Chapter 2.4

WHAT KNOWLEDGE DO TEACHERS NEED FOR TEACHING MATHEMATICS THROUGH APPLICATIONS AND MODELLING?

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Abstract: This paper begins by describing teachers' knowledge as the creation and development of increasingly sophisticated models or ways of interpreting the tasks of teaching. One study illuminates several ways that pre-service teachers perceive the processes of modelling and the limits of their experiences with stochastic models. Results from a second study indicate that teachers need to have a broad and deep understanding of the diversity of approaches that students might take with modeling tasks. The second study also suggests a reversal in the usual roles of teachers and students by engaging students as evaluators of models.

1. INTRODUCTION

The call for contributions to the ICMI *Study on Applications and Modelling in Mathematics Education* observes that only rarely do mathematics teacher education programs include an orientation to mathematical modelling or the use of modelling in prospective teachers' mathematics courses. This suggests that one reason for the limited use of applications and modelling at the primary and secondary levels of schooling is the lack of knowledge by those who are expected to teach mathematics through applications and modelling. However, the research base on the knowledge needed for teaching, at least in the United States, has established that subject matter knowledge alone, while necessary, is insufficient for quality teaching. This raises the issue, then, as to the scope of the knowledge that teachers need in order to be effective in using applications and modelling in their practice. As

I will argue below, the pedagogical knowledge for teaching modelling would appear to differ in some significant ways from traditional and reform-based methods for teaching mathematics.

In this paper, I will frame a discussion of the issues that applications and modelling raise for the mathematics education community when we focus our attention on teachers and teaching. First, I will describe a theoretical perspective on the nature and the development of teachers' knowledge. Second, I will provide results from my research on several aspects of the subject matter knowledge of pre-service teachers within the context of an undergraduate course in mathematical modelling. Third, I will provide an analysis of an example from a research project on teachers' pedagogical knowledge when teaching mathematics through modelling tasks. This example, drawn from the practice of an experienced secondary school teacher, illuminates the kinds of pedagogical knowledge that seem to be necessary when teaching from a modelling perspective. I will conclude with some comments about the challenges that this research raises about the knowledge that teachers need when teaching mathematics through applications and modelling.

2. THE NATURE OF TEACHERS' KNOWLEDGE

The starting point for conceptualizing the nature and development of the knowledge needed to teach mathematics through modelling and applications is that teaching is primarily about the creation and refinement of sophisticated models or ways of interpreting the tasks of teaching. These tasks include choosing appropriate modelling applications for students, knowing how students' models might develop over the course of several lessons or several applications, selecting activities and curricular materials that might further that development, and devising strategies for engaging students in the critical assessment of their models.

A modelling perspective on teachers' knowledge foregrounds the notion that teachers have models for teaching (Doerr & Lesh, 2003). These models are the systems of interpretation that teachers use to see students' ways of thinking, to respond to students' ideas, to differentiate the nuances of contexts in their practice, to see generalized understandings that cut across contexts, and to revise their own thinking in light of their experiences. In examining teachers' knowledge, we focus on how the teacher thinks about the context, what alternatives she considers, what purposes she has in mind, what elements of the situation she attends to and what meanings and relationships those elements have for her.

A central question for research on teacher knowledge is the examination of how teachers' models for teaching mathematics develop. It is clear that teachers come to their pre-service teacher programs with models of teaching already in place, based on years of apprenticeship as observers of practice. Furthermore, teachers' models of practice (or systems for interpreting practice) are significantly broader in scope and more complex than the kinds of models students develop. The results from two research projects that examined the subject matter knowledge of pre-service teachers and the complexities of the pedagogical knowledge of an experienced teacher illuminate some of the central characteristics of teachers' models for interpreting practice and provide some insight into the challenges inherent in the development of such models.

3. SUBJECT MATTER KNOWLEDGE IN PRE-SERVICE TEACHER EDUCATION

Few studies have directly addressed the knowledge of mathematical modelling that pre-service and in-service teachers hold and how that knowledge is acquired (e.g., Dugdale, 1994; Lingefjard, 2002; Zbiek, 1998). To examine the modelling knowledge of those preparing to teach, I designed an undergraduate course in mathematical modelling. The primary goals of the course were to introduce pre-service teachers (N=8) to some basic ideas and techniques in mathematical modelling by engaging them in the process of building mathematical models. The course content drew on problem situations from physics, biology, and mathematics itself. The course began with several empirical models and then moved to an analysis of discrete dynamical systems and stochastic models. We finished the course with some examples of continuous models. The technological tools included graphing calculators and calculator probes for data collection, Maple, spreadsheets, and a simple dynamic systems simulation language. The students worked in small groups and completed five modelling projects over the course of the semester. Several classes devoted time for students to work collaboratively on the projects and to present their findings.

The students' class work, their final projects, class discussions, and written assignments were the data corpus for this research study. The research questions focused on examining the nature of pre-service teachers' knowledge and perceptions about mathematical modelling. The analysis of the data yielded three significant findings. The first finding related to the mathematical knowledge of the pre-service teachers with respect to probabilistic situations. A serious misconception about binomial distributions and the probabilities of independent events occurred among the pre-service teachers in the same ways that I have found among secondary school students (Doerr, 2000). In particular, when creating a simulation for stochastic exponential growth using a random number generator, several pre-service teachers erroneously used a random number from a uniform distribution as an appropriate number in a context that called for a binomial distribution. A subsequent project involved creating a simulation for a stochastic logistic growth situation. In this context, the need for a random number from a binomial distribution was even less obvious; nearly all of the students made the error of choosing a random number from a uniform distribution. This finding confirmed results in the literature that would suggest that formal instruction in probability has limited impact on learners' abilities to reason probabilistically. However, it was also the case that all of the students were able to adjust their incorrect conceptions to mathematically correct ones through a process of explaining and justifying their models to each other. This suggests that mathematical modelling is a potentially powerful context for the mathematics learning of pre-service teachers.

The second finding directly addresses the perceptions and beliefs held by the pre-service teachers as to the nature of the modelling process. As part of the course, the students completed several readings that discussed modelling at a meta-level (Bassanezi, 1994; Weigand & Weller, 1998). Weigand and Weller (1998) present a description of modelling that involves a six-step process: analyzing (A), simulating (S), modelling with equations (M), working experimentally (W), interpreting (I), and explaining (E). Throughout the course, the pre-service teachers were asked to describe their own specific modelling processes in terms of these steps. Initially, the pre-service teachers saw Weigand and Weller's steps in the modelling process as an unproblematic description of how modelling was really done. They saw the steps as occurring in sequence. Early in the course, when asked to map the processes they had used to create a model, most students created maps similar to that in Fig. 2.4-1.

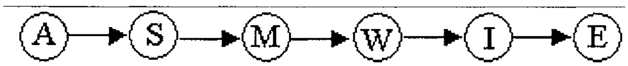

Figure 2.4-1. A sequential view of the modelling process

But later in the course, the pre-service teachers made a striking shift from seeing modelling as a fairly linear, sequential activity to seeing modelling as a non-linear, cyclic activity. A typical student map of the modelling process now looked more like the one shown in Fig. 2.4-2. In this case, the students saw their process as beginning with the simulation step (S) and then moving to interpreting (I), then modelling with equations (M) and so on.

The pre-service teachers engaged in extended discussions about the meaning of the terms that were used to identify each of these modelling steps. They began to give more nuanced meanings to the steps, describing their activities as "thinking about what is going on in the situation", "working and re-working the math equations to get them right", and asking questions such as "does everything we're doing make sense?" and "why do our ideas work?" This shift in the perceptions of the pre-service teachers came about as they reflected upon their experiences in developing models.

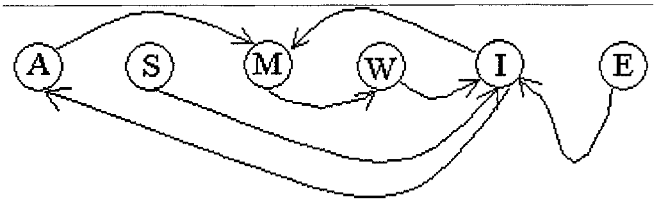

Figure 2.4-2. A non-linear, cyclic view of the modelling process

The third finding from the analysis contradicts the findings of Zbiek (1998), who in her study of pre-service secondary teachers found that many of the pre-service teachers tended to use regression analyses when available and that they often used curve-fitting uncritically in their approach to problem situations. We found no evidence to confirm these tendencies. Even though curve-fitting software was readily available, students rarely used it and when curve fitting was done, students always attended to the meaning of the resulting equations and coefficients in the problem tasks. This result does not suggest that these pre-service teachers were more sophisticated than those in Zbiek's study, but rather it argues that the nature of the modelling tasks, the range of tools available, the norms for argumentation, and the standards for quality of a solution were significant in influencing the types of modelling behavior that occurred in each setting.

Collectively, these findings suggest that pre-service teachers are likely to encounter greater difficulties in developing stochastic models than in developing models of deterministic phenomena. This result can in part be accounted for by the well-known misconceptions from the research on probabilistic reasoning and by the dominance in mathematics courses of continuous functions and their applications in physics. It leaves unanswered, however, questions about how to best approach the development of both kinds of models. The findings also suggest that by reflecting on their own modelling activity, pre-service teachers can come to understand the cyclic nature of the modelling process and appreciate the interconnectedness of the cognitive activities involved in the process. Finally, these results suggest that the use of regression models by pre-service teachers seems to be dependent on the kinds of modelling activities that they experience. This implies that pre-service teachers need to be exposed to a range of modelling activities that provide multiple opportunities for explanations and justifications of the modelling decisions that were made.

4. PEDAGOGICAL KNOWLEDGE IN ACTION

This set of results is drawn from the analysis of a teaching episode with an experienced secondary mathematics teacher who was using a sequence of modelling tasks related to exponential growth and decay. My intention in the analysis of the classroom data is to illustrate some of the pedagogical demands that are made on the teacher when a modelling approach is taken to the teaching and learning of mathematics. In this particular lesson with 16 – 17 year old students in a pre-calculus class, the students had been working on a task to model the doubling that occurs in bacteria growth. Finding appropriate graphs, equations, and tables to represent that growth was relatively straightforward for this group of students. One portion of the task focused the students' attention on the problematic issue of quantifying the growth rate and what units might be used to measure that rate:

> A biologist knows that the population of a bacteria culture doubles every 15 minutes. After 1 hour and 15 minutes, her assistant found that 80,000 bacteria were present.
>
> Examine the rate at which the bacteria culture is growing. How fast is the culture growing at 1 hour? At 1.5 hours? At 2 hours? How are you making these estimates? What are the units for this rate? Do your estimates make sense in terms of your graph?

The teacher, who had used these modelling tasks the previous year, knew that examining the rate of change would be problematic and challenging for her students. She recognized that the notion of rate of change was an important idea throughout the pre-calculus course. Understanding the changes in a model and ways of representing that change is a fundamental mathematical idea and one that foreshadows the development of important concepts in calculus. The teacher had chosen to focus on this particular aspect of the bacteria growth model because of the richness of the rate context.

Several groups of students presented their work on the board, including tables, graphs, and equations. The discussion of these solutions started out slowly with some comments on the tables and the different units for the rates and some comments on the functions, which were different as well. But the most interesting discussion occurred as the students talked about the rate of change. The teacher was able to pull in many student-to-student arguments as well as many elaborated student explanations. She was careful in listening and seeing how the students elaborated their rate concepts as the discussion evolved. The students had four different ways of presenting rates:

(1) Sara's method: Sara found the bacteria present at 1 hour. Her equation was $y = 2500 \cdot 2^{4x}$ where x was in hours. Her explanation of the "4" was that it took four quarters to double and hence four of these quarters

("4x") would give you one doubling. She then found the bacteria present at 1.00001 hours and divided the increase in the bacteria by .00001 and called this quantity "bacteria per hour". She then calculated the rate at 1.5 hours and 2.0 hours and insightfully observed that the rate itself is also increasing by a doubling factor! In her arguments in class, Sara pointed out that rate could be thought of as the slope of "the little line segments between the points" of the graph.

(2) Bryan's method: Bryan took the amount of bacteria present at 60 minutes and divided it by 60, yielding 40,000/60 bacteria per minute. His equation was $y = 2500 \cdot 2^{x/15}$. Bryan was adamant that his equation and his estimate for the rate were correct! Bryan said that he still didn't see what was wrong with his approach. This seemed to be both a need for resolution of multiple methods and a need to reconcile his view with the other competing views in class. The teacher made the decision to continue with the discussion. This brought Jack (another student) to Bryan's side, and he led other students to try to see Bryan's point of view. Sara and others appeared to appreciate what Bryan was saying but weren't convinced that it was correct, however they had difficulty in explaining a flaw in the reasoning. The graphic representation of Bryan's estimate could be seen as the slope of the line joining the point (60, 40000) and the origin. The teacher drew this segment on the graph as the discussion evolved.

(3) Peter's method: Peter found the bacteria present at 1 hour and then said since it is 40,000, that you should divide the 40,000, since that is also the amount that it will increase in the next interval, by 15 to get the rate of increase. The teacher was initially unclear about how Peter was finding the rate. It appeared that the 40,000, which was the amount of bacteria present at 1 hour, was being divided by the time interval. But it was clear that the student was thinking that the 40,000 was both the amount and the increase in the amount, and hence you could divide it by the time interval and get the rate. The teacher re-cast Peter's description into the language of the change in the amount of bacteria divided by the change in the time interval and wrote (80,000 – 40,000)/15. Later the teacher commented that last year, several students had taken this approach and she had had trouble grading their papers because the students had not made clear how they were thinking about the quantity.

(4) Mark's method: Mark had written $y = 2500 \cdot 2^{2x}$ as his equation. The discussion of his solution focused on the rate at 1.5 hours, or the 6th time interval. Mark had used x to represent the number of 15 minute time intervals, rather than the actual time in hours or minutes (as had been done by the other students). Mark changed the table interval on his calculator

to .001 and found 160,111 bacteria at 6.001. Mark had written the rate as (160,111 – 160,000)/.001 and then described the rate "as per 15 minute interval." The teacher asked, "How are we getting the 15 minutes?" Mark replied that he was using time as 15 minute intervals.

During the discussion of Mark's method, Peter commented that this was just finding the slope between two points. Later, Mark argued that if we thought of the graph of the bacteria population as a position graph, then what we are trying to find is its velocity graph. The teacher quickly picked up on this as the connection to early work that the students had done with a simulation environment (Kaput & Roschelle, 1997) in exploring the relationship between a velocity and a position graph. She then asked, "How do we find a velocity graph from a position graph?" and the students answered, "by finding the slope." Bryan however stayed strong in his position by arguing that he was finding the slope at a point and asked, "what does that mean?" and "why can't I do it that way?" As class ended, Bryan and Mark continued to argue this point. After class, the teacher indicated that she wanted to have the students "commit" to their ideas and to think about the concept of rate, before pursuing it further in class. In this way, the teacher saw how a central concept such as rate of change is not understood "all at once" but is revisited through a sequence of modelling tasks.

This teaching episode suggests two major implications for the pedagogical knowledge of the teacher when teaching with modelling tasks. First, the teacher needs to have a broad and deep understanding of the diversity of approaches that students might take. Trying to quickly grasp the mathematics presented in the four approaches described above, while simultaneously devising appropriate responses, is not an easy task for the teacher. The difficulties in doing this should not be underestimated. To acquire such understanding, the teacher must engage in listening to the students as they interpret and explain their models. In the case above, the teacher recognized the ambiguity in how one student (Peter) was finding the rate, since the value of the function at the particular point in time was also equal to the increase over the next time interval. The teacher cast the student's representation into the language of rate of change so as to clarify the underlying mathematics.

The teacher also needed to carefully listen to another student's description of the rate as being "per 15 minutes", an approach that the teacher had not expected. In this instance, the teacher probed the student's thinking and attempted to understand the mathematics being expressed by that student. Later, the teacher supported the development of the students' ideas by elaborating on the connections that the students made to earlier representations that they had used. Modelling tasks provide the opportunity for students to develop a diversity of approaches to expressing their interpretations of a given situation. While this created a rich source of mathematical discussion for the students, it also placed substantial pedagogical knowledge demands on the teacher. This case illustrates four characteristics of the teachers' knowledge: (1) to be able to listen for anticipated ambiguities, (2) to offer useful representations of student ideas, (3) to hear unexpected approaches, and (4) to support students in making connections to other representations. How teachers acquire this knowledge, both in their preparation programs and in practice, remains an open question for researchers.

The second implication for the pedagogical knowledge of the teacher is illuminated in the shift that occurs in giving explanations and justifications. Rather than the teacher giving explanations and justifications to the students, the discussion of the models created a learning context in which the students were giving explanations and justifications to each other and to the teacher. This shift signals an important aspect of learning that takes place when using applications and modelling: the task for the teacher becomes one of putting the students in situations where they can interpret, explain, justify and evaluate the "goodness" of their models. In the case of the competing models for finding the rate of growth, the teacher encouraged the students to share their thinking and make sense of the explanations that were given by others. At the end of the discussion, however, she chose to give the students time to "commit" to their own ideas, perhaps re-evaluate them, before continuing with class discussion. In this way, the teacher gave the students the task of refining and revising their models, rather than proceeding to evaluate them herself. This change in pedagogical strategy is a major shift from more traditional instruction in mathematics where a primary role of the teacher is to evaluate students' work.

5. CONCLUDING REMARKS

The brief synopses of research that I have presented here are intended to suggest some of the challenges for teacher education programs that are raised by the use of modelling and applications for the teaching and learning of mathematics. Teacher education programs need to address both the subject matter knowledge of teachers and the development of new kinds of pedagogical knowledge. In particular, pre-service teachers need to gain experiences in their preparation programs with stochastic models; such a change would imply a shift away from the current dominance of deterministic models in the mathematical preparation of teachers. The difficulties that all learners have with probabilistic concepts make such a shift especially challenging. Pre-service teachers need to encounter modelling experiences that provide for a range of contexts and tools and that engage them in meta-level analyses of their modelling activity.

Teaching mathematics through modelling provides substantial challenges to our current ideas about pedagogy. When engaged in such teaching, teachers are likely to encounter substantial diversity in student thinking. This places new demands on teachers for listening to students, responding with useful representations, hearing unexpected approaches, and making connections to other mathematical ideas. A modelling approach to teaching mathematics calls for a major reversal in the usual roles of teachers and students. Students need to do more evaluating of their own ideas and teachers need to create opportunities where this evaluation can productively occur. Current research in the preparation and development of teachers in taking on these new roles is limited. International research in this area could provide the needed coherence for the development of a knowledge base of effective pedagogies when teaching mathematics through applications and modelling.

REFERENCES

Bassanezi, R. (1994). Modelling as a teaching-learning strategy. *For the Learning of Mathematics, 14*(2), 31-35.

Doerr, H. M. (2000). How can I find a pattern in this random data? The convergence of multiplicative and probabilistic reasoning. *Journal of Mathematical Behavior, 18*(4), 431-454.

Doerr, H. M., & Lesh, R. A. (2003). A modeling perspective on teacher development. In R. A. Lesh & H. M. Doerr (Eds.), *Beyond constructivism: Models and modeling perspectives on mathematics problem solving, learning and teaching* (pp. 125-139). Mahwah, NJ: Lawrence Erlbaum Associates.

Dugdale, S. (1994). K-12 teachers' use of a spreadsheet for mathematical modeling and problem solving. *Journal of Computers in Mathematics and Science Teaching, 13*(1), 43-68.

Kaput, J., & Roschelle, J. (1997). Deepening the impact of technology beyond assistance with traditional formalism in order to democratize access to ideas underlying calculus. In E. Pehkonen (Ed.), *Proceedings of the 21st Conference of the International Group for the Psychology of Mathematics Education* (pp. 105-112). Lahti, Finland: University of Helsinki.

Lingefjard, T. (2002). Mathematical modeling for preservice teachers: A problem from anesthesiology. *International Journal of Computers for Mathematical Learning, 7*, 117-143.

Weigand, H. & Weller, H. (1998). Modelling real-life problems involving periodic processes with computer algebra. *The International Journal of Computer Algebra in Mathematics Education, 5*(4), 251-267.

Zbiek, R. M. (1998). Prospective teachers' use of computing tools to develop and validate functions as mathematical models. *Journal for Research in Mathematics Education, 29(2)*, 184-201.

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