5

The Foundation Principles of Classical Mechanics

> I would mention the experience that it is exceedingly difficult to expound to thoughtful hearers the very introduction to mechanics without being occasionally embarrassed, without feeling tempted now and again to apologize, without wishing to get as quickly as possible over the rudiments and on to examples which speak for themselves. I fancy that Newton himself must have felt this embarrassment....
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> *The Principles of Mechanics*

5.1. Introduction

Dynamics is the theory of motion and the forces and torques that produce it. This theory integrates our earlier studies of kinematics, the geometry of motion, with certain fundamental laws of nature that relate force, torque, and motion. In this chapter the primitive concepts of mass and force introduced in Chapter 1 are related to motion through some basic principles commonly known as *Newton's laws.* Sir Isaac Newton (1642–1727) in his *Philosophiae Naturalis Principia Mathematica* (Mathematical Principles of Natural Philosophy), often referred to as simply the *Principia,* published in 1687, formalized and extended earlier achievements of others by creating an axiomatic structure for the foundation principles of mechanics. By the organization of problems around his fundamental laws, Newton successfully demonstrated the application of his theory to the study of problems of mechanics of the solar system. He thus began the idea that the motions of bodies may be deduced from a few simple principles.

The formulation and application of Newton's laws entail the use of analytical methods of differential equations. Surprisingly, however, Newton never recorded or applied his laws in any general mathematical form; and historians (e.g. Truesdell) have found no evidence to suggest that he was able to set up differential equations for the mechanical systems he investigated. Others (e.g. Bixby) suggest that for the benefit of scholars, in those times well-versed in geometry, Newton's arguments were laboriously worked out by geometrical methods, rather than in terms of his emerging new calculus, so that mathematicians and scientists would be able to understand his new ideas on the motions of bodies. In fact, it was not until 1750 that Newton's laws for material points were first formulated more generally by Leonhard Euler (1707–1783) as differential equations relating force, torque, and motion for all bodies, including deformable bodies. Thus, it was not Newton; it was Euler who demonstrated countless times how to set up mechanical problems as definite mathematical problems formulated from basic, first principles. Therefore, it is not uncommon nowadays that the basic laws of mechanics are often referred to as *Euler's laws*. The classical, mathematical principles of mechanics created by Newton and Euler thus establish the fundamental laws governing the motions of all bodies. They provide the foundation for our study of dynamics—*the analysis of motion*.

The simplest kind of dynamical problem is to find the force needed to produce a specified motion of a particle. The converse problem of finding the motion arising from the application of known forces of various kinds is more difficult. This problem requires the solution of differential equations. Our earlier practice with simple integration methods applied in kinematics, therefore, will prove useful in the study of problems of this kind.

To formulate these types of problems, we need to know how to specify mathematically the nature of various kinds of forces that act between pairs of bodies. These forces are of two general kinds, *contact force* and *body force*. The weight of a body is a familiar example of a body force that arises from the mutual action between pairs of separated bodies in accordance with *Newton's law of universal gravitational attraction*. This basic body force law is studied in this chapter. Of course, two bodies may also interact by contact, i.e. by mutual touching. Everyone knows, for example, that when two blocks are pressed together, a force tangent to their common surfaces must be applied in order to slide one block on the other. But once the sliding has begun, the force needed to sustain the motion is somewhat smaller than that required to initiate it. The fundamental laws that characterize these familiar experiences are studied here too. These principles, called *Coulomb's laws,* relate the normal and the tangential components of the contact force that acts between two bodies to oppose their relative sliding motion. Other kinds of viscous, elastic, electromagnetic, and time varying forces are introduced in the next chapter. In addition, we are going to find that certain pseudoforces act on bodies having motion relative to an accelerating, rotating reference frame.

The effect of the motion of the frame of reference on the form of Newton's second law of motion is investigated. It turns out that our moving Earth frame is not the reference frame with respect to which Newton's laws hold. Therefore, we must learn how the governing laws are to be modified so that they may be applied to problems in any moving frame, including our Earth frame. In addition to aiding our understanding of the extent of the error that may be expected when the motion of the reference frame is neglected, the theory will also reveal in later applications some interesting and subtle physical phenomena that arise from the Earth's rotation. The idea that laws governing the internal forces between the parts of a system should be independent of any external reference frame used to describe them is expressed in the *principle of frame indifference* studied here in the context of the mutual force that acts between two particles and depends on only their spatial positions. Application of this idea leads to the most general form of the law of mutual action between two particles as a function of only their distance of separation.

Our main objective in this chapter is to study the foundation principles of classical mechanics. The *Newton–Euler laws of mechanics* are here formulated in a manner that parallels that introduced by Newton and generalized by Euler. The content, utility, and the predictive value of these rules in relation to special force laws, like those that govern gravitation and sliding friction for example, are explored in their application to physical theory and problems, and in some cases by comparison of their theoretical predictions with experimental observations. A few introductory illustrations of these qualities are investigated here; and many additional examples and practice problems and solution techniques for particle dynamics are presented in Chapter 6. Some other useful principles of momentum, work, and energy that derive from the primary Newton–Euler law for a particle or center of mass object are presented in Chapter 7. The structure used in these beginning chapters is extended in Chapter 8 to the motion of a system of particles. The moment of inertia tensor is introduced in Chapter 9; and then Euler's grand generalization of Newton's principles of mechanics are formulated for a rigid body in Chapter 10. Our study ends in Chapter 11 with an introduction to the methods of advanced dynamics. The formulation of Lagrange's equations and Hamilton's principle for analytical mechanics are explored there. This is the point where books on advanced dynamics usually begin. Construction of a foundation for these future studies begins with one particle. First, it is recommended that the reader review the primitive terms and concepts introduced on pages 3–7 in Chapter 1.

5.2. Mass, Momentum, and the Center of Mass

The mass of a particle, a system of particles, and a rigid body, and the corresponding principle of conservation of mass for each of these is introduced. The momentum of a particle, and the momentum and the center of mass of a system of particles and of a rigid body are defined. The latter ideas are then applied to learn how the momentum of a system of particles and of a rigid body are related to the momentum of their respective centers of mass. These preliminary concepts and results on the center of mass are important to our future study of the classical principles of mechanics.

5.2.1. Mass and Momentum of a Particle

We recall from Chapter 1 that the mass $m(P, t)$ of a particle P is a positive scalar measure of its material content. The physical dimension of mass is denoted by $[M]$. It is a postulate of Newton's mechanics that *the mass of a particle is invariant in time, that is,*

$$m(P, t) = m(P) \tag{5.1}$$

for all times t. This axiom is called *the principle of conservation of mass.* It emphasizes that the mass of a particle is an invariant measure of its material content alone. Of course, the mass of another particle may be different.

The *momentum* $\mathbf{p}(P, t)$ of a particle P in a reference frame Φ is a vector-valued function of time t defined by the product of the mass $m(P)$ of the particle and its velocity $\mathbf{v}(P, t)$ in Φ:

$$\mathbf{p}(P, t) \equiv m(P)\mathbf{v}(P, t). \tag{5.2}$$

Sometimes the momentum is called the *linear momentum* to distinguish it from the moment of momentum introduced later on. It is seen from (5.2) that the momentum vector has the physical dimensions $[\mathbf{p}] = [MV] = [MLT^{-1}]$. Specific measure units are reviewed in the Appendix following the References at the end of this chapter and in the Problems.

5.2.2. Mass, Momentum, and Center of Mass of a System of Particles

We recall from Chapter 1 that a body $\beta = \{P_k\}$ consisting of n discrete particles P_k having mass $m_k = m(P_k)$, $k = 1, 2, \ldots, n$, is called a *system of particles.* It is clear that mass is an additive scalar measure on β. *Hence, the mass* $m(\beta)$ *of the system of particles is defined by the sum of the masses* m_k *of the particles* P_k *of* β:

$$m(\beta) \equiv \sum_{k=1}^{n} m_k. \tag{5.3}$$

The principle of conservation of mass (5.1) requires that the mass of the system is constant: $dm(\beta)/dt = 0$. Clearly, in a system of particles the mass may vary from one particle to another; and the mass of another system may be different.

5.2.2.1. *Momentum of a System of Particles*

By (5.2), each particle P_k has a momentum $\mathbf{p}_k \equiv \mathbf{p}(P_k, t) = m_k\mathbf{v}_k$ for which $\mathbf{v}_k \equiv \mathbf{v}(P_k, t)$ denotes the velocity of P_k in Φ. Therefore, *the momentum of the*

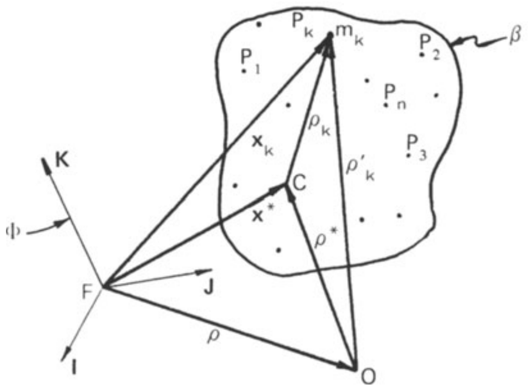


Figure 5.1. Schema for the center of mass properties of a system of particles.

system β in frame Φ is defined by

$$\mathbf{p}(\beta, t) \equiv \sum_{k=1}^{n} \mathbf{p}_k = \sum_{k=1}^{n} m_k \mathbf{v}_k. \tag{5.4}$$

5.2.2.2. Center of Mass of a System of Particles

The center of mass of a system of particles is an important concept that enables us to reduce the momentum (5.4) of the system to the momentum of a single, fictitious particle—a neat trick that proves most useful in future work. With this objective in mind, consider a system of particles shown in Fig. 5.1 in frame $\Phi = \{F; \mathbf{I}_j\}$, a set comprising an origin point F and an orthonormal vector basis $\mathbf{I}_j$, as defined in Chapter 1. Let $\mathbf{x}_k \equiv \mathbf{x}(P_k, t)$ denote at time t the position vector of a particle P_k whose mass is m_k. The *center of mass of a system of particles* $\beta = \{P_k\}$ is defined as the point in Φ whose position vector $\mathbf{x}^* \equiv \mathbf{x}^*(\beta, t)$ is determined by

$$m(\beta)\mathbf{x}^* = \sum_{k=1}^{n} m_k \mathbf{x}_k, \tag{5.5}$$

wherein we recall (5.3) for the mass $m(\beta)$ of the system. In this sense, the weighted-average motion of the particles of the system is described by the motion $\mathbf{x}^*(\beta, t)$ of a single, fictitious particle of mass $m(\beta)$, the mass of the system. Some properties of the center of mass are discussed next.

We first note that *the center of mass need not be a place occupied by a particle of β, but it may be.* Consider for example a system $\beta = \{P_1, P_2\}$ of two particles of equal mass $m_1 = m_2 = m$, one at the origin $\mathbf{x}_1 = \mathbf{0}$ and the other at an arbitrary place $\mathbf{x}_2 = \mathbf{d}$ in Φ at an instant t. Then by (5.3), we have $m(\beta) = 2m$; and (5.5) provides $2m\mathbf{x}^* = \sum_{k=1}^{2} m_k \mathbf{x}_k = m\mathbf{d}$. Hence, the center of mass of this system at the instant t is at the place $\mathbf{x}^* = \mathbf{d}/2$ in Φ—a place that is not occupied by either particle of β. On the other hand, consider a system of three particles of equal mass m; one at $\mathbf{x}_1 = \mathbf{0}$, one at $\mathbf{x}_2 = \mathbf{d}/2$, and the other at $\mathbf{x}_3 = \mathbf{d}$ in Φ at time t. In this case, (5.5) shows that the center of mass of the system at the instant t is at the place $\mathbf{x}^* = \mathbf{d}/2$ occupied by the particle P_2.

We show next that *the center of mass is a unique point whose definition is independent of the reference origin in* Φ. First consider the reference origin. Identify another reference point O at $\boldsymbol{\rho}$ from F in Φ in Fig. 5.1. Introduce $\mathbf{x}_k = \boldsymbol{\rho} + \boldsymbol{\rho}'_k$ and $\mathbf{x}^* = \boldsymbol{\rho} + \boldsymbol{\rho}^*$, where $\boldsymbol{\rho}'_k$ and $\boldsymbol{\rho}^*$ are the respective position vectors of the particle P_k and of the center of mass C from O. Then (5.5), with the aid of (5.3), becomes

$$m(\beta)(\boldsymbol{\rho} + \boldsymbol{\rho}^*) = \sum_{k=1}^{n} m_k \boldsymbol{\rho} + \sum_{k=1}^{n} m_k \boldsymbol{\rho}'_k = m(\beta)\boldsymbol{\rho} + \sum_{k=1}^{n} m_k \boldsymbol{\rho}'_k.$$

It thus follows that for an arbitrary point O,

$$m(\beta)\boldsymbol{\rho}^* = \sum_{k=1}^{n} m_k \boldsymbol{\rho}'_k$$

has the same form as (5.5). *Therefore, the definition* (5.5) *for the center of mass is independent of the choice of the reference origin in* Φ.

Now let us choose O at the center of mass C so that $\boldsymbol{\rho}^* = \boldsymbol{\rho}'_k - \boldsymbol{\rho}_k = \mathbf{0}$ in Fig. 5.1. Then relative to the center of mass, we have

$$\sum_{k=1}^{n} m_k \boldsymbol{\rho}_k = \mathbf{0}, \tag{5.6}$$

wherein $\boldsymbol{\rho}_k$ is the position vector of the particle P_k from C at time t. Clearly, (5.6) simply states that the position vector of the center of mass from itself is the zero vector.

It is now easy to prove that the center of mass is the only point with respect to which (5.6) holds for a system of particles. Indeed, suppose there exists another point C', say, at the place $\mathbf{r}$ from C such that (5.6) holds. Then $\sum_{k=1}^{n} m_k \mathbf{r}_k = \mathbf{0}$, where $\mathbf{r}_k$ is the position vector of P_k from C'. However, substitution of $\boldsymbol{\rho}_k = \mathbf{r} + \mathbf{r}_k$ into (5.6) shows that $\mathbf{r} = \mathbf{0}$; that is, the points C and C' coincide. *Therefore, at each instant, the center of mass of a given system of particles is the unique point for which* (5.6) *holds*. Plainly, if the system is altered in any way, so is its center of mass.

5.2.2.3. *Momentum of the Center of Mass of a System of Particles*

We now derive an important result relating the momentum of a system of particles to the momentum of its center of mass. Of course, the system of particles is generally in motion in Φ with momentum (5.4), in which $\mathbf{v}_k \equiv \dot{\mathbf{x}}_k$. We recall (5.5) and define $\mathbf{v}^* \equiv \dot{\mathbf{x}}^*(\beta, t)$, the velocity of the center of mass. Then, differentiation of (5.5) with respect to time in Φ, the mass of the system being conserved, and use of (5.4), yields the important result

$$\mathbf{p}^* \equiv m(\beta)\mathbf{v}^* = \sum_{k=1}^{n} m_k \mathbf{v}_k = \mathbf{p}(\beta, t). \tag{5.7}$$

The vector $\mathbf{p}^*$ defined by the first equation in (5.7) is the momentum of an imaginary particle of mass $m(\beta)$ that moves with the velocity $\mathbf{v}^*$ of the center of mass. This particle is named the *center of mass particle* (or *object*); and $\mathbf{p}^*$ is called briefly the *momentum of the center of mass*. The result (5.7) thus shows that *the momentum of a system of particles is equal to the momentum of its center of mass*: $\mathbf{p}(\beta, t) = \mathbf{p}^*(\beta, t)$.

Further, differentiation of (5.6) yields

$$\sum_{k=1}^{n} m_k \dot{\boldsymbol{\rho}}_k = \mathbf{0}. \tag{5.8}$$

Hence, the momentum of a system of particles relative to its center of mass particle is always zero.

Example 5.1. A system $\beta = \{P_1, P_2, P_3\}$ consists of three particles with mass $m_1 = m$, $m_2 = 2m$, $m_3 = 3m$ and having the respective constant velocities $\mathbf{v}_1 = v(6, -7, 0)$, $\mathbf{v}_2 = v(0, 2, -3)$, $\mathbf{v}_3 = v(2, -1, -2)$ in frame $\Phi = \{F; \mathbf{I}_k\}$. Determine the momentum of the system in Φ, find the velocity of each particle relative to the center of mass C, and thus confirm (5.8).

Solution. First recall (5.4) for the momentum of the system. The momentum of each particle is determined by (5.2); and from the assigned data, we obtain

$$\mathbf{p}_1 = m_1\mathbf{v}_1 = mv(6, -7, 0), \qquad \mathbf{p}_2 = m_2\mathbf{v}_2 = 2mv(0, 2, -3),$$
$$\mathbf{p}_3 = m_3\mathbf{v}_3 = 3mv(2, -1, -2). \tag{5.9a}$$

Then, by (5.4), the momentum of the system in Φ is given by

$$\mathbf{p}(\beta, t) = \mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 = 6mv(2\mathbf{I} - \mathbf{J} - 2\mathbf{K}). \tag{5.9b}$$

The velocity of the particle P_k relative to C is given by $\dot{\boldsymbol{\rho}}_k = \mathbf{v}_k - \mathbf{v}^*$, in which the velocity of C may be found from (5.7). Hence, with (5.3), the momentum of C is $\mathbf{p}^* = 6m\mathbf{v}^* = \mathbf{p}(\beta, t)$; and use of (5.9b) yields $\mathbf{v}^* = v\,(2\mathbf{I} - \mathbf{J} - 2\mathbf{K})$. Therefore,

$$\dot{\boldsymbol{\rho}}_1 = \mathbf{v}_1 - \mathbf{v}^* = v(4, -6, 2), \qquad \dot{\boldsymbol{\rho}}_2 = \mathbf{v}_2 - \mathbf{v}^* = v(-2, 3, -1),$$
$$\dot{\boldsymbol{\rho}}_3 = \mathbf{v}_3 - \mathbf{v}^* = \mathbf{0}, \tag{5.9c}$$

identify the velocity of each particle relative to C in Φ; and hence

$$\sum_{k=1}^{3} m_k \dot{\rho}_k = m_1\dot{\rho}_1 + m_2\dot{\rho}_2 + m_3\dot{\rho}_3 = mv(4, -6, 2) + 2mv(-2, 3, -1) + \mathbf{0} = \mathbf{0}, \tag{5.9d}$$

in agreement with the general result (5.8). □

5.2.3. Mass, Momentum, and Center of Mass of a Rigid Body

Let us consider a rigid body $\mathscr{B}$, and let $dm(P)$ denote an additive parcel (or element) of mass at the material point P. Then *the mass of the body* is defined by

$$m(\mathscr{B}) \equiv \int_{\mathscr{B}} dm(P) = \int_{\mathscr{B}} \rho(P)\, dV(P), \tag{5.10}$$

wherein $dV(P)$ is the elemental material volume of $\mathscr{B}$ at P, and $\rho(P) \equiv dm(P)/dV(P)$, the ratio of the element of mass at P to its element of volume at P, that is, the mass per unit volume of $\mathscr{B}$, is called the *mass density*. The subscript $\mathscr{B}$ on the integral sign, here and throughout this volume, means that the integration, with appropriate limits, is over the bounded region defined by the body $\mathscr{B}$. Neither the mass density nor the material volume of a rigid body can change with time, so the principle of balance of mass is satisfied: $dm(\mathscr{B})/dt = 0$. In general, however, the density may vary from one material point to another. A rigid body $\mathscr{B}$ is said to be *homogeneous* whenever its mass density is constant throughout $\mathscr{B}$. Thus, by (5.10), the mass of a homogeneous rigid body is simply the product of the mass density and the material volume of $\mathscr{B}$, namely, $m(\mathscr{B}) = \rho V(\mathscr{B})$, where $V(\mathscr{B}) = \int_{\mathscr{B}} dV(P)$.

5.2.3.1. Momentum of a Body

The momentum of a body element of mass $dm(P)$ at P in Φ is $dm(P)\mathbf{v}(P, t)$. Hence, the *momentum* $\mathbf{p}(\mathscr{B}, t)$ *of a body* in a reference frame Φ is defined by

$$\mathbf{p}(\mathscr{B}, t) \equiv \int_{\mathscr{B}} \mathbf{v}(P, t)\, dm(P). \tag{5.11}$$

In general, both the velocity and mass distributions must be known to effect the integration of (5.11). Consider, for example, a rigid body $\mathscr{B}$ having a uniform motion in the frame Φ. In this case, the velocity of every particle of $\mathscr{B}$ is a constant vector $\mathbf{v}(P, t) = \mathbf{v}$, so equations (5.11) and (5.10) yield $\mathbf{p}(\mathscr{B}, t) = \mathbf{v} \int_{\mathscr{B}} dm\,(P) = m(\mathscr{B})\mathbf{v}$. Hence, the momentum of a rigid body $\mathscr{B}$ having a uniform motion is the same as that of a single particle of mass $m(\mathscr{B})$ moving with the constant velocity $\mathbf{v}$. We shall see next that this imaginary particle is the center of mass of the body.

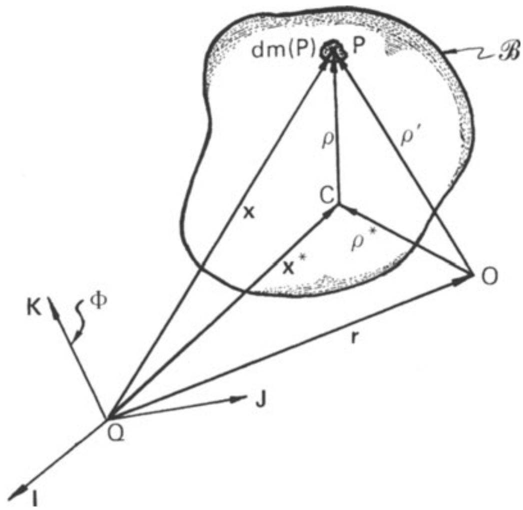


Figure 5.2. Schema for the center of mass properties of a body.

5.2.3.2. Center of Mass of a Body

We shall soon discover that the dynamics of a rigid body involves the motion of its center of mass, an important concept by which the momentum (5.11) of a body may be replaced by the momentum of a single, imaginary particle situated at its center of mass. With this in mind, let $\mathbf{x}(P, t)$ denote at time t the position vector of the material parcel $dm(P)$ of a body $\mathscr{B}$ in a spatial frame $\Phi = \{Q; \mathbf{I}_k\}$ shown in Fig. 5.2. The *center of mass of the body* $\mathscr{B}$ is the unique point in Φ whose position vector $\mathbf{x}^* \equiv \mathbf{x}^*(\mathscr{B}, t)$ at time t is determined by

$$m(\mathscr{B})\mathbf{x}^*(\mathscr{B}, t) = \int_{\mathscr{B}} \mathbf{x}(P, t)\, dm(P), \tag{5.12}$$

in which we recall (5.10). In this sense, the weighted-average motion of the particles of the body is described by the motion $\mathbf{x}^*(\mathscr{B}, t)$ of a single, fictitious particle of mass $m(\mathscr{B})$, called *the center of mass particle*. Some properties of the center of mass are described next.

It is easy to prove that *the definition* (5.12) *is independent of the choice of reference origin* Q *in* Φ. Therefore, relative to the center of mass point itself, (5.12) becomes

$$\int_{\mathscr{B}} \boldsymbol{\rho}(P, t) dm(P) = \mathbf{0}, \tag{5.13}$$

where $\boldsymbol{\rho}(P, t)$ is the position vector from the center of mass C to the parcel $dm(P)$ at P in frame Φ, as shown in Fig. 5.2. Thus, by an argument similar to that used for a system of particles, it follows that *at each instant t the center of mass is the unique point with respect to which* (5.13) *holds*. Indeed, its unique location in a rigid body is determined relative to a body reference frame $\varphi = \{O; \mathbf{i}_k\}$ with respect to which the position vectors in (5.12) and (5.13) are independent of time. Therefore, the center of mass of a rigid body is a unique point determined by the geometry and material content of that body alone—it always occupies the same place in the body reference frame relative to which (5.13) holds. The center of mass moves with the body, and, of course, its position vector with respect to different spatial reference frames will naturally vary.

Exercise 5.1. (a) Show that the definition (5.12) for the center of mass of a body is independent of the choice of reference origin. (b) Prove that the center of mass is the *unique* point for which (5.13) holds. □

The center of mass of a homogeneous body often may be easily identified. For a homogeneous body, the constant mass density may be eliminated from (5.12) to obtain at time t the familiar *formula for the geometrical centroid of $\mathscr{B}$*:

$$V(\mathscr{B})\mathbf{x}^*(\mathscr{B}, t) = \int_{\mathscr{B}} \mathbf{x}(P, t)\, dV(P), \tag{5.14}$$

wherein $V(\mathscr{B})$ is the material volume of $\mathscr{B}$. *Thus, the mass center of a homogeneous body coincides with its centroid.* Of course, very often, the centroid is easy to identify.

In general, the center of mass need not be a place occupied by a particle of $\mathscr{B}$. It is clear, for example, that the center of mass of a homogeneous, circular cylindrical tube is at the geometrical center on its axis—plainly a place that is not occupied by a particle of the tube. On the other hand, the center of mass of a similar solid cylinder has the same location. These assertions are evident from symmetry considerations. Nevertheless, it is instructive to review integration methods typically involved in the use of (5.12) or (5.14), because similar techniques are used for both homogeneous and nonhomogeneous bodies for which symmetry may not be so evident.

Example 5.2. (i) Compute the location of the center of mass of the homogeneous, cylindrical tube described in Fig. 5.3. (ii) Find the center of mass when the density varies linearly from the constant value ρ_o at $z = 0$ to $2\rho_o$ at $z = \ell$.

Solution of (i). The circular tube shown in Fig. 5.3 has an inner radius r_i, outer radius r_o, and length ℓ. Because the material is homogeneous, the center of mass is at the centroid determined by (5.14) in which

$$V(\mathscr{B}) = \pi\ell\left(r_o^2 - r_i^2\right) \tag{5.15a}$$

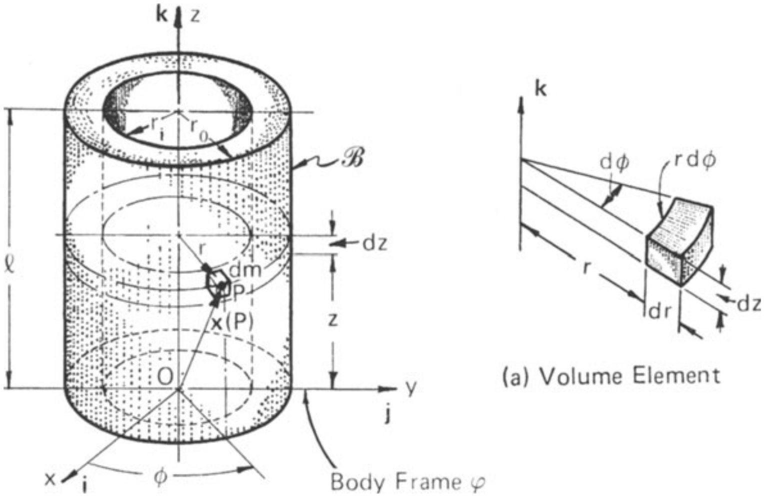


Figure 5.3. Geometry for determination of the center of mass of a tube.

is the material volume of the tube. It is natural to introduce cylindrical coordinates in the imbedded frame $\varphi = \{O; \mathbf{i}_k\}$, whose origin is at the base of the tube. Then the position vector $\mathbf{x}(P, t) \equiv \mathbf{x}(P)$ of a particle P of $\mathscr{B}$ and the elemental volume at P in Fig. 5.3 may be expressed as $\mathbf{x}(P) = r(\cos\phi\mathbf{i} + \sin\phi\mathbf{j}) + z\mathbf{k}$ and $dV(P) = rdrd\phi dz$. Hence, with (5.14), the center of mass location $\mathbf{x}^*(\mathscr{B}, t) \equiv \mathbf{x}^*(\mathscr{B})$ in φ is given by

$$V(\mathscr{B})\mathbf{x}^*(\mathscr{B}) = \int_0^{2\pi}\int_0^{\ell}\int_{r_i}^{r_o}(r\cos\phi\mathbf{i} + r\sin\phi\mathbf{j} + z\mathbf{k})rdrdzd\phi. \quad (5.15b)$$

The first two integrals in the angle ϕ vanish. Therefore, as anticipated from the symmetry, the center of mass lies on the axis of the tube. Integration of the remaining term in (5.15b) and use of (5.15a) yields $\mathbf{x}^*(\mathscr{B}) = \ell/2\mathbf{k}$, that is, the center of mass is at the center of the void. We notice also that $\mathbf{x}^*$ is independent of the radii of the tube, so the location of the center of mass in φ is the same for all radii. In particular, for a solid cylinder for which $r_i = 0$, $\mathbf{x}^*(\mathscr{B}) = \ell/2\mathbf{k}$ holds as well. Of course, whatever reference point may be used, the center of mass of the rigid tube remains at the same central position; and as the tube moves in space, its center of mass retains its central location in the moving, imbedded frame.

In problems of this kind it is often easier to simplify the integration in (5.15b) by use of the method of slices. The application of this method to the previous homogeneous problem is left as a review exercise for the reader to show that $\ell\mathbf{x}^*(\mathscr{B}) = \int_0^{\ell} z\mathbf{k}dz$, which yields $\mathbf{x}^*(\mathscr{B}) = \ell/2\mathbf{k}$, as before. We next apply this method to solve the variable density problem.

Solution of (ii). We are given that the mass density of the tube varies linearly from ρ_o at $z = 0$ to $2\rho_o$ at $z = \ell$, and hence $\rho = \rho_o (1 + z/\ell)$. Because ρ varies only along the tube's length, the simultaneous geometrical and mass distribution symmetries about the tube's axis imply that the center of mass is on the axis. Therefore, $x^* = y^* = 0$ and only the z^* component need be found. Hence, (5.12) yields

$$m(\mathscr{B})z^* = \int_{\mathscr{B}} z dm. \tag{5.15c}$$

The method of slices shows that for the annular ring in Fig. 5.3 the volume element $dV = Adz$, where $A = \pi\left(r_o^2 - r_i^2\right)$ is the constant area of the ring. The mass is then found by (5.10):

$$m(\mathscr{B}) = \rho_o A \int_0^{\ell} \left(1 + \frac{z}{\ell}\right) dz = \frac{3}{2} A \rho_o \ell,$$

and the right-hand side of (5.15c) becomes

$$\int_{\mathscr{B}} z dm = \rho_o A \int_0^{\ell} z\left(1 + \frac{z}{\ell}\right) dz = \frac{5}{6} \rho_o A \ell^2.$$

Therefore, by (5.15c), the center of mass is on the axis of the tube at $z^* = 5\ell/9$ from its base at O. Clearly, the center of mass is not the centroid, which is located at $z^* = \ell/2$ in accordance with (5.14). □

5.2.3.3. *Momentum of the Center of Mass of a Rigid Body*

We shall now derive an important result relating the momentum of a rigid body to the momentum of its center of mass. The body is generally in motion in Φ with momentum defined by (5.11), in which $\mathbf{v}(P, t) \equiv \dot{\mathbf{x}}(P, t)$. The motion of the center of mass is defined by (5.12), and hence $\mathbf{v}^*(\mathscr{B}, t) \equiv \dot{\mathbf{x}}^*(\mathscr{B}, t)$ defines the velocity of the center of mass. Thus, differentiation of (5.12) with respect to time and use of (5.11) for a rigid body yields the important result

$$\mathbf{p}^*(\mathscr{B}, t) \equiv m(\mathscr{B})\mathbf{v}^*(\mathscr{B}, t) = \int_{\mathscr{B}} \mathbf{v}(P, t)\, dm(P) = \mathbf{p}(\mathscr{B}, t). \tag{5.16}$$

The vector $\mathbf{p}^*(\mathscr{B}, t)$ defined by the first equation in (5.16) is the momentum of a fictitious particle of mass $m(\mathscr{B})$ that moves with the velocity $\mathbf{v}^*(\mathscr{B}, t)$ of the center of mass. Our imaginary particle is sometimes called the *center of mass particle* (or *object*). Hence, $\mathbf{p}^*$ is called briefly the *momentum of the center of mass*. The result (5.16) thus shows that *the momentum of a rigid body is equal to the momentum of its center of mass*: $\mathbf{p}(\mathscr{B}, t) = \mathbf{p}^*(\mathscr{B}, t)$.

Moreover, differentiation of (5.13) in the spatial frame Φ yields

$$\int_{\mathscr{B}} \dot{\boldsymbol{\rho}}(P, t)\, dm(P) = \mathbf{0}. \tag{5.17}$$

Hence, the momentum of a body relative to its center of mass in Φ is always zero.

The definitions (5.10) and (5.11) may be readily extended to a deformable body whose volume and density may vary with time, and for which similar center of mass properties hold at each instant. In this case, however, greater care must be exercised in differentiation of the integrals in (5.12) and (5.13) because the region $\mathscr{B}$ of the integration over the deforming body varies with time; and the location of the center of mass will vary with the deformation. Of course, the body region $\mathscr{B}$ for a rigid body is the same for all time. Deformable bodies are not studied in this text.

5.3. Moment of a Vector About a Point

The moment of a vector about a point occurs frequently in future work. This operation is first defined in general terms; and the transformation rule that describes the effect of a change of the reference point follows. The familiar idea of the moment of a force about a point is then reviewed; and the moment of momentum vector is introduced in the next section.

We start with the general idea. Let $\mathbf{x}_Q(P)$ be the position vector of a point P from a point Q, and let $\mathbf{u}(P)$ denote a vector quantity at P in Fig. 5.4. *The moment about Q of the vector $\mathbf{u}(P)$ is a vector entity $\boldsymbol{\mu}_Q(P)$ defined by the rule*

$$\boldsymbol{\mu}_Q(P) \equiv \mathbf{x}_Q(P) \times \mathbf{u}(P). \tag{5.18}$$

This vector is perpendicular to both $\mathbf{x}_Q(P)$ and $\mathbf{u}(P)$. It is represented in Fig. 5.4 as a vector line with an arrow turning about it in the right-hand sense of (5.18).

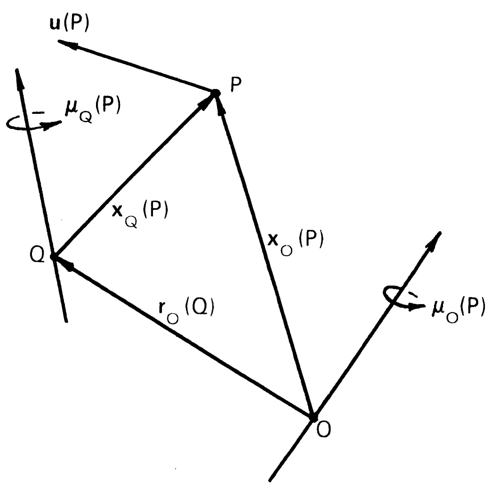


Figure 5.4. Schema for the moment of a vector about a point.

5.3.1. Reference Point Transformation Rule

The vector $\boldsymbol{\mu}_Q(P)$ depends on the choice of Q. The moment of the same vector $\mathbf{u}(P)$ about another reference point O in Fig. 5.4 is given by

$$\boldsymbol{\mu}_O(P) = \mathbf{x}_O(P) \times \mathbf{u}(P),$$

where $\mathbf{x}_O(P)$ is the position vector of P from O. It is seen in Fig. 5.4 that $\mathbf{x}_O(P) = \mathbf{r}_{OQ} + \mathbf{x}_Q(P)$, in which $\mathbf{r}_{OQ} \equiv \mathbf{r}_O(Q)$ is the position vector of Q from O. Hence, substitution of this relation into the previous equation and use of (5.18) yields *the transformation rule relating the moments of the same vector* $\mathbf{u}(P)$ *about the points* O *and* Q:

$$\boldsymbol{\mu}_O(P) = \boldsymbol{\mu}_Q(P) + \mathbf{r}_{OQ} \times \mathbf{u}(P). \tag{5.19}$$

It is seen that $\boldsymbol{\mu}_O(P) = \boldsymbol{\mu}_Q(P)$ when and only when the nonzero vector $\mathbf{r}_{OQ}$ is parallel to $\mathbf{u}(P)$.

5.3.2. Moment of a Force About a Point

We recall the familiar idea of the moment of a force about a point. In Fig. 5.4, let $\mathbf{u}(P) \equiv \mathbf{F}(P)$ denote a force acting on a particle P whose position vector from point Q is $\mathbf{x}_Q(P)$, and write $\boldsymbol{\mu}_Q(P) \equiv \mathbf{M}_Q(P)$. Then, by (5.18), *the moment about* Q *of the force* $\mathbf{F}(P)$ *is the vector* $\mathbf{M}_Q(P)$ *defined by the rule*

$$\mathbf{M}_Q(P) \equiv \mathbf{x}_Q(P) \times \mathbf{F}(P). \tag{5.20}$$

The moment vector is a measure of the turning or twisting effect of the force about the reference point. Hence, the moment of a force is also called the *torque*; its physical dimensions are $[\mathbf{M}_Q] = [FL]$.

If $\mathbf{a}$ is a vector from Q to *any* point A on the action line of $\mathbf{F}(P)$, the vector defined by $\mathbf{r} \equiv \mathbf{x}_Q(P) - \mathbf{a}$ is parallel to $\mathbf{F}(P)$. It thus follows from (5.20) that $\mathbf{M}_Q(P) = \mathbf{a} \times \mathbf{F}(P)$ holds for any point A on the action line of the force acting on P. *Therefore, the moment of the force* $\mathbf{F}(P)$ *about* Q *is independent of the actual point of application of the force along its line of action*; *and hence only the component of* $\mathbf{x}_Q(P)$ *that is perpendicular to* $\mathbf{F}(P)$ *determines the torque of* $\mathbf{F}(P)$ *about* Q. Thus, in abbreviated notation, the magnitude $|\mathbf{M}_Q| = |\mathbf{x}_Q|\,|\mathbf{F}| \sin < \mathbf{x}_Q, \mathbf{F} >$ of the moment vector $\mathbf{M}_Q$ is equal to the product of the magnitude of the force $F \equiv |\mathbf{F}|$ and the perpendicular distance $d \equiv |\mathbf{x}_Q| \sin < \mathbf{x}_Q, \mathbf{F} >$ from Q to the action line of $\mathbf{F}$, where $< \mathbf{x}_Q, \mathbf{F} >$ denotes the smaller angle between $\mathbf{x}_Q$ and $\mathbf{F}$, as usual; that is, $|\mathbf{M}_Q| = Fd$, a familiar elementary rule.

The definition (5.20) may be applied to each particle P_k of a system of particles. In this case, the total, or resultant, moment about a point Q of the several forces $\mathbf{F}_k = \mathbf{F}(P_k)$ that act on a system of n particles $\beta = \{P_k\}$ is defined by the sum of the moments about Q of all of the forces $\mathbf{F}_k$ that act on β:

$$\mathbf{M}_Q(\beta) \equiv \sum_{k=1}^{n} \mathbf{M}_Q(P_k) = \sum_{k=1}^{n} \mathbf{x}_{Qk} \times \mathbf{F}_k, \tag{5.21}$$

where $\mathbf{x}_{Qk} \equiv \mathbf{x}_Q(P_k)$ is the position vector of particle P_k from Q; and the total, or resultant, force is defined by $\mathbf{F}(\beta) \equiv \sum_{k=1}^{n} \mathbf{F}_k$.

The same rule may be applied to determine the total moment about a point Q of all the concentrated and distributed forces that act on a rigid body $\mathscr{B}$. For the elemental force distribution $d\mathbf{F}_d(P)$ acting on a material parcel at P, for example, the total torque about a point Q of the distributed force is defined by

$$\mathbf{M}_Q(\mathscr{B}) \equiv \int_{\mathscr{B}} \mathbf{x}_Q(P) \times d\mathbf{F}_d(P), \tag{5.22}$$

where $\mathbf{x}_Q(P)$ is the position vector from Q to the parcel at P. A formula similar to (5.21) holds for n concentrated forces $\mathbf{F}_k(\mathscr{B})$ acting on $\mathscr{B}$.

Now consider the point transformation rule. Clearly, the turning effect of a force about another reference point at O in Fig. 5.4 will be different from that about Q. The transformation rule (5.19) shows that *the moment of the same force about the reference point O is related to its moment* (5.20) *about the point Q by the rule*

$$\mathbf{M}_O(P) = \mathbf{M}_Q(P) + \mathbf{r}_{OQ} \times \mathbf{F}(P). \tag{5.23}$$

We recall that $\mathbf{r}_{OQ}$ is the position vector of point Q from O; and hence $\mathbf{r}_{OQ} \times \mathbf{F}(P)$ is the moment about O of the total force as though it were placed at Q.

The same point transformation rule applies to (5.21) and (5.22); thus,

$$\mathbf{M}_O(\mathscr{B}) = \mathbf{M}_Q(\mathscr{B}) + \mathbf{r}_{OQ} \times \mathbf{F}(\mathscr{B}), \tag{5.24}$$

where the total force acting on $\mathscr{B}$, namely, $\mathbf{F}(\mathscr{B}) = \mathbf{F}_d(\mathscr{B}) + \mathbf{F}_c(\mathscr{B})$, is the sum of the total distributed force $\mathbf{F}_d(\mathscr{B}) \equiv \int_{\mathscr{B}} d\mathbf{F}_d(P)$ and the total of all concentrated forces $\mathbf{F}_c(\mathscr{B}) \equiv \sum_{k=1}^{n} \mathbf{F}_k(\mathscr{B})$. Also, $\mathbf{M}_O(\mathscr{B})$ and $\mathbf{M}_Q(\mathscr{B})$ are the total moments about points O and Q of all of these forces. Therefore, by (5.24), *the total moment of force about a point O is equal to the total moment of force about point Q plus the moment about O of the total force placed at Q.*

The rule (5.24) relates the moments of the force about any two points. In particular, if O is the center of mass at C, then $\mathbf{r}_{OQ} = -\mathbf{r}_{QC} = -\mathbf{x}_Q^*(\mathscr{B})$ and (5.24) is written

$$\mathbf{M}_Q(\mathscr{B}) = \mathbf{M}_Q^*(\mathscr{B}) + \mathbf{M}_C(\mathscr{B}), \tag{5.25}$$

in which $\mathbf{M}_C(\mathscr{B})$ is the total moment of force about the center of mass and $\mathbf{M}_Q^*(\mathscr{B})$ is the moment about Q of the total force placed at the center of mass:

$$\mathbf{M}_Q^*(\mathscr{B}) = \mathbf{x}_Q^*(\mathscr{B}) \times \mathbf{F}(\mathscr{B}). \tag{5.26}$$

Thus, in physical terms (5.25) shows that *the total moment of force about any point Q is equal to the moment about Q of the total force placed at the center of mass plus the total moment of force about the center of mass.*

5.3.3. Equipollent Force Systems

Now consider two systems of forces and torques. These systems are said to be *equipollent* if and only if they have the same total force and the same total torque about the same point. That is, a system A with total force $\mathbf{F}^A$ and total torque $\mathbf{M}_Q^A$ about a point Q is equipollent to a system B with total force $\mathbf{F}^B$ and total torque $\mathbf{M}_Q^B$ about the same point Q when and only when

$$\mathbf{F}^A = \mathbf{F}^B \quad \text{and} \quad \mathbf{M}_Q^A = \mathbf{M}_Q^B. \tag{5.27}$$

It follows from the point transformation rule (5.23) or (5.24) that if two force systems are equipollent with respect to a point Q, they are equipollent with respect to any other point O.

We know from (5.20) that the moment about Q of a *single* force is perpendicular to the force and to the position vector from Q to its point of application. In general, however, this is not true for a *system* of forces—the total torque $\mathbf{M}_Q$ about a point Q of a system of forces generally is not perpendicular to the total force acting on the system. Here we focus on the special case when the system of forces A is such that $\mathbf{M}_Q^A \cdot \mathbf{F}^A = 0$; then, by (5.27) the same holds for the equipollent system B. Consider, for example, a distributed system of forces $\mathbf{F}^B(\mathscr{B}) = \mathbf{F}_d(\mathscr{B})$ with a total torque $\mathbf{M}_Q^B(\mathscr{B})$ equal to (5.22) such that $\mathbf{M}_Q^B(\mathscr{B}) \cdot \mathbf{F}^B(\mathscr{B}) = 0$. Then, this system is equipollent to a single force $\mathbf{F}^A(\mathscr{B}) = \mathbf{P}$ located at distance from Q such that

$$\mathbf{P} = \int_{\mathscr{B}} d\mathbf{F}_d(P) = \mathbf{F}_d(\mathscr{B}), \tag{5.28}$$

$$\mathbf{M}_Q^A(\mathscr{B}) \equiv \bar{\mathbf{x}}_Q \times \mathbf{P} = \int_{\mathscr{B}} \mathbf{x}_Q(P) \times d\mathbf{F}_d(P) = \mathbf{M}_Q^B(\mathscr{B}), \tag{5.29}$$

where the locus of the unknown vector $\bar{\mathbf{x}}_Q$ from Q traces the line of action of $\mathbf{P}$. Of course, $\bar{\mathbf{x}}_Q$ is necessarily perpendicular to $\mathbf{M}_Q^B(\mathscr{B})$. Now, bearing in mind that only the component of $\bar{\mathbf{x}}_Q$ perpendicular to the line of action of the force $\mathbf{P}$ influences the torque about Q, the relation (5.29) determines the place $\bar{\mathbf{x}}_Q^*$, say, on the line from Q perpendicular to $\mathbf{P}$, called the *center of force with respect to Q*, through which the force $\mathbf{P}$ must act to produce the same total torque about Q. Of course, the center of force with respect to another moment center at O, say, though also on the line of action of $\mathbf{P}$, will be different. Notice that (5.29) may be written as $\bar{\mathbf{x}}_Q \times \mathbf{F}_d(\mathscr{B}) = \int_{\mathscr{B}} \mathbf{x}_Q(P) \times d\mathbf{F}_d(P)$. Specifically, for any system of planar forces or for any system of parallel forces, the total moment of the forces about an arbitrary point is plainly perpendicular to the total force; therefore, in accordance with (5.28) and (5.29), each of these systems may be reduced to a single force

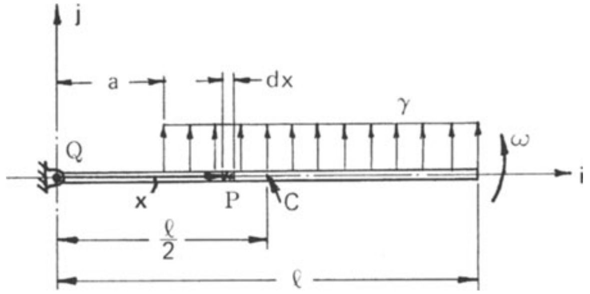


Figure 5.5. A homogeneous, thin rigid rod under a uniformly distributed load.

acting at its center of force. Clearly, for a system of discrete forces, the procedure is similar. (See Problem 5.35.) For further discussion on the reduction of force systems for the general case see the referenced texts on statics.

Example 5.3. A homogeneous, thin rigid rod of length ℓ is supported at one end by a smooth hinge at Q and is subjected to a load of magnitude γ per unit length distributed uniformly over the region $[a, \ell]$ shown in Fig. 5.5. (i) Find the force system with respect to Q that is equipollent to the distributed load. (ii) Determine the moment of the distributed load about the center of mass of the rod at C.

Solution of (i). The total force $\mathbf{F}^A = \mathbf{P}$ equipollent to the distributed load $\mathbf{F}^B = \mathbf{F}_d(\mathscr{B})$ for which $d\mathbf{F}_d(P) = \gamma dx\mathbf{j}$ is given by (5.28). Thus,

$$\mathbf{P} = \int_a^\ell \gamma dx\mathbf{j} = \gamma(\ell - a)\mathbf{j}. \tag{5.30a}$$

The total moment of the distribution about the hinge point Q is given by (5.22) in which $\mathbf{x}_Q(P) = x\mathbf{i} + y\mathbf{j}$;

$$\mathbf{M}_Q^B(\mathscr{B}) = \int_a^\ell x\mathbf{i} \times \gamma dx\mathbf{j} = \frac{\gamma}{2}(\ell^2 - a^2)\mathbf{k}. \tag{5.30b}$$

Of course, for the system B only the component $x\mathbf{i}$ of $\mathbf{x}_Q(P)$ that is perpendicular to the distribution contributes to the torque about Q.

Notice that this is a system of parallel forces, and $\mathbf{M}_Q^B(\mathscr{B})$ is perpendicular to $\mathbf{P}$. Thus, with $\bar{\mathbf{x}}_Q = \bar{x}\mathbf{i} + \bar{y}\mathbf{j}$ and (5.30a), we may write $\mathbf{M}_Q^A = \bar{\mathbf{x}}_Q \times \mathbf{P} = \bar{x}\gamma(\ell - a)\mathbf{k}$. Here we see that for the system A only the component $\bar{x}\mathbf{i}$ of $\bar{\mathbf{x}}_Q$ that is perpendicular to $\mathbf{P}$ contributes to the torque about Q. Thus, with (5.30b), (5.29) yields $\bar{x} = \frac{1}{2}(\ell + a)$; that is, with respect to Q, the center of force $\bar{\mathbf{x}}_Q^*$ for $\mathbf{P}$ is at

$$\bar{\mathbf{x}}_Q^* = \tfrac{1}{2}(\ell + a)\mathbf{i} = \left[a + \tfrac{1}{2}(\ell - a)\right]\mathbf{i}. \tag{5.30c}$$

The line of action of $\mathbf{P}$ is traced by $\bar{\mathbf{x}}_Q = \bar{\mathbf{x}}_Q^* + \bar{y}\mathbf{j}$ for all values of $\bar{y}$. Equation (5.30c) shows that the center of force for the uniformly distributed load is at the geometrical center of the loaded portion of the rod in Fig. 5.5. The force system consisting of the single force $\mathbf{P}$ acting at the center of force $\bar{\mathbf{x}}_Q^*$ in (5.30c) is equipollent to the assigned uniformly distributed force system; it consists of the same total force (5.30a) and produces the same total moment about Q in (5.30b).

Solution of (ii). The moment of the same distribution about point C may be found from the transformation rule (5.25). In accordance with (5.26), consider the load $\mathbf{P}$ placed at the center of mass of the homogeneous rod at $\mathbf{x}_Q^*(\mathscr{B}) = \frac{1}{2}\ell\mathbf{i}$, and recall (5.30a) to determine $\mathbf{M}_Q^*(\mathscr{B}) = \mathbf{x}_Q^* \times \mathbf{P} = \frac{1}{2}\gamma\ell(\ell - a)\mathbf{k}$. Then by (5.25) and (5.30b), we find

$$\mathbf{M}_C^B(\mathscr{B}) = \mathbf{M}_Q^B(\mathscr{B}) - \mathbf{M}_Q^*(\mathscr{B}) = \gamma\frac{a}{2}(\ell - a)\mathbf{k}. \tag{5.30d}$$

The same result may be obtained by our noting that the equipollent system consists of the single force (5.30a) acting at $\bar{\mathbf{x}}_Q^*$ in (5.30c). Hence, its moment about C at $\mathbf{x}_Q^* = \frac{1}{2}\ell\mathbf{i}$ is given by $(\bar{\mathbf{x}}_Q^* - \mathbf{x}_Q^*) \times \mathbf{P} = \frac{1}{2}\gamma a(\ell - a)\mathbf{k}$, which is the same as (5.30d). □

Finally, notice that if $\mathbf{F}(\mathscr{B}) \equiv \mathbf{0}$, then (5.24) shows that $\mathbf{M}_O(\mathscr{B}) = \mathbf{M}_Q(\mathscr{B})$ and hence the resultant moment is independent of the choice of reference point. In this case, the force system is called a *couple*. A force system consisting of a noncollinear pair of equal and oppositely directed forces is a familiar example. If both $\mathbf{F}(\mathscr{B}) \equiv \mathbf{0}$ and $\mathbf{M}_O(\mathscr{B}) \equiv \mathbf{0}$, then $\mathbf{M}_Q(\mathscr{B}) \equiv \mathbf{0}$ as well. In this case the resultant moment with respect to any reference point vanishes, and the force system is said to be *equipollent to zero*. It is an exercise for the reader to show that any force system can be reduced to a single force acting at an arbitrary point together with a couple. A torque $\mathbf{M}_Q$ induced by essentially twisting a body about an axis at a point Q is called a *concentrated couple*. Tightening a screw in a wooden body by twisting the screw about its axis is a physical example that may be modeled as a concentrated couple acting on the wooden body. We may think of a concentrated couple at Q as a pair of equal and opposite, noncollinear forces of very large intensity and having a very small moment arm, the perpendicular distance between the force pair, at Q.

None of the foregoing results for a body require that it be rigid. Moreover, although explicit dependence on time t is not indicated, it is clear that all of the foregoing vector entities also may vary with time. Another useful application of the moment of a vector about a point follows.

5.4. Moment of Momentum

Here we introduce an important vector quantity called the moment of momentum. The moment of momentum of a particle, a system of particles, and a body are defined in turn.

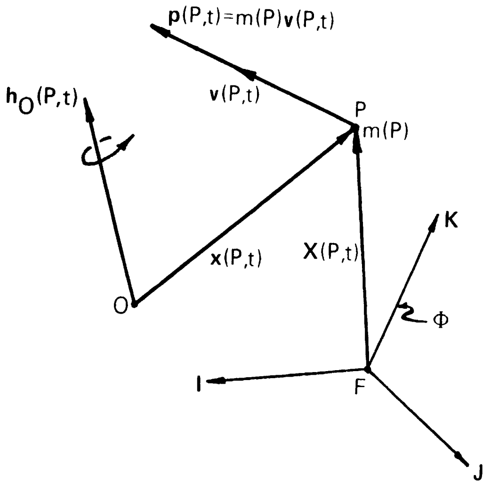


Figure 5.6. Schema for the moment about a point O of the momentum of a particle P relative to frame Φ.

5.4.1. Moment of Momentum of a Particle

Let $\mathbf{x}_O(P,t) = \mathbf{x}(P,t)$ denote the position vector of a particle P from an arbitrary spatial point O in a reference frame $\Phi = \{F; \mathbf{I}_k\}$ shown in Fig. 5.6. The velocity of P relative to Φ is given by $\mathbf{v}(P,t) = \dot{\mathbf{X}}(P,t)$, where $\mathbf{X}(P,t)$ is the position vector of P from F, as usual; and the momentum of P is defined by (5.2). In accordance with (5.18), *the moment about point O of the momentum of P relative to Φ, denoted by $\mathbf{h}_O(P,t)$, is a vector-valued function of time defined by*

$$\mathbf{h}_O(P,t) \equiv \mathbf{x}_O(P,t) \times \mathbf{p}(P,t) = \mathbf{x}(P,t) \times m(P)\,\mathbf{v}(P,t). \tag{5.31}$$

Notice that two reference points are involved in this definition, the origin F of frame Φ and the spatial point O. The moment about reference points O and Q of the same momentum vector $\mathbf{p}(P,t)$ are related by $\mathbf{h}_O(P,t) = \mathbf{h}_Q(P,t) + \mathbf{r}_{OQ} \times \mathbf{p}(P,t)$ in accordance with the transformation rule (5.19).

The moment of momentum is also known as the *angular momentum*, a term frequently used in other texts. It follows from (5.31) that moment of momentum has the physical dimensions $[\mathbf{h}_O] \equiv [H] = \left[ML^2T^{-1}\right]$.

5.4.2. Moment of Momentum of a System of Particles

Each particle of a system $\beta = \{P_k\}$ of n particles has a moment of momentum about point O given by (5.31), so that $\mathbf{h}_{Ok} \equiv \mathbf{h}_O(P_k,t) = \mathbf{x}_{Ok} \times \mathbf{p}_k$, where $\mathbf{x}_{Ok} \equiv \mathbf{x}_O(P_k,t)$ is the position vector of P_k from O, and $\mathbf{p}_k = m_k\mathbf{v}_k = m_k\dot{\mathbf{X}}_k$ is its momentum relative to Φ. *Relative to a frame $\Phi = \{F; \mathbf{I}_k\}$, the moment of* *momentum* $\mathbf{h}_O(\beta, t)$ *of a system of particles about a point* O *in* Φ *is defined by*

$$\mathbf{h}_O(\beta, t) \equiv \sum_{k=1}^{n} \mathbf{h}_{Ok} = \sum_{k=1}^{n} \mathbf{x}_{Ok} \times \mathbf{p}_k = \sum_{k=1}^{n} \mathbf{x}_{Ok} \times m_k \mathbf{v}_k. \tag{5.32}$$

Example 5.4. At an instant of interest t_0, the three particles described in Example 5.1, page 9, are situated at $\mathbf{x}_{O1} = (0, 0, -1)$, $\mathbf{x}_{O2} = (-3, -2, 2)$, and $\mathbf{x}_{O3} = (6, -2, -4)$ from a point O located at $\mathbf{B} = (2, -1, 3)$ from F in frame $\Phi = \{F; \mathbf{I}_k\}$. Compute the moment of momentum of the system about O at t_0.

Solution. The moment about O of the momentum of a particle is determined by (5.31). Thus, for the system of three particles with momenta (5.9a), we find

$$\mathbf{h}_{O1} = \mathbf{x}_{O1} \times \mathbf{p}_1 = -\mathbf{K} \times mv\,(6\mathbf{I} - 7\mathbf{J}) = mv\,(-7\mathbf{I} - 6\mathbf{J}),$$

$$\mathbf{h}_{O2} = \mathbf{x}_{O2} \times \mathbf{p}_2 = 2mv \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ -3 & -2 & 2 \\ 0 & 2 & -3 \end{vmatrix} = 2mv\,(2\mathbf{I} - 9\mathbf{J} - 6\mathbf{K}),$$

$$\mathbf{h}_{O3} = \mathbf{x}_{O3} \times \mathbf{p}_3 = 3mv \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ 6 & -2 & -4 \\ 2 & -1 & -2 \end{vmatrix} = 6mv\,(2\mathbf{J} - \mathbf{K}).$$

Then, by (5.32), the moment of momentum of the system about point O in Φ is

$$\mathbf{h}_O(\beta, t_0) = \mathbf{h}_{O1} + \mathbf{h}_{O2} + \mathbf{h}_{O3} = -3mv\,(\mathbf{I} + 4\mathbf{J} + 6\mathbf{K}).$$

□

Exercise 5.2. What is the moment of momentum about F in Φ for the system of particles described above? Derive the reference point transformation rule for the moment of momentum of a system of particles. □

5.4.3. Moment of Momentum of a Body

Consider a body $\mathscr{B}$ in Fig. 5.7 and recall that the momentum in Φ of a parcel of mass $dm(P)$ of $\mathscr{B}$ at P is defined by $\mathbf{v}(P, t)\,dm(P)$. Thus, *for a body* $\mathscr{B}$ *the moment of momentum about a point* O *in* $\Phi = \{F; \mathbf{I}_k\}$ is defined by

$$\mathbf{h}_O(\mathscr{B}, t) \equiv \int_{\mathscr{B}} \mathbf{x}_O\,(P, t) \times \mathbf{v}\,(P, t)\,dm\,(P). \tag{5.33}$$

Herein $\mathbf{x}_O\,(P, t) = \mathbf{x}\,(P, t)$ is the position vector of the material point P from the point O in Φ and $\mathbf{v}(P, t) = \dot{\mathbf{X}}(P, t)$ is its velocity relative to Φ. While in this book we shall be concerned only with bodies that are rigid, the definitions (5.11) for the momentum and (5.33) for the moment of momentum of a body hold more generally for all deformable solid and fluid bodies.

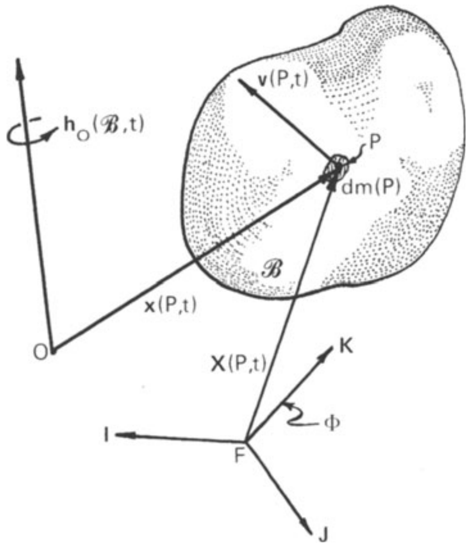


Figure 5.7. Schema for the Moment about a point O of the momentum of a body $\mathscr{B}$ relative to frame Φ.

Example 5.5. Find the moment about O in Φ of the momentum of a body having a constant translational acceleration relative to Φ.

Solution. Since $\mathbf{a}(P,t) = \mathbf{a}^*(\mathscr{B})$ is a constant vector for all particles of $\mathscr{B}$, the translational velocity $\mathbf{v}(P,t) = \mathbf{v}^*(\mathscr{B},t) = \mathbf{a}^* t + \mathbf{v}_0^*(\mathscr{B})$ is also the same for all particles of $\mathscr{B}$, where $\mathbf{v}(P,0) = \mathbf{v}_0^*(\mathscr{B})$ is the translational velocity of the center of mass of $\mathscr{B}$ initially. Hence, (5.33) may be written as

$$\mathbf{h}_O(\mathscr{B},t) = \int_{\mathscr{B}} \mathbf{x}_O(P,t)\, dm(P) \times \mathbf{v}^*(\mathscr{B},t).$$

Recalling (5.12) and (5.16), we obtain

$$\mathbf{h}_O(\mathscr{B},t) = \mathbf{x}_O^*(\mathscr{B},t) \times m(\mathscr{B})\,\mathbf{v}^*(\mathscr{B},t) = \mathbf{x}_O^*(\mathscr{B},t) \times \mathbf{p}^*(\mathscr{B},t),$$

in which $\mathbf{x}_O^*(\mathscr{B},t)$ is the position vector of the center of mass from O. This equation has the same form as (5.31) for a single particle. Thus, with respect to an arbitrary point O, the moment of momentum of a body having a uniform translational acceleration is equal to the moment of momentum of its center of mass. □

The forgoing concepts on the mass, momentum, and moment of momentum of a particle, a system of particles, and a body have been assembled here for future convenience and to emphasize their parallel definitions and structure. These ideas, including the notion of the center of mass of a system and a body, will also be helpful in our introduction and discussion of the basic laws of mechanics to be studied next. Their main thrust, however, will appear later as the theory unfolds leading eventually to the analysis of the motion of a system of particles and of a rigid body.

5.5. Newton's Laws of Motion

The structure of classical dynamics rests upon three foundation axioms introduced by Sir Isaac Newton in 1687. These are known as Newton's laws of motion. In their original form, however, Newton's principles are inadequate for the study of the motion of a rigid or a deformable body. These applications require a brilliant generalization introduced by Leonhard Euler in 1750 and thereafter. Here we follow the course of classical developments and begin with an introduction to the foundation principles of mechanics for a particle.* Principles for systems and continua are discussed briefly below and in greater detail in later chapters. In the meanwhile, we shall see in the following two chapters that our subject is rich with interesting and useful results that derive from the following principles of classical mechanics.

1. **The first law of motion:** *In every material universe, the motion of a particle in a preferential reference frame Φ is determined by the action of forces whose total vanishes for all times when and only when the velocity of the particle is constant in Φ. That is, a particle initially at rest or in uniform motion in the preferential frame Φ continues in that state unless compelled by forces to change it.*
2. **The second law of motion:** *There exists a material universe, called the world, wherein the total force $\mathbf{F}(P, t)$ exerted on a particle P in the preferential frame Φ is equal to the time rate of change of the momentum of P in Φ:*

$$\mathbf{F}(P, t) = \frac{d\mathbf{p}(P, t)}{dt} = \frac{d}{dt}\left[m(P)\,\mathbf{v}(P, t)\right]. \tag{5.34}$$

3. **The law of mutual action:** *To every action force $\mathbf{A}$ there corresponds an equal and oppositely directed reaction force $\mathbf{R}$. That is, the mutual actions of two particles, one on the other, are oppositely directed vectors:* $\mathbf{R} = -\mathbf{A}$.

These foundation principles characterize a material universe that is intended to model the physical world, the real world in which we live. Indeed, a large body of practical experience and the test of many experiments have shown that these

* In the statement of his laws, Newton uses the term "body" or "bodies". The least of these, however, is a single particle; and we shall see later on that for a body of finite size the laws may be stated in terms of its center of mass particle. Moreover, we recall that Newton's theory focuses principally on its applications to the motions of celestial bodies whose dimensions are small compared with their enormous distances of separation, so heavenly bodies are usually modeled as particles.

laws model very well mechanical phenomena in the real world. Therefore, they are employed universally with confidence in their predictive value. On the other hand, there may exist other material universes where these rules do not hold, or they hold only approximately. We shall say more about this later on. Let us look more closely at their content.

5.5.1. The Material Universe and Forces

In analytical terms, the material universe is the set $\mathscr{U} = \{O_k\}$ whose elements O_k are material objects; and a body $\mathscr{B}$ is a subset of $\mathscr{U}$, the least of which consists of a single particle P. Forces can exist only in the presence of pairs of bodies. A force acts on a body $\mathscr{B}$ only when there exists another body $\hat{\mathscr{B}}$ separate from $\mathscr{B}$ which is the source of the action. Moreover, the action of a force in one direction is not the same as its action in another direction. Thus, force is a vector-valued entity defined on pairs of separate bodies in $\mathscr{U}$.

The forces of interaction between pairs of material objects are classified as *contact forces* and *body forces.* Contact force arises from the mutual action of material objects that touch one another. Body force arises from the mutual action between a pair of separated objects, and for this reason body force is often called *action at a distance.* Gravitational, electrical, and magnetic forces are familiar examples of body forces. However, forces are not always what they seem to be. Artificial gravity, for example, can be created by the whirling motion of a human centrifuge used to train astronauts. This apparent gravity is felt by the astronaut as a contact force when pressed hard into the seat by the centrifuge motion; and everyone has witnessed the apparent increase and decrease in gravity while riding up and down, respectively, in a fast moving elevator. A similar feeling of artificial gravity would be experienced in an elevator in outer space moving "upward" with a constant acceleration. And we all know that astronauts experience "weightlessness" (actually the absence of contact force in a perpetual free fall within the spacecraft), because the gravitational force that continues to act on them is very nearly balanced by a certain pseudo-force that arises from the orbital motion of the rapidly moving spacecraft and its passengers.

Interaction between material objects in $\mathscr{U}$ may be internal or external to a subset $\mathscr{S}$ of $\mathscr{U}$. This is diagrammed in Fig. 5.8. A force exerted on part $\mathscr{P}$ (a subset) of a body $\mathscr{S} \subset \mathscr{U}$ by another disjoint part $\hat{\mathscr{P}}$ of the same body is called an *internal force.* The force exerted on a part $\mathscr{P}$ of a body $\mathscr{S} \subset \mathscr{U}$ due to another body $\hat{\mathscr{S}} \subset \mathscr{U}$ that is not contained in $\mathscr{S}$ is called an *external force.* The collection of forces that act on a body is assumed additive. We remember that a part $\mathscr{P}$ of a body is itself a body. Hence, the total force exerted on a body $\mathscr{P}$ in $\mathscr{S}$ is defined as the vector sum of all internal and external forces that act on $\mathscr{P}$. Since the first two laws apply only to a body $\mathscr{S}^*$ consisting of a single particle (see Fig. 5.8.), it follows that the total force in these laws is necessarily the total external force that acts on that particle. Whatever may be the physical nature

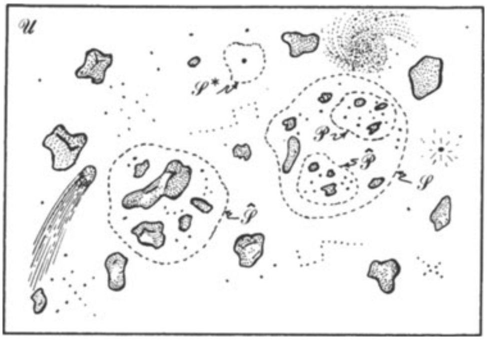


Figure 5.8. The material universe and its interacting parts.

of a force, its physical dimensions are defined on the basis of (5.34); namely, $[\mathbf{F}] \equiv [F] = [MVT^{-1}] = [MLT^{-2}]$. (See also the preface to the Problems for this chapter.) The three foundation laws are next discussed in turn.

5.5.2. The First Law of Motion

It is important to observe that Newton's laws hold only with respect to a certain preferential frame Φ. This special frame is called a *Newtonian* or *inertial reference frame.* The properties of the inertial frame will be studied later. For the time being, let us accept the idea that there exists in the universe an inertial frame that may serve as the preferred frame of Newton's laws, and continue.

The first law of motion postulates the existence of at least one preferred frame Φ and specifies that any disturbance of a particle P which is at rest or in uniform motion relative to this frame can occur only in response to force, while an arbitrary uniform motion or stationary state of P in Φ requires no force at all. So, explicitly, if $\mathbf{F}(P, t)$ denotes the total force acting on a particle P in any material universe whatever, the motion $\mathbf{x}(P, t)$ of P relative to Φ is determined by a certain functional relation (i.e., an equation in which the variable itself is a function or a set of functions) $\mathbf{x}(P, t) = \chi\,(\mathbf{F}(P, t))$, more commonly expressed in the standard form

$$\mathbf{F}(P, t) = \mathscr{F}(\mathbf{x}(P, t)). \tag{5.35}$$

Moreover, whatever its form, this general functional equation must satisfy the specified necessary and sufficient condition for a uniform motion in Φ, namely,

$$\mathbf{x}(P, t) = \mathbf{x}_0(P) + \mathbf{v}_0(P)t \Leftrightarrow \mathbf{F}(P, t) = \mathbf{0} \text{ for all } t, \tag{5.36}$$

wherein $\mathbf{x}_0$ and $\mathbf{v}_0$ are constant vectors. A rest state corresponds to the trivial case $\mathbf{v}_0 = \mathbf{0}$. Accordingly, the first law states that the unique solution of the equation $\mathbf{F}(P, t) = \mathscr{F}(\mathbf{x}(P, t)) = \mathbf{0}$ valid for all t in Φ is the uniform motion in (5.36). Or, conversely, if the motion is uniform in Φ, then $\mathbf{F}(P, t) = \mathscr{F}(\mathbf{x}_0 + \mathbf{v}_0 t) = \mathbf{0}$ for all t.

Alternatively, since a motion is uniform in Φ when and only when the acceleration in Φ is zero for all times t, (5.36) may be written as

$$\mathbf{F}(P, t) = \mathbf{0} \text{ for all } t \Leftrightarrow \mathbf{a}(P, t) = \mathbf{0} \text{ for all } t. \tag{5.37}$$

Because there is no inherent difference between a uniform motion and a state of rest, by definition, a stationary or uniform state of motion in the preferred frame Φ is called an *equilibrium state* in Φ. Thus, in accordance with (5.36) and (5.37), the first law specifies that *in every material universe, a condition necessary and sufficient for equilibrium of a particle P in an inertial reference frame is that the total force acting on P shall vanish for all times*:

$$\text{Equilibrium} \Leftrightarrow \mathbf{F}(P, t) = \mathbf{0} \Leftrightarrow \mathbf{a}(P, t) = \mathbf{0}. \tag{5.38}$$

Thus, Newton's first law postulates the general rule of determinism (5.35) and it specifies, by (5.36) or (5.38), a *universal principle of equilibrium* for a particle. It provides the foundation for the important special branch of dynamics called *statics*—the study of forces on bodies *at rest* in Φ.

The principle of equilibrium is the same in every material universe—it is a universal rule. However, when the motion is not uniform, the form of the functional equation (5.35) will depend upon the nature of the material universe it describes. In this respect, the first law is intentionally vague. The second law, on the other hand, is specific about the form of (5.35).

5.5.3. The Second Law of Motion

The second law of motion identifies a special material universe, called the *world,* for which the definite relation (5.34) between force and motion is introduced to describe the mechanical nature of things in the world. Of course, the abstract world of the second law is our analytical model of the real material universe, the real world where we live. However, the rule (5.34) must respect the conditions set in (5.36) or (5.37). Clearly, $\mathbf{F}(P, t) = \mathbf{0}$ for all t holds in (5.34) when and only when the momentum $\mathbf{p}(P, t) = m(P)\mathbf{v}(P, t) = \mathbf{p}_0(P)$ is a constant vector. *Hence, the motion is uniform if and only if the mass $m(P)$ is constant* (which it is).

On the other hand, imagine a different material universe in which (5.34) holds but now the mass varies with the particle speed. The second law would still support the conditions of the first law in this other material universe. In classical mechanics, however, the mass of a given body is an invariant, fixed property of the body—it is independent of the position, velocity, temperature, or any other influence acting on the body, so long as no part of the body disappears; that is, the mass of the body, or any part of the body, does not change in time. The principle of conservation of mass (5.1) invokes this condition for every motion of a particle. In consequence, from the rule (5.34), we obtain the basic formula popularly known as **Newton's**

equation of motion:

$$\mathbf{F}(P,t) = m(P)\mathbf{a}(P,t), \tag{5.39}$$

in which $\mathbf{a}(P,t)$ is the acceleration of P in the inertial frame.

The condition (5.37) imposed by the first law for every material universe strongly suggests that the simplest law of motion for the world is one for which $\mathbf{F} \propto \mathbf{a}$, so that $\mathbf{F} = \mathbf{0}$ implies a uniform motion in the inertial frame Φ, and conversely. This means we should have $\mathbf{F} = k\mathbf{a}$, where k is some constant characteristic of the particle. And what more appropriate constant might we select than the invariant mass of the object? Indeed, this is just the way it turned out in (5.39).

Thus, according to the first law, there may exist infinitely many material universes, or worlds, all having the same law of equilibrium but each characterized by a special equation of motion of its own, conceivably quite different from (5.34). The second law, however, provides a simple mathematical model to study the nature of most, though not all, physical phenomena in our world. Let us briefly look at its extension to a system of particles and to a continuum.

5.5.3.1. The Second Law for a System of Particles

The total force acting on a system of particles is defined as the sum of the forces that act on all of its particles. Let $\mathbf{F}_k = \mathbf{F}(P_k, t)$ denote the total force acting on the particle P_k of a system $\beta = \{P_k\}$ of n particles. Then, with (5.34) and (5.4), we *derive* **Newton's second law for a system of particles:** *The total force acting on a system of particles is equal to the time rate of change of the momentum of the system in the inertial frame, i.e.,*

$$\mathbf{F}(\beta,t) = \sum_{k=1}^{n} \mathbf{F}_k = \sum_{k=1}^{n} \frac{d\mathbf{p}_k}{dt} = \frac{d\mathbf{p}(\beta,t)}{dt}. \tag{5.40}$$

With the aid of (5.7) and the fact that mass is conserved, (5.40) may be cast in the same form as the basic equation of motion (5.39) for a single particle:

$$\mathbf{F}(\beta,t) = \frac{d\mathbf{p}^*(\beta,t)}{dt} = m(\beta)\mathbf{a}^*(\beta,t), \tag{5.41}$$

where $\mathbf{a}^*(\beta,t) = \dot{\mathbf{v}}^*(\beta,t)$ is the acceleration of the center of mass of the system. In words, *the total force acting on a system of particles is equal to the time rate of change of the momentum of its center of mass in the inertial frame Φ, and hence is equal to the product of the mass of the system and the acceleration of its center of mass in Φ*. The second law (5.41) for a system of particles thus aids the determination of the motion of the fictitious center of mass particle and external forces that control or constrain the motion of the system. In addition to (5.41), for a system of particles the auxiliary relations (5.5) through (5.8) are often needed in applications, as are the separate equations of motion of the particles. The equations of motion for a system of particles are discussed further in Chapter 8. Some further remarks on the equilibrium and interaction between the particles of the system follow shortly.

5.5.3.2. Introduction to Euler's Laws for a Continuum

We may visualize that as the number of particles of a system grows indefinitely, the system becomes a continuum $\mathscr{B}$ with momentum (5.11). In this case, the rule (5.34) is *replaced* by a more general principle known as **Euler's first law of motion:** *The total (external) force* $\mathbf{F}(\mathscr{B}, t)$ *acting on a body is equal to the time rate of change of its momentum in the preferred frame, i.e.,*

$$\mathbf{F}(\mathscr{B}, t) = \frac{d\mathbf{p}(\mathscr{B}, t)}{dt} = \frac{d}{dt}\int_{\mathscr{B}} \mathbf{v}(P, t)dm(P). \tag{5.42}$$

It is an amazing fact that this relation also may be written in the form of Newton's basic equation (5.39). We recall (5.16) and note that because the mass is conserved, Euler's first law (5.42) becomes

$$\mathbf{F}(\mathscr{B}, t) = \frac{d\mathbf{p}^*(\mathscr{B}, t)}{dt} = m(\mathscr{B})\mathbf{a}^*(\mathscr{B}, t). \tag{5.43}$$

Therefore, the total force acting on a body is equal to the time rate of change of the momentum of its center of mass, and hence is equal to product of the mass of the body and the acceleration $\mathbf{a}^*(\mathscr{B}, t)$ *of its center of mass in the inertial frame.* Euler's first law for a body thus relates the applied force to the motion of the center of mass.

Euler's second law has no counterpart among Newton's laws of motion. Euler's second principle relates the rotational part of the body's motion to the applied torque—the total moment of the applied forces about a fixed point in the inertial frame; and it also involves the moment of momentum (5.33) for a body. Thus, to study the general motion of a rigid body, besides (5.43), we shall need **Euler's second law of motion:** *With respect to a fixed point O in the inertial frame Φ, the total torque* $\mathbf{M}_O(\mathscr{B}, t)$ *that acts on a body is equal to the time rate of change in Φ of the total moment of momentum of the body about O:*

$$\mathbf{M}_O(\mathscr{B}, t) = \dot{\mathbf{h}}_O(\mathscr{B}, t) = \frac{d}{dt}\int_{\mathscr{B}} \mathbf{x}_O(P, t) \times \mathbf{v}(P, t)dm(P). \tag{5.44}$$

Euler's basic laws (5.42) and (5.44) are postulated for *all* bodies, including deformable solid and fluid bodies. Their application in this book, however, is restricted to rigid bodies. In this case, the velocity $\mathbf{v}(P, t)$ of an arbitrary body particle P may be expressed in terms of the angular velocity vector. This fact suggests that (5.44) relates the body's angular velocity and angular acceleration to the total applied torque about a fixed point in the inertial frame. We thus envision that Euler's second law is useful in determination of the rotational motion of the rigid body.

It follows from (5.43) and (5.44) that *equilibrium of a rigid body* requires two conditions necessary and sufficient in order that *every* particle of the body initially at rest or in uniform motion in the inertial frame shall continue in that state. With the initial conditions in mind, equilibrium requires that both the total force *and* the total torque acting on the rigid body about a fixed point must vanish for all time, i.e. the system of forces must be equipollent to zero:

$$\text{Equilibrium} \Leftrightarrow \mathbf{F}(\mathscr{B}, t) = \mathbf{0} \quad \text{and} \quad \mathbf{M}_O(\mathscr{B}, t) = \mathbf{0} \text{ for all } t. \tag{5.45}$$

This rule and Euler's laws are discussed further in Chapter 10.

The principle (5.43) that the mass center moves like a particle having mass equal to the mass of the body and acted upon by a force equal to the total force acting on the body means that the motion of the center of mass of a body often may be found by the methods of particle dynamics. Therefore, in our future study of the dynamics of a particle, it should be clear that it is correct to model a body of finite size by its center of mass particle. In general, however, because the equations of motion (5.43) and (5.44) for a body may be coupled, we cannot suppose that a problem of rigid body motion may be split into simple separate parts—a problem of particle dynamics and one of rotation of the body about an axis. In problems where rotational effects are absent, however, Euler's first law for a rigid body, or equivalently, Newton's second law for a particle, may be used to determine the motion of the center of mass particle and related unknown forces that drive or constrain that motion. The effects due to torques that may act on the body are studied later. Further discussion of (5.40) through (5.44) is reserved for their own place later; but, as we continue, we shall need to consider continua and systems of particles in discussion of their mutual interactions.

5.5.4. The Law of Mutual Action

Newton's third law admits that particles may exert *mutual forces* on one another to induce motion in accordance with the previous laws; however, whatever the nature of the force, the reaction of one particle in response to the action of another must be of equal, but oppositely directed intensity. Of course, this does not mean that these two forces will cancel from the equations of motion (5.39) for the particles, for the forces of action and reaction do not act on the same particle.

On the other hand, when the two particles are treated as a system, the mutual forces have no influence in the equation of motion (5.40) for the system. To see this, let us consider a system $\beta = \{P_1, P_2\}$ in which the particles P_1 and P_2 exert mutual force on one another. Let $\mathbf{F}_{12} = \mathbf{F}(P_1, t)$ be the force exerted on particle P_1 by particle P_2, and $\mathbf{F}_{21} = \mathbf{F}(P_2, t)$ the force exerted on particle P_2 by particle P_1. Then the third law requires that $\mathbf{F}_{12} = -\mathbf{F}_{21}$. These mutual forces are internal forces, and hence the total internal force is $\mathbf{F}(P_1, t) + \mathbf{F}(P_2, t) = \mathbf{0}$. Therefore, such mutual pairs of internal forces do not contribute to the total force

$\mathbf{F}(\beta, t)$ in the equation of motion (5.40), or (5.41), for the system. On the other hand, if only one particle P_1, say, is considered, then the mutual force $\mathbf{F}_{12}$ acts on this new "system", and it does not vanish in the equation of motion (5.39) for P_1.

This example shows the importance of carefully distinguishing the system being considered. The system chosen for study in a particular situation is called a *free body.* A drawing that shows all of the forces acting on the free body is called a *free body diagram,* a device introduced to facilitate the solution of a problem. To construct a free body diagram for any system, we need only recall that there are two classes of forces: contact forces and body forces. Therefore, we may begin by asking the question—What bodies are touching our free body? We then show in the free body sketch the appropriately directed contact forces exerted on the free body by each contacting body. Next, we ask—What bodies exert forces at a distance that are acting on our free body? And we show these appropriately directed body forces in the free body diagram. This simple but important initial procedure in the analysis of problems is illustrated many times in the sequel. It is essential that the student learn how to do this.

It is also important to mention that although the total internal force acting on a system of two particles is always zero, this does not imply that the system is in equilibrium. The particles could be moving with proportional acceleration vectors directed along the same line, or perhaps moving on distinct parallel lines. Also, particles of a system need not have the same uniform motion to be in equilibrium. On the other hand, for a system of two particles that separately are in equilibrium, the equal and oppositely directed mutual forces must be balanced by external forces so that the total force acting on each particle treated as a separate system is zero. Hence, the vanishing of the total force that acts on a system of particles is a necessary but not a sufficient condition for equilibrium. Moreover, if it is not required or otherwise established that mutual forces act along the line joining the particles, the force $\mathbf{F}_{12}$ exerted on P_1 by P_2 will have a definite turning effect on P_1 in moving it around P_2 as center. Newton's law of universal gravitational attraction assumes this collinearity, whereas, as shown later, the collinearity of mutual forces actually may be proved on the basis of a general rule governing the nature of mutual internal force that depends only on the locations of the two particles.

To advance further, however, we shall need to identify various kinds of forces. We begin by introducing the mutual gravitational force between two material objects.

5.6. Newton's Law of Gravitation

One kind of body force between two bodies is the mutual force of gravitational attraction, a basic force of nature that everyone knows as *gravity.* The theory of

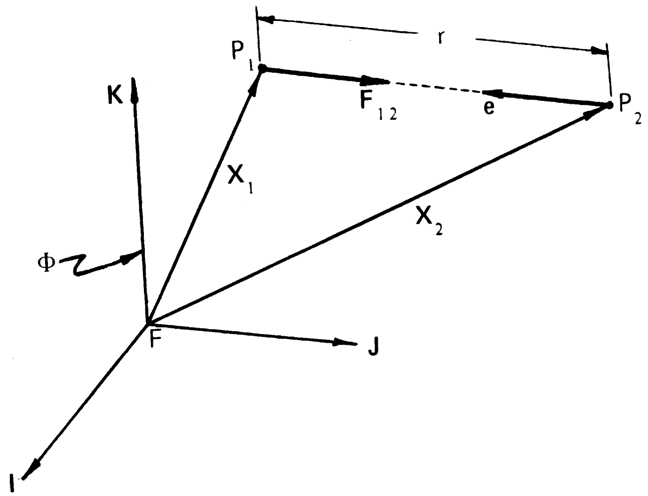


Figure 5.9. Schema for the mutual gravitational attraction of two particles.

gravitation invented by Newton to explain the motions of celestial bodies is studied here. The idea of a gravitational field created by the existence of matter is introduced to describe the gravitational field strength due to a particle, to a system of particles, and to a continuum; and the gravitational force exerted by these bodies on another particle, or body, is derived. We shall see that with regard to their gravitational attraction, bodies behave very much like particles, but not entirely. Our objective is to show that in all cases the gravitational force acting on a material object is equal to the product of its mass and the gravitational field strength it experiences. Afterwards, Newton's theory of gravitation is illustrated in a few examples. The gravitational attraction by an ideal planet is determined, and subsequently the definition of the weight of a body is introduced.

We begin with a pair of particles P_1, P_2 having mass m_1, m_2, respectively, and denote by $\mathbf{F}_{12}$ the force exerted on P_1 by P_2, as shown in Fig. 5.9. Let $\mathbf{e}$ be a unit vector directed from P_2, the source of the action, toward P_1; and write $r = |\mathbf{X}_2 - \mathbf{X}_1|$ for the distance between P_1 and P_2, wherein $\mathbf{X}_1$ and $\mathbf{X}_2$ are the respective distinct position vectors of P_1 and P_2 in any reference frame $\Phi = \{F; \mathbf{I}_k\}$. Clearly, only the relative position vector $\mathbf{r} \equiv r\mathbf{e}$ of P_1 from P_2 is important, so a reference frame is needed only for the solution of particular problems. These terms are used to state the following law of nature.

Newton's law of gravitation: *Between any two particles in the world, there exists a mutual gravitational force that is directly proportional to the product of their masses, inversely proportional to the square of their distance of separation, and directed in the sense of mutual attraction along their common line, i.e.,*

$$\mathbf{F}_{12} = -G\frac{m_1 m_2}{r^2}\mathbf{e} = -G\frac{m_1 m_2}{r^3}\mathbf{r}. \tag{5.46}$$

The positive constant G in (5.46) is named the *gravitational constant*, it is universal for all particles. Its physical dimensions consistent with (5.46) are $[G] = \left[FL^2/M^2\right] = \left[L^3/(MT^2)\right]$; its value will be given later. Of course, the roles of P_1 and P_2 are mutual and may be reversed. Hence, it is a consequence of the law itself that the mutual gravitational force exerted by P_1 on P_2 automatically respects the principle of mutual action, that is, $\mathbf{F}_{21} = -\mathbf{F}_{12}$.

Newton's law describes the gravitational interaction between any two particles in the world; and it has the same form in every reference frame—it depends only on the relative positions of the particles and their invariant masses. It is conceivable, however, that there may exist other material worlds where the law of gravity is different, or where Newton's law may hold only approximately. In fact, in the real world it has been known for a long time that the observed orbit of the planet Mercury differs very slightly from the path determined from calculations based on Newton's law. Indeed, the combined gravitational influence of all the known planets has failed to account for the observed shift in Mercury's perihelion.

In 1915, however, Einstein proposed a theory of relativity by which he showed that for bodies that move with speeds that are small compared with the speed of light Newton's theory of gravitation is a first approximation to a more general theory of gravitational fields. Unlike Newton's theory, which introduces the idea of mysterious forces at a distance, Einstein's theory is based on a special geometry of space and time—a theory whose formulation far exceeds the scope of our studies here.

Of course, practically all deviations from Newton's law that are predicted by Einstein's theory are so small that even with precision instruments they are difficult to measure. The precessional motion of the elliptical orbit of Mercury is a model case for which measurements of the rotation of its major axis, about 43 arc-seconds each century, agree precisely with Einstein's prediction. The deflection of light by the gravitational field of a star, the influence of gravitational field strength on the frequency of emitted light, and an explanation of the expanding motion of galactic systems are other effects predicted by Einstein's theory of relativity and confirmed by observations. These delicate, fascinating phenomena cannot be explained by Newton's theory. There are, however, countless other phenomena in the world that are perfectly and more easily modeled by Newton's simpler theory of gravitation described by (5.46). The discovery of Neptune based on an incredibly tedious year long calculation in 1846 by Urbain Jean Joseph Le Verrier, for example, was an exceptional accomplishment of Newton's theory.

Irregularity in the orbit of the planet Uranus was also known for a long time.† Calculations by the astronomer Le Verrier of the path of a hypothetical planet, whose gravitational attraction in accordance with Newton's theory would produce the observed discrepancy in Uranus's orbit, predicted the position of a new body in the sky. And when eventually a telescope was focussed on this place, the new

† Historical details of the discovery of Neptune and the search for the putative planet Vulcan are provided in articles by J. D. Fernie cited in the References.

planet Neptune was discovered very close to its predicted position. The same trick was used by Le Verrier to try to account for the discrepancies in Mercury's orbit. But his hypothetical planet named Vulcan has never been found. Rather, it was Einstein's theory of gravitation in 1915 that eventually accounted for the orbital discrepancies of Mercury, and it predicted similar effects for other planets, including the Earth. These are impressive theoretical results. Nevertheless, it is fair to say that in general Newton's simpler law of gravitation provides an exceptionally good mathematical model for studying the nature of many, though certainly not all, gravitational phenomena in the world; and we may use it with confidence in its predictive value. The idea of a gravitational field based on Newton's theory is introduced next.

5.6.1. The Gravitational Field of a Particle

A *gravitational field* $\mathscr{G}$ is said to exist in all of space due to the mass m_o whenever a force of attraction is felt by another "test" particle placed anywhere in $\mathscr{G}$. Hence, m_o is named the *origin*, or *source*, of the gravitational field. The attractive force due to m_o, per unit mass of the test particle, is called the *strength* of the field $\mathscr{G}$. Let $\mathbf{g}(\mathbf{X})$ denote the field strength at $\mathbf{X}$. Then, in accordance with (5.46),

$$\mathbf{g}(\mathbf{X}) = -\frac{Gm_o}{r^2}\mathbf{e}, \tag{5.47}$$

where $\mathbf{e}$ is the unit vector directed from the source m_o to the field point $\mathbf{X}$ whose distance from m_o is r, as shown in Fig. 5.10. Since $\mathbf{g}$ is the gravitational force that a particle of unit mass will experience when placed at $\mathbf{X}$ in $\mathscr{G}$, the gravitational force $\mathbf{F}(P;\mathbf{X})$ exerted on a particle P of mass m at $\mathbf{X}$ is given by

$$\mathbf{F}(P;\mathbf{X}) = m(P)\mathbf{g}(\mathbf{X}). \tag{5.48}$$

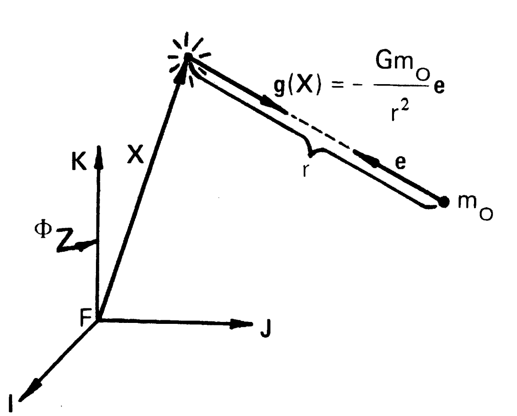


Figure 5.10. Gravitational field strength $\mathbf{g}(\mathbf{X})$ due to the mass point m_o.

Alternatively, with (5.47), $\mathbf{F}(P;\mathbf{X}) = -Gm_o m\mathbf{e}/r^2$, which is the same as (5.46). Observe again that the gravitational force is independent of the reference frame that may be used to identify the place $\mathbf{X}$.

5.6.2. The Gravitational Field of a System of Particles

The law of gravitation (5.46), hence also its alternate form (5.48), applies only to two particles. To find the gravitational force exerted on a particle P by a system of particles $\beta = \{P_k\}$, we use the fact that the field strength is a vector measure of force per unit mass. Since forces are vectorially additive, the separate field strengths of all particles of β must be vectorially additive. We suppose that the internal forces between the particles of β remain equal and opposite and in no way alter the individual field strengths $\mathbf{g}_k(\mathbf{X})$ due to the separate particles P_k of β. Then, with the aid of (5.48), the resultant gravitational force exerted on P by the totality of particles that comprise β is given by $\mathbf{F}(P;\mathbf{X}) = \sum_{k=1}^{n} \mathbf{F}_k(P;\mathbf{X}) = \sum_{k=1}^{n} m(P)\mathbf{g}_k(\mathbf{X})$, wherein $\mathbf{F}_k(P;\mathbf{X})$ is the gravitational force exerted by P_k on the particle P at $\mathbf{X}$. Thus, use of (5.47) for each source mass m_k in β yields the *resultant field strength* $\mathbf{g}(\mathbf{X})$ *for a system of n particles*:

$$\mathbf{g}(\mathbf{X}) \equiv \sum_{k=1}^{n} \mathbf{g}_k(\mathbf{X}) = -\sum_{k=1}^{n} \frac{Gm_k}{r_k^2}\mathbf{e}_k. \tag{5.49}$$

The interpretation of r_k and $\mathbf{e}_k$ is evident from Fig 5.11 in which the resultant field strength at $\mathbf{X}$ for a two particle system is illustrated. Hence, use of (5.49) yields the *resultant gravitational force on a particle P due to a system of particles*: $\mathbf{F}(P;\mathbf{X}) = m(P)\mathbf{g}(\mathbf{X})$, which has the same form as (5.48). Of course, the particle P exerts an equal but oppositely directed gravitational force on β. (See Problem 5.14.)

The direction of $\mathbf{F}(P;\mathbf{X})$ will depend on the direction of $\mathbf{g}(\mathbf{X})$, which is determined by the system β. In general, the resultant gravitational field strength (5.49), and hence the resultant gravitational force, does not pass through the center

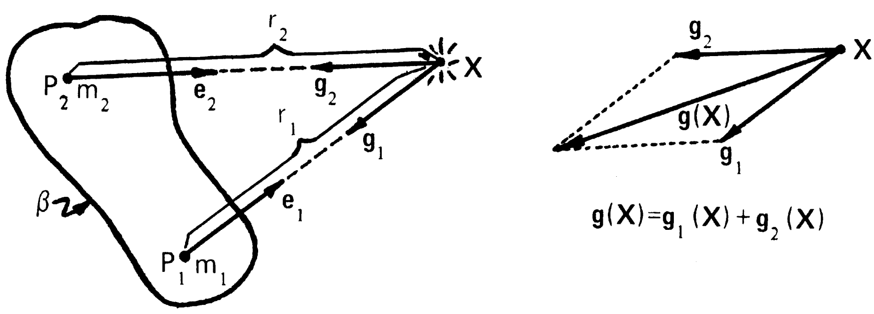


Figure 5.11. Resultant field strength $\mathbf{g} = \mathbf{g}_1 + \mathbf{g}_2$ of a system of particles $\beta = \{P_1, P_2\}$.

of mass of the field source β. Indeed, the field strength $\mathbf{g}^*(\mathbf{X})$ due to the center of mass particle at the place $\mathbf{r}^* = -r^*\mathbf{e}^*$ from $\mathbf{X}$ and having mass $m^* \equiv m(\beta)$ is given by (5.47). Writing $\mathbf{r}_k = -r_k\mathbf{e}_k$ (no sum) for the position vector of P_k from $\mathbf{X}$, as suggested in Fig. 5.11, and recalling (5.5) for the center of mass, we see by (5.47) and (5.49) that

$$\mathbf{g}^*(\mathbf{X}) = \frac{G}{r^{*3}} m^* \mathbf{r}^* = \sum_{k=1}^{n} \frac{Gm_k}{r^{*3}} \mathbf{r}_k \neq \sum_{k=1}^{n} \frac{Gm_k}{r_k^3} \mathbf{r}_k = \mathbf{g}(\mathbf{X}).$$

In general, therefore, $\mathbf{g}(\mathbf{X})$ is not parallel to $\mathbf{r}^*$, and hence the resultant gravitational force does not pass through the center of mass of β. Consequently, the gravitational force on P has a moment about the center of mass of the system. On the other hand, it may be seen that $\mathbf{g}(\mathbf{X}) = \mathbf{g}^*(\mathbf{X})$, very nearly, when the particle P is sufficiently far from the neighborhood of β so that the distance r_k of each particle P_k from $\mathbf{X}$ is equal, very nearly, to the distance r^* of the center of mass of β from $\mathbf{X}$. Precise demonstration of this statement based on the last relation above is left for the reader.

We have found that the formula for the resultant gravitational force on a particle due to a system of particles has the same form as the basic rule (5.48) for the gravitational force due to one particle. Derivation of a similar result for the gravitational interaction of two separate systems is left for the reader. The procedure and consequences are similar to those described below for two continuous bodies.

5.6.3. The Gravitational Field of a Body

The gravitational force due to a continuum acting on a particle may be found in a parallel manner. In this case, we generalize the particle theory by considering a gravitational field whose strength due to a parcel of mass dm_o of the body $\mathscr{B}_o$ is defined by $-(G dm_o/r^2)\mathbf{e}$, where $\mathbf{e}$ is the unit vector directed from the source dm_o to the field point $\mathbf{X}$ shown in Fig. 5.12 at a distance r from dm_o. Then the *resultant field strength at* $\mathbf{X}$ *due to the body* $\mathscr{B}_o$ is defined by

$$\mathbf{g}(\mathbf{X}) \equiv -G \int_{\mathscr{B}_o} \frac{\mathbf{e}}{r^2(\mathbf{X})} dm_o. \tag{5.50}$$

Both $\mathbf{e}$ and r will vary in the integration over the source body $\mathscr{B}_o$, so they cannot be taken outside the integral. The *resultant gravitational force exerted by the body* $\mathscr{B}_o$ *on a particle* P *of mass* m *at* $\mathbf{X}$ is determined by $\mathbf{F}(P; \mathbf{X}) = m(P)\mathbf{g}(\mathbf{X})$, which has the same representation as the basic rule (5.48) for the attraction between two particles. Of course, the particle exerts an equal and oppositely directed gravitational force on the body.

The direction of the resultant force $\mathbf{F}(P; \mathbf{X})$ is the same as that of $\mathbf{g}(\mathbf{X})$, which is determined by the body $\mathscr{B}_o$. It may be seen that the resultant gravitational field strength (5.50), and hence the resultant gravitational force usually does not pass through the center of mass of $\mathscr{B}_o$. The proof is parallel to that for a system of particles. Hence, in general, the resultant gravitational force exerted on P by the

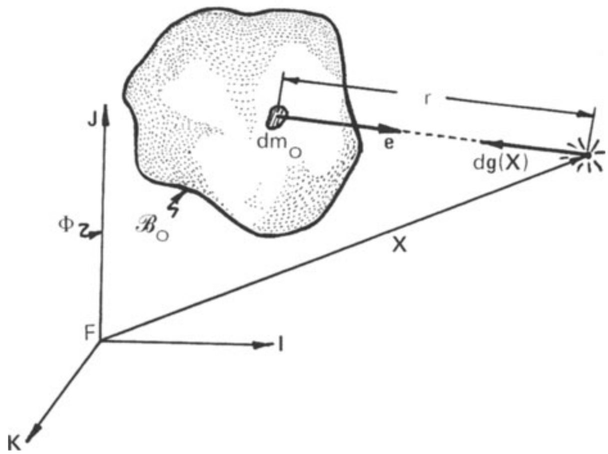


Figure 5.12. Elemental gravitational field strength $d\mathbf{g}(\mathbf{X})$ due to a parcel of mass dm_o of a body $\mathscr{B}_o$.

body is not the same as the gravitational force exerted on P by its center of mass particle. In consequence, the gravitational force on P exerts a torque about the center of mass of the body. Of course, when the particle P at $\mathbf{X}$ is sufficiently far from the neighborhood of $\mathscr{B}_o$ so that the distance of each of its particles from $\mathbf{X}$ is equal very nearly to the distance r^* of the center of mass of $\mathscr{B}_o$ from $\mathbf{X}$, the two field strengths are very nearly equal.

Finally, let us suppose that P is a material parcel $dm(P)$ of another body $\mathscr{B}$ with mass $m(\mathscr{B})$. Then use of the field strength (5.50) in integration over $\mathscr{B}$ determines the *resultant gravitational force* $\mathbf{F}(\mathscr{B})$ *exerted on* $\mathscr{B}$ *by* $\mathscr{B}_o$, namely,

$$\mathbf{F}(\mathscr{B}) = \int_{\mathscr{B}} \mathbf{g}(\mathbf{X})dm(P) \equiv m(\mathscr{B})\,\hat{\mathbf{g}}\,(\mathscr{B}). \tag{5.51}$$

The quantity $\hat{\mathbf{g}}\,(\mathscr{B})$ defined by (5.51) is named the *average,* or *mean field strength* due to $\mathscr{B}_o$. (See Problems 5.23 and 5.24.)

The gravitational force exerted by $\mathscr{B}$ on $\mathscr{B}_o$ is necessarily equal and oppositely directed to $\mathbf{F}(\mathscr{B})$; but the forces need not be collinear, nor pierce the center of mass of either body. Thus, with respect to an arbitrary reference point, in general the source body $\mathscr{B}_o$ will exert a *gravitational torque* on the body $\mathscr{B}$. If $\mathbf{x}_Q(P)$ is the position vector from a reference point Q to an element of mass $dm(P)$ of $\mathscr{B}$, the moment about Q of the gravitational force distribution exerted on $\mathscr{B}$ by the field source $\mathscr{B}_o$, in accordance with (5.22), is

$$\mathbf{M}_Q(\mathscr{B}) = \int_{\mathscr{B}} \mathbf{x}_Q(P) \times \mathbf{g}(\mathbf{X})dm(P). \tag{5.52}$$

This is illustrated in a subsequent Exercise 5.5, page 42, that includes discussion of the equipollent force and couple for the gravitational force system (5.51) and

(5.52). Of course, the gravitational torque may vanish and the mutual gravitational forces may pierce the centers of mass in special cases. This happens, for example, when $\mathscr{B}$ is sufficiently far from the source body $\mathscr{B}_o$.

Observations of the kind described above will be helpful in understanding the approximations assumed in our future studies of particle dynamics in which bodies of finite size occur in many of the problems. We have seen that with regard to the equation of motion, a body may be replaced by its corresponding center of mass object; and as regards the gravitational force acting on a body, there is presently only one rule that need concern us here. In sum, *regardless of the nature of the field source, the gravitational force* $\mathbf{F}(\mathscr{O})$ *acting on a material object* $\mathscr{O}$ *is equal to the product of its mass* $m(\mathscr{O})$ *and the total gravitational field strength* $\mathbf{g}(\mathscr{O})$ experienced by $\mathscr{O}$; *that is, in contracted notation,*

$$\mathbf{F}(\mathscr{O}) = m(\mathscr{O})\mathbf{g}(\mathscr{O}). \tag{5.53}$$

5.7. Some Applications of Newton's Theory of Gravitation

The application of Newton's theory of gravitation is illustrated next in two examples. The gravitational interaction between a wire ring and a particle, and between a wire ring and a thin rod are studied. It is confirmed that when a material object is sufficiently far from the field source, the gravitational interaction reduces to the fundamental law (5.46) for two particles. The gravitational torque exerted on a rod by a semicircular wire is then described in an exercise. We begin with the gravitational interaction between a solid body and a particle.

Example 5.6. *Interaction between a wire ring and a particle.* A homogeneous, thin circular wire $\mathscr{B}_o$ of radius R and mass m_o is shown in Fig. 5.13.

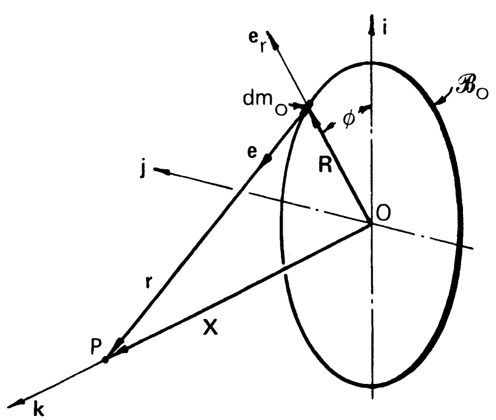


Figure 5.13. Geometry for the gravitational interaction between a wire ring and a particle.

Determine the gravitational field strength of the wire ring at a point P on the normal axis through its center O. Show that the resultant gravitational force exerted by $\mathscr{B}_o$ on a particle of mass m placed at P reduces to the gravitational force (5.46) between two particles when P is far enough from O such that $|\mathbf{X}| \gg R$.

Solution. The resultant field strength of the circular wire $\mathscr{B}_o$ at the place $\mathbf{X} = Z\mathbf{k}$ is determined by (5.50) in which the relative position vector $\mathbf{r}(\mathbf{X})$ of the point P from the parcel of mass dm_o of $\mathscr{B}_o$ is given by $\mathbf{r}(\mathbf{X}) = r\mathbf{e} = \mathbf{X} - \mathbf{R} = Z\mathbf{k} - R\mathbf{e}_r$ in terms of the cylindrical reference variables shown in Fig. 5.13. With $r^2 = Z^2 + R^2$, the integrand in (5.50) may be written as

$$\frac{\mathbf{e}}{r^2} = \frac{Z\mathbf{k} - R\mathbf{e}_r}{\left(Z^2 + R^2\right)^{\frac{3}{2}}}. \tag{5.54a}$$

Introducing $\sigma = m_o/2\pi R$, the mass per unit length of the homogeneous wire, and $ds = Rd\phi$, its elemental length, we have $dm_o = \sigma ds = \frac{1}{2\pi} m_o d\phi$. Then, with (5.54a) in (5.50), noting that both Z and R are fixed quantities, and setting the limits of integration over $\mathscr{B}_o$, we obtain the resultant field strength of the circular wire at $\mathbf{X}$:

$$\mathbf{g}(\mathbf{X}) = -\frac{Gm_o}{2\pi(Z^2 + R^2)^{\frac{3}{2}}}\left(Z\mathbf{k}\int_0^{2\pi} d\phi - R\int_0^{2\pi} \mathbf{e}_r d\phi\right), \tag{5.54b}$$

in which $\mathbf{e}_r = \cos\phi\mathbf{i} + \sin\phi\mathbf{j}$. The last term vanishes; and the gravitational field strength at the place $\mathbf{X}$ due to the circular wire is thus given by

$$\mathbf{g}(\mathbf{X}) = -\frac{Gm_o Z}{\left(Z^2 + R^2\right)^{\frac{3}{2}}}\mathbf{k}. \tag{5.54c}$$

The field strength at the place P is directed toward the center of the ring.

A particle of mass m placed at $\mathbf{X}$ in the field (5.54c) experiences an attractive gravitational force given by (5.48) in accordance with the rule (5.53), namely,

$$\mathbf{F}(P; \mathbf{X}) = m(P)\mathbf{g}(\mathbf{X}) = -\frac{Gmm_o Z}{\left(Z^2 + R^2\right)^{\frac{3}{2}}}\mathbf{k}, \tag{5.54d}$$

directed through the center of mass of $\mathscr{B}_o$. Notice that if P is placed at the center O where $Z = 0$, the resultant, mutual gravitational force on P is zero.

Finally, suppose that P is far enough from O so that $R/Z \ll 1$, hence negligible. Then $r = Z$, $\mathbf{k} = \mathbf{e}$, approximately, and (5.54d) may be written as $\mathbf{F}(P; \mathbf{X}) = -Gmm_o\mathbf{e}/r^2$, which has the same form as Newton's law (5.46) for the gravitational force between two particles of mass m and m_o, respectively placed at P and O. □

We next study an application of (5.51) for the gravitational attraction between two solid bodies.

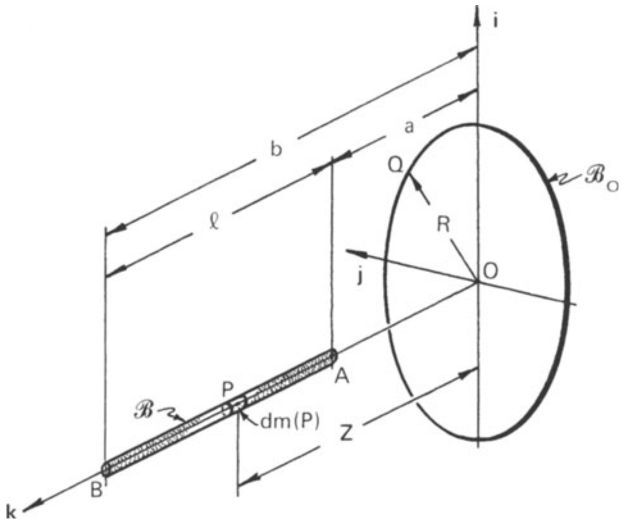


Figure 5.14. Gravitational interaction between a wire ring and a thin rod.

Example 5.7. *Interaction between a wire ring and a thin rod.* A homogeneous, thin rod $\mathscr{B}$ of length ℓ and mass $m(\mathscr{B})$ is placed along the normal axis of the wire ring described in the last example. What is the resultant gravitational force exerted by the rod on the ring? Find the mean field strength due to the ring.

Solution. Since the gravitational field strength of the wire ring is known by (5.54c), it is convenient to first find the resultant force that the ring exerts on the rod, and afterwards obtain the opposite force acting on the ring. The rod is placed along the central axis with its ends A and B at the respective distances a and b from the center O, as shown in the Fig. 5.14. For the homogeneous, thin rod, the parcel of mass at $\mathbf{X} = Z\mathbf{k}$ from O is $dm(P) = m(\mathscr{B})dZ/\ell$. Hence, the substitution into (5.51) of the gravitational field strength vector (5.54c) acting on $dm(P)$ determines the resultant gravitational force on the rod. Introducing the integration limits for the rod $\mathscr{B}$ and noting that $2ZdZ = d(Z^2 + R^2)$, we obtain

$$\mathbf{F}(\mathscr{B}) = -\frac{Gm_o m(\mathscr{B})}{2\ell}\mathbf{k}\int_a^b \frac{d\left(Z^2 + R^2\right)}{(Z^2 + R^2)^{\frac{3}{2}}}. \tag{5.55a}$$

This yields the resultant gravitational force on the rod $\mathscr{B}$ due to the wire ring $\mathscr{B}_o$:

$$\mathbf{F}(\mathscr{B}) = -\frac{Gm_o m(\mathscr{B})}{\ell}\mathbf{k}\left(\left(R^2 + a^2\right)^{-\frac{1}{2}} - \left(R^2 + b^2\right)^{-\frac{1}{2}}\right). \tag{5.55b}$$

The resultant gravitational force exerted on the ring is now given by $\mathbf{F}(\mathscr{B}_o) = -\mathbf{F}(\mathscr{B})$. This force pierces the mass centers of both homogeneous solids.

When the center of the rod is at $O, b = a = \ell/2$ and the mutual resultant gravitational force vanishes. When the rod is sufficiently far from the ring so that R/a and ℓ/a are both $\ll 1$, (5.55b) for the gravitational attraction between the two bodies reduces to (5.46) for two particles.

To determine the mean field strength due to the ring, first observe that

$$a_o \equiv \sqrt{R^2 + a^2}, \qquad b_o \equiv \sqrt{R^2 + b^2}, \tag{5.55c}$$

are the respective distances from any point Q on the ring to the end points A and B of the rod. Then, with (5.55b), the mean gravitational field strength due to the ring, in accordance with (5.51), is

$$\hat{\mathbf{g}}(\mathscr{B}) = \frac{\mathbf{F}(\mathscr{B})}{m(\mathscr{B})} = -\frac{Gm_o}{a_o b_o}\left(\frac{b_o - a_o}{\ell}\right)\mathbf{k}. \tag{5.55d}$$

□

The reader will find it informative to work through the following exercises. These review the previous examples in the solution of a similar problem for a semicircular wire. In addition, the gravitational torque effect is illustrated.

Exercise 5.3. *Interaction between a semicircular wire and a thin rod.* Suppose that the ring in the previous example is replaced by a semicircular wire of radius R in the upper half plane so that $\phi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ (see Fig. 5.13), while the rod retains the configuration shown in Fig. 5.14. Recall the sequence of equations (5.54a) through (5.54d). Show that the semicircular wire produces on the normal axis through O at $\mathbf{X} = Z\mathbf{k}$ a resultant gravitational field strength given by

$$\mathbf{g}(\mathbf{X}) = \frac{Gm_o}{\pi\left(Z^2 + R^2\right)^{\frac{3}{2}}}(2R\mathbf{i} - \pi Z\mathbf{k}); \tag{5.56a}$$

and hence the resultant gravitational force exerted by the wire on the rod is

$$\mathbf{F}(\mathscr{B}) = \frac{Gm_o m}{\pi\ell a_o b_o}\left[\frac{2}{R}(ba_o - ab_o)\mathbf{i} - \pi(b_o - a_o)\mathbf{k}\right], \tag{5.56b}$$

where a_o and b_o are defined in (5.55c). □

Exercise 5.4. *Gravitational torque exerted by a semicircular wire on a thin rod.* It is seen in (5.56b) that the resultant gravitational force on the rod has a vertical component that has a moment about the center point O, for example. Therefore, the gravitational force distribution on the rod gives rise to a gravitational torque (5.52). Let Q be the reference point at O so that $\mathbf{x}_O = Z\mathbf{k}$ in (5.52). Show that the gravitational torque about O exerted on the rod by the semicircular wire is

$$\mathbf{M}_O(\mathscr{B}) = \frac{2RGm_o m}{\pi\ell a_o b_o}(b_o - a_o)\mathbf{j}. \tag{5.56c}$$

□

Exercise 5.5. The force system (5.56b) and (5.56c) is equipollent to a *gravitational force* $\mathbf{F}(\mathscr{B})$ at a certain point Q and a *gravitational couple* $\mathbf{C}(\mathscr{B}) \equiv \bar{\mathbf{x}}_O \times \mathbf{F}(\mathscr{B})$, where $\bar{\mathbf{x}}_O = (\bar{x}, \bar{y}, \bar{z})$ is a position vector from O to any point on the line of action of $\mathbf{F}(\mathscr{B})$. Hence, Q is an arbitrary point on this line; and $\mathbf{C}(\mathscr{B}) = \mathbf{M}_O(\mathscr{B})$ provides the equation of the line of action of the equipollent force. Find the equation of the line of action of the equipollent force for the gravitational force system exerted by the semicircular wire on the rod. Determine its intercepts $(\bar{x}_o, \bar{y}_o, \bar{z}_o)$ with the axes, and thus show that the line of action of the equipollent force acting on the rod pierces the center of mass of the homogeneous semicircular wire, but not that of the rod. Consequently, the rod exerts on the wire a gravitational force $\mathbf{F}(\mathscr{B}_o) = -\mathbf{F}(\mathscr{B})$ and a gravitational couple $\mathbf{C}(\mathscr{B}_o) = -\mathbf{C}(\mathscr{B})$ at its center of mass. □

5.8. Gravitational Attraction by an Ideal Planet

Though enormous in size compared to ordinary material things, heavenly bodies are separated by great distances, so the ratios d/D of their diameters d to their distances of separation D are small quantities. Consequently, as regards their gravitational interactions, the heavenly bodies typically are modeled as particles. Here we examine this hypothesis for an ideal planet and show that its gravitational field strength is the same as the field strength of a particle of equal mass placed at its center.

Every material object in the vicinity of the Earth experiences a gravitational attraction that arises principally from the attractive force exerted by all parts of the Earth on every part of the object. Of course, the dimensions of ordinary bodies are infinitesimal in comparison with the size of the Earth, so even when these bodies may be on or very near the Earth, it seems sensible in a first approximation to model the body in its relationship to the Earth as a particle or, more precisely, as a center of mass object of mass m. Since the mass of a planet like the Earth is so considerably greater than the mass of even the largest structures, like an aircraft, a ship, or a skyscraper, the mutual gravitational attractions of these bodies obviously are small in comparison with the total gravitational force due to the Earth. Indeed, in all of our experience we have suffered no apparent propulsion toward these objects, nor they toward one another. But when we have the misfortune to tumble from even the slightest height, it hurts! The effect would be the same if it happened on the Moon, but with much reduced intensity due to the Moon's smaller size and mass. (See Problems 5.25 and 5.26.) In any case, ignoring other bodies, we want to know—What is the gravitational force on a body due to the Earth?

To model the shape of a typical planet and its mass distribution, let us assume that (i) the planet is a sphere of radius A, and (ii) its mass density $\rho = \rho(R)$ varies only with the distance R from its center. The total gravitational force exerted by the sphere on an external material point P at a distance X from its center may be determined by use of (5.50) in (5.48), and cast in the spherical coordinates

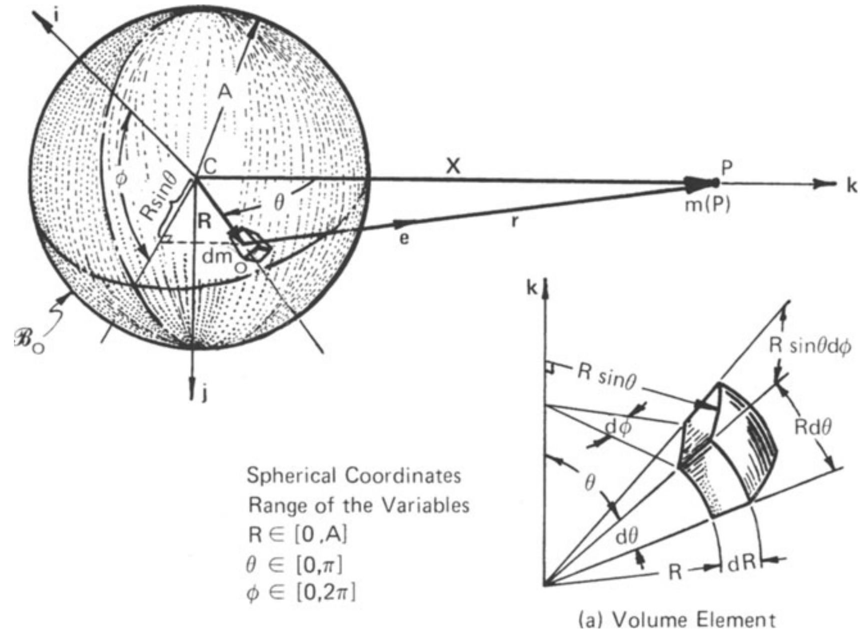


Figure 5.15. Geometry for the gravitational attraction of a particle P due to an ideal spherical planet.

(R, θ, ϕ) shown in Fig. 5.15. The material volume element is shown in Fig. 5.15a. Hence, the spherical element of mass is $dm_o = \rho(R)R^2 \sin\theta \; dR d\theta d\phi$. Also, $-\mathbf{e}/r^2 = (\mathbf{R} - \mathbf{X})/r^3$, wherein the unit source vector $\mathbf{e}$ is directed from the parcel dm_o at $\mathbf{R} = R\sin\theta(\cos\phi\mathbf{i} + \sin\phi\mathbf{j}) + R\cos\theta\mathbf{k}$ to the particle P at $\mathbf{X} = X\mathbf{k}$, and $r = (R^2 + X^2 - 2RX\cos\theta)^{\frac{1}{2}}$. Collecting these terms into (5.50) and setting the limits of integration over the sphere $\mathscr{B}_o$, we find the gravitational force (5.48) exerted by the ideal planet on a particle P of mass m at $\mathbf{X}$ is given by

$$\frac{\mathbf{F}(P;\mathbf{X})}{mG} = \int_0^A \int_0^\pi \int_0^{2\pi} \left(\frac{R\sin\theta(\cos\phi\mathbf{i} + \sin\phi\mathbf{j}) + (R\cos\theta - X)\mathbf{k}}{(R^2 + X^2 - 2RX\cos\theta)^{\frac{3}{2}}} \right)$$

$$\times\, \rho(R)R^2 \sin\theta dR d\theta d\phi.$$

The integrations are not so formidable as may appear. In fact, integration with respect to ϕ yields $\int_0^{2\pi}(\cos\phi\mathbf{i} + \sin\phi\mathbf{j})d\phi = \mathbf{0}$ and $\int_0^{2\pi} d\phi = 2\pi$. Therefore, it follows, as one might expect from symmetry, that *the resultant gravitational force exerted by an ideal planet on a particle of mass m is directed toward the center of the sphere*:

$$\mathbf{F}(P; X) = -2\pi mG\mathbf{k} \int_0^A \int_0^\pi \frac{(X - R\cos\theta)\rho(R)R^2 \sin\theta dR d\theta}{\left(R^2 + X^2 - 2RX\cos\theta\right)^{\frac{3}{2}}}. \tag{5.57}$$

The reader may show directly that (5.57) may be obtained by use of symmetry about the **k**-axis and by considering the attraction of a thin ring of radius $R\sin\theta$ and thickness dR at a central angle 2θ. So far the result (5.57) actually holds more generally for $\rho = \rho(R,\theta)$. To continue, however, we need $\rho = \rho(R)$.

Returning to (5.57) and integrating the functions in θ, being careful to observe that the particle at **X** lies *outside the sphere*, i.e. $X \geq R$, we eventually find the important result

$$\mathbf{F}(P;\mathbf{X}) \equiv -G\frac{m(P)m(\mathscr{B}_o)}{X^2}\mathbf{k} = m(P)\mathbf{g}(\mathbf{X}) \text{ for } X \geq R, \tag{5.58}$$

wherein $m(\mathscr{B}_o)$ is the mass of the sphere and $\mathbf{g}(\mathbf{X})$ is its field strength at $\mathbf{X}$:

$$m(\mathscr{B}_o) = \int_{\mathscr{B}_o} dm_o = \int_0^A 4\pi\rho(R)R^2 dR, \qquad \mathbf{g}(\mathbf{X}) = -\frac{Gm(\mathscr{B}_o)}{X^2}\mathbf{k}. \tag{5.59}$$

The gravitational force (5.58) has precisely the same form as (5.46) for the gravitational attraction between two particles; and the gravitational field strength in (5.59) is the same as the field strength (5.47) of a particle of equal mass $m(\mathscr{B}_o)$ placed at the center of the sphere. *Therefore, as regards its gravitational attraction, a sphere of mass density $\rho(R)$ behaves like a particle having mass $m_o = m(\mathscr{B}_o)$, the mass of the sphere, and located at its center.* Thus, any planet that is essentially spherical and has an average density variation that depends only on the distance from its center will attract a particle of mass m with the central directed force (5.58) characteristic of a source particle located at its center. Plainly, our hypothetical planet does not represent accurately the true features of the Earth, nor any other real planet. This analysis provides only a simple first approximation of the field strength due to the Earth, or any similar body.

5.9. Gravitational Force on an Object Near an Ideal Planet

Let us consider the field strength in the vicinity of our ideal planet. The radius vector from its center to an object P in the neighborhood of its surface may be written as $\mathbf{X} = (A + \varepsilon)\mathbf{k}$, where ε, the normal distance of P from the surface, is very small compared with the planet's radius A. Then (5.58) may be written as

$$\mathbf{F}(P;\mathbf{A}+\varepsilon) = -\frac{Gmm_o\mathbf{k}}{A^2(1+\varepsilon/A)^2}, \tag{5.60}$$

where $m_o \equiv m(\mathscr{B}_o)$ denotes the planet's mass. When $\varepsilon = 0$, we obtain the *gravitational force on P at the planet's surface*:

$$\mathbf{F}(P;\mathbf{A}) = m(P)\mathbf{g}(\mathbf{A}) \quad \text{with} \quad \mathbf{g}(\mathbf{A}) = -g\mathbf{k} \equiv -\frac{Gm_o}{A^2}\mathbf{k}. \tag{5.61}$$

The constant $g \equiv Gm_o/A^2$ is known as the *acceleration of gravity*; its value plainly depends upon the size and mass distribution of the planet. Although $\mathbf{g}$, as its name implies, has the physical dimensions of acceleration, it is not a kinematical quantity; it is not the derivative of a velocity vector.

To determine the error committed by our neglecting the term ε/A in (5.60), the relation (5.61) and the binomial expansion of $(1+\varepsilon/A)^{-2}$ are used to obtain

$$\mathbf{F}(P;\mathbf{A}+\varepsilon) = \mathbf{F}(P;\mathbf{A})(1 - \frac{2\varepsilon}{A} + \frac{3\varepsilon^2}{A^2} - \cdots).$$

The first approximation $\varepsilon/A = 0$ yields (5.61). Therefore, the next term $2\varepsilon/A$ is a measure of the error committed when this term is ignored. For example, for an aircraft flying at an altitude of $\varepsilon = 10$ mile (16 km) above the Earth, whose average radius is 3960 mile (6373 km), $2\varepsilon/A = 0.005$, whereas for a spacecraft at an altitude of 100 mile (161 km), $2\varepsilon/A = 0.05$. In the first instance we commit an error of about 0.5% when using the estimate (5.61), in the second we err by nearly 5%. Thus, so long as the object P does not stray too far from the planet, to a close approximation, the gravitational force $\mathbf{F} = m\mathbf{g}$ is a constant vector given by (5.61). The extent to which this approximation may be useful depends on the particular application. In situations where gravitational variations with the altitude are important, the estimate (5.61) is not to be used. (See Problem 5.22.)

5.10. Weight of a Body and its Center of Gravity

The gravitational force exerted by a body $\mathscr{B}_1$ on another body $\mathscr{B}_2$ is called the *weight of* $\mathscr{B}_2$ *relative to* $\mathscr{B}_1$. The gravitational field strength of a body $\mathscr{B}_o$ is given by (5.50), and the gravitational force it exerts on an object $\mathscr{O}$ is described by (5.53). This is the weight of $\mathscr{O}$ relative to $\mathscr{B}_o$. Thus, specifically, the weight $\mathbf{W}(P;\mathbf{X})$ at $\mathbf{X}$ of a particle P of mass $m(P)$, relative to $\mathscr{B}_o$, is defined by

$$\mathbf{W}(P;\mathbf{X}) \equiv m(P)\mathbf{g}(\mathbf{X}). \tag{5.62}$$

The universal law of gravitation (5.46), hence (5.50), involves invariant quantities that are independent of the reference frame—it is the same for all observers. Therefore, the weight of an object is the same for all observers; but it varies with the relative gravitational source. The weight of a particle P near the Earth is estimated by the constant force (5.61). The weight of the same particle in the neighborhood of the Moon, say, is also estimated by (5.61), but its value differs from its weight relative to the Earth. (See Problem 5.25.) In both cases, however, the mass $m(P)$ is the same—mass is an invariant property of a body; its weight is not. Henceforward, unless stated otherwise, the *weight of a body* shall mean its weight relative to the Earth. Thus, by (5.61) and (5.62), the weight $\mathbf{W}$ of a body modeled as a particle of mass m is an attractive body force abbreviated by $\mathbf{W} = W\mathbf{n} = mg\mathbf{n} = m\mathbf{g}$, where $\mathbf{n}$ is a unit vector directed toward the center of the Earth. In accordance with (5.53), the weight of a system of particles and a continuum are regarded similarly.

5.10.1. The Local Acceleration of Gravity—An Estimate

It is known from experimental measurements that the gravitational constant has the value $G = 6.67 \times 10^{-11}\ \mathrm{N \cdot m^2 \cdot kg^{-2}} = 3.43 \times 10^{-8}\ \mathrm{lb \cdot ft^2 \cdot slug^{-2}}$. The estimated average mass density of the Earth is $\rho = 5520\ \mathrm{kg/m^3}$, and its average radius is $A = 6373$ km, very nearly. Hence, the constant acceleration of gravity in the vicinity of the Earth *estimated* by (5.61) is $g = 9.824\ \mathrm{m/sec^2} = 32.23\ \mathrm{ft/sec^2}$. These values are reviewed and refined later on. In most engineering applications, however, it is customary to use the estimate $g = 9.8\ \mathrm{m/sec^2} = 32.2\ \mathrm{ft/sec^2}$.

Since the gravitational constant G is so very small, even when two bodies may be very close to one another, the gravitational force between them, though measurable (as demonstrated in experiments to measure G), is insignificant unless the mass of at least one of the bodies, like the Earth, is enormous. Therefore, the mutual attractive forces of neighboring bodies other than the Earth are ignored, and hence the total attractive gravitational force on an object is its weight. (See Problem 5.26.)

5.10.2. Center of Gravity of a Body

So far as a particle may be concerned there is no ambiguity as to where the weight vector acts—it acts on the particle. But when the total weight of a system of particles or of a body is introduced, the place relative to their material points at which the total weight of these bodies may be supposed to act is not evident. The concept of the center of gravity is introduced to clarify this question. We shall discuss the center of gravity for a body and leave as an exercise the parallel development for a system of particles.

The weight of a material parcel of mass $dm(P)$ at a point P of a body $\mathscr{B}$ is $\mathbf{g}(P)dm(P)$, where $\mathbf{g}(P)$ is the gravitational field strength at P due to the Earth. In accordance with the first equation in (5.51), the *weight* of $\mathscr{B}$ is defined by

$$\mathbf{W}(\mathscr{B}) = \int_{\mathscr{B}} \mathbf{g}(P)dm(P). \tag{5.63}$$

If the gravitational field strength is uniform over $\mathscr{B}$ so that $\mathbf{g}(P) = \mathbf{g}$, a constant vector, the weight of $\mathscr{B}$ is simply the product of $\mathbf{g}$ and its mass $m(\mathscr{B})$: $\mathbf{W}(\mathscr{B}) = m(\mathscr{B})\mathbf{g}$.

Since the body over which the gravitational field acts is small compared with the Earth, the Earth's field, though directed approximately toward its center, may be modeled as a parallel field over the body region, so that $\mathbf{g}(P) = g(P)\mathbf{n}$, where

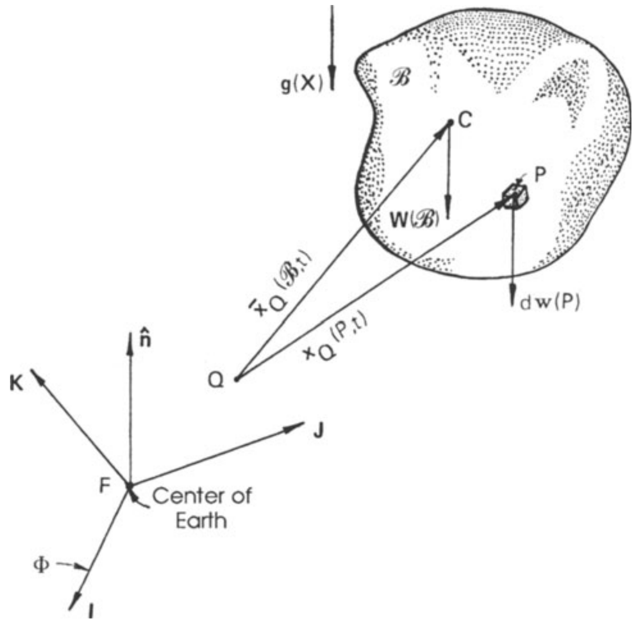


Figure 5.16. Schema for the equipollent moment condition in a parallel, variable gravity field.

$\mathbf{n}$ is a constant unit vector radially directed toward the Earth. Hence, (5.63) yields

$$W(\mathscr{B}) = \int_{\mathscr{B}} dw(P) = \int_{\mathscr{B}} g(P)dm(P), \tag{5.64}$$

in which $dw(P) \equiv g(P)dm(P)$ is the elemental weight of the parcel $dm(P)$.

By (5.63), the distribution of the weight of a body in a parallel, but variable gravitational field is equipollent to the single force $\mathbf{W}(\mathscr{B})$. In addition, for any assigned point Q in the Earth frame $\Phi = \{F; \mathbf{I}_k\}$ shown in Fig. 5.16, the moment $\mathbf{M}_Q$ about Q of the weight distribution $d\mathbf{w}(P) = \mathbf{n}dw(P)$ in the parallel Earth field is equipollent to the moment about Q of the total weight $\mathbf{W}(\mathscr{B}) = \mathbf{n}W(\mathscr{B})$ acting at a point C along its line of action in Φ. The unknown position vector of C from Q is denoted by $\bar{\mathbf{x}}_Q(\mathscr{B})$ in Fig 5.16. Thus, the equipollent moment condition (5.29) is

$$\mathbf{M}_Q = \bar{\mathbf{x}}_Q(\mathscr{B}) \times \mathbf{W}(\mathscr{B}) = \int_{\mathscr{B}} \mathbf{x}_Q(P) \times d\mathbf{w}(P), \tag{5.65}$$

wherein $\mathbf{x}_Q(P)$ is the position vector of a material parcel of weight $d\mathbf{w}(P)$ at P. With $d\mathbf{w} = dw\mathbf{n}$ and use of (5.64), (5.65) yields $W(\mathscr{B})\bar{\mathbf{x}}_Q \times \mathbf{n} = \int_{\mathscr{B}} \mathbf{x}_Q(P)dw(P) \times \mathbf{n}$. For simplicity, let us discard the subscript Q, and note that in general the position vectors may vary with time t, as suggested in Fig. 5.16. Then, with these adjustments, since Q may be chosen arbitrarily and $\mathbf{n}$ is a fixed direction, we may satisfy this equation by choosing the point at $\bar{\mathbf{x}}$ *defined by*

$$W(\mathscr{B})\bar{\mathbf{x}}(\mathscr{B}, t) = \int_{\mathscr{B}} \mathbf{x}(P, t)dw(P) \tag{5.66}$$

to provide the location from Q of the point C at which the weight of $\mathscr{B}$ acts to produce a moment about Q equal to that of its distribution. The point of the body $\mathscr{B}$ defined by $\bar{\mathbf{x}}(\mathscr{B}, t)$ in (5.66) is called the *center of gravity* of $\mathscr{B}$.

The location of the center of gravity will depend on the variable gravitational field strength $g(P)$ and the orientation of the body, which also might be nonhomogeneous. So, if the body is moved to a different configuration at another place in a variable gravity field, the center of gravity generally is not at the same place in the body frame; and hence the center of gravity generally is not a unique point in the body frame.

Example 5.8. A homogeneous cylinder $\mathscr{B}$ of height h and its base at the distance a from the Earth's center F in frame $\Phi = \{F; \mathbf{I}_k\}$ is shown in Fig. 5.17. Show that the center of gravity in a variable gravity field is not an invariant point in the body reference frame.

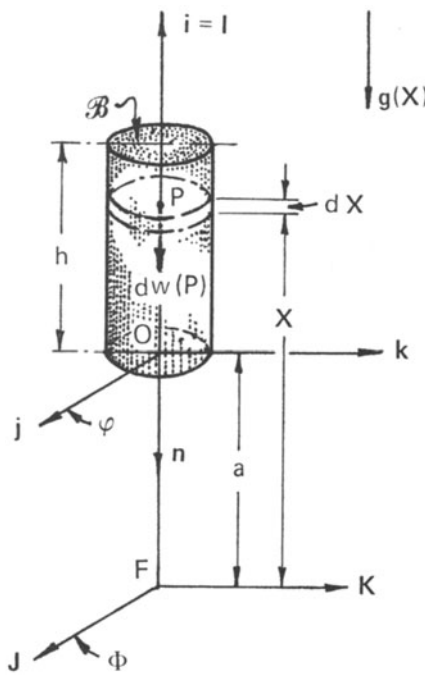


Figure 5.17. Schema for evaluation of the center of gravity of a uniform cylinder in a variable, parallel gravitational field.

Solution. The second equation in (5.59) gives the variable gravitational field strength $g(P) = MG/X^2$ at P due to the Earth. The Earth's mass is $M = m(\mathscr{B}_o)$ and X is the distance from F to a material parcel at P having weight $dw(P) = g(P)dm(P) = (MG/X^2)\sigma dX$, where $\sigma = m/h$ is the mass per unit length of $\mathscr{B}$. Integration in accordance with (5.64) shows that the weight of the cylinder in the given configuration will vary with the distance a from the Earth:

$$W(\mathscr{B}) = \frac{mMG}{h}\int_a^{a+h}\frac{dX}{X^2} = \frac{mMG}{a(a+h)}. \tag{5.67a}$$

The location $\bar{\mathbf{x}}(\mathscr{B}) = \bar{X}\mathbf{I} + \bar{Y}\mathbf{J} + \bar{Z}\mathbf{K}$ of the center of gravity from F is given by (5.66). With $\mathbf{x}(P) = X\mathbf{I} + Y\mathbf{J} + Z\mathbf{K}$ in Fig. 5.17, we find by symmetry about the $\mathbf{I}$-axis that $\bar{Y} = \bar{Z} = 0$ and

$$W(\mathscr{B})\bar{X} = \frac{MmG}{h}\int_a^{a+h}\frac{dX}{X} = \frac{MmG}{h}\ln\left(\frac{a+h}{a}\right). \tag{5.67b}$$

Using (5.67a) and introducing $\bar{x} \equiv \bar{X} - a$, we obtain the location $\bar{x}$ of the center of gravity in the body frame $\varphi = \{O; \mathbf{i}_k\}$ in Fig. 5.17:

$$\bar{x} = a\left[\frac{1+h/a}{h/a}\ln\left(1+\frac{h}{a}\right) - 1\right]. \tag{5.67c}$$

This result shows that the center of gravity in the body frame varies with a, the vertical distance of O from the center of the Earth. If the body is moved vertically to another place, the location $\bar{x}$ of the center of gravity in the body frame will change. Hence, in contrast with the invariant center of mass of the same body, the center of gravity generally is not a unique point in the body reference frame φ. The center of gravity is not an invariant property of the body.

On the other hand, the variable gravity effect on the position of the center of gravity of an ordinary body usually may be considered negligible. Because the body's height h is small compared to the radial distance a from the center of the Earth, we may ignore in the last formula all terms of order greater than the first in $h/a \ll 1$. We recall the series expansion $ln(1+z) = z - \frac{1}{2}z^2 + \frac{1}{3}z^3 - \cdots$ valid for $0 < z = h/a < 1$ and thus obtain the *unique, approximate location* $\bar{x} = h/2$ of the center of gravity in the body frame φ. □

In most practical cases of interest, the gravitational field throughout a body $\mathscr{B}$ that is small compared with the Earth may be approximated by a constant field of strength g throughout that body. Hence, the center of gravity of a body $\mathscr{B}$, even in a variable, parallel gravitational field, is the unique point in the body frame φ whose position vector $\bar{\mathbf{x}}$ from any assigned point Q in Φ is given by (5.66), very nearly. Therefore, so far as its weight is concerned, the body may be replaced by a particle of weight $\mathbf{W}(\mathscr{B})$ located at its center of gravity. Of course, the center of gravity particle need not be a material point of $\mathscr{B}$, but it may be. In a fixed configuration of the body, the definition (5.66) is independent of the choice of the reference point Q in Φ, and hence in a locally, constant gravity field, the center of gravity is the unique point C in the body frame relative to which

$$\int_{\mathscr{B}} \boldsymbol{\rho}(P, t) dw(P) = \mathbf{0}. \tag{5.68}$$

Here $\boldsymbol{\rho}(P, t)$ is the position vector from C to the parcel of weight $dw(P)$ at P. Equation (5.68) states that the moment of the weight distribution of the body about its center of gravity vanishes. The foregoing construction does not specify that the body be rigid. For a rigid body, however, $\boldsymbol{\rho}(P, t)$ is independent of time in a body frame.

We have learned that in general the center of gravity is not an invariant property of the body—it varies with the gravitational field strength in the region of space that the body currently occupies. However, because the field strength due to the Earth varies insignificantly over ordinary bodies, it is quite reasonable to replace the variable, parallel gravitational field by a locally uniform, parallel field. In this case, (5.66) reduces to (5.12) so that $\bar{\mathbf{x}} = \mathbf{x}^*$. *Thus, in a locally uniform gravitational field, the center of gravity and the center of mass of a body coincide, in which case the center of gravity shares all of the properties of the center of mass.*

Finally, we recall that sometimes the *weight density* $\gamma(P) \equiv \rho(P) g(P)$, the weight per unit volume of $\mathscr{B}$, is used in engineering analysis. In this case, we have $dw(P) = \gamma(P) dV(P)$. Thus, if the weight density of a body is constant, the weight of the body is the product of its weight density and its volume $V(\mathscr{B})$: $W(\mathscr{B}) = \gamma V(\mathscr{B})$. Hence, from (5.66) and (5.14), *the center of gravity of a body of uniform weight density is at its centroid. For a homogeneous body in a locally uniform, parallel gravitational field, ρ, g, and $\gamma = \rho g$ are constants, and hence in this important special case the center of gravity, the center of mass, and the centroid of the body are coincident points.* In general, however, they are not.

5.11. Coulomb's Laws of Friction

So far, our study has focused on one important kind of body force, the familiar force of gravity. We now consider a familiar kind of contact force, the frictional force that arises between pairs of separate bodies in their pending or relative sliding motion. Two physical laws, known as *Coulomb's laws*, govern the nature of this frictional force.

The first law of friction was known for a long time before Charles Coulomb (1736–1806), a senior captain in the French Royal Corps Engineers, verified it in 1781 during investigation of mechanical improvements for military gear. Historians, however, discovered long ago a statement of the first law in the notebooks of the famous Italian artist and inventor Leonardo da Vinci (1452–1519). From simple experiments, da Vinci concluded that *the amount of friction is proportional* *to the normal pressure between the contacting bodies and is independent of their area of contact.* Da Vinci's empirical proposition thus provided the first record in scientific writings of a law for sliding friction, an important contribution to mechanical science that was lost for nearly three centuries!

The notebooks, for several reasons, were virtually unknown prior to 1797. Translation of the manuscripts, language aside, was hampered by da Vinci's habit of writing in a reversed, left-directed fashion that required reading from a mirror, certainly an uninviting prospect. Though da Vinci apparently planned to assemble his voluminous notes for publication, this never happened. Upon his death in 1519, the encoded notebooks were passed to a close friend who guarded and preserved them until his own death in 1570; and from that time onward the manuscripts passed many hands, some parts being lost forever. Thirteen volumes survived and eventually were collected in the Ambrosian Library at Milan. But in the invasion of Italy in 1796, the documents were seized by Napoleon Bonaparte and carried to Paris, where for the first time they were studied by J. B. Venturi who later described them in an essay published in 1797. (See Hart, Chapters I and VII.)

It is no surprise, therefore, that da Vinci's law of friction was unknown to the French engineer Guillaume Amontons, who rediscovered it in 1699, nearly 200 years after da Vinci. It is astonishing, however, that the French Academy of Sciences, which expressed disbelief of the independence of the area of contact, received Amontons's rule with skepticism. Yet later, in 1781, the Academy awarded Coulomb a prize for essentially the same thing, though presented more thoroughly and in broader terms. (See Deresiewicz.) Coulomb's exemplary experiments established, not one, but two basic laws of friction that express a clear distinction between static friction and dynamic friction that went unnoticed by all others. These principles characterize the nature of the contact force between surfaces at rest and in relative sliding motion; they are the focus of the discussion that follows.

5.11.1. Contact Force between Bodies

A contact force is the mutual force acting at the interface between separate bodies that touch one another. At each interface point $\mathbf{q}$, the contact force $\boldsymbol{\gamma}(\mathbf{q})$, say, exerted by one body upon the other may be separated into component forces $\boldsymbol{\eta}(\mathbf{q})$ and $\boldsymbol{\tau}(\mathbf{q})$, respectively, normal and tangent to the interface at $\mathbf{q}$, so that $\boldsymbol{\gamma}(\mathbf{q}) = \boldsymbol{\eta}(\mathbf{q}) + \boldsymbol{\tau}(\mathbf{q})$. The normal component describes the mutual pulling (tension) or pushing (compression) of one body by the other perpendicular to the interface; it is called the *normal force*. If the contacting bodies $\mathscr{B}_1$ and $\mathscr{B}_2$ are subsets of the same body $\mathscr{B} = \mathscr{B}_1 \cup \mathscr{B}_2$ separated by an imaginary material surface $\mathscr{A}$, the tangential component $\boldsymbol{\tau}(\mathbf{q})$ characterizes the mutual resistance to shearing of the two parts along $\mathscr{A}$, so that $\boldsymbol{\tau}(\mathbf{q})$ is named the *shear force*. These particular contact forces play a paramount role in the mechanics of deformable solids and fluids. We shall encounter them in a different setting in various problems ahead. If we wish, for example, to determine how the tension in the string of a pendulum varies as the pendulum swings to and fro, it is necessary to introduce an imaginary cut in the string, and show in its place in a free body diagram of the pendulum bob, the normal (tensile) force that the string exerts on the bob. On the other hand, when two bodies $\mathscr{B}_1$ and $\mathscr{B}_2$ are physically separate, to maintain their contact the normal component of the contact force must be compressive; and its tangential component characterizes the mutual resistance to sliding of one body surface over the other, a natural effect that everybody knows as *friction.* In this case, the tangential component is called the *frictional force.*

The description of the frictional force is far more complicated than suggested above. The normal and tangential components of the interfacial force are distributed over the area of contact. But the actual area of contact is unknown. Indeed, even the most carefully polished surfaces look under magnification like miniature mountain ranges with hills and valleys that are much larger than molecular dimensions, and the contacting surfaces press upon these tiny mountains. Therefore, the actual area of contact may be much smaller than the apparent area of contact described by the macroscopic dimensions of the interfacial region.

Although interlocking effects of the surface asperities play a role in the overall complex mechanism[‡] of sliding friction, it is known from sophisticated measurements that frictional force arises mainly from the force required to shear the mountain peaks. Moreover, these experiments reveal that the actual area of contact, accounting for the deformation, depends on the intensity of the normal force. This area, however, is very nearly independent of the apparent interfacial area of the sliding bodies. The intense pressure at the contact points increases the area of contact until it is large enough to support the load. But in observation of the frictional resistance, the growth in the real area of contact manifests itself through the increase in the applied normal thrust, and hence is independent of the apparent interfacial area of contact. These measurements confirm da Vinci's primary observations and support his law of sliding friction; and they are the foundation for Coulomb's laws.

The distribution of the contact force is also unknown. But information about this force is required before any problem that involves friction can be solved. To hurdle this obstacle, we adopt the advance strategy that the normal and tangential distributions of the contact force, whatever the actual area of contact may be, are equipollent to a resultant normal force **N** and a resultant frictional force **f** that acts to oppose the relative motion of two contacting separate bodies $\mathscr{B}_1$ and $\mathscr{B}_2$. Thus, if $\boldsymbol{\eta}(\mathbf{q})$ and $\boldsymbol{\tau}(\mathbf{q})$ denote the normal and tangential force distributions per unit area a of the apparent contact area A, then $\mathbf{N} = \int_A \boldsymbol{\eta}(\mathbf{q})da$, $\mathbf{f} = \int_A \boldsymbol{\tau}(\mathbf{q})da$; and the resultant contact force $\mathbf{R} = \int_A \boldsymbol{\gamma}(\mathbf{q})da$ exerted by $\mathscr{B}_1$ on $\mathscr{B}_2$ is

$$\mathbf{R} = \mathbf{N} + \mathbf{f}. \tag{5.69}$$

[‡] See the classical treatise by Bowden and Tabor. Contemporary molecular theories of friction and modern surface measurement techniques are discussed in the referenced article by Krim.

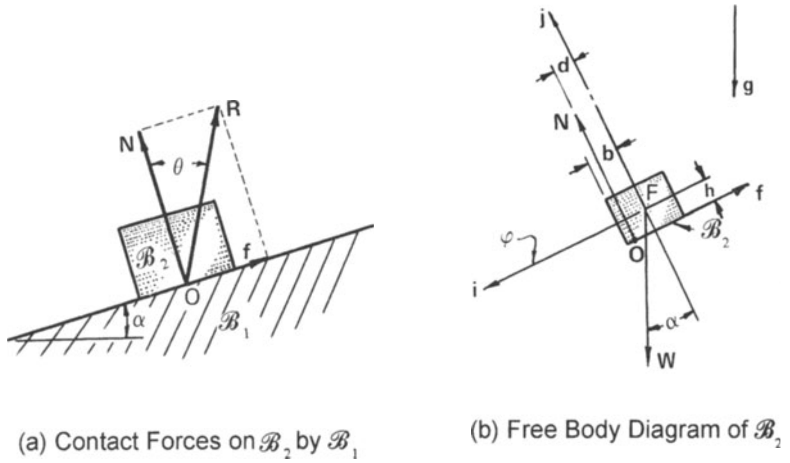


Figure 5.18. The contact forces exerted by the body $\mathscr{B}_1$ on the body $\mathscr{B}_2$, and the free body diagram of $\mathscr{B}_2$.

Thus, instead of having to deal with the unknown surface load distributions, we may work with their resultants in (5.69). The resultant contact forces exerted by a body $\mathscr{B}_1$ on another body $\mathscr{B}_2$ are shown in Fig. 5.18a. Other contact and body forces may act on $\mathscr{B}_2$, but these are not shown here. Of course, the contact forces exerted by $\mathscr{B}_2$ on $\mathscr{B}_1$ are opposite to those exerted by $\mathscr{B}_1$ on $\mathscr{B}_2$.

5.11.2. Governing Principles of Sliding Friction

Perfectly smooth, frictionless surfaces do not exist. Nonetheless, sometimes the surface asperity is so fine that the surface feels perfectly smooth to our sensation of touch. Therefore, in situations where frictional effects may be considered negligible or unimportant, we may sometimes consider an ideal model of smooth contacting surfaces that offer no sliding resistance whatever, a model that brings to mind the seemingly effortless, graceful motion of a skater on virtually frictionless ice. In this ideal case, the frictional force is zero and the contact force is normal to the interfacial tangent plane, that is, $\mathbf{f} = \mathbf{0}$ and $\mathbf{R} = \mathbf{N}$ in (5.69). This ideal property characterizes a so-called *smooth surface.*

When the surfaces are not perfectly smooth, it is intuitively clear that if the angle α of the inclination of the plane surface of the body $\mathscr{B}_1$ shown in Fig. 5.18a is sufficiently small, the body $\mathscr{B}_2$ will remain at rest on the plane. But as α is gradually increased, the magnitude of the frictional force must also increase gradually to restrain $\mathscr{B}_2$. Eventually, the *angle of repose* α will exceed a certain critical, maximum value α_c at which the frictional force can no longer sustain the equilibrium of $\mathscr{B}_2$, and $\mathscr{B}_2$ will begin to slide down the plane. Thus, the magnitude $f = |\mathbf{f}|$ of the *static frictional force* between the bodies eventually will reach a critical value $f = f_c$, called the *critical force, at which slip is imminent.* Of course, after sliding begins, friction continues to act between the contacting surfaces to oppose the relative motion. Clearly, this *dynamic frictional force* f_d cannot exceed the static, critical force; in fact, experiments show that $f_d < f_c$. These critical values of the frictional forces depend on the intensity of the normal contact force between the bodies. Indeed, we see readily that when lightly pressed together, our hands can be slid easily one upon the other; but when pressed tightly together, their relative sliding is rendered more difficult. The values of the critical, static and dynamic frictional forces are most simply related to the magnitude of the normal contact force between the bodies in accordance with the following basic and ideal principles of friction commonly known as **Coulomb's laws of friction**.

1. **The law of static friction:** *The critical magnitude f_c of the static frictional force between dry or lightly wetted surfaces that are at the verge of slipping relative to each other is proportional to the intensity N of the mutual, resultant normal force between them*:
$$f_c = \mu N. \tag{5.70}$$
The constant μ, called the coefficient of static friction, is independent of the interfacial contact area; it depends only on the nature of the contacting surfaces.
2. **The law of dynamic friction:** *The magnitude f_d of the frictional force between two dry or lightly wetted surfaces sliding relative to one another is proportional to the intensity N of the mutual, resultant normal force between them*:
$$f_d = \nu N. \tag{5.71}$$
The constant ν, named the coefficient of dynamic friction, is less than the static coefficient for the same conditions, $\nu < \mu$. Moreover, ν is independent of the interfacial area of contact and of the relative sliding speed of the surfaces; it depends only on the nature of the contacting surfaces.

The first law determines the *greatest* frictional force that can develop between contacting surfaces *before* sliding occurs, whereas the second law determines the magnitude of the frictional force that acts *during* the relative sliding motion. If a sliding motion between two bodies has not occurred and is not imminent, then the magnitude f of the frictional force is always less than the critical force f_c and may be determined by equilibrium considerations. These remarks are summarized schematically in Fig. 5.19 to illustrate the relations

$$\text{Static: } 0 \le f \le f_c = \mu N; \tag{5.72}$$
$$\text{Dynamic: } 0 \le f = f_d = \nu N < f_c. \tag{5.73}$$

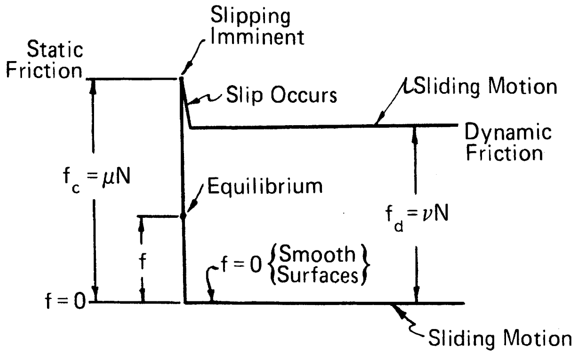


Figure 5.19. Graphical interpretation of Coulomb's laws of static and dynamic friction.

Note that $f = f_c$ holds in (5.72) only when relative slip is imminent; and $f = f_d$ holds in (5.73) only while sliding occurs. Further, $f = 0$ holds only for ideal, perfectly smooth surfaces for which $\mu = \nu = 0$. Otherwise, since $\nu < \mu$, once slip is achieved, a smaller force acts to retard the motion. These effects are assumed to be independent of the interfacial area of contact and of the relative speed of the dry or lightly wetted surfaces.

The static and dynamic coefficients of friction will depend only on the nature of the contacting material surfaces, that is, on the materials of which the bodies are made, their surface roughness quality, their degree of lubrication, their temperature, perhaps their chemical characteristics, and some other less important things. Clearly, the values of both μ and ν must be found by experiments. Also, when one body rolls on another, there is very little interfacial slip; but the bodies still experience mutual resistance to rolling, which is called *rolling friction*. Everyone knows that it is easier to roll than to slide a body on a flat surface; hence, rolling friction is considerably smaller than sliding friction. Further, when a layer of fluid, such as air or water, separates two surfaces, there is a resisting force exerted by the fluid which is called *drag* or *viscous friction*. Both rolling and viscous friction are determined by laws that are entirely different from Coulomb's rules of sliding friction. The effects of viscous friction are discussed in Chapter 6. The interested reader may consult the sources cited at the end of this chapter for details on these additional matters. We now turn to some examples.

5.11.3. Equilibrium of a Block on an Inclined Plane

Let us consider the familiar, elementary problem of equilibrium of a rigid block $\mathscr{B}_2$ shown in Fig. 5.18a at rest on an inclined plane $\mathscr{B}_1$. Our focus is on the general procedure for setting up and solving this problem. In addition, some elementary results of static friction are also reviewed.

First, choose $\mathscr{B}_2$ as a free body (the system to be investigated). Now identify all of the contact and body forces that act on $\mathscr{B}_2$ alone. We may ignore the contact force of the surrounding air. (Why?) Then the only body that touches $\mathscr{B}_2$ is the body $\mathscr{B}_1$, so the total contact force acting on $\mathscr{B}_2$ consists of the equipollent normal force $\mathbf{N}$ and frictional force $\mathbf{f}$ due to $\mathscr{B}_1$, or the equivalent reaction force $\mathbf{R}$. The Earth is the only body that exerts a significant body force on $\mathscr{B}_2$, hence the total body force acting on $\mathscr{B}_2$ is its weight $\mathbf{W}$. All of the forces that act on $\mathscr{B}_2$, whether it be in equilibrium or in motion in an assigned inertial frame are shown in the free body diagram in Fig. 5.18b. The direction of these forces must be consistent with the physical situation. In particular, $\mathbf{f}$ must act to retard the potential motion of $\mathscr{B}_2$, $\mathbf{N}$ must support $\mathscr{B}_2$, and $\mathbf{W}$ must be directed toward the center of the Earth. The vector $\mathbf{g}$ denotes in the figure the direction of the gravitational attraction of the Earth.

Any inertial frame may be introduced to formulate the problem, but one choice may be mathematically more convenient than another. The inertial frame $\varphi = \{F; \mathbf{i}_k\}$ shown in Fig. 5.18b is a good choice because the forces are most easily related to it. The free body diagram shows that the total force acting on the block $\mathscr{B}_2$ is

$$\mathbf{F}(\mathscr{B}_2, t) = \mathbf{W} + \mathbf{f} + \mathbf{N}. \tag{5.74a}$$

Next, express these forces in terms of their components in φ:

$$\mathbf{W} = W(\sin\alpha\mathbf{i} - \cos\alpha\mathbf{j}), \quad \mathbf{f} = -f\mathbf{i}, \quad \mathbf{N} = N\mathbf{j}. \tag{5.74b}$$

Here $W = mg$, f, and N denote the magnitudes of these forces. This completes the primary phase in the problem formulation.

5.11.3.1. The Force Equilibrium Relations

Since the block is in equilibrium in φ, in accordance with (5.45), the total force (5.74a) and the total moment about a point fixed in φ of the forces in (5.74b) must vanish for all times. First, consider the forces. Substitute (5.74b) into (5.74a), and write the force equilibrium equation,

$$\mathbf{F}(\mathscr{B}_2, t) = (W\sin\alpha - f)\mathbf{i} + (N - W\cos\alpha)\mathbf{j} = \mathbf{0}. \tag{5.74c}$$

Consequently, each vector component must be zero; and so the normal and frictional forces on $\mathscr{B}_2$ are determined by

$$N = W\cos\alpha, \qquad f = W\sin\alpha. \tag{5.74d}$$

5.11.3.2. The Moment Equilibrium and No Tip Conditions

The zero moment equation $\mathbf{M}_O(\mathscr{B}_2, t) = \mathbf{0}$, the second of the equilibrium conditions for a rigid body in (5.45), will fix the location d of the line of action of the resultant normal force $\mathbf{N}$. Since the resultants $\mathbf{N}$ and $\mathbf{f}$ in Fig. 5.18b are concurrent at a certain point O in the interfacial plane, it is clear that the moment $\mathbf{M}_O$ of the forces (5.74b) taken about this fixed point in φ will vanish if and only if the line of action of $\mathbf{W}$ also passes through O. Indeed, it is easily shown in general that three concurrent forces acting on a particle in equilibrium must be coplanar. (Is the same generally true for a particle in motion?) Hence, the concurrent forces $\mathbf{W}$, $\mathbf{N}$, and $\mathbf{f}$ lie in the vertical plane containing the interface point O and the center of gravity of the block at the height h above the plane surface. Clearly, the block will not tumble forward so long the line of action of $\mathbf{W}$ falls within the distance b to the leading edge of the block, and hence for $d \leq b$ in Fig. 5.18b. The line of action of $\mathbf{W}$, and hence the point O, is at the leading edge when the angle $\alpha = \alpha_t = \tan^{-1} b/h$. Therefore, if the plane's angle of inclination may be increased to the angle α_t without exciting slip so that $\alpha_t < \alpha_c$, the critical angle of sliding friction, the block will be at the verge of tipping over rather than sliding down the plane. The slightest further increase in the inclined angle α_t pushes the line of action of $\mathbf{W}$ ahead of the leading edge and the block will topple down the plane before sliding impends, because the moment of $\mathbf{W}$ about O at the leading edge of the block is no longer balanced. Henceforward, we shall suppose that the moment equilibrium condition $\alpha \leq \alpha_t$ for no tipping of the block is respected. (See Problem 5.28.)

5.11.3.3. The No Slip Condition

From (5.74a), $\mathbf{R} = \mathbf{f} + \mathbf{N} = -\mathbf{W}$; and hence the resultant contact force must be opposite to the weight $\mathbf{W}$. Indeed, the zero moment condition for equilibrium in (5.45) shows that $\mathbf{R}$ and $\mathbf{W}$ must be collinear. Hence, in consequence of equilibrium, the angle θ that $\mathbf{R}$ makes with the normal to the inclined plane surface in Fig. 5.18a is equal to the plane's angle α. From (5.74d) and (5.72), it is seen that the angle of repose must satisfy the inequality

$$\tan\alpha = \frac{f}{N} \leq \mu, \tag{5.74e}$$

in order that f shall be less than the critical force f_c at which $\mathscr{B}_2$ will be at the verge of slipping. The greatest angle α_c for which (5.74e) holds is called the *critical angle of friction*; it is given by Coulomb's law (5.70) as expressed in (5.74e), that is,

$$\tan\alpha_c = \frac{f_c}{N} = \mu. \tag{5.74f}$$

Thus, the tangent of the angle of repose is critical when it reaches a value equal to the coefficient of static friction, a value that is independent of the weight of the block $\mathscr{B}_2$. When $\alpha = \alpha_c < \alpha_t$, the block will not topple over, but the slightest further increase in the plane's inclination will cause the block to slide, and our equilibrium analysis, no longer valid, must be replaced by analysis of the block's motion. □

The basic *free body formulation* procedure used above is applied almost invariably in the formulation of all problems in both statics and dynamics. Sometimes problems may be solved easily in direct vector notation, so the decomposition of the forces into components may not be necessary, but more often than not, for simplicity, it is. It cannot be too strongly emphasized that *the free body formulation for the total force is the same for both a statics and a dynamics problem; and it is important that the student become thoroughly familiar with this method.* The analysis of the block's motion follows.

5.11.4. Motion of a Block on an Inclined Plane

We now encounter our first application of dynamics in the analysis of the sliding motion of a block down an inclined plane. Let us continue from where we left off above and suppose that the plane's angle of inclination exceeds the critical angle of friction. Then the block slides down the plane without tumbling provided that $\alpha_c < \alpha < \alpha_t$ holds. The free body diagram for $\mathscr{B}_2$ is shown in Fig. 5.18b; it is the same as before. Consequently, the free body formulation for the dynamics problem is the same as that for the statics problem and leads again to (5.74a); but this time the block has a translational motion down the plane, and the appropriate dynamical equations of motion must be decided. Since the body $\mathscr{B}_2$ is rigid and does not tumble, its motion is determined by the Newton–Euler equation (5.43) for its center of mass:

$$\mathbf{F}(\mathscr{B}_2, t) = \mathbf{W} + \mathbf{f} + \mathbf{N} = m(\mathscr{B}_2)\mathbf{a}^*(\mathscr{B}_2, t). \tag{5.75a}$$

The next step is the formulation of the appropriate kinematics. Since the motion is along a straight line on and down the plane,

$$\mathbf{a}^*(\mathscr{B}_2, t) = \ddot{\mathbf{x}}^*(\mathscr{B}_2, t) = \ddot{x}^*\mathbf{i}, \tag{5.75b}$$

where $\mathbf{x}^*(\mathscr{B}_2, t) = x^*\mathbf{i}$ is the position vector of the center of mass of $\mathscr{B}_2$ from the fixed origin F in φ. Collecting the first equation in (5.74c) and (5.75b) in (5.75a), we have

$$(W \sin\alpha - f)\mathbf{i} + (N - W\cos\alpha)\mathbf{j} = m\ddot{x}^*\mathbf{i}. \tag{5.75c}$$

This yields the component equations of motion

$$m\ddot{x}^* = W\sin\alpha - f \quad \text{and} \quad N - W\cos\alpha = 0 \tag{5.75d}$$

to be solved for the normal force N and for the rectilinear motion $x^*(\mathscr{B}_2, t)$ of the center of mass of $\mathscr{B}_2$.

The second equation of (5.75d) determines the normal force $N = W\cos\alpha$, and Coulomb's second law (5.71) for the sliding motion gives

$$f = f_d = \nu N = \nu W\cos\alpha. \tag{5.75e}$$

Then with (5.75b) and $W = mg$, the first relation in (5.75d) yields

$$\mathbf{a}^* = \ddot{\mathbf{x}}^* = g(\sin\alpha - \nu\cos\alpha)\mathbf{i}. \tag{5.75f}$$

Thus, the acceleration of the center of mass, indeed the acceleration of every particle of the block in its parallel translation down the plane, is a constant vector.

The velocity and the motion of the center of mass point are now easily obtained by integration of (5.75f), subject to specified initial conditions. Let us suppose that the block is released from rest in φ so that $\mathbf{v}^*(\mathscr{B}_2, 0) = \mathbf{0}$ and $\mathbf{x}^*(\mathscr{B}_2, 0) = \mathbf{0}$ at $t = 0$. Then integration of (5.75f) yields

$$\mathbf{v}^*(\mathscr{B}_2, t) = gt(\sin\alpha - \nu\cos\alpha)\mathbf{i}, \qquad \text{then} \qquad \mathbf{x}^*(\mathscr{B}_2, t) = \tfrac{1}{2}gt^2(\sin\alpha - \nu\cos\alpha)\mathbf{i}. \tag{5.75g}$$

We thus find that *the sliding motion is independent of the mass of the body—it is the same for all bodies, both large and small, so long as* (5.71) *holds and the no tip constraint is satisfied.* This completes the analysis of the sliding translational motion of the block, but some additional points are noted in the exercise.

Exercise 5.6. Equation (5.75a) shows that in the dynamics problem the resultant contact force $\mathbf{R}$ on the block is *not opposite* to the weight $\mathbf{W}$. Consider at time t the moment equation (5.44) for the applied forces about the fixed origin F at the initial position of the center of mass of the block. (a) Prove that $\mathbf{M}_F = \mathbf{0}$, and thus show that $\mathbf{R}$ is *concurrent* with $\mathbf{W}$ through the center of mass. (See Example 5.5, page 23.) Therefore, in the absence of rotation, the moment of the forces about the moving center of mass point also vanishes. (b) Show that the same result follows when the fixed point F is in the contact plane at the initial position. What is $\dot{\mathbf{h}}_F$ in this case? (See Problem 5.28.) □

Our sliding block example illustrates for a simple translational motion the more complex nature of the motion analysis of bodies and the importance of the center of mass. The translational motion of the block is described completely by the motion of its center of mass particle, regardless of its location in the body. Notice that the actual identity of the center of mass was unimportant in (5.75f), and it remained anonymous in (5.75g)—its location (actually the center of gravity in this case) was important only in the discussion of potential rotational effects expressed by the no tip condition derived from the moment equation. The anonymity of the center of mass is typical of many rigid body problems in which rotational effects are absent.

5.12. Applications of Coulomb's Laws

Two problems that use Coulomb's laws in demonstration of the predictive value of the principles of mechanics are studied. The first example illustrates the

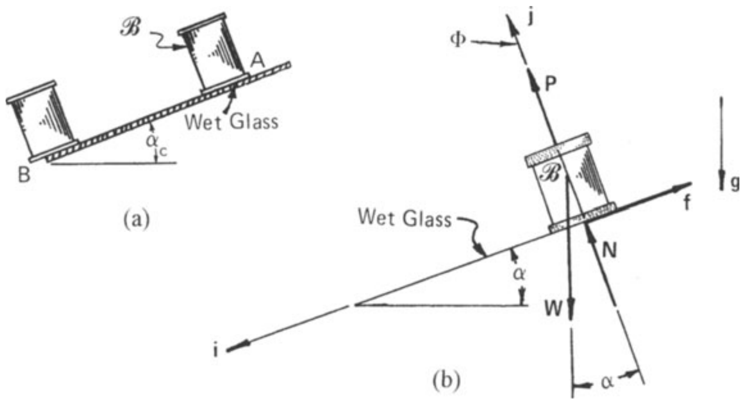


Figure 5.20. A simple experiment demonstrating the pressure induced, friction reduction principle.

phenomenon of pressure-induced friction reduction useful in a variety of engineering applications. The second example demonstrates the application of basic principles in providing the solution to a major technical problem during World War II.

5.12.1. The Sliding Can Experiment

An empty beverage can§ $\mathscr{B}$ having identical top and bottom rims is shown in Fig. 5.20a. The can is placed at A on a sheet of slightly wetted glass, which is then gradually tilted until the critical angle α_c is attained at which sliding of the can is initiated. Since the can slides on its narrow rim, the critical angle is independent of whether the open or the closed end of the can is upward. Of course, upon reaching the edge of the glass at B, the can falls off. The experiment is conducted at room temperature and the measured critical angle of friction is about 17°. Coulomb's laws hold for slightly wetted surfaces, and (5.74f) thus determines the coefficient of static friction $\mu = \tan 17^\circ = 0.30$.

The empty can is then chilled and the test repeated by first placing the can on the wetted glass with its *open end upward.* The critical angle is found to be the same as before, thus showing for this case that μ is independent of the temperature. Finally, the can is chilled to the same temperature as before and placed on the wetted surface with its *open end downward.* Surprisingly, the can starts to slide when the critical angle α_c is only 1° or 2°; and it slides down the entire length of the glass held at this very small inclination. But it *stops* rather abruptly when the open end extends just beyond the edge of the sheet at B in Fig. 5.20a.

§ Adapted from the article by M. K. Hubbert and W. W. Rubey cited in the chapter references. See also the related articles by M. B. Karelitz and by B. Noble reported therein.

This curious phenomenon occurs because after a few seconds the cold, trapped air expands as it begins to warm, causing the internal air pressure to increase. Because the surface area of the closed end of the can is greater than that of its open end, there is a resultant uplifting, internal normal pressure on the closed end that partially supports the weight of the can, and thus reduces the normal surface reaction force between the can and the glass. The can stops suddenly at the edge of the sheet because the pressure is abruptly released. To prove this hypothesis, we analyze the phenomenon.

5.12.1.1. The Equilibrium Analysis

We begin by showing in Fig. 5.20b the free body diagram of the chilled can placed on the glass with its open end downward in the inertial frame Φ. The body force is the weight $\mathbf{W}$ of the can. In addition to the normal and frictional contact forces $\mathbf{N}$ and $\mathbf{f}$, there is also a resultant internal contact force $\mathbf{P}$ on the closed end of the can due to excess of the internal air pressure over the outside air pressure. Thus, the total force acting on the can $\mathscr{B}$ is

$$\mathbf{F}(\mathscr{B}, t) = \mathbf{W} + \mathbf{N} + \mathbf{f} + \mathbf{P}. \tag{5.76a}$$

Introducing in (5.76a) the component representations for $\mathbf{W}$, $\mathbf{N}$, and $\mathbf{f}$ given in (5.74b), noting that $\mathbf{P} = P\mathbf{j}$, and equating each component to zero in the equilibrium equation $\mathbf{F}(\mathscr{B}, t) = \mathbf{0}$, we find the contact forces

$$f_d = W \sin\alpha_d, \qquad N_d = W\cos\alpha_d - P, \tag{5.76b}$$

in which the subscript notation should be evident. We see that N_d, the normal surface reaction force when the open end is down, is indeed reduced by the excess internal contact force P.

The case when the open end of the can is upward follows from (5.76b) in which we set $P = 0$, adjust the subscripts accordingly, and thus recover (5.74d). When α_u is increased gradually until sliding is imminent, (5.74f) yields

$$\mu \equiv f_{cu}/N_u = \tan\alpha_{cu}, \tag{5.76c}$$

α_{cu} denoting the critical angle of friction when the open end of the can is upward. This gives the coefficient of static friction μ between the can and the glass.

Now let us return to the case when the open end of the can is downward, and rewrite (5.76b) to obtain

$$\tan\alpha_d = (1 - p(\alpha_d))\,\frac{f_d}{N_d}, \tag{5.76d}$$

in which $p(\alpha_d) \equiv P/W\cos\alpha_d$ is the ratio of the uplifting force P to the normal component $W\cos\alpha_d$ of the weight of the can. Hence, $0 \le p(\alpha_d) \le 1$. Suppose that α_d is gradually increased to the angle α_{cd} at which the can is at the verge of sliding down the plane. Now remember that in both instances the coefficient of static friction in (5.70) is defined by the ratio of the tangential surface frictional force to the normal surface reaction force; and since the coefficient of friction must be the same as before, by (5.76c), $f_{cd}/N_d = f_{cu}/N_u = \tan\alpha_{cu}$ holds, and (5.76d) yields the following relation for the *apparent critical angle* α_{cd} when the open end is downward:

$$\tan\alpha_{cd} = (1 - p(\alpha_{cd}))\tan\alpha_{cu}, \qquad \text{with} \quad p(\alpha_{cd}) \equiv P/W\cos\alpha_{cd}. \tag{5.76e}$$

Because $1 - p(\alpha_{cd}) < 1$, it follows that $\alpha_{cd} < \alpha_{cu}$, that is, the apparent critical angle of sliding when the open end of the can is downward is smaller, perhaps much smaller, than the actual critical angle when its open end is upward. Now, we know from the experimental data that $\mu = \tan\alpha_{cu} = \tan 17^\circ$ and the largest critical angle $\alpha_{cd} = 2^\circ$; therefore, (5.76e) yields $p(2^\circ) = 1 - \tan 2^\circ/\tan 17^\circ = 0.886$, that is, the normal internal force on the closed end is very nearly 89% of the can's weight. The result (5.76e), therefore, confirms the hypothesis explaining the sliding beverage can phenomenon—the frictional effect is reduced due to the uplifting, internal air pressure.

To continue from here in the static case, we shall need to know the weight of a typical can, and then compare the predicted force $P = 0.89W$ with the value computed from thermodynamics on the basis of the volume and the initial temperature of the air trapped in the chilled can at room temperature. Without getting into this, however, we may ask instead—What can be learned about the subsequent motion of $\mathscr{B}$?

5.12.1.2. *The Motion Analysis*

The observation that the can stops abruptly when the open end extends just at the edge of the sheet is investigated. Singularity functions are used to describe the discontinuous behavior of $\mathbf{P}$ when the trapped air suddenly escapes. A similar analysis may be carried out without the use of singularity functions, an exercise left for the reader.

Let ℓ_o be the distance moved by the center of the can from its initial rest position at $x = 0$ to its position at B in Fig. 5.20a, where the trapped air is released. Afterwards the can will continue to move so that it extends beyond the edge of the glass an amount say, δ, but it does not fall off. To determine the value of δ compared with ℓ_o, we first find the speed of the can as a function of its position along the sheet.

Let $x^* = x$ denote the center of mass coordinate in the inertial frame Φ, and begin with the force analysis. The free body diagram of the can is shown in Fig. 5.20b. We suppose that the internal pressure is "turned on" at $x = 0$ when the can is placed on the glass with its open end downward, and later "shut off" at $x = \ell_o$ as the air suddenly escapes when the can reaches the edge of the sheet. Then, with the aid of the unit step function (1.117), we have

$$\mathbf{P} = \left[P < x - 0 >^0 - P < x - \ell_o >^0\right]\mathbf{i}. \tag{5.77a}$$

The total force on the can throughout its motion is given by (5.76a), and hence with (5.74b) and (5.77a), the equation of motion $\mathbf{F}(\mathscr{B}, t) = m\mathbf{a}^* = m\ddot{x}\mathbf{i}$ yields the scalar component relations for the sliding motion at the critical angle α_{cd}:

$$m\ddot{x} = W\sin\alpha_{cd} - f_{dd}, \qquad N_d = W\cos\alpha_{cd} - P\left(< x - 0 >^0 - < x - \ell_o >^0\right), \tag{5.77b}$$

wherein by Coulomb's second law (5.71), $f_{dd} = \nu N_d$ during the sliding motion. Then with $W = mg$ and $p(\alpha_{cd})$ in (5.76e), (5.77b) yields the equation of motion:

$$\ddot{x} = g\cos\alpha_{cd}\left[\tan\alpha_{cd} - \nu + \nu p(\alpha_{cd})(< x - 0 >^0 - < x - \ell_o >^0)\right]. \tag{5.77c}$$

To find the speed $\dot{x} = v(x)$ as a function of x, we write $\ddot{x} = v dv/dx = d(v^2/2)/dx$, and recall (1.132) for integration of the unit step function. Then use of the initial data $v(0) = 0$ at $x = 0$ in the integration of (5.77c) yields the squared speed of the can at its current position $x(t)$:

$$v^2(x) = 2g\cos\alpha_{cd}\left[x\left(\tan\alpha_{cd} - \nu\right) + \nu p(\alpha_{cd})\left(< x - 0 >^1 - < x - \ell_o >^1\right)\right]. \tag{5.77d}$$

Now consider the case when the can slides beyond the edge of the glass and stops at $x = \ell > \ell_o$. Recalling (1.127) for the unit slope function, setting $v(\ell) = 0$, and introducing $p(\alpha_{cd})$ from (5.76e), we find from (5.77d) the relation for δ/ℓ_o:

$$\frac{\ell}{\ell_o} = 1 + \frac{\delta}{\ell_o} = \frac{1 - (\tan\alpha_{cd})/\mu}{1 - (\tan\alpha_{cd})/\nu}, \tag{5.77e}$$

wherein $\delta \equiv \ell - \ell_o$ is the overhang distance at the edge of the sheet. The solution thus shows that the overhang δ is proportional to the length ℓ_o, and hence our analysis discloses an oversight in the experimental description. If the sheet were too long, δ might exceed the can's radius r, the critical overhang when the can slides beyond the edge of the glass; and the can would then fall off. An estimate of the critical length $\hat{\ell}_o$ of the sheet, i.e. the maximum initial distance of $\mathscr{B}$ from the edge in order that the can will not slide off the end, may be obtained from (5.77e) at $\delta = r$; we find

$$\frac{\ell_o}{\delta} = \frac{\hat{\ell}_o}{r} = \mu\frac{\nu - \tan\alpha_{cd}}{(\mu - \nu)\tan\alpha_{cd}}. \tag{5.77f}$$

Since $\tan\alpha_{cd}$ and $(\mu - \nu)$ are small quantities, it follows that the critical length may be rather large. Hence, for most practical experimental circumstances, our theoretical analysis predicts that the can generally will stop abruptly and not fall from the edge.

To get an idea of the size of $\hat{\ell}_o$, suppose that $\nu = 0.25 < \mu = 0.3$. Then for $\alpha_{cd} = 2^\circ$, say, the critical length to can radius ratio, by (5.77f), is $\hat{\ell}_o/r = 36.95$, and for the same parameters the can's overhang ratio is $\delta/\ell_o = 0.027$. Thus, for a can of radius $r = 3.3$ cm (1.3 in), the critical distance would be about $\hat{\ell}_o = 1.22$ m (4. 00 ft). For a plate of length $\ell_o = 25$ cm (about 10 in.), say, the overhang will be $\delta = 0.68$ cm (0.27 in.), and for $\ell_o = 1$ m (39.4 in.), a value close to the length of plate reported for the experiment, $\delta = 2.7$ cm (1.1 in.). Both example values are much smaller than the can's radius. For a larger value of ν, or a smaller value of α_{cd}, the overhang will be even smaller while the critical length of the plate will grow larger. Thus, starting at a practicable distance from the edge, the can will travel beyond the edge only a small distance compared with its radius and will indeed stop rather suddenly.

5.12.1.3. Technical Applications of the Friction Reduction Principle

The idea that frictional effects may be reduced by an uplifting internal pressure has been applied to study other phenomena. The spectacular geological phenomenon in which huge masses of nearly horizontal rock formations are displaced great distances, sometimes as much as 10 to 50 miles or more, is an example. For sufficiently high interstitial fluid pressure in porous rock, fault blocks of rock may be pushed over a nearly horizontal subsurface. Like our can experiment, due to uplifting fluid pressure, the fault blocks slide under their own weight over very much smaller slopes than otherwise would be possible.

Another striking application of pressure induced friction reduction occurred in the mechanical design of bearings for the 200 inch telescope at the Mount Palomar Observatory. Frictional forces opposing the steady, precise rotation of the telescope in tracking the apparent motion of the stars relative to the Earth had to be very much less than those that would be produced by conventional bearings. Moreover, for these bearing devices, the torque required to turn the telescope would demand considerable horsepower, and the required loading would cause excessive deformation of the telescope's mounting yoke. The problem of supporting and moving precisely such a massive structure was solved by floating the telescope on a thin film of oil under pressure. The entire weight of the telescope, roughly one million pounds (455, 000 kg), was supported by bearing surfaces separated by a thin film of oil 0.005 in. (0.013 mm) thick and under pressure ranging from 200 to 500 psi (1.4 to 3.4×10^6 N/m^2). This design concept reduced considerably the power required to drive the massive telescope to only $1/12$ horsepower!

These examples underscore the utility of the friction reduction principle illustrated by the sliding can experiment. Our next example applies the principles of mechanics to explain critical U.S. Navy torpedo failures during World War II.

5.12.2. Damn the Torpedoes!

U.S. Navy submarine operations[¶] in the early months of World War II reported recurring instances of frustrating torpedo malfunction and detonation failures.

¶ This narrative is adapted from the referenced articles by A. A. Bartlett, D. Murphy, and the book by T. Roscoe. All discuss the problem of torpedo failures in U.S. Navy submarine operations. See also S. E. Morison.

Faced with a shortage of torpedoes and state-of-the-art magnetic detonators that proved greatly unreliable, Admiral Charles A. Lockwood in Pearl Harbor ordered the magnetic detonators replaced with impact detonators. But in no time at all worrisome reports of torpedo failures continued to come in. More than a year passed with no solution in sight when good fortune in disguise appeared unexpectedly.

On July 24, 1943, the U.S. submarine *Tinosa* was patrolling west of Truk with 16 torpedoes aboard when Lieutenant Commander Lawrence R. Daspit sighted the unescorted oil tanker *Tonan Maru* No. 3, one of the largest in the Japanese fleet, at an unfavorable great range of 4000 yards (3658 m). Four torpedoes were fired in a fan pattern oblique to the tanker, actually an unfavorable angle of attack. Two found their target and exploded near the tanker's stern to slow the great ship. Two more were released. Daspit at the periscope, witnessed two explosions that brought the *Tonan Maru* to a stop, dead in the water, smoking and starting to settle by the stern, but not sinking. At the ideal range of about 875 yards (800 m) and now stationed for a perfect shot at 90° off the tanker's bow, Daspit setup for the kill. The *Tinosa* fired a single torpedo that struck normal to the side, nearly amidships of the giant tanker. The torpedo was heard to make a normal run, followed by silence. Daspit witnessed only a spray at the point of impact. The torpedo was a dud! Two more perfect shots followed—both duds. The remaining "tin fish" were pulled from their tubes and their settings checked, all in good order. Over the next few hours, six additional torpedoes were launched one at a time. Each failed to explode on impact. Damn the torpedoes—all duds! A frustrated Daspit returned to Pearl Harbor with his last torpedo, and Japanese salvage vessels from their naval base at Truk saved the *Tonan Maru*. The fact that many similar torpedo failures in the early months of the war slowed U.S. efforts to contain Japanese advances across the South Pacific islands and the Philippines, underscores the significance of this major technical problem.

The Germans experienced similar frustration with magnetic influence torpedo failures, many exploded prematurely, others missed their target, or failed to explode on impact. A particularly significant incident occurred on the morning of October 30, 1939, the day before Sir Winston Churchill's scheduled meeting aboard the battleship *Nelson* with Admiral Sir Charles Forbes, Commander-in-Chief, and Admiral of the Fleet Sir Dudley Pound. Two weeks earlier on October 14, the German U-boat commander, Lieutenant Commander Gunther Prien, slipped his *U-47* into the center of Britain's main naval harbor at the supposedly impregnable Scapa Flow. Prien maneuvered there on the surface, undetected, and around 1 a.m. attacked and sunk at anchor the magnificent British battleship *HMS Royal Oak*, afterwards escaping to become a celebrated naval hero.[‖] Following this disaster in

‖ On March 8, 1941, the destroyer *Wolverine* while escorting a convoy in the North Atlantic, sighted the *U-47* running initially on the surface, and attacked and sank her by depth charges. The remarkable and daring Lieutenant Commander Gunther Prien, age 33, and his entire crew lost their lives. See the book by G. S. Snyder in the chapter references for the full story of the *Royal Oak* disaster, including many tales of German submarine commander frustration with torpedo failures.

which 833 officers and men lost their lives, an urgent conference was arranged for October 31, between Churchill and his admirals aboard the *Nelson*, the flagship of Admiral Forbes. But another disaster was unfolding during the morning hours of the 30th, when *U-56*, commanded by Lieutenant Wilhelm Zahn, sighted the battleships *Nelson* and *Rodney,* accompanied by the battle cruiser *Hood* and a screen of ten destroyers. Zahn maneuvered within range and released a spread of three torpedoes on *Nelson.* Three impacting thumps against the battleship's side were heard in *U-56*, but no detonation. All duds! The angry Zhan turned away and reported his aborted attack to U-boat Command, unaware of the true significance of his failed attempt to sink the *Nelson*. Nearly every U-boat commander, including the celebrated "ace" Gunther Prien, reported torpedo failures; sometimes every "eel", whether set to explode on impact or set for magnetic detonation, was a dud.

5.12.2.1. Identifying the Problem

What was wrong with the German torpedoes? A special Torpedo Commission discovered that the fault was not with the torpedoes themselves, but with the depth at which they were set to pass beneath the target's hull, the point at which the magnetic pull of the victim was supposed to trigger the warhead. Errors of design caused the weapon to run too deep, and countermeasures applied by the British also may have contributed to the German problem. The delicate magnetic exploders eventually were replaced with dependable impact exploders. By the time the U.S. entered the war in Europe, the U-boats were scoring hit after hit with shocking efficiency. (I do not know of any studies on German torpedo defects responsible for impact failures reported above.)

What was wrong with the U.S. Navy's torpedoes? The torpedo returned by Daspit to Pearl Harbor, checked and later test fired at underwater cliffs of Kahoolawe Island in Hawaii, also was a dud. Examination of the torpedo's detonator mechanism revealed that the firing pin that would set off the warhead had released, but it failed to strike the primer cap with sufficient force to trigger it. Impact experiments were conducted to study the problem. To model a normal impact against the side of a ship, torpedoes loaded with cinder concrete rather than explosives were dropped from about 90 ft (27 m) onto a steel plate. Seventy percent of the tests revealed the same kind of trigger failure on normal impact. In actual submarine operations, however, an oblique impact was believed more likely to occur. To simulate this condition, the steel plate was set at an angle so that the torpedo would strike a glancing blow. It was found that the exploder mechanism generally functioned properly. The investigation now focused on the firing pin design, a small device weighing several ounces. When released, a spring drove the pin along parallel guide rods perpendicular to the torpedo axis. The perpendicular impact force of deceleration was found to be about $500g$'s, that is, 500 times the force of gravity, per unit mass. This force produced a guide rod Coulomb frictional component of nearly 190 lbs on the firing pin. The trigger spring was unable to overcome the frictional force and drive the firing pin with sufficient force against the primer

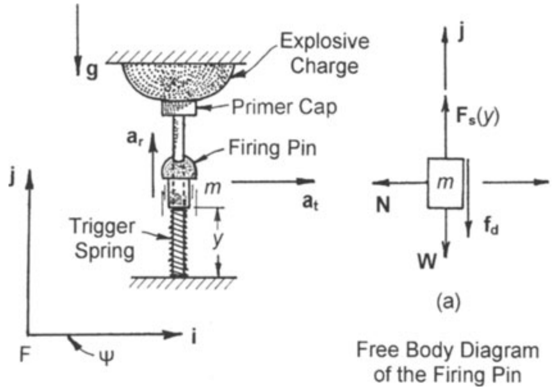


Figure 5.21. Model of a torpedo exploder mechanism.

cap. In an oblique, glancing impact, the frictional effect was less severe and the torpedoes often exploded on impact. So, nearly 2 years after the start of the war, between July and September 1943, as a fortuitous consequence of Daspit's failed attack on the *Tonan Maru*, the torpedo exploder mechanism problem was finally identified and solved.[§§]

5.12.2.2. The Model Analysis

The problem of U.S. Navy torpedo failures was finally explained by elementary principles of mechanics involving Coulomb friction. To explore this, consider the simple model of the exploder mechanism shown in Fig. 5.21. The free body diagram of the firing pin modeled as a block of weight $\mathbf{W} = m\mathbf{g}$ is shown in Fig. 5.21a. The actual direction of $\mathbf{g}$ may vary from that chosen in the example. The trigger spring driving force from its precompressed state is a known function $\mathbf{F}_s(y)$ of the firing pin displacement y; $\mathbf{N}$ denotes the normal (impulsive reaction) force exerted by the guide rods, and $\mathbf{f}_d$ is the dynamic friction force. So, the total force on the block in its sliding motion is $\mathbf{F} = \mathbf{F}_s + \mathbf{N} + \mathbf{W} + \mathbf{f}_d = -N\mathbf{i} + (F_s(y) - W - f_d)\mathbf{j}$, in which $f_d = \nu N$ and $W = mg$.

Here we have a motion of the mass m relative to the rapidly decelerating torpedo frame. Therefore, the total acceleration of m in the inertial frame $\Psi = \{F; \mathbf{i}_k\}$ is given by $\mathbf{a} = \mathbf{a}_r + \mathbf{a}_0 = \ddot{y}\mathbf{j} - a_t\mathbf{i}$, in which $\mathbf{a}_r \equiv \delta^2\mathbf{x}/\delta t^2 = \ddot{y}\mathbf{j}$ is the relative acceleration of the firing pin in the moving torpedo frame, and

§§ The *Tinosa* soon returned to the hunt, and by the end of the war she had sunk 16 Japanese vessels, 64,655 tons in all, and survived. In both the number of ships and tonnage sunk in the Pacific theater, she ranked 19^{th} among the top 25 *pig boats* in the list of leading individual submarine scores. (See Roscoe, p. 446. According to this expert (p. 442), "submarines played the leading role in Japan's defeat. They wrecked Japan's merchant marine. They sank a sizeable chunk of the Imperial Navy. They bankrupted Japan's home economy with a blockade which established a new adage: viz., an island is a body of land surrounded by submarines.")

$\mathbf{a}_0 \equiv \mathbf{a}_t = -a_t\mathbf{i}$ is the rigid body deceleration of the torpedo in Ψ. Therefore, the corresponding scalar components in Newton's law (5.39) are $-N = -ma_t$ and $F_s(y) - mg - \nu N = m\ddot{y}$, from which the relative acceleration of the firing pin during the rapid deceleration period is given by

$$\ddot{y} = \frac{1}{m}F_s(y) - (g + \nu a_t).$$

This equation essentially determines the force with which the firing pin will strike the primer cap to detonate the warhead—it reveals both the problem and its easy solution. The contribution of g is negligible compared to νa_t. The spring force that drives the firing pin is effectively reduced by the increased frictional force arising from the large deceleration of the torpedo in its normal impact. Therefore, because of its reduced relative acceleration $\ddot{y}$, the firing pin is unable to strike the primer cap with sufficient force to trigger the warhead. The simplest direct solution is to increase the spring force, reduce the firing pin's mass and, if possible, reduce the coefficient of friction. The predictive value of the principles of mechanics demonstrated in this and in previous examples is repeated many times in future problems.

5.13. What is the Inertial Frame?

In addition to specifying a law of equilibrium for every material universe, Newton's first law provides the criterion for deciding whether a reference frame is an inertial frame. The inertial frame in Newton's laws is an undefined entity, a primitive concept, but its choice is not arbitrary; it must be a reference frame relative to which a uniform motion can be sustained without force. Otherwise, the laws are not applicable, in fact, they have no meaning until the inertial frame itself is identified. But the first law does not tell us which reference frame is the preferred referential frame, it merely assumes that such a reference frame exists. Therefore, what physical reference frame (or body) in the real world may be identified as Newton's preferential frame?

Plainly, every motion can be determined in a reference frame that is absolutely at rest. But a body can be identified as fixed in space only relative to other bodies known to be fixed in space, an evident irresolvable tautology. So, the idea of an inertial reference frame (or body) being fixed in space is meaningless. In its place, our most natural choice appears to be the Earth frame. We know, however, that the Earth's principal motion has a subtle, but demonstrable effect on the oscillations of a pendulum and on the trajectories of shells and falling bodies. Such relative motion effects preclude the possibility of an arbitrary uniform motion of a particle relative to the Earth without intervention of a controlling force, as we shall see shortly. Then what is the reference frame relative to which the Earth's motion may be referred, and under what circumstances may the Earth frame be used as a Newtonian frame?

It appears time after time that the remote stars visible in the night sky always are in their same place relative to the Sun. And these "fixed stars" are used to obtain a navigational fix on our motion. While sophisticated measurements reveal that the distant stars are, in fact, not fixed relative to each other, the so-called "fixed stars" are chosen as a physical model of an inertial reference system for the real world, because the remote stars comprise a set of objects (bodies) whose perceptible mutual distances have not changed significantly over countless centuries. Therefore, the astronomical frame of the fixed stars is a prime *candidate* for a reference system that may *approximate* an inertial frame to a precision sufficient for our needs. To evaluate the accuracy of this *assumption*, we may compare the observed physical behavior of bodies with theoretical predictions of that behavior based on Newton's laws in the astronomical frame. Well, it happens that theoretical predictions of the effects of the Earth's rotation on the swing of Foucault's pendulum, on the motion of missiles and falling bodies, and various other phenomena in the world, stand in sharp agreement with observations. Therefore, the real world, physical reference frame that corresponds to the ideal, abstract inertial reference frame in Newton's laws may be tentatively identified as a reference frame in the distant stars. The motion of the Earth relative to the astronomical frame is known, so we are now in a position to evaluate the effects of using the Earth as a first approximation to an inertial frame. The effect of the motion of a reference frame on the form of the laws of motion is described next.

5.14. The Second Law of Motion in a Noninertial Frame

Now, we are, after all, concerned mainly with motion relative to the noninertial Earth frame, or perhaps another convenient moving reference frame. Therefore, we shall need to express Newton's second law in terms of the acceleration $\delta^2\mathbf{x}/\delta t^2$ apparent to a moving observer. We thus recall (4.48) for the total acceleration of a particle referred to a moving frame and rewrite the second law (5.39) to obtain *the equation of motion for a particle of mass m having a motion relative to a moving frame* φ:

$$m\mathbf{a}_{\varphi}(P,t) = \mathbf{F} - m\left(\mathbf{a}_O + \boldsymbol{\omega}_f \times (\boldsymbol{\omega}_f \times \mathbf{x}) + \dot{\boldsymbol{\omega}}_f \times \mathbf{x} + 2\boldsymbol{\omega}_f \times \mathbf{v}_{\varphi}\right). \quad (5.78)$$

Here $\mathbf{F} = m\mathbf{a}_P$ is the force acting on the particle P whose absolute acceleration in the Newtonian frame Φ is $\mathbf{a}_P = \mathbf{a}(P,t)$; and $\mathbf{a}_{\varphi}(P,t) \equiv \delta^2\mathbf{x}/\delta t^2$ and $\mathbf{v}_{\varphi} = \mathbf{v}_{\varphi}(P,t) \equiv \delta\mathbf{x}/\delta t$ are the respective acceleration and velocity of P relative to φ.

The form of Newton's second law (5.78), in addition to the total force $\mathbf{F}$, exposes several "fictitious" forces apparent only to the moving observer in φ, to whom it appears that the particle is acted upon by a total force

$$\mathbf{F}_{\varphi} \equiv \mathbf{F} - m\left(\mathbf{a}_O + \boldsymbol{\omega}_f \times (\boldsymbol{\omega}_f \times \mathbf{x}) + \dot{\boldsymbol{\omega}}_f \times \mathbf{x} + 2\boldsymbol{\omega}_f \times \mathbf{v}_{\varphi}\right), \quad (5.79)$$

called the *apparent force*. The pseudoforces $-m\boldsymbol{\omega}_f \times (\boldsymbol{\omega}_f \times \mathbf{x})$ and $-2m\boldsymbol{\omega}_f \times \mathbf{v}_\varphi$ are called the *centrifugal force* and the *Coriolis force,* respectively. The total of the pseudoforces, namely,

$$\mathbf{F}_I \equiv -m\left(\mathbf{a}_O + \boldsymbol{\omega}_f \times (\boldsymbol{\omega}_f \times \mathbf{x}) + \dot{\boldsymbol{\omega}}_f \times \mathbf{x} + 2\boldsymbol{\omega}_f \times \mathbf{v}_\varphi\right), \tag{5.80}$$

is called the *inertial force.* Use of (5.79) in (5.78) now yields *Newton's second law of motion relative to any moving frame* φ, *including the Earth frame*:

$$\mathbf{F}_\varphi = m\mathbf{a}_\varphi(P,t) = m\frac{\delta^2\mathbf{x}}{\delta t^2}. \tag{5.81}$$

The basic difference between (5.81) and (5.39) is that the force $\mathbf{F}_\varphi$ in (5.79) is not the total of forces due purely to the interaction between pairs of bodies in the universe. The additional inertial force (5.80) arises solely from the motion of the moving observer's frame of reference. Therefore, to a moving observer, the actual forces that act on a body are not always what they may seem to be.

We are now positioned to show that there exists relative to the inertial frame infinitely many moving reference frames with respect to which Newton's laws hold unchanged. Hence, each of these frames is an inertial reference frame. Indeed, we need characterize only those frames for which the inertial force (5.80) vanishes for all motions relative to φ, i.e. those frames for which

$$\mathbf{a}_O + \boldsymbol{\omega}_f \times (\boldsymbol{\omega}_f \times \mathbf{x}) + \dot{\boldsymbol{\omega}}_f \times \mathbf{x} + 2\boldsymbol{\omega}_f \times \mathbf{v}_\varphi = \mathbf{0}, \tag{5.82}$$

for all $\mathbf{x}(P,t)$ and $\mathbf{v}_\varphi(P,t)$. This is possible when and only when both $\mathbf{a}_O \equiv \mathbf{0}$ and $\boldsymbol{\omega}_f \equiv \mathbf{0}$, that is, if and only if φ has a uniform translational motion relative to the inertial frame Φ. In this case, from (5.79) and (5.81), $\mathbf{F}_\varphi = \mathbf{F} = m\mathbf{a}_\varphi(P,t)$ holds for all motions of the particle P in the moving frame φ. In particular, $\mathbf{F}_\varphi = \mathbf{0}$ holds, if and only if the particle P has a uniform motion relative to φ, and hence φ is an inertial frame.

Now let us return momentarily to (5.81) and extend the definition of an equilibrium state to a particle in a moving frame φ. A particle P is in *equilibrium relative to* φ if and only if P is at rest or in uniform motion relative to φ. Then, by (5.81),

$$\text{equilibrium in } \varphi \Leftrightarrow \mathbf{a}_\varphi(P,t) = \mathbf{0} \Leftrightarrow \mathbf{F}_\varphi(P,t) = \mathbf{0}. \tag{5.83}$$

In this case, by (5.79), the force $\mathbf{F}$ applied to P to control its uniform motion in φ is balanced by the inertial force (5.80): $\mathbf{F} + \mathbf{F}_I = \mathbf{0}$. Hence, the frame φ is not an inertial frame. In general, a particle in equilibrium in φ will not be in equilibrium in the inertial frame Φ, and vice versa. In fact, by (5.38), the particle P may be in equilibrium simultaneously in Φ if and only if $-\mathbf{F}_I = \mathbf{F} = \mathbf{0}$ so that (5.82) holds for all uniform motions $\mathbf{x} = \mathbf{x}_0 + \mathbf{v}_0 t$ relative to φ, where $\mathbf{x}_0$ and $\mathbf{v}_0 \equiv \mathbf{v}_\varphi(P)$ are constant vectors; but (5.82) holds when and only when frame φ has a uniform translational motion relative to the inertial frame in the distant stars.

In sum, *every nonrotating, uniformly translating reference frame is a Newtonian reference frame in which Newton's laws may be applied. Moreover, a particle*

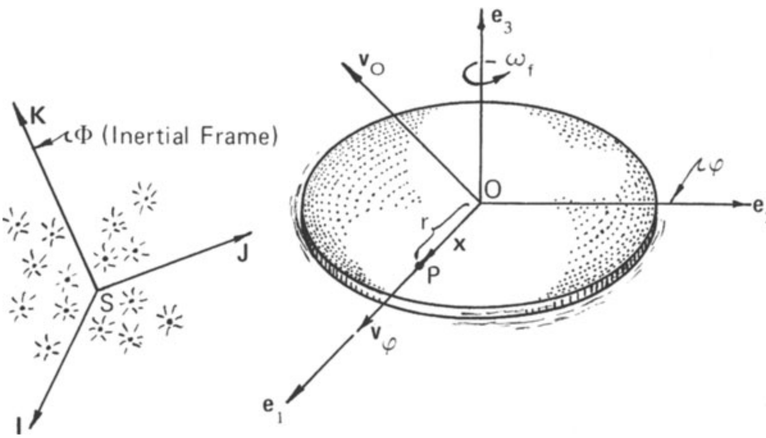


Figure 5.22. Uniform motion of a particle P relative to a moving frame $\varphi = \{O; \mathbf{e}_k\}$.

that is in equilibrium in one inertial frame Φ is in equilibrium in every frame φ having only a uniform motion of translation relative to Φ.

Example 5.9. A particle P in Fig. 5.22 has a radially directed, uniform motion relative to a frame $\varphi = \{O; \mathbf{e}_k\}$ that is rotating with angular velocity $\boldsymbol{\omega}_f$ relative to the inertial frame Φ fixed in the distant stars. The origin O has a constant velocity $\mathbf{v}_O$ in Φ. What is the force acting on the particle, and under what conditions does it vanish?

Solution. We wish to find $\mathbf{F} = \mathbf{F}(P, t)$ in (5.78). Since the motion of P relative to φ is uniform, the particle is in equilibrium relative to φ. Hence, (5.78) and (5.83) yield

$$\mathbf{F} = m\left(\mathbf{a}_O + \boldsymbol{\omega}_f \times (\boldsymbol{\omega}_f \times \mathbf{x}) + \dot{\boldsymbol{\omega}}_f \times \mathbf{x} + 2\boldsymbol{\omega}_f \times \mathbf{v}_\varphi\right). \tag{5.84a}$$

Moreover, the origin O has a constant velocity, so $\mathbf{a}_O = \mathbf{0}$. Further, with $\mathbf{x} = r\mathbf{e}_1$, we have $\mathbf{v}_\varphi = \delta\mathbf{x}/\delta t = \dot{r}\mathbf{e}_1$, which is constant relative to frame $\varphi = \{O; \mathbf{e}_k\}$, shown in Fig. 5.22. Thus, noting that $\boldsymbol{\omega}_f = \omega\mathbf{e}_3$ and $\dot{\boldsymbol{\omega}}_f = \dot{\omega}\mathbf{e}_3$ in the astronomical frame $\Phi = \{S; \mathbf{I}_k\}$, we find by (5.84a) the force that acts on the particle to control its uniform motion in the moving frame φ:

$$\mathbf{F}(P, \mathbf{t}) = -mr\omega^2\mathbf{e}_1 + m(r\dot{\omega} + 2\omega\dot{r})\mathbf{e}_2. \tag{5.84b}$$

Therefore, frame φ is not an inertial frame—the uniform motion in φ cannot be sustained without the application of force in the inertial frame Φ. Clearly, $\mathbf{F} = \mathbf{0}$ in Φ if and only if $\omega \equiv 0$, that is, when and only when the frame φ has a uniform translational motion with velocity $\mathbf{v}_O$ in the inertial frame Φ, then φ is an inertial frame too. □

5.15. Newton's Law in the Noninertial Earth Frame

Now let us consider the influence of the Earth's motion on the form of Newton's equation of motion for a particle moving relative to the noninertial Earth frame. Introduce an inertial frame $\Phi = \{F; \mathbf{A}, \mathbf{B}, \mathbf{C}\}$ fixed relative to the distant stars (see Fig. 5.23), and recall the notation used in (4.92) where $\boldsymbol{\omega}_f = \boldsymbol{\Omega}$ approximates the constant total angular velocity of the Earth frame $\varphi = \{O; \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}\}$ relative to Φ, $\mathbf{x} = \mathbf{r}$ is the position vector from the Earth's center C to a particle P moving on or near the Earth's surface, and $\mathbf{a}_O = \mathbf{a}_C$ denotes the acceleration of C in Φ. Then the apparent force (5.79) acting on P in its motion relative to φ becomes

$$\mathbf{F}_\varphi = \mathbf{F} - m\left(\mathbf{a}_C + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r}) + 2\boldsymbol{\Omega} \times \mathbf{v}_\varphi\right). \tag{5.85}$$

First, determine $\mathbf{a}_C$ by using the second law in which the total force acting on the Earth as a center of mass object of mass m_E at C is to be estimated. All bodies in the universe exert a gravitational pull on the Earth, whose mass is estimated at 5.98×10^{24} kg (1.36×10^{22} tons). In view of the result (5.58) for spherical bodies, the gravitational actions of all bodies—the Sun and the Earth, the Earth and the Moon, the Earth and an apple—are modeled as the attractions of particles. Therefore, an estimate of the total gravitational force acting on the Earth may be obtained by regarding the Earth E as a free body, in Fig. 5.23, acted upon by the particle P, the moon M, and the sun S. The equation of motion for the center of mass of the Earth is thus given by

$$m_E \mathbf{a}_C = m_E(\mathbf{g}_S + \mathbf{g}_M + \mathbf{g}_P) + \mathbf{F}_E, \tag{5.86}$$

in which $\mathbf{g}_S,\ \mathbf{g}_M$, and $\mathbf{g}_P$ are the respective gravitational field strengths at C due to the principal surrounding bodies $S,\ M$, and P; and $\mathbf{F}_E$ is the resultant of all other forces that may act on E, including other weak gravitational forces and the contact force exerted by the Earth's atmosphere, for example. This estimates $\mathbf{a}_C$ in (5.85).

Now consider the free body diagram of the object P in Fig. 5.23. The total force acting on P is $\mathbf{F} = m(\mathbf{g}_1 + \mathbf{g}_2 + \mathbf{g}_3) + \mathbf{F}_O$, where $\mathbf{g}_1,\ \mathbf{g}_2,\ \mathbf{g}_3$ are the field strengths at P due to the bodies $S,\ M$, and E, respectively, $\mathbf{F}_O$ is the total of all other forces acting on P and $m_E \mathbf{g}_P = -m\mathbf{g}_3$ is the mutual gravitational force between E and P. Use of these relations and (5.86) in (5.85) yields the equation of motion (5.81) for the object P in the Earth frame φ:

$$\begin{aligned} m\mathbf{a}_\varphi = \mathbf{F}_O &+ m\mathbf{g}_3\left(1 + \frac{m}{m_E}\right) + m\left(\mathbf{g}_1 - \mathbf{g}_S\right) + m\left(\mathbf{g}_2 - \mathbf{g}_M\right) \\ &- \frac{m}{m_E}\mathbf{F}_E - m\left(\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}) + 2\boldsymbol{\Omega} \times \mathbf{v}_\varphi\right). \end{aligned} \tag{5.87}$$

In view of the great distances separating the principal bodies, some further approximations are introduced to simplify (5.87). For the motion of P on or near the Earth's surface, we have $|\mathbf{r}| = r_3$ in Fig. 5.23. Hence, the other distances shown there may be approximated by $r_1 = r_S$ and $r_2 = r_M$ so that $\mathbf{g}_1 = \mathbf{g}_S$ and

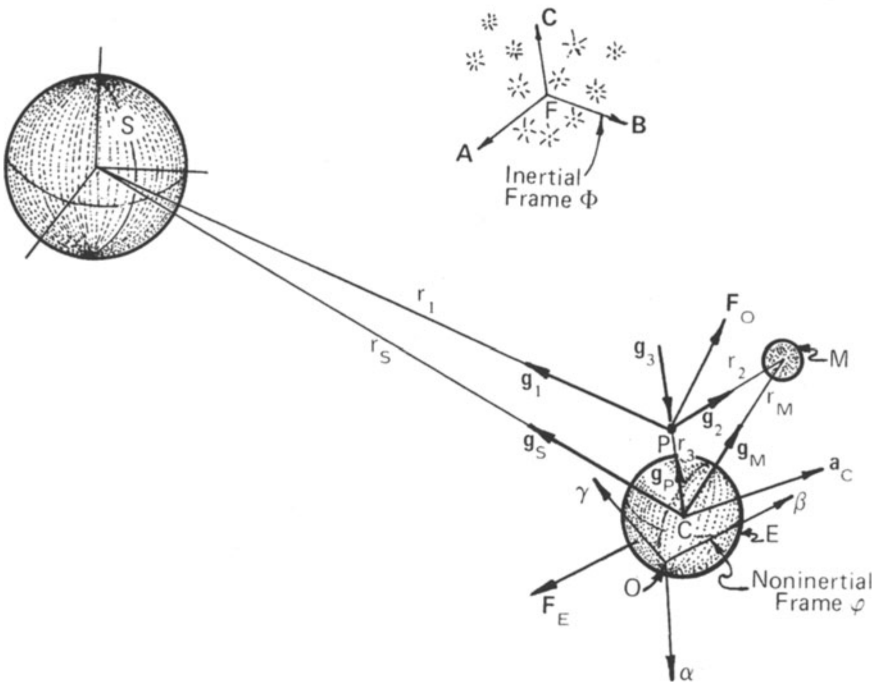


Figure 5.23. Free body diagram of a particle P and principal interacting bodies—the Earth, the Moon, and the Sun.

$\mathbf{g}_2 = \mathbf{g}_M$, very nearly. Clearly, the ratio m/m_E is infinitesimal, hence negligible compared with unity, and even though $|\mathbf{F}_E|$ may be large, we may assume that $m\,|\mathbf{F}_E|\,/m_E \ll |\mathbf{F}_O|$. Use of these further approximations in (5.87) yields the final reduced form of *Newton's equation of motion for a particle in the noninertial Earth frame*:

$$m\mathbf{a}_\varphi = m\mathbf{g}_3 + \mathbf{F}_O - m\left(\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{r}) + 2\boldsymbol{\Omega}\times\mathbf{v}_\varphi\right), \tag{5.88}$$

where $m\mathbf{g}_3$ is the gravitational force on P due to the Earth, $\mathbf{F}_O$ is the total of all contact and nongravitational body forces that act on P, and the other terms are inertial forces due to the Earth's rotation.

5.16. The Apparent Gravitational Field Strength of the Earth

The Earth's gravitational field strength apparent to an Earth observer is affected by the Earth's rotation and by the variation in its shape. To understand this and to learn how the real and apparent gravitational field strengths are related, let us consider an object P at rest relative to the Earth so that $\mathbf{v}_\varphi = \mathbf{0}$ and $\mathbf{a}_\varphi = \mathbf{0}$. Then (5.88) reduces to the equation of relative static equilibrium:

$$\mathbf{F}_O + m\,(\mathbf{g}_3 - \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r})) = \mathbf{0}. \tag{5.89}$$

Suppose P rests on the smooth, horizontal surface of a highly polished desk. Then $\mathbf{F}_O$ is the normal, desk top reaction force on P, and $\mathbf{F}_O + m\mathbf{g}_3 = (-F_O + mg_3)\mathbf{n}$, wherein $\mathbf{g}_3 = g_3\mathbf{n}$ and $\mathbf{n}$ is the central directed, unit normal vector to the Earth's spherical surface. Note, however, that the centrifugal force term in (5.89) is directed outward and perpendicular to the Earth's rotation vector $\boldsymbol{\Omega}$, so it has components both normal *and* tangential to the Earth's surface at the colatitude θ, namely,

$$-m\,\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}) = mr\Omega^2 \sin\theta(\cos\theta\ \mathbf{t} - \sin\theta\ \mathbf{n}), \tag{5.90}$$

where $\mathbf{t}$ is the southward directed, unit tangent vector to the surface at P. Because there is no other tangential component in (5.89) to balance the tangential component of the centrifugal force, we find the contradictory result $\Omega = 0$; otherwise, the relative equilibrium of an object at ease on a polished desk is not possible!

This dichotomy implies that the general equation (5.88) for the motion of a particle relative to the Earth cannot be a correct approximation. Review of earlier estimates used to obtain (5.88), however, suggests that the problem is more subtle than the possibility of error introduced by our treating the Earth, the Sun, and the Moon as particles separated by great distances and neglecting small terms in m/m_E. Suppose, on the other hand, that the gravitational force in (5.89) must have a small tangential component that balances the tangential centrifugal force component in (5.90). Though this correction addresses objections raised here, it implies that our spherical model of the Earth is inaccurate.

Let us consider the revised model shown in Fig. 5.24. Suppose that the attractive force $m\mathbf{g}_3$ of the Earth on P has a small northerly directed, tangential component $-mg_3 \sin\alpha\mathbf{t}$ to balance the tangential centrifugal force component $mr^2\Omega \sin\theta \cos\beta\mathbf{t}$ shown in Fig. 5.24a. If the gravitational force exerted by the Earth is directed toward its center C, while $\mathbf{F}_O$ is normal to its surface, as shown in Fig. 5.24, then the Earth must flatten somewhat at the poles and bulge slightly at the equator. In fact, geophysical theory and measurements show that the Earth is an oblate spheroid with a mean equatorial radius $r_E = 3963$ mile (6378 km) and a smaller mean polar radius $r_P = 3950$ mile (6357 km), approximately. The accepted international value for the *amount of flattening* at the pole is $\mu \equiv (r_E - r_P)/r_E = 1/297$. The centrifugal force arising from the Earth's rotation thus produces a measurable equatorial bulge of the Earth. Therefore, to derive a more precise equilibrium result and resolve earlier contradictions, it is necessary to account for the Earth's oblateness in computing the gravitational field strength for a spheroid. This is a difficult problem that we shall not need to discuss here. The interested reader may consult the chapter references by Heiskanen and Meinesz and by Ramsey for further details.

To account for polar flattening, let us suppose that the direction of the actual gravitational force $m\mathbf{g}_3$ due to the Earth is still directed toward its center C in

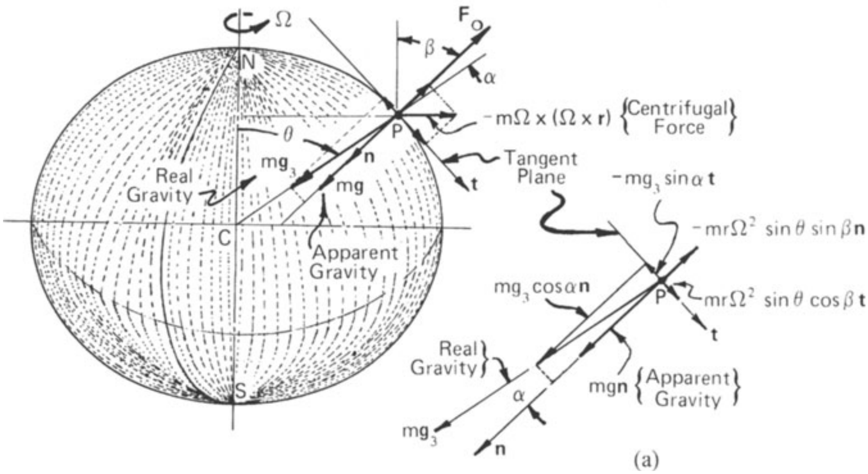


Figure 5.24. The real and apparent gravitational forces acting on a particle P at rest relative to a spheroidal Earth model.

Fig. 5.24. For equilibrium of P relative to the oblate spheroidal Earth, (5.89) now yields

$$\begin{aligned}&\left(-F_O + mg_3 \cos\alpha - mr\Omega^2 \sin\theta \sin\beta\right)\mathbf{n}\\&+\left(-mg_3 \sin\alpha + mr\Omega^2 \sin\theta \cos\beta\right)\mathbf{t} = \mathbf{0}.\end{aligned} \tag{5.91}$$

In this equation, β is the *geographical colatitude angle*, the angle between the polar axis of rotation and the outward, normal vector to the surface; θ is the *geocentric colatitude angle*, the angle between the polar axis and the radial line through the Earth's center; and $\alpha \equiv \theta - \beta$ is their *angle of deviation*. (See Fig. 5.24.) Thus, the normal reaction force $\mathbf{F}_O$ in (5.89) balances the *apparent weight* $m\mathbf{g}$ of P, which varies slightly over the surface of the Earth. That is, $\mathbf{F}_O + m\mathbf{g} = \mathbf{0}$, wherein the *apparent gravitational field strength* $\mathbf{g}$ is defined by

$$\mathbf{g} \equiv \mathbf{g}_3 - \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}). \tag{5.92}$$

This rule shows the effect of the Earth's rotation on the real gravitational field strength $\mathbf{g}_3$. The tangential component of $\mathbf{g}$ vanishes in accordance with (5.91):

$$-g_3 \sin\alpha + r\Omega^2 \sin\theta\ \cos(\theta - \alpha) = 0; \tag{5.93}$$

and (5.92) becomes

$$\mathbf{g} = g\mathbf{n} = \left(g_3 \cos\alpha - r\Omega^2 \sin\theta \sin(\theta - \alpha)\right)\mathbf{n}, \tag{5.94}$$

in which $\mathbf{n}$ is the inward directed unit normal vector to the Earth's surface. (See Fig. 5.24a.) This is named the *apparent acceleration of gravity*; it is the gravitational field strength apparent to an observer stationed at a point on the surface of the Earth at the geographic colatitude $\beta = \theta - \alpha$.

The apparent acceleration of gravity is always perpendicular to the Earth's surface. This is the direction $\mathbf{n}$ along which a plumb bob is attracted when freely suspended by a string. In this case, $\mathbf{F}_O$ is the tension in the line. The angle α of the plumb line's deviation from the direction of the real gravitational vector $\mathbf{g}_3$ in Fig. 5.24 may be determined by (5.93), but we must remember that g_3, r, and α vary with the angle θ. It can be shown by (5.93) and (5.94) that

$$g_3 = g\left(\cos\alpha + A\sin^2\theta\right), \tag{5.95}$$

in which $\sin\alpha = A\sin\theta\cos\theta$ and $A \equiv r\Omega^2/g$, with $r \equiv r(\theta) \in [r_P, r_E]$. Since A is very small (see (4.89)), the angle α is a very small quantity. Retaining only terms to the second order in α in (5.95), we derive the estimates

$$g_3 = g\left(1 + A\sin^2\theta - \frac{A^2}{8}\sin^2 2\theta\right), \qquad \alpha = \frac{A}{2}\sin 2\theta. \tag{5.96}$$

A final simplification of (5.96) in which terms of order A^2 and αA are omitted and r is approximated by its mean value R, say, is given by

$$g_3 = g + R\Omega^2\cos^2\lambda = g_E - R\Omega^2\sin^2\lambda, \qquad \alpha = \frac{R\Omega^2}{2g}\sin 2\lambda, \tag{5.97}$$

where $\lambda = \frac{\pi}{2} - \theta + \alpha$ is the *geographic latitude*, the angle between the equatorial plane and the outward normal to the Earth's surface. This simple estimate relates the values of the real and apparent field strengths as functions of the latitude λ, and it gives an estimate of the angle of deviation. In particular, $g_3 = g$ at the poles and $g_3 = g_E = g + R\Omega^2$ at the equator. Although g_3 is closely approximated by the apparent gravitational field strength g, we have not determined the variation of g as a function of θ or λ. This is given accurately by the international gravity formula.

When r and g are known as functions of θ or λ, the real gravitational field strength and its angle of deviation from the normal to the Earth's surface may be found. A more advanced analysis in potential theory is used to determine $g(\lambda)$, and ellipsoidal geometry is applied to determine $r(\lambda, \mu)$ in terms of the geographic latitude λ and the flatness factor μ. These details need not concern us. It turns out that the general formulas for the *Earth's variable radius* r and for the *apparent acceleration of gravity* g are given as

$$r(\lambda, \mu) = r_E\left(1 - \mu\sin^2\lambda + \frac{5\mu^2}{8}\sin^2 2\lambda\right), \tag{5.98}$$

$$g(\lambda) = g_E(1 + a\sin^2\lambda - b\sin^2 2\lambda), \tag{5.99}$$

wherein a and b are certain constants. It is seen that $r(0, \mu) = r_E$ and $g(0) = g_E$ are the respective equatorial values of $r(\lambda, \mu)$ and $g(\lambda)$. See the text by Heiskanen and Meinesz in the References.

The ellipsoidal shape function with $\mu = 1/297$ adopted by the International Geodetic Association at Madrid in 1924 is given by the *international ellipsoid formula*:

$$r = 6378.388(1 - 0.0033670 \sin^2 \lambda + 0.0000071 \sin^2 2\lambda) \text{ km}. \quad (5.100)$$

The constants g_E and a in (5.99) are obtained empirically from gravity measurements, but b is derived theoretically. The accepted values of these constants adopted by the General Assembly of the International Union of Geodesy and Geophysics at Stockholm in 1930, and reaffirmed unanimously at the Toronto, Canada Assembly in 1957, appear in the *international gravity formula*:

$$g = 9.780490(1 + 0.0052884 \sin^2 \lambda - 0.0000059 \sin^2 2\lambda) \text{ m/sec}^2. \quad (5.101)$$

This formula provides the apparent local acceleration of gravity as a function of the geographic latitude. The value of g varies from 9.83 m/sec^2 at the poles to 9.78 m/sec^2 at the equator. Our earlier rough calculation based on (5.61) for an ideal spherical Earth, namely, $g = 32.23$ ft/sec^2 $= 9.824$ m/sec^2, stands in excellent agreement with these extremes. The *standard value* adopted internationally for the apparent acceleration of gravity at sea level and at latitude $\lambda = 45°$ is $g = 32.1740$ ft/sec^2 $= 9.80665$ m/sec^2. It is customary to use the rounded value $g = 32.2$ ft/sec^2 $= 9.80$ m/sec^2 in numerical examples. In the sequel, however, we shall sometimes use $g = 32$ ft/sec^2 to simplify a numerical illustration.

The *apparent weight* $m\mathbf{g}$ of a body $\mathscr{B}$ is its weight apparent to an observer on the Earth; it is the weight, for example, that one measures when standing on a bathroom scale! We thus witness again that to a moving observer the actual force acting on a body is not always what it may seem to be. The difference between the apparent weight of $\mathscr{B}$ and its *absolute,* or *real weight relative to the Earth* in the inertial reference frame is quite small. Nevertheless, it is our custom to measure the weight of a body relative to our moving Earth frame, so no confusion should arise if, henceforward, the apparent weight of a body $\mathscr{B}$ relative to the Earth is called, briefly, the *weight* of $\mathscr{B}$. Then $\mathbf{g} = g\mathbf{n}$ in (5.62) is the apparent acceleration of gravity, and the weight of $\mathscr{B}$ is $\mathbf{W} = m\mathbf{g} = mg\mathbf{n}$, where $\mathbf{n}$ is the inward directed, unit normal vector to the Earth's surface.

5.17. Newton's Law in the Earth Frame

The foregoing analysis of the effect of the Earth's motion on the real weight of a body is based on static considerations. It is clear, however, that the terms in (5.92) are independent of the particle's motion relative to a fixed point Q on the Earth's surface, and the same terms may always be grouped in the same way in the dynamical equation (5.88) in which $\mathbf{r}$ is replaced by the current position vector of P from C, written as $\mathbf{x} = \mathbf{r} + \boldsymbol{\rho}$, where $\boldsymbol{\rho}$ is the position vector of P from Q. Thus, for motion on or near the Earth's surface $|\boldsymbol{\rho}| \ll |\mathbf{r}|$, and hence the additional centripetal acceleration term $|\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \rho)| \ll |\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times r)|$ is negligible in comparison with all other terms in the equation. Therefore, in all cases of motion on or near the Earth's surface, the form of Newton's second law of motion (5.88) relative to the Earth frame simplifies to

$$m\mathbf{a}_\varphi = \mathbf{F} - 2m\boldsymbol{\Omega} \times \mathbf{v}_\varphi, \tag{5.102}$$

in which the total force $\mathbf{F}$ acting on the particle P includes its apparent weight $\mathbf{W} = m\mathbf{g}$ and the total $\mathbf{F}_O$ of all other forces that act on P. The Coriolis force in (5.102) is the only term that reflects directly the influence of the Earth's motion. Its maximum value, however, is about 1.6×10^{-4} sec^{-1} times the magnitude of the relative momentum $m\left|\mathbf{v}_\varphi\right|$, so its contribution is generally small in comparison with all other forces in (5.102). Consequently, very often the approximation of (5.102) to the classical Newtonian law in a noninertial Earth frame is used in engineering practice. Indeed, our examples demonstrate that excellent analytical predictions can be obtained by taking the Earth frame as the preferred frame. Nevertheless, Coriolis effects are sometimes surprising and difficult to predict without careful analysis, so use of (5.102) for the motion of a particle relative to the Earth is of interest. Some examples are explored in the next chapter.

In general, however, in problems of motion referred to a noninertial reference frame φ, Newton's law (5.81) may be used in φ provided that the total "force" $\mathbf{F}_\varphi$ defined in (5.79) includes all inertial forces and all applied forces. The inertial forces can be significant in noninertial frames other than the Earth frame, and they should never be thoughtlessly ignored.

This concludes the introduction to the foundation principles of classical mechanics created by great mathematicians of the seventeenth and eighteenth centuries. More about this grand and bountiful heritage follows in the chapters ahead. We end this chapter with an advanced topic borrowed from continuum mechanics. Here we focus on its application to the problem of the internal interaction between two particles. The result is useful in our study of the internal potential energy of a system of particles in Chapter 8. Study of this topic requires familiarity with the material in Chapters 3 and 4, the relevant parts of which are sketched below. The reader who may have omitted this material in a first reading, however, will suffer no significant loss of continuity in moving on to the next chapter.

5.18. Frame Indifference and the Law of Mutual Internal Action

Consider two reference frames $\varphi = \{O; \mathbf{i}_k\}$ and $\Phi = \{F; \mathbf{I}_k\}$, the frame Φ being the preferred frame so that $\mathbf{I}_k$ are independent of t, though this is not really essential. Recall the basis transformation tensor $\mathbf{Q}(t) = \mathbf{i}_k(t) \otimes \mathbf{I}_k$ so that $\mathbf{i}_k(t) = \mathbf{Q}(t)\mathbf{I}_k$ is the Euler rotation of the basis of frame Φ into the basis of frame φ. Of course, to an observer in frame φ, the bases vectors $\mathbf{i}_k$ are independent of t, as discussed in Chapter 4. Let $\mathbf{x}_\varphi(P,t)$ denote the position vector in Φ of a particle P from the origin of φ, but referred to φ so that $\mathbf{x}_\varphi(P,t) = x_k(P,t)\mathbf{i}_k(t) = \mathbf{Q}(t)[x_k(P,t)\mathbf{I}_k]$. We define the relative position vector $\mathbf{x}_\Phi(P,t) \equiv x_k(P,t)\mathbf{I}_k$ referred to Φ, and thus obtain the transformation rule relating the relative position vectors:

$$\mathbf{x}_\varphi(P,t) = \mathbf{Q}(t)\mathbf{x}_\Phi(P,t). \tag{5.103}$$

The relative position vectors have the same time dependent components $x_k(P,t)$ in both frames. Therefore, a transformation of this kind is said to be *frame indifferent*, or *objective.* (Here and below, see Chapter 4, pages 313–317.)

5.18.1. Change of Reference Frame

A change of reference frame is characterized by an orthogonal linear transformation that preserves distances and angles, and for which all observers use the same universal clock so that trivial, constant time shifts may be ignored. The change of frame is exhibited in terms of the position vectors $\mathbf{X}_\Phi(P,t)$ and $\mathbf{x}_\varphi(P,t)$ of the same particle from the origins F and O of the respective frames Φ and φ in accordance with

$$\mathbf{X}_\Phi(P,t) = \mathbf{B}_\Phi(O,t) + \mathbf{x}_\varphi(P,t) = \mathbf{B}_\Phi(O,t) + \mathbf{Q}(t)\mathbf{x}_\Phi(P,t), \tag{5.104}$$

where $\mathbf{B}_\Phi(O,t)$ is the position vector of O from F and we recall (5.103). Henceforward, for simplicity of notation, let us write $\mathbf{x}'(P,t) \equiv \mathbf{X}_\Phi(P,t)$, $\mathbf{c(t)} \equiv \mathbf{B}_\Phi(O,t)$, and $\mathbf{x}(P,t) \equiv \mathbf{x}_\Phi(P,t)$ so that the change of reference frame is given by

$$\mathbf{x}'(P,t) = \gamma(\mathbf{x},t) \equiv \mathbf{c}(t) + \mathbf{Q}(t)\mathbf{x}(P,t). \tag{5.105}$$

Thus, $\mathbf{c}(t)$ is the position vector of O in frame Φ and $\mathbf{Q}(t)$ is an orthogonal tensor that specifies the rigid rotation of frame φ relative to frame Φ. It is easy to verify that the change of frame preserves distance between points and angles between lines.

From now on, let us consider (5.105) as a general change of reference frame mapping $\varphi = \{O; \mathbf{e}_k\}$ into $\varphi' = \{O'; \mathbf{e}'_k\}$. Then $\mathbf{x}$ and $\mathbf{x}'$ are the respective position vectors of the same particle P from the origins O and O' at time t, and $\gamma(\mathbf{x},t) : \varphi \to \varphi'$ is shorthand for the right-hand side of (5.105). We may exclude trivial rigid body rotations of $2n\pi$ rad, for $n = 1, 2, \ldots$. For all of these and for a null rotation, $\mathbf{Q} = \mathbf{1}$. A pure translation is thus described by $\mathbf{Q} \equiv \mathbf{1}$ so that $\gamma(\mathbf{x},t) = \mathbf{c}(t) + \mathbf{x}(P,t)$. Also, we recall from (3.88) that a rotation tensor $\mathbf{Q}$ preserves the axis of rotation $\mathbf{e}$, and hence all points $\mathbf{u} = u\mathbf{e}$ along that axis, that is, $\mathbf{Qu} = \mathbf{u}$. Therefore, $\mathbf{Qv(u)} = \mathbf{v(u)}$ holds if and only if the vector $\mathbf{v(u)}$ is parallel to $\mathbf{u}$, and hence $\mathbf{v(u)} = g(\mathbf{u})\mathbf{u}$, where $g(\mathbf{u})$ is a scalar-valued function of $\mathbf{u}$. These results are needed below.

5.18.2. The Principle of Material Frame Indifference

It is commonly assumed, without actually saying so, that the internal force in a spring is independent of the particular situation in which the spring might be used. We take for granted that the same extension of the same spring in a fixed reference frame and in any other reference frame having an arbitrary motion, gives rise to the same internal spring force and vice versa. Accordingly, the internal force-extension law of the spring (introduced in the next chapter) is the same at the top of a high mountain, the bottom of a deep mine, in fact at any place of rest, and on a rotating table in a laboratory or in a vehicle speeding along a tortuous highway. In fact, the idea of invariance of the spring law under translations was adopted by Hooke in 1675 in a proposal to use the spring to measure gravity. Thus, it is commonly assumed that the law relating the internal force to the extension depends only on the extension of the spring relative to itself, and it is not affected in any manner by arbitrary superimposed rigid body motions of translation and rotation, the latter altering only the relative direction of the spring force. This is an example of the important classical principle of invariance of internal material response to external superimposed rigid body motions, called, briefly, the *principle of material frame indifference*. The principle** has been widely applied in works on material response of deformable bodies, though often indirectly. In 1955, however, the general principle of material frame indifference for deformable bodies was given new motivation by Noll in its application to the constitutive theory of materials in continuum mechanics. This rule is stated in Noll's terms†† as follows.

The principle of material frame indifference: *The constitutive laws governing the internal interactions between the parts of a physical system do not depend on whatever external frame of reference is used to describe them.*

It is emphasized that the principle applies only to *internal interactions between parts of a system*, not to actions of the external world on the system and its parts. It does not apply to actions on a body that arise, for example, from inertial forces induced by the motion of the reference frame. These are frame dependent actions of the external environment on the system, actions that arise as a consequence of the noninertial nature of the reference frame, and which vanish only when an inertial frame is used. The choice of the external frame of reference is a matter of convenience. The internal interactions may be mechanical, gravitational, thermodynamical, electromagnetic, for example. Here we apply the principle to study the nature of the internal force between a pair of particles, an illustration due to Noll.

** A history of this principle is traced in the remarkable treatise by Truesdell and Noll cited in the References.

†† The presentation below, in somewhat different notation and without use of the language and mathematical rigor of finite dimensional spaces, parallels that due to W. Noll in unpublished articles described in the References. I thank Professor Noll for providing a copy of his papers and for his permission to use the example.

5.18.3. The Law of Mutual Internal Action

Newton's law of universal gravitational interaction between any two particles in (5.46) postulates that the force exerted by one particle on another at any given instant depends only on their positions, such that (i) the force is directed along their common line; and (ii) the magnitude of the force depends only on the distance between them. We are going to show, as Noll proved, that both conditions are consequences of the principle of material frame indifference.

Consider a system of two particles P_1 and P_2 at a given fixed time t; and let us assume that the mutual force $\mathbf{F}_{21}$ exerted on the particle P_2 by P_1 depends only on the positions $\mathbf{y}$ and $\mathbf{x}$ of the two particles at that instant, so that

$$\mathbf{F}_{21} = \hat{\mathbf{F}}(\mathbf{x}, \mathbf{y}). \tag{5.106}$$

Of course, we consider only distinct material points: $\mathbf{x} \neq \mathbf{y}$. Now, after a change of frame (5.105), or an equivalent superimposed rigid body motion of the system, the particles appear at the positions $\mathbf{x}' = \gamma(\mathbf{x})$, $\mathbf{y}' = \gamma(\mathbf{y})$ and the force appears to be rotated into $\mathbf{F}'_{21} = \mathbf{Q}\mathbf{F}_{21}$, where $\mathbf{Q}$ is the orthogonal tensor in (5.105). Then according to the principle of frame indifference, the same function $\hat{\mathbf{F}}$ should also describe the dependence of the force on the positions $\mathbf{x}'$, $\mathbf{y}'$ after the change of frame, so that $\mathbf{Q}\mathbf{F}_{21} = \mathbf{F}'_{21} = \hat{\mathbf{F}}(\mathbf{x}', \mathbf{y}')$. This means that the function $\hat{\mathbf{F}}$ must satisfy

$$\mathbf{Q}\hat{\mathbf{F}}(\mathbf{x}, \mathbf{y}) = \hat{\mathbf{F}}(\mathbf{x}', \mathbf{y}'), \tag{5.107}$$

for every change of frame (5.105) and for all points $\mathbf{x}$ and $\mathbf{y} \in \varphi$ at the instant t.

Let $\mathbf{x}, \mathbf{y}$ be given, choose a point at $\mathbf{q} \in \varphi$ arbitrarily, and consider a pure translation for which $\mathbf{Q} = \mathbf{1}$ and $\gamma(\mathbf{x}) = \mathbf{x} + \mathbf{c}$ translates $\mathbf{x}$ to $\mathbf{x}' = \mathbf{q}$. Then $\mathbf{y}' = \mathbf{c} + \mathbf{y} = \mathbf{q} + (\mathbf{y} - \mathbf{x})$, and hence (5.107) reduces in a pure translation to

$$\hat{\mathbf{F}}(\mathbf{x}, \mathbf{y}) = \hat{\mathbf{F}}(\mathbf{q}, \mathbf{q} + (\mathbf{y} - \mathbf{x})).$$

In particular, we may take $\mathbf{q} = \mathbf{0}$, which is equivalent to our choosing $\mathbf{c} = -\mathbf{x}$. This relation, however, must hold regardless of what $\mathbf{q}$ may be chosen. Therefore, we find that the function $\hat{\mathbf{F}}$ must have the form

$$\hat{\mathbf{F}}(\mathbf{x}, \mathbf{y}) = \hat{\mathbf{G}}(\mathbf{y} - \mathbf{x}), \tag{5.108}$$

for all $\mathbf{x}, \mathbf{y}$. Returning to (5.107) and using (5.108), we have

$$\mathbf{Q}\hat{\mathbf{G}}(\mathbf{y} - \mathbf{x}) = \hat{\mathbf{G}}(\mathbf{y}' - \mathbf{x}'), \tag{5.109}$$

for all orthogonal $\mathbf{Q}$ and for all positions $\mathbf{x}, \mathbf{y}$.

Recalling from (5.105) that $\mathbf{y}' - \mathbf{x}' = \mathbf{Q}(\mathbf{y} - \mathbf{x})$ holds for all rotations $\mathbf{Q}$ and for all $\mathbf{x}, \mathbf{y}$, by (5.109), we have $\mathbf{Q}\hat{\mathbf{G}}(\mathbf{y} - \mathbf{x}) = \hat{\mathbf{G}}(\mathbf{Q}(\mathbf{y} - \mathbf{x}))$, that is, with $\mathbf{r} \equiv \mathbf{y} - \mathbf{x}$, the position vector of the particle P_2 relative to the particle P_1,

$$\mathbf{Q}\hat{\mathbf{G}}(\mathbf{r}) = \hat{\mathbf{G}}(\mathbf{Q}\mathbf{r}). \tag{5.110}$$

This must hold for all vectors $\mathbf{r}$ and for all orthogonal transformations $\mathbf{Q}$. Given $\mathbf{r}$, (5.110) must hold, in particular, for all rotations $\mathbf{Q}$ about the axis $\mathbf{r}$ so that $\mathbf{Q}\mathbf{r} = \mathbf{r}$.

Then, by (5.110), $\mathbf{Q}\hat{\mathbf{G}}(\mathbf{r}) = \hat{\mathbf{G}}(\mathbf{r})$, and hence these $\mathbf{Q}$ leave $\hat{\mathbf{G}}(\mathbf{r})$ unchanged. This means that $\hat{\mathbf{G}}(\mathbf{r})$ must be parallel to $\mathbf{r}$, the axis of rotation. Hence, there exists a scalar-valued function $g(\mathbf{r})$ such that

$$\hat{\mathbf{G}}(\mathbf{r}) = g(\mathbf{r})\mathbf{r}, \tag{5.111}$$

for all $\mathbf{r}$. But the condition (5.110) requires that $g(\mathbf{r})\mathbf{Q}\mathbf{r} = g(\mathbf{Q}\mathbf{r})\mathbf{Q}\mathbf{r}$, that is,

$$g(\mathbf{r}) = g(\mathbf{Q}\mathbf{r}) \quad \text{for all orthogonal } \mathbf{Q}. \tag{5.112}$$

Given $\mathbf{r} = r\mathbf{e}$, where $r = |\mathbf{r}| = \sqrt{\mathbf{r}\cdot\mathbf{r}}$, introduce $\mathbf{e}' = \mathbf{Q}\mathbf{e}$ and note that $\mathbf{Q}\mathbf{r} = \mathbf{Q}r\mathbf{e} = r\mathbf{e}'$. Then, by (5.112), $g(r\mathbf{e}) = g(r\mathbf{e}')$ for an arbitrary direction $\mathbf{e}'$. Thus, choose $\mathbf{e}' = -\mathbf{e}$ to obtain $g(r\mathbf{e}) = g(-r\mathbf{e})$. Therefore, the scalar-valued function $g(\mathbf{r})$ must be an even function of $\mathbf{r}$ and independent of its direction. Hence, $g(\mathbf{r})$ is a scalar-valued function of r alone, defined by $g(\mathbf{r}) \equiv h(r)$, and now (5.111) becomes

$$\hat{\mathbf{G}}(\mathbf{r}) = h(r)\mathbf{r}. \tag{5.113}$$

Recalling (5.108) and noting in (5.113) that $\mathbf{r} = \mathbf{y} - \mathbf{x}$, we have

$$\hat{\mathbf{F}}(\mathbf{x}, \mathbf{y}) = h(|\mathbf{y} - \mathbf{x}|)(\mathbf{y} - \mathbf{x}). \tag{5.114}$$

We thus find that the dependence of the force $\mathbf{F}_{21}$ in (5.106) on the positions $\mathbf{x}$ and $\mathbf{y}$ must reduce to the specific form

$$\mathbf{F}_{21} = h(r)\mathbf{r}, \tag{5.115}$$

where $r = |\mathbf{r}|$ and $\mathbf{r} = r\mathbf{e} = \mathbf{y} - \mathbf{x}$ is the position vector of particle P_2 from P_1. *This is the most general form of the law of mutual internal action that satisfies the principle of material frame indifference.* Moreover, from (5.114), $\hat{\mathbf{F}}(\mathbf{y}, \mathbf{x}) = -\hat{\mathbf{F}}(\mathbf{x}, \mathbf{y})$, that is, $\mathbf{F}_{12} = -\mathbf{F}_{21}$. This is Newton's third law of mutual action. Thus, the principle of frame indifference applied to the internal force between two particles that depends only on their positions, shows that their mutual internal force is a function of the distance of their separation and is directed along their common line.

Exercise 5.7. Begin with (5.115) and show that (5.107) is satisfied for an arbitrary change of frame (5.105). This will conclude the proof of Noll's theorem: *The internal force* (5.106) *between two particles that depends only on their positions is frame indifferent if and only if it has the form* (5.115). □

Newton's law (5.46) for the mutual gravitational attraction of a pair of particles is obtained from (5.115) with $h(r) \equiv -Gm_1m_2/r^3$. Similarly, Coulomb's law for the electrostatic force between two particles with electrical charges q_1 and q_2, studied in the next chapter, follows from (5.115) with $h(r) \equiv kq_1q_2/r^3$, in which k is a constant. The general rule (5.115) is also useful in characterizing the total internal potential energy of a system of particles in Chapter 8.

References

1. Bartlett, A. A., Are we overlooking something? 1981 Robert A. Millikan Lecture, Steven's Point, Wisconsin, June 17, 1981. *American Journal of Physics* **49**, 1105–19 (1981). The problem of U.S. Navy torpedo failures and its simple explanation are described. See also Bartlett's earlier article, An effect of friction in an accelerated system, *American Journal of Physics* **37**, 665–6 (1969).
2. Berlinski, D., *Newton's Gift*, Free Press, New York, London, 2000. The author begins with the ordinary aspects of Newton's childhood, sketches his student years, and describes his professional life at Cambridge. The ultimate focus, however, is on Newton's creation of the calculus and his remarkable masterpiece, his supreme gift—the *Principia*, published in 1687—wherein the principles of rational mechanics and the law on universal gravitation are created and the results demonstrated in describing the motions of planets. Remarks on Newton's theory of light, his experiments on optics, his development of the reflective telescope, and his quarrel with Robert Hooke provide additional interesting reading.
3. Bixby, W., *The Universe of Galileo and Newton*, American Heritage, 1st Edition, New York, 1964. The writer presents a well-illustrated sketch of the life and scientific accomplishments of Galileo Galilei (1564–1642) and Isaac Newton (1642–1727) in describing and explaining the motions of bodies in the universe.
4. Bowden, F. P., and Tabor, D., *The Friction and Lubrication of Solids* (2 volumes), Clarendon, Oxford, 1950. A much abbreviated introduction to this major treatise on the mechanism of friction may be found in their more elementary monograph *Friction and Lubrication,* Methuen's Monographs on Physical Subjects, Methuen and Co., London, 1967.
5. Cajori, F., *Newton's Principia.* A well-known English translation of *Mathematical Principles of Natural Philosophy* by Isaac Newton, 1687. University of California Press, Berkeley, 1947. The wide recognition of Newton's monumental work derives mainly from the famous axioms, or laws of motion, presented in the first of the three books. Newton's style anticipates readers having a strong foundation in geometry, even though his calculus had been developed earlier.
6. Chandrasekhar, S., *Newton's Principia for the Common Reader*, Oxford University Press, New York, 1995. The author presents a comprehensive study of the *Principia* in contemporary language and the methods of analysis by the calculus to take the reader through the chain of argument leading to Newton's formulation of his universal law of gravitation.
7. Deresiewicz, H., Amontons and Coulomb, friction's founding fathers. In: *Approaches to Modelling Friction and Wear*, editors F. F. Ling and C. H. T. Pao, Springer-Verlag, New York, 1986, pp. 50–60. Amontons's report to the French Academy and its response are outlined, and Coulomb's comprehensive investigation is discussed briefly. References to original works may be found here.
8. Durrell, C. V., *Readable Relativity*, Harper and Brothers, New York, 1960. Written by a teacher of mathematics and a master of the subject, this little book presents an entertaining introduction to relativity. Simple mathematical arguments provide the general reader with a clear exposition of the basic principles of Einstein's theory.
9. Einstein, A., *Relativity, The Special and the General Theory*, Crown Publishers, New York, 1961. A simply written, popular exposition on the relation of the theory of relativity to Newton's classical theory of motion and gravitation. The special theory compares observer experiences in frames having uniform relative motions, while the general theory admits all possible frames having nonuniform and curvilinear relative motions. The special theory is based upon two axioms: (1) the laws of nature in every material universe have the same form for all observers in uniform relative motion, and (2) the speed of light emitted by a body at rest or in uniform motion is the same for all observers.
10. Fernie, J. D., The neptune affair. *American Scientist* **83**, No. 2, 116–9 (1995). The intrigue of Neptune's discovery is reviewed here. Though credited with the discovery of Neptune, Le Verrier was not the first to predict its location. In September 1845, the British mathematician John Couch Adams presented calculations that predicted within less than 2° the place in the sky where the planet might be found as of October 1, 1845. The incredible arrogance of Adams and the Astronomer

Royal, George Airy, blocked further communication and publication of Adams's more precise prediction of Neptune's location, while details of Le Verrier's work appeared later in the *Comptes Rendus* of the French Academy of Sciences. Le Verrier's final paper on the topic reached England on September 29, 1846; it gave Neptune's mass and coordinates within only a few degrees of Adams's prediction. The concluding irony of the story, however, is that Galileo had twice recorded in his notebooks during the period December 1612 to January 1613, almost 234 years earlier, diagrams of telescopic observations that show a "fixed star" drawn on a directed line from Jupiter in the plane of its satellites. A more recent review by astronomers in 1980 of Galileo's observations revealed that he had actually discovered Neptune. Due to the poor resolution of his telescope, however, he identified it as a star.

11. FERNIE, J. D., In pursuit of Vulcan. *American Scientist* **82**, No. 5, 412–5 (1994). The history of the search for Vulcan, the hypothetical planet thought to be somewhere between Mercury and the Sun, is outlined.
12. FRANK, P., *Philosophy of Science—The Link Between Science and Philosophy*, Prentice Hall, 1957. The logical structure of Newton's laws of motion and their related operational foundation are discussed in Chapter 4.
13. HART, I. B., *The Mechanical Investigations of Leonardo da Vinci*, University of California Press, Berkeley, 1963. This is a fascinating historical study of da Vinci, not as an artist but as a student of mechanics. Leonardo's personality, strange habits, and his creative genius in conceiving unusual applications of the mechanics of his time, and in some measure contributing to it through his mechanical investigations, are graphically described. A wealth of interesting historical references may be found here. See also the critical Forward by E. A. Moody. Leonardo's notes on friction are described in Chapter VI. The quotation on page 1 above is cited in Chapter V, p. 78
14. HEISKANEN, W. A., AND VENING MEINESZ, F. A., *The Earth and its Gravity Field*, McGraw-Hill, New York, 1958. The gravity field, the shape and other features of the Earth, the international gravity formula, and various static and dynamic gravity-measuring methods are described in Chapters 3 and 4.
15. HERTZ, H., *The Principles of Mechanics*. Dover, New York, 1956. The author focuses on providing a philosophical and mathematical foundation for the kinematics and dynamics of material systems in development of the theory of mechanics. There are no worked examples in the text.
16. HUBBERT, M. K., AND RUBEY, W. W., Role of fluid pressure in mechanics of overthrust faulting. I. Mechanics of fluid filled porous solids and its application to overthrust faulting, *Bulletin of the Geological Society of America* **70**, 115–66 (1959). The sliding can experiment, attributed to M. A. Biot, is related to gravitational sliding of fault blocks due to interstitial fluid pressure that allows low angle thrust faults to occur under shear stress of only moderate magnitude. The application of the principle to the design of bearings for the Mount Palomar telescope is described by Karelitz, M. B., Oil pad bearings and driving gears of the 200-in. telescope, *Mechanical Engineering* **60,** 541–4 (1938). Several further references may be found here. See also Noble, B., *Applications of Undergraduate Mathematics in Engineering*, Macmillan, New York, 1967. Equilibrium analysis of the sliding can experiment is described in Chapter 8, pp. 166–8.
17. KANE, T. R., *Analytical Elements of Mechanics*, Academic, New York, 1959. Volume 1 concerns statics. A parallel discussion of gravitational and contact forces, including several worked examples, may be found in Chapter 4.
18. KRIM, J., Friction at macroscopic and microscopic length scales. *American Journal of Physics* **70**, 890–7 (2002). This paper provides a useful summary and extensive guide to a variety of resources on the fundamentals of friction. Traditional concepts on the origin of friction and modern molecular theories are presented.
19. LONG, R., *Engineering Science Mechanics*, Prentice-Hall, Englewood Cliffs, New Jersey, 1963. The author adopts Mach's approach in setting down the foundation postulates of Newton's mechanics. Coulomb friction, gravitational attraction by a sphere, Newton's law in the noninertial Earth frame, and other topics are presented.
20. MORISON, Admiral S. E., *The Two-Ocean War, A Short History of the United States Navy in the Second World War*, Little-Brown, Boston, Toronto, 1963. This book, based on the author's fifteen-volume *History of the United States Naval Operations in World War II*, covers the most

important battles and campaigns of the two-ocean war. Brilliant contributions of engineers, mathematicians, and scientists to the design, development and efficient use of military devices and equipment for the American and British Armed Forces are described, pp. 125–130. Such questions as—"What pattern of depth charges at what settings has the best mathematical chance of killing a submarine?"—were addressed. During the critical early months of the Pacific war, however, the Navy learned the hard way that its torpedoes were gravely defective and that Japanese torpedoes were more powerful, faster, had a greater range, and were more efficient machines of destruction than its own, pp. 12–13.

21. Murphy, D., Hit or Miss. *American Heritage of Invention and Technology* **13**, No. 4, 56–63 (1998). The frustrating effects of U.S. Navy torpedo failures and the solution of the problem is described.
22. Noll, W., *On Material Frame-Indifference*, Carnegie Mellon University, Center for Nonlinear Analysis, Department of Mathematics, Technical Report No. 95-NA-022, November 1995, 10 pages. See also Chapter 1 (pages 13–22) of the same title in Noll's recent comprehensive work *5 Contributions to Natural Philosophy*, Center for Nonlinear Analysis, Department of Mathematical Sciences, Research Report No. 04-CNA-018, October 2004, 73 pages. In these reports the author presents a thorough, lucid discussion of the principle of material frame indifference, a lexicon of earlier terms used to characterize it is included, and its application to the internal interaction between two particles is illustrated. The text illustration parallels that due to Noll. A similar derivation was first reported in Noll's article *On the Foundation of the Mechanics of Continuous Media*, Carnegie Institute of Technology, Department of Mathematics, Technical Report No. 17, June 1957, 68 pages (see pages 38–40).
23. Ramsey, A. S., *Newtonian Attraction*, Cambridge University Press, Cambridge, Massachusetts 1961. This is an intermediate level introduction to the theory of potential based on Newton's law of gravitation. Attraction by a sphere is studied in Chapter III, and the difficult problem of attraction by an ellipsoid is described in Chapter VII. Additional examples and problems of scaled difficulty may be found here.
24. Roscoe, T., *Pig Boats*, Bantam Books, New York, 1982. This is a comprehensive account of U.S. Pacific Fleet submarine operations during World War II written principally for the Bureau of Naval Personnel based on both U.S. and Japanese naval records and submarine patrol reports. Countless torpedo malfunctions were reported during the critical early months of the war. Chapter 9 recounts, for example, the submarine *Grenadier's* sinking of the large Japanese passenger liner *Taiyo Maru* that was transporting a group of Japanese scientists and other experts, most of whom went down with the ship en route to conquered territories in the East Indies. Lieutenant Commander "Pilley" Lent, however, reported that two of the four magnetically triggered torpedoes fired on *Taiyo Maru* were duds or malfunctioned. Frequent problems of unreliable state-of-the-art magnetic exploders that sometimes exploded prematurely, and sometimes not at all, ultimately led to the use of older impact exploders. Chapter 11, Torpedo!, tells of the serious effects of disturbing torpedo impact failures (including *Tinosa's* failed attack on the *Tonan Maru*), their cause and ultimate elimination. These major technical problems of defective magnetic and impact torpedoes and the drop-test experiments leading eventually to the solution of the latter are depicted, though somewhat loosely, in the 1951 Warner Brothers film *Operation Pacific*, starring John Wayne and Patricia Neal.
25. Snyder, G. S., *The Royal Oak Disaster*, William Kimber, Great Britain, 1976; reprinted by Presido Press of San Rafael, California, 1978. An American journalist describes in this book one of the worst disasters in the history of the Royal Navy, which occurred during the early months of World War II. The story is a fascinating account of the sinking of the great battleship *Royal Oak* by the German submarine U-47, under Lieutenant Commander Gunther Prien, following his remarkable intrusion into the supposedly impregnable British naval base at Scapa Flow.
26. Synge, J. L., and Griffith, B. A., *Principles of Mechanics*, 2nd Edition, McGraw-Hill, New York, 1949. Chapters 5 and 13 study effects of the Earth's rotation on the apparent gravitational field strength and on the equation of motion.
27. Truesdell, C., *Essays in the History of Mechanics*, Springer-Verlag, Berlin-Heidelberg-New York, 1968. This is a fascinating collection of lectures by a renowned scholar, applied mathematician, and historian of mechanics—highly recommended to all readers. Chapter 1 describes and critiques the notebooks and the mechanics of Leonardo da Vinci, including his law of friction and study of

motion on an inclined plane. The development of the foundation principles of classical mechanics in the 17th and 18th centuries due to Newton (1687), Euler (1750), Lagrange (1788), and others is detailed in Chapter 2. See also Reactions of late Baroque mechanics to success, conjecture, error, and failure in Newton's *Principia.* In: *Mechanics*, editor N. C. Lind, American Academy of Mechanics, University Park, Pennsylvania pp. 1–47, 1970. Euler's papers of 1744–1750 are sketched in *The Rational Mechanics of Flexible or Elastic Bodies 1638–1788. Introduction to Leonardi Euleri Opera Omnia,* Vol. 10 and 11, 2nd Series, pages 222–9, 250–4, Orell Füssli Turici, Switzerland, 1960. This is a historical study of the mechanics of deformable bodies ideal for all students of engineering and applied mathematics.

28. TRUESDELL C., AND NOLL, W., *The Non-Linear Field Theories of Mechanics*, Flügge's Handbuch der Physik, Vol. III/3, New York, 1st Edition 1965, 2nd Edition Springer-Verlag, Berlin, Heidelberg, 1992. The principle of material frame indifference is presented in Sections 17 through 19, pages 41–5, and the history of the principle is traced in Section 19A, pages 45–7, of the first edition.
29. YEH, H., AND ABRAMS, J. I., *Principles of Mechanics of Solids and Fluids.* Vol. 1, *Particle and Rigid Body Mechanics*, McGraw-Hill, New York, 1960. Equipollent force systems are discussed in Chapters 4 and 5.

Appendix: Measure Units in Mechanics

In numerical examples, exercises, and problems where measure units are not explicit, consistent measure units always are understood. It makes no difference in theoretical mechanics what measure units may be used to express numerical results. But all countries throughout the world have agreed to adopt in scientific work the International System of Units, called SI units. Some SI units used in mechanics are listed in the Table 5.1.

Table 5.1. Systems of measure units

Measure	SI units	Engineering units	English units
Mass	kilogram (kg)	slug ($lb \cdot sec^2/ft$)	pound (lb_m)
Length	meter (m)	feet (ft)	feet (ft)
Time	second (sec)	second (sec)	second (sec)
Force	Newton (N)	pound (lb)	poundal (lb_l)

Universal conversion to the SI system, even at this date, is incomplete, and, of course, many important earlier reference works employ other systems of units, including the Engineering system which still enjoys wide use throughout the United States and to a lesser extent in Great Britain. The English system, now largely abandoned, is another scheme that has been used by engineers in these countries. Table 5.1 identifies for these systems the measure units of force based on Newton's second law:

$$1\ \text{N} = 1\ \text{kg} \cdot \text{m/sec}^2, \qquad 1\ \text{lb} = 1\ \text{slug} \cdot \text{ft/sec}^2, \qquad 1\ \text{lb}_l = 1\ \text{lb}_m \cdot\ \text{ft/sec}^2\,.$$

The following conversion factors may be used to relate SI and engineering units:

$$1\ \text{N} = 0.225\ \text{lb}, \qquad 1\ \text{m} = 3.281\ \text{ft}, \qquad 1\ \text{slug} = 14.58\ \text{kg}.$$

The Engineering and the English units of mass are related by a dimensionless conversion factor g_o whose numerical value is equal to the standard value of the acceleration of gravity at a specified point on the Earth. By definition, the mass of a *standard one pound body* is 1 lb_m and its weight is 1 lb, thus $W = 1$ lb $= 1$ slug $\cdot$ ft/sec^2 $= mg_o = 1\ \mathrm{lb}_m \cdot g_o$ ft/sec^2. Then with $g_o = 32.2$, say, 1 slug $= 32.2\ \mathrm{lb}_m$. Similarly, the pound is defined as the unit of force that will impart to a 1 lb_m an acceleration equal to g_o. Then with force measured in pounds (engineering units) and mass measured in pounds mass (English units), Newton's law would become $\mathbf{F} = m\mathbf{a}/g_o$. We may be thankful that this practice is no longer fashionable. Though we shall have no need in this book to prefer one system over another, in numerical work only Engineering and SI units are used.

Problems

It is essential that throughout the study of this text the student should work a variety of problems in order to grow familiar with use of the notation, concepts, and definitions; to cultivate, test, and expand one's understanding of the subject matter; to learn the general methods of mechanics; and to master various techniques of problem solving. Moreover, it is important that the problems be approached in a spirit and manner similar to that expressed in the examples, namely by the use of vector methods so far as may be reasonable and, in large measure, without the aid of a computing device. Instances where use of a computer is desirable to promote practice with some numerical calculations will be evident. In general, however, numerical values usually will serve only to simplify an analysis and to lay bare the relevant aspects of the illustration. Therefore, the majority of problems in this book have been constructed to avoid senseless use of a computer so that the student's skills with direct calculations and with manipulations of analytical relations may be reinforced and sharpened to further develop the student's ability to handle fundamental aspects of analytic geometry, trigonometry, calculus, vector methods, and differential equations, all essential to the modern demands of engineering practice.

5.1. Three particles of mass $m_1 = 3$ kg, $m_2 = 2$ kg, $m_3 = 5$ kg are initially located in $\Phi = \{F; \mathbf{i}_k\}$ at $\mathbf{x}_1 = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ m, $\mathbf{x}_2 = 2\mathbf{i} - 3\mathbf{k}$ m, $\mathbf{x}_3 = -\mathbf{i} + 4\mathbf{k}$ m, respectively, and their corresponding initial velocities are given by $\mathbf{v}_1 = \mathbf{i} - 2\mathbf{k}$ m/sec, $\mathbf{v}_2 = 2\mathbf{i} - 3\mathbf{j}$ m/sec, $\mathbf{v}_3 = -2\mathbf{k}$ m/sec. Determine for the initial instant (a) the position and velocity of the center of mass and (b) the momentum of the system.

5.2. Consider a system $\beta = \{P_k\}$ of n particles P_k with mass m'_k, and introduce the normalized mass $m_k \equiv m'_k/m(\beta)$ in which $m(\beta)$ is the mass of the system. Let $\mathbf{x}_k = \mathbf{x}^* + \hat{\mathbf{x}}_k$ and $\hat{\mathbf{x}}_k$ denote the respective position vectors of P_k from point O and from the center of mass C in frame $\Psi = \{F; \mathbf{e}_k\}$. Then, by (5.5), the position vector of C from O is given by

$$\mathbf{x}^* = \sum_{k=1}^{n} m_k \mathbf{x}_k \quad \text{with} \quad \sum_{k=1}^{n} m_k = 1. \tag{P5.2a}$$

Lagrange observed that the location of the center of mass C of a system of particles is determined uniquely by their relative positions, that is, by their mutual distances of separation d_{jk}. He thus posed the problem of finding C in terms of only these mutual distances. To see how this may be done, first (a) prove *Lagrange's Lemma* (1783):

$$\frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n} m_j m_k d_{jk}^2 = \sum_{k=1}^{n} m_k \hat{d}_k^2, \tag{P5.2b}$$

where $\hat{d}_k$ is the distance from C to the particle P_k and d_{jk} denotes the distance between the particles with mass m_j and m_k. Hint: Note that the vector $\hat{\mathbf{x}}_k - \hat{\mathbf{x}}_j = \mathbf{x}_k - \mathbf{x}_j$ from m_j to m_k determines the squared distance $d_{jk}^2 = d_{kj}^2$. (b) Apply (P5.2b) to prove *Lagrange's Theorem*[‡‡] *on the center of mass* (1783):

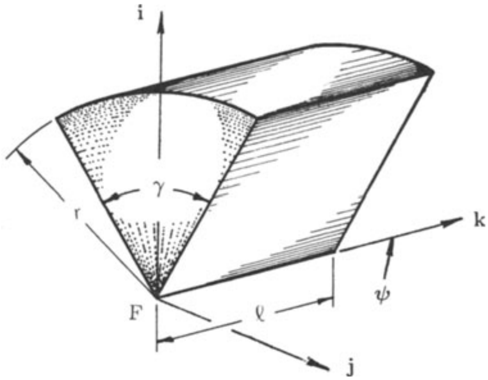
$$d_C^2 = \sum_{k=1}^{n} m_k d_k^2 - \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n} m_j m_k d_{jk}^2, \tag{P5.2c}$$

wherein d_k and d_C are the respective distances of the particle P_k and of the center of mass C from any specified point O. Hint: Determine $\sum_{k=1}^{n} m_k \hat{d}_k^2 = \sum_{k=1}^{n} m_k(\hat{\mathbf{x}}_k \cdot \hat{\mathbf{x}}_k)$. The result follows from here. Because O is an arbitrary point, it may be chosen at any of the particle locations so that the distance of C from any three noncoplanar and noncoaxial particles can be found from (P5.2c). Therefore, the location of C may be found when *only* the mutual distances of separation of the particles are known.

5.3. Lagrange's method described in the previous problem generally involves some rather tedious calculations in its application, but it gives an easy solution in some cases. To grasp the idea of the theorem, consider a system of two identical particles separated by a distance a. Apply Lagrange's theorem to find the center of mass, and describe carefully how its location is fixed.

5.4. Four identical particles are situated at the vertices of an equilateral pyramid with edge lengths a and height h. Find the center of mass C of the system (a) by use of Lagrange's theorem in Problem 5.2 and (b) by the usual method expressed in the normalized form (P5.2a). (c) Show that C is the intersection point of the pyramid altitude lines at distance $d_C = 3h/4$ from each particle.

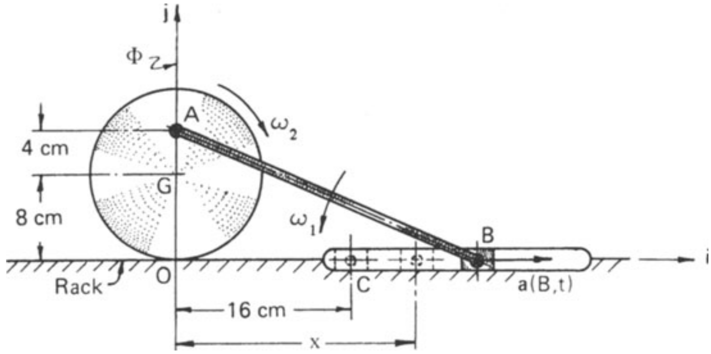
5.5. Find the center of mass of a homogeneous right circular cone of base radius r and height h. What is the mass of the cone?



Problem 5.6.

5.6. A homogeneous cylindrical wedge of radius r, length ℓ, and central angle γ is shown in the figure. Determine the mass of the wedge, and find its center of mass in $\psi = \{F; \mathbf{i}_k\}$. Locate the center of mass of a homogeneous half cylinder.

‡‡ A special case of Lagrange's theorem applied to a molecular chain configuration of n atoms of equal mass is presented by P. J. Flory, *Statistical Mechanics of Chain Molecules*, Hanser, New York, pp. 5, 383–4, 1988. See also M. F. Beatty, Lagrange's theorem on the center of mass of a system of particles, *American Journal of Physics* **40**, 205–7 (1972).

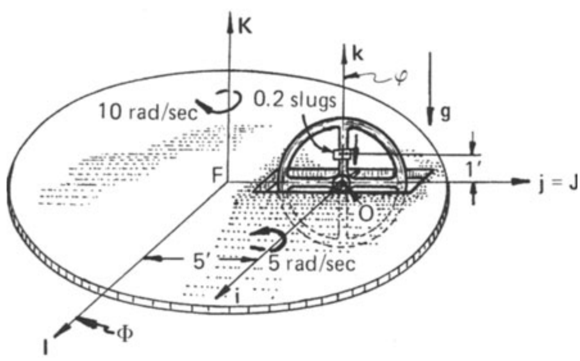


Problem 5.7.

5.7. One end of a connecting link AB is hinged at A to a gear G of radius 8 cm; the other end is hinged at B to a slider block of mass $m = 100$ gm. The gear rolls on a fixed horizontal rack. In 2 sec, the slider block moves from its initial rest position at C in frame $\Phi = \{O; \mathbf{i}_k\}$ to the position shown in the diagram. During the interval of interest, the slider has acceleration $\mathbf{a}_B = 18\sqrt[3]{(x-16)/3}\,\mathbf{i}$ cm/sec^2 in Φ. Determine the momentum of the slider block at the instant shown in the figure. What is the moment of momentum of B about points at O and A at the instant of interest?

5.8. At a moment of interest t_0, a particle P of mass 2 kg has the velocity $\mathbf{v}(P, t_0) = 16\mathbf{i} + 4\mathbf{j} - 12\mathbf{k}$ m/sec at the place $\mathbf{X}(P, t_0) = 2\mathbf{i} - \mathbf{j} + 4\mathbf{k}$ m in frame $\Phi = \{F; \mathbf{i}_k\}$. (a) Determine the momentum of P and find its moment about F at the time t_0. (b) What is the instantaneous moment of momentum of P about the point O at $\mathbf{r} = 2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$ m in Φ when (i) O is fixed in Φ and (ii) O is moving in Φ with the velocity $\mathbf{v}_O = 4\mathbf{i} - 6\mathbf{j}$ m/sec?

5.9. Water issuing from the nozzles of the garden sprinkler described in Problem 4.66, Volume 1, causes it to turn with an angular velocity $\omega(t)$ as shown. Compute the moment of momentum about O of a fluid particle P of mass m as it exits the nozzle at E with a constant speed v relative to the nozzle. What is the absolute time rate of change of the moment of momentum of P at E?



Problem 5.10.

5.10. The flywheel shown in the figure has a constant, counterclockwise angular speed of 5 rad/sec relative to a platform turning with a constant angular speed of 10 rad/sec, as indicated. A small slider block of mass 0.2 slug is moving along a wheel spoke toward the center O. At the instant t_o shown, the slider block is 1 ft from O and has a speed of 20 ft/sec that is increasing at the rate of 10 ft/sec^2 relative to the flywheel frame $\varphi = \{O; \mathbf{i}_k\}$. (a) What is the linear momentum of the block in the ground frame $\Phi = \{F; \mathbf{I}_k\}$ at t_o? (b) What is its corresponding moment of momentum about O? (c) Determine at t_o the moment of momentum of the slider about F in Φ.

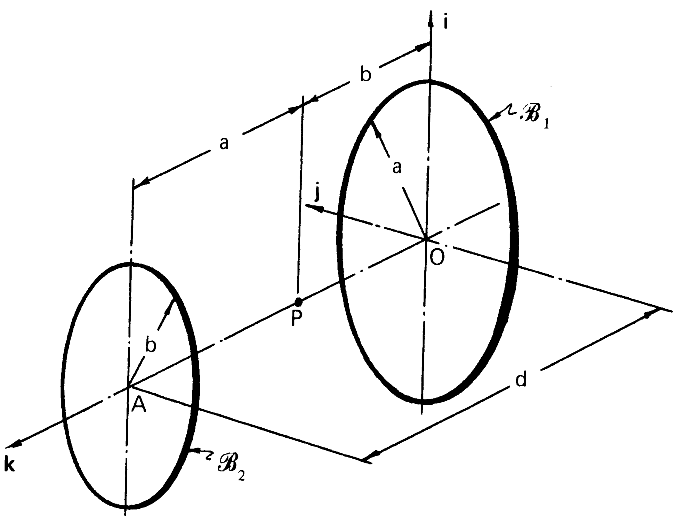
5.11. For the data provided in Problem 5.1, determine for the initial instant the moment of momentum of the system about F. What is the moment of momentum of the system about another fixed point O at $\mathbf{X}_O = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ m in Φ? How is the moment about O of the momentum of the system affected when O has the initial velocity $\mathbf{v}_O = 4\mathbf{i} - 13\mathbf{j} + 6\mathbf{k}$ m/sec?

5.12. Three particles of mass m, $2m$, and $3m$ occupy the respective initial positions $\mathbf{x}_1 = 6\mathbf{j}$ ft, $\mathbf{x}_2 = \mathbf{0}$, $\mathbf{x}_3 = -2\mathbf{j}$ ft, and they have the constant velocities $\mathbf{v}_1 = 6\mathbf{i} + 3\mathbf{j}$, $\mathbf{v}_2 = 6\mathbf{i} - 3\mathbf{j}$, $\mathbf{v}_3 = 4\mathbf{i} - 5\mathbf{j} + 2\mathbf{k}$ (all in ft/sec), respectively, in frame $\Phi = \{O; \mathbf{i}_k\}$. Determine (a) the velocity of the center of mass particle and (b) the momentum of the system. (c) Find the motion of the center of mass particle as a function of time t, and describe its path. (d) What is the moment of momentum of the system about O initially? (e) What is the moment about O of the momentum of the center of mass particle?

5.13. A loaded balloon of total weight W is falling vertically with a constant acceleration $\mathbf{a}$. Neglect wind effects and air resistance, but account for the buoyant force of the air, and find the amount of ballast weight w that must be discarded to give the balloon an upward acceleration $-\mathbf{a}$.

5.14. Three particles of mass m, $2m$, and $3m$ are stationary at the respective points (0, 0, 0), (1, 2, 3), and (3, 2, 1) in frame $\Phi = \{O; \mathbf{i}_k\}$. Find the resultant gravitational force exerted on the particle of mass m.

5.15. A particle P of mass μ is at the central point of a homogeneous, semicircular, thin wire of radius b and mass density σ per unit length. Determine the gravitational force exerted on P by the wire.



Problem 5.16.

5.16. Two thin, homogeneous circular wires $\mathscr{B}_1$ and $\mathscr{B}_2$ of radii a and b, respectively, are positioned in parallel planes distance d apart. The mass density of $\mathscr{B}_2$, per unit length, is twice that of $\mathscr{B}_1$. A particle P of unit mass is situated as shown in the figure on the normal line OA through their centers. (a) Apply (5.54d) to find the total gravitational force on P due to both rings. (b) Show that the gravitational force due to $\mathscr{B}_1$ alone vanishes at the center of the ring at O and at infinity, hence a maximum value of this force exists. Find the location b^* of P where the intensity of the gravitational force of $\mathscr{B}_1$ on P is greatest. (c) Repeat part (a) for the case $b = b^*$. What is the mass ratio m_2/m_1 of the rings?

5.17. A thin, flat annular body $\mathscr{B}$ has an inner radius R_1, an outer radius R_2, and uniform mass density σ per unit of area. (a) What gravitational field strength does $\mathscr{B}$ produce at a point P on the line normal to the plane of $\mathscr{B}$ through its center O, at distance X from O? (b) Determine the field strength at O due to $\mathscr{B}$. (c) Show that for $X \gg R_2$ the field strength of $\mathscr{B}$ is $\mathbf{g(X)} = -Gm/X^2\mathbf{k}$, wherein $m = m(\mathscr{B})$, and hence in its gravitational attraction at a sufficiently great distance $\mathbf{X}$, the ring behaves like a particle in accordance with (5.47).

5.18. A particle P of mass β is situated at a distance $X > a$ from the center, and along the axis of a homogeneous thin rod of length $2a$ and mass density σ per unit length. Find the gravitational force acting on P due to the rod.

5.19. A particle P of mass β is located at a distance X on the center line perpendicular to the axis of a homogeneous thin rod of mass m and length $2a$, both lying in the xz-plane. The origin is at the center of the rod with its axis along $\mathbf{k}$. Show that the gravitational force that the rod exerts on P is

$$\mathbf{F}(P;\mathbf{X}) = -\frac{Gm\beta}{X\sqrt{X^2+a^2}}\mathbf{i}. \tag{P5.19}$$

5.20. A particle of mass m is placed at an external point on the axis of a homogeneous, right circular cylinder at a distance α from one end. (Choose a frame with origin at the particle and the cylinder axis as $\mathbf{k}$.) The cylinder has radius R, length L, and mass M. Find the attractive force it exerts on the particle.

5.21. Determine the gravitational field strength at the central point Q of a homogeneous, thin hemispherical shell of radius R and mass m. What is the field strength at Q for a complete spherical shell?

5.22. Show that the gravitational field strength of a spherical Earth model with radius R and mass density $\rho = \rho(R)$ varies with the normal altitude h from its surface in accordance with the relation

$$\mathbf{g(X)} \equiv \hat{\mathbf{g}}(h) = \frac{\mathbf{g(R)}}{(1+h/R)^2}, \tag{P5.22}$$

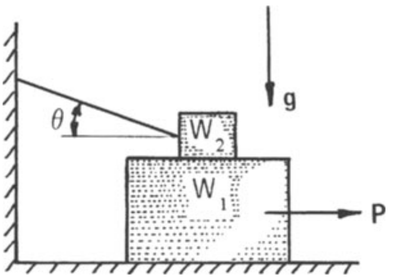
where $\mathbf{g(R)}$ denotes the field strength at the surface.

5.23. A homogeneous thin rod R_1 of length $2b$ and mass M is placed with its axis along the center line perpendicular to the axis of a similar rod R_2 of mass m and length $2a$, in the xy-plane. The center of R_1 is at $\mathbf{c} = c\mathbf{j}$ from the center of R_2. Determine the gravitational force that the rod R_2 exerts on R_1. (See Problem 5.19.) Tables of integrals may be needed.

5.24. A homogeneous, thin rod of length ℓ and mass m is positioned with its axis on the line through the center O and perpendicular to the plane of a homogeneous, thin circular disk of radius R and mass m. The end of the rod near the disk is at $\mathbf{a} = a\mathbf{k}$ from point O. Find the total gravitational force exerted on the rod by the disk. What gravitational force does the rod exert on the disk?

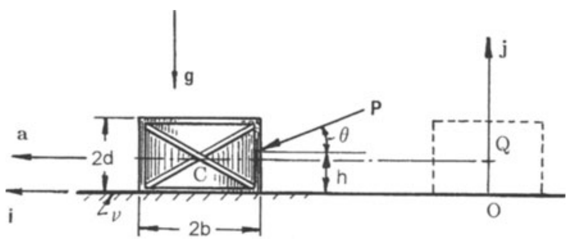
5.25. The moon has a mean diameter of about 2160 miles, while that of the Earth is roughly 7910 miles. The ratio of the mass of the moon to that of the Earth is about 3/250. What is the acceleration of gravity on or near the surface of the moon? Compare your weight relative to the Earth and the Moon.

5.26. Determine the gravitational force between two identical spheres of diameter d when they touch each other. What is the ratio of the magnitude W_o of their mutual attraction to the magnitude W of the attractive force exerted on each of them by the Earth? Evaluate the result for lead spheres with $d = 2$ ft and $\rho = 22.5$ slug/ft^3.



Problem 5.27.

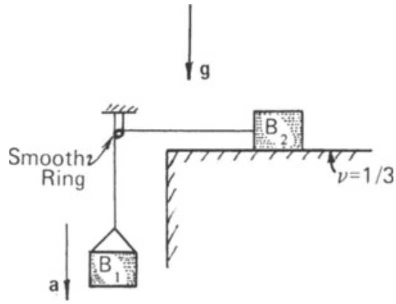
5.27. A block of weight W_1 supports a smaller block of weight $W_2 = \frac{1}{2}W_1$ constrained by a light wire inclined at an angle θ, as shown. (a) Find the horizontal force **P** required to just start the block of weight W_1 moving toward the right. (b) Find the tension in the cable after slip has occurred. Assume that all surfaces have the same coefficients of static and dynamic friction, and express the results in terms of $\tan\theta$.



Problem 5.28.

5.28. A homogeneous crate of mass m rests on a horizontal surface where the coefficient of dynamic friction is ν. (a) Find the magnitude of the inclined force **P** required to give the crate a constant acceleration **a** in the direction shown. (b) Apply Euler's second law (5.44) to find the distance from the center of mass to the line of action of the normal surface reaction force **N**. Do this in three ways. (i) Prove that $\mathbf{M}_Q = \mathbf{0}$ about a fixed point Q at the initial position of the center of mass of the crate, and thus solve for the location of **N**. (ii) Repeat the analysis for the torque $\mathbf{M}_O = \dot{\mathbf{h}}_O$ about a fixed point O in the contact plane at the initial position. (iii) Prove that the total torque $\mathbf{M}_C$ about the moving center of mass must vanish and thus locate the action line of **N**. (c) What is the critical angle θ_c for impending tip expressed in terms of assigned quantities only?

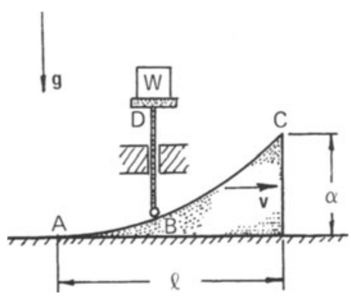
5.29. The wedge body $\mathscr{B}_1$ in Fig 5.18a, page 53, is accelerated at a constant rate a toward the right. The block $\mathscr{B}_2$ maintains contact with the plane throughout the motion. The gravitational force acts downward in the figure. Show that $\mathscr{B}_2$ will slide down the inclined surface if $a > g\tan(\alpha - \psi)$, where $\tan\alpha = \mu$ is the coefficient of static friction for the two surfaces and $\psi < \alpha$.



Problem 5.30.

5.30. The figure shows a block B_1 of weight W_1 attached by an inextensible cable to a block B_2 of weight W_2. The weight ratio $W_1/W_2 = 5/6$. The cable is supported by a smooth ring, and B_2 rests on a rough horizontal surface where $\mu = 2/5$ and $\nu = 1/3$. (a) Determine the critical weight ratio W_1/W_2 for which motion is imminent, and thus show that the system must move if released from rest. (b) Find the acceleration **a** of the block B_1 as a function of the weight ratio, and determine its value for the assigned data.

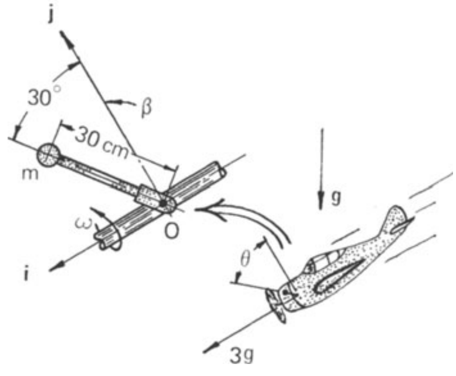
5.31. A body P of mass $m = 5$ slug has weight $\mathbf{W} = 160\mathbf{j}$ lb relative to the planet Φ. (a) Suppose that P is at rest on a scale in a nonrotating frame $\varphi = \{O; \mathbf{i}_k\}$ which has an acceleration $\mathbf{a}_O = 20\mathbf{j}$ ft/sec^2 relative to Φ. What is the weight of P apparent to an observer in φ? What is its apparent weight when φ has the opposite acceleration $\mathbf{a}_O = -20\mathbf{j}$ ft/sec^2 in Φ? Find the acceleration of φ for which the apparent weight of P vanishes. (b) Now suppose that P is dropped from a state of rest in Φ so that the only force that acts on P is its weight relative to Φ. Address the previous question for the observer in φ. (c) Discuss the results and compare the observations in φ with those in Φ.



Problem 5.32.

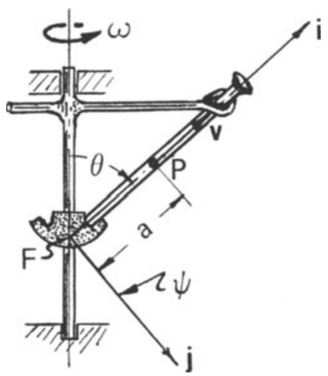
5.32. During an interval of interest, the vertical motion of a load W is controlled by a parabolic cam ABC that moves horizontally with a constant velocity **v** directed as shown. Draw a free body diagram of the block. Determine the compressive force in the push rod BD in terms of the load and the assigned quantities. Neglect friction.

5.33. A part in an aircraft engine consists of a 0.10 kg mass m attached by a 30 cm rod to the propeller drive shaft. The shaft turns, as shown, with an angular velocity $\boldsymbol{\omega} = 100\omega(t)\mathbf{i}$ rad/sec. During a dive, the aircraft accelerates at $3g$, and the rod is inclined at a fixed angle $\theta = 30°$ in the frame $\beta = \{O; \mathbf{i}_k\}$ fixed in the propeller shaft. Determine the total force acting on m.



Problem 5.33.

5.34. A test tube is held at a fixed angle θ in a centrifuge spinning, as shown, with a constant angular velocity $\boldsymbol{\omega}$ about a fixed vertical axis. A fluid particle of mass m, initially near the bottom at F, is moving outward in the tube with a constant relative velocity $\mathbf{v} = v\mathbf{i}$. Identify the time dependent variables, and determine as a function of time the total force that acts on P, referred to the tube frame $\psi = \{F; \mathbf{i}_k\}$.



Problem 5.34.

5.35. A system of three forces $\mathbf{F}_1 = 6\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$ N, $\mathbf{F}_2 = -2\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$ N, $\mathbf{F}_3 = 5\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$ N act at the respective points $\mathbf{x}_1 = (1, 0, 0)$ m, $\mathbf{x}_2 = (0, 1, 0)$ m, $\mathbf{x}_3 = (0, 0, 1)$ m in frame $\Phi = \{Q; \mathbf{i}_k\}$. (a) Find the equipollent system with force $\mathbf{F}^A = \mathbf{P}$ and torque $\mathbf{M}_Q^A$ with respect to Q. (b) Is $\mathbf{F}^A \cdot \mathbf{M}_Q^A = 0$? (c) Find the equations that describe the line of action of the single force $\mathbf{P}$. (d) Determine the center of force $\mathbf{x}_Q^*$ with respect to the origin Q. (e) Determine the center of force $\mathbf{x}_O^*$ with respect to the point O at $\mathbf{x}_1$ in Φ, and confirm your result by showing that $\mathbf{x}_O^* \times \mathbf{P} = \mathbf{M}_O^A$ for the original system of forces.