Chapter 1
From Assenting to Asserting

Promoting Active Learning

John Mason

The issue addressed in this chapter is age-old: How can learners be stimulated to move from *assenting* (passively and silently accepting what they are told, doing what they are shown how to do) to *asserting* (actively taking initiative, by making, testing and modifying conjectures, and by taking responsibility for making subject pertinent choices). How can learners be provoked into actively working on and making sense of the ideas and techniques that they encounter, and how can this cultural ethos be fostered and sustained?

I use the term *asserting* because of the assonance with *assenting*, but also because it signals that the learner is taking initiative and making significant choices. It is not intended to indicate that learners become either arrogant or garrulous. Much of the most desirable *assertive* behaviour is internal, and need not have visibly overt external behaviour. It involves taking initiative, taking control, making choices, and becoming independent.

One of the reasons often given for assenting or passive stances taken by students is the fact that they are immersed in a culture of testing, so that their focus is on being told what they have to do to pass the next test. In other words, it is to be expected that in a culture of testing you get test-oriented (assenting, passive) behaviour. However, in this chapter the claim is made that it is perfectly possible to develop a culture of seeking to understand, and more, a culture of enquiry, within any testing regime. The way to do this is to evoke, support and develop learners' use of their own natural powers of sense making.

The chapter describes some of the most fundamental powers which all learners possess and have already used before coming to class, but which may have been put to one side due to previous experiences of being taught which failed to make use of these powers. Identification of these powers comes from reflecting on personal experience informed by the seminal insights of Pólya (1962) and Gattegno (1970) and developed over many years since Mason et al. (1982). The chapter also suggests tactics which have been used by teachers to get learners to make use of those powers, in contrast to teachers unwittingly doing the real work for learners, whether in plenary sessions, in tutorials or in informal learner discussions through an unconscious

J. Mason
Open University, United Kingdom

O. Skovsmose et al. (eds.), *University Science and Mathematics Education in Transition*,

decision to "push things along a bit", "get to the end of the lesson on time", or "helping learners to understand". The tactics are equally applicable to lectures, tutorials, problem-based sessions and the use of Information Communication Technology (ICT). In each case there are forces acting to increase passivity that need to be countered through suitable choice of teaching tactics and the development of a culture that overtly values learners taking initiative.

Promoting active rather than passive learning can be expressed in terms of the development of personal agency and identity. The chapter draws upon studies of similarities and differences between Confucian and Western views of education to try to locate a common core of personal agency, which straddles the two traditions.

1.1 Assumptions

Before describing powers and strategies, it may be helpful to articulate some salient assumptions on which the chapter is based.

1.1.1 Assumptions About Human Beings

Human beings are seen here as naturally active construers of the sense impressions which they experience. They possess not only fundamental powers for dealing with these impressions, but also a fundamental curiosity or drive to want to make sense of them. When learners are encountered who seem to want to sit passively and learn by rote, then it is because they have been trained or acculturated to act this way. Nevertheless it is still possible to engage such learners, to provoke them into activity, by refusing to amplify their tendency to want things packaged and painlessly injected. Instead of trying to "do the learning for them", teaching can be exciting and stimulating by taking as its core purpose the provocation of learners to activate, become aware of, and develop their natural powers through encounters with pervasive mathematical and scientific themes.

This chapter takes an overtly constructivist stance, combining aspects of both psychological and social constructivism. Learners are seen as active agents (though the level of activity may be below desired thresholds!) whose psyche consist of the interweaving of behaviour (enactive), emotional (affective) and intellectual (cognitive) strands through the exercise of will, and who are embedded in a social milieu in which colleagues and teachers display practices which are adopted and adapted by learners.

1.1.2 Assumptions About Cultural Heritage

Covert activity, in which the learner is assembling and internalizing what is being offered so as to engage deeply with it subsequently is all too easily confused with actual passivity, in the sense of "waiting to be told what to do and explicitly how to do it", and "taking as little initiative as possible". For example, in the Confucian tradition, "human self-perfection [is] pursued as the highest purpose of life through personal commitment to learning" (Lin, 2004, pp. 129–130). Whereas in the West, rote memorization is seen as an end in itself, an attempt to minimize effort and simply regurgitate what has been "taught", the Confucian heritage sees commitment to memory as simply the first of a sequence of four phases. Memorized passages are interrogated for intention, style and meaning, and then applied to other situations in order to test it out in experience. This moves into critique and modification so that what was "memorized" is internalized and integrated into the learner's functioning (Lin, 2004, p. 131).

At the heart of both Western and Confucian heritage approaches to teaching and learning lies initiation into productive "ways of working" on science and mathematics, whether in the classroom or in the world outside. The tactics proposed in this chapter serve to activate and intensify activity corresponding to the later phases of the Confucian approach, but right from the beginning without depending on an initial phase of memorization.

1.1.3 Assumptions About Mathematics and Science and Their Didactics

Mathematics and science, as bodies of knowledge occupy what Popper (1972) called the "third world" of accounts lodged in libraries. However, these accounts are at best the expression of how someone else's attention was altered in its structure. They indicate subject specific sensitivities to notice and dispositions to act, and characteristic ways of acting. What the authors attended to, and how they attended, are evidenced only through what they were then able to express as a result, through the manifestation of their behaviour, itself usually refined and distilled. To make use of these accounts it is necessary for readers to experience transformations in the structure of their own attention, even if only vicariously and in their imagination.

Mathematics and science are both seen here as activities, as ways of thinking and acting in the world. Facts and figures are accumulated and rehearsed while engaging in pertinent activity. While learners sit passively, accepting and assenting to what they are told or shown, taking notes to "learn" later, they are not actually doing mathematics or doing science. Certainly they are not developing their powers of thinking mathematically and scientifically, to make mathematical and scientific sense of the world.

How then can learners be provoked into taking initiative, into making use of and developing their natural powers, into anticipating what is coming rather than rushing to catch up with what is being written on a screen in front of them by an expert? How can they be acculturated into developing both disposition for, and familiarity with productive ways of working on scientific and mathematical phenomena?

1.2 Phenomena

An incident or situation can only become an instance of a phenomenon when some boundary is drawn around it, when some aspect is discerned and fore-grounded. This stressing and consequent ignoring is the basis for abstraction, as Gattegno (1987) pointed out. It is one of the basic powers which human beings possess, but it can be all too easily subverted and parked if not used effectively. Mathematics, physics and chemistry all arise from becoming aware of characteristic situations as a phenomenon, of discerning details and relationships which comprise that phenomenon, and seeking underlying mechanisms to explain or account for the apparent relative invariance which constitutes that phenomenon.

1.2.1 Phenomena Encountered in Teaching

The following generalized incidents are intended to be generic examples of experiences in teaching, which, if they speak to experience, constitute phenomena encountered in teaching.

> Some learners attend a laboratory and follow the instructions as a recipe, finally writing up what they measured or observed. They can only assent to what they are told to do, and try to do it as faithfully as possible.

- For example, I discovered as a first year undergraduate that my father had 25 years earlier attended the same physics laboratory sessions and written up the identical experiments. Indeed neither the descriptions nor even the equipment had changed! Consequently I was able to find out in advance what each experiment was supposed to show and what the results were supposed to look like. I learned nothing about doing or constructing experiments.

In some cases, however, learners are required to design (at least parts of) a laboratory experiment for themselves in order to resolve some stated question. They are not permitted into the lab until the description is deemed acceptable from a safety as well as a scientific standpoint. This is becoming standard practice in science, and supports a move to asserting rather than mere assenting.

> Some learners sit in rows waiting to be told what mathematical or scientific procedures and facts they need to know. They naturally want to minimise their investment of energy, particularly when the 'subject' is peripheral or seen as at best a necessary tool rather than as a central element of their discipline. They are in a transition phase waiting to acquire a 'license to practise' in their discipline in the world beyond education. They assent to what they are given and asked to do, but have no thought to take initiative or to do more than requested.
>
> In some cases, however, learners are actively engaged in trying to make sense of phenomena. They see their courses as a means to achieve this, and they apply themselves actively to everything they encounter. They assert themselves in and through their engagement. Nevertheless, sometimes the going gets tough, the ideas complex, and the techniques unclear or complicated and multi-phased. Learners revert to minimising effort, looking for 'what they need to do' to pass the course. They assent to the practices of the system.

Underlying these phenomena is a pervasive theory of learning: if learners complete most of the tasks they are set by the teacher, to a reasonable standard, then the requisite learning will have taken place. Educationalists have railed against this attitude for as long as teaching has been written about. For example:

- Plato (Laws VII 819b-c, in Hamilton and Cairns, 1961) complained about the teaching of arithmetic, and many of his dialogues are explorations into what learning is and how it can be promoted.
- Spencer (1878, p. 28) complained about generalizations such as definitions and rules preceding exposure to cases, and that telling or showing does not teach how to observe.
- Whitehead (1932, p. 7) coined the expression *inert knowledge* to describe the central problem of education, which he saw to be the problem of keeping knowledge alive (assertive).
- Brousseau (1984, 1997) elaborated the notion of an implicit *didactic contract* which produces an inescapable tension: the more precisely the teacher indicates the specific behaviour being sought, the easier it is for learners to display that behaviour without actually generating it from understanding.
- Harré and van Langenholve (1999) used *positioning theory* to articulate the ways in which people are both positioned and position themselves into taking different roles at different times, through sometimes delicate social interaction.

The task of teaching is to create conditions in which the doing of set tasks will lead to activity that promotes active construal rather than simply "getting answers". Inner pondering and solitary work, discussion occasions this construal with colleagues, and interaction with a teacher (relative expert). The teacher's task is to call upon and make use of learners' natural powers of sense making in any or all of these milieux. The next section elaborates some of those powers and proposes ways of constructing tasks, which can lead to fruitful and pedagogically effective activity and interaction between learners and between learners and teacher. The suggestions sketched briefly here are elaborated in Mason (2002a).

1.2.2 Learning Phenomena

One of the observable outcomes of effective learning, and hence of effective teaching, is an enhanced and enriched disposition to notice phenomena in the material world to which mathematical or scientific thinking can be applied. For example:

- People often beat egg whites to create a foam, but what happens when you beat them, and why does it sometimes collapse and sometimes not?
- People often take showers where there is a soft curtain, and as the water is turned on, the shower curtain moves inwards towards the water flow and the person's legs, but why not ask yourself why?
- People are aware that the moon is sometimes visible and sometimes not, but when can you see a vertical half-moon (the terminator is parallel to your body as you stand and look) and can you ever see a horizontal half-moon?

- People often go through double doors, but why is it that if you open both doors at the same time for someone, the gap takes longer to become wide enough for them to get through than if you only open one door?
- People expect businesses to have pricing policies, but themselves are only interested in the price they are offered: the entrepreneur uses algebra, the customer uses arithmetic.

Incidents like these only become phenomena when someone becomes aware of a (potential) repetition and then isolates those features that are common to various incidents. Becoming sensitized to notice such opportunities, even if they are not taken up, is to become more mathematical or scientifically assertive, in the sense of engaged with and contributing to the community. Some forms of teaching are more effective than others at promoting such a disposition (Boaler, 1997). Lin (2004, pp. 134–135) points to differences between Western and Confucian heritage learners in this respect. The former emphasize activity as the origins of learning, through problem solving, and depend on suitable and compatible teaching resources. The latter value and emphasize "heart and mind wanting to learn", "seeking knowledge" and opting for both breadth and depth. Both are aimed at active agentive conditions rather than passivity, but they are expressed differently.

1.3 Powers

The idea of people possessing powers has ancient roots in Indian psychology. In the west, Spencer (1878), Montessori (1912), Gattegno (1970) and Krutetskii (1976) among others all referred to them in the development of their specific approaches to teaching and learning. My observations have been inspired and informed by Pólya (1962), and confirmed and developed by my own observations and reflections over a period of nearly 40 years. The powers elaborated here have been chosen because they are, in my experience, the core powers that are used both to make sense of mathematics and science and to make mathematical and scientific sense of the world. My aim is not to be comprehensive, because such a list would then become unwieldy. Rather my aim is to draw attention to key powers, which, though on the one hand perfectly obvious to any mathematician and probably to any scientist, are on the other hand, often overlooked and not made use of fully in pedagogic and didactic settings.

- Imagining and Expressing
- Specializing and Generalizing
- Conjecturing and Convincing
- Focusing and De-Focusing Attention
- Organizing and Classifying

Gattegno (1988) proposed the notion that, in a manner of speaking, by responding to sense impressions the soma constructs its brain, or as others might say, brain structure co-emerges as sense impressions and soma co-operate. Connections are made as the soma attends, and attention becomes possible due to brain connections forming. Mason (2007) elaborates the claim that these powers are demonstrated by newborns, and form the basis for effective participation in and action upon the material world. The aim here is to elaborate how these powers are the key to getting learners to engage with mathematics and science.

Sense-making powers are activated when learners experience some sort of disturbance: expectations are revealed when something happens which produces surprise or when some task presents a challenge because it cannot be accomplished through the routine use of habits. This notion of disturbance as the impetus to learning also has a long history. Piaget (1971) used a biological metaphor (assimilation, accommodation and equilibrium seeking), Wertheimer (1945) recognized disturbance explicitly as the source for activity leading to learning, and Festinger (1957) made disturbance the core of his notion of *cognitive dissonance*.

1.3.1 Imagining and Expressing

The power to imagine and to express what is imagined through movement, pictures, words and symbols is fundamental to human functioning. This power permits us to imagine what is not present even when not being physically possible. It also permits the use of sophisticated language. Mental imagery is also the means we use to harness the energies produced by our emotions, by mentally imagining ourselves in future situations and carrying out actions as we intend. Indeed Norretranders (1998) concludes that consciousness lags behind and interprets behaviour rather than directing it, reinforcing ancient psychological insights that mental imagery as a planning device acts as the reins to direct emotions. It is what we use in order to plan and prepare for the future. It is how we tell stories and how we communicate with others. "Imagining" is taken here to encompass virtual sensations of any mode or combination of modes, including sight, sound, taste and smell, touch, and a more generalized "sense-of" which is not directly attributable to any particular mode.

Expressing what we imagine produces communication of many different forms, including both science and art. Education is constantly confounded by trying to identify what is imagined (understood, appreciated) with what is expressed, for example in the act of assessing learners. Just because someone does something in one situation (say on a test), it does not follow that they either can or will think to do it, much less do it, in another context. Hence the notion of situated cognition (Lave and Wenger, 1991), and situated abstraction (Noss and Hoyles, 1996), in which emphasis is placed on the important role of the situation or situation-type in which encounters and hence learning take place. Conversely, just because someone does not do something in a situation, it does not follow that they could not, or that they will not in some other situation. As archaeologists are fond of saying, "absence of evidence is not evidence of absence".

Language contributes to imagination (you can strengthen some images by repeating inner speech as reinforcement of images) but imagination also provides the source of what is expressed in language. Becoming articulate in speech and writing, whether in primary or in tertiary level, is as much about connecting language with inner experience and thought as it is about learning the relevant and appropriate format. Language also strengthens the power to imagine, because when one person expresses what they are imagining they can stimulate imagination in others. For example diagrams are static images, which provide a stable background on which to superimpose variation and further construction. A diagram or photograph of a cell, a picture of a moment during an experiment, or a geometrical figure can all serve in this way. The learner can then imagine further details, other aspects, and what happens before and after. Similarly dynamic images, such as a video or animation, provide what Salomon (1979) called *supplanted images:* images that can be used as eidetic components in the construction of further mental images.

However, images, like text, can be interpreted in many ways. There are likely to be relationships which are particular to the specific diagram and not intended to be included as part of the generality being illustrated. Fischbein (1987, 1993) coined the expression *figural concepts* to describe the concepts constructed by learners from attending to unintended features. The issue of how people learn general concepts from particular examples has exercised psychologists for some time (Bruner et al., 1956; Rissland, 1991; Renkl et al. 1998) and is of abiding interest in computer science and A.I. (Winston, 1975) as well as in education (Sweller and Cooper, 1985).

Experience with dynamic geometry software has also highlighted the fact that in a diagram or picture there are local relationships, which are not directly observable (Laborde, 1995). In geometry it is vital to become aware of these relationships, which are often what the diagram is really about. Something similar applies to any picture or diagram that is supposed to illustrate some relationship, property, some process, principle or technique. Furthermore, relationships may be detected among discerned elements in a particular case without being perceived as properties, which can apply in many different situations, or in the case of figural concepts, properties may be inappropriately or unintentionally assumed to be essential. This is what gives rise to undesirable situatedness. Thus some subtle shifts in the structure of attention may be required, from discerning relevant items to recognizing relationships, to perceiving these as properties, before learners can be expected to engage in reasoning on the basis of those properties, and so displaying understanding (Mason 2003, 2004).

In science as in mathematics, being invited to imagine (guided by the teacher) the steps in carrying out a task or in thinking about a phenomenon, which the topic can help explain, can be a powerful introduction to a lesson or suite of lessons. For example:

- Imagine a beaker, half filled with pure water. Put it in a beaker of ice. Insert a thermometer and watch the temperature…
- Imagine the graph of a function. Imagine a point not on the curve. Imagine a line passing through that point and slowly rotating. Sometimes it intersects the curve, sometimes it does not, and if these are both possible, somewhere in between it touches the curve at two coincident points (it is tangent there) as well as, perhaps crossing the curve elsewhere…

A learner who is trying to follow what they are being told or shown is at a disadvantage, because they are always one step behind, reacting to what has happened rather than participating in real time. This is often evident when learners who are copying notes from a board or screen fall behind the production of those notes. They cannot therefore be concentrating fully on what is being said, so much of it goes over their heads, literally and figuratively. By contrast, learners who are able to anticipate what is about to happen are actively engaged and in a much better position to experience disturbance or at least contrast and surprise when their expectation is contradicted. This is the moment when a crucial decision can be made, for if the exposition flows ever onward, the disturbance may only be suppressed or parked. The learner who falls behind is in a much less favourable position to anticipate, and is driven into a position of severely restricted and situated assertion.

1.3.1.1 Making Use of Imagining and Expressing

The power to imagine can be strengthened and developed, through use. But that use has to be active rather than passive. One of the dangers of the sea of images imposed on our senses by television and earphones is the pollution and over stimulation of our senses. Bombarding people with sense impressions in a competitive free-for-all is unlikely to result in the development of learners' powers. Rather, learners can be called upon to anticipate, to imagine what something will look like or what will happen next, before being shown.

Say What You See

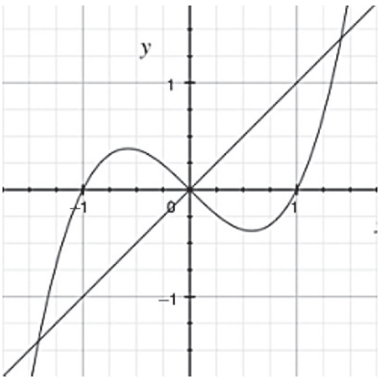
Learners can be called upon to *Say What You See* to others in order to discover when listening to others that there are other details to discern, other relationships to recognize, other properties to perceive than the ones that may come to attention immediately. A variant is to be asked *what is the same and what is different?* Or *what is changing and what is invariant?*

- For example, observing a fountain, it can be seen simply as "fountain". However, if someone endeavours to say what they see, trying not to make unwarranted assumptions, they will find their attention directed to what is changing (the apparent path of the water droplets, the movement on the surface of the water) and what is invariant (the basin etc.). In the case of the second fountain shown, the spout is itself rotating. Nevertheless, even though individual drops follow their own unique path, there is the appearance of a path being formed by the water (a relative invariance or an invariant relationship). *Say What You See*, informed by seeking what is (relatively) invariant and what is changing is likely to invoke curiosity: what is the actual path? Are the apparent paths actually paths, and are they also parabolae?

Walker (1975) provides descriptions of numerous everyday phenomena, which can be explained by relatively elementary physics. If learners are not acculturated into seeing the material world through the eyes of a curious physicist, then they are not learning to be physicists, and arguably, not even learning physics. They are merely being trained in the use of formulae. The same applies to chemistry. Why is it difficult to open a freezer door immediately after closing it (assuming it is functioning!)? How does soap work? What happens when you heat up ice and it turns into water or heat up water and it turns into steam? Curiously, phase change can be a helpful metaphor for teaching, in which effort sometimes goes in to explaining and correcting with no visible effect, and then suddenly the learner acts as if they have actually learned something.

Say What You See is an effective pedagogic device for provoking learners to probe beneath the surface of situations, to detect an underlying phenomenon, and to locate details that might be used to explain or account for what is observed. *Say What You See* can also be used with abstract entities such as appear in mathematics. For example, *Say What You See* can be used with either of the following objects or both together for comparisons, in order to draw attention to different component elements and what they signify, and to indicate which features are irrelevant and which are essential to some problem or task type.

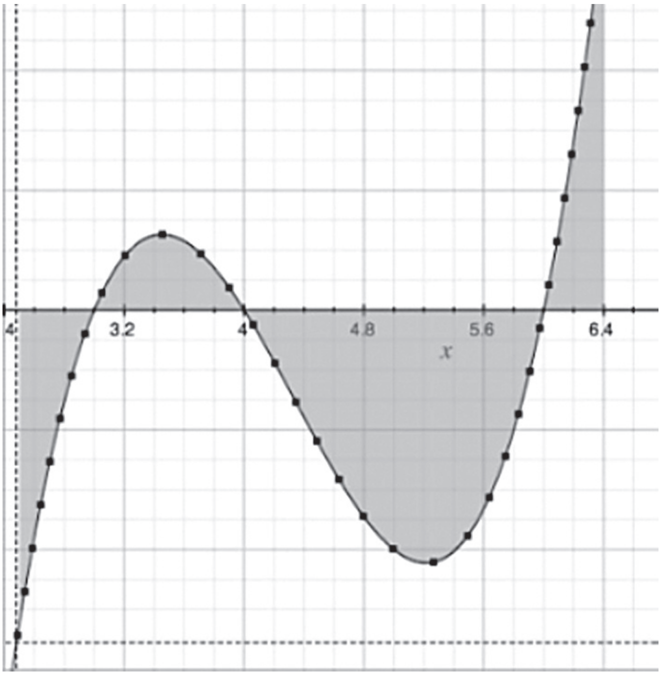
$$\int_{\frac{1-\sqrt{5}}{2}}^{\frac{1+\sqrt{5}}{2}} (x(x^2-1)-x)dx$$



Through being exposed to how others attend, learners can discover that they too can choose to direct their attention, to perceive differently, and to make different sense.

Indicate When You Can See …

Learners can also have their attention directed explicitly so that they discern relevant details (*look at this; hear that*) using verbal and physical pointing. More subtly, they can be invited to recognize significant relationships through being invited to indicate when they can detect a particular relationship (Thompson, 2002). For example, after gazing at this diagram,



can you see:

area under a curve?	four areas?	an area-so-far function?
an integral to calculate a mathematical area?	four integrals to calculate a total area?	An instance of the fundamental theorem?

Indicate When You Can See … is useful for prompting learners to become aware that it is both necessary and useful to see one object from multiple perspectives. Of course the question *"Can You See"* on its own has only a "yes/no" answer, so it is much more effective to ask learners to indicate when they can see. The time delay then may suggest whether more time could usefully be spent on the particular issue. The question is also useful simply for directing attention to relevant details before drawing attention to relationships that are instances of general properties.

It is vital for a teacher to be able to direct learner attention appropriately, and this requires becoming aware that learner attention may not spontaneously be directed as the teacher expects. Not only can attention be focused on different features, causing break down in communication, but, even where there is a common focus, learners and teacher may actually attend differently. Some may be occupied by discerning details, others may be seeking relationships between particular discerned components, yet others may be perceiving relationships as instances of more general properties, and some may be prepared to reason about the phenomenon only on the basis of explicit and agreed properties, and not just on anything and everything they think they know.

1.3.2 Specializing and Generalizing

Pólya (1962) used the term *specializing* to refer to all aspects of trying out a particular case of something in order to "see" what is going on, to experience the underlying structure, and hence to re-generalize for oneself. Specializing includes ordinary familiar examples as well as extreme or "special" cases. People do this entirely naturally, as when they ask for an example, or resort to using diagrams or physical objects as a model of a more complex situation. The interesting thing from a teacher's point of view is that when learners get stuck on a problem or with a concept, they often do not think of using their natural power to specialize for themselves (Mason et al., 1982). Consequently it is important pedagogically to display and enact specializing and generalizing, to call upon learners to use those powers themselves, and to draw learner attention to the fact that they have used those powers. These are the three aspects that contribute to noticing opportunities to act in the future and so to inform practice (Mason, 2002b): acting, reflecting, imagining.

1.3.2.1 Making Use of Specializing and Generalizing

Halmos (1994, p. 852) stresses concentrating "attention on the definite, the concrete, the specific". As Whitehead (1932, p. 4) put it, "To see what is general in what is particular and what is permanent in what is transitory is the aim of scientific thought". I prefer a rephrasing as "to see the general through the particular and the particular in the general" and "to be aware of what is invariant in the midst of change" is how human beings cope with the sense- impressions that form their experience.

Scaffolding and Fading

Wood et al. (1976) introduced the metaphor of *scaffolding* as a description of the way in which a teacher can act as "consciousness for two", augmenting the learners' awareness and reminding them of things that are not coming to mind because their attention is fully taken up with matters which will ultimately become integrated and subordinated into their functioning (Gattegno, 1970). For example, when a learner is stuck on a problem, asking them to specialize (give me an example …) can be seen as an act of scaffolding. However, teacher intervention can create dependency as well as independence. It all depends on *fading*, that is, reducing the support being offered (Brown et al., 1989; Love and Mason, 1992). A useful way to do this is to move from direct prompts (have you got an example?) to increasingly more indirect prompts (What did you do last time? or What question do you think I am going to ask you?) until learners have taken up those prompts and internalized them.

Learner Constructed Examples

Given the problems which arise when learners try to use examples to work out what features are intended to be exemplary and what features are in fact specific or special, a useful additional strategy turns out to be to get learners to construct their own examples (Mason and Watson, 2001, 2005). Not only does this reveal some of the richness or poverty of what learners have access to, but as an action it actually serves to extend and enrich the space of examples to which learners have access in future.

For example,

Can you construct a cubic whose inflection tangent does not cross the curve?	Can you find a situation involving a gas in which temperature changes but neither pressure nor volume change?
Sketch a cubic graph;	Can you find a situation in which change of temperature produces a change of pressure?
and another;	And another?
and another	And another?
Construct a cubic;	Find a situation in which change of pressure produces a change of temperature;
Construct a cubic which has only one real root;	
Construct a cubic which has only one real root and which has a local minimum;	A change of volume produces a change of pressure;
Construct a cubic, which has only one real root, which has a local minimum and whose inflection slope is positive.	A change of volume produces a change of temperature

The structure of the second task is intended to illustrate how being asked for another and then another (not all at once, but in sequence) directs most people to begin to explore the full range of generality from which to make a choice, rather than sticking with the first idea that comes to mind. In this way their personal example space becomes enriched (Watson and Mason, 2005).

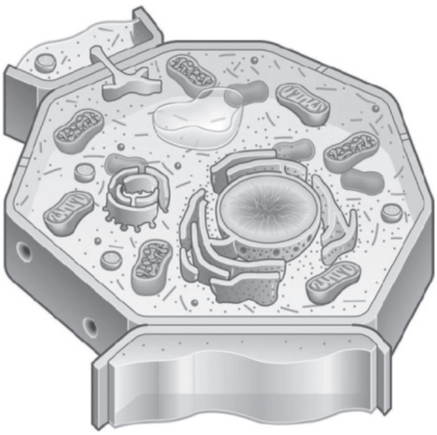
The third task is intended to illustrate how treating constraints one at a time, and paying attention to the full generality at each stage can be more instructive than trying to meet all constraints at once. The constraints are chosen to try to force people to reconsider their current example. At each stage choices have to be made, but sticking with a generality is more fruitful than trying to modify a single example.

Example Spaces

The central idea being explored herehas been articulated by Ference Marton and colleagues (Johansson et al., 1985; Marton and Booth, 1997; Marton and Trigwell, 2000) as *dimensions of variation*. To understand or appreciate a concept means to be aware of what aspects (dimensions) can vary (change) in an example and still it remains an example. A mathematical version of this is the theme of invariance in the midst of change, which lies at the heart of mathematics and science (Whitehead, op. cit.) and perhaps of all learning. Marton goes further, proposing that learners become aware of a dimension of possible variation through exposure to variation in sufficiently quick succession to be attended to *as* variation. Watson and Mason (2005) go further in drawing attention to the importance of becoming aware of the permissible range of change in which the invariance is preserved. Very often learners are aware of a subset of the dimensions of possible variation, and even when they are aware of a particular feature that can be varied, their sense of the range of permissible variation is often attenuated. For example, they may think only of integers instead of real numbers. Consequently when the teacher uses a concept, the learners' sense of what is being said may be so greatly restricted as to nullify its significance.

A classic example in mathematics is the function $x \rightarrow |x|$ which is the standard example of a continuous function differentiable everywhere except at one point. It is well known that many students simply monster-bar this example and continue to think of it as merely an aberration (MacHale, 1980). Until learners think of exploring possibilities such as functions which are non-differentiable at 2, 3, … points, and even the set of points at which functions could be non-differentiable, they have not appreciated what the counterexample is telling them. By becoming familiar and confident with example tinkering, they enrich both their appreciation of a concept and their accessible example space.

In science, a diagram of a cell, no matter how beautiful the graphics, suffers from the possibility that learners will attend differently to the ways intended. They



may take some specific details as generic when in fact they are arbitrary, and some details as arbitrary when in fact they are specific.

The nucleus need not be in the position shown, nor even near the centre; other components displayed three times appear hundreds or thousands of times, and so on. A learners' sense of what can vary and what is invariant nor necessary can be highly attenuated and yet they can answer many questions correctly and appear to understand the topic.

Watson and Mason (2005) have found that there is a tendency for learners of mathematics to have a restricted range of examples to use as touchstone and on which to experiment. Their example spaces may be much more restricted and impoverished than teachers realize. Asking learners to construct one example, then another, and then yet another often has the effect of expanding learners' awareness of the range of possibilities from which a choice can be made when looking for a suitable example. It is this sense of choice, of potential (if not actual) infinity that enriches and educates their awareness and provides access to an expanded example space in the future. The confidence gained by having a sense of control over the potential complexity, which comes from being able to construct your own variations, also enriches the possibilities when a learner wants to specialize in order to make sense of a generality.

1.3.3 Conjecturing and Convincing

Mathematics and science proceed and develop through conjectures, proofs, and refutations (Lakatos, 1976). Convincing themselves, friends, and finally, other sceptics is what mathematicians and scientists do. But so does everyone, although in less formal and explicit ways. Every action in and on the world is a conjecture; when the response is unexpected, that disturbance generates energy, which can be used to try to return to equilibrium, to make sense, to accommodate or to find awareness and sensitivity enriched.

Human beings are also natural story-tellers, or as Bruner (1996) said, *narrative animals*, for we try to account for our experience, weaving it into a story which justifies our actions and explains why things happen the way they do. diSessa (1987) uses the label *phenomenological primitives* for core explanations that people use to account for phenomena, to justify their conjectures about why things happen. The term *primitive* is used to indicate that the explanation itself is not probed any further. For example, when asked what happens to the sound of a vacuum cleaner motor when the tube is blocked, "working harder" is often used to account for claims both that the pitch rises and that it falls.

Scientists and mathematicians have developed formal practices in order to try to convince each other, although these practices do change and develop over time as both questions and reasoning become more sophisticated or political. Learning to adopt these practices, to take up the relevant discourse, is one of the challenges for learners in any topic.

1.3.3.1 Making Use of Conjecturing and Convincing

If learners are to become mathematicians and scientists, they too need to engage in overt conjecturing and convincing. They need the time, opportunity and impetus to try to articulate their reasoning to others in order to convince themselves. They also need to learn to be sceptical about someone else's reasoning, developing the disposition to look for counter-examples, to test out special cases in an effort to find flaws. In this way they develop their power to convince not only colleagues, but also a more sceptical "other" or "enemy" (Mason et al., 1982). In science, many learners arrive with assumptions and partial stories to account for observations about the material world that need to be challenged in order that more appropriate scientific perspectives are adopted. For example, as diSessa (1987) showed, even very intelligent students can adopt a split view that Newtonian mechanics is what you use in class while retaining an Aristotelian view of forces. He asked undergraduates to predict the path of a particle emerging from a circular track at speed, and found many considered the circularity to continue after release. It is vital that learners become aware of assumptions they are making, convert these into conjectures, and begin to construct (even if only mentally) experiments to test out those conjectures.

Because of the strong effect of years of schooling, conjecturing does not just "happen" in classrooms. Establishing a conjecturing atmosphere or ethos takes time. Attention needs to be drawn to the status of assertions and assumptions as conjectures needing confirmation and justification. Those who are uncertain need to be encouraged to try to articulate their conjectures and be overtly respected for modifying their conjecture as a result; those who are confident or certain need to be encouraged to ask helpful questions of others, to try to find counter-examples to other people's conjectures, and to probe themselves and others for justifications and reasoning. Most importantly, conjecturing is most effective as a collaborative rather than a competitive interaction with others. Everybody learns when someone modifies a conjecture or proposes a counter-example.

Overtly calling upon learners to make a conjecture before they test it on an example or against a phenomenon puts them in a more active stance. When a conjecture is refuted, learners have more personal commitment to sorting it out, modifying, augmenting or even radically changing their conjecture. For example:

> For what conjectures could the function $f(x) = \begin{cases} x^2 \text{ when } x \geq 0 \\ -x^2 \text{ when } x < 0 \end{cases}$ be used as a counterexample?
>
> For what conjectures could the fact that the ratio of the masses of a substance before and after burning is independent of the sample size be used as a counterexample?

It is so easy to sit back passively and wait for the "correct" answer to emerge, when in fact "learning" is the result of struggling to articulate ideas and reasons rather than actually getting the particular answer. Most lectures have a few people attending who fool themselves into thinking that "I could have got that, done that, thought of that" when they don't make an act of commitment through formulating a conjecture. Pólya (1962) recommended making conjectures and then disbelieving them. As long as ideas and possibilities toss around in your mind like clothes in a tumble drier, they are hard to pin down and likely to dissipate. By recording a conjecture and then immediately looking for counter-examples the learner engages in mathematical and scientific thinking through taking an active stance. There is of course a close connection between anticipation and conjecturing, setting up expectations which might then be confounded by examples or by the phenomenon, but which can then lead to scientific advance.

Legrand (1990, 1998/2000) has developed *scientific debate* as a format or structure within which learners are encouraged to conjecture and justify, challenge and try to convince each other, in order to develop their understanding of a concept or theorem, as well as to develop their powers to engage in mathematical-scientific practices (Warfield, 2005). Mathematical thinking in its full sense can really only thrive in a conjecturing atmosphere, where everything said is taken as a conjecture, and everyone is encouraged to test it and to offer potential modifications where necessary.

To establish an appropriate atmosphere and collection of practices the teacher needs to be able to hold back and listen (Davis, 1996, p. xxiv). Disturbance felt by learners is not often assuaged simply by assertions from teachers, especially when the teacher jumps in at the first opportunity to correct what a learner has offered. It does not matter that wrong conjectures are made. What does matter is the establishment of a conjecturing atmosphere, an ethos of enquiry; what does matter is that learners encounter counter-examples, and that they modify their conjectures until they are either put to one side for further exploration some other time, or justified using mathematical or scientific reasoning. Effective teaching has taken place when learners are moved (motivated) to go away and re-construct for themselves mathe matical objects as examples of concepts, applications of theorems, and illustrations of techniques.

1.3.4 Focusing and De-Focusing Attention

Attention is partly under conscious control, and partly not. People can have their attention attracted by sudden movement or sound, by the appeal of something with particular meaning for them, by having attention attracted or directed by a respected "other" such as a teacher, or by direct application of their own will-power. Careful observation of your own attention soon reveals that noticing begins with a sudden rush of concentrated attention, a sharpening of awareness, which then gradually dissipates until it is either refreshed and renewed or overlaid with some other noticing.

Attention can be highly focused on some detail, but it can also be very diffuse as when gazing at a diagram or into space waiting for inspiration. Attention can be uni-focal but also multi-focal as when you think about preparing supper while listening to a colleague and looking at something out the window.

Teaching can only be effective when learners and teacher are attending to the same thing and in similar ways. There are subtle but important shifts not only in what it is worth attending to, but how, which learners need to experience repeatedly as they begin to perceive and think in ways characteristic of the discipline and the particular topic. Sometimes it is useful to *hold wholes*, that is to gaze at the whole of a phenomenon, situation, object or experience. Sometimes it is important to work at *discerning details*, which includes locating boundaries that distinguish and define sub-wholes. Sometimes it is relevant to *recognise relationships* between particular details, either in the same whole or between wholes. Often it is valuable to *perceive properties* as generalizations of specific relationships, which might hold amongst other objects or other details elsewhere in the current situation or in the future. This is part of the move to generalization and abstraction. Relationships may arise as particular instances of perceiving a property to be relevant, and sometimes perceiving properties arises as an abstraction from relationships between particular details. This is another way of thinking about "seeing the general through the particular, and the particular in the general". Justification and explanation is usually based on articulating *reasoning* (based on intuition arising from gazing), which is only possible when it proceeds *on the basis of properties* that are accepted, and acknowledged, and agreed. These five subtly different structures of attention lie at the heart of what learners are doing when being taught. Although there is considerable overlap and consonance with the van Hiele levels articulated by van Hiele (1986) there is a significant difference when applied to attention itself. Rather than forming levels through which a learner might be expected to move, self-observation reveals that attention often shifts very quickly between structures, back and forth, round and round, while at other times one or other structure becomes quite stable.

One of the values of sensitizing yourself to these subtle shifts in structure is that they can be used to account for learner difficulties. For example, during exposition, learners gazing at a whole may not even hear distinctions being drawn by a teacher or be aware of relationships being identified. If they are occupied with discerning details they may not be able to make sense of descriptions of relationships amongst those details. If they are seeking out relationships in the situation before them they may not be aware of those relationships as instances of more general properties, or even the potential for those relationships to be generalized and perceived as properties. If they are coming to grips with properties as general qualities which may explain, capture, or indicate specific relationships, they may not be in a position to benefit from the teacher's use of certain properties as the basis of reasoning.

One of the reasons that scientific and mathematical reasoning might be so difficult to teach is that learners are rarely in an appropriate state with their attention encompassing the perception of properties, which can be used as the basis of reasoning. There is a subtle difference between using language (which is inherently general) to talk about specific relationships recognized in a situation, and reasoning about that and similar situations on the basis of properties. The same language is often used in both cases, but learner experience is quite different. This would account for the otherwise mystifying experience of learners who seem to be reasoning quite generally but who suddenly resort to particulars or fail to use that reasoning in a new situation. For example, Rowland (2001) reports the case of students claiming to follow and appreciate a number theory proof carried out in a particular case, but unwilling to accept the implied generality of the proof.

1.3.4.1 Making Use of Attention

Becoming aware of the structure of one's own attention, of not only what is being attended to but in what manner, improves sensitivity to learner experience. Awareness of one's own attention makes it more likely to notice mismatches with that of learners, and then to be motivated to construct tasks which provide the time necessary for learners to engage with different structures of attention, and which prompt learners to shift from one structure to another. Awareness of one's own attention also makes it possible to refer explicitly to the fact that it is possible and desirable to attend to certain things in certain ways. A number of frameworks have been developed which help as reminders when planning a session or when in the midst of interacting with learners (Floyd et al., 1981; Mason and Johnston-Wilder, 2004, 2004/2006).

See, Experience, Master

It is rare for learners to make a great deal of a concept or technique the first time they encounter it. With continuing exposure they develop experience, and as the practices and ways of thinking and perceiving become integrated into their functioning, they move towards mastery. The label *See–Experience–Master* serves as a warning, a trigger not to expect too much, and to construct tasks that provide requisite time and experience.

Manipulating – Getting-A-Sense-Of – Articulating

As many educators have suggested since Plato, having something tangible to manipulate can often assist in recognizing significant relationships and so making sense of some generality. Specializing or particularizing is another way to refer to the entirely natural act of seeking out a familiar and confidence-inspiring example or object through which to try to see the general. Getting learners to construct their own examples is one way to support learners in developing a rich repertoire of objects to which they can refer when they want to test out a conjecture or try to get a sense of some underlying structure or phenomenon. Attempting to articulate that "sense" can assist with the sense making, especially when it takes place within a supportive conjecturing atmosphere.

1.3.5 Organizing and Classifying

There is a sense of satisfaction and energy release when some sort of order or orderliness is brought to a collection of disparate and scattered items. The whole effect of language is to provide labels for collections of objects according to shared properties to support organizing and characterizing. Thus language has the effect of organizing experiences. To recognize suddenly that two or more apparently different situations are actually examples of one phenomenon makes future sense making much easier on the one hand, while on the other hand it sets up the possibility of failing to attend sufficiently to differences. Thus every act of organization has the potential to inform but also to mislead.

Much of mathematics and of science can be seen as a process of classifying phenomena, characterizing the conditions under which the phenomena come about or can be recognized. Theorems asserting that such and such a condition is equivalent to some other condition, or that all objects of one sort also have some additional property all serve to organize example spaces and characterize objects in those spaces.

1.3.5.1 Making Use of Organizing and Classifying

What Is the Same, What Is Different About?

Offering learners two or more objects, situations or phenomena and asking them what is the same and what is different about them directs their attention from specific details of individual cases to relationships which are common to several instances and helps create in the learner a sense of phenomenon (an identifiable repeatable experience) and hence generality. It also adds to their enculturation into becoming aware of what features are worth attending to and are of significance in the particular discipline, for each discipline consists of a collection of characteristic awarenesses and sensitivities to notice. Learners are most likely to become attuned to these when their attention is suitably directed and when they are in the presence of a relative expert whose attention is similarly directed.

The notion of invariance lies at the heart of elementary science. If you burn a quantity of substance and compare the masses before and after burning, there will be an invariant ratio. This is due to the fact that it is the nature of the substance, not its size, which matters in terms of physical and chemical properties.

Sorting and Ordering Tasks

Offering learners collections of objects (descriptions of specific experiments, states or phases in a single experiment, a collection of mathematical exercises or other objects) and asking them to sort them in some way that seems sensible, or in some way which illustrates a collection of concepts, or to order them in some suitable fashion invites learners to stand back from doing calculations and to think about structure, relationships and properties. Once they start thinking in terms of “types” they are organizing and classifying, as well as preparing themselves to be tested on an examination by recognizing the type of a question.

1.4 Concluding Comments

Teaching consists of acts which take place *in* time, but learning takes place *over* time, usually during sleep when the impressions of the day are selected, rejected, and organized. Where learners have been actively engaged, taking initiative and asserting rather than merely assenting, where they have engaged all five of the aspects of their psyche: enaction, affect, cognition, attention and will, they are more likely to have something salient to process during sleep, to link to other experiences, and so, to learn.

To promote an active rather than a passive stance, to get learners asserting themselves and becoming more intentionally agentive rather than merely assenting requires the teacher to establish an ethos, to set up conditions (the tasks set), and to interact with learners in ways which evoke and promote the use of learners' own powers. It requires developing confidence that learners will indeed make use of their powers if given the opportunity. It requires not so much "getting the sage off the stage" (King, 1993) as sensitizing yourself to notice opportunities for learners to take responsibility, to make subject-significant choices, to become active learners rather than mere receivers.

The powers suggested here play a central role in any mathematical or scientific activity. They can be used to inform the planning and conduct of workshops, lectures and tutorials, the design of course curricula, and the writing of texts. Adopting an approach to pedagogy and didactics which aims to evoke and develop learners' use of their own powers frees the teacher to work on the direction of learner attention rather than on rehearsing well-known content in front of assenting learners. It opens the way to having fruitful interactions with learners about the core aspect of mathematical and scientific thinking.

Constructs and frameworks such as those suggested here are only of value if they are integrated with personal experience. Integration is achieved through looking for examples in your own experience which highlight or illuminate the general, and mentally imagining yourself in a future teaching situation with a label coming to mind which triggers you to act in a fresh rather than habitual manner. It all depends on noticing an opportunity in the moment, whether when planning or when teaching, and the possibility of noticing freshly is enhanced by preparing for the future, based on reflecting on the past (Mason, 2002b).

References

Boaler, J. (1997). *Experiencing School Mathematics: Teaching Styles, Sex and Setting*. Buckingham: Open University Press.

Brousseau, G. (1984). The Crucial Role of the Didactical Contract in the Analysis and Construction of Situations in Teaching and Learning Mathematics. In H. Steiner (Ed.), *Theory of Mathematics Education*, Paper 54 (pp. 110–119). Institut fur Didaktik der Mathematik der Universitat Bielefeld.

Brousseau, G. (1997). *Theory of Didactical Situations in Mathematics: Didactiques des mathématiques, 1970–1990*, N. Balacheff, M. Cooper, R. Sutherland & V. Warfield (Trans. & Eds.). Dordrecht: Kluwer.

Brown, S., Collins, A. & Duguid, P. (1989). Situated Cognition and the Culture of Learning. *Educational Researcher*, 18(1), 32–41.

Bruner, J. (1996). *The Culture of Education*. Cambridge: Harvard University Press.

Bruner, J., Goodnow, J. & Austin, A. (1956). *A Study of Thinking*. New York: Wiley.

Davis, B. (1996). *Teaching Mathematics: Toward a Sound Alternative*. London: Garland.

diSessa, A. (1987). Phenomenology and the Evolution of Intuition. In C. Janvier (Ed.), *Problems of Representation in the Teaching and Learning of Mathematics* (pp. 83–96). Mahwah: Erlbaum.

Festinger, L. (1957). *A Theory of Cognitive Dissonance*. Stanford: Stanford University Press.

Fischbein, E. (1987). *Intuition in Science and Mathematics: An Educational Approach*. Dordecht: Reidel.

Fischbein, E. (1993). The Theory of Figural Concepts. *Educational Studies in Mathematics*, 24(2), 139–162.

Floyd, A., Burton, L., James, N. & Mason, J. (1981). *EM235: Developing Mathematical Thinking*. Milton Keynes: Open University.

Gattegno, C. (1970). *What We Owe Children: The Subordination of Teaching to Learning*. London: Routledge & Kegan Paul.

Gattegno, C. (1987). *The Science of Education Part I: Theoretical Considerations*. New York: Educational Solutions.

Gattegno, C. (1988). *The Mind Teaches the Brain*. New York: Educational Solutions.

Halmos, P. (1994). What is Teaching? *American Mathematical Monthly*, 101(9), 848–854.

Hamilton, E. & Cairns, H. (Eds.) (1961). *Plato: The Collected Dialogues Including the Letters. Bollingen Series LXXI*. Princeton: Princeton University Press.

Harré, R. & van Langenhove, L. (1999). Introducing Positioning Theory. In R. Harré & L. van Langenhove (Eds.), *Positioning Theory* (pp. 14–31). Oxford: Blackwell.

Johansson, B., Marton, F. & Svensson, L. (1985). An Approach to Describing Learning as Change Between Qualitatively Different Conceptions. In L. West & A. Pines (Eds.), *Cognitive Structure and Conceptual Change* (pp. 233–257). New York: Academic.

King, A. (1993). From Sage on the Stage to Guide on the Side. *College Teaching*, 41(1), 30–35.

Krutetskii, V.A. (1976). J. Teller, Trans.) In Kilpatrick, J. & Wirszup, I. (Eds.), *The Psychology of Mathematical Abilities in School Children*. Chicago: University of Chicago Press.

Laborde, C. (1995). Designing Tasks for Learning Geometry in a Computer Environment. In L. Burton & B. Jaworski (Eds.), *Technology in Mathematics Teaching: A Bridge Between Teaching and Learning* (pp. 35–67). London: Chartwell-Bratt.

Lakatos, I. (1976). *Proofs and Refutations: The Logic of Mathematical Discovery*. Cambridge: Cambridge University Press.

Lave, J. & Wenger, E. (1991). *Situated Learning: Legitimate Peripheral Participation*. Cambridge: Cambridge University Press.

Legrand, M. (1990). Débat scientifique en cours de mathématiques. *Repères-IREM*, 10, 123–158.

Legrand, M. (1998/2000). Scientific Debate in Mathematics Course. *La Lettre de la Preuve Novembre/Décembre 2000* on-line text. Retrieved May 2008 Http://europa.eu.int/comm/research/conferences/2004/univ/pdf/univ_sciencediss_mathematics_290704_en.pdf.

Lin, J. (2004). A Chinese Cultural Model of Learning. In F. Lianghuo, N.-Y. Wong, J. Cai & S. Li. (Eds.), *How Chinese Learn Mathematics: Perspectives from Insiders* (pp. 124–156). Singapore: World Scientific.

Love, E. & Mason, J. (1992). *Teaching Mathematics: Action and Awareness*. Milton Keynes: Open University.

MacHale, D. (1980). The Predictability of Counterexamples. *American Mathematical Monthly*, 87, 752.

Marton, F. & Booth, S. (1997). *Learning and Awareness*. Hillsdale, USA: Lawrence Erlbaum.

Marton, F. & Trigwell, K. (2000). 'Variatio est Mater Studiorum'. *Higher Education Research and Development*, 19(3), 381–395.

Mason, J. (2002a). *Mathematics Teaching Practice: A Guide for University and College Lecturers*. Chichester: Horwood Publishing.
Mason, J. (2002b). *Researching Your Own Practice: The Discipline of Noticing*. London: RoutledgeFalmer.
Mason, J. (2003). On the Structure of Attention in the Learning of Mathematics. *AAMT*, 59(4), 17–25.
Mason, J. (2004). Doing ≠ Construing and Doing + Discussing ≠ Learning: The importance of the structure of attention. *Regular Lecture at the 10th International Congress on Mathematics Education*. Copenhagen, July 2004.
Mason, J. (2007). Making Use of Children's Powers to Produce Algebraic Thinking. In D. Carraher, J. Kaput & M. Blanton (Eds.), *Algebra in the early grades*. Hillsdale: Erlbaum.
Mason, J., Burton, L. & Stacey, K. (1982). *Thinking Mathematically*. London: Addison Wesley.
Mason, J. & Johnston-Wilder, S. (2004). *Fundamental Constructs in Mathematics education*. London: RoutledgeFalmer.
Mason, J. & Johnston-Wilder, S. (2004/2006). *Designing and Using Mathematical tasks*. Milton Keynes: Open University, republished (2006). St. Albans: Tarquin.
Mason, J. & Watson, A. (2001). Stimulating Students to Construct Boundary Examples, *Quaestiones Mathematicae*, 24 (Suppl 1), 123–132.
Montessori, M. (1912 reprinted 1964). (George, A. Trans.) *The Montessori Method*. New York: Schocken Books.
Norretranders, T. (1998). (J. Sydenham, Trans.). *The User Illusion: Cutting Consciousness Down to Size*. London: Allen Lane.
Noss, R. & Hoyles, C. (1996). *Windows on Mathematical Meanings: Learning Cultures and Computers*. Dordrecht: Kluwer.
Piaget, J. (1971). *Biology and Knowledge*. Chicago: University of Chicago Press.
Pólya, G. (1962). *Mathematical Discovery: On Understanding, Learning, and Teaching Problem Solving*. New York: Wiley.
Popper, K. (1972). *Objective Knowledge: An Evolutionary Approach*. Oxford: Oxford University Press.
Renkl, A., Stark, R., Gruber, H. & Mandl, H. (1998). Learning from Worked-out Examples: The Effects of Example Variability and Elicited Self-explanations. *Contemporary Educational Psychology*, 23, 90–108.
Rissland, E.L. (1991). Example-based Reasoning. In J.F. Voss, D.N. Parkins & J.W. Segal (Eds.), *Informal Reasoning in Education* (pp. 187–208). Hillsdale, NJ: Lawrence Erlbaum.
Rowland, T. (2001). Generic Proofs in Number Theory. In S. Campbell & R. Zazkis (Eds.), *Learning and Teaching Number Theory: Research in Cognition and Instruction* (pp. 157–184). Westport, CT: Ablex.
Salomon, G. (1979). *Interaction of Media, Cognition and Learning*. London: Jossey-Bass.
Spencer, H. (1878). *Education: Intellectual, Moral, and Physical*. London: Williams and Norgate.
Sweller, J. & Cooper, G.A. (1985). The Use of Worked Examples as a Substitute for Problem Solving in Learning Algebra. *Cognition and Instruction,* 2, 59–85.
Thompson, P. (2002). Didactic Objects and Didactic Models in Radical Constructivism. In K. Gravemeijer, R. Lehrer, B. van Oers & L. Verschaffel (Eds.), *Symbolizing, Modeling and Tool Use in Mathematics Education* (pp. 191–212). Dordrecht: Kluwer.
van Hiele, P. (1986). *Structure and Insight: A Theory of Mathematics Education*. London: Academic Press.
Walker, J. (1975). *The Flying Circus of Physics with Answers*. New York: Wiley.
Warfield, V. (2005). (accessed June 05). Calculus by Scientific Debate as an Application of *didactique*. On line document. Retrieved May 2008. Http://www.math.washington.edu/warfield/articles/Calc&Didactique.html
Watson, A. & Mason, J. (2005). *Mathematics as a Constructive Activity: The Role of Learner-Generated Examples*. Mahwah: Erlbaum.
Wertheimer, M. (1945). *Productive Thinking*. New York: Harper.

Whitehead, A. (1932). *The Aims of Education and Other Essays*. London: Williams and Norgate.

Winston, P. (1975). Learning Structural Descriptions from Examples. In P. Winston (Ed.), *The Psychology of Computer Vision* (pp. 157–210). New York: McGraw-Hill.

Wood, D., Bruner, J. & Ross, G. (1976). The Role of Tutoring in Problem Solving. *Journal of Child Psychology*, 17, 89–100.