CHAPTER 10

Convergence

In this chapter we will give an introduction to several of the convergence ideas for Riemannian manifolds. The goal is to understand what it means for a sequence of Riemannian manifolds, or more generally metric spaces, to converge to a space. In the first section we develop the weakest convergence concept: Gromov-Hausdorff convergence. We then go on to explain some of the elliptic regularity theory we need for some of the later developments that use stronger types of convergence. In section 3 we develop the idea of norms of Riemannian manifolds. This is a concept developed by the author in the hope that it will make it easier to understand convergence theory as a parallel to the easier Hölder theory for functions (as is explained in section 2.) At the same time, we also feel that it has made some parts of the theory more concise. In this section we examine some stronger convergence ideas that were developed by Cheeger and Gromov and study their relation to the norms of manifolds. These preliminary discussions will enable us in subsequent sections to establish the convergence theorem of Riemannian geometry and its generalizations by Anderson and others. These convergence theorems contain the Cheeger finiteness theorem stating that certain very general classes of Riemannian manifolds contain only finitely many diffeomorphism types.

The idea of measuring the distance between subspaces of a given space goes back to Hausdorff and was extensively studied in the Polish and Russian schools of topology. The more abstract versions we use here seem to begin with Shikata's proof of the differentiable sphere theorem. In Cheeger's thesis, the idea that abstract manifolds can converge to each other is also evident. In fact, as we shall see below, he proved his finiteness theorem by showing that certain classes of manifolds are precompact in various topologies. After these two early forays into convergence theory it wasn't until Gromov bombarded the mathematical community with his highly original approaches to geometry that the theory developed further. He introduced a very weak kind of convergence that is simply an abstract version of Hausdorff distance. The first use of this new idea was to prove a group-theoretic question about the nilpotency of groups with polynomial growth. Soon after the introduction of this weak convergence, the earlier ideas on strong convergence by Cheeger resurfaced. There are various conflicting accounts on who did what and when. Certainly, the Russian school, notably Nikolaev and Berestovskii, deserve a lot of credit for their work on synthetic geometry, which could and should have been used in the convergence context. It appears that they were concerned primarily with studying generalized metrics in their own right. By contrast, the western school studied convergence and thereby developed an appreciation for studying Riemannian manifolds with little regularity, and even metric spaces.

1. Gromov-Hausdorff Convergence

1.1. Hausdorff Versus Gromov Convergence. At the beginning of the twentieth century, Hausdorff introduced what we call the *Hausdorff distance* between subsets of a metric space. If (X,d) is the metric space and $A,B\subset X$, then we define

$$\begin{aligned} d(A,B) &= \inf\{d(a,b) : a\in A, b\in B\},\\ B(A,\varepsilon) &= \{x\in X : d(x,A)<\varepsilon\},\\ d_H(A,B) &= \inf\{\varepsilon : A\subset B(B,\varepsilon), B\subset B(A,\varepsilon)\}.\end{aligned}$$

Thus, $d(A,B)$ is small if some points in these sets are close, while the *Hausdorff distance* $d_H(A,B)$ is small iff every point of A is close to a point in B and vice versa. One can easily see that the Hausdorff distance defines a metric on the closed subsets of X and that this collection is compact when X is compact.

We shall concern ourselves only with compact metric spaces and *proper* metric spaces. The latter have by definition proper distance functions, i.e., all closed balls are compact. This implies, in particular, that the spaces are separable, complete, and locally compact.

Around 1980, Gromov extended this concept to a distance between abstract metric spaces. If X and Y are metric spaces, then an *admissible* metric on the disjoint union $X\amalg Y$ is a metric that extends the given metrics on X and Y. With this we can define the *Gromov-Hausdorff distance* as

$$d_{G-H}(X,Y)=\inf\{d_H(X,Y) : \text{admissible metrics on } X\amalg Y\}.$$

Thus, we try to put a metric on $X\amalg Y$ such that X and Y are as close as possible in the Hausdorff distance, with the constraint that the extended metric restricts to the given metrics on X and Y. In other words, we are trying to define distances between points in X and Y without violating the triangle inequality.

EXAMPLE 53. *If Y is the one-point space, then*

$$\begin{aligned} d_{G-H}(X,Y) &\leq \mathrm{rad}X\\ &= \inf_{y\in X}\sup_{x\in X} d(x,y)\\ &= \textit{radius of smallest ball covering } X.\end{aligned}$$

EXAMPLE 54. *By defining $d(x,y)=D/2$, where* $\mathrm{diam}X$, $\mathrm{diam}Y\leq D$ *and* $x\in X$, $y\in Y$ *we see that*

$$d_{G-H}(X,Y)\leq D/2.$$

Let $(\mathcal{M},d_{G-H})$ denote the collection of compact metric spaces. We shall study this class as a metric space in its own right. To justify this we must show that only isometric spaces are within distance zero of each other.

PROPOSITION 42. *If X and Y are compact metric spaces with $d_{G-H}(X,Y)=0$, then X and Y are isometric.*

PROOF. Choose a sequence of metrics d_i on $X\amalg Y$ such that the Hausdorff distance between X and Y in this metric is $<i^{-1}$. Then we can find (possibly discontinuous) maps

$$\begin{aligned} I_i &: X\to Y, \text{ where } d_i(x,I_i(x))\leq i^{-1},\\ J_i &: Y\to X, \text{ where } d_i(y,J_i(y))\leq i^{-1}.\end{aligned}$$

Using the triangle inequality and that d_i restricted to either X or Y is the given metric d on these spaces yields

$$\begin{aligned} d\left(I_i\left(x_1\right), I_i\left(x_2\right)\right) &\leq 2i^{-1} + d\left(x_1, x_2\right), \\ d\left(J_i\left(y_1\right), J_i\left(y_2\right)\right) &\leq 2i^{-1} + d\left(y_1, y_2\right), \\ d\left(x, J_i \circ I_i\left(x\right)\right) &\leq 2i^{-1}, \\ d\left(y, I_i \circ J_i\left(y\right)\right) &\leq 2i^{-1}. \end{aligned}$$

We construct $I : X \to Y$ and $J : Y \to X$ as limits of these maps in the same way the Arzela-Ascoli lemma is proved. For each x the sequence $\left(I_i\left(x\right)\right)$ in Y has an accumulation point since Y is compact. As in the Arzela-Ascoli lemma select a dense countable set $A \subset X$. Using a diagonal argument select a subsequence I_{i_j} such that $I_{i_j}\left(a\right) \to I\left(a\right)$ for all $a \in A$. The first inequality now shows that I is distance decreasing on A. In particular, it is uniformly continuous and therefore has a unique extension to a map $I : X \to Y$, which is also distance decreasing. In a similar fashion we also get a distance decreasing map $J : Y \to X$.

The last two inequalities imply that I and J are inverses to each other. It then follows that both I and J are isometries. □

Both symmetry and the triangle inequality are easily established for d_{G-H}. Thus, $(\mathcal{M}, d_{G-H})$ is a pseudometric space, and if we consider equivalence classes of isometric spaces it becomes a metric space. In fact, as we shall see, this metric space is both complete and separable. First we show how spaces can be approximated by finite metric spaces.

EXAMPLE 55. *Let X be compact and $A \subset X$ a finite subset such that every point in X is within distance ε of some element in A, i.e., $d_H\left(A, X\right) \leq \varepsilon$. Such sets A are called* ε-dense *in X. It is then clear that if we use the metric on A induced by X, then also $d_{G-H}\left(X, A\right) \leq \varepsilon$. The importance of this remark is that for any $\varepsilon > 0$ we can in fact find such finite subsets of X, since X is compact.*

EXAMPLE 56. *Suppose we have ε-dense subsets*

$$\begin{aligned} A &= \{x_1, \ldots, x_k\} \subset X, \\ B &= \{y_1, \ldots, y_k\} \subset Y, \end{aligned}$$

with the further property that

$$\left|d\left(x_i, x_j\right) - d\left(y_i, y_j\right)\right| \leq \varepsilon, \ 1 \leq i, j \leq k.$$

Then $d_{G-H}\left(X, Y\right) \leq 3\varepsilon$. We already have that the finite subsets are ε-close to the spaces, so by the triangle inequality it suffices to show that $d_{G-H}\left(A, B\right) \leq \varepsilon$. For this we must exhibit a metric d on $A \amalg B$ that makes A and B ε-Hausdorff close. Define

$$\begin{aligned} d\left(x_i, y_i\right) &= \varepsilon, \\ d\left(x_i, y_j\right) &= \min_k \left\{d\left(x_i, x_k\right) + \varepsilon + d\left(y_j, y_k\right)\right\}. \end{aligned}$$

Thus, we have extended the given metrics on A and B in such a way that no points from A and B get identified, and in addition the potential metric is symmetric. It *then remains to check the triangle inequality. Here we must show*

$$\begin{aligned} d(x_i, y_j) &\leq d(x_i, z) + d(y_j, z), \\ d(x_i, x_j) &\leq d(y_k, x_i) + d(y_k, x_j), \\ d(y_i, y_j) &\leq d(x_k, y_i) + d(x_k, y_j). \end{aligned}$$

It suffices to check the first two cases as the third is similar to the second. In the first one we can assume that $z = x_k$. Then we can find l such that

$$d(y_j, x_k) = \varepsilon + d(y_j, y_l) + d(x_l, x_k).$$

Hence,

$$\begin{aligned} d(x_i, x_k) + d(y_j, x_k) &= d(x_i, x_k) + \varepsilon + d(y_j, y_l) + d(x_l, x_k) \\ &\geq d(x_i, x_l) + \varepsilon + d(y_j, y_l) \\ &\geq d(x_i, y_j). \end{aligned}$$

For the second case select l, m with

$$\begin{aligned} d(y_k, x_i) &= d(y_k, y_l) + \varepsilon + d(x_l, x_i), \\ d(y_k, x_j) &= d(y_k, y_m) + \varepsilon + d(x_m, x_j). \end{aligned}$$

Then, using our assumption about the comparability of the metrics on A and B, we have

$$\begin{aligned} d(y_k, x_i) + d(y_k, x_j) &= d(y_k, y_l) + \varepsilon + d(x_l, x_i) + d(y_k, y_m) + \varepsilon + d(x_m, x_j) \\ &\geq d(x_k, x_l) + d(x_l, x_i) + d(x_k, x_m) + d(x_m, x_j) \\ &\geq d(x_i, x_j). \end{aligned}$$

EXAMPLE 57. *Suppose $M_k = S^3/\mathbb{Z}_k$ with the usual metric induced from $S^3(1)$. Then we have a Riemannian submersion $M_k \to S^2(1/2)$ whose fibers have diameter $2\pi/k \to 0$ as $k \to \infty$. Using the previous example, we can therefore easily check that $M_k \to S^2(1/2)$ in the Gromov-Hausdorff topology.*

One can similarly see that the Berger metrics $\left(S^3, g_\varepsilon\right) \to S^2(1/2)$ as $\varepsilon \to 0$. Notice that in both cases the volume goes to zero, but the curvatures and diameters are uniformly bounded. In the second case the manifolds are even simply connected. It should also be noted that the topology changes rather drastically from the sequence to the limit, and in the first case the elements of the sequence even have mutually different fundamental groups.

PROPOSITION 43. *The "metric space" $(\mathcal{M}, d_{G-H})$ is separable and complete.*

PROOF. To see that it is separable, first observe that the collection of all finite metric spaces is dense in this collection. Now take the countable collection of all finite metric spaces that in addition have the property that all distances are rational. Clearly, this collection is dense as well.

To show completeness, select a Cauchy sequence $\{X_n\}$. To show convergence of this sequence, it suffices to check that some subsequence is convergent. Select a subsequence $\{X_i\}$ such that $d_{G-H}(X_i, X_{i+1}) < 2^{-i}$ for all i. Then select metrics $d_{i,i+1}$ on $X_i \amalg X_{i+1}$ making these spaces 2^{-i}-Hausdorff close. Now define a metric $d_{i,i+j}$on $X_i \amalg X_{i+j}$ by

$$d_{i,i+j}(x_i, x_{i+j}) = \min_{\{x_{i+k} \in X_{i+k}\}} \left\{ \sum_{k=0}^{j-1} d(x_{i+k}, x_{i+k+1}) \right\}.$$

We have then defined a metric d on $Y = \amalg_i X_i$ with the property that in this metric $d_H(X_i, X_{i+j}) \leq 2^{-i+1}$. This metric space is not complete, but the "boundary" of the completion is exactly our desired limit space. To define it, first consider

$$\hat{X} = \{\{x_i\} : x_i \in X_i \text{ and } d(x_i, x_j) \to 0 \text{ as } i, j \to \infty\}.$$

This space has a pseudometric defined by

$$d(\{x_i\}, \{y_i\}) = \lim_{i\to\infty} d(x_i, y_i).$$

Given that we are only considering Cauchy sequences $\{x_i\}$, this must yield a metric on the quotient space X, obtained by the equivalence relation

$$\{x_i\} \sim \{y_i\} \text{ iff } d(\{x_i\}, \{y_i\}) = 0.$$

Now we can extend the metric on Y to one on $X \amalg Y$ by declaring

$$d(x_k, \{x_i\}) = \lim_{i\to\infty} d(x_k, x_i).$$

Using that $d_H(X_j, X_{j+1}) \leq 2^{-j}$, we can for any $x_i \in X_i$ find a sequence $\{x_{i+j}\} \in \hat{X}$ such that $x_{i+0} = x_i$ and $d(x_{i+j}, x_{i+j+1}) \leq 2^{-j}$. Then we must have $d(x_i, \{x_{i+j}\}) \leq 2^{-i+1}$. Thus, every X_i is 2^{-i+1}-close to the limit space X. Conversely, for any given sequence $\{x_i\}$ we can find an equivalent sequence $\{y_i\}$ with the property that $d(y_k, \{y_i\}) \leq 2^{-k+1}$ for all k. Thus, X is 2^{-i+1}-close to X_i. $\square$

From the proof of this theorem we get the useful information that Gromov-Hausdorff convergence can always be thought of as Hausdorff convergence. In other words, if we know that $X_i \to X$ in the Gromov-Hausdorff sense, then after possibly passing to a subsequence, we can assume that there is a metric on $X \amalg (\amalg_i X_i)$ in which X_i Hausdorff converges to X. With such a selection of a metric, it then makes sense to say that $x_i \to x$, where $x_i \in X_i$ and $x \in X$. We shall often use this without explicitly mentioning a choice of ambient metric on $X \amalg (\amalg_i X_i)$.

There is an equivalent way of picturing convergence. For a compact metric space X, let $C(X)$ denote the continuous functions on X, and $L^\infty(X)$ the bounded measurable functions with the sup-norm (not the essential sup-norm). We know that $L^\infty(X)$ is a Banach space. When X is bounded, we construct a map $X \to L^\infty(X)$, by sending x to the continuous function $d(x, \cdot)$. This is usually called the *Kuratowski embedding* when we consider it as a map into $C(X)$. From the triangle inequality, we can easily see that this is in fact a distance-preserving map. Thus, any compact metric space is isometric to a subset of some Banach space $L^\infty(X)$. The important observation now is that two such spaces $L^\infty(X)$ and $L^\infty(Y)$ are isometric if the spaces X and Y are Borel equivalent (there exists a measurable bijection). Also, if $X \subset Y$, then $L^\infty(X)$ sits isometrically as a linear subspace of $L^\infty(Y)$. Now recall that any compact metric space is Borel equivalent to some subset of $[0,1]$. Thus all compact metric spaces X are isometric to some subset of $L^\infty([0,1])$. We can then define

$$d_{G-H}(X, Y) = \inf d_H(i(X), j(Y)),$$

where $i : X \to L^\infty([0,1])$ and $j : Y \to L^\infty([0,1])$ are distance-preserving maps.

1.2. Pointed Convergence. So far, we haven't really dealt with noncompact spaces. There is, of course, nothing wrong with defining the Gromov-Hausdorff distance between unbounded spaces, but it will almost never be finite. In order to change this, we should have in mind what is done for convergence of functions on unbounded domains. There, one usually speaks about convergence on compact subsets. To do something similar, we first define the pointed Gromov-Hausdorff distance

$$d_{G-H}\left(\left(X,x\right),\left(Y,y\right)\right)=\inf\left\{d_H\left(X,Y\right)+d\left(x,y\right)\right\}.$$

Here we take as usual the infimum over all Hausdorff distances and in addition require the selected points to be close. The above results are still true for this modified distance. We can then introduce the Gromov-Hausdorff topology on the collection of proper pointed metric spaces $\mathcal{M}_*=\{(X,x,d)\}$ in the following way: We say that

$$(X_i,x_i,d_i)\to(X,x,d)$$

in the *pointed Gromov-Hausdorff topology* if for all R, the closed metric balls

$$\left(\bar{B}\left(x_i,R\right),x_i,d_i\right)\to\left(\bar{B}\left(x,R\right),x,d\right)$$

converge with respect to the pointed Gromov-Hausdorff metric.

1.3. Convergence of Maps. We shall also have recourse to speak about *convergence of maps*. Suppose we have

$$\begin{array}{rcl} f_k & : & X_k\to Y_k, \\ X_k & \to & X, \\ Y_k & \to & Y. \end{array}$$

Then we say that f_k converges to $f:X\to Y$ if for every sequence $x_k\in X_k$ converging to $x\in X$ we have that $f_k\left(x_k\right)\to f\left(x\right).$ This definition obviously depends in some sort of way on having the spaces converge in the Hausdorff sense, but we shall ignore this. It is also a very strong kind of convergence for if we assume that $X_k=X$, $Y_k=Y$, and $f_k=f$, then f can converge to itself only if it is continuous.

Note also that convergence of functions preserves such properties as being distance preserving or submetries.

Another useful observation is that we can regard the sequence of maps f_k as one continuous map

$$F:\left(\amalg_i X_i\right)\to Y\amalg\left(\amalg_i Y_i\right).$$

The sequence converges iff this map has an extension

$$X\amalg\left(\amalg_i X_i\right)\to Y\amalg\left(\amalg_i Y_i\right),$$

in which case the limit map is the restriction to X. Thus, a sequence is convergent iff the map

$$F:\left(\amalg_i X_i\right)\to Y\amalg\left(\amalg_i Y_i\right)$$

is uniformly continuous.

A sequence of functions as above is called *equicontinuous,* if for every $\varepsilon>0$ there is an $\delta>0$ such that

$$f_k\left(B\left(x_k,\delta\right)\right)\subset B\left(f_k\left(x_k\right),\varepsilon\right)$$

for all k and $x_k \in X_k$. A sequence is therefore equicontinuous if, for example, all the functions are Lipschitz continuous with the same Lipschitz constant. As for standard equicontinuous sequences, we have the Arzela-Ascoli lemma:

LEMMA 45. *An equicontinuous family $f_k : X_k \to Y_k$, where $X_k \to X$, and $Y_k \to Y$ in the (pointed) Gromov-Hausdorff topology, has a convergent subsequence. When the spaces are not compact, we also assume that f_k preserves the base point.*

PROOF. The standard proof carries over without much change. Namely, first choose dense subsets

$$A_i = \left\{a_1^i, a_2^i, \ldots\right\} \subset X_i$$

such that the sequences

$$\left\{a_j^i\right\} \to a_j \in X.$$

Then also, $A = \{a_j\} \subset X$ is dense. Next, use a diagonal argument to find a subsequence of functions that converge on the above sequences. Finally, show that this sequence converges as promised. □

1.4. Compactness of Classes of Metric Spaces. We now turn our attention to conditions that ensure convergence of spaces. More precisely we want some good criteria for when a collection of (pointed) spaces is precompact (i.e., closure is compact).

For a compact metric space X, define the capacity and covering as follows

$$\begin{aligned}
\mathrm{Cap}(\varepsilon) &= \mathrm{Cap}_X(\varepsilon) = \text{ maximum number of disjoint } \frac{\varepsilon}{2}\text{-balls in } X,\\
\mathrm{Cov}(\varepsilon) &= \mathrm{Cov}_X(\varepsilon) = \text{ minimum number of } \varepsilon\text{-balls it takes to cover } X.
\end{aligned}$$

First, we observe that $\mathrm{Cov}(\varepsilon) \leq \mathrm{Cap}(\varepsilon)$ .To see this select disjoint balls $B\left(x_i, \frac{\varepsilon}{2}\right)$, then consider the collection $B(x_i, \varepsilon)$. In case the latter do not cover X there exists $x \in X - \cup B(x_i, \varepsilon)$. This would imply that $B\left(x, \frac{\varepsilon}{2}\right)$ is disjoint from all of the balls $B\left(x_i, \frac{\varepsilon}{2}\right)$. Thus showing that the former balls do not form a maximal disjoint family.

Another important observation is that if two compact metric spaces X and Y satisfy $d_{G-H}(X, Y) < \delta$, then it follows from the triangle inequality that:

$$\begin{aligned}
\mathrm{Cov}_X(\varepsilon + 2\delta) &\leq \mathrm{Cov}_Y(\varepsilon),\\
\mathrm{Cap}_X(\varepsilon) &\geq \mathrm{Cap}_Y(\varepsilon + 2\delta).
\end{aligned}$$

With this information we can now characterize precompact classes of compact metric spaces.

PROPOSITION 44. (M. Gromov, 1980) *For a class $\mathcal{C} \subset (\mathcal{M}, d_{G-H})$, the following statements are equivalent:*

(1) $\mathcal{C}$ is precompact, i.e., every sequence in $\mathcal{C}$ has a subsequence that is convergent in $(\mathcal{M}, d_{G-H})$.

(2) There is a function $N_1(\varepsilon) : (0, \alpha) \to (0, \infty)$ such that $\mathrm{Cap}_X(\varepsilon) \leq N_1(\varepsilon)$ *for all $X \in \mathcal{C}$.*

(3) There is a function $N_2(\varepsilon) : (0, \alpha) \to (0, \infty)$ such that $\mathrm{Cov}_X(\varepsilon) \leq N_2(\varepsilon)$ *for all $X \in \mathcal{C}$.*

Proof. (1) $\Rightarrow$ (2): If $\mathcal{C}$ is precompact, then for every $\varepsilon > 0$ we can find $X_1, \ldots, X_k \in \mathcal{C}$ such that for any $X \in \mathcal{C}$ we have that $d_{G-H}(X, X_i) < \frac{\varepsilon}{4}$ for some i. Then

$$\operatorname{Cap}_X(\varepsilon) \leq \operatorname{Cap}_{X_i}\left(\frac{\varepsilon}{2}\right) \leq \max_i \operatorname{Cap}_{X_i}\left(\frac{\varepsilon}{2}\right).$$

This gives a bound for $\operatorname{Cap}_X(\varepsilon)$ for each $\varepsilon > 0$.

(2) $\Rightarrow$ (3) Use $N_2 = N_1$.

(3) $\Rightarrow$ (1): It suffices to show that $\mathcal{C}$ is totally bounded, i.e., for each $\varepsilon > 0$ we can find finitely many metric spaces $X_1, \ldots, X_k \in \mathcal{M}$ such that any metric space in $\mathcal{C}$ is within ε of some X_i in the Gromov-Hausdorff metric. Since

$$\operatorname{Cov}_X\left(\frac{\varepsilon}{2}\right) \leq N\left(\frac{\varepsilon}{2}\right),$$

we know that any $X \in \mathcal{C}$ is within $\frac{\varepsilon}{2}$ of a finite subset with at most $N\left(\frac{\varepsilon}{2}\right)$ elements in it. Using the induced metric we think of these finite subsets as finite metric spaces. Next, observe that

$$\operatorname{diam} X \leq 2\delta \operatorname{Cov}_X(\delta)$$

for any fixed δ. This means that these finite metric spaces have no distances that are bigger than $\varepsilon N\left(\frac{\varepsilon}{2}\right)$. The metric on such a finite metric space then consists of a matrix (d_{ij}), $1 \leq i, j \leq N\left(\frac{\varepsilon}{2}\right)$, where each entry satisfies $d_{ij} \in \left[0, \varepsilon N\left(\frac{\varepsilon}{2}\right)\right]$. From among all such finite metric spaces it is then possible to select a finite number of them such that any of the matrices (d_{ij}) is within $\frac{\varepsilon}{2}$ of one matrix from the finite selection of matrices. This means that the spaces are within $\frac{\varepsilon}{2}$ of each other. We have then found the desired finite collection of metric spaces. $\square$

As a corollary we can also get a precompactness theorem in the pointed category.

Corollary 30. *A collection $\mathcal{C} \subset \mathcal{M}_*$ is precompact iff for each $R > 0$ the collection*

$$\{B(x, R) : B(x, R) \subset (X, x) \in \mathcal{C}\} \subset (\mathcal{M}, d_{G-H})$$

is precompact.

Using the relative volume comparison theorem we can now show

Corollary 31. *For any integer $n \geq 2$, $k \in \mathbb{R}$, and $D > 0$ we have that the following classes are precompact:*

(1) The collection of closed Riemannian n-manifolds with $\operatorname{Ric} \geq (n-1)k$ *and* $\operatorname{diam} \leq D$.

(2) The collection of pointed complete Riemannian n-manifolds with $\operatorname{Ric} \geq (n-1)k$.

Proof. It suffices to prove (2). Fix $R > 0$. We have to show that there can't be too many disjoint balls inside $B(x, R) \subset M$. To see this, suppose $B(x_1, \varepsilon), \ldots, B(x_\ell, \varepsilon) \subset B(x, R)$ are disjoint. If $B(x_i, \varepsilon)$ is the ball with the smallest volume, we have

$$\ell \leq \frac{\operatorname{vol} B(x, R)}{\operatorname{vol} B(x_i, \varepsilon)} \leq \frac{\operatorname{vol} B(x_i, 2R)}{\operatorname{vol} B(x_i, \varepsilon)} \leq \frac{v(n, k, 2R)}{v(n, k, \varepsilon)}.$$

This gives the desired bound. $\square$

It seems intuitively clear that an n-dimensional space should have $\mathrm{Cov}(\varepsilon) \sim \varepsilon^{-n}$ as $\varepsilon \to 0$. In fact, the Minkowski dimension of a metric space is defined as

$$\dim X = \limsup_{\varepsilon \to 0} \frac{\log \mathrm{Cov}(\varepsilon)}{-\log \varepsilon}.$$

This definition will in fact give the right answer for Riemannian manifolds. Some fractal spaces might, however, have nonintegral dimension. Now observe that

$$\frac{v(n,k,2R)}{v(n,k,\varepsilon)} \sim \varepsilon^{-n}.$$

Therefore, if we can show that covering functions carry over to limit spaces, then we will have shown that manifolds with lower curvature bounds can only collapse in dimension.

LEMMA 46. *Let $\mathcal{C}(N(\varepsilon))$ be the collection of metric spaces with* $\mathrm{Cov}(\varepsilon) \leq N(\varepsilon)$. *Suppose N is continuous. Then $\mathcal{C}(N(\varepsilon))$ is compact.*

PROOF. We already know that this class is precompact. So we only have to show that if $X_i \to X$ and $\mathrm{Cov}_{X_i}(\varepsilon) \leq N(\varepsilon)$, then also $\mathrm{Cov}_X(\varepsilon) \leq N(\varepsilon)$. This follows easily from

$$\mathrm{Cov}_X(\varepsilon) \leq \mathrm{Cov}_{X_i}(\varepsilon - 2d_{G-H}(X, X_i)) \leq N(\varepsilon - 2d_{G-H}(X, X_i)),$$

and

$$N(\varepsilon - 2d_{G-H}(X, X_i)) \to N(\varepsilon) \text{ as } i \to \infty.$$

□

2. Hölder Spaces and Schauder Estimates

First, we shall define the Hölder norms and Hölder spaces. We will then briefly discuss the necessary estimates we need for elliptic operators for later applications. The standard reference for all the material here is the classic book by Courant and Hilbert [**30**], especially chapter IV, and the thorough text [**44**], especially chapters 1-6. A more modern text that also explains how PDE's are used in geometry, including some of the facts we need, is [**90**], especially vol. III.

2.1. Hölder Spaces. Let us fix a bounded domain $\Omega \subset \mathbb{R}^n$. The bounded continuous functions from Ω to $\mathbb{R}^k$ are denoted by $C^0(\Omega, \mathbb{R}^k)$, and we use the sup-norm, denoted by

$$\|u\|_{C^0} = \sup_{x \in \Omega} |u(x)|,$$

on this space. This makes $C^0(\Omega, \mathbb{R}^k)$ into a Banach space. We wish to generalize this so that we still have a Banach space, but in addition also take into account derivatives of the functions. The first natural thing to do is to define $C^m(\Omega, \mathbb{R}^k)$ as the functions with m continuous partial derivatives. Using multi-index notation, we define

$$\partial^i u = \partial_1^{i_1} \cdots \partial_n^{i_n} u = \frac{\partial^l u}{\partial (x^1)^{i_1} \cdots \partial (x^n)^{i_n}},$$

where $i = (i_1, \ldots, i_n)$ and $l = |i| = i_1 + \cdots + i_n$. Then the C^m-norm is

$$\|u\|_{C^m} = \sup_{x \in \Omega} |u(x)| + \sum_{1 \leq |i| \leq m} \sup_\Omega \left|\partial^i u\right|.$$

This norm does result in a Banach space, but the inclusions

$$C^m\left(\Omega,\mathbb{R}^k\right)\subset C^{m-1}\left(\Omega,\mathbb{R}^k\right)$$

do not yield closed subspaces. For instance, $f(x)=|x|$ is in the closure of

$$C^1([-1,1],\mathbb{R})\subset C^0([-1,1],\mathbb{R}).$$

To accommodate this problem, we define for each $\alpha\in(0,1]$ the C^α-pseudonorm of $u:\Omega\to\mathbb{R}^k$ as

$$\|u\|_\alpha=\sup_{x,y\in\Omega}\frac{|u(x)-u(y)|}{|x-y|^\alpha}.$$

When $\alpha=1$, this gives the best Lipschitz constant for u.

Define the *Hölder space* $C^{m,\alpha}\left(\Omega,\mathbb{R}^k\right)$ as being the functions in $C^m\left(\Omega,\mathbb{R}^k\right)$ such that all mth-order partial derivatives have finite C^α-pseudonorm. On this space we use the norm

$$\|u\|_{C^{m,\alpha}}=\|u\|_{C^m}+\sum_{|i|=m}\left\|\partial^i u\right\|_\alpha.$$

If we wish to be specific about the domain, then we write

$$\|u\|_{C^{m,\alpha},\Omega}.$$

We can now show

LEMMA 47. *$C^{m,\alpha}\left(\Omega,\mathbb{R}^k\right)$ is a Banach space with the $C^{m,\alpha}$-norm. Furthermore, the inclusion*

$$C^{m,\alpha}\left(\Omega,\mathbb{R}^k\right)\subset C^{m,\beta}\left(\Omega,\mathbb{R}^k\right),$$

where $\beta<\alpha$ is always compact, i.e., it maps closed bounded sets to compact sets.

PROOF. We only need to show this in the case where $m=0$; the more general case is then a fairly immediate consequence.

First, we must show that any Cauchy sequence $\{u_i\}$ in $C^\alpha\left(\Omega,\mathbb{R}^k\right)$ converges. Since it is also a Cauchy sequence in $C^0\left(\Omega,\mathbb{R}^k\right)$ we have that $u_i\to u\in C^0$ in the C^0-norm. For fixed $x\neq y$ observe that

$$\frac{|u_i(x)-u_i(y)|}{|x-y|^\alpha}\to\frac{|u(x)-u(y)|}{|x-y|^\alpha}.$$

As the left-hand side is uniformly bounded, we also get that the right-hand side is bounded, thus showing that $u\in C^\alpha$.

Finally select $\varepsilon>0$ and N so that for $i,j\geq N$ and $x\neq y$

$$\frac{|(u_i(x)-u_j(x))-(u_i(y)-u_j(y))|}{|x-y|^\alpha}\leq\varepsilon.$$

If we let $j\to\infty$, this shows that

$$\frac{|(u_i(x)-u(x))-(u_i(y)-u(y))|}{|x-y|^\alpha}\leq\varepsilon.$$

Hence $u_i\to u$ in the C^α-topology.

Now for the last statement. A bounded sequence in $C^\alpha\left(\Omega,\mathbb{R}^k\right)$ is equicontinuous so the inclusion

$$C^\alpha\left(\Omega,\mathbb{R}^k\right)\subset C^0\left(\Omega,\mathbb{R}^k\right)$$

is compact. We then use

$$\frac{|u(x)-u(y)|}{|x-y|^{\beta}}=\left(\frac{|u(x)-u(y)|}{|x-y|^{\alpha}}\right)^{\beta/\alpha}\cdot|u(x)-u(y)|^{1-\beta/\alpha}$$

to conclude that

$$\|u\|_{\beta}\leq\left(\|u\|_{\alpha}\right)^{\beta/\alpha}\cdot\left(2\cdot\|u\|_{C^0}\right)^{1-\beta/\alpha}.$$

Therefore, a sequence that converges in C^0 and is bounded in C^{α}, also converges in C^{β}, as long as $\beta<\alpha\leq 1$. □

2.2. Elliptic Estimates. We now turn our attention to *elliptic operators*. We shall consider equations of the form

$$Lu=a^{ij}\partial_i\partial_j u+b^i\partial_i u=f,$$

where $a^{ij}=a^{ji}$. The operator is called *elliptic* if the matrix $\left(a^{ij}\right)$ is positive definite. Throughout we assume that all eigenvalues for $\left(a^{ij}\right)$ lie in some interval $\left[\lambda,\lambda^{-1}\right]$, $\lambda>0$, and that the coefficients are bounded

$$\begin{aligned}\left\|a^{ij}\right\|_{\alpha}&\leq\lambda^{-1},\\ \left\|b^{i}\right\|_{\alpha}&\leq\lambda^{-1}.\end{aligned}$$

Let us state without proof the a priori estimates, usually called the *Schauder estimates,* or *elliptic estimates*, that we shall need.

THEOREM 70. *Let $\Omega\subset\mathbb{R}^n$ be an open domain of diameter $\leq D$ and $K\subset\Omega$ a subdomain such that $d(K,\partial\Omega)\geq\delta$. Moreover assume $\alpha\in(0,1)$, then there is a constant $C=C(n,\alpha,\lambda,\delta,D)$ such that*

$$\begin{aligned}\|u\|_{C^{2,\alpha},K}&\leq C\left(\|Lu\|_{C^{\alpha},\Omega}+\|u\|_{C^{\alpha},\Omega}\right),\\ \|u\|_{C^{1,\alpha},K}&\leq C\left(\|Lu\|_{C^{0},\Omega}+\|u\|_{C^{\alpha},\Omega}\right).\end{aligned}$$

Furthermore, if Ω has smooth boundary and $u=\varphi$ on $\partial\Omega$, then there is a constant $C=C(n,\alpha,\lambda,D)$ such that on all of Ω we have

$$\|u\|_{C^{2,\alpha},\Omega}\leq C\left(\|Lu\|_{C^{\alpha},\Omega}+\|\varphi\|_{C^{2,\alpha},\partial\Omega}\right).$$

One way of proving these results is to establish them first for the simplest operator:

$$Lu=\Delta u=\delta^{ij}\partial_i\partial_j u.$$

Then observe that a linear change of coordinates shows that we can handle operators with constant coefficients:

$$Lu=\Delta u=a^{ij}\partial_i\partial_j u.$$

Finally, Schauder's trick is that the assumptions about the functions a^{ij} imply that they are almost constant locally. A partition of unity type argument then finishes the analysis.

The first-order term doesn't cause much trouble and can even be swept under the rug in the case where the operator is in divergence form:

$$Lu=a^{ij}\partial_i\partial_j u+b^i\partial_i u=\partial_i\left(a^{ij}\partial_j u\right).$$

Such operators are particularly nice when one wishes to use integration by parts, as we have

$$\int_{\Omega}\left(\partial_i\left(a^{ij}\partial_j u\right)\right)h=-\int_{\Omega}a^{ij}\partial_j u\partial_i h$$

when $h = 0$ on $\partial\Omega$. This is interesting in the context of geometric operators, as the Laplacian on manifolds in local coordinates looks like

$$\begin{aligned} Lu &= \Delta_g u \\ &= \frac{1}{\sqrt{\det g_{ij}}} \partial_i \left(\sqrt{\det g_{ij}} \cdot g^{ij} \cdot \partial_j u \right). \end{aligned}$$

The above theorem has an almost immediate corollary.

Corollary 32. *If in addition we assume that* $\left\| a^{ij} \right\|_{C^{m,\alpha}}, \left\| b^i \right\|_{C^{m,\alpha}} \leq \lambda^{-1}$, *then there is a constant* $C = C(n, m, \alpha, \lambda, \delta, D)$ *such that*

$$\|u\|_{C^{m+2,\alpha},K} \leq C \left(\|Lu\|_{C^{m,\alpha},\Omega} + \|u\|_{C^{\alpha},\Omega} \right).$$

And on a domain with smooth boundary,

$$\|u\|_{C^{m+2,\alpha},\Omega} \leq C \left(\|Lu\|_{C^{m,\alpha},\Omega} + \|\varphi\|_{C^{m+2,\alpha},\partial\Omega} \right).$$

The Schauder estimates can be used to show that the Dirichlet problem always has a unique solution.

Theorem 71. *Suppose* $\Omega \subset \mathbb{R}^n$ *is a bounded domain with smooth boundary, then the Dirichlet problem*

$$\begin{aligned} Lu &= f, \\ u|_{\partial\Omega} &= \varphi \end{aligned}$$

always has a unique solution $u \in C^{2,\alpha}(\Omega)$ *if* $f \in C^{\alpha}(\Omega)$ *and* $\varphi \in C^{2,\alpha}(\partial\Omega)$.

Observe that uniqueness is an immediate consequence of the maximum principle. The existence part requires a bit more work.

2.3. Harmonic Coordinates. The above theorem makes it possible to introduce *harmonic coordinates* on Riemannian manifolds.

Lemma 48. *If* (M, g) *is an* n*-dimensional Riemannian manifold and* $p \in M$, *then there is a neighborhood* $U \ni p$ *on which we can find a harmonic coordinate system*

$$x = \left(x^1, \ldots, x^n\right) : U \to \mathbb{R}^n,$$

i.e., a coordinate system such that the functions x^i *are harmonic with respect to the Laplacian on* (M, g).

Proof. First select a coordinate system $y = \left(y^1, \ldots, y^n\right)$ on a neighborhood around p such that $y(p) = 0$. We can then think of M as being an open subset of $\mathbb{R}^n$ and $p = 0$. The metric g is written as

$$g = g_{ij} = g(\partial_i, \partial_j) = g\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j} \right)$$

in the standard Cartesian coordinates $\left(y^1, \ldots, y^n\right)$. We must then find a coordinate transformation $y \to x$ such that

$$\Delta x^k = \frac{1}{\sqrt{\det g_{ij}}} \partial_i \left(\sqrt{\det g_{ij}} \cdot g^{ij} \cdot \partial_j x^k \right) = 0$$

To find these coordinates, fix a small ball $B(0,\varepsilon)$ and solve the Dirichlet problem

$$\begin{aligned} \Delta x^k &= 0 \\ x^k &= y^k \text{ on } \partial B(0,\varepsilon) \end{aligned}$$

We have then found n harmonic functions that should be close to the original coordinates. The only problem is that we don't know if they actually are coordinates. The Schauder estimates tell us that

$$\begin{aligned} \|x-y\|_{C^{2,\alpha},B(0,\varepsilon)} &\leq C\left(\|\Delta(x-y)\|_{C^{\alpha},B(0,\varepsilon)} + \left\|(x-y)_{|\partial B(0,\varepsilon)}\right\|_{C^{2,\alpha},\partial B(0,\varepsilon)}\right) \\ &= C\|\Delta y\|_{C^{\alpha},B(0,\varepsilon)}. \end{aligned}$$

If matters were arranged such that

$$\|\Delta y\|_{C^{\alpha},B(0,\varepsilon)} \to 0 \text{ as } \varepsilon \to 0,$$

then we could conclude that Dx and Dy are close for small ε. Since y does form a coordinates system, we would then also be able to conclude that x formed a coordinate system.

Now we just observe that if y were chosen as exponential Cartesian coordinates, then we would have that $\partial_k g_{ij} = 0$ at p. The formula for Δy then shows that $\Delta y = 0$ at p. Hence, we have

$$\|\Delta y\|_{C^{\alpha},B(0,\varepsilon)} \to 0 \text{ as } \varepsilon \to 0.$$

Finally recall that the constant C depends only on an upper bound for the diameter of the domain aside from α, n, λ. Thus,

$$\|x-y\|_{C^{2,\alpha},B(0,\varepsilon)} \to 0 \text{ as } \varepsilon \to 0.$$

□

One reason for using harmonic coordinates on Riemannian manifolds is that both the Laplacian and Ricci curvature tensor have particularly nice formulae in such coordinates.

LEMMA 49. *Let (M,g) be an n-dimensional Riemannian manifold and suppose we have a harmonic coordinate system $x : U \to \mathbb{R}^n$. Then*

(1) $\Delta u = \frac{1}{\sqrt{\det g_{st}}}\partial_i\left(\sqrt{\det g_{st}} \cdot g^{ij} \cdot \partial_j u\right) = g^{ij}\partial_i\partial_j u.$

(2) $\frac{1}{2}\Delta g_{ij} + Q(g,\partial g) = -\mathrm{Ric}_{ij} = -\mathrm{Ric}(\partial_i,\partial_j)$. *Here Q is some universal analytic expression that is polynomial in the matrix g, quadratic in ∂g, and a denominator term depending on $\sqrt{\det g_{ij}}$.*

PROOF. (1) By definition, we have that

$$\begin{aligned}
0 &= \Delta x^k \\
&= \frac{1}{\sqrt{\det g_{st}}}\partial_i\left(\sqrt{\det g_{st}}\cdot g^{ij}\cdot\partial_j x^k\right) \\
&= g^{ij}\partial_i\partial_j x^k + \frac{1}{\sqrt{\det g_{st}}}\partial_i\left(\sqrt{\det g_{st}}\cdot g^{ij}\right)\cdot\partial_j x^k \\
&= g^{ij}\partial_i\delta_j^k + \frac{1}{\sqrt{\det g_{st}}}\partial_i\left(\sqrt{\det g_{st}}\cdot g^{ij}\right)\cdot\delta_j^k \\
&= 0 + \frac{1}{\sqrt{\det g_{st}}}\partial_i\left(\sqrt{\det g_{st}}\cdot g^{ik}\right) \\
&= \frac{1}{\sqrt{\det g_{st}}}\partial_i\left(\sqrt{\det g_{st}}\cdot g^{ik}\right).
\end{aligned}$$

Thus, it follows that

$$\begin{aligned}
\Delta u &= \frac{1}{\sqrt{\det g_{st}}}\partial_i\left(\sqrt{\det g_{st}}\cdot g^{ij}\cdot\partial_j u\right) \\
&= g^{ij}\partial_i\partial_j u + \frac{1}{\sqrt{\det g_{st}}}\partial_i\left(\sqrt{\det g_{st}}\cdot g^{ij}\right)\cdot\partial_j u \\
&= g^{ij}\partial_i\partial_j u.
\end{aligned}$$

(2) Recall that if u is harmonic, then the Bochner formula for ∇u is

$$\Delta\left(\frac{1}{2}|\nabla u|^2\right) = |\mathrm{Hess}u|^2 + \mathrm{Ric}\left(\nabla u,\nabla u\right).$$

Here the term $|\mathrm{Hess}u|^2$ can be computed explicitly and depends only on the metric and its first derivatives. In particular,

$$\frac{1}{2}\Delta g\left(\nabla x^k,\nabla x^k\right) - \left|\mathrm{Hess}x^k\right|^2 = \mathrm{Ric}\left(\nabla x^k,\nabla x^k\right).$$

Polarizing this quadratic expression gives us an identity of the form

$$\frac{1}{2}\Delta g\left(\nabla x^i,\nabla x^j\right) - g\left(\mathrm{Hess}x^i,\mathrm{Hess}x^j\right) = \mathrm{Ric}\left(\nabla x^i,\nabla x^j\right).$$

Now use that

$$\nabla x^k = g^{ij}\partial_j x^k\partial_i = g^{ik}\partial_i$$

to see that $g\left(\nabla x^i,\nabla x^j\right) = g^{ij}$. We then have

$$\frac{1}{2}\Delta g^{ij} - g\left(\mathrm{Hess}x^i,\mathrm{Hess}x^j\right) = \mathrm{Ric}\left(\nabla x^i,\nabla x^j\right),$$

which in matrix form looks like

$$\frac{1}{2}\left[\Delta g^{ij}\right] - \left[g\left(\mathrm{Hess}x^i,\mathrm{Hess}x^j\right)\right] = \left[g^{ik}\right]\cdot\left[\mathrm{Ric}\left(\partial_k,\partial_l\right)\right]\cdot\left[g^{lj}\right].$$

This is, of course, not the promised formula. Instead, it is a similar formula for the inverse of (g_{ij}). One can now use the matrix equation $[g_{ik}]\cdot\left[g^{kj}\right] = \left[\delta_i^j\right]$ to conclude that

$$\begin{aligned}
0 &= \Delta\left([g_{ik}]\cdot\left[g^{kj}\right]\right)\\
&= [\Delta g_{ik}]\cdot\left[g^{kj}\right]+2\left[\sum_k g\left(\nabla g_{ik},\nabla g^{kj}\right)\right]+[g_{ik}]\cdot\left[\Delta g^{kj}\right]\\
&= [\Delta g_{ik}]\cdot\left[g^{kj}\right]+2\left[\nabla g_{ik}\right]\cdot\left[\nabla g^{kj}\right]+[g_{ik}]\cdot\left[\Delta g^{kj}\right]
\end{aligned}$$

Inserting this in the above equation yields

$$\begin{aligned}
[\Delta g_{ij}] &= -2\left[\nabla g_{ik}\right]\cdot\left[\nabla g^{kl}\right]\cdot[g_{lj}]-[g_{ik}]\cdot\left[\Delta g^{kl}\right]\cdot[g_{lj}]\\
&= -2\left[\nabla g_{ik}\right]\cdot\left[\nabla g^{kl}\right]\cdot[g_{lj}]\\
&\quad -2\left[g_{ik}\right]\cdot\left[g\left(\mathrm{Hess}x^k,\mathrm{Hess}x^l\right)\right]\cdot[g_{lj}]\\
&\quad -2\left[g_{ik}\right]\cdot\left[g^{ks}\right]\cdot\left[\mathrm{Ric}\left(\partial_s,\partial_t\right)\right]\cdot\left[g^{tl}\right]\cdot[g_{lj}]\\
&= -2\left[\nabla g_{ik}\right]\cdot\left[\nabla g^{kl}\right]\cdot[g_{lj}]-2\left[g_{ik}\right]\cdot\left[g\left(\mathrm{Hess}x^k,\mathrm{Hess}x^l\right)\right]\cdot[g_{lj}]\\
&\quad -2\left[\mathrm{Ric}\left(\partial_i,\partial_j\right)\right].
\end{aligned}$$

Each entry in these matrices then satisfies

$$\begin{aligned}
\frac{1}{2}\Delta g_{ij}+Q_{ij}\left(g,\partial g\right) &= -\mathrm{Ric}_{ij},\\
Q_{ij} &= -2\sum_{k,l} g\left(\nabla g_{ik},\nabla g^{kl}\right)\cdot g_{lj}\\
&\quad -2\sum_{k,l} g_{ik}\cdot g\left(\mathrm{Hess}x^k,\mathrm{Hess}x^l\right)\cdot g_{lj}.
\end{aligned}$$

□

It is interesting to apply this formula to the case of an Einstein metric, where $\mathrm{Ric}_{ij}=(n-1)\,kg_{ij}.$ In this case, it reads

$$\frac{1}{2}\Delta g_{ij}=-\left(n-1\right)kg_{ij}-Q\left(g,\partial g\right).$$

This formula makes sense even when g_{ij} is only $C^{1,\alpha}$. Namely, multiply by some test function, integrate, and use integration by parts to obtain a formula that uses only first derivatives of g_{ij}. If now g_{ij} is $C^{1,\alpha}$, then the left-hand side lies in C^{α}; but then our elliptic estimates show that g_{ij} must be in $C^{2,\alpha}$. This can be continued until we have that the metric is C^{∞}. In fact, one can even show that it is analytic. We can therefore conclude that any metric which in harmonic coordinates is a weak solution to the Einstein equation must in fact be smooth. We have obviously left out a few details about weak solutions. A detailed account can be found in [**90**, vol. III].

3. Norms and Convergence of Manifolds

We shall now explain how the $C^{m,\alpha}$ norm and convergence concepts for functions generalize to Riemannian manifolds. We shall also see how these ideas can be used to prove various compactness and finiteness theorems for classes of Riemannian manifolds.

3.1. Norms of Riemannian Manifolds. Before defining norms for manifolds, let us discuss which spaces should have norm zero. Clearly Euclidean space is a candidate. But what about open subsets of Euclidean space and other flat manifolds? If we agree that all open subsets of Euclidean space also have norm zero, then any flat manifold becomes a union of manifolds with norm zero and should therefore also have norm zero. In order to create a useful theory, it is often best to have only one space with zero norm. Thus we must agree that subsets of Euclidean space cannot have norm zero. To accommodate this problem, we define a family of norms of a Riemannian manifold, i.e., we use a function $N : (0, \infty) \to (0, \infty)$ rather than just a number. The number $N(r)$ then measures the degree of flatness on the scale of r, where the standard measure of flatness on the scale of r is the Euclidean ball $B(0, r)$. For small r, all flat manifolds then have norm zero; but as r increases we see that the space looks less and less like $B(0, r)$, and therefore the norm will become positive unless the space is Euclidean space.

For the precise definition, suppose A is a subset of a Riemannian n-manifold (M, g). We say that the $C^{m,\alpha}$*-norm on the scale of* r of $A \subset (M, g)$:

$$\|A \subset (M, g)\|_{C^{m,\alpha},r} \leq Q,$$

if we can find charts

$$\varphi_s : B(0, r) \subset \mathbb{R}^n \longleftrightarrow U_s \subset M$$

such that

(n1) Every ball $B\left(p, \frac{1}{10}e^{-Q}r\right), p \in A$ is contained in some U_s.
(n2) $|D\varphi_s| \leq e^Q$ on $B(0, r)$ and $\left|D\varphi_s^{-1}\right| \leq e^Q$ on U_s.
(n3) $r^{|j|+\alpha}\left\|D^j g_{s\cdot\cdot}\right\|_\alpha \leq Q$ for all multi indices j with $0 \leq |j| \leq m$.
(n4) $\left\|\varphi_s^{-1} \circ \varphi_t\right\|_{C^{m+1,\alpha}} \leq (10 + r)\, e^Q$.

Here $g_{s\cdot\cdot}$ is the matrix of functions of metric coefficients in the φ_s coordinates regarded as a matrix on $B(0, r)$.

First, observe that we think of the charts as maps from the fixed space $B(0, r)$ into the manifold. This is in order to have domains for the functions which do not refer to M itself. This simplifies some technical issues and makes it more clear that we are trying to measure how different the manifolds are from the standard objects, namely, Euclidean balls. The first condition says that we have a Lebesgue number for the covering of A. The second condition tells us that in the chosen coordinates the metric coefficients are bounded from below and above (in particular, we have uniform ellipticity for the Laplacian). The third condition gives us bounds on the derivatives of the metric. The fourth condition is included to ensure that the bounds for the metric in individual coordinates don't vary drastically in places where coordinates overlap. This last condition can be eliminated in many cases. We shall give another norm concept below that does this.

It will be necessary on occasion to work with Riemannian manifolds that are not smooth. The above definition clearly only requires that the metric be $C^{m,\alpha}$ in the coordinates we use, and so there is no reason to assume more about the metric. Some of the basic constructions, like exponential maps, then come into question, and indeed, if $m \leq 1$ these items might not be well-defined. We shall therefore have to be a little careful in some situations.

When it is clear from the context where A is, we shall merely write $\|A\|_{C^{m,\alpha},r}$, or for the whole space, $\|(M, g)\|_{C^{m,\alpha},r}$ or $\|M\|_{C^{m,\alpha},r}$. If A is precompact in M, then it is clear that the norm is bounded for all r. For unbounded domains or manifolds the norm might not be finite.

EXAMPLE 58. *Suppose* (M,g) *is a complete flat manifold. Then* $\|(M,g)\|_{C^{m,\alpha},r} = 0$ *for all* $r \leq \operatorname{inj}(M,g)$. *In particular,* $\|(\mathbb{R}^n, \mathrm{can})\|_{C^{m,\alpha},r} = 0$ *for all* r. *We shall later see that these properties characterize flat manifolds and Euclidean space.*

3.2. Convergence of Riemannian Manifolds. Now for the convergence concept that relates to this new norm. As we can't subtract manifolds, we have to resort to a different method for defining this. If we fix a closed manifold M, or more generally a precompact subset $A \subset M$, then we say that a sequence of functions on A converges in $C^{m,\alpha}$, if they converge in the charts for some fixed finite covering of coordinate patches. This definition is clearly independent of the finite covering we choose. We can then more generally say that a sequence of tensors converges in $C^{m,\alpha}$ if the components of the tensors converge in these patches. This then makes it possible to speak about convergence of Riemannian metrics on compact subsets of a fixed manifold.

A sequence of pointed complete Riemannian manifolds is said to *converge in the pointed* $C^{m,\alpha}$ *topology* $(M_i, p_i, g_i) \to (M, p, g)$ if for every $R > 0$ we can find a domain $\Omega \supset B(p,R) \subset M$ and embeddings $F_i : \Omega \to M_i$ for large i such that $F_i(\Omega) \supset B(p_i, R)$ and $F_i^* g_i \to g$ on Ω in the $C^{m,\alpha}$ topology. It is easy to see that this type of convergence implies pointed Gromov-Hausdorff convergence. When all manifolds in question are closed, then we have that the maps F_i are diffeomorphisms. This means that for closed manifolds we can speak about unpointed convergence. In this case, convergence can therefore only happen if all the manifolds in the tail end of the sequence are diffeomorphic. In particular, we have that classes of closed Riemannian manifolds that are precompact in some $C^{m,\alpha}$ topology contain at most finitely many diffeomorphism types.

A warning about this kind of convergence is in order here. Suppose we have a sequence of metrics g_i on a fixed manifold M. It is possible that these metrics might converge in the sense just defined, without converging in the traditional sense of converging in some fixed coordinate systems. To be more specific, let g be the standard metric on $M = S^2$. Now define diffeomorphisms F_t coming from the flow corresponding to the vector field that is 0 at the north and south poles and otherwise points in the direction of the south pole. As t increases, the diffeomorphisms will try to map the whole sphere down to a small neighborhood of the south pole. The metrics $F_t^* g$ will therefore in some fixed coordinates converge to 0 (except at the poles). They can therefore not converge in the classical sense. If, however, we pull these metrics back by the diffeomorphisms F_{-t}, then we just get back to g. Thus the sequence (M, g_t), from the new point of view we are considering, is a constant sequence. This is really the right way to think about this as the spaces $\left(S^2, F_t^* g\right)$ are all isometric as abstract metric spaces.

3.3. Properties of the Norm. Let us now consider some of the elementary properties of norms and their relation to convergence.

PROPOSITION 45. *If* $A \subset (M,g)$ *is precompact, then*

(1) $\|A \subset (M,g)\|_{C^{m,\alpha},r} = \left\|A \subset \left(M, \lambda^2 g\right)\right\|_{C^{m,\alpha},\lambda r}$ *for all* $\lambda > 0$.

(2) The function $r \to \|A \subset (M,g)\|_{C^{m,\alpha},r}$ *is continuous and converges to* 0 *as* $r \to 0$.

(3) Suppose $(M_i, p_i, g_i) \to (M, p, g)$ *in* $C^{m,\alpha}$. *Then for a precompact domain* $A \subset M$ *we can find precompact domains* $A_i \subset M_i$ *such that*

$$\|A_i\|_{C^{m,\alpha},r} \to \|A\|_{C^{m,\alpha},r} \text{ for all } r > 0$$

When all the manifolds are closed, we can let $A = M$ *and* $A_i = M_i$.

Proof. (1) If we change the metric g to $\lambda^2 g$, then we can change the charts $\varphi_s : B(0,r) \to M$ to

$$\varphi_s^\lambda(x) = \varphi_s\left(\lambda^{-1}x\right) : B(0, \lambda r) \to M.$$

Since we scale the metric at the same time, the conditions n1-n4 will still hold with the same Q.

(2) Suppose, as above, we change the charts

$$\varphi_s : B(0,r) \to M$$

to

$$\varphi_s^\lambda(x) = \varphi_s\left(\lambda^{-1}x\right) : B(0, \lambda r) \to M,$$

without changing the metric g. If we assume that

$$\|A \subset (M,g)\|_{C^{m,\alpha},r} < Q,$$

then

$$\|A \subset (M,g)\|_{C^{m,\alpha},\lambda r} \le \max\left\{Q + |\log\lambda|, Q \cdot \lambda^2\right\}.$$

Denoting

$$N(r) = \|A \subset (M,g)\|_{C^{m,\alpha},r},$$

we therefore obtain

$$N(\lambda r) \le \max\left\{N(r) + |\log\lambda|, N(r) \cdot \lambda^2\right\}.$$

By letting $\lambda = \frac{r_i}{r}$, where $r_i \to r$, we see that this implies

$$\limsup N(r_i) \le N(r).$$

Conversely, we have that

$$\begin{aligned} N(r) &= N\left(\frac{r}{r_i} r_i\right) \\ &\le \max\left\{N(r_i) + \left|\log\frac{r}{r_i}\right|, N(r_i) \cdot \left(\frac{r}{r_i}\right)^2\right\}. \end{aligned}$$

So

$$\begin{aligned} N(r) &\le \liminf N(r_i) \\ &= \liminf \max\left\{N(r_i) + \left|\log\frac{r}{r_i}\right|, N(r_i) \cdot \left(\frac{r}{r_i}\right)^2\right\}. \end{aligned}$$

This shows that $N(r)$ is continuous. To see that $N(r) \to 0$ as $r \to 0$, just observe that any coordinate system around a point $p \in M$ can, after a linear change, be assumed to have the property that the metric $g_{ij} = \delta_{ij}$ at p. In particular $|D\varphi|_p| = \left|D\varphi^{-1}|_p\right| = 1$. Using these coordinates on sufficiently small balls will therefore give the desired charts.

(3) We fix $r > 0$ in the definition of $\|A \subset (M,g)\|_{C^{m,\alpha},r}$. For the given $A \subset M$, pick a domain $\Omega \supset A$ such that for large i we have embeddings $F_i : \Omega \to M_i$ with the property that: $F_i^* g_i \to g$ in $C^{m,\alpha}$ on Ω. Define $A_i = F_i(A)$.

For $Q > \|A \subset (M,g)\|_{C^{m,\alpha},r}$, choose appropriate charts $\varphi_s : B(0,r) \to M$ covering A, with the properties n1-n4. Then define charts in M_i by

$$\varphi_{i,s} = F_i \circ \varphi_s : B(0,r) \to M_i.$$

Condition n1 will hold just because we have Gromov-Hausdorff convergence. Condition n4 is trivial. Conditions n2 and n3 will hold for constants $Q_i \to Q$, since $F_i^* g_i \to g$ in $C^{m,\alpha}$. We can therefore conclude that

$$\limsup \|A_i\|_{C^{m,\alpha},r} \leq \|A\|_{C^{m,\alpha},r}.$$

On the other hand, for large i and $Q > \|A_i\|_{C^{m,\alpha},r}$, we can take charts $\varphi_{i,s} : B(0,r) \to M_i$ and then pull them back to M by defining $\varphi_s = F_i^{-1} \circ \varphi_{i,s}$. As before, we then have

$$\|A\|_{C^{m,\alpha},r} \leq Q_i,$$

where $Q_i \to Q$. This implies

$$\liminf \|A_i\|_{C^{m,\alpha},r} \geq \|A\|_{C^{m,\alpha},r},$$

and hence the desired result. □

3.4. Compact Classes of Riemannian Manifolds. We are now ready to prove the result that is our manifold equivalent of the Arzela-Ascoli lemma. This theorem is essentially due to J. Cheeger, although our use of norms makes the statement look different.

THEOREM 72. (Fundamental Theorem of Convergence Theory) *For given $Q > 0$, $n \geq 2$, $m \geq 0$, $\alpha \in (0,1]$, and $r > 0$ consider the class $\mathcal{M}^{m,\alpha}(n,Q,r)$ of complete, pointed Riemannian n-manifolds (M,p,g) with $\|(M,g)\|_{C^{m,\alpha},r} \leq Q$. $\mathcal{M}^{m,\alpha}(n,Q,r)$ is compact in the pointed $C^{m,\beta}$ topology for all $\beta < \alpha$.*

PROOF. We proceed in stages. First, we make some general comments about the charts we use. We then show that $\mathcal{M} = \mathcal{M}^{m,\alpha}(n,Q,r)$ is pre-compact in the pointed Gromov-Hausdorff topology. Next we prove that $\mathcal{M}$ is closed in the Gromov-Hausdorff topology. The last and longest part is then devoted to the compactness statement.

Setup: First fix $K > Q$. Whenever we select an $M \in \mathcal{M}$, we shall assume that it comes equipped with an atlas of charts satisfying n1-n4 with K in place of Q. Thus we implicitly assume that all charts under consideration belong to these atlases. We will consequently only prove that limit spaces (M,p,g) satisfy $\|(M,g)\|_{C^{m,\alpha},r} \leq K$. But as K was arbitrary, we still get that $(M,p,g) \in \mathcal{M}$.

(1) Every chart $\varphi : B(0,r) \to U \subset M \in \mathcal{M}$ satisfies
 (a) $d(\varphi(x_1), \varphi(x_2)) \leq e^K |x_1 - x_2|$
 (b) $d(\varphi(x_1), \varphi(x_2)) \geq \min\{e^{-K}|x_1 - x_2|,\ e^{-K}(2r - |x_1| - |x_2|)\}$.

Here, d is distance measured in M, and $|\cdot|$ is the usual Euclidean norm.

The condition $|D\varphi| \leq e^K$, together with convexity of $B(0,r)$, immediately implies the first inequality. For the other, first observe that if any segment from x_1 to x_2 lies in U, then $|D\varphi^{-1}| \leq e^K$ implies, that

$$d(\varphi(x_1),\ \varphi(x_2)) \geq e^{-K}|x_1 - x_2|.$$

So we may assume that $\varphi(x_1)$ and $\varphi(x_2)$ are joined by a segment $\sigma : [0,1] \to M$ that leaves U. Split σ into $\sigma : [0,t_1) \to U$ and $\sigma : (t_2, 1) \to U$ such that $\sigma(t_i) \notin U$.

Then we clearly have

$$\begin{aligned} d(\varphi(x_1),\varphi(x_2)) &= L(\sigma) \geq L(\sigma|_{[0,t_1)}) + L(\sigma|_{(t_2,1]}) \\ &\geq e^{-K}(L(\varphi^{-1}\circ\sigma|_{[0,t_1)}) + L(\varphi^{-1}\circ\sigma|_{(t_2,1]})) \\ &\geq e^{-K}(2r - |x_1| - |x_2|). \end{aligned}$$

The last inequality follows from the fact that $\varphi^{-1}\circ\sigma(0) = x_1$ and $\varphi^{-1}\circ\sigma(1) = x_2$, and that $\varphi^{-1}\circ\sigma(t)$ approaches the boundary of $B(0,r)$ as $t \nearrow t_1$ or $t \searrow t_2$.

(2) Every chart

$$\varphi : B(0,r) \to U \subset M \in \mathcal{M},$$

and hence any δ-ball $\delta = \frac{1}{10}e^{-K}r$ in M can be covered by at most N balls of radius $\delta/4$. Here, N depends only on n, K, r.

Clearly, there exists an $N(n,K,r)$ such that $B(0,r)$ can be covered by at most N balls of radius $e^{-K}\cdot\delta/4$. Since $\varphi : B(0,r) \to U$ is a Lipschitz map with Lipschitz constant $\leq e^K$, we get the desired covering property.

(3) Every ball $B(x,\ell\cdot\delta/2) \subset M$ can be covered by $\leq N^\ell$ balls of radius $\delta/4$.

For $\ell = 1$ we just proved this. Suppose we know that $B(x,\ell\cdot\delta/2)$ is covered by $B(x_1,\delta/4)$, ..., $B(x_{N^\ell},\delta/4)$. Then

$$B(x,\ell\cdot\delta/2 + \delta/2) \subset \cup B(x_i,\delta).$$

Now each $B(x_i,\delta)$ can be covered by $\leq N$ balls of radius $\delta/4$, and hence $B(x,(\ell+1)\delta/2)$ can be covered by $\leq N\cdot N^\ell = N^{\ell+1}$ balls of radius $\delta/4$.

(4) $\mathcal{M}$ is precompact in the pointed Gromov-Hausdorff topology.

This is equivalent to asserting, that for each $R > 0$ the family of metric balls

$$B(p,R) \subset (M,p,g) \in \mathcal{M}$$

is precompact in the Gromov-Hausdorff topology. This claim is equivalent to showing that we can find a function $N(\varepsilon) = N(\varepsilon,R,K,r,n)$ such that each $B(p,R)$ can contain at most $N(\varepsilon)$ disjoint ε-balls. To check this, let $B(x_1,\varepsilon)$, ..., $B(x_s,\varepsilon)$ be a collection of disjoint balls in $B(p,R)$. Suppose that

$$\ell\cdot\delta/2 < R \leq (\ell+1)\delta/2.$$

Then

$$\begin{aligned} \mathrm{vol}B(p,R) &\leq (N^{(\ell+1)}) \cdot (\text{maximal volume of } \frac{\delta}{4}\text{-ball}) \\ &\leq (N^{(\ell+1)}) \cdot (\text{maximal volume of chart}) \\ &\leq N^{(\ell+1)}\cdot e^{nK}\cdot \mathrm{vol}B(0,r) \\ &\leq V(R) = V(R,n,K,r). \end{aligned}$$

As long as $\varepsilon < r$ each $B(x_i,\varepsilon)$ lies in some chart $\varphi : B(0,r) \to U \subset M$ whose preimage in $B(0,r)$ contains an $e^{-K}\cdot\varepsilon$-ball. Thus

$$\mathrm{vol}B(p_i,\varepsilon) \geq e^{-2nK}\mathrm{vol}B(0,\varepsilon).$$

All in all, we get

$$\begin{aligned} V(R) &\geq \mathrm{vol}B(p,R) \\ &\geq \sum \mathrm{vol}B(p_i,\varepsilon) \\ &\geq s\cdot e^{-2nK}\cdot\mathrm{vol}B(0,\varepsilon). \end{aligned}$$

Thus,

$$s \leq N(\varepsilon) = V(R) \cdot e^{2nK} \cdot (\mathrm{vol}B(0,\varepsilon))^{-1}.$$

Now select a sequence (M_i, g_i, p_i) in $\mathcal{M}$. From the previous considerations we can assume that $(M_i, g_i, p_i) \to (X, d, p)$ converge to some metric space in the Gromov-Hausdorff topology. It will be necessary in many places to pass to subsequences of (M_i, g_i, p_i) using various diagonal processes. Whenever this happens, we shall not reindex the family, but merely assume that the sequence was chosen to have the desired properties from the beginning. For each (M_i, p_i, g_i) choose charts

$$\varphi_{is} : B(0, r) \to U_{is} \subset M_i$$

satisfying n1-n4. We can furthermore assume that the index set $\{s\} = \{1, 2, 3, 4, \cdots\}$ is the same for all M_i, that $p_i \in U_{i1}$, and that the balls $B(p_i, \ell \cdot \delta/2)$ are covered by the first N^ℓ charts. Note that these N^ℓ charts will then be contained in $\bar{B}\left(p_i, \ell \cdot \delta/2 + [e^K + 1]\delta\right)$. Finally, for each ℓ the sequence $\bar{B}(p_i, \ell \cdot \delta/2)$ converges to $\bar{B}(p, \ell \cdot \delta/2) \subset X$, so we can choose a metric on the disjoint union

$$Y_\ell = \left(\bar{B}(p, \ell \cdot \delta/2) \coprod \left(\coprod_{i=1}^{\infty} \bar{B}(p_i, \ell \cdot \delta/2)\right)\right)$$

such that

$$\begin{array}{rcl} p_i & \to & p, \\ \bar{B}(p_i, \ell \cdot \delta/2) & \to & \bar{B}(p, \ell \cdot \delta/2) \end{array}$$

in the Hausdorff distance inside this metric space.

(5) (X, d, p) is a Riemannian manifold of class $C^{m,\alpha}$ with norm $\leq K$.

Obviously, we need to find bijections

$$\varphi_s : B(0, r) \to U_s \subset X$$

satisfying n1-n4. For each s, consider the maps

$$\varphi_{is} : B(0, r) \to U_{is} \subset Y_{\ell'}$$

for some fixed $\ell' >> \ell$. From 1 we have that this is a family of equicontinuous maps into the compact space $Y_{\ell'}$. The Arzela-Ascoli lemma shows that this sequence must subconverge (in the C^0 topology) to a map

$$\varphi_s : B(0, r) \subset Y_{\ell'}$$

that also has Lipschitz constant e^K. Furthermore, the inequality

$$d(\varphi(x_1), \varphi(x_2)) \geq \min\{e^{-K}|x_1 - x_2|,\ e^{-K}(2r - |x_1| - |x_2|)\}$$

will also hold for this map, as it holds for all the φ_{is} maps. In particular, φ_s is one-to-one. Finally, since $U_{is} \subset \bar{B}(p_i, \ell')$ and $\bar{B}(p_i, \ell')$ Hausdorff converges to $\bar{B}(p, \ell') \subset X$, we see that

$$\varphi_s(B(0, r)) = U_s \subset X.$$

A simple diagonal argument yields that we can pass to a subsequence of (M_i, g_i, p_i) having the property that $\varphi_{is} \to \varphi_s$ for all s. In this way, we have constructed (topological) charts

$$\varphi_s : B(0, r) \to U_s \subset X,$$

and we can easily check that they satisfy n1. Since the φ_s also satisfy 1(a) and 1(b), they would also satisfy n2 if they were differentiable (equivalent to saying that the transition functions are C^1). Now the transition functions $\varphi_{is}^{-1} \circ \varphi_{it}$ approach

$\varphi_s^{-1} \circ \varphi_t$, because $\varphi_{is} \to \varphi_s$. Note that these transition functions are not defined on the same domains, but we do know that the domain for $\varphi_s^{-1} \circ \varphi_t$ is the limit of the domains for $\varphi_{is}^{-1} \circ \varphi_{it}$, so the convergence makes sense on all compact subsets of the domain of $\varphi_s^{-1} \circ \varphi_t$. Now,

$$\|\varphi_{is}^{-1} \circ \varphi_{it}\|_{C^{m+1,\alpha}} \leq (10 + r)\, e^K,$$

so a further application (and subsequent passage to subsequences) of Arzela-Ascoli tells us that

$$\|\varphi_s^{-1} \circ \varphi_t\|_{C^{m+1,\alpha}} \leq (10 + r)\, e^K,$$

and that we can assume $\varphi_{is}^{-1} \circ \varphi_{it} \to \varphi_s^{-1} \circ \varphi_t$ in the $C^{m+1,\beta}$ topology. This then establishes n4. We now construct a compatible Riemannian metric on X that satisfies n2 and n3. For each s, consider the metric $g_{is} = g_{is\cdot\cdot}$ written out in its components on $B(0, r)$ with respect to the chart φ_{is}. Since all of the $g_{is\cdot\cdot}$ satisfy n2 and n3, we can again use Arzela-Ascoli to insure that also $g_{is\cdot\cdot} \to g_{s\cdot\cdot}$ on $B(0, r)$ in the $C^{m,\beta}$ topology to functions $g_{s\cdot\cdot}$ that also satisfy n2 and n3. The local "tensors" $g_{s\cdot\cdot}$ satisfy the right change of variables formulae to make them into a global tensor on X. This is because all the $g_{is\cdot\cdot}$ satisfy these properties, and everything we want to converge, to carry these properties through to the limit, also converges. Recall that the rephrasing of n2 gives the necessary C^0 bounds and also shows that $g_{s\cdot\cdot}$ is positive definite. We have now exhibited a Riemannian structure on X such that the

$$\varphi_s : B(0, r) \to U_s \subset X$$

satisfy n1-n4 with respect to this structure. This, however, does not guarantee that the metric generated by this structure is identical to the metric we got from X being the pointed Gromov-Hausdorff limit of (M_i, p_i, g_i). However, since Gromov-Hausdorff convergence implies that distances converge, and we know at the same time that the Riemannian metric converges locally in coordinates, it follows that the limit Riemannian structure must generate the "correct" metric, at least locally, and therefore also globally.

(6) $(M_i, p_i, g_i) \to (X, p, d) = (X, p, g)$ in the pointed $C^{m,\beta}$ topology.

We assume that the setup is as in 5, where charts φ_{is}, transitions $\varphi_{is}^{-1} \circ \varphi_{it}$, and metrics $g_{is\cdot\cdot}$ converge to the same items in the limit space. First, let us agree that two maps F_1, F_2 between subsets in M_i and X are $C^{m+1,\beta}$ close if all the coordinate compositions $\varphi_s^{-1} \circ F_1 \circ \varphi_{it}$, $\varphi_s^{-1} \circ F_2 \circ \varphi_{it}$ are $C^{m+1,\beta}$ close. Thus, we have a well-defined $C^{m+1,\beta}$ topology on maps from M_i to X. Our first observation is that

$$\begin{aligned} f_{is} &= \varphi_{is} \circ \varphi_s^{-1} : U_s \to U_{is}, \\ f_{it} &= \varphi_{it} \circ \varphi_t^{-1} : U_t \to U_{it} \end{aligned}$$

"converge to each other" in the $C^{m+1,\beta}$ topology. Furthermore,

$$(f_{is})^* g_i|_{U_{is}} \to g|_{U_s}$$

in the $C^{m,\beta}$ topology. These are just restatements of what we already know. In order to finish the proof, we construct maps

$$F_{i\ell} : \Omega_\ell = \bigcup_{s=1}^{\ell} U_s \to \Omega_{i\ell} = \bigcup_{s=1}^{\ell} U_{is}$$

that are closer and closer to the f_{is}, $s = 1, \ldots, \ell$ maps (and therefore all f_{is}) as $i \to \infty$. We will construct $F_{i\ell}$ by induction on ℓ and large i depending on ℓ. For this purpose we shall need a partition of unity (λ_s) on X subordinate to (U_s). We can find such a partition, since the covering (U_s) is locally finite by choice, and we can furthermore assume that λ_s is $C^{m+1,\beta}$.

For $\ell = 1$ simply define $F_{i1} = f_{i1}$.

Suppose we have $F_{i\ell} : \Omega_\ell \to \Omega_{i\ell}$ for large i that are arbitrarily close to f_{is}, $s = 1, \ldots, \ell$ as $i \to \infty$. If $U_{\ell+1} \cap \Omega_\ell = \emptyset$, then we just define $F_{i\ell+1} = F_{i\ell}$ on $\Omega_{i\ell}$, and $F_{i\ell+1} = f_{i\ell+1}$ on $U_{\ell+1}$. In case $U_{\ell+1} \subset \Omega_\ell$, we simply let $F_{i\ell+1} = F_{i\ell}$. Otherwise, we know that $F_{i\ell}$ and $f_{i\ell+1}$ are as close as we like in the $C^{m+1,\beta}$ topology as $i \to \infty$. So the natural thing to do is to average them on $U_{\ell+1}$. Define $F_{i\ell+1}$ on $U_{\ell+1}$ by

$$\begin{aligned} & F_{i\ell+1}(x) \\ = \; & \varphi_{i\ell+1} \circ \left(\left(\sum_{s=\ell+1}^{\infty} \lambda_s(x) \right) \cdot \varphi_{i\ell+1}^{-1} \circ f_{i\ell+1}(x) + \left(\sum_{s=1}^{\ell} \lambda_s(x) \right) \cdot \varphi_{i\ell+1}^{-1} \circ F_{i\ell}(x) \right) \\ = \; & \varphi_{i\ell+1} \circ (\mu_1(x) \cdot \varphi_{i\ell+1}^{-1} \circ f_{i\ell+1}(x) + \mu_2(x) \cdot \varphi_{i\ell+1}^{-1} \circ F_{i\ell}(x)). \end{aligned}$$

This map is clearly well-defined on $U_{\ell+1}$, since $\mu_2(x) = 0$ on $U_{\ell+1} - \Omega_\ell$. Moreover, as $\mu_1(x) = 0$ on Ω_ℓ it is a smooth $C^{m+1,\beta}$ extension of $F_{i\ell}$. Now consider this map in coordinates

$$\begin{aligned} \varphi_{i\ell+1}^{-1} \circ F_{i\ell+1} \circ \varphi_{\ell+1}(y) &= \big(\mu_1 \circ \varphi_{\ell+1}(y)\big) \cdot \varphi_{\ell+1}^{-1} \circ f_{i\ell+1} \circ \varphi_{\ell+1}(y) \\ &\quad + \big(\mu_2 \circ \varphi_{\ell+1}(y)\big) \cdot \varphi_{i\ell+1}^{-1} \circ F_{i\ell} \circ \varphi_{\ell+1}(y) \\ &= \tilde{\mu}_1(y) F_1(y) + \tilde{\mu}_2(y) F_2(y). \end{aligned}$$

Then

$$\begin{aligned} \|\tilde{\mu}_1 F_1 + \tilde{\mu}_2 F_2 - F_1\|_{C^{m+1,\beta}} &= \|\tilde{\mu}_1 (F_1 - F_1) + \tilde{\mu}_2 (F_2 - F_1)\|_{C^{m+1,\beta}} \\ &\leq \|\tilde{\mu}_2\|_{k+1+\beta} \cdot \|F_2 - F_1\|_{C^{m+1,\beta}}. \end{aligned}$$

This inequality is valid on all of $B(0, r)$, despite the fact that F_2 is not defined on all of $B(0, r)$, since

$$\tilde{\mu}_1 \cdot F_1 + \tilde{\mu}_2 \cdot F_2 = F_1$$

on the region where F_2 is undefined. By assumption

$$\|F_2 - F_1\|_{C^{m+1,\beta}} \to 0 \text{ as } i \to \infty,$$

so $F_{i\ell+1}$ is $C^{m+1,\beta}$-close to f_{is}, $s = 1, \ldots, \ell + 1$ as $i \to \infty$.

Finally we see that the closeness of $F_{i\ell}$ to the coordinate charts shows that it is an embedding on all compact subsets of the domain. □

COROLLARY 33. *The subclasses of $\mathcal{M}^{m,\alpha}(n, Q, r)$, where the elements in addition satisfy* diam $\leq D$, *respectively* vol $\leq V$, *are compact in the $C^{m,\beta}$ topology. In particular, they contain only finitely many diffeomorphism types.*

PROOF. We use notation as in the fundamental theorem. If $\operatorname{diam}(M, g, p) \leq D$, then clearly $M \subset B(p, k \cdot \delta/2)$ for $k > D \cdot 2/\delta$. Hence, each element in $\mathcal{M}^{m,\alpha}(n, Q, r)$ can be covered by $\leq N^k$ charts. Thus, $C^{m,\beta}$-convergence is actually in the unpointed topology, as desired.

If instead, $\operatorname{vol} M \leq V$, then we can use part 4 in the proof to see that we can never have more than

$$k = V \cdot e^{2nK} \cdot (\operatorname{vol} B(0, \varepsilon))^{-1}$$

disjoint ε-balls. In particular, diam $\leq 2\varepsilon \cdot k$, and we can use the above argument.

Finally, compactness in any $C^{m,\beta}$ topology implies that the class cannot contain infinitely many diffeomorphism types. $\square$

COROLLARY 34. *The norm $\|A \subset (M,g)\|_{C^{m,\alpha},r}$ for compact A is always realized by some charts $\varphi_s : B(0,r) \to U_s$ satisfying n1-n4, with $\|(M,g)\|_{C^{m,\alpha},r}$ in place of Q.*

PROOF. Choose appropriate charts

$$\varphi_s^Q : B(0,r) \to U_s^Q \subset M$$

for each $Q > \|(M,g)\|_{C^{m,\alpha},r}$, and let $Q \to \|(M,g)\|_{C^{m,\alpha},r}$. If the charts are chosen to conform with the proof of the fundamental theorem, we will obviously get some limit charts with the desired properties. $\square$

COROLLARY 35. *M is a flat manifold if $\|(M,g)\|_{C^{m,\alpha},r} = 0$ for some r, and M is Euclidean space with the canonical metric if $\|(M,g)\|_{C^{m,\alpha},r} = 0$ for all $r > 0$.*

PROOF. The proof works even if $m = \alpha = 0$. As in the previous corollary and part (1) of the theorem, M can be covered by charts $\varphi : B(0,r) \to U \subset M$ satisfying

$$\begin{aligned}
&\text{(a)} \quad d(\varphi(x_1), \varphi(x_2)) \leq e^{Q}|x_1 - x_2| \\
&\text{(b)} \quad d(\varphi(x_1), \varphi(x_2)) \geq \min\{e^{-Q}|x_1 - x_2|,\ e^{-Q}(2r - |x_1| - |x_2|)\}.
\end{aligned}$$

for each $Q > 0$. By letting $Q \to 0$, we can then use Arzela-Ascoli to find a covering of charts such that

$$\begin{aligned}
&\text{(a)} \quad d(\varphi(x_1), \varphi(x_2)) \leq |x_1 - x_2| \\
&\text{(b)} \quad d(\varphi(x_1), \varphi(x_2)) \geq \min\{|x_1 - x_2|,\ (2r - |x_1| - |x_2|)\}.
\end{aligned}$$

This shows that the maps φ are locally distance preserving and injective. Hence they are distance preserving maps. This shows that they are also Riemannian isometries. This finishes the proof. $\square$

3.5. Alternative Norms. Finally, we should mention that all properties of this norm concept would not change if we changed n1-n4 to say

(n1') U_s has Lebesgue number $f_1(n,Q,r)$.
(n2') $|D\varphi_s|, \left|D\varphi_s^{-1}\right| \leq f_2(n,Q)$.
(n3') $r^{|j|+\alpha} \cdot \|\partial^j g_{s\cdot\cdot}\|_\alpha \leq f_3(n,Q), \quad 0 \leq |j| \leq m$.
(n4') $\|\varphi_s^{-1} \circ \varphi_t\|_{C^{m+1,\alpha}} \leq f_4(n,Q,r)$.

As long as the f_is are all continuous, $f_1(n,0,r) = 0$, and $f_2(n,0) = 1$. The key properties we want to preserve are continuity of $\|(M,g)\|$ with respect to r, the fundamental theorem, and the characterization of flat manifolds and Euclidean space.

Another interesting thing happens if in the definition of $\|(M,g)\|_{C^{m,\alpha},r}$ we let $m = \alpha = 0$. Then n3 no longer makes sense, because $\alpha = 0$, but aside from that, we still have a C^0-norm concept. Note also that n4 is an immediate consequence of n2 in this case. The class $\mathcal{M}^0(n,Q,r)$ is now only precompact in the pointed Gromov-Hausdorff topology, but the characterization of flat manifolds is still valid. The subclasses with bounded diameter, or volume, are also only precompact with respect to the Gromov-Hausdorff topology, and the finiteness of diffeomorphism types apparently fails. It is, however, possible to say more. If we investigate the proof of the fundamental theorem, we see that the problem lies in constructing the maps $F_{ik} : \Omega_k \to \Omega_{ik}$, because we now have convergence of the coordinates only in the C^0 (actually $C^\alpha, \alpha < 1$) topology, and so the averaging process fails as it is described. We can, however, use a deep theorem from topology about local contractibility of homeomorphism groups (see [**35**]) to conclude that two C^0-close topological embeddings can be "glued" together in some way without altering them too much in the C^0 topology. This makes it possible to exhibit topological embeddings $F_{ik} : \Omega \hookrightarrow M_i$ such that the pullback metrics (not Riemannian metrics) converge. As a consequence, we see that the classes with bounded diameter or volume contain only finitely many homeomorphism types. This is exactly the content of the original version of Cheeger's finiteness theorem, including the proof as we have outlined it. But, as we have pointed out earlier, Cheeger also considered the easier to prove finiteness theorem for diffeomorphism types given better bounds on the coordinates.

Notice that we cannot easily use the fact that the charts converge in $C^\alpha (\alpha < 1)$. But it is possible to do something interesting along these lines. There is an even weaker norm concept called the *Reifenberg norm* which is related to the Gromov-Hausdorff distance. For a metric space (X, d) we define the n-dimensional norm on the scale of r as

$$\|(X,d)\|_r^n = \frac{1}{r} \sup_{p \in X} d_{G-H} \left(B\left(p,r\right), B\left(0,r\right)\right),$$

where $B(0,R) \subset \mathbb{R}^n$. The the r^{-1} factor insures that we don't have small distance between $B(p,r)$ and $B(0,r)$ just because r is small. Note also that if $(X_i, d_i) \to (X, d)$ in the Gromov-Hausdorff topology then

$$\|(X_i,d_i)\|_r^n \to \|(X,d)\|_r^n$$

for fixed n, r.

For an n-dimensional Riemannian manifold one sees immediately that

$$\lim_{r \to 0} \|(M,g)\|_r^n \to 0 = 0.$$

Cheeger and Colding have proven a converse to this (see [**25**]). There is an $\varepsilon(n) > 0$ such that if $\|(X,d)\|_r^n \le \varepsilon(n)$ for all small r, then X is in a weak sense an n-dimensional Riemannian manifold. Among other things, they show that for small r the α-Hölder distance between $B(p,r)$ and $B(0,r)$ is small. Here the α-Hölder distance $d_\alpha(X,Y)$ between metric spaces is defined as the infimum of

$$\log \max \left\{ \sup_{x_1 \neq x_2} \frac{d\left(F\left(x_1\right), F\left(x_2\right)\right)}{\left(d\left(x_1,x_2\right)\right)^\alpha}, \sup_{y_1 \neq y_2} \frac{d\left(F^{-1}\left(y_1\right), F^{-1}\left(y_2\right)\right)}{\left(d\left(y_1,y_2\right)\right)^\alpha} \right\},$$

where $F : X \to Y$ runs over all homeomorphisms. They also show that if $(M_i, g_i) \to (X, d)$ in the Gromov-Hausdorff distance and $\|(M_i,g_i)\|_r^n \le \varepsilon(n)$ for all i and small r, then $(M_i, g_i) \to (X, d)$ in the Hölder distance. In particular, all of the M_is have to be homeomorphic (and in fact diffeomorphic) to X for large i.

This is enhanced by an earlier result of Colding (see [**29**]) stating that for a Riemannian manifold (M, g) with $\mathrm{Ric} \ge (n-1)k$ we have that $\|(M,g)\|_r^n$ is small iff and only if

$$\mathrm{vol} B(p,r) \ge (1-\delta)\,\mathrm{vol} B(0,r)$$

for some small δ. Relative volume comparison tells us that the volume condition holds for all small r if it holds for just one r. Thus the smallness condition for the norm holds for all small r provided we have the volume condition for just some r.

4. Geometric Applications

We shall now study the relationship between volume, injectivity radius, sectional curvature, and the norm.

First let us see what exponential coordinates can do for us. Let (M,g) be a Riemannian manifold with $|\mathrm{sec}M| \leq K$ and $\mathrm{inj}M \geq i_0$. On $B(0,i_0)$ we have from chapter 6 that

$$\max\left\{\left|D\exp_p\right|, \left|D\exp_p^{-1}\right|\right\} \leq \exp\left(f\left(n,K,i_0\right)\right)$$

for some function $f(n,K,i_0)$ that depends only on the dimension, K, and i_0. Moreover, as $K \to 0$ we have that $f(n,K,i_0) \to 0$. This implies

Theorem 73. *For every $Q > 0$ there exists $r > 0$ depending only on i_0 and K such that any complete (M,g) with $|\mathrm{sec}M| \leq K$, $\mathrm{inj}M \geq i_0$ has $\|(M,g)\|_{C^0,r} \leq Q$. Furthermore, if (M_i,p_i,g_i) satisfy $\mathrm{inj}M_i \geq i_0$ and $|\mathrm{sec}M_i| \leq K_i \to 0$, then a subsequence will converge in the pointed Gromov-Hausdorff topology to a flat manifold with $\mathrm{inj} \geq i_0$.*

The proof follows immediately from our previous constructions.

This theorem does not seem very satisfactory, because even though we have assumed a C^2 bound on the Riemannian metric, we get only a C^0 bound. To get better bounds under the same circumstances, we must look for different coordinates. Our first choice for alternative coordinates uses distance functions, i.e., distance coordinates.

Lemma 50. *Given a Riemannian manifold (M,g) with $\mathrm{inj} \geq i_0$, $|\mathrm{sec}| \leq K$, and $p \in M$, then the distance function $d(x) = d(x,p)$ is smooth on $B(p,i_0)$, and the Hessian is bounded in absolute value on the annulus $B(p,i_0) - B(p,i_0/2)$ by a function $F(n,K,i_0)$.*

Proof. From chapter 6 we know that in polar coordinates

$$\sqrt{K}\cot\left(\sqrt{K}r\right) g_r \leq \mathrm{Hess}d \leq \sqrt{K}\coth\left(\sqrt{K}r\right) g_r.$$

Thus, we get the desired estimate as long as $r \in (i_0/2, i_0)$. □

Now fix (M,g), $p \in M$, as in the lemma, and choose an orthonormal basis $e_1,\ldots,e_n$ for T_pM. Then consider the geodesics $\gamma_i(t)$ with $\gamma_i(0) = p$, $\dot\gamma_i(0) = e_i$, and together with those, the distance functions

$$d_i(x) = d\left(x, \gamma_i\left(i_0 \cdot \left(4\sqrt{K}\right)^{-1}\right)\right).$$

These distance functions will then have uniformly bounded Hessians on $B(p,\delta)$, $\delta = i_0 \cdot \left(8\sqrt{K}\right)^{-1}$. Define

$$\varphi(x) = (d_1(x),\ldots,d_n(x))$$

and recall that $g^{ij} = g(\nabla d_i, \nabla d_j)$.

Theorem 74. (The Convergence Theorem of Riemannian Geometry) *Given i_0, $K > 0$, there exist $Q, r > 0$ such that any (M,g) with*

$$\begin{aligned} \mathrm{inj} &\geq i_0, \\ |\mathrm{sec}| &\leq K \end{aligned}$$

has $\|(M,g)\|_{C^1,r} \leq Q$. *In particular, this class is compact in the pointed* C^α *topology for all* $\alpha < 1$.

PROOF. The inverse of φ is our potential chart. First, observe that $g_{ij}(p) = \delta_{ij}$, so the uniform Hessian estimate shows that $|D\varphi_p| \leq e^Q$ on $B(p,\varepsilon)$ and $\left|(D\varphi_p)^{-1}\right| \leq e^Q$ on $B(0,\varepsilon)$, where Q, ε depend only on i_0, K. The proof of the inverse function theorem then tells us that there is an $\hat{\varepsilon} > 0$ depending only on Q, n such that $\varphi : B(0,\hat{\varepsilon}) \to \mathbb{R}^n$ is one-to-one. We can then easily find r such that

$$\varphi^{-1} : B(0,r) \to U_p \subset B(p,\varepsilon)$$

satisfies n2. The conditions n3 and n4 now immediately follow from the Hessian estimates, except, we might have to increase Q somewhat. Finally, n1 holds since we have coordinates centered at every $p \in M$. $\square$

Notice that Q cannot be chosen arbitrarily small, as our Hessian estimates cannot be improved by going to smaller balls. This will be taken care of in the next section by using a different set of coordinates. This convergence result, as stated, was first proven by M. Gromov. The reader should be aware that what Gromov refers to as a $C^{1,1}$-manifold is in our terminology a manifold with $\|(M,h)\|_{C^{0,1},r} < \infty$, i.e., $C^{0,1}$-bounds on the Riemannian metric.

Using the diameter bound in positive curvature and Klingenberg's estimate for the injectivity radius from chapter 6 we get

COROLLARY 36. (J. Cheeger, 1967) *For given* $n \geq 1$ *and* $k > 0$, *the class of Riemannian* $2n$*-manifolds with* $k \leq \sec \leq 1$ *is compact in the* C^α *topology and consequently contains only finitely many diffeomorphism types.*

A similar result was also proven by A. Weinstein at the same time. The hypotheses are the same, but Weinstein only showed that the class contained finitely many homotopy types.

Our next result shows that one can bound the injectivity radius provided that one has lower volume bounds and bounded curvature. This result is usually referred to as Cheeger's lemma. With a little extra work one can actually prove this lemma for complete manifolds. This requires that we work with pointed spaces and also to some extent incomplete manifolds as it isn't clear from the beginning that the complete manifolds in question have global lower bounds for the injectivity radius.

LEMMA 51. (J. Cheeger, 1967) *Given* $n \geq 2$ *and* $v, K \in (0,\infty)$ *and a compact* n*-manifold* (M,g) *with*

$$\begin{aligned} |\sec| &\leq K, \\ \mathrm{vol}B(p,1) &\geq v, \end{aligned}$$

for all $p \in M$, *then* $\mathrm{inj}M \geq i_0$, *where* i_0 *depends only on* n, K, *and* v.

PROOF. The proof goes by contradiction using the previous theorem. So assume we have (M_i, g_i) with $\mathrm{inj}M_i \to 0$ and satisfying the assumptions of the lemma. Find $p_i \in M_i$ such that $\mathrm{inj}_{p_i} = \mathrm{inj}(M_i, g_i)$, and consider the pointed sequence $(M_i, p_i, \bar{g}_i)$, where $\bar{g}_i = (\mathrm{inj}M_i)^{-2} g_i$ is rescaled so that

$$\begin{aligned} \mathrm{inj}(M_i, \bar{g}_i) &= 1, \\ |\sec(M_i, \bar{g}_i)| &\leq (\mathrm{inj}(M_i, g_i))^2 \cdot K = K_i \to 0. \end{aligned}$$

The two previous theorems, together with the fundamental theorem, then implies that some subsequence of $(M_i, p_i, \bar{g}_i)$ will converge in the pointed $C^\alpha, \alpha < 1$, topology to a flat manifold (M, p, g).

The first observation about (M, p, g) is that $\mathrm{inj}(p) \leq 1$. This follows because the conjugate radius for $(M_i, \bar{g}_i) \geq \pi/\sqrt{K_i} \to \infty$, so Klingenberg's estimate for the injectivity radius implies that there must be a geodesic loop of length 2 at $p_i \in M_i$. Since $(M_i, p_i, \bar{g}_i) \to (M, p, g)$ in the pointed C^α topology, the geodesic loops must converge to a geodesic loop in M based at p of length 2. Hence, $\mathrm{inj}(M) \leq 1$.

The other contradictory observation is that $(M, g) = (\mathbb{R}^n, \mathrm{can})$. Recall that $\mathrm{vol}B(p_i, 1) \geq v$ in (M_i, g_i), so relative volume comparison shows that there is a $v'(n, K, v)$ such that $\mathrm{vol}B(p_i, r) \geq v' \cdot r^n$, for $r \leq 1$. The rescaled manifold $(M_i, \bar{g}_i)$ therefore satisfies $\mathrm{vol}B(p_i, r) \geq v' \cdot r^n$, for $r \leq (\mathrm{inj}(M_i, g_i))^{-1}$. Using again that $(M_i, p_i, \bar{g}_i) \to (M, p, g)$ in the pointed C^α topology, we get $\mathrm{vol}B(p, r) \geq v' \cdot r^n$ for all r. Since (M, g) is flat, this shows that it must be Euclidean space.

This last statement requires some justification. Let M be a complete flat manifold. As the elements of the fundamental group act by isometries on Euclidean space, we know that they must have infinite order (any isometry of finite order is a rotation around a point and therefore has a fixed point). Therefore, if M is not simply connected, then there is an intermediate covering $\hat{M}$:

$$\mathbb{R}^n \to \hat{M} \to M,$$

where $\pi_1\left(\hat{M}\right) = \mathbb{Z}$. This means that $\hat{M}$ looks like a cylinder. Hence, for any $p \in \hat{M}$ we must have

$$\lim_{r \to \infty} \frac{\mathrm{vol}B\left(p, r\right)}{r^{n-1}} < \infty.$$

The same must then also hold for M itself, contradicting our volume growth assumption. □

This lemma was proved with a more direct method by Cheeger. We have included this, perhaps more convoluted, proof in order to show how our convergence theory can be used. The lemma also shows that the convergence theorem of Riemannian geometry remains true if the injectivity radius bound is replaced by a lower bound on the volume of 1-balls. The following result is now immediate.

Corollary 37. (J. Cheeger, 1967) *Let $n \geq 2$, $\Lambda, D, v \in (0, \infty)$ be given. The class of closed Riemannian n-manifolds with*

$$\begin{aligned} |\sec| &\leq \Lambda, \\ \mathrm{diam} &\leq D, \\ \mathrm{vol} &\geq v \end{aligned}$$

is precompact in the C^α topology for any $\alpha \in (0, 1)$ and in particular, contains only finitely many diffeomorphism types.

This convergence theorem can be generalized in another interesting direction, as observed by S.-h. Zhu.

Theorem 75. *Given $i_0, k > 0$, there exist Q, r depending on i_0, k such that any manifold (M, g) with*

$$\begin{aligned} \sec &\geq -k^2, \\ \mathrm{inj} &\geq i_0 \end{aligned}$$

satisfies $\|(M,g)\|_{C^{1},r} \leq Q$.

PROOF. It suffices to get a Hessian estimate for distance functions $d(x) = d(x,p)$. We have, as before, that

$$\mathrm{Hess}d(x) \leq k \cdot \coth(k \cdot d(x))g_r.$$

Conversely, if $d(x_0) < i_0$, then $d(x)$ is supported from below by $f(x) = i_0 - d(x, y_0)$, where $y_0 = \gamma(i_0)$ and γ is the unique unit speed geodesic that minimizes the distance from p to x_0. Thus, $\mathrm{Hess}d(x) \geq \mathrm{Hess}f$ at x_0. But

$$\mathrm{Hess}f \geq -k \cdot \coth(d(x_0, y_0) \cdot k)g_r = -k \cdot \coth(k(i_0 - r(x_0)))g_r$$

at x_0. Hence, we have two-sided bounds for $\mathrm{Hess}d(x)$ on appropriate sets. The proof can then be finished as before. $\square$

This theorem is interestingly enough optimal. Consider rotationally symmetric metrics $dr^2 + f_\varepsilon^2(r)d\theta^2$, where f_ε is concave and satisfies

$$f_\varepsilon(r) = \begin{cases} r & \text{for } 0 \leq r \leq 1 - \varepsilon, \\ \frac{3}{4}r & \text{for } 1 + \varepsilon \leq r. \end{cases}$$

These metrics have $\sec \geq 0$ and $\mathrm{inj} \geq 1$. As $\varepsilon \to 0$, we get a $C^{1,1}$ manifold with a $C^{0,1}$ Riemannian metric (M, g). In particular, $\|(M,g)\|_{C^{0,1},r} < \infty$ for all r. Limit spaces of sequences with $\mathrm{inj} \geq i_0$, $\sec \geq k$ can therefore not in general be assumed to be smoother than the above example.

With a more careful construction, we can also find g_ε with

$$g_\varepsilon(r) = \begin{cases} \sin r & \text{for } 0 \leq r \leq \frac{\pi}{2} - \varepsilon, \\ 1 & \text{for } \frac{\pi}{2} \leq r. \end{cases}$$

Then the metric $dr^2 + g_\varepsilon^2(r)d\theta^2$ satisfies $|\sec| \leq 4$ and $\mathrm{inj} \geq \frac{1}{4}$. As $\varepsilon \to 0$, we get a limit metric that is $C^{1,1}$. So while we may suspect (this is still unknown) that limit metrics from the convergence theorem are $C^{1,1}$, we prove only that they are $C^{0,1}$. In the next section we shall show that they are in fact $C^{1,\alpha}$ for all $\alpha < 1$.

5. Harmonic Norms and Ricci curvature

To get better estimates on the norms, we must use some more analysis. The idea of using harmonic coordinates for similar purposes goes back to [**33**]. In [**57**] it was shown that manifolds with bounded sectional curvature and lower bounds for the injectivity radius admit harmonic coordinates on balls of an a priori size. This result was immediately seized by the geometry community and put to use in improving the theorems from the previous section. At the same time, Nikolaev developed a different, more synthetic approach to these ideas. For the whole story we refer the reader to Greene's survey in [**45**]. Here we shall develop these ideas from a different point of view initiated by Anderson.

5.1. The Harmonic Norm. We shall now define another norm, called the *harmonic norm* and denoted

$$\|A \subset (M,g)\|^{harm}_{C^{m,\alpha},r}.$$

The only change in our previous definition is that condition n4 is replaced by the requirement that $\varphi_s^{-1} : U_s \to \mathbb{R}^n$ be harmonic with respect to the Riemannian metric g on M. Recall that this is equivalent to saying that for each j

$$\frac{1}{\sqrt{\det g_{st}}}\partial_i\left(\sqrt{\det g_{st}}\cdot g^{ij}\right)=0$$

We can use the elliptic estimates to compare this norm with our old norm. Namely, recall that in harmonic coordinates $\Delta = g^{ij}\partial_i\partial_j$, conditions n2 and n3 insure that these coefficients are bounded in the required way. Therefore, if $u : U \to \mathbb{R}$ is any harmonic function, then we get that on compact subsets $K \subset U \cap U_s$,

$$\|u\|_{C^{m+1,\alpha},K} \leq C\,\|u\|_{C^{\alpha},U}\,.$$

Using a coordinate function φ_t^{-1} as u then shows that we can get bounds for the transition functions on compact subsets of their domains. Changing the scale will then allow us to conclude that for each $r_1 < r_2$, there is a constant $C = C(n, m, \alpha, r_1, r_2)$ such that

$$\|A \subset (M,g)\|_{C^{m,\alpha},r_1} \leq C\,\|A \subset (M,g)\|^{harm}_{C^{m,\alpha},r_2}\,.$$

We can then show the harmonic analogue to the fundamental theorem.

Corollary 38. *For given $Q > 0$, $n \geq 2$, $m \geq 0, \alpha \in (0,1]$, and $r > 0$ consider the class of complete, pointed Riemannian n-manifolds (M,p,g) with $\|(M,g)\|^{harm}_{C^{m,\alpha},r} \leq Q$. This class is closed in the pointed $C^{m,\alpha}$ topology and compact in the pointed $C^{m,\beta}$ topology for all $\beta < \alpha$.*

The only issue to worry about is whether it is really true that limit spaces have $\|(M,g)\|^{harm}_{C^{m,\alpha},r} \leq Q$. But one can easily see that harmonic charts converge to harmonic charts. This is also discussed in the next proposition.

Proposition 46. (M. Anderson, 1990) *If $A \subset (M,g)$ is precompact, then:*

(1) $\|A \subset (M,g)\|^{harm}_{C^{m,\alpha},r} = \left\|A \subset \left(M,\lambda^2 g\right)\right\|^{harm}_{C^{m,\alpha},\lambda r}$ for all $\lambda > 0$.

(2) The function $r \to \|A \subset (M,g)\|^{harm}_{C^{m,\alpha},r}$ is continuous. Moreover, when $m \geq 1$, it converges to 0 as $r \to 0$.

(3) Suppose $(M_i, p_i, g_i) \to (M,p,g)$ in $C^{m,\alpha}$ and in addition that $m \geq 1$. Then for $A \subset M$ we can find precompact domains $A_i \subset M_i$ such that

$$\|A_i\|^{harm}_{C^{m,\alpha},r} \to \|A\|^{harm}_{C^{m,\alpha},r}$$

for all $r > 0$. When all the manifolds are closed, we can let $A = M$ and $A_i = M_i$.

(4) $\|A \subset (M,g)\|^{harm}_{C^{m,\alpha},r} = \sup_{p\in A}\|\{p\} \subset (M,g)\|^{harm}_{C^{m,\alpha},r}$.

Proof. Properties (1) and (2) are proved as for the regular norm. For the statement that the norm goes to zero as the scale decreases, just solve the Dirichlet problem as we did when existence of harmonic coordinates was established. Here it was necessary to have coordinates around every point $p \in M$ such that in these coordinates the metric satisfies $g_{ij} = \delta_{ij}$ and $\partial_k g_{ij} = 0$ at p. If $m \geq 1$, then it is easy to show that any coordinates system around p can be changed in such a way that the metric has the desired properties.

(3) The proof of this statement is necessarily somewhat different, as we must use and produce harmonic coordinates. Let the set-up be as before. First we show the easy part:

$$\liminf \|A_i\|^{harm}_{C^{m,\alpha},r} \geq \|A\|^{harm}_{C^{m,\alpha},r}\,.$$

To this end, select $Q > \liminf \|A_i\|^{harm}_{C^{m,\alpha},r}$. For large i we can then select charts $\varphi_{i,s} : B(0,r) \to M_i$ with the requisite properties. After passing to a subsequence, we can make these charts converge to charts

$$\varphi_s = \lim F_i^{-1} \circ \varphi_{i,s} : B(0,r) \to M.$$

Since the metrics converge in $C^{m,\alpha}$, the Laplacians of the inverse functions must also converge. Hence, the limit charts are harmonic as well. We can then conclude that $\|A\|^{harm}_{C^{m,\alpha},r} \leq Q$.

For the reverse inequality

$$\limsup \|A_i\|^{harm}_{C^{m,\alpha},r} \leq \|A\|^{harm}_{C^{m,\alpha},r},$$

select $Q > \|A\|^{harm}_{C^{m,\alpha},r}$. Then, from the continuity of the norm we can find $\varepsilon > 0$ such that also $\|A\|^{harm}_{C^{m,\alpha},r+\varepsilon} < Q$. For this scale, select charts

$$\varphi_s : B(0, r+\varepsilon) \to U_s \subset M$$

satisfying the usual conditions. Now define

$$U_{i,s} = F_i(\varphi_s(B(0, r+\varepsilon/2))) \subset M_i.$$

This is clearly a closed disc with smooth boundary

$$\partial U_{i,s} = F_i(\varphi_s(\partial B(0, r+\varepsilon/2))).$$

On each $U_{i,s}$ solve the Dirichlet problem

$$\begin{aligned} \psi_{i,s} &: U_{i,s} \to \mathbb{R}^n, \\ \Delta_{g_i}\psi_{i,s} &= 0, \\ \psi_{i,s} &= \varphi_s^{-1} \circ F_i^{-1} \text{ on } \partial U_{i,s}. \end{aligned}$$

The inverse of $\psi_{i,s}$, if it exists, will then be a coordinate map $B(0,r) \to U_{i,s}$. On the set $B(0, r+\varepsilon/2)$ we can now compare $\psi_{i,s} \circ F_i \circ \varphi_s$ with the identity map I. Note that these maps agree on the boundary of $B(0, r+\varepsilon/2)$. We know that $F_i^* g_i \to g$ in the fixed coordinate system φ_s. Now pull these metrics back to $B\left(0, r+\frac{\varepsilon}{2}\right)$ and refer to them as $g\,(= \varphi_s^* g)$ and $g_i\,(= \varphi_s^* F_i^* g_i)$. In this way the harmonicity conditions read $\Delta_g I = 0$ and $\Delta_{g_i}\psi_{i,s} \circ F_i \circ \varphi_s = 0$. In these coordinates we have the correct bounds for the operator

$$\Delta_{g_i} = g_i^{kl}\partial_k\partial_l + \frac{1}{\sqrt{\det g_i}}\partial_k\left(\sqrt{\det g_i} \cdot g_i^{kl}\right)\partial_l$$

to use the elliptic estimates for domains with smooth boundary. Note that this is where the condition $m \geq 1$ becomes important, so that we can bound

$$\frac{1}{\sqrt{\det g_i}}\partial_k\left(\sqrt{\det g_i} \cdot g_i^{kl}\right)$$

in C^α. The estimates then imply

$$\begin{aligned} \left\|I - \psi_{i,s} \circ F_i \circ \varphi_s\right\|_{C^{m+1,\alpha}} &\leq C\left\|\Delta_{g_i}\left(I - \psi_{i,s} \circ F_i \circ \varphi_s\right)\right\|_{C^{m-1,\alpha}} \\ &= C\left\|\Delta_{g_i} I\right\|_{C^{m-1,\alpha}}. \end{aligned}$$

However, we have that

$$\begin{aligned}\|\Delta_{g_i} I\|_{C^{m-1,\alpha}} &= \left\|\frac{1}{\sqrt{\det g_i}}\partial_k\left(\sqrt{\det g_i}\cdot g_i^{kl}\right)\right\|_{C^{m-1,\alpha}}\\ &\to \left\|\frac{1}{\sqrt{\det g}}\partial_k\left(\sqrt{\det g}\cdot g^{kl}\right)\right\|_{C^{m-1,\alpha}}\\ &= \|\Delta_g I\|_{C^{m-1,\alpha}} = 0.\end{aligned}$$

In particular, we must have

$$\left\|I - \psi_{i,s}\circ F_i\circ\varphi_s\right\|_{C^{m+1,\alpha}} \to 0.$$

It is now evident that $\psi_{i,s}$ must become coordinates for large i. Also, these coordinates will show that $\|A_i\|^{harm}_{C^{m,\alpha},r} < Q$ for large i.

(4) Since there is no transition function condition to be satisfied in the definition of $\|A\|^{harm}_{C^{m,\alpha},r}$, it is obvious that

$$\|A\cup B\|^{harm}_{C^{m,\alpha},r} = \max\left\{\|A\|^{harm}_{C^{m,\alpha},r}, \|B\|^{harm}_{C^{m,\alpha},r}\right\}.$$

This shows that the norm is always realized locally. □

5.2. Ricci Curvature and the Harmonic Norm. The most important feature about harmonic coordinates is that the metric is apparently controlled by the Ricci curvature. This is exploited in the next lemma, where we show how one can bound the harmonic $C^{1,\alpha}$ norm in terms of the harmonic C^1 norm and Ricci curvature.

LEMMA 52. (M. Anderson, 1990) *Suppose that a Riemannian manifold (M,g) has bounded Ricci curvature $|\mathrm{Ric}|\le\Lambda$. For any $r_1<r_2$, $K\ge\|A\subset(M,g)\|^{harm}_{C^1,r_2}$, and $\alpha\in(0,1)$ we can find $C(n,\alpha,K,r_1,r_2,\Lambda)$ such that*

$$\|A\subset(M,g)\|^{harm}_{C^{1,\alpha},r_1}\le C(n,\alpha,K,r_1,r_2,\Lambda).$$

Moreover, if g is an Einstein metric $\mathrm{Ric}=kg$, then for each integer m we can find a constant $C(n,\alpha,K,r_1,r_2,k,m)$ such that

$$\|A\subset(M,g)\|^{harm}_{C^{m+1,\alpha},r_1}\le C(n,\alpha,K,r_1,r_2,k,m).$$

PROOF. We just need to bound the metric components g_{ij} in some fixed harmonic coordinates. In these coordinates we have that $\Delta = g^{ij}\partial_i\partial_j$. Given that $\|A\subset(M,g)\|^{harm}_{C^1,r_2}\le K$, we can conclude that we have the necessary conditions on the coefficients of $\Delta = g^{ij}\partial_i\partial_j$ to use the elliptic estimate

$$\|g_{ij}\|_{C^{1,\alpha},B(0,r_1)}\le C(n,\alpha,K,r_1,r_2)\left(\|\Delta g_{ij}\|_{C^0,B(0,r_2)}+\|g_{ij}\|_{C^\alpha,B(0,r_2)}\right).$$

Now use that

$$\Delta g_{ij} = -2\mathrm{Ric}_{ij} - 2Q(g,\partial g)$$

to conclude that

$$\|\Delta g_{ij}\|_{C^0,B(0,r_2)}\le 2\Lambda\|g_{ij}\|_{C^0,B(0,r_2)}+\hat{C}\|g_{ij}\|_{C^1,B(0,r_2)}.$$

Using this we then have

$$\begin{aligned}\|g_{ij}\|_{C^{1,\alpha},B(0,r_1)} &\le C(n,\alpha,K,r_1,r_2)\left(\|\Delta g_{ij}\|_{C^0,B(0,r_2)}+\|g_{ij}\|_{C^\alpha,B(0,r_2)}\right)\\ &\le C(n,\alpha,K,r_1,r_2)\left(2\Lambda+\hat{C}+1\right)\|g_{ij}\|_{C^1,B(0,r_2)}.\end{aligned}$$

For the Einstein case we can use a bootstrap method, as we get $C^{1,\alpha}$ bounds on the Ricci tensor from the Einstein equation $\text{Ric} = kg$. Thus, we have that Δg_{ij} is bounded in C^{α} rather than just C^0. Hence,

$$\begin{aligned}\|g_{ij}\|_{C^{2,\alpha},B(0,r_1)} &\leq C(n,\alpha,K,r_1,r_2)\left(\|\Delta g_{ij}\|_{C^{\alpha},B(0,r_2)} + \|g_{ij}\|_{C^{\alpha},B(0,r_2)}\right) \\ &\leq C(n,\alpha,K,r_1,r_2,k)\cdot C\cdot \|g_{ij}\|_{C^{1,\alpha},B(0,r_2)}.\end{aligned}$$

This gives $C^{2,\alpha}$ bounds on the metric. Then, of course, Δg_{ij} is bounded in $C^{1,\alpha}$, and thus the metric will be bounded in $C^{3,\alpha}$. Clearly, one can iterate this until one gets $C^{m+1,\alpha}$ bounds on the metric. □

Combining this with the fundamental theorem gives a very interesting compactness result.

COROLLARY 39. *For given $n \geq 2, Q, r, \Lambda \in (0,\infty)$ consider the class of Riemannian n-manifolds with*

$$\begin{aligned}\|(M,g)\|^{harm}_{C^1,r} &\leq Q,\\ |\text{Ric}| &\leq \Lambda.\end{aligned}$$

This class is precompact in the pointed $C^{1,\alpha}$ topology for any $\alpha \in (0,1)$. Moreover, if we take the subclass of Einstein manifolds, then this class is compact in the $C^{m,\alpha}$ topology for any $m \geq 0$ and $\alpha \in (0,1)$.

We can now prove our generalizations of the convergence theorems from the last section.

THEOREM 76. (M. Anderson, 1990) *Given $n \geq 2$ and $\alpha \in (0,1)$, Λ, $i_0 > 0$, one can for each $Q > 0$ find $r(n,\alpha,\Lambda,i_0) > 0$ such that any complete Riemannian n-manifold (M,g) with*

$$\begin{aligned}|\text{Ric}| &\leq \Lambda,\\ \text{inj} &\geq i_0\end{aligned}$$

satisfies $\|(M,g)\|^{harm}_{C^{1,\alpha},r} \leq Q$.

PROOF. The proof goes by contradiction. So suppose that there is a $Q > 0$ such that for each $i \geq 1$ there is a Riemannian manifold (M_i, g_i) with

$$\begin{aligned}|\text{Ric}| &\leq \Lambda,\\ \text{inj} &\geq i_0,\\ \|(M_i,g_i)\|^{harm}_{C^{1,\alpha},i^{-1}} &> Q.\end{aligned}$$

Using that the norm goes to zero as the scale goes to zero, and that it is continuous as a function of the scale, we can for each i find $r_i \in \left(0, i^{-1}\right)$ such that $\|(M_i,g_i)\|^{harm}_{C^{1,\alpha},r_i} = Q$. Now rescale these manifolds: $\bar{g}_i = r_i^{-2} g_i$. Then we have that $(M_i, \bar{g}_i)$ satisfies

$$\begin{aligned}|\text{Ric}| &\leq r_i^2\Lambda,\\ \text{inj} &\geq r_i^{-1} i_0,\\ \|(M_i,\bar{g}_i)\|^{harm}_{C^{1,\alpha},1} &= Q.\end{aligned}$$

We can then select $p_i \in M_i$ such that

$$\|p_i \in (M_i, \bar{g}_i)\|^{harm}_{C^{1,\alpha},1} \in \left[\frac{Q}{2}, Q\right].$$

The first important step is now to use the bounded Ricci curvature of $(M_i, \bar{g}_i)$ to conclude that in fact the $C^{1,\gamma}$ norm must be bounded for any $\gamma \in (\alpha, 1)$. Then we can assume by the fundamental theorem that the sequence $(M_i, p_i, \bar{g}_i)$ converges in the pointed $C^{1,\alpha}$ topology, to a Riemannian manifold (M, p, g) of class $C^{1,\gamma}$. Since the $C^{1,\alpha}$ norm is continuous in the $C^{1,\alpha}$ topology we can conclude that

$$\|p \in (M, g)\|^{harm}_{C^{1,\alpha},1} \in \left[\frac{Q}{2}, Q\right].$$

The second thing we can prove is that $(M, g) = (\mathbb{R}^n, \mathrm{can})$. This clearly violates what we just established about the norm of the limit space. To see that the limit space is Euclidean space, recall that the manifolds in the sequence $(M_i, \bar{g}_i)$ are covered by harmonic coordinates that converge to harmonic coordinates in the limit space. In these harmonic coordinates the metric components satisfy

$$\frac{1}{2}\Delta \bar{g}_{kl} + Q(\bar{g}, \partial \bar{g}) = -\mathrm{Ric}_{kl}.$$

But we know that

$$|-\mathrm{Ric}| \le r_i^{-2}\Lambda \bar{g}_i$$

and that the $\bar{g}_{kl}$ converge in the $C^{1,\alpha}$ topology to the metric coefficients g_{kl} for the limit metric. We can therefore conclude that the limit manifold is covered by harmonic coordinates and that in these coordinates the metric satisfies:

$$\frac{1}{2}\Delta g_{kl} + Q(g, \partial g) = 0.$$

The limit metric is therefore a weak solution to the Einstein equation $\mathrm{Ric} = 0$ and must therefore be a smooth Ricci flat Riemannian manifold. It is now time to use that: $\mathrm{inj}(M_i, \bar{g}_i) \to \infty$. In the limit space we have that any geodesic is a limit of geodesics from the sequence $(M_i, \bar{g}_i)$, since the Riemannian metrics converge in the $C^{1,\alpha}$ topology. If a geodesic in the limit is a limit of segments, then it must itself be a segment. We can then conclude that as $\mathrm{inj}(M_i, \bar{g}_i) \to \infty$ any finite length geodesic must be a segment. This, however, implies that $\mathrm{inj}(M, g) = \infty$. The splitting theorem then shows that the limit space is Euclidean space. □

From this theorem we immediately get

Corollary 40. (M. Anderson, 1990) *Let $n \ge 2$ and Λ, D, $i \in (0, \infty)$ be given. The class of closed Riemannian n-manifolds satisfying*

$$\begin{aligned} |\mathrm{Ric}| &\le \Lambda, \\ \mathrm{diam} &\le D, \\ \mathrm{inj} &\ge i \end{aligned}$$

is precompact in the $C^{1,\alpha}$ topology for any $\alpha \in (0, 1)$ and in particular contains only finitely many diffeomorphism types.

Notice how the above theorem depended on the characterization of Euclidean space we obtained from the splitting theorem. There are other similar characterizations of Euclidean space. One of the most interesting ones uses volume pinching.

5.3. Volume Pinching. The idea is to use the relative volume comparison theorem rather than the splitting theorem. We know from the exercises to chapter 9 that Euclidean space is the only space with

$$\begin{aligned} \text{Ric} &\geq 0, \\ \lim_{r\to\infty} \frac{\text{vol}B(p,r)}{\omega_n r^n} &= 1, \end{aligned}$$

where $\omega_n r^n$ is the volume of a Euclidean ball of radius r. This result has a very interesting gap phenomenon associated with it, when one assumes the stronger hypothesis that the space is Ricci flat.

Lemma 53. (M. Anderson, 1990) *For each $n \geq 2$ there is an $\varepsilon(n) > 0$ such that any complete Ricci flat manifold (M,g) that satisfies*

$$\text{vol}B(p,r) \geq (\omega_n - \varepsilon) r^n$$

for some $p \in M$ is isometric to Euclidean space.

Proof. First observe that on any complete Riemannian manifold with $\text{Ric} \geq 0$, relative volume comparison can be used to show that

$$\text{vol}B(p,r) \geq (1-\varepsilon)\,\omega_n r^n$$

as long as

$$\lim_{r\to\infty} \frac{\text{vol}B(p,r)}{\omega_n r^n} \geq (1-\varepsilon).$$

It is then easy to see that if this holds for one p, then it must hold for all p. Moreover, if we scale the metric to $(M, \lambda^2 g)$, then the same volume comparison still holds, as the lower curvature bound $\text{Ric} \geq 0$ can't be changed by scaling.

If our assertion were not true, then we could for each integer i find Ricci flat manifolds (M_i, g_i) with

$$\begin{aligned} \lim_{r\to\infty} \frac{\text{vol}B(p_i,r)}{\omega_n r^n} &\geq \left(1 - i^{-1}\right), \\ \|(M_i, g_i)\|^{harm}_{C^{1,\alpha},r} &\neq 0 \text{ for all } r > 0. \end{aligned}$$

By scaling these metrics suitably, it is then possible to arrange it so that we have a sequence of Ricci flat manifolds $(M_i, q_i, \bar{g}_i)$ with

$$\begin{aligned} \lim_{r\to\infty} \frac{\text{vol}B(q_i,r)}{\omega_n r^n} &\geq \left(1 - i^{-1}\right), \\ \|(M_i, \bar{g}_i)\|^{harm}_{C^{1,\alpha},1} &\leq 1, \\ \|q_i \in (M_i, \bar{g}_i)\|^{harm}_{C^{1,\alpha},1} &\in [0.5, 1]. \end{aligned}$$

From what we already know, we can then extract a subsequence that converges in the $C^{m,\alpha}$ topology to a Ricci flat manifold (M, q, g). In particular, we must have that metric balls of a given radius converge and that the volume forms converge. Thus, the limit space must satisfy

$$\lim_{r\to\infty} \frac{\text{vol}B(q,r)}{\omega_n r^n} = 1.$$

This means that we have maximal possible volume for all metric balls, and thus the manifold must be Euclidean. This, however, violates the continuity of the norm in the $C^{1,\alpha}$ topology, as the norm for the limit space would then have to be zero. □

Corollary 41. *Let $n \geq 2$, $-\infty < \lambda \leq \Lambda < \infty$, and D, $i_0 \in (0,\infty)$ be given. There is a $\delta = \delta\left(n, \lambda \cdot i_0^2\right)$ such that the class of closed Riemannian n-manifolds satisfying*

$$\begin{aligned} (n-1)\Lambda &\geq \mathrm{Ric} \geq (n-1)\lambda, \\ \mathrm{diam} &\leq D, \\ \mathrm{vol}B(p, i_0) &\geq (1-\delta)\, v(n, \lambda, i_0) \end{aligned}$$

is precompact in the $C^{1,\alpha}$ topology for any $\alpha \in (0,1)$ and in particular contains only finitely many diffeomorphism types.

Proof. We use the same techniques as when we had an injectivity radius bound. Observe that if we have a sequence $(M_i, p_i, \bar{g}_i)$ where $\bar{g}_i = k_i^2 g_i$, $k_i \to \infty$, and the (M_i, g_i) lie in the above class, then the volume condition now reads

$$\begin{aligned} \mathrm{vol}B_{\bar{g}_i}(p_i, i_0 \cdot k_i) &= k_i^n \mathrm{vol}B_{g_i}(p_i, i_0) \\ &\geq k_i^n (1-\delta)\, v(n, \lambda, i_0) \\ &= (1-\delta)\, v\left(n, \lambda \cdot k_i^{-2}, i_0 \cdot k_i\right). \end{aligned}$$

From relative volume comparison we can then conclude that for $r \leq i_0 \cdot k_i$ and very large i,

$$\mathrm{vol}B_{\bar{g}_i}(p_i, r) \geq (1-\delta)\, v\left(n, \lambda \cdot k_i^{-2}, r\right) \sim (1-\delta)\,\omega_n r^n.$$

In the limit space we must therefore have

$$\mathrm{vol}B(p, r) \geq (1-\delta)\,\omega_n r^n \text{ for all } r.$$

This limit space is also Ricci flat and is therefore Euclidean space. The rest of the proof goes as before, by getting a contradiction with the continuity of the norms. □

5.4. Curvature Pinching. Let us now turn our attention to some applications of these compactness theorems. One natural subject to explore is that of *pinching* results. Recall that we showed earlier that complete constant curvature manifolds have a uniquely defined universal covering. It is natural to ask whether one can in some topological sense still expect this to be true when one has close to constant curvature. Now, any Riemannian manifold (M, g) has curvature close to zero if we multiply the metric by a large scalar. Thus, some additional assumptions must come into play.

We start out with the simpler problem of considering Ricci pinching and then use this in the context of curvature pinching below. The results are very simple consequences of the convergence theorem we have already presented.

Theorem 77. *Given $n \geq 2$, i, $D \in (0,\infty)$, and $\lambda \in \mathbb{R}$, there is an $\varepsilon = \varepsilon(n, \lambda, D, i) > 0$ such that any closed Riemannian n-manifold (M, g) with*

$$\begin{aligned} \mathrm{diam} &\leq D, \\ \mathrm{inj} &\geq i, \\ |\mathrm{Ric} - \lambda g| &\leq \varepsilon \end{aligned}$$

is $C^{1,\alpha}$ close to an Einstein metric with Einstein constant λ.

Proof. We already know that this class is precompact in the $C^{1,\alpha}$ topology no matter what ε we choose. If the result were not true, we could therefore find a sequence $(M_i, g_i) \to (M, g)$ that converges in the $C^{1,\alpha}$ topology to a closed Riemannian manifold of class $C^{1,\alpha}$, where in addition, $|\mathrm{Ric}_{g_i} - \lambda g_i| \to 0$. Using harmonic coordinates as usual we can therefore conclude that the metric on the limit space must be a weak solution to

$$\frac{1}{2}\Delta g + Q\left(g, \partial g\right) = -\lambda g.$$

But this means that the limit space is actually Einstein, with Einstein constant λ, thus, contradicting that the spaces (M_i, g_i) were not close to such Einstein metrics. □

Using the compactness theorem for manifolds with almost maximal volumes we see that the injectivity radius condition could have been replaced with an almost maximal volume condition. Now let us see what happens with sectional curvature.

THEOREM 78. *Given $n \geq 2$, $v, D \in (0, \infty)$, and $\lambda \in \mathbb{R}$, there is an $\varepsilon = \varepsilon\left(n, \lambda, D, i\right) > 0$ such that any closed Riemannian n-manifold (M, g) with*

$$\begin{aligned} \operatorname{diam} &\leq D, \\ \operatorname{vol} &\geq v, \\ \left|\sec - \lambda\right| &\leq \varepsilon \end{aligned}$$

is $C^{1,\alpha}$ close to a metric of constant curvature λ.

PROOF. In this case we first observe that Cheeger's lemma gives us a lower bound for the injectivity radius. The previous theorem then shows that such metrics must be close to Einstein metrics. We now have to check that if $(M_i, g_i) \to (M, g)$, where $\left|\sec_{g_i} - \lambda\right| \to 0$ and $\operatorname{Ric}_g = (n-1)\lambda g$, then in fact (M, g) has constant curvature λ. To see this, it is perhaps easiest to observe that if

$$M_i \ni p_i \to p \in M,$$

then we can use polar coordinates around these points to write $g_i = dr^2 + g_{r,i}$ and $g = dr^2 + g_r$. Since the metrics converge in $C^{1,\alpha}$, we certainly have that $g_{r,i}$ converge to g_r. Using the curvature pinching, we conclude from chapter 6 that

$$\operatorname{sn}^2_{\lambda+\varepsilon_i}(r)\, ds^2_{n-1} \leq g_{r,i} \leq \operatorname{sn}^2_{\lambda-\varepsilon_i}(r)\, ds^2_{n-1},$$

where $\varepsilon_i \to 0$. In the limit we therefore have

$$\operatorname{sn}^2_{\lambda}(r)\, ds^2_{n-1} \leq g_r \leq \operatorname{sn}^2_{\lambda}(r)\, ds^2_{n-1}.$$

This implies that the limit metric has constant curvature λ. □

It is interesting that we had to go back and use the more geometric estimates for distance functions in order to prove the curvature pinching, while the Ricci pinching could be handled more easily with analytic techniques using harmonic coordinates. One can actually prove the curvature result with purely analytic techniques, but this requires that we study convergence in a more general setting where one uses L^p norms and estimates. This has been developed rigorously and can be used to improve the above results to situations were one has only L^p curvature pinching rather than the L^∞ pinching we use here (see [**79**], [**80**], and [**32**]).

When the curvature λ is positive, some of the assumptions in the above theorems are in fact not necessary. For instance, Myers' estimate for the diameter makes the diameter hypothesis superfluous. For the Einstein case this seems to be as far as we can go. In the positive curvature case we can do much better. In even dimensions, we already know from chapter 6, that manifolds with positive curvature have both bounded diameter and lower bounds for the injectivity radius, provided that there is an upper curvature bound. We can therefore show

COROLLARY 42. *Given* $2n \geq 2$, *and* $\lambda > 0$, *there is an* $\varepsilon = \varepsilon(n, \lambda) > 0$ *such that any closed Riemannian* $2n$*-manifold* (M, g) *with*

$$|\sec - \lambda| \leq \varepsilon$$

is $C^{1,\alpha}$ *close to a metric of constant curvature* λ.

This corollary is, in fact, also true in odd dimensions. This was proved by Grove-Karcher-Ruh in [**49**]. Notice that convergence techniques are not immediately applicable because there are no lower bounds for the injectivity radius. Their pinching constant is also independent of the dimension.

Also recall the quarter pinching results in positive curvature than we proved in chapter 6. There the conclusions were much weaker and purely topological. In a similar vein there is a nice result of Micaleff-Moore in [**66**]stating that any manifold with positive isotropic curvature has a universal cover that is homeomorphic to the sphere. However, this doesn't generalize the above theorem, for it is not necessarily true that two manifolds with identical fundamental groups and universal covers are homotopy equivalent.

In negative curvature some special things also happen. Namely, Heintze has proved that any complete manifold with $-1 \leq \sec < 0$ has a lower volume bound when the dimension ≥ 4 (see also [**46**] for a more general statement). The lower volume bound is therefore an extraneous condition when doing pinching in negative curvature. Unlike the situation in positive curvature, the upper diameter bound is, however, crucial. See, e.g., [**48**] and [**38**] for counterexamples.

This leaves us with pinching around 0. As any compact Riemannian manifold can be scaled to have curvature in $[-\varepsilon, \varepsilon]$ for any ε, we do need the diameter bound. The volume condition is also necessary, as the Heisenberg group from the exercises to chapter 3 has a quotient where there are metrics with bounded diameter and arbitrarily pinched curvature. This quotient, however, does not admit a flat metric. Gromov was nevertheless able to classify all n-manifolds with

$$\begin{aligned} |\sec| &\leq \varepsilon(n), \\ \operatorname{diam} &\leq 1 \end{aligned}$$

for some very small $\varepsilon(n) > 0$. More specifically, they all have a finite cover that is a quotient of a nilpotent Lie group by a discrete subgroup. For more on this and collapsing in general, the reader can start by reading [**39**].

6. Further Study

Cheeger first proved his finiteness theorem and put down the ideas of C^k convergence for manifolds in [**21**]. They later appeared in journal form [**22**], but not all ideas from the thesis were presented in this paper. Also the idea of general pinching theorems as described here are due to Cheeger [**23**]. For more generalities on convergence and their uses we recommend the surveys by Anderson, Fukaya, Petersen, and Yamaguchi in [**45**]. Also for more on norms and convergence theorems the survey by Petersen in [**50**] might prove useful. The text [**47**] should also be mentioned again. It was probably the original french version of this book that really spread the ideas of Gromov-Hausdorff distance and the stronger convergence theorems to a wider audience. Also, the convergence theorem of Riemannian geometry, as stated here, appeared for the first time in this book.

We should also mention that S. Peters in [**77**] obtained an explicit estimate for the number of diffeomorphism classes in Cheeger's finiteness theorem. This also seems to be the first place where the modern statement of Cheeger's finiteness theorem is proved.

7. Exercises

(1) Find a sequence of 1-dimensional metric spaces that Hausdorff converge to the unit cube $[0,1]^3$ endowed with the metric coming from the maximum norm on $\mathbb{R}^3$. Then find surfaces (jungle gyms) converging to the same space.

(2) C. Croke has shown that there is a universal constant $c(n)$ such that any n-manifold with $\mathrm{inj} \geq i_0$ satisfies $\mathrm{vol}B(p,r) \geq c(n) \cdot r^n$ for $r \leq \frac{i_0}{2}$. Use this to show that the class of n-dimensional manifolds satisfying $\mathrm{inj} \geq i_0$ and $\mathrm{vol} \leq V$ is precompact in the Gromov-Hausdorff topology.

(3) Develop a Bochner formula for $\mathrm{Hess}\left(\frac{1}{2}g(X,Y)\right)$ and $\Delta\frac{1}{2}g(X,Y)$, where X and Y are vector fields with symmetric ∇X and ∇Y. Discuss whether it is possible to devise coordinates where $\mathrm{Hess}(g_{ij})$ are bounded in terms of the full curvature tensor. If this were possible we would be able to get $C^{1,1}$ bounds for manifolds with bounded curvature. It is still an open question whether this is possible.

(4) Show that in contrast with the elliptic estimates, it is not possible to find C^α bounds for a vector field X in terms of C^0 bounds on X and $\mathrm{div}X$.

(5) Define $C^{m,\alpha}$ convergence for incomplete manifolds. On such manifolds define the boundary ∂ as the set of points that lie in the completion but not in the manifold itself. Show that the class of incomplete spaces with $|\mathrm{Ric}| \leq \Lambda$ and $\mathrm{inj}(p) \geq \min\{i_0, i_0 \cdot d(p,\partial)\}$, $i_0 < 1$, is precompact in the $C^{1,\alpha}$ topology.

(6) Define a *weighted norm* concept. That is, fix a positive function $\rho(R)$, and assume that in a pointed manifold (M,p,g) the distance spheres $S(p,R)$ have norm $\leq \rho(R)$. Prove the corresponding fundamental theorem.

(7) Suppose we have a class that is compact in the $C^{m,\alpha}$ topology. Show that there is a function $f(r)$ depending on the class such that $\|(M,g)\|_{C^{m,\alpha},r} \leq f(r)$ for all elements in this class, and also, $f(r) \to 0$ as $r \to 0$.

(8) The *local models* for a class of Riemannian manifolds are the types of spaces one obtains by scaling the elements of the class by a constant $\to \infty$. For example, if we consider the class of manifolds with $|\sec| \leq K$ for some K, then upon rescaling the metrics by a factor of λ^2, we have the condition $|\sec| \leq \lambda^{-2}K$, as $\lambda \to \infty$, we therefore arrive at the condition $|\sec| = 0$. This means that the local models are all the flat manifolds. Notice that we don't worry about any type of convergence here. If, in this example, we additionally assume that the manifolds have $\mathrm{inj} \geq i_0$, then upon rescaling and letting $\lambda \to \infty$ we get the extra condition $\mathrm{inj} = \infty$. Thus, the local model is Euclidean space. It is natural to suppose that any class that has Euclidean space as it only local model must be compact in some topology.

Show that a class of spaces is compact in the $C^{m,\alpha}$ topology if when we rescale a sequence in this class by constants that $\to \infty$, the sequence subconverges in the $C^{m,\alpha}$ topology to Euclidean space.

(9) Consider the singular Riemannian metric $dt^2 + (at)^2\, d\theta^2$, $a > 1$, on $\mathbb{R}^2$. Show that there is a sequence of rotationally symmetric metrics on $\mathbb{R}^2$ with $\sec \leq 0$ and $\mathrm{inj} = \infty$ that converge to this metric in the Gromov-Hausdorff topology.

(10) Show that the class of spaces with $\mathrm{inj} \geq i$ and $\left|\nabla^k \mathrm{Ric}\right| \leq \Lambda$ for $k = 0, \ldots, m$ is compact in the $C^{m+1,\alpha}$ topology.

(11) (S.-h. Zhu) Consider the class of complete or compact n-dimensional Riemannian manifolds with

$$\begin{aligned} \mathrm{conj.rad} &\geq r_0, \\ |\mathrm{Ric}| &\leq \Lambda, \\ \mathrm{vol} B(p,1) &\geq v. \end{aligned}$$

Using the techniques from Cheeger's lemma, show that this class has a lower bound for the injectivity radius. Conclude that it is compact in the $C^{1,\alpha}$ topology.

(12) Using the Eguchi-Hanson metrics from the exercises to chapter 3 show that one cannot in general expect a compactness result for the class

$$\begin{aligned} |\mathrm{Ric}| &\leq \Lambda, \\ \mathrm{vol} B(p,1) &\geq v. \end{aligned}$$

Thus, one must assume either that v is large as we did before or that there a lower bound for the conjugate radius.

(13) The *weak (harmonic) norm* $\|(M,g)\|^{weak}_{C^{m,\alpha},r}$ is defined in almost the same way as the norms we have already worked with, except that we only insist that the charts $\varphi_s : B(0,r) \to U_s$ are *immersions*. The inverse is therefore only locally defined, but it still makes sense to say that it is harmonic.
   (a) Show that if (M,g) has bounded sectional curvature, then for all $Q > 0$ there is an $r > 0$ such that $\|(M,g)\|^{weak}_{C^{1,\alpha},r} \leq Q$. Thus, the weak norm can be thought of as a generalized curvature quantity.
   (b) Show that the class of manifolds with bounded weak norm is precompact in the Gromov-Hausdorff topology.
   (c) Show that (M,g) is flat iff the weak norm is zero on all scales.