i**-directed) Coriolis deflection of the projectile normal to its east directed range line, toward the south in the northern hemisphere and toward the north in the southern hemisphere. To find the deflection, we need the projectile's time of flight t^* given by $z(t^*) = 0$ in (6.116a). To the first order in Ω, we find for $V\Omega/g \ll 1$,

$$t^* = \frac{2V\sin\beta}{g}\left(1 + \frac{2V\Omega\cos\beta\cos\lambda}{g}\right). \quad (6.116d)$$

The lateral deflection $x^* \equiv x(t^*)$ and the range $r^* \equiv y(t^*)$ are now determined by the remaining components in (6.116a). The projectile's Coriolis deflection to first order in Ω, with $V\Omega/g \ll 1$, is thus given by

$$x^* = \frac{4\Omega V^3 \sin^2\beta\cos\beta\sin\lambda}{g^2}. \quad (6.116e)$$

The reader will explore the range effect in the exercise. □

Exercise 6.9. Show that to the first order in Ω, the variation $\delta r = r^* - r$ in the range due to the Earth's rotation when the gun is fired eastward with muzzle speed V at an elevation angle β and at latitude λ is

$$\begin{aligned}\delta r &= \frac{4\Omega V^3}{g^2}\cos\lambda\sin\beta\left(1-\frac{4}{3}\sin^2\beta\right)\\ &= \Omega\cos\lambda\sqrt{\frac{2r^3\cot\beta}{g}}\left(1-\frac{1}{3}\tan^2\beta\right).\end{aligned} \tag{6.116f}$$

□

Notice that $\delta r = 0$ when $\beta = 60^\circ$; therefore, in the absence of air resistance, the Earth's rotation has no first order effect on the projectile's range when fired eastward at an elevation angle $\beta = 60^\circ$. Otherwise, the Coriolis effect is to increase the range when $\beta < 60^\circ$ and decrease it when $\beta > 60^\circ$. The effect is the same in both hemispheres, and it may be considerable for high velocity projectiles or missiles. Large naval guns operate at fairly small angles of elevation, usually less than 15°; so the Earth's rotational effect is to increase their eastward directed range.

Finally, consider the Coriolis deflection (6.116e). In the northern hemisphere, $\sin\lambda > 0$ and $x^* > 0$; therefore, the projectile's lateral deflection from its eastward directed firing line is toward the right, southward. In the southern hemisphere, however, the deflection is toward the left, northward. *A correction for the effect in the northern hemisphere by directing the line of fire northward by an amount* x^* *without subsequent readjustment in the southern hemisphere at the opposite latitude would roughly double the northward deflection from the eastward directed line of fire.* While the projectile suffers no lateral deflection at the equator, $\lambda = 0$, the variation in the range with latitude given by (6.116f) is greatest there. Ballistic accuracy, therefore, requires that the Coriolis effect be accounted for in fire control and inertial guidance designs for long range, high velocity projectiles or missiles.

6.18.2. The Battle of the Falkland Islands

In late October 1914, Germany's (East Asiatic) China Squadron under the command of Vice Admiral von Spee patrolling in the Pacific Ocean was underway toward Cape Horn to harass British bases and shipping in the South Atlantic before attempting to return up the Atlantic to Germany.[††] Two heavy cruisers, von Spee's flagship, the *Scharnhorst*, and her sister ship, the *Gneisenau*, each mounting eight rapid-firing 8.2-in. guns, were accompanied by the three light cruisers *Nürnberg*, *Leipzig*, and *Dresden*, each with ten 4.1-in. batteries. The German gunners were well-trained, experienced, and most efficient.

[††] This account is adapted from the referenced reports by D. Howarth and Major R. N. Spafford, that by Howarth being more detailed. There are, however, a few minor discrepancies between them.

A British Squadron of older, slower ships, manned mostly by inexperienced reservists, based at the Falkland Islands under the command of Rear Admiral Sir Christopher Cradock, was at sea off the Pacific coast of Chile in search of von Spee. Cradock's flagship *Good Hope* mounted two 9.2-in. and sixteen 6-in. batteries; two light cruisers, the *Monmouth* and the *Glasgow*, each with several 6-in. guns; and an armored merchant ship, the *Otranto*, carried eight 4.7-inchers. A dilapidated battleship *Canopus* with four 12-inchers was too slow to keep pace with the others.

In the evening of November 1, Spee was found at Coronel off the coast of Chile. In heavy seas with winds near hurricane force, Cradock decided to run a parallel course and wait for an opportunity; but by 7 PM. Spee seized the initiative and engaged the British. The German light cruisers were outgunned and retreated from action, but Spee's two armored cruisers provided overwhelming rapid-fire power far superior to Cradock's. The *Scharnhorst* scored 35 hits on the *Good Hope*; the last struck the ship's magazine. An enormous explosion followed. Ablaze from stem to stern, almost instantly, the *Good Hope*, with Rear Admiral Sir Christopher Cradock and all 900 officers and crew, was gone. Later that night, following a relentless barrage by the *Gneisenau*, the *Monmouth* sank with all 754 hands. The *Glasgow* and *Otranto* fled southward to escape in the darkness and join the old battleship *Canopus*.

Not one man among the 1654 on board the two British cruisers survived the battle royal, while the Germans suffered only two wounded and six minor hits in the exchange. When word of this great tragedy and crushing defeat of the Royal Navy reached the British Admiralty, a superior British Squadron of eight warships was ordered to the Falklands to arrive on December 7, 1914. The dreadnoughts *Invincible* and *Inflexible*, two of the first heavily armored British battleships to have a large battery of eight 12-in. guns capable of being fired simultaneously in the same direction, five light cruisers, and an armed merchant vessel were directed to avenge the humiliating defeat at Coronel. The order: "Find Spee and destroy him!"

At dawn the next morning, December 8, the *Gneisenau* and the *Nürnberg* arrived at the Falklands to reconnoiter for a raid on the strategic coaling and wireless station at Port Stanley, expecting to find no ships of any importance stationed there. They were met instead with fire from the old *Canopus*, intentionally grounded in the harbor mud to serve as a Falkland fortress. One 12-in. shell hit the *Gneisenau*. Realizing the circumstances, von Spee's ships turned toward the open seas of the South Atlantic, unaware that any major British ships were in the area. Dreadnoughts with superior speed and fire power suddenly appeared on the horizon at the harbor entrance. At that moment, Spee realized his pending doom. At 12.45 PM that afternoon, in a calm sea with a clear sky, von Spee's *Scharnhorst, Gneisenau, Nürnberg*, and *Dresden* were overtaken and attacked by the superior British Squadron. Eight hours later, the fury ended. Vice Admiral Sir F. C. Doveton Sturdee's Royal Navy Squadron reported 6 killed and 19 wounded, while the Germans lost Vice Admiral Maximilian Graf von Spee, the Danish born

Pioneer of the German Navy, and 2260 other courageous officers and men.[††] The heavy cruiser *Scharnhorst*, the *Gneisenau*, and the light cruisers *Nürnberg* and *Leipzig* all sunk. Only the light cruiser *Dresden* escaped the British rage. Three months later, she was found at a small island off the Pacific coast of Chile. During negotiations for surrender and while flying the white flag from her foremast, the *Dresden* was scuttled by her crew on March 14, 1915.

Marion and Spafford report[§§] that at the start of this horrific battle, the British shells completely missed the German ships. Marion suggests that this was due to the double Coriolis effect, but precise details are not provided. It is a fact, however, that the British Isles are situated near 50° N latitude and the Falklands near 50° S latitude.

For south directed fire at an angle of elevation α, the transverse Coriolis deflection $y^* \equiv y(t^*)$ obtained from (6.115b) to first order in Ω is approximated by

$$y^* = -\frac{4\Omega V^3}{g^2}\sin^2\alpha\left(\frac{1}{3}\cos\lambda\sin\alpha + \sin\lambda\cos\alpha\right), \tag{6.117}$$

which varies with the latitude. On the other hand, we find no variation δr in the range at any latitude, when a projectile is fired either southward or northward, which may explain why the combatants ran a parallel course toward the east, firing toward the north and south. Notice that the Coriolis deflection (6.117) in a south directed shot is not symmetric in λ, so there is a slight difference in the magnitudes of the westward, northern hemisphere and eastward, southern hemisphere deflections. The maximum angle of elevation for large naval guns is about 15°. To estimate the Earth's rotational effect on a projectile's motion based only partially on circumstances reported for the Falklands engagement, let us suppose that at $\lambda = 50°$ N latitude a shell from a 12-in. gun is fired southward with a muzzle speed $V = 1800$ ft/sec (1227 mph) at an angle $\beta = 13°$. The reader will find that the range, which is given by the classical rule in (6.116c), is approximately 8.3 miles.

[††] The gallant Spee, his *Scharnhorst* seriously crippled and listing, rejected surrender to Sturdee. The *Scharnhorst* sank with Spee and all 765 hands. One hundred and ninety of the 850 man crew of the *Gneisenau* and only 23 sailors from both the *Nürnberg* and *Leipzig*, all sunk, were rescued from the frigid waters of the South Atlantic; but many of them subsequently died from their battle wounds or shock.

[§§] Marion (page 348) remarks on the Coriolis effect but provides no reference or calculation to support his claim that the British salvos fell 100 yards east of their southward targets. See the References and Spafford's report mentioned below.

The muzzle speeds used in the example presented below equation (6.117) and in Problem 6.76 are estimates obtained from general naval records: for a 5-in. gun, $V = 2650$ ft/sec, and for a 12-in. gun $V = 1800$ ft/sec and greater, depending on the model design. The range and latitude (actually closer to 51.5°S) are estimated from battle data described by Major R. N. Spafford, whose sketch of the battle plan of December 8, 1914 shows the British heading east, running a parallel course, 14,000 yards (8 miles) north of the Germans. By Spafford's account, *initial fire was exchanged but without effect*, except for a single German round that struck the *Invincible.* At the ideal range of 15,000 yards (8.5 miles) for the 12-in. guns of his battle cruisers, Sturdee found the target first and bombarded Spee's squadron.

Under these conditions, the deflection, according to (6.117), will be about 19 yards to the right, westward of the line of fire in the northern hemisphere and roughly 22 yards eastward in the southern hemisphere. A fire control system that corrects for the deflection only in the northern hemisphere (by pointing its sights eastward), when fired southward at 50° S latitude, will direct a shell about 41 yards to the left, east of its south directed target. These deviations increase substantially for larger muzzle speeds. (See Problem 6.76 and the remarks in the last footnote above.)

The weight of a projectile may vary considerably from roughly 70 lb for a 5-in. shell to about 1800 lb for a 16-in. shell. Since in this analysis gravitational force is the only force acting on the projectile, however, it is seen that the results are independent of the mass of the projectile or any of its design features. Introduction of drag force and aerodynamic body features would bring these additional characteristics into view. Of course, variations in the results arise from the lack of more precise data for the parameters, and the motion of the ships has been ignored.

Even though our model is not precise, it shows for a simple case that if initially the range of the British guns was erroneously set to correct for a westward Coriolis deflection (appropriate for battle in the northern hemisphere in the vicinity of the British Isles), when fired southward in similar circumstances at the opposite latitude in the South Atlantic Ocean, the barrage would fall to the left of its target, eastward, by a distance nearly double that deflection. If our simplified model is typical of the real circumstances, the actual gross effect must have appeared surprising to the British gunners when, in the situation described by Marion, their "accurately" aimed, southward directed salvos fell 100 yards to the east of the German ships.

6.18.3. Concluding Remarks

There are other kinds of subtle but measurable Coriolis effects. Instead of a single particle model, we may consider a stream of river particles flowing from the north toward the south, like the great Mississippi. The Coriolis force on a fluid particle in the Earth frame is directed westward. We thus see, if only heuristically, that the water will exert greater pressure on the west bank than the east. Geographers have established that this pressure causes greater erosion on the west bank and further that the water level also is slightly but measurably higher on the west bank. The same flow from north to south in the southern hemisphere would induce greater erosion and a higher water level on the river's east bank. The extent of the effect varies, of course, with the geographic latitude. A similar effect occurs for other directions of flow. The Coriolis effect on ocean and tidal currents is similar; the effect on atmospheric air flow and cyclonic motion is more pronounced. All of these measurable effects arise from the fact that the Earth is not an inertial reference frame, and all are predictable from Newton's basic principles of mechanics.

We have seen that the effects due to the Coriolis acceleration, though usually small, certainly are not always negligible. For the sake of simplicity and because the moving Earth frame closely approximates an inertial reference frame, henceforward, unless specified otherwise, the Earth's rotational motion is ignored in future applications. It is nonetheless important that the engineering analyst be aware of potential Coriolis effects and evaluate whether these should be safely excluded in problems of motion relative to the Earth.

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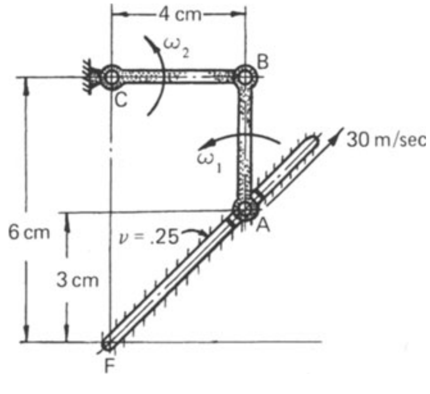
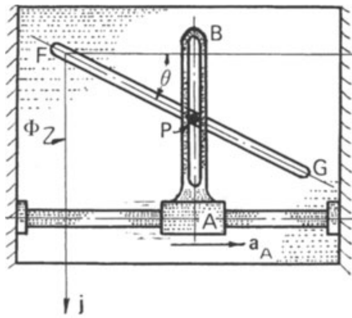
Problems

6.1. The slider block A of a mechanism has mass $m = 400$ gm and moves in the horizontal plane in a straight track with a dynamic coefficient of friction $\nu = 0.25$. At an instant of interest, the links AB and BC are in the positions shown in the figure, and A has a speed of 30 m/sec which is increasing at the rate of 20 m/sec^2. An instrument indicates that link AB is under tension. Find the forces that act on A in the plane of its motion at the moment of interest.



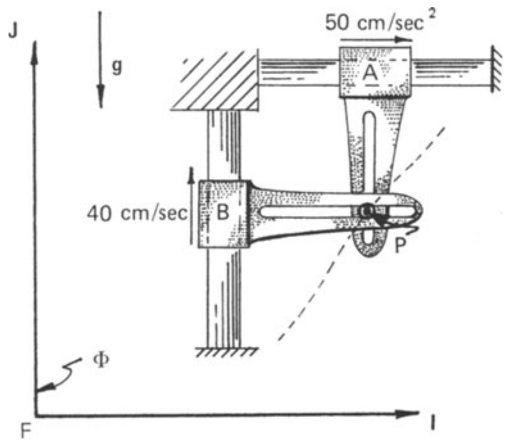
Problem 6.1.

6.2. A small pin P of mass m is constrained to move in a smooth, straight slot FG milled at an angle θ in a flat plate fixed in the horizontal plane. The motion of P is controlled by a smooth, slotted link AB that moves during an interval of interest with constant acceleration $\mathbf{a}_A$ in $\Phi = \{F; \mathbf{i}_k\}$, as shown in the figure. Find as functions of θ the force exerted on P by each slot, and show that the ratio of their magnitudes is a simple function of the angle θ alone. What is the acceleration of P relative to A?



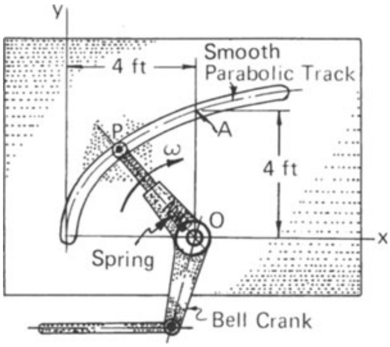
Problem 6.2.

6.3. Two slotted links shown in the figure move on smooth guide rails fixed at right angles to one another, their motion being controlled by a smooth pin of mass $m = 0.04$ kg. At a moment of interest in the machine frame $\Phi = \{F; \mathbf{I}_k\}$, the link A has an acceleration $\mathbf{a}_A = 50\mathbf{I}$ cm/sec^2, and the link B is moving upward with a speed of 40 cm/sec which is decreasing at the rate of 100 cm/sec^2. (a) What is the total force acting on P at the instant of interest? (b) Determine the force that each link exerts on P.



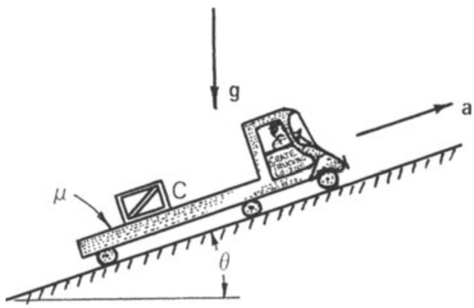
Problem 6.3.

6.4. A small guide pin P of mass 0.2 slug is attached to a spring loaded telescopic arm OP of a bell crank lever hinged at O. The guide pin moves in a smooth, horizontal parabolic track shown in the figure. At the track point A, the pin has a speed of 20 ft/sec, a rate of change of speed of 10 ft/sec^2, and the telescopic arm exerts a uniaxial compressive force on P. Determine for the instant of interest the magnitudes of all forces exerted on the pin P at A.



Problem 6.4.

6.5. The truck shown in the figure moves from rest with a constant acceleration **a** up an incline of angle θ. What is the greatest speed v that the truck can acquire in a distance d, if the crate C is not to slip on the truck bed? The coefficient of static friction is μ.

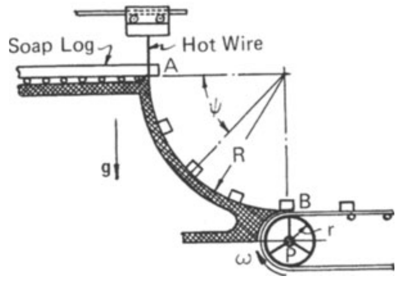


Problem 6.5.

6.6. A mass m is suspended from a point O by an inextensible cord of length ℓ, in a gravity field $\mathbf{g} = -g\mathbf{k}$ directed along the vertical axis through O. The mass rotates about the vertical axis with a constant angular velocity $\boldsymbol{\omega} = \omega\mathbf{k}$. (a) Apply cylindrical coordinates to determine the tension in the string and the vertical distance d from point O to the plane of motion of m. (b) Solve the problem by application of appropriate spherical coordinates.

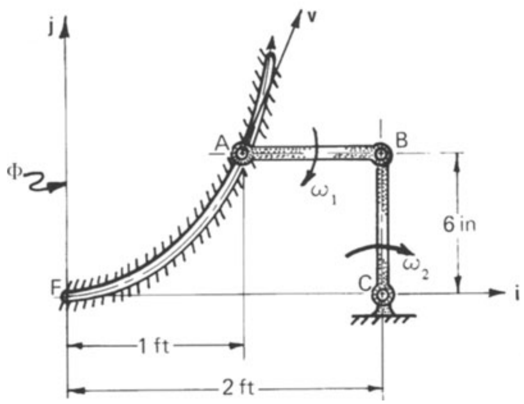
6.7. A small block of mass m rests on a rough, horizontal circular table that spins with a constant angular speed ω about its fixed central axis. What is the largest value that ω may have if the block is to remain at rest at the radial distance r from the center? Explain how this device might be used as an instrument to measure the coefficient of static friction.

6.8. Small bars of soap of equal weight W are cut from a continuous rectangular log by a moving hot wire at the point A in a packaging machine shown in the figure. Each bar is released from rest at A and slides down a smooth, circular chute of radius R to a conveyor belt at B. (a) Determine the constant angular speed of the belt pulley P so that continuous transfer of the bars to the conveyor will occur smoothly without sliding. (b) Find the contact force exerted on a bar of soap as a function of ψ and W. What force will a bar exert on the chute at the point B?



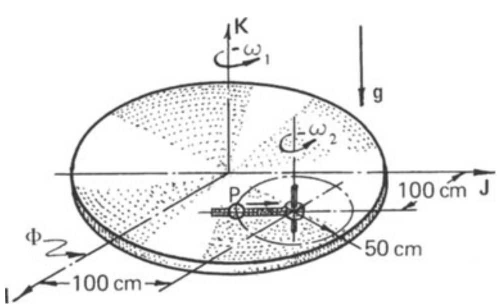
Problem 6.8.

6.9. A mechanism slider A of mass 0.2 slug moves in a horizontal plane in a smooth, parabolic slot defined by $2y = x^2$. At the instant shown in the diagram, A has a speed of 40 ft/sec, decreasing at the rate of $20\sqrt{2}$ ft/sec^2. All joints and surfaces are smooth. Find the plane forces that act on A.



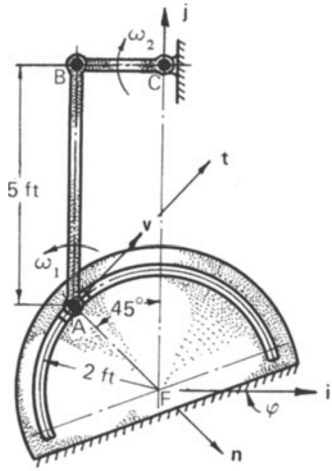
Problem 6.9.

6.10. A particle P weighing 1 N is free to slide on a smooth, rigid wire that rotates at a constant angular speed $\omega_2 = 30$ rad/sec relative to a platform. At the instant shown, the platform has an angular speed $\omega_1 = 15$ rad/sec that is decreasing at the rate of 5 rad/sec^2 relative to the ground frame Φ. The particle is initially at rest on the wire. What central directed force **F** is needed to impart to P an instantaneous initial acceleration of 1 m/sec^2 relative to the wire?



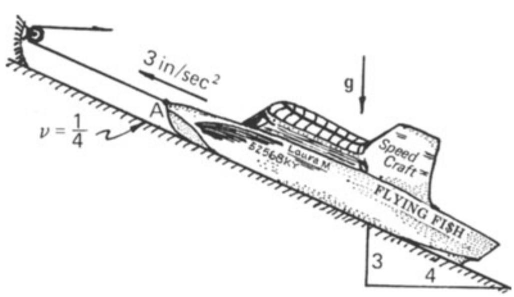
Problem 6.10.

6.11. The slider block A shown in the figure moves in a smooth, circular slot of radius 2 ft milled in a horizontal plate. The slider has a speed of 10 ft/sec, increasing at the rate of 20 ft/sec^2 at the instant when the links AB and BC are perpendicular. The link AB exerts a uniaxial tensile force on A, whose mass $m = 0.10$ slug. (a) Find the forces in the horizontal plane that act on A at this instant. (b) Suppose that the circular slot is rough with coefficient of dynamic friction $\nu = 0.30$, all other conditions being the same as before. Find the forces that act on A at the moment of interest.



Problem 6.11.

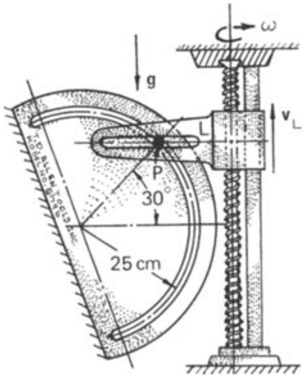
6.12. A 2560 lb boat is being dragged from its place of rest with a constant acceleration of 3 in./sec^2 up a steep inclined boat ramp shown in the figure. The dynamic coefficient of friction is $\nu = 1/4$ and at this place $g = 32$ ft/sec^2. (a) Find the tension in the cable at the connector A. (b) After 8 sec, the connector breaks. How much farther will the boat move up the plane?



Problem 6.12.

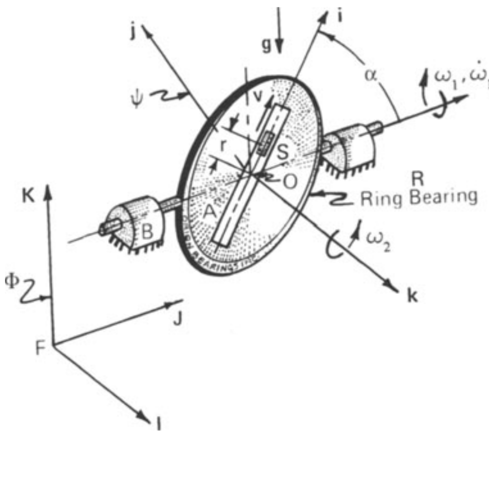
6.13. A guide link L is controlled by a drive screw to move a pin P of mass 50 gm in a circular slot in the vertical plane. The screw has a right-handed pitch $p = 5$ mm and is turning at a constant rate $\omega = 120$ rpm, as described in the figure. Ignore friction. What is the magnitude of the force exerted by the circular slot on the pin at the position shown?

6.14. Suppose that the drive screw described in the previous problem is turning at the rate $\omega = 150$ rpm, as shown, but is slowing down at the rate of 30 rpm each second. Calculate the magnitude of the force exerted by the circular slot on the pin at the position shown. What is the intensity of the force exerted on the pin by the guide link?



Problem 6.13.

6.15. A slider block S of mass $m = 0.5$ slug is constrained to move within a straight cylindrical tube attached to a large disk A supported in a ring bearing R. The machine is situated on the planet Vulcan where $g = 20$ ft/sec^2. The slider maintains a constant speed $v = 2$ ft/sec relative to A, which has a constant angular speed $\omega_2 = \dot{\alpha} = 2$ rad/sec relative to R. At an instant of interest shown in the figure, $\alpha = \tan^{-1}(3/4)$, $r = 2$ ft, and the ring bearing is turning about its horizontal shaft B with angular speed $\omega_1 = 10$ rad/sec and angular acceleration $\dot{\omega}_1 = 5$ rad/sec^2 in the inertial frame $\Phi = \{F; \mathbf{I}_k\}$. Determine the instantaneous value of the total contact force $\mathbf{F}_c$ exerted on S, referred to the frame $\psi = \{O; \mathbf{i}_k\}$ fixed in A.

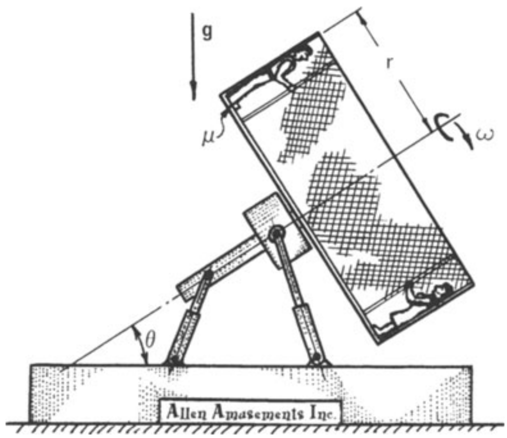


Problem 6.15.

6.16. A small object of mass m rests at the top of a smooth cylinder of radius r. Under the influence of gravity, a negligible disturbance causes the object to slide down the side of cylinder. Determine the angle ϕ_0 and the speed at which the object leaves the cylinder.

6.17. A constant total force $\mathbf{F} = 35\mathbf{i}$ lb acts for 3 sec at the center of mass particle C of a body $\mathscr{B}$ that weighs 161 lb. The initial velocity of C is $\mathbf{v}_0 = 9\mathbf{i} + 40\mathbf{j}$ ft/sec at $\mathbf{x}_0 = 16\mathbf{j}$ ft in the inertial frame $\Phi = \{O; \mathbf{i}_k\}$. (a) Find the velocity and the motion of C as functions of time in Φ. (b) Determine the speed of C after 3 sec. (c) What is its location 2 sec later, and how far did C move during that time? (d) Solve the problem for the same details by application of singularity functions. See Volume 1, Chapter 1, page 47.

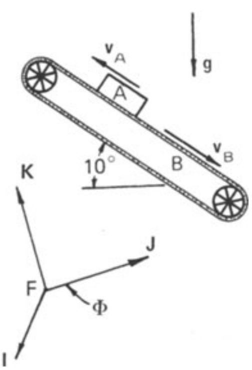
6.18. An amusement park centrifuge shown in the figure consists of a large circular cylindrical cage of radius r that rotates about its axis. People stand against the cylindrical wall, and after the cage has reached a certain constant angular speed ω, to further excite the riders the cage is rotated from its initial horizontal position to an inclined position at an angle θ. Determine the minimum angular speed in order that a passenger will not fall when reaching the highest point in the motion. The coefficient of static friction at the floor is μ.



Problem 6.18.

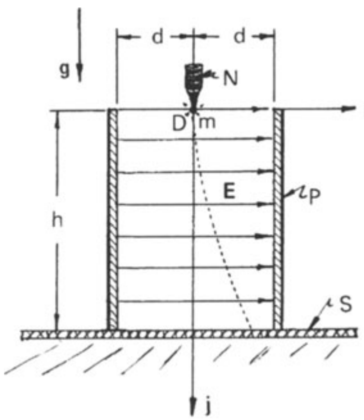
6.19. A small object of mass m rests on a smooth conical surface having an apex angle 2β. The cone turns about its vertical axis with a constant angular velocity $\boldsymbol{\omega} = \omega \mathbf{k}$ in a gravity field $\mathbf{g} = -g\mathbf{k}$. The object is restrained from sliding by an inextensible string of length ℓ attached at the apex on the axis of rotation. Identify appropriate spherical coordinates and apply (6.5) to determine the critical angular speed at which the object will leave the surface. What is the tension in the string at the critical speed?

6.20. The figure shows a box A moving upward on a loading belt B inclined at 10° and moving downward with a constant speed of 150 cm/sec relative to the ground frame $\Phi = \{F; \mathbf{I}_k\}$. At the initial instant, the speed of A is 50 cm/sec relative to Φ; and the coefficient of friction between the sliding bodies is $\nu = 0.3$. How long does it take to reduce the relative speed between the bodies to 25 cm/sec? Frame Φ should be suitably oriented for convenience.



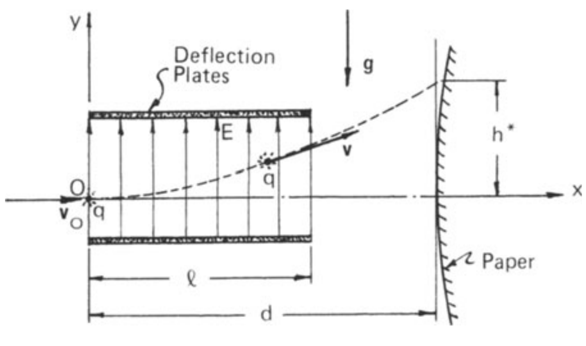
Problem 6.20.

6.21. An electrically conducting droplet of paint D of mass m and charge q, initially at rest at the tip of a nozzle N, falls through a uniform electric field of strength $\mathbf{E}$, directed as shown in the figure, and ultimately impacts a flat sheet S beneath it. The field deflection plates P have width $2d$ and height h. (a) Derive the equation of the path traveled by D. What is the maximum intensity of $\mathbf{E}$ that will still allow a droplet to impact S? (b) If the droplet D has an initial speed of 40 cm/sec, what electric field strength, directed as before, must be applied to produce a motion $\mathbf{x}(D,t) = 6t^2\mathbf{i} + (\beta t + \gamma t^2)\mathbf{j}$ cm, in which β and γ are constants? Determine β and γ. (c) Find the free fall droplet trajectory when the apparatus is tilted counterclockwise to an angle θ from the vertical axis.



Problem 6.21.

6.22. The deflection plates of an ink jet printer are arranged as shown in the figure. A charged ink droplet q enters the constant electric field $\mathbf{E}$ with the initial horizontal velocity $\mathbf{v}_0$ at point O. Find the trajectory $y = y(x)$ of q for $0 \le x \le d$ and determine the droplet deflection h^* at the paper surface, approximated as a plane. Show that, independent of g, the deflection h derived in (6.28d) for the case when $\ell = d$ is larger than the deflection h^* by an amount $h - h^* = (cE/2v_0^2)(d-\ell)^2$, where $c = q/m$.



Problem 6.22.

6.23. A bullet of mass m is fired directly into a fluid that exerts on the bullet a drag force that is proportional to its linear momentum. The gun has a muzzle velocity $\mathbf{v}_0$. Neglect gravity and other fluid forces. Determine the total distance traveled by the bullet.

6.24. Water exerts a drag force on a boat which is proportional to the cube of its speed. When the power is cut off, the boat's speed decreases from v_0 to $v(t)$ in time t. Find the distance traveled by the boat and determine t.

6.25. Consider a particle Q initially at rest at O in frame $\Phi = \{O; \mathbf{i}_k\}$ and acted upon by a constant force $\mathbf{f} = 4\mathbf{i} - \mathbf{j} + 32\mathbf{k}$ lb/unit mass and by a drag force $\mathbf{f}_D = -\dot{x}\mathbf{i} - 2\dot{y}\mathbf{j} - 0.4\dot{z}\mathbf{k}$ lb/unit mass. Find the velocity and the place in Φ occupied by Q after 2 sec.

6.26. A shell fired vertically upward from the ground with a muzzle velocity $\mathbf{v}_0$ experiences air resistance proportional to the square of its speed. (a) Determine the shells speed and altitude as functions of time. (b) What is the maximum altitude attained by the shell? (c) Find the time t^* required to reach the maximum height and show that no matter how large $\mathbf{v}_0$ may be, t^* cannot exceed $\pi\tau/2$. Identify the time constant τ.

6.27. A ball dropped from rest at the origin experiences air resistance proportional to the square of its speed. (a) Find its speed after the ball has fallen a distance h. What is its terminal speed? (b) Determine as functions of time the speed and the distance through which the ball has fallen. Sketch and label a nondimensionalized graph of the speed versus time and describe the results in a manner similar to Example 6.11, page 120.

6.28. Consider the following integral

$$u(t) = \int_{g(t)}^{h(t)} F(\tau; t)d\tau, \tag{P6.28a}$$

wherein $F(\tau; t)$ is an integrable function of τ and also depends continuously on a parameter t. Notice that the limits of integration are continuous functions $g(t)$ and $h(t)$ of t. (a) Use the definition of the derivative of a function $u(t)$, namely,

$$\frac{du(t)}{dt} = \operatorname*{limit}_{\Delta t \to 0} \frac{u(t + \Delta t) - u(t)}{\Delta t}, \tag{P6.28b}$$

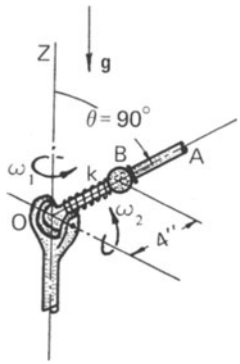
apply the mean value theorem of integral calculus, and derive Leibniz's formula for the derivative of the integral (P6.28a):

$$\frac{du(t)}{dt} = F(h(t); t)\frac{dh(t)}{dt} - F(g(t); t)\frac{dg(t)}{dt} + \int_{g(t)}^{h(t)} \frac{dF(\tau; t)}{dt} d\tau. \tag{P6.28c}$$

(b) Apply this rule to show that (6.47) is a particular solution of the differential equation (6.39).

6.29. Derive from (6.47) the particular solution (6.45b) of the differential equation (6.39) when $h(t)$ is given by (6.45a).

6.30. A ball governor of a speed control device consists of an arm *OA* hinged at O to a vertical shaft *OZ* that rotates relative to the machine with a constant angular speed $\omega_1 = 9$ rad/sec, as shown. At the same time, *OA* is elevated at a constant angular rate $\omega_2 = 3$ rad/sec relative to the shaft and a ball B of mass 0.02 slug slides on the smooth arm. The ball is attached to a spring, the other end of which is fastened to the arm. Design criteria specify that the shut-off position at which the ball comes to rest on *OA* must be 4 in. from O; and for the position shown at $\theta = 90°$, the spring must elongate 2 in. to achieve shut-off. Find the spring constant in units of lb/in. that will satisfy the design shut-off criteria. What force is exerted by the spring in this position?



Problem 6.30.

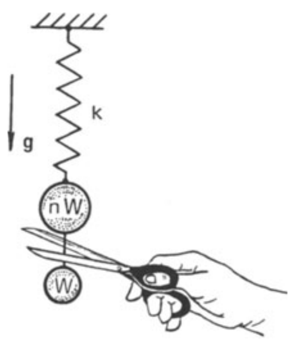
6.31. Consider a rigid body rotating through an angle $\theta(t)$ about a fixed axis with unit direction $\boldsymbol{\alpha}$. Notice that the velocity vector $\dot{\mathbf{x}}(P,t) = \dot{\theta}\boldsymbol{\alpha} \times \mathbf{x}$ of a body point P at $\mathbf{x}(P,t)$ from a point O on $\boldsymbol{\alpha}$ yields the equation $d\mathbf{x}/d\theta = \boldsymbol{\alpha} \times \mathbf{x}$ relating $\mathbf{x}$ and θ. (a) If initially $\mathbf{x}(P,0) = \mathbf{x}_0$ and $\theta(0) = 0$, prove that $\mathbf{x} \cdot \boldsymbol{\alpha}$ is a constant, and derive the relation $d^2\mathbf{x}/d\theta^2 + \mathbf{x} = (\boldsymbol{\alpha} \cdot \mathbf{x}_0)\boldsymbol{\alpha}$. Hint: Notice that $d\mathbf{x}/d\theta$ is perpendicular to $\boldsymbol{\alpha}$. (b) Determine the general solution of this vector differential equation. This involves two constant vectors of integration, say **A** and **B**. (c) Find **A** and **B** and thus show that the solution yields (2.7), Volume 1, for the displacement of a particle P of a rigid body in its finite rotation about the fixed line.

6.32. The motion of a particle P initially at rest at the origin is governed by the equation $\ddot{x} - q^2x = e^{qt}$. Find the motion of P.

6.33. The motions of two particles P and Q are governed by the following scalar equations of motion: $\ddot{x}(P,t) + p^2x(P,t) = g$ and $\ddot{x}(Q,t) - p^2x(Q,t) = g$, in which p and g are constants. Initially, each particle is started separately at the place $x(0) = x_0$ with a speed v_0. Find the motions of P and Q and discuss their physical nature. Determine their common motion when $p = 0$.

6.34. A linear spring-mass system shown in its natural state in Fig. 6.13, page 134, is given an instantaneous initial speed $v_0 = 3$ ft/sec on a smooth horizontal surface. The mass $m = 8$ lb$_m$ and the spring stiffness $k = 3$ lb/in. Suppose that $g = 32$ ft/sec^2. What is the maximum displacement of m? Caution: See Chapter 5 remarks on measure units, page 86.

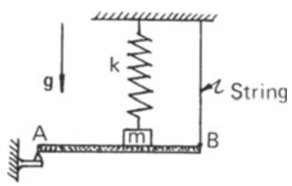
6.35. A linear spring of stiffness k supports weights W and nW connected by a cord, as shown. Initially, the system is at rest. (a) Determine the acceleration of the load nW immediately



Problem 6.35.

after the cord supporting the load W is cut. (b) Find the motion $z(t)$ of the load nW from the undeformed natural state of the spring. (c) Determine the motion $x(t)$ from the initial stretched state of the spring. (d) What is the motion $\xi(t)$ from the static equilibrium state of the load nW? (e) Which of the three motions is the simpler? Are they equivalent? How are they related?

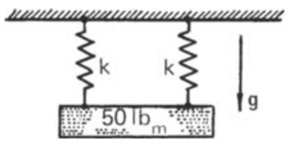
6.36. The figure shows an unstretched linear spring of stiffness k attached to a small block of mass m at rest on a horizontal board simply supported at A and suspended by a string at B. The string is cut and the board falls clear of m. Derive the equation of motion for the mass and determine its subsequent motion. How long does it take for m to return to its initial position?



Problem 6.36.

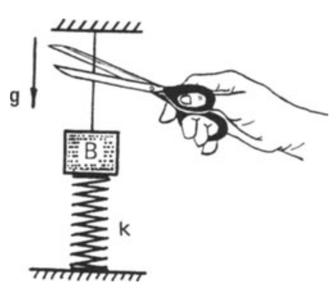
6.37. A certain simple harmonic oscillator has mass $m = 2$ slug and an equivalent spring constant $k_e = 600$ lb/in. The load is released at $u_0 = 2$ in. with a speed $\dot{u}_0 = -20$ in./sec directed toward its equilibrium position. Determine the frequency, amplitude, and initial phase of its motion $u(t)$.

6.38. A load $m = 50$ lb$_m$ is supported as shown by two linear springs having the same elasticity $k = 25$ lb/in. (a) Find the static stretch of each spring from its natural state and determine the stiffness of a single equivalent spring that may replace the parallel pair. (b) The mass is given an additional 2 in. displacement and released. Find its maximum speed and determine its greatest height from the equilibrium position. How long does it take to first attain these states? Compare these times with the period of the vibration.



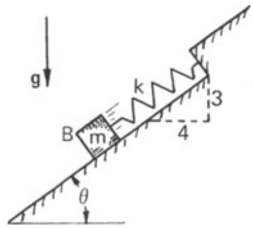
Problem 6.38.

6.39. The figure shows a block B weighing 25 N suspended by a string and attached to a linear spring of stiffness $k = 20$ N/cm in its natural state. Determine the amplitude, the frequency, and the position about which the vibration will occur when the string is suddenly cut.



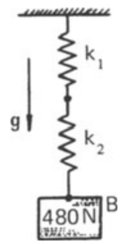
Problem 6.39.

6.40. A small block B of mass $m = 0.25$ slug is attached to a linear spring of stiffness $k = 16$ lb/ft in a gravity field of strength $g = 32$ ft/sec^2. The spring is compressed 6 in. from its natural state and the mass is released to execute oscillations on a smooth plane inclined as shown in the figure. Find the motion as a function of time and determine its frequency and amplitude.



Problem 6.40.

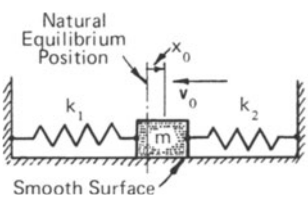
6.41. The figure shows a box B of weight 480 N supported by uniaxial linear springs having constant elasticities $k_1 = 40$ N/cm and $k_2 = 60$ N/cm. (a) Find the static displacement δ of B and determine the stiffness of a single equivalent spring that may replace the series pair. (b) The box is displaced an additional 5 cm from δ and released. What is the period of its vibration? (c) What is the location of the box from its static state 2 sec after its release?



Problem 6.41.

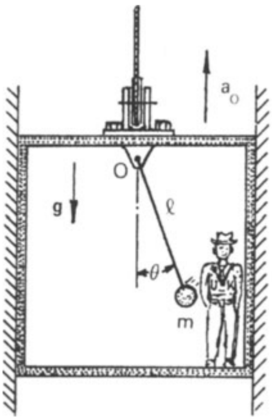
6.42. A particle P of mass m and charge q moves in an electromagnetic field of constant field strengths $\mathbf{E} = E\mathbf{i}$ and $\mathbf{B} = B\mathbf{k}$ in an inertial frame $\Phi = \{O; \mathbf{i}_k\}$. Initially, P is at rest at O. Find the motion $\mathbf{x}(P, t)$ of P in Φ and characterize its path. Neglect gravity.

6.43. Two unstretched, linear springs having moduli k_1 and k_2 are fastened, as shown, to a slider mass m that rests on a smooth horizontal surface. The slider is displaced a distance x_0 and released with speed v_0 directed toward the natural state. (a) What are the circular frequency, the period, and the amplitude of the vibration? (b) Derive the subsequent motion of m. Sketch the motion as a function of $\theta = pt$ and label its major features.



Problem 6.43.

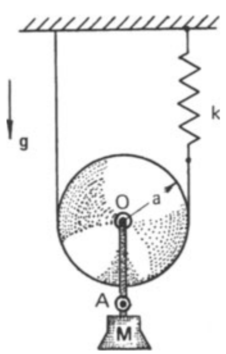
6.44. A simple pendulum shown in the figure is supported by a light, hinged rod hung from the ceiling in an elevator which moves upward with a constant acceleration $\mathbf{a}_O$. A curious person displaces the bob a finite angular amount θ_0 and releases it. (a) Find as a function of θ the ratio of the tension in the rod to the weight of the bob. (b) For small placements θ_0, what will be the circular frequency and the period of the pendulum motion witnessed by the person? (c) How will these results be changed if the elevator accelerates downward at the same rate? Describe any potentially unusual effects.



Problem 6.44.

6.45. According to elasticity theory, the infinitesimal circumferential engineering strain ϵ of a homogeneous, thin circular ring undergoing pure radial oscillations in its horizontal plane is given by $\epsilon = u/r$, where r denotes the undeformed radius and u is the infinitesimal radial displacement of the ring. The ring has uniform cross sectional area A and mass density μ per unit length. (a) Consider a circumferential ring element of mass $dm(P)$ at a material point P. Apply Hooke's law for the uniform circumferential engineering stress $\sigma = E\epsilon$, where E is Young's modulus, and derive the equation for the radial motion. Recall that the circumferential force F is defined by $F = \sigma A$. (b) Determine the circular frequency and period. (c) What is the stiffness of an equivalent linear spring-mass system that will produce the same vibrational frequency of a load equal to total mass of the ring?

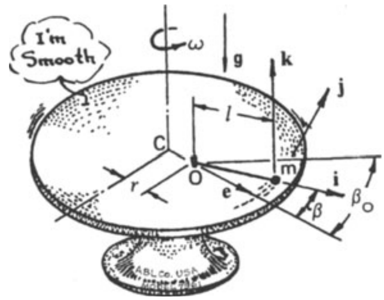
6.46. The spring and pulley system shown in the figure supports a load of mass M. The spring has stiffness k and the masses of the cable, pulley, and load support bar are negligible.



Problem 6.46.

Neglect friction and determine the circular frequency and period of the free vibration of the load in its vertical displacement $x(t)$ from the static equilibrium state of the system.

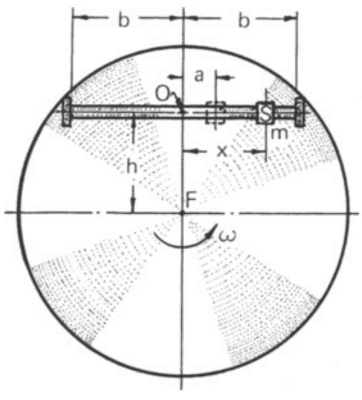
6.47. A pendulum bob of mass $m = 0.01$ kg is fastened by a string of length $l = 16$ cm to a hinge pin at $r = 4$ cm from the center of a smooth horizontal table on which the bob rests. The table turns with a constant angular speed ω, as shown in the figure. Relative to an observer in the table reference frame, the pendulum executes oscillations of small amplitude β_O and period $\tau = 0.5$ sec. Find the angular speed of the table and compute the string tension T when $\beta = \beta_O$.



Problem 6.47.

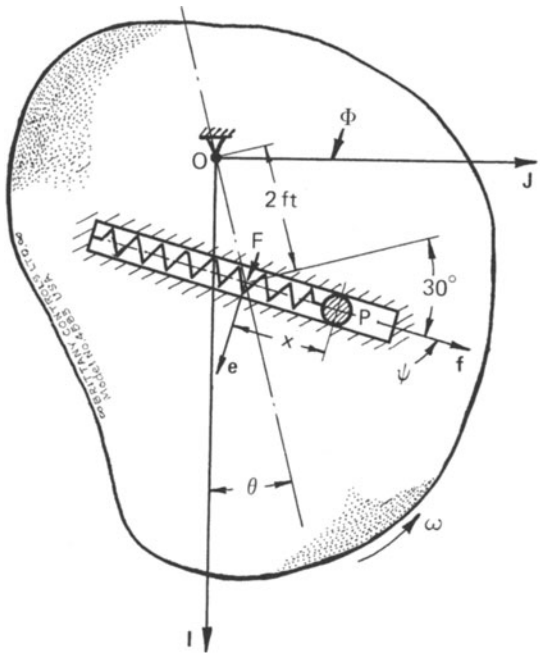
6.48. Gravitational attraction by a fixed, homogeneous, thin ring of radius R and mass M induces a particle P of mass m to move along its normal central axis, as shown in Fig. 5.13. (See Example 5.6, page 38.) (a) Derive the differential equation of motion for P. (b) Show that for sufficiently small displacements $\mathbf{X}(P, t)$ from the center O, the motion of P is simple harmonic. What is the frequency of its small oscillations?

6.49. A smooth, rigid rod of length $2b$ is attached to a table that turns in the horizontal plane with a constant angular velocity ω, as shown. A slider block S of mass m is released from rest relative to the rod at a distance a from its midpoint O. (a) Determine the horizontal force $\mathbf{R}$ exerted by the rod on the slider as a function of its distance x from O. (b) Find of the motion of S relative to the table.



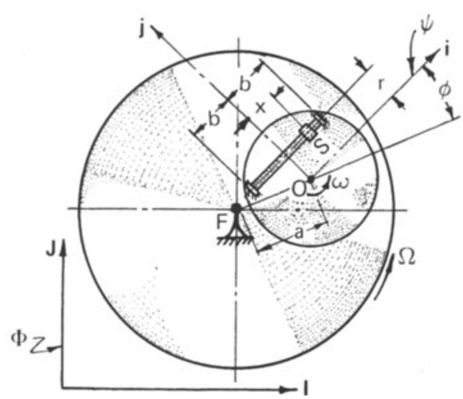
Problem 6.49.

6.50. A small ball P of mass m slides in a smooth slot cut in a flat plate, as described in the diagram. The plate rotates in the horizontal plane with a constant angular speed $\dot{\theta} = \omega$ about an axle at O in frame $\Phi = \{O; \mathbf{I}, \mathbf{J}\}$ fixed in the plane space. The ball is attached to a linear spring of modulus k, which initially is unstretched when P is released from rest relative to the plate at F. (a) Find the motion $\mathbf{x}(P, t)$ of P relative to the plate for all constant values of the angular speed ω. (b) Determine the force exerted on P by the slot as a function of x and as a function of t. (c) Characterize all physical aspects of the motion of P for all values of ω. Refer all quantities to the plate frame $\psi = \{F; \mathbf{e}, \mathbf{f}\}$.



Problem 6.50.

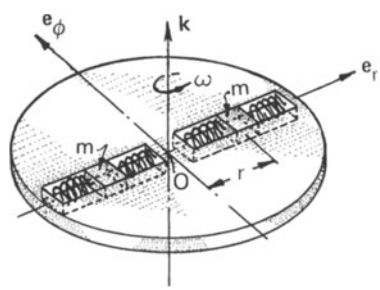
6.51. A block S of mass m is free to slide on a smooth rod of length $2b$ shown in the figure. The rod is fastened to a circular disk that rotates about an axle at O with a constant angular velocity



Problem 6.51.

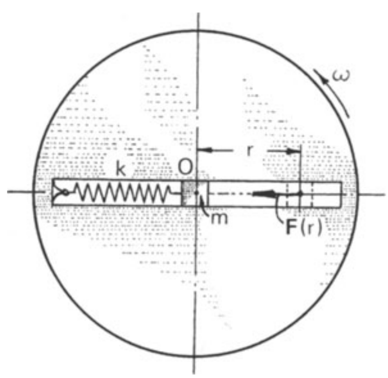
$\boldsymbol{\omega} = \omega \mathbf{k}$ relative to a turntable. The turntable spins in the horizontal plane with a constant angular velocity $\boldsymbol{\Omega} = \Omega \mathbf{k}$ about an axle at F in the ground frame. (a) Account for all forces that act on S and derive its scalar equations of motion referred to the disk frame $\psi = \{O; \mathbf{i}_k\}$. What unknown quantities do these equations determine? (b) Suppose that S is initially at ease at $x = 0$. Determine the unknown quantities as functions of time.

6.52. The diagram shows two slider blocks of equal mass m attached to precompressed springs of equal stiffness $k/2$. The blocks are confined to slide horizontally in smooth radial slots in a table that spins counterclockwise with a constant angular speed ω. Each block is positioned at a distance ℓ from the center O when $\omega = 0$. If each spring is always under compression, determine the equilibrium position $r = r_S$ of each block relative to the table. Examine the stability of this relative equilibrium state.



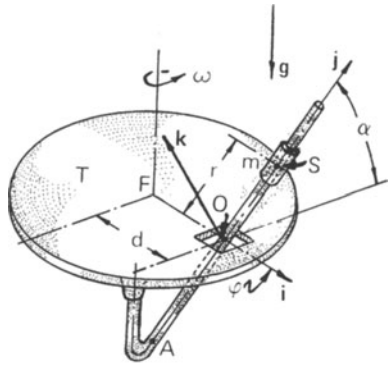
Problem 6.52.

6.53. The figure shows a slider block of mass m attached to a spring of stiffness k in its natural state at the center of a smooth rotating table upon which it rests in the horizontal plane. The table turns with a constant anticlockwise angular speed ω. (a) Determine the equation for the motion $r(t)$ of the slider and examine the stability of the relative equilibrium states. (b) Note that Hooke's spring law (6.64) is the same in every reference frame and for every observer, that is, the same extension of the spring in a fixed reference frame and in any other reference frame having an arbitrary motion gives rise to the same force. The spring force is an internal action. The inertial forces induced by the motion of the frame are external actions of the environment on the system. The rotating observer, however, may perceive a pseudo-spring force $\mathbf{F}(r)$ with stiffness k^* that includes these inertial effects of the environment. What pseudo-spring force and apparent stiffness are perceived by an observer in the table frame? (c) Discuss the character of the motion as ω is gradually varied.



Problem 6.53.

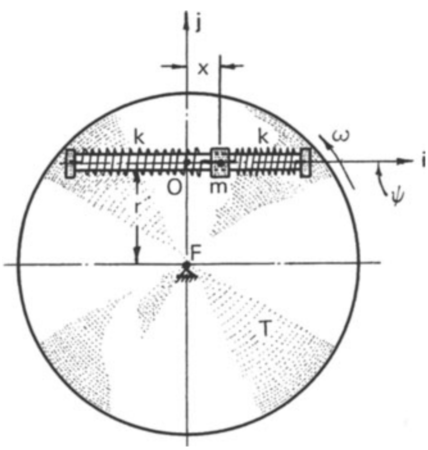
6.54. A block S of mass m slides freely on a smooth rigid rod inclined at an angle α with the horizontal plane of a rotating table T to which the rod is fastened, as shown in the figure. The table turns with a constant angular velocity $\boldsymbol{\omega}$ about a fixed vertical axis. If S is projected upward from point O in the plane of T with an initial speed v_0 relative to T, determine its subsequent position as a function $r(t)$. Find the initial force exerted on S by the rod. Refer all quantities to the frame $\varphi = \{O; \mathbf{i}_k\}$ fixed in the rod.



Problem 6.54.

6.55. Suppose in the previous problem that a coaxial spring of elasticity k is attached to the smooth rod at point A and to the block S. The spring is unstretched when S is at O where its initial speed is v_0, as before. (a) Determine the relative equilibrium positions r_S of S. (b) Find the motion $r(t)$ of S relative to the table for all values of the angular speed ω. (c) Discuss the stability of the relative equilibrium states of S. Refer all quantities to the rod frame $\varphi = \{O; \mathbf{i}_k\}$.

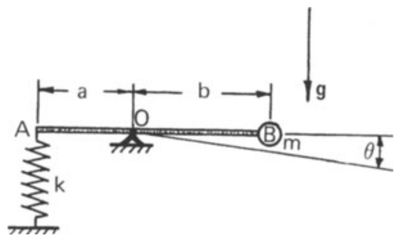
6.56. A smooth rigid rod, whose geometry is described in the figure, is attached to a table T that rotates in the horizontal plane with a constant angular velocity $\boldsymbol{\omega} = \omega\mathbf{k}$. A slider block of mass m, supported symmetrically by identical springs of elasticity k, is released from rest relative to the rod at a distance a from the natural state at point O. (a) Determine the rod reaction force on m as a function of its distance x from O. (b) Determine the critical angular speed ω^*



Problem 6.56.

of the table for which a simple harmonic motion is not possible. (c) Find the motion $x(m, t)$ for the three cases for which $\omega < \omega^*$, $\omega = \omega^*$, and $\omega > \omega^*$. What are the period and the amplitude of the motion of m in the oscillatory case? Use the table frame $\psi = \{O; \mathbf{i}_k\}$ as reference.

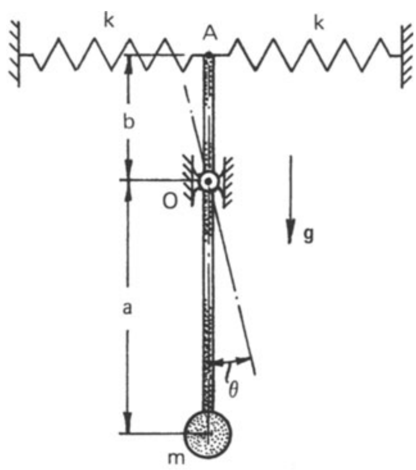
6.57. A mass m is attached to one end of a rigid rod supported by a smooth hinge at O and by a spring of stiffness k at A. The rod has negligible mass and the system is in equilibrium in the horizontal position shown in the figure. The mass is given a small angular placement and released. Apply the moment of momentum principle to derive the equation for the angular motion $\theta(t)$ of m and find the frequency of its small oscillations.



Problem 6.57.

6.58. Apply the moment of momentum relation (6.80) for a moving point O to derive the equation of motion of the pendulum bob in Problem 6.47.

6.59. A pendulum bob of mass m is attached to one end of a thin, rigid rod suspended vertically by a smooth hinge at an intermediate point O. The rod is fastened at its other end to identical springs of stiffness k, shown in the figure in their undeformed configuration. The pendulum is given a small angular placement θ_0 and released with a small angular speed ω_0 toward the vertical equilibrium state. Ignore the mass of the rod. Find the motion $\theta(t)$ of the bob and describe its physical characteristics.

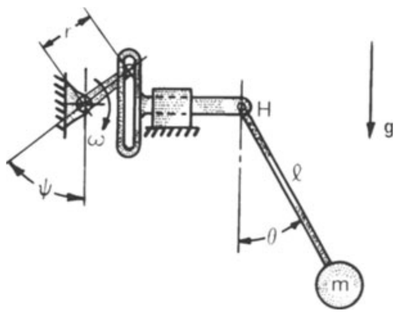


Problem 6.59.

6.60. Problem 4.48 in Volume 1 illustrates a simple pendulum of mass m and length ℓ hung from a sliding support that oscillates vertically with a motion $x(S, t) = a + b \sin pt$, where a, b, p are constants. Derive the scalar equations of motion for the bob. What quantities do these equations determine? This is a difficult nonlinear problem whose exact solution is unknown. For small amplitude pendulum oscillations, however, the motion $\theta(t)$ is described by Mathieu's linear differential equation, whose analysis, though well-studied, is not elementary. Let $2z = pt + \pi/2$ and thus show that the Mathieu form of the equation of motion for small angular placements is

$$\frac{d^2\theta}{dz^2} + \left(\frac{4g}{p^2\ell} - \frac{4b}{\ell}\cos 2z\right)\theta = 0. \tag{P6.60}$$

6.61. The hinge support H for a simple pendulum of mass m and length ℓ is attached to a Scotch mechanism. The crank has radius r and turns with a constant angular speed ω, as illustrated. (a) Derive the differential equation of motion for the bob m. (b) This equation has no known exact solution. Show, however, that for a small angular motion $\theta(t)$ the differential equation reduces to the equation of motion for the forced vibration of an undamped, harmonic oscillator. Find its solution when the pendulum is released at a small angle θ_0 with $\dot{\theta}(0) = 0$.

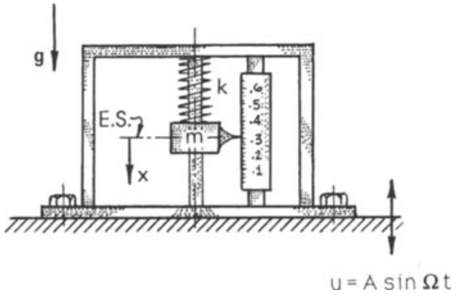


Problem 6.61.

6.62. Discuss the free vibrational motion (6.86l) of the heavily damped oscillator in relation to Fig. 6.22, page 156. Show that if the mass is released from rest, it can only creep back to its equilibrium position at $z = 0$ as $t \to \infty$ (similar to Curve 2). However, if released with initial velocity v_0, it is possible that the load may cross its equilibrium position at one and only one instant t_o, as suggested in Fig. 6.22. Find t_o.

6.63. Repeat the details of the last problem for the critically damped, free vibrational motion described in (6.86n).

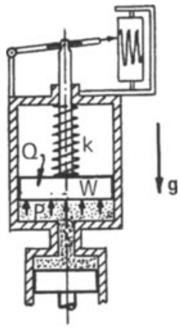
6.64. The pointer of a vibration instrument has mass m and is supported vertically by a spring of stiffness k. The base is subjected to a vertical motion $u = A \sin \Omega t$. (a) Derive the



Problem 6.64.

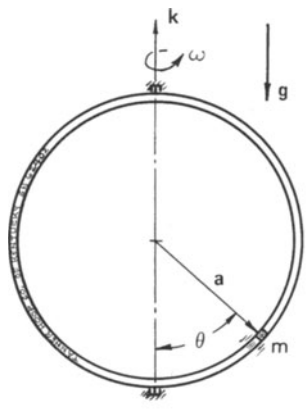
equation for the steady-state motion $x(t)$ of the pointer relative to the instrument and determine its amplitude. (b) Let $k = 5$ N/mm, $m = 2$ kg, and suppose that the pointer moves between the 0.35 and 0.45 scale marks when the base motion has frequency $\Omega = 100$ rad/sec. Determine the amplitude of the base motion. (c) Now suppose further that the base motion frequency is doubled while its amplitude is unchanged. What will be the response range of the pointer? Is the pointer amplitude increased or decreased? (d) Is the system operating above or below its resonant frequency?

6.65. The steam pressure indicator shown in the figure is an instrument that records the time varying cylinder pressure generated in an engine. The piston Q, with surface area A, is restrained by a spring of stiffness $k = 100$ lb/in. on one side and subjected to a periodically varying engine cylinder pressure $P = P_0 \cos \omega t$ on the other. The pressure produces forced vibrations of the piston which are recorded on a uniformly rotating drum. The design requires that the natural, free vibrational frequency p of the piston and recording pen assembly, which has a total effective weight W, shall be much greater than the cylinder pressure fluctuation frequency ω. Frictional effects may be considered negligible. Derive the equation of motion for the piston assembly relative to its static equilibrium position and estimate the weight limit of the assembly if the pressure fluctuation frequency is not to exceed 10 Hz.



Problem 6.65.

6.66. A heavy bead of mass m slides freely in a smooth circular tube of radius a in the vertical plane. The tube spins with constant angular speed about the vertical axis, as shown.

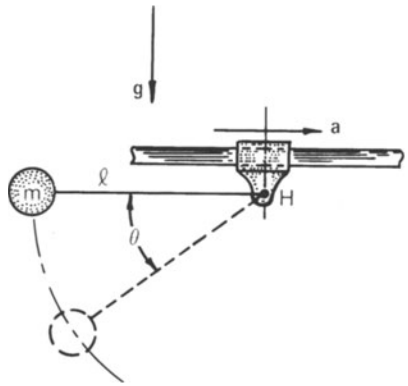


Problem 6.66.

(a) Derive the equation of motion two ways: (i) by use of the moment of momentum principle and (ii) by application of the Newton–Euler law. (b) Examine the infinitesimal stability of all relative equilibrium positions of the bead.

6.67. Experiment shows that the undamped, forced horizontal motion of the system shown in Fig. 6.20, page 152, has a steady-state amplitude H_1 when the driving frequency is Ω_1. When the machine is speeded up to double the driving frequency, the amplitude is reduced to 20% of its previous value. What is the resonant frequency of the system? Was the test data obtained above or below the resonant frequency?

6.68. The supporting hinge H of a simple pendulum of mass m and length ℓ is attached to a horizontal slider that has a constant acceleration **a**. The pendulum is released from rest in a horizontal position relative to the slider, as shown in the figure. (a) Find the pendulum string tension $T(\theta)$ as a function of its angular displacement θ. (b) Show that the other extreme position of the pendulum is given by $\theta_e = 2\tan^{-1}(g/a)$ and determine the string tension in terms of $a = |\mathbf{a}|$ at both extremes. (c) Derive an equation for the time t_e required to attain the position θ_e. (d) Determine all positions of relative equilibrium and examine their infinitesimal stability in terms of the assigned parameters only. Refer all quantities to the natural intrinsic frame for m.

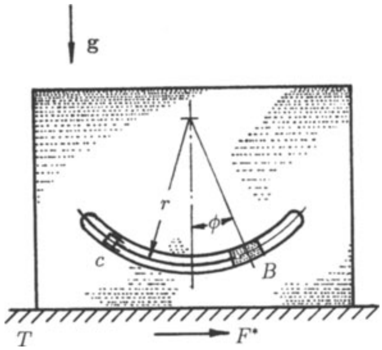


Problem 6.68.

6.69. A slider block B of mass m oscillates in a smooth circular groove of radius r milled in a plate in the vertical plane. The slider is attached to a linear viscous damper of circular design and damping coefficient c. The assembly is mounted on a shaker table T that exerts a horizontal driving force $F^* = F_0 \sin \Omega t$, as shown. (a) Derive the differential equation for the finite amplitude motion of B about its vertical equilibrium position and find an equation for the force exerted by the groove on the slider. (b) Now suppose that the shaker table is arrested and the damper is removed so that $\Omega = 0$ and $c = 0$ in the equation of motion. The block is then released from rest at a finite angle $\phi(0) = \phi_0$. Derive an exact integral relation that determines the period of the finite motion as a function of ϕ_0. What is the period of the small amplitude motion?

6.70. Consider the shaker table (Problem 6.69) for the case when the angular placement $\phi(t)$ of the slider is small. (a) Find the steady-state and transient parts of the motion $\phi(t)$. (b) What is the resonant frequency of the system? (c) What is the amplitude at the resonant frequency? (d) Identify the amplitude factor for the system.

6.71. A small cylindrical block B of unit mass oscillates with a simple harmonic motion $y = a\cos pt$ in a smooth, straight cylindrical tube oriented in the east–west direction on the Earth's surface at north latitude λ. The parameters a and p are constants. Show that in addition

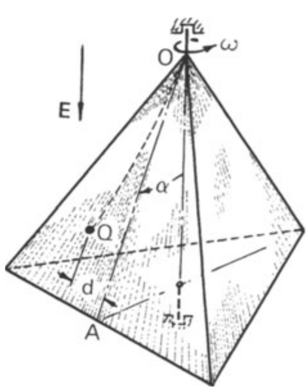


Problem 6.69.

to the weight of B, the Earth's rotation induces a tube reaction force on B which has both a north–south component and a vertical, radially directed component. Although these additional force components are very small compared with the weight of B, over a period of time they eventually induce wear of the tube surface, for example.

6.72. The motion of a particle on a smooth plane inclined at an angle γ is determined by the coupled equations $\ddot{x} - 2\dot{y}\omega\cos\gamma = g\sin\gamma$, $\ddot{y} + 2\dot{x}\omega\cos\gamma = 0$, in which ω is a small constant for which terms of $O(\omega^2)$ may be neglected and g is the acceleration of gravity. If the particle starts from rest at the origin of the inclined plane frame $\psi = \{O; \mathbf{i}, \mathbf{j}\}$, show that after a time t the particle has been deflected a distance $d(t) = \frac{1}{6}g\omega t^3 \sin 2\gamma$ from the **i**-axis.

6.73. A particle Q of mass m and charge $q > 0$ moves in outer space down the side of a smooth, right pyramid that rotates with a small, constant angular speed ω about its fixed vertical axis in a constant electric field **E** directed as shown in the figure. Show that if the particle starts from rest at the apex O, its trajectory suffers a deflection $d(t)$ from the normal, altitude line OA which, to the first order in ω, is given by $d(t) = \frac{1}{6}\omega c E t^3 \sin 2\alpha$. Herein α denotes the surface inclination from the vertical axis and $c \equiv q/m$.



Problem 6.73.

6.74. A projectile is fired from the ground at north latitude λ with an initial velocity $\mathbf{v}_0$ directed skyward and it attains the ultimate altitude h. Neglect air resistance; assume that h is sufficiently small that effects due to altitude variations in **g** may be ignored; and include only first order effects of the Earth's rotation rate Ω. (a) Determine the Coriolis deflection $d^*(h)$ when the projectile reaches the height h. (b) Show that the projectile strikes the ground to the west of its launching site at a distance $d = \frac{8}{3}\Omega h \cos\lambda\sqrt{2h/g}$. (c) Find expressions for d^* and d in terms of the initial speed of the projectile.

6.75. A person seated at the wall in a cylindrical amusement park centrifuge of radius a tosses a ball B straight upward into the sky. The centrifuge has a constant angular velocity $\boldsymbol{\omega}$ relative to the Earth at north latitude λ. Derive the scalar equations of motion for B referred to the centrifuge frame $\psi = \{O; \mathbf{i}_k\}$. Include the effects of the Earth's rotation and identify the appropriate initial data. Check your result against the text solution in (6.109)–(6.111) for the free fall case when $\boldsymbol{\omega} = \mathbf{0}$. Show that when $\boldsymbol{\Omega} = \mathbf{0}$, $\dot{x}^2 + \dot{y}^2 = \omega^2(r^2 - a^2)$, where $r(t)$ is the radial distance of B from the centrifuge axis at time t.

6.76. British battle maps for the Falkland Islands conflict of 1914 show that the British directed their fire on the Germans from the north, almost directly southward, while heading east at a constant flank speed v^*. (a) Derive equation (6.117), for the Coriolis deflection relative to the ship, at 50 ° N and S latitudes; and find the projectile range and its Coriolis variation. (b) Determine the range and the Coriolis deflection for a shell fired with a muzzle velocity $V = 2650$ ft/sec at an angle of elevation $\alpha = 10°$. If the gun sight design corrected for the Coriolis effect only near 50 °N latitude, what is the total deflection by which the British shells would miss a German cruiser when fired at 50° S latitude? (c) Discuss any situations where the deflection may vanish when $\Omega \neq 0$.