15 POLYHEDRAL APPROACHES TO THE DESIGN OF SURVIVABLE NETWORKS

Bernard Fortz[1] and Martine Labbé[2]

[1]Institut d'Administration et de Gestion
Université Catholique de Louvain
B-1348 Louvain-la-Neuve, Belgium
fortz@poms.ucl.ac.be

[2]Département d'Informatique
Université Libre de Bruxelles
1050 Bruxelles, Belgium
mlabbe@ulb.ac.be

Abstract: Long-term planning of backbone telephone networks has been an important area of application of combinatorial optimization over the last few years. In this chapter, we review polyhedral results for models related to these problems. In particular, we study classical survivability requirements in terms of k-connectivity of the network, then we extend the survivability model to include the notion of *bounded rings* that limit the length of the rerouting path in case of link failure.
Keywords: Network design, combinatorial optimization, branch-and-cut.

15.1 INTRODUCTION

Recently, the nature of services and the volume of demand in the telecommunication industry has changed drastically, with the replacement of analog transmission and traditional copper cables by digital technology and fiber optic transmission equipment. Moreover, we see an increasing competition among providers of telecommunication services, and the development of a broad range of new services for users, combining voice, data, graphics and video. Telecommunication network planning has thus become an important problem area for developing and applying optimization models.

Telephone companies have initiated extensive modeling and planning efforts to expand and upgrade their transmission facilities, which are, for most national telecommunication networks, divided in three main levels (see Balakrishnan et al. (1991)), namely,

- The *long-distance* or *backbone* network that typically connects city pairs through *gateway nodes*;
- The *inter-office* or *switching center* network within each city, that interconnects *switching centers* in different subdivisions (clusters of customers) and provides access to the gateway(s) node(s);
- The *local access* network that connects individual subscribers belonging to a cluster to the corresponding switching center.

These three levels differ in several ways including their design criteria. Ideally, the design of a telecommunication network should simultaneously account for these three levels. However, to simplify the planning task, the overall planning problem is decomposed by considering each level separately.

In this chapter, we study models and techniques for *long-term planning* in the first level of the hierarchy, i.e. the *backbone network.*

Planning in the backbone network is divided in two different stages : mid-term and long-term planning. Mid-term planning consists in dimensioning the network. More precisely, given a forecast of the demand matrix for this period and the current topology of the network, we have to compute how the expected demands will be routed as well as the necessary capacities of the cables. In some models, the addition of new edges is allowed. These problems involve, at the same time, survivable design criteria and routing constraints. A survey on these models can be found in De Jongh (1998).

Long-term planning involves a longer period of time so that demand data are not reliable enough, and we only deal with topological aspects. The goal is then to determine a set of cables connecting all nodes under some survivability criteria. In this context, the telephone network is seen as a given set of nodes and a set of possible fiber links that have to be placed between these nodes to achieve connectivity and survivability at minimum cost.

In traditional backbone networks, the limited capacity of copper cables resulted in highly diverse routing between offices. The developments in fiber-optic technology have led to components that are cheap and reliable, having an almost unlimited capacity. The introduction of such a technology has made hierarchical routing and bundling of traffic very attractive. This approach has resulted in sparse, even treelike network topologies with larger amounts of traffic carried by each link.

Two main issues appear in the planning process of fiber-optic networks: economy and survivability. Economy refers to the construction cost, which is expressed as the sum of the edge costs, while survivability refers to the restoration of services in the event of node or link failure. Trees satisfy the primary goal of minimizing the total cost while connecting all nodes. However, only one node or edge breakdown causes a tree network to fail in its main objective of enabling communication between all pairs of nodes.

This means that some survivability constraints have to be considered while building the network. Losing end-to-end customer service could lead to dramatic loss of revenue for commercial providers of telecommunication services. Constructing network topologies that provide protection against cable or office failures has become one of the most important problems in the field of telecommunications network design.

The most studied models deal with k-connectivity requirements, i.e. the ability to restore network service in the event of a failure of at most $k-1$ components of the network. Among them, the minimum-cost two-connected spanning network problem consists in finding a network with minimal total cost for which two node-disjoint paths are available between every pair of nodes. This means that two-connected networks are able to deal with a single link or node failure. two-connected networks have been found to provide a sufficient level of survivability in most cases, and a considerable amount of research has focused on so-called *low-connectivity constrained* network design problems, i.e. problems for which each node j is characterized by a requirement $r_j \in \{0,1,2\}$ and $\min\{r_i, r_j\}$ node-disjoint paths between every pair of nodes i, j are required. Section 15.3 presents a survey of the literature on these models.

Two-connectivity seems a sufficient level of survivability for most networks, since the probability of dealing with two simultaneous failures is very low. However, it turns out that the optimal solution of this problem is often a Hamiltonian cycle. Hence, any edge failure implies that the flow that passed through that edge must be rerouted, using all the edges of the network, an obviously undesirable feature. This led us to examine a new model for limiting the region of influence of the traffic which it is necessary to reroute : the *Two-Connected Network with Bounded Rings problem (2CNBR)*. This problem is studied in Section 15.4. In addition to the classical two-connectivity constraints, we require in this model that each edge belongs to at least one cycle (or *ring*) whose length is bounded by a given constant. It also finds its motivation in the emerging technology of *self-healing rings*. These are cycles in the network equipped in such a way that any link failure in the ring is automatically detected by the link end nodes and the traffic rerouted along the alternative path in the cycle. When such a strategy is chosen, rings must cover the network and their size is limited. These two requirements are fulfilled by our model.

In the case where edge lengths are equal to one, i.e. the edge (or node) cardinality of the rings is bounded, there exist more structural properties and polyhedral results. In particular, a lower bound on the number of edges in any feasible solution can be derived. These results are presented in Section 15.5. The chapter ends with a review of recent works on closely related models.

15.2 NOTATION AND DEFINITIONS

The aim in long-term planning of the backbone network is to determine a set of cables connecting given nodes and satisfying some survivability criteria that we will describe later. The given set of nodes and possible cable connections can be represented by an undirected graph $G = (V, E)$ where V is the set of *nodes* and E is the set of *edges* that represent the possible pairs of nodes between which a direct transmission link (cable) can be placed. The graph G may have parallel edges but should not contain loops. Graphs without parallel edges and without loops are called *simple*. If there exists an edge $e := \{i,j\}$ between two nodes i and j, these two nodes are called *adjacent*, and e is *incident* to i and j. Throughout this chapter, $n := |V|$ and $m := |E|$ will denote the number of nodes and edges of G.

Given the graph $G = (V,E)$ and $W \subset V$, the edge set

$$\delta(W) := \{\{i,j\} \in E \mid i \in W,\ j \in V \backslash W\}$$

is called the *cut* induced by W, and its size is given by $|\delta(W)|$. We write $\delta_G(W)$ to make clear – in case of possible ambiguities – with respect to which graph the cut induced by W is considered. For a single node $v \in V$, we denote $\delta(v) := \delta(\{v\})$. The *degree* of a node v is the cardinality of $\delta(v)$. The set

$$E(W) := \{\{i,j\} \in E \mid i \in W,\ j \in W\}$$

is the set of edges having both end nodes in W. We denote by $G(W) = (W, E(W))$ the subgraph induced by edges having both end nodes in W. If $E(W)$ is empty, W is an *independent set*. G/W is the graph obtained from G by contracting the nodes in W to a new node w (retaining parallel edges). Given two subsets of nodes W_1 and W_2, $W_1 \cap W_2 = \phi$, the subset of edges having one endpoint in each subset is denoted by

$$[W_1 : W_2] := \{\{i,j\} \in E \mid i \in W_1,\ j \in W_2\}.$$

We denote by $V - z := V \backslash \{z\}$ and $E - e := E \backslash \{e\}$ the subsets obtained by removing one node or one edge from the set of nodes or edges. $G - z$ denotes the graph $(V - z, E \backslash \delta(z))$, i.e. the graph obtained by removing a node z and its incident edges from G. This is extended to a subset $Z \subset V$ of nodes by the notation $G - Z := (V \backslash Z, E \backslash (\delta(Z) \cup E(Z)))$.

Each edge $e := \{i,j\} \in E$, has a *fixed cost* $c_e := c_{ij}$ representing the cost of establishing the direct link connection, and a *length* $d_e := d_{ij} := d(i,j)$. It is assumed throughout this work that these edge lengths satisfy the *triangle inequality*, i.e.

$$d(i,j) + d(j,k) \geq d(i,k) \quad \text{for all } i,j,k \in V.$$

The cost of a network $N = (V,F)$ where $F \subseteq E$ is a subset of possible edges is denoted by $c(F) := \sum_{e \in F} c_e$. The *distance* between two nodes i and j in this network is denoted by $d_F(i,j)$ and is given by the length of a shortest path linking these two nodes in F.

Without loss of generality, all costs are assumed to be nonnegative, because an edge e with a negative cost c_e will be contained in any optimum solution.

For any pair of distinct nodes $s,t \in V$, an *$[s,t]$-path* P is a sequence of nodes and edges $(v_0, e_1, v_1, e_2, \ldots, v_{l-1}, e_l, v_l)$, where each edge e_i is incident to the nodes v_{i-1} and v_i $(i = 1, \ldots, l)$, where $v_0 = s$ and $v_l = t$, and where no node or edge appears more than once in P. A collection $P_1, P_2, \ldots, P_k$ of $[s,t]$-paths is called *edge-disjoint* if no edge appears in more than one path, and is called *node-disjoint* if no node (other than s and t) appears in more than one path. A *cycle* (containing s and t) is a set of two node-disjoint $[s,t]$-paths.

A *Hamiltonian cycle* is a cycle using each node of the network exactly once. The problem of determining if a graph contains a Hamiltonian cycle is NP-complete. The corresponding optimization problem – the *traveling salesman problem (TSP)* – has been well studied. We refer to Lawler et al. (1985) for an in depth treatment of this problem.

A graph $G = (V, E)$ is *k-edge-connected* (resp., *k-node-connected*) if, for each pair s, t of distinct nodes, G contains at least k edge-disjoint (resp., node-disjoint) $[s,t]$-paths.

When the type of connectivity is not mentioned, we assume node-connectivity. The *edge connectivity* (resp., *node-connectivity*) of a graph is the maximal k for which it is k-edge-connected (resp., k-node-connected). A 1-edge-connected network is also 1-node-connected, and we call it simply *connected.* A cycle-free graph is a *forest* and a connected forest is a *tree*. A *connected component* of a graph is a maximal connected subgraph. If $G - e$ has more connected components than G for some edge e, we call e a *bridge*. Similarly, if Z is a node set and $G - Z$ has more connected components than G, we call Z an *articulation set* of G. If a single node forms an articulation set, the node is called *articulation point.*

Node and edge-disjoint $[s,t]$-paths are related to cuts and articulation sets by Menger's theorem (Menger, 1927).

Theorem 15.1 (Menger)

1. *In a graph $G = (V, E)$, there is no cut of size $k - 1$ or less disconnecting two given nodes s and t, if and only if there exist at least k edge-disjoint $[s,t]$-paths in G.*

2. *Let s and t be two nonadjacent nodes in G. Then there is no articulation set Z of size $k - 1$ or less disconnecting s and t, if and only if there exist at least k node-disjoint $[s,t]$-paths in G.*

We will also use the following definitions arising from polyhedral theory (see e.g. Nemhauser and Wolsey (1988)). Given a polyhedron P, the *dimension* $\dim(P)$ of P is defined as the maximum number of affinely independent elements in P minus one. An inequality $a^T x \leq \alpha$ is *valid* with respect to P if $P \subseteq \{x : a^T x \leq \alpha\}$. The set $F_a := \{x \in P : a^T x = \alpha\}$ is called the *face* of P defined by $a^T x \leq \alpha$. If $dim(F_a) = \dim(P) - 1$ and $F_a \neq \phi$, then F_a is a *facet* of P and $a^T x \leq \alpha$ is called *facet-inducing* or *facet-defining*. A vector x is a *vertex* of P if it cannot be written as a non-trivial convex combination of points in P.

The convex hull of a set of points S will be denoted by $\text{conv}(S)$. We also denote by e_i the i-th unit vector in $\mathbb{R}^n$.

15.3 LOW-CONNECTIVITY CONSTRAINED NETWORK DESIGN PROBLEMS

Throughout this chapter, a (backbone) telephone network is seen as a set of gateway nodes (or telephone offices) and fiber links that are placed between nodes. In this context, survivability refers to the restoration of services in the event of office or link failure, or, in other words, a network is survivable if there exists a prespecified number of node-disjoint or edge-disjoint paths between any two offices. The only costs considered are construction costs, like the cost of digging trenches and placing a fiber cable into service.

In this framework, a considerable amount of research has focused on low connectivity constrained network design problems. Following the terminology used by Monma and Shallcross (1989) and Stoer (1992), these models can be described informally as follows : we are given a set of telephone offices that have to be connected by a network. The offices may be classified according to importance, namely the

- *special offices*, for which a "high" degree of survivability has to be ensured in the network to be constructed;
- *ordinary offices*, which have to be simply connected to the network;
- *optional offices*, which may not be part of the network at all.

Given are also the pairs of offices between which a direct transmission link can be placed, and the associated cost of placing the fiber cable and putting it into service. The problem now consists in determining where to place fiber cables so that the construction cost, i.e. the sum of the fiber cable costs, is minimized and certain survivability constraints are ensured. For instance, we may require that

- the destruction of any single link may not disconnect any two special offices, or
- the destruction of any single office may not disconnect any two special offices.

These requirements are equivalent to ask that there exist

- at least two edge-disjoint paths, or
- at least two node-disjoint paths

between any two special offices.

Higher survivability levels may be imposed by requiring the existence of three or more paths between certain pairs of offices according to their importance class. However, up to now, low-connectivity requirements have been found to provide a sufficient level of survivability for telephone companies. For high-connectivity requirements, the reader is referred to Grötschel et al. (1995b); Stoer (1992).

In graph-theoretic language, the set of offices and possible link connections can be represented by an undirected graph $G = (V,E)$. The survivability requirement or importance of a node is modeled by node types. In particular, each node $s \in V$ has an associated nonnegative integer r_s, the *type* of s. Sometimes, we also write $r(s)$ instead of r_s. A network $N = (V,F)$, where $F \subseteq E$ is a subset of the possible links, is said to satisfy the *node-connectivity requirements*, if, for each pair $s,t \in V$ of distinct nodes, N contains at least

$$r(s,t) := \min\{r_s, r_t\}$$

node-disjoint $[s,t]$-paths.

Similarly, we say that N satisfies the *edge-connectivity requirements*, if, for each pair $s,t \in V$ of distinct nodes, N contains at least $r(s,t)$ edge-disjoint $[s,t]$-paths. If all node types have the same value k, it is equivalent to request that N is k-node-connected or k-edge-connected.

We restrict here to low-connectivity requirements, i.e. node types $r_s \in \{0,1,2\}$. Using our previous classification of nodes,

- special offices are represented by nodes of type 2,
- ordinary offices by nodes of type 1, and
- optional offices by nodes of type 0.

To shorten notation, we extend the type function r to sets by setting

$$\begin{aligned} r(W) &:= \max\{r_s \mid s \in W\} \text{ for all } W \subseteq V, \text{ and} \\ \operatorname{con}(W) &:= \max\{r(s,t) \mid s \in W, t \in V \backslash W\} \\ &= \min\{r(W), r(V \backslash W)\} \\ &\qquad \text{for all } W \subseteq V, \phi \neq W \neq V. \end{aligned}$$

We write $\operatorname{con}_G(W)$ to make clear with respect to which graph $\operatorname{con}(W)$ is considered.

In order to formulate network design problems as integer linear programs, we associate with every subset $F \subseteq E$ an *incidence vector* $\mathbf{x}^{\mathbf{F}} = (x_e^F)_{e \in E} \in \{0,1\}^{|E|}$ by setting

$$x_e^F := \begin{cases} 1 & \text{if } e \in F, \\ 0 & \text{otherwise}. \end{cases}$$

Conversely, each vector $\mathbf{x} \in \{0,1\}^{|E|}$ induces a subset

$$F^{\mathbf{x}} := \{e \in E \mid x_e = 1\}$$

of the edge set E. For any subset of edges $F \subseteq E$ we define

$$x(F) := \sum_{e \in F} x_e.$$

We can now formulate the connectivity constrained network design problem as the following integer linear program:

$$\begin{aligned} \min \quad & \textstyle\sum_{e \in E} c_e x_e \\ \text{s.t.} \quad & \\ x(\delta(W)) &\geq \operatorname{con}(W) & & W \subset V,\ \phi \neq W \neq V, & (15.1) \\ x(\delta_{G-z}(W)) &\geq \operatorname{con}_{G-z}(W) & & z \in V,\ W \subset V \backslash \{z\}, & \\ & & & \phi \neq W \neq V \backslash \{z\}, & (15.2) \\ x_e &\in \{0,1\} & & e \in E. & (15.3) \end{aligned}$$

It follows from Menger's Theorem that, for any feasible solution $\mathbf{x}$ of this program, the subgraph $N = (V, F^{\mathbf{x}})$ of G defines a network satisfying the node-connectivity requirements. Removing (15.2), we obtain an integer linear program for edge-connectivity requirements. Inequalities (15.1) are called *cut inequalities*, while inequalities (15.2) are called *node cut inequalities*.

The remainder of this section is devoted to a review of the work on these models, describing exact solution methods for more general or more specialized problems. Much of the material is taken from references cited in the surveys of Christofides and Whitlock (1981); Winter (1986b); Stoer (1992).

15.3.1 *Structural properties and particular cases*

A lot of research has focused on the survivability model presented in the previous section. The next sections survey these results. We begin by looking at the complexity of the problem, before considering some polynomially solvable cases. We also present work on particular cases, either with restricted connectivity requirements or restricted costs.

15.3.1.1 Complexity. The connectivity constrained network design problem is NP-hard in general. In particular :

- If $r_s \in \{0,1\}$, $\forall s \in V$, it reduces to the well-known NP-hard Steiner tree problem in networks. Winter (1987) made an in depth survey of these problems.
- If $r_s = 2$, $\forall s \in V$, it consists in determining a minimum cost two-connected network. This last problem is NP-hard even if the graph is complete and costs satisfy the triangle inequality, since with an algorithm for this problem, one could decide whether a graph has a Hamiltonian cycle by associating a cost equal to 1 to all graph edges and cost equal to 2 to all non-graph edges (see Eswaran and Tarjan (1976)).

However, for some particular connectivity requirements or costs, or when the underlying graph G is restricted, the problem may become polynomially solvable. We now review these cases.

15.3.1.2 Restricted connectivity requirements. By restricting the connectivity requirements r_s, the connectivity constrained network design problem reduces to some well-known polynomially solvable problems :

- If $r_s = 1$, $\forall s \in V$, the problem reduces to the minimum spanning tree problem. The most famous polynomial time algorithms for solving it are those from Kruskal (1956) and Prim (1957).
- If $r_s = 1$ for exactly two nodes of V and $r_s = 0$ for all the other nodes, the problem becomes a shortest path problem, solvable e.g. by the algorithms of Bellman (1958) or Dijkstra (1959).
- If $r_s = k$, $k \geq 2$, for exactly two nodes of V and $r_s = 0$ for all the other nodes, the problem becomes a k-shortest paths problem. This problem was studied by Suurballe (1974) and Suurballe and Tarjan (1984).
- If $r_s \in \{0,1\}$, $\forall s \in V$, the problem reduces to the Steiner tree problem in networks. This problem is NP-hard in general, but Lawler (1976) solved it in polynomial time in the case where either the number of nodes of type 0 or the number of nodes of type 1 is restricted.

15.3.1.3 Restricted costs. Under uniform or 0/1 costs, certain classes of connectivity constrained network design problems are polynomially solvable. We now examine these choices of costs.

Under uniform costs, the underlying graph G can be seen as a complete graph, and the problem turns into the construction of a sufficiently highly connected graph with a minimum number of edges. Chou and Frank (1970) solved this problem for edge connectivity requirements by producing a feasible graph where each node has degree r_s, except possibly for one node that has degree $r_s + 1$. Since these are the lowest possible degrees under the given connectivity requirements, the graph has the minimum number of edges. This proves the following lemma.

Lemma 15.1 *Given node types $r_s \geq 2$ for a set V of nodes, the minimum number of edges of a graph satisfying the edge-connectivity requirements given by r is*

$$\left\lceil \frac{1}{2} \sum_{s \in V} r_s \right\rceil .$$

The use of parallel edges is allowed in the construction.

Stoer (1992) describes a polynomial algorithm similar to that of Chou and Frank which also handles nodes of type 1. Frank and Chou (1970) also solved the problem when no parallel edges but extra nodes are allowed in the solution.

Unfortunately, to our knowledge, no general solution for the node connectivity version of the problem is available in the literature. But more can be said about uniform connectivity requirements $r_s = k$ for some $k \geq 2$.

An early work by Fulkerson and Shapley (1971) – written in 1961 but published ten years later – proved Lemma 15.1 for the edge-connectivity problem with uniform requirements, but without using parallel edges. Harary (1962) showed with the help of a polynomial algorithm that the same result holds for the node-connectivity problem with uniform requirements, leading to the following lemma.

Lemma 15.2 *Given $k \geq 2$ and $n \geq k+1$, the minimum number of edges in a k-node-connected graph on n nodes without parallel edges is*

$$\left\lceil \frac{kn}{2} \right\rceil .$$

Now, one may guess that Lemma 15.1 also holds for general node-connectivity requirements, but this conjecture is not true and a counter-example can be found in Stoer (1992).

We now turn to problems with 0/1 costs. These are known in the literature as augmentation problems, since these correspond to the problem of augmenting a graph $G = (V, E)$ by a minimum number of edges in $V \times V$, so that it meets connectivity requirements.

These augmentation problems were brought up by Eswaran and Tarjan (1976) for two-edge and two-node-connected graphs. Rosenthal and Goldner (1977) studied the augmentation to two-node-connected graphs. Their linear time algorithm contains an error that was corrected by Hsu and Ramachandran (1993), who also proposed a parallel implementation of their algorithm. Hsu and Ramachandran (1991) also developed a linear time algorithm for the augmentation to 3-node-connected networks.

The augmentation to k-edge-connected graphs was studied by Watanabe and Nakamura (1987), Ueno et al. (1988) and Cai and Sun (1989). The fastest known algorithm for this problem is the one by Naor et al. (1990). Frank (1992) solved the augmentation problem completely for general edge-connectivity requirements. All solution procedures allow the use of parallel edges, except those of Eswaran and Tarjan (1976) and Rosenthal and Goldner (1977). Again, the problem of augmentation to a node-connected graph is open in most cases.

15.3.1.4 Other polynomially solvable cases. Other cases of connectivity constrained network design problems are polynomially solvable if the underlying graph G is restricted to certain graph classes.

Among these, the class of *series-parallel graphs* has received a lot of attention. Series-parallel graphs are created from a single edge by two operations :

- addition of parallel edges,
- subdivision of edges by insertion of nodes.

Works on various connectivity requirements for these graphs can be found in the literature:

- $r_s \in \{0,1\}$, $\forall s \in V$ (Steiner tree problem):
  The problem was solved in linear time by Wald and Colbourn (1983). Goemans (1994) gave a complete description of the polytope associated with the solutions of the problem.

- $r_s = k$, $k \geq 2$, $\forall s \in V$, with edge connectivity requirements (k-edge-connected network problem):
  Mahjoub (1994) gave a complete description of the polytope associated with the solutions of the case when $k = 2$. This work was extended to any $k \geq 2$ by Didi Biha and Mahjoub (1996).

- $r_s \in \{0,2\}$, $\forall s \in V$ (two-connected Steiner subgraph problem):
  The problem was solved in linear time by Winter (1986a), both for edge and node-connectivity requirements. Coullard et al. (1991) gave a complete description of the polytope associated with the solutions of the node-connectivity case.

Winter has also developed linear-time algorithms for the case $r_s \in \{0,2\}$ in outerplanar (Winter, 1985b) and Halin graphs (Winter, 1985a). He also mentions in Winter (1987) that he solved the problem in linear time for $r_s \in \{0,3\}$ in Halin graphs. Coullard et al. solved the problem with $r_s \in \{0,2\}$ in W_4-free graphs (Coullard et al., 1993) and gave a complete description of the dominant of the corresponding polytope (Coullard et al., 1996).

Dominant of the polytopes of k-edge-connected networks where parallel edges are allowed were completely described by Cornuéjols et al. (1985) for k even and G series-parallel and by Chopra (1994) for k odd and G outerplanar.

15.3.2 Polyhedral studies and exact algorithms

Most Branch-and-Cut algorithms for these problems are based on the linear programming formulation (15.1)-(15.3). The first method for solving the problem exactly was developed by Christofides and Whitlock (1981). Their algorithm is based on the linear relaxation obtained by replacing integrality constraints (15.3) by $0 \leq x_e \leq 1$, and keeping only cut constraints (15.1) corresponding to subsets W such that $|W| = 1$. These particular cut constraints are called *degree constraints*. The starting linear program is thus

$$\begin{aligned} &\min \ \textstyle\sum_{e \in E} c_e x_e \\ &\text{s.t.} \\ &\quad x(\delta(v)) \geq r(v) \quad v \in V, \\ &\quad 0 \leq x_e \leq 1 \qquad e \in E. \end{aligned}$$

Given a solution to this LP, they impose the edge-connectivity requirements by adding violated cut constraints. These can be found in polynomial time by computing the minimum cut in the graph $G = (V,E)$ with edge capacities equal to the values of the corresponding variables in the solution of the current LP – using e.g. the Gomory-Hu algorithm (Gomory and Hu, 1961). When all cut constraints are satisfied, if some variables have fractional values, a branch-and-bound procedure is applied. Christofides and Whitlock (1981) mention that this algorithm is able to solve problems with "well over a hundred nodes" for edge-connectivity requirements.

If one wants to add node-connectivity requirements, they propose to check the node-connectivity each time an integer solution satisfying edge-connectivity requirements is found. If some node-connectivity requirements are violated, the corresponding node-cut constraints (15.2) are added to the LP.

Grötschel, Monma and Stoer studied in detail network design problems with connectivity constraints. A survey of their work can be found in (Grötschel et al., 1995a) and (Stoer, 1992).

In their earliest work on the subject, Grötschel and Monma (1990) introduced a general model mixing edge and node survivability requirements. They examined the dimension of the associated polytope and proved facet results for cut and node-cut inequalities.

They also described completely the polytope of the (1-)connected network problem, based on the work of Cornuéjols et al. (1985). This is done by the introduction of *partition inequalities*, that generalize cut inequalities (15.1). Given a partition $W_1, W_2, \ldots, W_p$ $(p \geq 2)$ of V into p nonempty subsets, the inequality

$$\frac{1}{2} \sum_{i=1}^{p} x(\delta(W_i)) \geq p - 1$$

is valid for the polytope of connected networks.

Based on partition inequalities for connected networks, Grötschel and Monma introduced the *node-partition inequalities* for k-node-connected networks. These inequalities come from the fact that the deletion of $k-1$ nodes from a k-node-connected network leaves a connected graph. Thus, if $Z \subseteq V$ is a node set with exactly $k-1$ nodes and $W_1, W_2, \ldots, W_p$ $(p \geq 2)$ is a partition of $V \backslash Z$ into p nonempty subsets, the inequality

$$\frac{1}{2}\sum_{i=1}^{p} x(\delta_{G-Z}(W_i)) \geq p-1$$

is valid for the polytope of k-node-connected networks. It is obvious that these node-partition inequalities are a generalization of node cut inequalities (15.2).

Grötschel, Monma and Stoer then attacked low-connectivity constrained problems (with $r_s \in \{0,1,2\}$), deriving new facets (Grötschel et al., 1992b) and implementing some of these into a Branch-and-Cut algorithm (Grötschel et al., 1992a). They generalized partition and node-partition inequalities, and introduced lifted 2-cover and comb inequalities. These results were extended to higher survivability requirements in Grötschel et al. (1995b).

More inequalities for two-edge-connected network problems were found by Boyd and Hao (1993) (complemented comb inequalities), and by Boyd and Zhang (1994) (clique tree inequalities). Baïou et al. (2000) and Kerivin and Mahjoub (2002) studied an extension of partition inequalities, the F-partition inequalities, first introduced by Mahjoub (1994), and showed these prove helpful for solving low survivability network design problems where edge-connectivity only is considered.

15.4 TWO-CONNECTED NETWORKS WITH BOUNDED RINGS

It turns out that the optimal solution of the two-connected network problem is often a Hamiltonian cycle. Hence, any edge failure implies that the flow that passed through that edge must be rerouted, using all the edges of the network, an obviously undesirable feature.

It is therefore necessary to add extra constraints to limit the region of influence of the traffic which is necessary to reroute if a connection is broken. Imposing a limit on the length of the rerouting can be done by limiting the length of the shortest cycle including each edge. Such a condition has also a direct implication in networks using the technology of *self-healing rings*. Self-healing rings are cycles in the network equipped in such a way that any link failure in the ring is automatically detected and the traffic rerouted by the alternative path in the cycle. It is natural to impose a limited length of these rings. This is equivalent to set a bound on the length of the shortest cycle including each edge.

The problem of designing a minimum *cost* network N with the following constraints:

1. The network N contains at least two node-disjoint paths between every pair of nodes (*2-connectivity constraints*),
   and

2. each edge of N belongs to at least one cycle whose *length* is bounded by a given constant K (*ring constraints*).

This problem is called the *Two-Connected Network with Bounded rings (2CNBR) problem*. It was first studied by Fortz et al. (2000). More polyhedral results can be found in (Fortz, 2000; Fortz and Labbé, 2002; 2004). Recently, Fortz et al. (2003a) studied the edge connectivity version of the problem.

A useful tool to analyze feasible solutions of 2CNBR is the *restriction of a graph to bounded rings*. Given a graph $G = (V, E)$ and a constant $K > 0$, we define for each subset of edges $F \subseteq E$ its restriction to bounded rings F_K as

$$F_K := \left\{ e \in F : \begin{array}{l} e \text{ belongs to at least one cycle} \\ \text{of length less than or equal to } K \text{ in } F \end{array} \right\}.$$

The subgraph $G_K = (V, E_K)$ is the *restriction of G to bounded rings*. Note that an edge $e \in E \backslash E_K$ will never belong to a feasible solution of 2CNBR.

Further we denote by $\mathcal{D}_{G,K}$ the set of incidence vectors x^F with $F \subseteq E$ such that

1. F is two-connected,

2. $F = F_K$.

Then, the 2CNBR problem consists in

$$\min \left\{ \sum_{e \in E} c_e x_e : x \in \mathcal{D}_{G,K} \right\}.$$

Checking that G_K is two-connected, i.e. that $\mathcal{D}_{G,K}$ is nonempty, can be done in polynomial time. We therefore assume in the remainder of this chapter that there always exists a feasible solution to the problem.

Since all costs c_e, $e \in E$ are assumed to be nonnegative, there always exists an optimal solution of 2CNBR whose induced graph is minimal with respect to inclusion. More precisely, if F_K is two-connected, as $F \supseteq F_K$, F is also two-connected and the cost of F is greater than or equal to the cost of F_K. We can thus relax the constraints and just require that F_K is two-connected for a set of edges F to be feasible. Hence, 2CNBR can be equivalently formulated as

$$\min \left\{ \sum_{e \in E} c_e x_e : x \in \{0,1\}^{|E|} \text{ and } F_K^x \text{ is two-connected} \right\}.$$

We denote by

$$\mathcal{P}_{G,K} := \text{conv}\{x \in \{0,1\}^{|E|} : F_K^x \text{ is two-connected}\}$$

the polyhedron associated to the 2CNBR problem.

Several formulations have been proposed for this problem. The first formulation using only design variables was proposed in Fortz and Labbé (2002). If a subset of edges $S \subseteq E$ is such that $(G - S)_K$ is not two-connected, then $G - S$ does not contain a feasible solution, and therefore each feasible solution contains at least one edge from S. As we are only interested in minimal feasible solutions, this is sufficient to formulate the 2CNBR problem as the following integer linear program :

$$\min \quad \sum_{e\in E} c_e x_e$$

s.t.

$$x(S) \geq 1 \qquad S \subseteq E,\ (G-S)_K \text{ is not two-connected,} \tag{15.4}$$

$$x_e \in \{0,1\} \qquad e \in E. \tag{15.5}$$

Constraints (15.4) are called *subset constraints.*

15.4.1 Cut and ring-cut inequalities

Fortz and Labbé (2002) studied under which conditions cut constraints (15.1) are facet-defining for 2CNBR. Given a subset of nodes $W \subseteq V$, $\emptyset \neq W \neq V$, the cut constraint imposes that there are at least two edges leaving W, i.e.

$$x(\delta(W)) \geq 2.$$

To characterize which cut constraints define facets, it is useful to know, for any pair of edges $e, f \in \delta(W)$, if there exists a vector of $\mathcal{P}_{G,K}$ lying in the face $x(\delta(W)) = 2$ whose corresponding graph contains e and f. This is the case if and only if the incidence vector of

$$C_{e,f} := E(W) \cup E(V \backslash W) \cup \{e, f\}$$

belongs to $\mathcal{P}_{G,K}$, i.e. if $(C_{e,f})_K$ is two-connected. A useful tool to represent and analyze the vectors belonging to the face defined by a cut constraint is the *ring-cut graph* defined below.

Definition 15.1 (Ring-cut graph)
Let $G = (V,E)$ be a graph, $K > 0$ a given constant, and $W \subseteq V$ a subset of nodes, $\emptyset \neq W \neq V$.
The ring-cut graph $RCG_{W,K} := (\delta(W), RCE_{W,K})$ induced by W is the graph defined by associating one node to each edge in $\delta(W)$ and by the set of edges

$$RCE_{W,K} = \left\{ \{e,f\} \subseteq \delta(W) : (C_{e,f})_K \text{ is two-connected} \right\}.$$

With the help of the ring-cut graph, we can characterize which cut constraints are facet-defining.

Theorem 15.2 *Let $G = (V,E)$ be a graph, $K > 0$ a given constant, and $W \subseteq V$ a subset of nodes, $\emptyset \neq W \neq V$. The inequality*

$$x(\delta(W)) \geq 2$$

defines a facet of $\mathcal{P}_{G,K}$ if and only if

1. *for all $e \in \delta(W)$, there exists $f \in \delta(W)$ such that $(C_{e,f})_K$ is two-connected;*

2. *in each connected component of $RCG_{W,K}$, there exists a cycle of odd cardinality;*

3. *for all* $e \in E(W) \cup E(V \setminus W)$, *there exist* $f, g \in \delta(W)$ *such that* $(C_{f,g} \setminus \{e\})_K$ *is two-connected.*

Moreover, Fortz and Labbé (2002) used the ring-cut graph to derive new valid inequalities. Let $G = (V, E)$ be a graph, $K > 0$ a given constant, $W \subseteq V$ a subset of nodes, $\emptyset \neq W \neq V$. If $S \subseteq \delta(W)$ is an independent subset in the ring-cut graph $RCG_{W,K}$, then

$$x(S) + 2x(\delta(W)\backslash S) \geq 3 \tag{15.6}$$

is a valid inequality for the 2CNBR problem. Inequalities (15.6) are called *ring-cut inequalities*. Fortz and Labbé (2002) also provide necessary conditions for these inequalities to be facet-defining.

15.4.2 Node-partition inequalities

In Section 15.3.2, we mentioned that node-partition inequalities (Grötschel and Monma, 1990) are valid for the two-connected network polytope.

Given a node $z \in V$ and a partition $W_1, W_2, \ldots, W_p$ $(p \geq 2)$ of $V\backslash\{z\}$, the node-partition inequality for two-connected networks is

$$\frac{1}{2}\sum_{i=1}^{p} x(\delta_{G-z}(W_i)) \geq p - 1.$$

Since $\mathcal{P}_{G,K}$ is included in this polytope, node-partition inequalities are also valid for the 2CNBR problem. Fortz and Labbé (2002) give sufficient conditions for node-partition inequalities to define facets of $\mathcal{P}_{G,K}$.

15.5 RINGS OF BOUNDED CARDINALITY

An important application of ring constraints appears in topologies using the recent technology of *self-healing rings*. Self-healing rings are cycles in the network equipped in such a way that any link failure in the ring is automatically detected and the traffic rerouted by the alternative path in the cycle. Due to technological constraints, the length of self-healing rings must be limited. This is equivalent to set a bound on the length of the shortest cycle including each edge. In practice, the length of the ring is computed as the number of *hops*, i.e., the number of nodes that compose the ring. This corresponds to the particular case of 2CNBR that arises when a unit length is given to each edge. This model is only a first step in solving the self-healing ring network design problem, as it only ensures the presence of feasible rings in the network. The next step is dimensioning the rings, taking into account the demands and the additional cost for inter-ring transfer. A heuristic for the self-healing ring network design problem was proposed by Fortz et al. (2003b).

In this section, we present additional properties for this particular case, coming from Fortz et al. (2003a) and Fortz and Labbé (2004). We first describe a new class of valid inequalities, the cycle inequalities, that can be used to provide an alternative formulation of this special case. Another important result is a lower bound on the number of edges in any feasible solution of 2CNBR. This result is useful for showing that the problem is NP-complete for any fixed $K \geq 3$ and for deriving new valid inequalities.

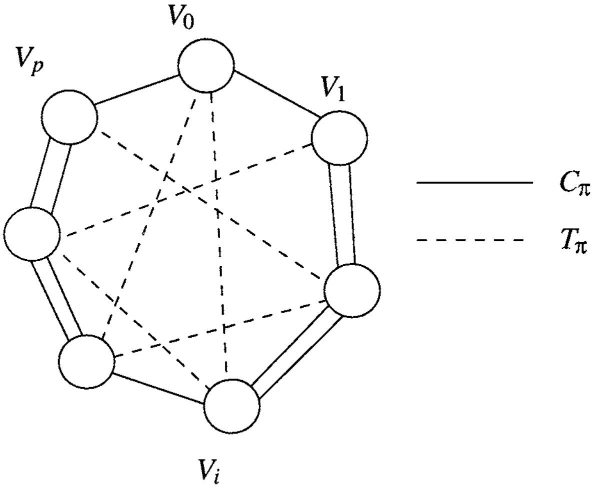

Figure 15.1 C_π and T_π

15.5.1 Cycle and metric inequalities

Let $G = (V,E)$ be a graph and $K \geq 3$. Let $\pi = (V_0,\ldots,V_p)$ be a partition of V such that $p \geq K$ and let $e \in [V_0,V_p]$. Moreover, let $C_\pi = \cup_{i=0}^{p-1}[V_i,V_{i+1}] \cup [V_0,V_p]$ and $T_\pi = \delta(V_0,\ldots,V_p) \setminus C_\pi$. Then, the inequality

$$x(T_\pi^e) \geq x_e \tag{15.7}$$

is valid for 2CNBR, with $T_\pi^e := T_\pi \cup ([V_0,V_p] \setminus \{e\})$, as illustrated in Figure 15.1. Inequalities (15.7) will be called *cycle inequalities*. Fortz et al. (2003a) showed that a formulation of 2CNBR is obtained by node-cut constraints, cycle inequalities and trivial inequalities.

Cycle inequalities are a special case of metric inequalities, that were studied by Fortz et al. (2000). Consider an edge $e := \{i,j\} \in E$ and a set of node potentials $(\alpha_k)_{k\in V}$ satisfying

$$\alpha_i - \alpha_j > K - 1.$$

Then

$$\sum_{f\in E-e} v_f x_f \geq x_e \tag{15.8}$$

is a valid inequality for $\mathcal{P}(G,K)$ where

$$v_f = \min\left(1, \max\left(0, \frac{|\alpha_l - \alpha_k| - 1}{\alpha_i - \alpha_j + 1 - K}\right)\right) \tag{15.9}$$

for all $f := \{k,l\} \in E - e$.

15.5.2 Cyclomatic inequalities

Theorem 15.3 *Let $G = (V,E)$ be a two-connected network with $n = |V|$ nodes and $m = |E|$ edges, such that there exists a covering of the network by cycles using at most K nodes. Then,*

$$m \geq M(n,K) := n + \min\left(\left\lceil \frac{n-K}{K-2} \right\rceil, \left\lceil \frac{n}{K-1} \right\rceil\right), \tag{15.10}$$

i.e., G contains at least $M(n,K)$ edges.

From this result, the complexity of the problem for K fixed can be established.

Problem 15.4 (R2CNBR) *Let $G = (V,E)$ be a graph, $K \geq 3$ a given constant and $B \geq 0$ an integer. To each edge $e \in E$ is associated a cost c_e and a unit length $d_e = 1$. Does there exists a subset $F \subseteq E$ of edges such that F_K is two-connected and $c(F) \leq B$?*

Theorem 15.5 *R2CNBR is NP-complete for any $K \geq 3$.*

Moreover, the result also applies to partitions of V, leading to a new class of valid inequalities:

Proposition 15.1 *Let $G = (V,E)$ be a graph with $n = |V|$ nodes, $K \geq 3$ a given constant, and $W_1, W_2, \ldots, W_p$ $(p \geq 2)$ a partition of V. Then*

$$\frac{1}{2}\sum_{i=1}^{p} x(\delta(W_i)) \geq M(p,K) \tag{15.11}$$

is a valid inequality for $\mathcal{P}_{G,K}$.

Inequalities (15.11) are called *cyclomatic inequalities*. The inequality bounding the total number of edges (i.e., $p = n$) is facet-defining for complete graphs.

15.6 RELATED HOP-CONSTRAINED MODELS

Other network design problems with limits on the lengths of paths in the network have been studied. In most of these models, there must exist a path between any pair of nodes, or between a given root and any other node, using at most L links (hops). The hop-constrained minimum spanning tree problem was studied by Gouveia (1996); Gouveia and Magnanti (2003). Shortest paths with hop constraints have also received attention. The L-path polytope – the convex hull of incidence vectors of st-paths with no more than L edges – was first studied by Dahl (1999). Recently, Nguyen (2003) gave a complete description of this polytope. The directed version of the problem was studied by Dahl and Gouveia (2004); Dahl et al. (2004).

The hop-constrained network design problem (HCNDP) consists in finding at minimum cost a subgraph such that each pair of terminals is connected by at least K edge-disjoint paths using at most L links, where K and L are fixed constants. Balakrishnan and Altinkemer (1992) studied the problem for $K = 1$ within the framework of a more general model. The case $K = 1$ and $L = 2$ was considered by Dahl and Johannessen (2004). Huygens et al. (2004) consider a single pair of terminals with $K = 2$ and $L = 3$, and provide a complete description of the associated polytope.

Bibliography

M. Baïou, F. Barahona, and A.R Mahjoub. Separation of partition inequalities. *Mathematics of Operations Research*, 25:243–254, 2000.

A. Balakrishnan and K. Altinkemer. Using a hop-constrained model to generate alternative communication network design. *ORSA Journal on Computing*, 4(2):192–205, 1992.

A. Balakrishnan, T.L. Magnanti, A. Shulman, and R.T. Wong. Models for planning capacity expansion in local access telecommunication networks. *Annals of Operations Research*, 33:239–284, 1991.

R.E. Bellman. On a routing problem. *Q. Appl. Math.*, 16:87–90, 1958.

S.C. Boyd and T. Hao. An integer polytope related to the design of survivable communication networks. *SIAM J. Discrete Math.*, 6(4):612–630, 1993.

S.C. Boyd and F. Zhang. Transforming clique tree inequalities to induce facets for the 2-edge connected polytope. Technical Report TR-94-13, Department of Computer Science, University of Ottawa, 1994.

G.-R. Cai and Y.-G. Sun. The minimum augmentation of any graph to a k-edge-connected graph. *Networks*, 19:151–172, 1989.

S. Chopra. The k-edge-connected spanning subgraph polyhedron. *SIAM J. Discrete Math.*, 7(2):245–259, 1994.

W. Chou and H. Frank. Survivable communication networks and the terminal capacity matrix. *IEEE Transactions on Circuit Theory*, CT-17:192–197, 1970.

N. Christofides and C.A. Whitlock. Network synthesis with connectivity constraints — a survey. In J.P. Brans, editor, *Operational Research '81*, pages 705–723. North-Holland Publishing Company, 1981.

G. Cornuéjols, F. Fonlupt, and D. Naddef. The traveling salesman problem on a graph and some related integer polyhedra. *Mathematical Programming*, 33:1–27, 1985.

C.R. Coullard, A. Rais, R.R. Rardin, and D.K. Wagner. The 2-connected-Steiner-subgraph polytope for series-parallel graphs. Technical Report CC-91-32, Purdue University, 1991.

C.R. Coullard, A. Rais, R.R. Rardin, and D.K. Wagner. The dominant of the 2-connected-Steiner-subgraph polytope for W_4-free graphs. *Discrete Applied Mathematics*, 66:195–205, 1996.

C.R. Coullard, A. Rais, D.K. Wagner, and R.L. Rardin. Linear-time algorithms for the 2-connected Steiner subgraph problem on special classes of graphs. *Networks*, 23, 1993.

G. Dahl. Notes on polyhedra associated with hop-constrained paths. *Operations Research Letters*, 25:97–101, 1999.

G. Dahl and L Gouveia. On the directed hop-constrained shortest path problem. *Operations Research Letters*, 32:15–22, 2004.

G. Dahl and B. Johannessen. The 2-path network problem. *Networks*, 43:190–199, 2004.

G. Dahl, Foldnes N., and L Gouveia. A note on hop-constrained walk polytopes. *Operations Research Letters*, 32:345–349, 2004.

A. De Jongh. *Uncapacitated network design with bifurcated routing*. PhD thesis, Université Libre de Bruxelles, 1998.

M. Didi Biha and A.R. Mahjoub. *k*-edge connected polyhedra on series-parallel graphs. *Operations Research Letters*, 19:71–78, 1996.

E.W. Dijkstra. A note on two problems in connection with graphs. *Numer. Math.*, 1: 269–271, 1959.

K.P. Eswaran and R.E. Tarjan. Augmentation problems. *SIAM Journal on Computing*, 5:653–665, 1976.

B. Fortz. *Design of Survivable Networks with Bounded Rings*, volume 2 of *Network Theory and Applications*. Kluwer Academic Publishers, 2000.

B. Fortz and M. Labbé. Polyhedral results for two-connected networks with bounded rings. *Mathematical Programming*, 93(1):27–54, 2002.

B. Fortz and M. Labbé. Two-connected networks with rings of bounded cardinality. *Computational Optimization and Applications*, 27(2):123–148, 2004.

B. Fortz, M. Labbé, and F. Maffioli. Solving the two-connected network with bounded meshes problem. *Operations Research*, 48(6):866–877, 2000.

B. Fortz, A.R. Mahjoub, S.T. Mc Cormick, and P. Pesneau. Two-edge connected subgraphs with bounded rings: Polyhedral results and branch-and-cut. IAG Working

Paper 98/03, Université Catholique de Louvain, 2003a. To appear in *Mathematical Programming*.

B. Fortz, P. Soriano, and C. Wynants. A tabu search algorithm for self-healing ring network design. *European Journal of Operational Research*, 151(2):280–295, 2003b.

A. Frank. Augmenting graphs to meet edge-connectivity requirements. *SIAM J. on Discrete Mathematics*, 5(1):22–53, 1992.

H. Frank and W. Chou. Connectivity considerations in the design of survivable networks. *IEEE Transactions on Circuit Theory*, CT-17:486–490, 1970.

D.R. Fulkerson and L.S. Shapley. Minimal k-arc connected graphs. *Networks*, 1: 91–98, 1971.

M.X. Goemans. Arborescence polytopes for series-parallel graphs. *Discrete Applied Mathematics*, 51:277–289, 1994.

R.E. Gomory and T.C. Hu. Multi-terminal network flows. *SIAM J. Appl. Math.*, 9: 551–570, 1961.

L Gouveia. Multicommodity flow models for spanning trees with hop constraints. *European Journal of Operational Research*, 95:178–190, 1996.

L. Gouveia and T.L. Magnanti. Network flow models for designing diameter-constrained minimum-spanning and steiner trees. *Networks*, 41(3):159–173, 2003.

M. Grötschel and C.L. Monma. Integer polyhedra arising from certain design problems with connectivity constraints. *SIAM J. Discrete Math.*, 3:502–523, 1990.

M. Grötschel, C.L. Monma, and M. Stoer. Computational results with a cutting plane algorithm for designing communication networks with low-connectivity constraints. *Operations Research*, 40(2):309–330, 1992a.

M. Grötschel, C.L. Monma, and M. Stoer. Facets for polyhedra arising in the design of communication networks with low-connectivity constraints. *SIAM J. Optimization*, 2(3):474–504, 1992b.

M. Grötschel, C.L. Monma, and M. Stoer. *Design of Survivable Networks*, volume 7 on Network models of *Handbooks in OR/MS*, chapter 10, pages 617–672. North-Holland, 1995a.

M. Grötschel, C.L. Monma, and M. Stoer. Polyhedral and computational investigations for designing communication networks with high survivability requirements. *Operations Research*, 43(6):1012–1024, 1995b.

F. Harary. The maximum connectivity of a graph. In *Proceedings of the National Academy of Sciences*, volume 48, pages 1142–1146, USA, 1962.

T.-S. Hsu and V. Ramachandran. A linear time algorithm for triconnectivity augmentation. In *Proc. 32nd Annual IEEE Symposium on Foundations of Computer Science*, pages 548–559, 1991.

T.-S. Hsu and V. Ramachandran. On finding a minimum augmentation to biconnect a graph. *SIAM Journal on Computing*, 22:889–891, 1993.

D. Huygens, A.R. Mahjoub, and P. Pesneau. Two edge-disjoint hop-constrained paths and polyhedra. *SIAM Journal on Discrete Mathematics*, 18(2):287–312, 2004.

H. Kerivin and A.R. Mahjoub. Separation of the partition inequalities for the (1,2)-survivable network design problem. *Operations Research Letters*, 30:265–268, 2002.

J.B. Kruskal. On the shortest spanning subtree of a graph and the traveling salesman problem. *Proc. Amer. Math. Soc.*, 7:48–50, 1956.

E.L. Lawler. *Combinatorial Optimization : Networks and Matroids*. Holt, Rinehart and Wilson, New-York, 1976.

E.L. Lawler, J.K. Lenstra, A.H.G. Rinnooy Kan, and D.B. Shmoys, editors. *The Traveling Salesman Problem*. John Wiley & Sons, New-York, 1985.

A.R. Mahjoub. Two-edge connected spanning subgraphs and polyhedra. *Mathematical Programming*, 64:199–208, 1994.

K. Menger. Zur allgemeinen kurventheorie. *Fundamenta Mathematicae*, 10:96–115, 1927.

C.L. Monma and D.F. Shallcross. Methods for designing communications networks with certain two-connected survivability constraints. *Operations Research*, 37(4): 531–541, 1989.

D. Naor, D. Gusfield, and Ch. Martel. A fast algorithm for optimally increasing the edge-connectivity. In *Proceedings of the Foundation of Computer Science '90*, pages 698–707, St. Louis, 1990.

G.L. Nemhauser and L.A. Wolsey. *Integer and combinatorial optimization*. Wiley-Interscience series in discrete mathematics and optimization. Wiley, 1988.

V.-H Nguyen. A complete description for the k-path polyhedron. Technical report, LIP6, 2003.

R.C. Prim. Shortest connection networks and some generalizations. *Bell System Tech. J.*, 36:1389–1401, 1957.

A. Rosenthal and A. Goldner. Smallest augmentation to biconnect a graph. *SIAM Journal on Computing*, 6:55–66, 1977.

M. Stoer. *Design of Survivable Networks*, volume 1531 of *Lecture Notes in Mathematics*. Springer-Verlag, 1992.

J.W. Suurballe. Disjoint paths in a network. *Networks*, 4:125–145, 1974.

J.W. Suurballe and R.E. Tarjan. A quick method for finding shortest pairs of disjoint paths. *Networks*, 14:325–336, 1984.

S. Ueno, Y. Kajitani, and H. Wada. Minimum augmentation of a tree to a k-edge-connected graph. *Networks*, 18:19–25, 1988.

J.A. Wald and C.J. Colbourn. Steiner trees, partial 2-trees, and minimum ifi networks. *Networks*, 13:159–167, 1983.

T. Watanabe and A. Nakamura. Edge-connectivity augmentation problems. *Computer and System Sciences*, 35:96–144, 1987.

P. Winter. Generalized Steiner problem in halin networks. In *Proc. 12th International Symposium on Mathematical Programming*. MIT, 1985a.

P. Winter. Generalized Steiner problem in outerplanar networks. *BIT*, 25:485–496, 1985b.

P. Winter. Generalized Steiner problem in series-parallel networks. *Journal of Algorithms*, 7:549–566, 1986a.

P. Winter. Topological network synthesis. In B. Simeone, editor, *Combinatorial Optimization - Como 1986*, volume 1403 of *Lecture Notes in Mathematics*, pages 282–303. Springer-Verlag, 1986b.

P. Winter. Steiner problems in networks : a survey. *Networks*, 17:129–167, 1987.