Mathematics Teacher Education 6

Orit Zaslavsky
Peter Sullivan
Editors

Constructing Knowledge for Teaching Secondary Mathematics

Tasks to Enhance Prospective and Practicing Teacher Learning

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Constructing Knowledge for Teaching Secondary Mathematics

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Setting the Stage: A Conceptual Framework for Examining and Developing Tasks for Mathematics Teacher Education

Orit Zaslavsky and Peter Sullivan

This book is about tasks that teacher educators might use with prospective or practicing secondary mathematics teachers. There is a substantial literature that has established the critical role that tasks play in the teaching and learning process for school mathematics classes. Kilpatrick et al. (2001), for example, claim that the quality of teaching depends on whether teachers select cognitively demanding tasks, and whether these tasks unfold in the classroom in ways that allow the students to elaborate on the tasks and learn through those tasks. The basic argument is that it is through and around tasks that teachers and students communicate and learn mathematical ideas, so the tasks used by the teachers become the mediating tools. Christiansen and Walther (1986), drawing on the work of Leont'ev (1978), argued that the tasks set and the associated activity form the basis of the interaction between teaching and learning. Other authors who have similarly emphasized the critical role of tasks in creating learning opportunities for school students as well as the significant influences tasks have on what students actually learn include Stein and Lane (1996), Brousseau (1997), Hiebert and Wearne (1997), and Boaler (2002). This book is contributing to a related literature on the important role of tasks in teacher education.

One of the goals of teacher education is to help prospective and practicing teachers develop from novice possibly uncritical perspectives on teaching and learning to more knowledgeable, adaptable, judicious, insightful, resourceful, reflective and competent professionals ready to address the challenges of teaching secondary mathematics. These ambitious goals present great demands on teacher educators, who are responsible for facilitating learning opportunities for teachers to develop and become capable of working towards these goals. We take the stand that simi-

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larly to students, teacher learning occurs largely through engagement in effective tasks, along with reflection on the experience of working on the tasks.

By tasks, we refer to problems or activities that are posed to prospective and practicing teachers by teacher educators. Such teachers are expected to engage in these tasks actively, collaboratively, and intellectually with an open mind and an orientation to future practice. The tasks might be similar to those used by classroom teachers (e.g., the analysis of a graphing problem) or idiosyncratic to teacher education (e.g., critique of videotaped practice).

There is an evolving body of literature indicating the subtleties involved in developing worthwhile tasks for secondary mathematics teacher education. The process of designing, evaluating, and refining tasks for mathematics teacher education is iterative and mostly occurs over a long period of time (Zaslavsky 2007, 2008). In this book, we offer a collection of chapters that constitute a rich resource for mathematics teacher educators. These chapters provide exemplary tasks for mathematics teachers, which have been tested and proven effective in facilitating teacher learning. They also provide analyses of the affordances and limitations of these tasks, descriptions of ways to implement them, evidence of teacher learning from engagement in these tasks, insights into design issues, and links to theoretical and practical perspectives. The types of tasks vary and address several aspects of teacher knowledge and skills that may be constructed through them.

In planning this edited volume, we developed a conceptual framework encompassing eight unifying themes of tasks used in secondary mathematics teacher education. These themes reflect goals for mathematics teacher education, and are closely related to various aspects of knowledge required for teaching secondary mathematics. The following are eight themes we had in mind:

1. Developing adaptability
2. Fostering awareness of similarities and differences
3. Coping with conflicts and dilemmas
4. Designing and solving problems for use in mathematics classroom
5. Learning from the study of practice
6. Selecting and using (appropriate) tools and resources for teaching
7. Identifying and overcoming barriers to students' learning
8. Sharing and revealing self, peer, and student dispositions

Our intention was to have a section with 2–3 chapters for each theme. Thus, for each theme, we invited at least two authors to contribute a chapter to that particular theme. However, when reading the final chapters we became aware that several authors focused on more than one of the above themes, thus, the original structure needed rethinking and adjustment. We believe this reflects the complexity of the field and its interconnectedness. We still find our conceptual framework useful in reflecting on the various chapters, and turn to an elaborative description of the eight themes.

Theme 1: Developing Adaptability. A unifying theme in many aspects of teacher education is the development in teachers of an orientation to being adaptable, to considering variations to questions, tasks, and intended curriculum, to searching for alternatives to unsuccessful approaches, and to adapting existing resources to intended goals. This kind of orientation can be considered as *adaptability* and concurs with Cooney's (1994) ideas of *adaptation*, as well as the notion of *contingency* discussed by Rowland et al. (2005). Especially in a teaching and learning environment that encourages active learning by students, there is a need for teachers to be prepared to make active responses that cannot be planned in advance. Thus, adaptability is inter-related to flexibility (Leikin and Dinur 2007). Indeed, it is not only a desirable orientation, but also a desirable personal quality. Teacher adaptability can be useful in diverse situations. Often teacher adaptability and flexibility are closely connected to knowing to act in the moment (Mason and Spence 1999).

From a teacher educator perspective and the role of designer of tasks, in order to enhance teacher adaptability one needs to be able to design tasks that address this issue from several perspectives; particularly from pedagogical, curricular, and mathematical perspectives.

Tasks enhancing pedagogical adaptability could include: varying types of questions, and the specific questions themselves, to catering for students experiencing difficulty and for students for whom the work lacks challenge, both before the class, and during the class "on your feet"; and when finding the students to be not yet ready for a lesson as planned, adapting the plan and delivering an alternate lesson.

Tasks that promote teacher curriculum adaptability could include: adapting existing context based resources to a current context relevant to their class; and identifying connections across the curriculum, and designing ways to connect different topics in various ways.

Tasks that promote teacher mathematical variability could include: taking a successful game or other activity, using a particular content, and extending it to different content and level of demand; and examining dimensions and domains of possible variation (of tasks, of examples), as described in Watson and Mason (2006).

In order to help teachers become adaptable, teacher educators must themselves be adaptable and exhibit this quality. Thus, the role of the teacher educator is to model flexibility and the ability to vary and consider alternatives and at the same time provide experiences for teachers to engage in such activities.

Theme 2: Fostering Awareness to Similarities and Differences. Noticing similarities and differences, in the broad sense, is at the heart of learning and teaching (Mason 1998). It is well known that the gradual process of associating concepts with categories is a critical aspect of learning. Classification of different objects according to various criteria may enhance awareness of ways in which they are related to each other (Silver 1979). This process requires the identification of similarities and differences between objects along several dimensions; this type of activity is considered fundamental to mathematical thinking. The awareness associated with comparing and contrasting is also needed in order to identify patterns and make connections between and across topics, contexts, types of problems and approaches. This theme can also be seen from pedagogical, curricular and mathematical perspectives.

Tasks aimed at developing teachers' awareness to similarities and differences from a pedagogical perspective could focus on students' thinking, for example by sorting student errors according to their possible sources/ways of thinking. Such tasks can also raise teachers' appreciation of, and ability to analyse, different learning opportunities for their students, such as by comparing and contrasting various classroom situations.

Tasks promoting curricular awareness could include: learning to distinguish between structure and surface features by analysing and sorting various textbook tasks (e.g., according to their levels of cognitive demand and solution processes); and analysing and comparing textbooks' presentations of a topic.

Tasks drawing attention to sameness from a mathematical perspective could include: revealing and promoting teachers' mathematical understanding by comparing mathematical objects (e.g., graphs) in various ways; designing sorting tasks involving mathematical objects; and generating examples according to common features and examining their differences.

Designing tasks involving sorting is rather demanding and often requires careful consideration of the choice of the specific examples/objects with respect to numerous criteria by which they may be grouped. Such tasks are open-ended in nature and may provide a rich context for eliciting many viewpoints. They are "low risk", as different learners may approach them in different ways, some attending to more immediate features, and others to deep structure ones. They have the potential of drawing attention and raising awareness, generating much discussion on a wide range of issues, including common features of various families of objects, different representations of mathematical objects, and connections between them (Zaslavsky and Leikin 2004). The special nature of this family of tasks makes them accessible and applicable to various communities of practice (students, teachers, and teacher educators). They may be used to identify learners' mathematical thinking as well as educators' pedagogical knowledge.

Clearly, in order to help teachers develop a tendency to notice and an ability to identify similarities and differences as a state of mind, and particularly in classroom situations, the teacher educator must exhibit such awareness, not only in the planning stage, but also in-the-moment decisions and interactions with teachers.

Theme 3: Coping with Conflicts and Dilemmas. Teachers constantly face dilemmas and need to make decisions and choices under conflicting constraints, and deal with uncertainty and complexity (Sullivan and Mousley 2001; Sullivan 2006). Thus, it is imperative that teachers are prepared for dealing with this complex terrain, both as teachers, as designers and orchestrators of such situations for their students.

The grounds for creating learning situations that involve uncertainty and doubt are rooted in Dewey's (1933) notion of reflective thinking. According to Fischbein (1987) the need for certitude is a strong driving force for learning. Engaging teachers in tasks evoking conflict has two main goals: first, the process of resolving the conflict may lead to insights and to refining teachers' understandings; second, when encouraged to reflect on their personal experience, teachers are likely to gain appreciation of the use of certain tasks for their students, as well as awareness of the problematic aspects that such tasks present.

Perplexity, confusion and doubt are often associated with and evoked by cognitive conflict. Most research on cognitive conflict has been inspired directly or indirectly by Piaget's (1985) notion of equilibration. Generally, cognitive conflict for teachers may be evoked in a number of ways (Zaslavsky 2005). The main use of tasks evoking cognitive conflict is to overcome misconceptions and/or challenge intuitions and to confront teachers with inconsistencies they may hold (e.g., Tirosh and Graeber 1990). The basic idea in this approach is to probe for anticipated misconceptions held by the learner by presenting tasks that provide evidence that contradicts these misconceptions. Thus, the tasks are designed to contradict existing beliefs. In tasks that involve cognitive conflict, social interactions play a particularly significant role both in evoking conflict and in leading to its resolution. This context lends itself to authentic argumentation and debate, requiring personal articulation and explanation.

This theme can be addressed from several perspectives, including pedagogical, curricular and mathematical. Tasks that deal with classroom dilemmas could include: dealing with and resolving teachers' dilemmas in different ways; anticipating conflicts in the classroom, analysing their origins, and considering their connection to relevant theories and literature; designing tasks that potentially involve uncertainty and conflict for students, implementing them, and reflecting on students' ways of dealing with such tasks; and attending to affective aspects of students who are faced with cognitive conflict.

Tasks aimed at identifying and dealing with curricular discrepancies or incoherency could include: examining curricular documents and textbooks for inconsistencies, ambiguity, and impossibilities; and taking apparently conflicting curriculum statements and seeking a reconciliation that allows the achievement of diverse goals.

Tasks involving coping with cognitive conflict in mathematics could include: evoking teachers' cognitive conflict that are followed by accounts of mathematical understandings they gain in the course of its resolution.

One of the most significant and challenging roles of a mathematics teacher educator is to prepare teachers for creating an ongoing, genuine, mathematical discourse in their classroom (Lampert 1990; Cobb et al. 1993). To do so, a teacher educator must demonstrate and reflect on ways to generate some degree of uncertainty or even confusion, a condition that Brown (1993) considers critical for learning mathematics, and Leinhardt (2001) considers a fruitful prompt for explanation of both the teacher and the student. Such experiences for teachers constitute opportunities for them to observe how a teacher educator may deal with dilemmas and conflicts that arise in real situations.

Theme 4: Designing and Solving Problems for Use in Mathematics Classrooms. Learning to attend to and enhance students' problem solving skills and strategies is a significant goal for prospective and practicing mathematics teachers. Teachers need to be problem solvers in the broad sense of this term, and enhance their students' ability to solve mathematical problems. Teacher education should aim at enhancing teachers' learning to incorporate in the classroom exploration of problems that have multiple solutions or solution-strategies (Leikin and Levav-Waynberg 2009), learning to analyze and evaluate students' problem solving strategies, learning to elevate the degree of openness of textbook problems, and learning to use mathematical games as a problem solving context.

Tasks that involve a focus on pedagogical actions to support problem solving could include: those that have multiple solutions and/or multiple pathways to solution; those that can be accessed at different levels requiring some adaptation by teachers to address the needs of particular learners (Sullivan et al. 2006); those which require the use of materials and tools; and those that are amenable to solution only after persistence by the student (Dweck 2000).

Tasks that can be used to foster engagement and prompt communication to support problem solving could include: problems that have an explicit social dimension which require some realistic interpretation as part of their solution (Peled 2008); those that involve creating a mathematical model of an authentic situation; and those in which the group members must take particular roles in order to arrive at a group solution.

Tasks that include a mathematical focus to support problem solving could include: those that involve connecting together different representations, those that use known mathematical principles to derive a new principle or concept; those that involve comparing and contrasting similar mathematical ideas; and those that involve proving or justifying a solution or identifying inaccuracies in the reasoning of others (Stein and Lane 1996).

It is important that teachers engage in mathematical problem solving and problem formulation. Tasks that can do this include: promoting teachers' problem solving skills by solving problems in multiple approaches and/or strategies; engaging teachers in exploration of open-ended problems; and structuring teachers' metacognitive approaches to problem solving and problem posing. In addition to being competent problem solvers and familiar with the relevant content and pedagogy, teacher educators are expected to be confident enough to engage teachers in open-ended problem situations to which the possible solutions and new questions that may arise are not necessarily known to them in advance. A teacher educator must be open minded and willing to accept and explore in real time unexpected approaches and ideas that teachers may suggest. This is similar to the demands on teachers to exhibit the same approaches with their students.

Theme 5: Learning from the Study of Practice. Teacher education is sometimes characterized by extremes. On the one hand, in response to criticisms of the remoteness of the content of teacher education programs and the positive reports that graduate teachers give to their practicum experience, there are calls for more teacher education activities to take place in schools. On the other hand, an orientation to learning from practice requires much more than time spent in unreflective field based experiences, and school based programs in the absence of research-informed teacher educator perspectives. Many teacher education programs are seeking ways to enhance the practical relevance of their curriculum, while allowing prospective teachers opportunities to review key theoretical perspectives, and ultimately to develop a career long orientation to learning from the study of their own teaching or the teaching of others.

It seems that there are opportunities for prospective teacher learning in the intensive and intelligent study of practice; the critique of practice both within its own context and within the light of other factors; the encouragement of critical reflection; the development of orientations toward moving beyond merely describing practice to analysing actions, responses, and pedagogical practices. Experience in schools is clearly necessary for the practical orientation to the study of teaching but it is not sufficient.

Worthwhile teacher education tasks are those that are motivated by desire to foster the orientation in prospective teachers to the study of practice. A unifying characteristic of such tasks is that they provide simulated access to practice in an environment that fosters critical analysis of practice. Examples of such tasks include: the realistic simulations offered by videotaped study of exemplary lessons (Clarke and Hollingsworth 2000); interactive study of recorded exemplars (e.g., Merseth and Lacey 1993); case methods of teaching dilemmas that problematise aspects of teaching (e.g., Stein et al. 2000); and Lesson Study that engages teachers in thinking about their long-term goals for students, developing a shared teaching-learning plan, encountering tasks that are intended for the students, and finally observing a lesson and jointly discussing and reflecting on it (e.g., Lewis et al. 2004; Fernandez and Yoshida 2004). Each of these requires appropriate prompts for critical analysis to be effective. In each case, the teacher learning is through the opportunity to view and review exemplars, to discuss with peers interpretations of the exemplars, to engage in critical dialog on the experience, and to hear informed analysis of both the practice and the experience of critique.

Tasks facilitating learning from the study of pedagogical, curriculum and mathematical practice (actual and hypothetical) could include: teachers learning from the study of videotaped lessons (including interactive DVD or on-line exemplars) emphasizing pedagogical, curricula, and mathematical challenges; teachers learning from the study of real or simulated (e.g. microteaching) exemplars, including “lesson study” approaches, emphasizing pedagogical, curricula, and mathematical challenges; and teachers learning from the study of specifically prepared cases of pedagogical, curricula, and mathematical challenges.

Numerous teacher educators have written on these and other approaches to teacher education that emphasise the learning opportunities in the study of practice (e.g., Sullivan 2002), and many have reported successful implementation of the delivery of teacher education in school settings that facilitates and fosters direct linking of theory and practice. Clearly, teacher educators who design and implement these experiences are presented with great challenges. They need to be able to capture problematic and insightful classroom situations, and translate them into challenging cases for teachers to ponder. Fostering critical discussions regarding such cases requires high level metacognitive and mentoring skills.

Theme 6: Selecting and Using (Appropriate) Tools and Resources for Teaching. Selecting appropriate tools for mathematics teaching and using them effectively is a major challenge for teachers. Tools can be text books, additional readings, manipulatives, construction and measuring devices, transparencies, graphical calculators, and other technological environments. Making educated choices about what tools to use for certain purposes and how to use them requires familiarity with a wide range of tools from both a learner's and a teacher's perspective. It also requires awareness of the potential and limitations of each tool, for various purposes and contexts, and confidence in using it for teaching. Over the past decade there has been an increasing demand on teachers to become competent with the use of advanced technological tools for teaching mathematics (e.g., Yerushalmy and Chazan 2002)

Tools can be seen in their broadest sense, to include many different kinds of resources, including human and cultural resources, in particular language and time, as Adler (2000) argues that increasing attention should be given to resources in mathematics teacher education from two aspects:

> First, mathematics teacher education programmes need to work with teachers to extend common-sense notions of resources beyond material objects and include human and cultural resources such as language and time as pivotal in school mathematical practice. Second, attention in professional development activities needs to shift from broadening a view of *what* such resources are to *how* resources function as an extension of the mathematics teacher in the teaching-learning process. (p. 207).

Tasks that help teachers become component and comfortable in using tools of the trade as learners could include: experiencing dynamic geometry environments; exploring graphical technologies; using physical technological models (and other micro-worlds); and encountering hands-on activities (e.g., with transparent paper, dice, visual aids).

Tasks addressing learning about the strengths and limitations of various tools for teaching could include: developing expertise in selecting and adjusting appropriate tools for the classroom; designing and incorporating suitable tasks for students in accordance to the implemented tools; experiencing a problem solving process with different tools; and entering time constraints into the picture by taking time into consideration in choosing tools.

From a mathematics teacher educator perspective, designing tasks for enhancing teachers' competence in selecting and effectively using tools for teaching requires a sound knowledge of a wide range of available tools and their potential for accomplishing various goals. It also requires great sensitivity to teachers' reluctance to incorporate unfamiliar innovative tools in their teaching.

Theme 7: Identifying and Overcoming Barriers to Students' Learning. Education and schooling strive to redress the advantages of privilege, and create opportunities for all students, especially those who would not otherwise have those opportunities. There is a need though to overcome some real, and in some cases substantial, barriers that would otherwise inhibit the realization of the opportunities. One of the challenges for teacher education is to educate prospective and practicing teachers about the existence and sources of barriers, and of strategies that can be effective in assisting students to overcome those barriers (Sullivan et al. 2003).

The literature is replete with identification and analysis of factors that create barriers to learning or engagement or success for some students. The barriers might be due to: epistemological aspects of mathematics (e.g., informal vs. formal approaches; modes of representation; missed prior learning opportunities; learning styles); cultural factors including community expectations, gender, school/home aspirational mismatches; language barriers and usage; physical and other disabilities; socio-economic factors including geographical considerations (rural vs. city); and family income and parental occupation.

It is important to understand teacher actions that facilitate successful lessons, defined as those that engage all students, especially those who may sometimes feel alienated from mathematics and schooling, in productive and successful mathematical thinking and learning. An underlying assumption is that lessons can seek to build a sense in students that their experience has elements in common with the rest of the class and that this can be done through attention to particular aspects of mathematical and socio-mathematical goals. This can be achieved by using open-ended tasks, preparing prompts to support students experiencing difficulty, and posing extension tasks to students who finish the set tasks quickly.

From a pedagogical perspective, tasks aiming at identifying and overcoming pedagogical barriers and developing sensitivity to student thinking could focus on drawing teacher attention to the impact of factors that may operate differentially on students such as geography, gender, socioeconomic status, cultural background, language and learning style, and on ways of addressing some barriers including through attention to aspect of pedagogy and building communities of learners.

From a mathematical perspective, tasks aiming at identifying and overcoming mathematical barriers and becoming sensitive to student inventive ideas could focus on developing teacher awareness of barriers to learning resulting from particular task types, societal expectations, conventions, forms of representation, and inappropriate formality, and on ways of overcoming barriers through effective scaffolding, appropriate sequencing, and considering prior learning opportunities.

A teacher educator, who intends to address this theme and offer teachers opportunities to become aware of, and able to identify, barriers to students' learning, must be aware of such barriers not just for students learning but also for teacher learning. One way to enhance teachers' appreciation of barriers to student learning is to experience overcoming of barriers to their own learning. To do this, a teacher educator must understand the causes of such barriers and their nature (e.g., mathematical, representational, communicational), and be knowledgeable with respect to possible productive interventions. He or she needs to be able to address any prejudices or knowledge mismatches within the prospective or practicing teachers, and design experiences that can assist teachers in intervening effectively to overcome barriers for their own learning as well as for their students.

Theme 8: Sharing and Revealing Self, Peer, and Student Dispositions. In the multidimensional endeavour of teaching and learning mathematics, and learning to teach mathematics, a key dimension is the disposition of the (prospective and practicing) teacher as a learner, the teacher as a teacher, and the pupil as a learner.

The dimension of disposition is itself multifaceted. It can include the following overlapping categories:

- beliefs about: the nature of mathematics; the utility of mathematics; the way mathematics is learned; and one's own ability to learn mathematics;
- self-regulatory behaviours such as persistence, self-efficacy, motivation, and resilience; and
- attitudes such as: liking for mathematics; enjoyment of mathematics; and mathematical anxiety.

Indeed almost all aspects of teacher education have an attitudinal or dispositional dimension that should be considered.

Tasks addressing dispositions toward learning, and learning to teach mathematics, could include consideration of specific tasks used in teacher education that address aspects of the disposition of prospective teachers, and allow consideration of the ways that their disposition influences their own learning, and learning to teach, and how the disposition of prospective and practicing teachers might influence their own teaching. Teachers should also engage in relating to their own dispositions and becoming familiar with ways to prompt and reveal peers' and students' dispositions.

Tasks dealing with motivation and self regulation could include examination and consideration of self-regulatory behaviours and the effects of dispositions such as self fulfilling prophecy (e.g., Brophy 1983) and openness to change. This can be done through reflection on individual and group actions and interactions and designing differential learning experiences that reduce mathematical anxiety by promoting learner's success.

Tasks aiming at developing positive dispositions toward mathematics could include building on surprises (e.g., Movshovitz-Hadar 1988), showing the beauty and usefulness in mathematics, and connecting mathematics and the learning of mathematics to real world experiences.

It follows that a teacher educator needs to know about the multifaceted dimension of beliefs and dispositions and their effects on various aspects of learning and teaching mathematics. Moreover, it is important for a mathematics teacher educator to exhibit positive dispositions and enthusiasm towards mathematics and learning mathematics.

As discussed earlier, we modified the structure of the book. This was done by grouping together the chapters that were intended for Themes 1, 2, 3, and 4. They all address a broader theme, which we term *Designing and Solving Pedagogical and Mathematical Problems* (Sect. 1). We also grouped together chapters intended for Themes 7 and 8 under a broader theme which we term *Dealing with Students' Barriers and Dispositions* (Sect. 4). We discuss the chapters according to the four broad themes. The way authors addressed the various themes provided an opportunity for us to crystallize our thinking and make some helpful distinctions.

Part 1 includes seven chapters that focus on designing and solving pedagogical and mathematical problems.

Challenging teachers' existing beliefs and conceptions of what mathematics or doing mathematics is and how this may look in the classroom is a major undertaking of teacher educators. The chapters in this section all address this challenge. Most of them do it through problems that are closely related to school mathematics.

One chapter goes beyond school mathematics. The mathematical problems are used in a broader context that encourage pedagogical considerations, reflections and discussions. Teachers move from experiencing mathematics as learners to reflecting on their experiences as teachers. Many of the tasks in this section mirror the kinds of activities, approaches and prompts teachers are expected to use with their students. Five of the chapters deal with prospective teachers. It appears that the theme of this section is naturally included in programs for prospective teachers (opposed, for example, to learning from the study of practice, which requires more experience on the part of the teachers). Interestingly, in spite of the diversity of goals and approaches, all chapters build to some extent on creating uncertainty, surprise, conflict, or tension. Three chapters explicitly advocate classification/sorting tasks as means for teacher learning to teach mathematics.

The first chapter, by Daniel Chazan, Patricio Herbst, and Hagit Sela, titled "Instructional alternatives via a virtual setting: Rich media supports for teacher development" describes the use of animations of classroom interactions in algebra to support conversations between prospective teachers and their mentors about the practice of teaching mathematics. This chapter focuses on the practice regarding students' multiple answers (correct and incorrect) to algebra word problems. The discussion analyzed in this chapter focuses on two alternative ways in which teachers can deal with a diversity of student answers to algebra word problems. One approach, which the authors consider more standard, encourages teachers to attend to the right answer and the correct solution method; the other, which the authors consider non-standard, encourages them to facilitate students' discussion about the reasonableness of each of the different answers in the context of the specific word problem. Analysis of actual conversations indicates that the comparison between the two approaches, the discussion of the merits and limitation of each, and the connection to the participants' own experiences is helpful in preparing prospective teachers to teach. In addition to the specific issue in the context of word problems in algebra, the act of considering alternative approaches to teaching (a particular topic) is in itself a desirable habit for teachers to adopt.

The second chapter, by John Mason, titled "Classifying and characterising: Provoking awareness of the use of a natural power in mathematics and in mathematical pedagogy" provides an overarching examination and illustration of what classifying and characterising entail in mathematical activities. These cognitive activities are natural powers, which children exhibit before entering formal education. Classifying and characterising constitute a significant process and key element of mathematics, as well as mathematical learning, thinking, and pedagogy. In addition to providing a rich and diverse collection of mathematical classification and characterization tasks, Mason offers special lens through which to look at mathematics. Accordingly, most theorems can be seen as classifying mathematical objects into those that satisfy certain properties versus those that do not. Moreover, for a particular solution strategy or method, there is both merit and challenge in classifying all the problems which lend themselves to the same method. Mason maintains that this fundamental process in mathematics is a key aspect of learning mathematics in a way that enhances interconnectedness and the appreciation of various techniques as well as the concepts on which they draw. Clearly, teachers need to become aware of and draw attention to students' use of these powers, throughout their schooling.

In the third chapter, by Malcolm Swan, titled "Designing tasks that challenge values, beliefs and practices: A model for the professional development of practicing teachers", the author describes a four-stage model for the professional development of practicing secondary and adult education mathematics teachers. This chapter provides a rich collection of specific tasks and prompts organized according to the four stages of the model: (1) recognizing existing values, beliefs and practices; (2) Analyzing discussion-based practices; (3) Suspending disbelief and adopting new practices; (4) Reflecting on experience. The tasks were used with teachers to evoke tension, conflict, and discussion that challenge their existing beliefs and practices by exposing their ways of thinking, observing contrasting practices, and reflecting on actual classroom experiences (e.g., through videotaped lessons). Swan presents several types of tasks by articulating their purpose and illustrating each type with a sample task. Some tasks are typical of teacher learning, however, the mathematical tasks can serve to promote students' learning as well. One type is a classification task, in the spirit that Mason discusses. Swan's work, similar to Chazan et al., provides opportunities for teachers to consider alternative approaches to teaching mathematics.

Irit Peled and Anat Suzan, in the fourth chapter titled "Pedagogical, mathematical, and epistemological goals in designing cognitive conflict tasks for teacher education", offer a fresh and in-depth look at the design and implementation of tasks that elicit cognitive conflict. Similar to the previous chapter, this work focuses on tasks that have the potential of changing (prospective) mathematics teachers' conceptions and beliefs by creating tension and conflict. The authors analyze the potential contribution of cognitive conflict tasks in promoting teacher learning according to three dimensions of knowledge for teaching mathematics: pedagogical; mathematical; and epistemological. The authors provide a detailed account of a sample of three tasks, which includes the motivation and underlining design principles that guided their construction, as well as some description of task implementations. Analysis of and comparison between the three tasks indicate that in spite of the common features of such tasks, they differ in ways that lead to different kinds of changes in teachers' conceptions and beliefs.

The fifth chapter, by Anne Watson and Liz Bills, titled "Working mathematically on teaching mathematics: Preparing graduates to teach secondary mathematics" illustrates an approach to using mathematical tasks with prospective teachers aimed at promoting complex thought about what it means to do and learn mathematics. The authors challenge prospective teachers' existing conceptions and beliefs by challenging their spontaneous responses to mathematical tasks that relate directly to school curriculum. Often a specific problem is followed by a related though dramatically more difficult problem, that requires re-thinking, debating, comparing, and/or resolving uncertainty. The mathematical tasks are designed in order to support the development of teachers' habits of probing mathematical meaning and connectedness as the starting point for thinking about teaching.

The sixth chapter, by Jarmila Novotná and Bernard Sarrazy, titled "Didactical variability in teacher training", sets forth as a goal for teacher education to enhance teachers' *didactical variability*—a construct they present and discuss. The authors build on their earlier work about the construct of didactical variability. Accordingly, their assumption is that students benefit more from teachers with strong didactical variability. More specifically, students of teachers with weak didactical variability in the domain of word problems are strong in standard problems, while those of teachers with advanced didactical variability can apply their mathematical knowledge in new contexts, thus, are stronger in solving non-routine, non-algorithmic problems. The chapter illustrates through one mathematical activity how it lends itself to developing prospective teachers' variability. Similar to previous chapters, the authors elicit teachers' considerations of alternative solutions, and build to some extent on cognitive conflict.

The seventh chapter, by Nitsa Movshovitz-Hadar, titled "Bridging between mathematics and education courses: Strategy games as generators of problem solving and proving tasks" describes the rationale and structure of one course of a series of four independent problem solving courses for prospective teachers that intertwine mathematics and pedagogy. The main goal of these courses is to provide rich hands-on experiences that facilitate prospective teachers' appreciation of the nature of mathematics as an engaging discipline, the core of which consists of problem posing, conjecturing and proving. This chapter focuses on a course on mathematical strategy games, and illustrates its merits through detailed account of two sample tasks, that includes design principles, classroom management considerations, mathematical analysis, and examples of students' responses to the tasks. Through her analysis, the author conveys how in addition to the ultimate goal of the course, the participants develop an enthusiastic attitude towards communicating to high-school students their realizations about the culture of mathematics, its beauty, and the intellectual fulfillment it offers.

Part 2 includes three chapters that focus on learning from the study of practice. Our experience, surely like that of our readers, is that many secondary teachers, both experienced and neophyte, have fixed views of what mathematics teaching is, and once they are comfortable in enacting that view, seek to replicate their approach in all of their teaching. Our fundamental assumption is that all teachers (whether at school or university level) can improve, but this requires an acceptance that improvement is possible and a commitment to processes for improvement. This section of this book describes three processes for improvement, two with practicing teachers and one with prospective teachers. All three chapters are based on eliciting an inquiry stance on the part of the teachers with whom the researchers worked, and indeed for themselves as well. It seems that this orientation to inquiry is the essential ingredient for improvement.

The first chapter, by Barbara Jaworski, Simon Goodchild, Stig Eriksen and Espen Daland, titled "Mediating Mathematics Teaching Development and Pupils' Mathematics Learning: The Life Cycle of a Task" describes a developmental collaborative project between university staff and school teachers. The project focused attention onto different aspects of inquiry: into doing mathematics; into the design of tasks; and into the process of researching teacher learning. They describe a particular classroom task and its use. They use activity theory as a way of describing actions and analysing the interaction between elements of the project: the actors; the tasks; and the activity. The task is clear and easy to pose, and is even deceptively simple, but the authors illustrate how the task might be used in classrooms across the whole school age range, including quite sophisticated adaptations. The inquiry stance is evident in the responses of the teachers to the experience.

The second chapter, by Catherine Lewis, Shelley Friedkin, Elizabeth Baker and Rebecca Perry titled "Learning from the Key Tasks of Lesson Study" illustrates how the well known Japanese Lesson Study process was adapted to a Western setting. They described five cycles of inquiry, described as tasks, that range from the initial choice of the theme, through anticipating classroom responses, to developing a plan, to data collection and the review of the process. Using a rich investigative task as the basis of the description, they outline the way that the five stages of their lesson study approach are enacted, and the ways that their participants respond. The whole approach assumes an orientation to learning from inquiry, as is described in this chapter.

The third chapter, "Mathematical problem solving: Linking theory and practice" by Berinderjeet Kaur and Toh Tin Lam, describes their approach to introducing prospective secondary teachers to problem solving, and orienting those teachers to an inquiry stance to that problem solving. They draw on the pentagonal model that has been used to inform the teaching and emphasis on problem solving in Singapore since 1990. As an aside, this stability in focus, evident in many Asian systems, is the envy of western educators who are coping with systems which seem to change foci and models capriciously. Kaur and Toh first introduce their prospective teachers to some literature on problem solving, then teachers analyse mathematically rich realistic tasks, and then engage in inquiry into problem solving processes by choosing, solving and analysing tasks chosen from a list of tasks. The authors create the potential for these teachers to commence their careers understanding that inquiry into teaching and learning is possible.

Part 3 includes three chapters that focus on designing, selecting, and using tools for teaching mathematics. The chapters in this section share three central themes. First, all three build on socio-cultural theories and set forth to involve participants in active collaborate work. Interactions between teachers (as learners) as well as between teachers and teacher educators is considered critical for desirable learning outcomes about the potential role and use of physical or digital devices in teaching mathematics. The second theme relates to the challenges and demands the choice and use of tools for teaching puts on a teacher. Most teachers have little or no earlier relevant experience and are not as competent in the use of technology. Moreover, the kind of use that the chapters in this section introduce requires a shift in the teacher's role from 'telling' to listening, observing, facilitating, and guiding. Finally, the third theme deals with learning to use such tools in an educated way requires teachers to experience as learners the kinds of activities they would offer their students and reflect on their own learning experiences with teaching lens.

The first chapter in this section, by Michal Yerushalmy and Galit Botzer, titled “Guiding mathematical inquiry in mobile settings”, introduces innovative exploratory work using mobile devices with prospective and practicing teachers. The goal is to enhance teachers’ mathematical knowledge, and at the same time prepare them to teach in ways that resonate with social constructivist views of and commitments to teaching; that is, guiding active inquiry; teaching skills; and covering the curriculum. The chapter presents tasks for mobile inquiry, each followed by a teaching scenario and discussion of possible integration in teacher education sessions. These specially designed tasks are related to the goals of teaching the construction of mathematical models in algebra and calculus. The tasks illustrate guided exploration of real-life phenomena, collaborative group discussions, and personal experience. The authors suggest that the learning encounters created by engaging in these tasks offer opportunities for identifying teachers’ needs and open new directions for research of mobile learning and teaching.

In the second chapter, by Merrilyn Goos, titled “Technology integration in secondary mathematics: Enhancing the professionalisation of prospective teachers” the author addresses the need for teachers to become more effective and confident users of technology to support student learning. Similar to the previous chapter, this chapter draws on socio-cultural theories; however, the emphasis here is on using the concept of *community of practice* for understanding how teachers develop professional knowledge through more experienced members of the community. The author describes and analyzes an assessment task that is part of an undergraduate technology seminar and has proven successful in preparing prospective teachers to use digital technologies in secondary school mathematics classrooms and to share this work with the broader professional community of mathematics teachers. Through the analysis of three examples of prospective teachers’ responses to the task, the author illustrates some difficulties as well as strengths. The author distinguishes between using technology as a *servant* (e.g., carrying out calculations) versus as a *partner* for building understanding, advocating for the latter. The chapter concludes with the author’s reflections on her role as teacher educator in promoting the partner approach to incorporating technology in teaching mathematics.

The third chapter, by Michela Maschietto and Maria G. Bartolini Bussi, titled “Mathematical machines: From history to mathematics classroom”, draws on the work done in a laboratory of mathematical machines that contains more than two hundred working reconstructions of ancient mathematical artifacts taken from the history of geometry. A laboratory in this context is, on one hand, an approach to teaching that consists of learning activities that involve the use of tools and, on the other hand, rely heavily on interactions between the participants who are expected to work collaboratively on the task. The authors present and discuss three ways of introducing a mathematical machine for the purpose of drawing an elliptical trajectory: a discussion and interpretation of a historical text describing an ancient artifact; a physical manipulative exploration based on the ancient artifact; a production of a digital simulation of the ancient artifact. The potential and challenge of such activities in developing teachers’ experience, confidence, and expertise in choosing and modifying suitable tools for the mathematics classroom are discussed.

Part 4 focuses on issues associated with tasks that can assist teachers in addressing barriers students experience in their mathematics learning or their disposition to that learning. The chapters in this part have some unifying themes. One is that the approaches to teaching that are recommended, and the tasks that are associated with those approaches, emphasise the potential for teachers to learn about teaching from the reflective and thoughtful study of practice. Another is a theme that the diversity of readiness in mathematics classes creates challenges and that teachers need support in addressing those challenges. A third theme is that, through experiencing challenges, both practicing and prospective teachers can appreciate the characteristics of challenge, and pedagogical features that make challenge productive rather than debilitating.

The first chapter in this part, by João Pedro da Ponte is titled "Using video episodes to reflect on the role of the teacher in mathematical discussions". João addresses a challenge faced by teacher educators across the world through the pedagogical changes associated with models of teaching based on student activity, and in particular the review of that activity. The teachers with whom he works first analyse a task, then observe and critique classroom action on that task, with particular focus on actions that facilitate the review. This approach recognises the complexity of practice and emphasises benefits to be gained from thoughtful study of that practice.

The second chapter, titled "Sensitivity to student learning: A possible way to enhance teachers' and students' learning" by Ulla Runesson, Angelika Kullberg and Tuula Maunula describes an approach similar to Japanese Lesson Study which focuses on the learning of the students. They term it "learning study". Runesson and her colleagues choose a topic that secondary students find difficult everywhere, operations with integers, and use this as the basis of the study of learning goals, anticipating challenges students might experience, and focusing on actions of students and their responses while they are learning. Ultimately it is the teachers' reflection on the students' challenges that prompt the teacher learning.

The potential of dynamic geometry software in prompting prospective teacher learning is described in the chapter "Overcoming pedagogical barriers associated with exploratory tasks in a college geometry course" by Norma Presmeg. Norma outlines the use of such software, particularly with exploratory tasks and the challenges that this creates, and the potential that the approach offers for allowing key, albeit sophisticated, concepts associated with the teaching of geometry to be raised. The chapter focuses on the issue of proof, and particularly that the software can have the effect of seemingly reducing in the prospective teachers' minds the need for, or importance of, learning about the nature of proof. Reflection of their experience provides powerful potential for learning about teaching by these prospective teachers.

In the fourth chapter in this part by Peter Sullivan, Robyn Zevenbergen and Judith Mousley, titled "Using a model For planning and teaching lessons as part of mathematics teacher education", they argue that it is important to recognise the complexity of converting tasks to lessons, that creating lessons is engaging for teachers, but that hypothetical models of teaching proposed may be different from the common models experienced by both prospective and even practicing teachers.

They outline tasks that can be posed to both prospective and practicing teachers to focus their attention to these key issues. A key aspect of their tasks is that they focus attention on ways of addressing differences in readiness among the students.

Gaye Williams in the fifth chapter in this part, in a chapter titled "Building optimism in prospective mathematics teachers: Psychological characteristics enabling flexible pedagogy", also uses a geometrical environment to pose challenging tasks to her prospective teachers. The tasks not only expose them to the pedagogical challenges of teaching difficult concepts such as proof, but also to their own experience as they work through tasks that are challenging for them. She describes a framework based on optimism and how the experience of success becomes sustaining and sustainable. The approach creates the potential for prospective teachers to extend this experience to develop an orientation to flexibility in their pedagogy.

These chapters, like all the others in this handbook, can serve as a model, resources and reference for mathematics teacher educators wherever they work.

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Part I

Instructional Alternatives via a Virtual Setting: Rich Media Supports for Teacher Development

Daniel Chazan, Patricio Herbst and Hagit Sela

Introduction

Thought Experiments in Mathematics Teaching (ThEMaT), a U.S. National Science Foundation funded project, has created two-dimensional representations of cartoon characters engaged in classroom mathematics. The classroom stories are presented as "live" animations and printed comic strips. By using these stories in meetings of study groups of teachers, the project's initial thesis has proven itself—rich media technologies can be used to represent classroom stories that stimulate practitioners to engage in revealing conversations about practice; practitioners respond to these stories by producing alternative stories, stories that have happened or could happen in their own classrooms (see Herbst and Nachlieli 2007; Herbst and Miyakawa 2008; Miyakawa and Herbst 2007a, b; Weiss and Herbst 2007).

Based on findings in the literature and on the teaching experience of project staff, we created a model of teaching word problems, word problems of the sort typically encountered in a US Algebra 1 classroom. Our model of the *instructional situation* (Herbst 2006) of doing word problems describes the responsibilities of the students and teacher and the nature of the objects of trade between them. Our model serves as a baseline to interpret 'usual' instructional moves and 'alternative' instructional moves. We used the model's hypotheses to invent the two alternatives for the discussion of one problem, which we then represented in the comic strips and animations.

Representing customary and non-standard teaching in a virtual space has particular affordances when it comes to supporting teachers' reflection on instruction-

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al alternatives. Whereas sampling from a large corpus of video records may help teachers conceive of alternative moves a teacher might carry out in a given situation, any one video case narrates one story forcefully and provides few supports for alternative stories that might have happened instead or could have happened in another setting. After all, what happened, happened! And, in addition, video is limited to what has actually happened; it can be hard to find examples of teachers trying non-standard instructional moves. It can be especially hard to find a teacher who might want to teach in her classroom exactly what a teacher educator would like prospective teachers to consider, let alone to be able to videotape the exact occasion on which they were able to carry out this instruction.

Animations and comic strips of cartoon characters are a more malleable medium. In this medium, the Animated Teacher can carry out an instructional alternative that teachers would be unlikely to try. This Animated Teacher can also do and say exactly what a teacher educator might wish, though if misused the result may read more like a fable or a fairy tale than a representation of practice. In addition, in this medium, animation designers can create alternative responses, for example, to a student comment. Each of these alternatives exists on the same existential plane and represent as faithfully what might have happened. Inasmuch as they only sketch stories, these virtual representations of teaching invite the formulation of alternatives, the second-guessing of moves, and the projection of the circumstances and settings of viewers (Herbst and Chazan 2006). Such representations of teaching can support teachers by providing opportunities to ponder how different instructional alternatives might play out, as opposed merely to considering alternatives to a particular classroom video. When one of the alternatives includes a non-standard instructional move, there is an opportunity to consider the potential costs and benefits of such an instructional move.

In this chapter, we present an instructional story with variants and illustrate how groups of teachers have used two alternative enactments of a classroom task to reflect on what it means to help students learn to solve word problems.

Using Animations in Teacher Preparation

One of the key challenges in teacher education is to help future teachers imagine possibilities for instructional interaction that they themselves have not seen or experienced. In the service of providing future teachers with images of alternative practice, many teacher educators use videos or visits to classrooms of teachers using such practices. In this chapter, we explore the potential of animations of classroom practice as a new "technology" for teacher preparation. While animations are clearly "authored" texts, without the authenticity of videotaped events that have actually occurred, they have other affordances that may offset this liability. In particular, within the hypothetical space of an animated classroom, as suggested earlier, prospective teachers can observe the "same" teacher carry out the same lesson in different ways with the "same" group of students. The ways in which particular teacher moves might play out differently can then be the focus of conversations about pedagogical actions.

We explore the potential of this new technology for teacher preparation by examining an animation involving a word problem. The students in this animated class generate four potential answers to the word problem. The teacher is faced with the challenge of this diversity in student responses. In one alternative, the teacher asks the students to use their knowledge of the problem to determine which of these answers is a reasonable response to the question and which of these answers cannot possibly be correct. In the second alternative, the teacher asks one student to show how they obtained the correct answer and uses this response to convince students that the other answers are not correct.

This pairing of animations was used with a group of prospective teachers and their mentors (supervising teachers) in after-school meetings during the student teaching semester of the program. The prospective teachers were in their final semester; they had already completed the coursework for their B.A. in mathematics and had taken two "methods" classes in the College of Education. During this semester, they had teaching responsibilities in their mentors' classrooms. In bringing together prospective teachers and mentors, the research project sought to test a model for understanding student and teacher responsibilities in the solving of word problems. However, for participants, this was not the focus of the sessions; with the participants, we sought to have conversations around specific incidents in teaching where prospective teachers might ask mentors (explicitly or tacitly) why teaching works the way that it does, and where mentors might explain to prospective teachers why, in their view, it does work this way. We call the knowledge at play in these questions and responses practical rationality (Herbst and Chazan 2003), what Schön (1983) calls 'knowledge-in-action' and 'reflection-in-action.' This is the knowledge that enables practitioners to do what they do; such rationality is common to people who perform the same job. Thus, we hoped that the rationality of the practice of teaching of mathematics would come to the fore around examples of teaching that were not the teaching of any particular mentor or prospective teacher, in a hypothetical classroom onto which the experiences of prospective teachers and mentors might be projected. In one of the sections, we analyze the conversation the prospective teachers and mentors had around these two alternatives to illustrate the kinds of conversations about teaching that can be stimulated by the use of animations in the service of teacher education (an analysis of teacher and student responsibilities when doing word problems can be found in Chazan, Sela, & Herbst, in review).

The "Stories" in the Animated Alternatives

The two animated alternatives we focus on involve a class doing an algebra word problem. The problem is a standard motion problem (Yerushalmy and Gilead 1999). The interaction in the classroom is in many ways consistent with Gerofsky's (2004) notion of word problems as a genre of problem and the interaction around them as a genre of classroom interaction (which we call "instructional situation", Herbst 2006). The following development is common to the two alternatives:

Narrator: the class was working on the following problem:

> A and B are 280 miles apart. A truck and a cab started traveling at the same time towards each other on the same road. The cab traveled from A towards B at an average speed of 80 miles per hour. The truck traveled from B towards A at an average speed of 60 miles per hour. How long after they set off did the two vehicles meet?

Both of the animated alternatives around this problem begin with students sharing the answers they had gotten for this problem. While the correct answer for the problem is 2 hours, the answers that were offered by the students are: 14, negative 14, 2, 2 hours and 20 minutes. We present with a comic strip the beginning of the story, which is common to the two alternatives (Fig. 1).

The teacher at this point in the animation is faced with a question of how to proceed. How does a teacher deal with multiple student answers in the context of a word problem in an algebra class? Most concretely, on whom should the teacher call? Should the teacher call on a student who has the correct answer and focus on the correct answer and how it was obtained? Will this help the students who did not get that answer? Should students with the incorrect answers be called upon? Should their answers be addressed in the public space, if so, as answers, or as outcomes of solution methods whose flaws can be brought to the attention of the class and thus learned from? How does the teacher help students with incorrect answers understand both that their answers are incorrect and why their answers are incorrect?

From this point in the animated story, there are two diverging alternatives. In one alternative, the teacher makes what, in our model for the teaching of word problems, is a non-standard move. In this alternative, the teacher focuses on the answers as answers, not on the solution methods that led to them. Rather than have students show how they have solved the problem, the teacher focuses on the answers that students obtained and asks students to decide whether or not these answers are reasonable given the circumstances described in the problem. The students resist this move by repeatedly asking for the teacher to tell them which answer is correct and asking the teacher to show how to solve the problem. The other alternative, one that is more standard according to our model, focuses first on the correct answer and moves swiftly to understanding how students got their answers; the question of judging the reasonableness of the answers is set aside, the focus is on how the students solved the problems, whether the solution method is correct or not, and, if not, how it might be corrected.

As teachers of algebra, we tend to focus on the solution method by which a solution was derived. In the context of doing word problems, teachers of an algebra class typically would like to see their students solve the problem by writing an equation and then solving the equation. After all, the function of the word problem in the curriculum is often two-fold, to provide a rationale for the importance of equation solving by illustrating its potential to answer questions embedded in a non-mathematical context and to provide a setting for the practice of equation solving. Such a

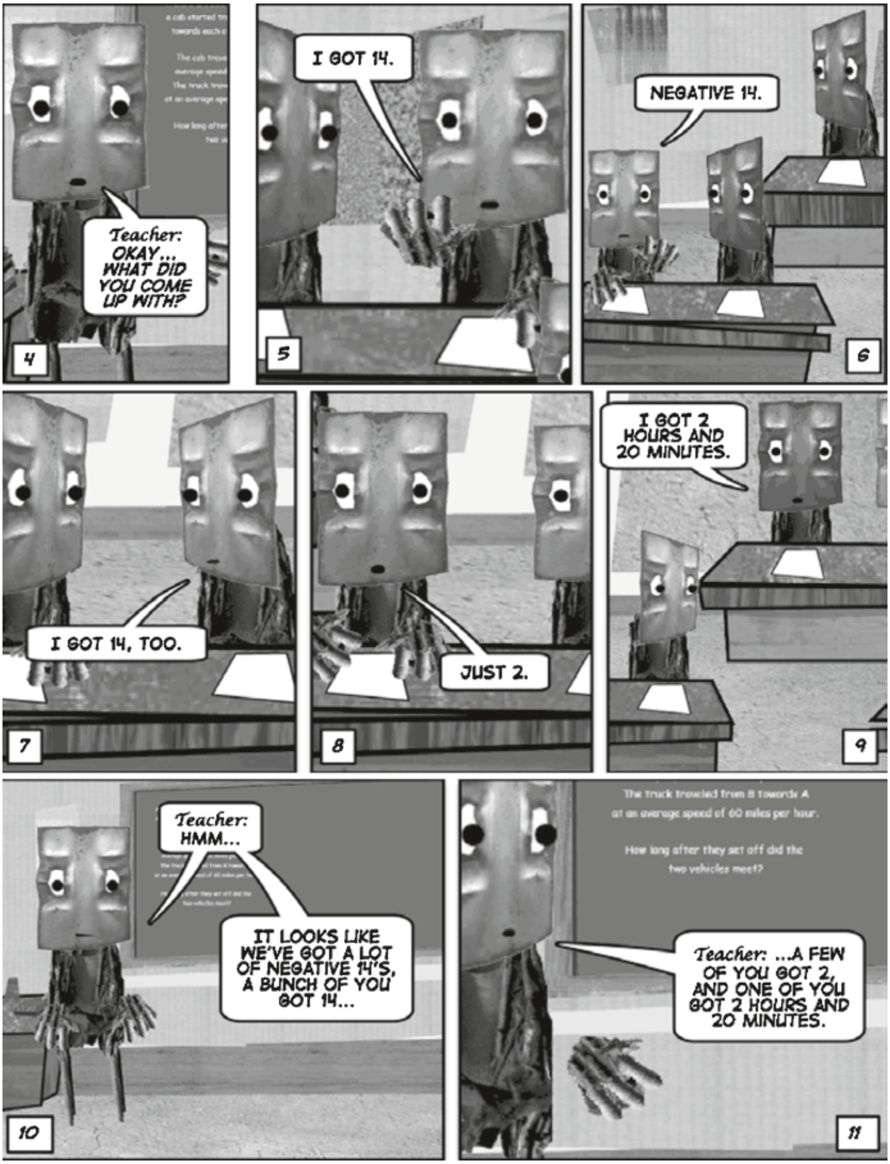

Fig. 1 Common introduction to the two alternatives. (The character set ThExpians M and the comic strip are © University of Maryland and University of Michigan)

focus on solution method however can obscure other important mathematical lessons, like learning to check the reasonableness of one's solutions. Particularly when it comes to solving applied problems, it is important to verify that the solution one has come to mathematically indeed is a reasonable solution to the problem. There are algebraic solution techniques that sometimes can lead a person to a solution to an equation that is not the solution to the problem as articulated in a context. Thus, on occasion, it seems valuable to consider asking students to argue for the reasonableness of the solutions they have found. An important task for teacher education is to help prospective teachers consider how and when to engage students in discussion about the reasonableness of answers. In the context of these word problem animations, this general lesson is bound up with the question of whom to call when there are multiple answers, and why?

Below, we present the two alternatives in detail, reviewing key elements of the alternative stories and providing commentary. For each alternative we suggest specific sets of questions that might be addressed to both prospective and practicing teachers about these stories.

The Alternative "A Correct Solution Method" and Questions for Teacher Discussion

In this alternative, the Animated Teacher focuses on the right answer by choosing to call on Orange, a student who had the right answer, to come up to the board and show what he did.

We discuss this alternative in three parts. In the first part of this alternative, Orange uses a common 'method', one that the teacher has presumably taught: a "Distance equals Rate times Time" chart[1] to generate an equation to solve. Orange writes on a board a chart and an equation, and does not explain what he has written (Fig. 2):

The teacher encourages him "That looks good. Well done, Orange". Questions that might be addressed to teachers for this part of the animation include:

- Why might a teacher want a class to examine Orange's work? What are the pros and cons of choosing Orange's work for examination?
- Is Orange's work sufficient? Orange did not speak, do we as teachers, or do the other students in the class, need further explanation?

	v	*t*	*s*
Cab	*80*	*x*	*80x*
Truck	*60*	*x*	*60x*

$$80x + (60x) = 280$$
$$140x = 280$$
$$x = 2$$

Fig. 2 Orange's solution

[1] See Hall et al. (1989) though rather than writing d=r * t, they write v * t=s.

- Is the chart an important component of the solution process? How is the equation generated from the chart? Do you think this is a useful tool for teaching students how to solve word problems, why or why not?
- Should the teacher affirm Orange's work (at this stage) as correct, why or why not?

In the second part of this alternative, Purple takes the initiative and has the courage to ask what is wrong with his solution:[2] "I got 2 hours and 20 minutes. I don't know what I did wrong. Can I show what I did?" The teacher does not take up the request to spend classroom time on understanding Purple's solution. The teacher says: "Why don't you just be sure to copy the correct answer from the board and try to see what went wrong in yours. If you're still having trouble, we can talk after class."

Questions that might be addressed to teachers at this part of the animation include:

- Is Purple's question a typical question, why or why not? Is it a good question for a student to ask? Would you like your students to ask this sort of question, why or why not?
- Do you like how the teacher responded to Purple or not? If not, how should the teacher react to Purple's question? What effect might the teacher's response have on Purple? On other students in class?
- What might have been some reasons for the teacher's response?

In the third part of this alternative, after setting aside Purple's question, the teacher decides to question the students who though the answer was 14: "I'm curious how so many of you got 14 as an answer. Can somebody show us how you got that answer?" Red comes to the board and uses Orange's table to show the class that thinking about the speeds as directed quantities (the velocity of the truck is negative) results in another equation (one that does not adequately capture the situation) and a solution (Fig. 3).

Orange asks why Red wrote negative 60. Blue joins Red: "Yeah, that's right. I did the same thing, but I had 80 as negative…so I got $x=-14$." Red replies that it's because they traveled in opposite directions. The teacher praises the students that thought about negative speed, but says it isn't appropriate to use a negative number for speed here.

$$80x + (-60x) = 280$$
$$20x = 280$$
$$x = 14$$

Fig. 3 Red's solution

[2] Note that this is often something that students might ask when they have an incorrect answer and when they do not understand why what they did was incorrect, see Office of Educational Research and Improvement (1998).

Questions that might be addressed to teachers at this part of the animation include:

- Given that the correct answer had been given, why might a teacher continue the discussion of the problem?
- If a teacher wanted the class to discuss 14 as a solution, should this discussion occur before Orange presented a correct solution, why or why not?
- Why might a teacher be curious about the 14 and not about the 2 hour and 20 minutes?

The Alternative "Reasonableness of Answers" and Questions for Teacher Discussion

In this alternative, instead of having students share how they solved the problem, the Animated Teacher focuses on the reasonableness of the answers obtained. We discuss this alternative in two parts.

In the first part of this alternative, the students are asked to work in groups in order to figure out which one of the solutions: 14, negative 14, 2 or 2 and 20 minutes, is correct, and also what they need to do to show that the correct answer is indeed correct. A few minutes later, Green presents one group's findings: "In our group, we couldn't decide which one was correct; all of them seemed okay for different reasons. Can we solve it together on the board?" The Animated Teacher returns the question to the class: "Let's go back and try to see how you decided which one of the solutions is correct." One student wants to rule out the negative 14 answer by saying: "Time can't be negative." The teacher does not acknowledge this bid initially and directs the discussion to the meaning of 14 in this context by asking questions like: "What does the number 14 represent?"

Questions that might be addressed for this part of the animation are:

- Is this a good context/problem for group work?
- Should the teacher have said that one (and only one) of these answers is correct?
- How could Green's group decide that each solution is reasonable?
- How might a teacher react now?

In the second part of this alternative, the teacher continues to ask students to decide whether or not a given answer is reasonable and fits with the given situation. The conversation becomes heated when there is a return to the answer negative 14. The teacher asks: "What does it mean when we get negative 14 as an answer? Can you explain with the story what negative 14 might mean?" Students are not able to answer these questions to the teacher's satisfaction and the teacher continues to probe. The students begin to lose patience, saying things like: "Just tell us what the answer is! Show us how to get it! Let's just solve it together on the board." The teacher settles the class down, saying it is important to understand how the numbers they get are related to the story, and that it allows them to decide if a solution is reasonable, "You have to use your commonsense to see if your answer is correct. Without doing that, you could write down an answer that couldn't possibly be true". The teacher makes one more pass at negative 14 and then relents and indicates that negative 14 is ruled out, but still wants to continue to discuss the other answers. When one of the students says he is sure that 2 is the right answer, the teacher asks if it is the only correct answer, and asks to explain by using the story, and not by writing an equation and solving it. Eventually, one student articulates that 14 couldn't possibly work because the cab and truck would travel too far, then another student illustrates how with 2 hours the cab and truck would both be between A and B and the sum of the distances traveled would be 280. Even after this progress, some students seem perturbed by the fact that they haven't solved the problem. Blue comments: "I don't understand why we are spending so much time on this one problem without even solving it".

Questions that might be addressed here include:

- Why might students start to lose their patience?
- What are students' expectations about solving word problems? About time allocation?
- How is the questioning of the students influencing their participation and their understanding?
- Is it a problem if classroom conversation gets heated?

Comparing the Two Alternatives

After having seen both alternatives and having discussed them both, in discussions with teachers, one might consider the following questions:

- What is the difference between the approaches taken in the two alternatives?
- What are the advantages of each of them? Disadvantages? Constraints?
- Could a teacher enact both of these strategies in one classroom session? If so, would there need to be a particular order, or could either order work?

Mentor/Prospective Teacher Discussion of the Animations

While the previous section focused on how the animation might be used with teachers, this section illustrates our use of this animation with a particular group that consisted both of prospective teachers and their mentors. The meeting took place during the student teaching semester when prospective teachers have mentors in whose class they are teaching. Convening a group of pairs, each pair consists of an experienced teacher and his or her student teacher, to discuss the animation and its two alternatives, had some additional affordances beyond those described earlier. The study group described here consisted of thirteen teachers: six mentors and seven prospective teachers during senior's year in a four-year undergraduate secondary teacher preparation program. They watched and discussed the alternative "reasonableness of answers" first, and then they dealt with the alternative "a correct solution method". The alternatives could have been used in a different order. Our choice was to start with a less-typical teacher response figuring that then there would be more vigorous response around the more typical teacher response when it was shown.

The study group was not a class. No grades were given; participants were given financial incentives for attending. The broad goal of the discussions was, in the context of "soft" professional development, creating conversation between prospective teachers and mentors where the prospective teachers could ask mentors how teaching is supposed to go and where mentors could share their wisdom of practice. The facilitator did not have an explicit agenda to convince participants of particular understandings of teaching.

In accordance with the project's goals, we did not use the specific questions suggested in the previous section to probe the participants' thinking. Specifically, the conversation about this animation and its two alternatives targeted the following key questions:

- How do teachers talk with students about how to decide whether answers to word problems are correct or incorrect?
- To what degree should teachers teach students to use the context of the word problem as a check on solutions, or is the check tied up with methods for generating and solving an equation?

Two people facilitated the group discussions. One person took the lead in directing the conversation. The other asked for points of clarification around the specific teaching aspects in each animation.

Typically, after watching an animation, the teachers in the study group were invited to share their thoughts with the group. The overall tenor of the discussions was serious and positive. The teachers treated the animations as instances of teaching practice and shared their thoughts with the group.

Discussion of the Alternative "Reasonableness of Answers"

The discussion of this alternative encouraged the prospective teachers to ask their mentors about the importance of checking the reasonableness of answers in the context of solving word problems. The mentors used this occasion to share their experience by responding to the comments that the prospective teachers raised.

The initial part of the discussion between prospective teachers and mentors was about the importance of determining if an answer is reasonable. It began when one of the mentors (Ralph[3]) was enthusiastic about the Animated Teacher's spending time teaching students to judge the reasonableness of answers. Ralph seems to appreciate what the Animated Teacher did, even though he considered the teaching unusual.

[3] This name and other teacher names are pseudonyms assigned to protect confidentiality.

> Ralph: You know what I like about this video [the animation they just watched]? Because, I'm always telling the kids to "check the reasonableness of your answer", but I liked it that the teacher actually spent time **teaching**[4] checking the reasonableness of the answer. Because that −14, I said [to myself], "Oh please, that's something my kids would do" [laughter]. And it would make perfect sense to them that that [time cannot be a negative number] was the correct answer.

After eight turns, a prospective teacher (Pat) makes what Sfard (2001) calls "pro-active utterance", an utterance that calls for a group reaction. She expresses her concern about teaching students to test the reasonableness of their answers: "I didn't really like the way she [the Animated Teacher] rejected the equation". The contrast between Pat's remark, as a prospective teacher, and Ralph's, the mentor, initial re-mark, implicitly asks: "Why did you like this teaching? And, why do you think reasonableness is so important?" Pat continues her thought by wondering why the Animated Teacher discouraged the using of equations:

> Pat: I think she [the Animated Teacher] should have encouraged the fact that they [the students] used an equation and said, "If we have time after our reasoning process, maybe we'll go back to that [equation] at the end of class". Or, like, because they [the students] are going to be asked to set up equations, so discouraging that [the use of equations], I don't think is beneficial, necessarily, especially if that [writing equations] helps them organize their thoughts.

Other mentors reacted to Pat's concern by arguing for the importance of teaching to check the reasonableness of answers. They did it by describing alternative ways, which are, to their mind, more effective in accomplishing this goal. For example, Floyd offered a different activity structure, that of preparing students for exams with multiple-choice questions. He suggested that it is better to deal with reasonableness within a context where students do not have solution methods, only answers. This way they can reason about the answers without thinking about the method they have used and without being biased by the solution that they have produced:

> Floyd: I wasn't against the reasonableness of the solution, but I probably wouldn't have used it [reasonableness] in that particular problem. It [the problem I would use] would've been more like a multiple choice. I always tell my students there are 4 choices, 3 of them wrong. Now explain to me why… Some days I do, instead of saying, 'What is the right answer?', I say, 'Why are the other 3 answers wrong?' [Ralph nods with agreement].

From there, everyone seemed to accept this line of reasoning. In contrast to his first remark that suggests he rarely teaches students to judge the reasonableness of their answers, the first mentor (Ralph) then says that in the context of multiple choice questions, he does that kind of work all the time; his (prospective teacher) mentee even confirms he had done something of that sort on that very day. So, the sense of the group seemed to be, this is a valuable focus, but, in response to Pat, not in this sort of problem context. Mary, a prospective teacher, reinforced Floyd's point. She hypothesizes that when students first work on a problem themselves, they are too focused on their own solutions, so they cannot think of other answers:

[4] Oral stress in vocal track.

> Mary: I thought maybe the kids [in the animation] didn't really want to [spend time on reasonableness], because everyone was very focused on their answer. Like, the one kid, I think it was Blue, kept saying "No, my answer's right, my answer's right." I don't think he got anything there talking about reasonableness, because he just wanted to sit and say "My answer's right, I know my answer's right, I don't have to explain it." Some of the kids who, like, had answers were more focused on their answer than checking what the other answers were.

Discussion of the Alternative "A Correct Solution Method"

While in the alternative "Reasonableness of Answers" the Animated Teacher focuses on the reasonableness of each answer without paying attention to the 'method' used to produce it, in this alternative the teacher focuses on the right answer. The Animated Teacher calls on a student who shows the 'method' (chart/table and equation), without an explanation (i.e. how one produces the equation) and without checking the reasonableness of the answer.

We expected the study group teachers to react to this alternative in light of the previous one, saying that reasonableness is important. Indeed, while watching the animation, the teachers were very surprised by the Animated Teacher's turn, "Copy the correct answer from the board". They reacted with an uproar, loud laughter and sharp movements: two mentors clapped their hands, one mentor held his head in his hands, and one mentor got up from her chair; one prospective teacher also got up from his chair.

Similar to the initial responses to the previous alternative, prospective teachers wanted the Animated Teacher to ask Orange to explain his solution. Also similar to the previous alternative, mentors focused on explaining why the teacher's act represents the reality of classrooms. Lea (a prospective teacher) was the first to say that the Animated Teacher was wrong to not have Orange explain his answer:

> Lea: If you let the first guy [Orange] explain [how he constructed the equation], the guy [Orange] just, like, put it [the chart and the equation] up the board and he's [the Teacher] like [said to Orange] "Okay, good, sit down".

Lea thinks that explaining the answer could help other students understand why they were wrong:

> Lea: Maybe if he [Orange] explained what he was doing… then the guy who got 2 hours and 20 minutes would have been like, "Oh, I see what he did [and therefore I understand what I should have done]". And then he [the student who got 2 hour and 20 minutes] could have come back to his [solution] and been [thinking] like, "Oh, I see why I'm wrong".

Pat (a prospective teacher) suggests that the exact place where an explanation is needed is the equation because this is the place that she herself had difficulty with: "I think… I wouldn't have thought to do $280 = 80t + 60t$."

Mentors' reactions then came in with Ralph making a sharp shift in the conversation back to the Animated Teacher's response to the student who wanted an explanation about 2 hours and 20 minutes. His turn seemed to shift the overall tenor of the discussion. He looked very decisive, striking the table with his hands and looking at his colleagues around the table and addressing the prospective teachers:

> Ralph: But you [the prospective teachers] will [act as the Animated Teacher did by asking students to copy the right answer], and I know you will. Next year you're going to be crankin' through your lesson, and some kid's going to have an off-the-wall answer [like 2 hours and 20 minutes] and you're going to say [like the Animated Teacher did], "I'm sorry, I don't have time to explain it", because you [as a teacher] have to cover all this stuff by the end of the year.

Similar to his role in the discussion around the alternative "reasonableness of answers", Ralph talks about the distinction between theory and practice. His teaching experience allows him to contend that the prospective teachers are paying too little attention to actual classroom constraints. According to Ralph, explaining answers is great, but having to cover content does not allow teachers to spend class time on explaining each answer. It is interesting to note here that while the prospective teachers were upset that Orange did not explain the correct answer, Ralph's response focused on not explaining other wrong answers like Purple's. It seems that Ralph agreed with the prospective teachers that Orange's answer should have been explained. But he wants the prospective teachers to realize that sometimes as a teacher you cannot deal with every answer a student puts forth.

Craig (another mentor) was concerned with "how" the Animated Teacher acted, more than "what" that teacher did. He felt that the Animated Teacher acted poorly and should have said that there was no time to explain.

> Craig: Seriously, if the teacher had just said that [not having the time to explain it], I would have said OK [It is fine not to spend time on Purple's wrong answer].

Craig's response suggests that there are circumstances under which not explaining the answers is appropriate. But the prospective teachers still remain troubled. Mary and Lea remind the group of the Animated Teacher's decision to ignore Purple's answer (2 hours and 20 minutes), but to address another answer (14) without any explicit reason. Mary: "And then they [the Animated Teacher] went and put another wrong answer on the board." Lea: "One wrong answer was better than another wrong answer".

Darcy, a mentor, agrees: "[the Animated Teacher said] I don't really want to see why you got it (2 hours and 20 minutes), but I want to see the 14". The study group members reply to this concern by relating to circumstances in which it makes sense to address the 14 but not to address the 2 hours and 20 minutes. Pat (prospective teacher) suggests that the Animated Teacher might have anticipated the 14, but not the 2 hours and 20 minutes. Two mentors and another prospective teacher propose that many students in the classroom got 14, which led to the decision to focus on the common answer. Ralph (mentor) suggests another circumstance, which causes teachers not to deal with an answer:

> Ralph: That kid [who asks the teacher to look at his wrong answer] might have been Brandon [a student in his class], who purposely goes on the board. He knows he has the wrong answers, but he wants the attention from the class for about 5 minutes, so he'll volunteer to go do the problem on the board.

Concluding Remarks

The discussions between mentors and prospective teachers around the two alternatives of the story brought forth some ideas that both mentors and prospective teachers seem to agree upon:

- Teaching students to judge the reasonableness of their answers in the way that the Animated Teacher did is a non-standard act. What makes it non-standard, and not viable in the reality of classrooms, is that when students have solution methods as well as answers, and when it is those methods that are what must be learned, focusing simply on the answers is a counter-cultural act, because the answers are not what is important. Reasonableness of answers is better focused on in other contexts.
- Choosing to deal with one wrong answer and not with others, emphasizes the difference between the desirability of responding to all students and the reality of what actually can happen: during the lesson, the teacher has to be flexible and make in-the-moment decisions according to circumstances, and cannot act only according to generalities about what is desirable.

In terms of the mentor/prospective teacher interaction during this conversation, it is interesting to note that a prospective teacher raised concerns about the Animated Teacher's action (checking the reasonableness of answers), while the mentors appreciated an aspect of that same action. As mentors work to identify what they value in the teacher action, they use their knowledge of the practice and are reminded of a teaching context in which examining the reasonableness of answers is easier to accomplish. This difference in point of view between prospective teachers and mentors about a non-standard teaching move supported a meaningful conversation where they could examine teaching practice as a group of experienced and novice teachers. In general, across all of our study group sessions, the prospective teachers and mentors use the animations to think about their own teaching and talk about themselves and their own teaching, and to say what teachers should and should not do. Indeed, use of the animations created a venue for mentors sharing the wisdom of the practice and for prospective teachers to ask questions about teaching that are on their mind.

In terms of the affordances of this animation for teacher preparation, we think this particular animation can be used in the context of "methods" courses to have prospective teachers explore issues of teaching like which student to call when students have multiple answers. The questions we proposed earlier are designed to support the use of this animation for such a purpose.

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Classifying and Characterising: Provoking Awareness of the Use of a Natural Power in Mathematics and in Mathematical Pedagogy

John Mason

Introduction

Mathematically, most theorems can be seen as classifying those mathematical objects which satisfy certain properties, in terms of other, usually more manageable properties. Thus Pythagoras' theorem classifies right-angled triangles as those triangles for which the sum of the squares on two sides is the square on the third, while the law of cosines defines a property which holds for all triangles. Whenever a method is devised for solving a particular problem, there is an immediate challenge (and value) to classify all those problems which succumb to the same method. This is a fundamental process in mathematics, and a key aspect of learning mathematics in order to appreciate each technique and the concepts on which it draws.

Classifying and characterising are natural powers which children display long before they get to school. They are also a core component of mathematical pedagogy. For mathematical thinking, it is important that learners are provoked to use their own powers to classify and characterise so that these are developed explicitly throughout their mathematical schooling, which means teachers being aware of and drawing attention to their use, whether actual or potential. This chapter elaborates on these claims. Since anything which is powerful can have negative as well as positive consequences, mention is made of situations in which it is possible to misuse these powers.

Human beings have natural powers for dealing with the complex world of sense-impressions which impact on them moment by moment. It is worth noting in passing that David Hume's basic assumption of sense-impressions as the basis of experience (Hume 1793) is recorded in the frozen idiom of 'sense-making'. To 'make sense' is to develop a narrative based on sensory experience (current and previous), however abstracted and rarefied. Many authors have thought in terms of powers, including Whitehead (1932), Gattegno (1987) and Bruner (1996). Examples of

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natural powers for sense-making of particular value in mathematics include imagining and expressing, specialising and generalising, conjecturing and convincing, stressing and ignoring, and classifying and characterising (Mason et al. 2005). This chapter aims to develop the last of these pairs and to point out both positive and negative consequences of the use of this power.

Classifying and Characterising Manifested in Natural Language

If every sense-impression that struck us had to be dealt with *ab initio* with no reference to previous experience, then human beings would be very slow processors and rather weak thinkers. In fact, as Nørretranders (1998) shows, the quantity of sense-impressions impacting on our brains is so immense that it forces our somatic system to pre-process them before we are consciously aware. In a sense, this is what brains are designed to do. Thus even before we become aware of a sense-impression, considerable classifying and characterising has already taken place. The next layer of classifying takes place in language, which Maturana (1988, p. 47) defined as 'the consensual coordination of the consensual coordination of action'. People coordinate actions through interacting, and they use language (including words, gestures, posture, etc.) to coordinate the way they interact. Actions are initiated because of perceived similarity of current conditions to conditions associated with that action. Minsky (1975) went so far as to try to describe human functioning in terms of default values for frames which 'fire' when their input values are all instantiated. So situations are effectively classified by the way in which metonymic associations and metaphoric resonances activate habits.

Verbal language itself is unavoidably general and so based on classification: nouns such as cup, foot, triangle are difficult to instantiate without gestural pointing or lengthy verbal description ("the cup on the table here where I am standing…"). Their generality arises because they apply to a large class of objects. Similarly, verbs are general because they apply to a large range of actions. As Lakoff (1987) pointed out, drawing on the work of Rosch (1977), many classification systems depend not on equivalence, as in mathematics, but on having central or paradigmatic examples, and examples of varying degrees of peripheral-ness. Thus a log is not seen as a chair when inside a house, but may be seen as one at a campfire; a stool rarely comes to mind when the word 'furniture' is used.

In mathematics definitions may seem to make concepts precise, but their power arises from the generality afforded by language. In order to be useful a definition has to be sufficiently general to apply in several or many different situations and contexts. Axioms are derived through abstraction of properties by omitting contexts. Powerful concepts and axiom systems are those which admit multiple interpretations or instantiations of the abstract relationships expressed in the definition or in the axioms. Classifying all the objects which satisfy a theorem or which possess a property is second nature to mathematicians.

Notation, which is a form of language, similarly carries power within its ambiguity when it can be interpreted in many different ways. For example, the term *number* can be used to refer to whole numbers, integers, fractions, decimals and beyond;

multiplication can be used with each of these types of numbers as well as with other types of objects, because they all share certain core properties. The symbols 2/3 can be interpreted as a ratio, as a division, as the answer to a division, as a fraction, as the value of a fraction, and as an operator. Much of the power of mathematics comes from the curious relationship between precision, on the one hand, and ambiguity, on the other. Thus mathematical theorems are about relationships between properties: one property may be a consequence of another, they may be mutually equivalent, or those objects having both properties can be characterised in yet another way. Mathematical concepts are defined and characterised, often in several different ways. Yet it is the context-independence, the underlying generality, the ambiguity due to lack of specificity, which provides power through generalisation.

Classifying and Characterising in Mathematics

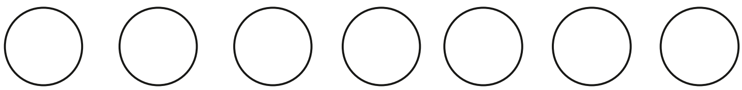
To discuss mathematics and mathematical pedagogy without actually engaging in mathematics is in my view a mistake. So much more can be learned from directly experiencing what is being talked about than can be garnered from mere descriptions. Consequently I begin this section with some mathematical tasks chosen to highlight mathematical aspects of classifying and characterising (Fig. 1).

For most people the issue immediately arises as to what constitutes ‘different’. Many people find that as they pursue the task, their sense of what is different actually changes, so that by the end they conclude that there is only one way to do it, because all available methods appear to be variations of starting at the centre, drawing a non-self-intersecting curve to the boundary, and then rotating that about the centre through 90, 180 and 270°. More difficult to articulate is the condition on the curve so that the rotated copies do not intersect each other.

This task captures the essence of classifying and characterising in mathematics. Different dissections are classified as being ‘essentially the same’ through different features being stressed while other features are ignored. This is the process of generalisation, as Gattegno (1987) pointed out: ignoring some aspects opens up the possibility of other aspects being shared with a wider class of objects, hence ‘clas-

Circular Division

Divide the first of the circles shown into four congruent pieces in the sense of a jigsaw puzzle.



Now do it again, differently, in another circle. And again, differently. Keep going, each time trying to find a different way to divide a circle into four congruent pieces.

Fig. 1 First task: circular division sification'. Language is the great classifier, for it is the use of one word in several situations or instances which prefigures the notion of the class of objects to which the word applies.

Mathematics typically goes beyond classification to characterisation. In mathematics it is desirable to prove that all and only the anticipated or imagined objects have the specified properties. The desire to articulate precisely a condition on a curve from the centre of the circle to the boundary so that when copies are rotated there will be no intersections, reflects the mathematician's desire to characterise the set of all such curves in some helpful way.

Classifying and Definitions

A definition identifies a class of objects that satisfy the definition. Mathematics abounds with theorems that provide alternative classifications by characterising objects satisfying a definition in terms of some other property (Fig. 2).

Which if any of the following statements are always true?

The diagonals of a parallelogram bisect each other

If the diagonals of a quadrilateral bisect each other, the quadrilateral is a parallelogram.

The diagonals of a square bisect each other and are perpendicular

If the diagonals of a quadrilateral bisect each other and are perpendicular, the quadrilateral is a square.

The diagonals of a kite are perpendicular, and one is bisected by the other

If the diagonals of a quadrilateral are perpendicular and one is bisected by the other, the quadrilateral is a kite.

The diagonals of a dart are perpendicular, and when extended so that they meet, one is bisected by the other

If the diagonals of a quadrilateral are perpendicular, and if when extended, one bisects the other, then the quadrilateral is a dart.

The diagonals of a rectangle bisect each other and are equal

If the diagonals of a quadrilateral bisect each other and are equal, then the quadrilateral is a rectangle.

Which of these classes contain quadrilaterals in which the diagonals do not intersect?

Which of these classes contain quadrilaterals which self-intersect?

Fig. 2 Classifying and characterising quadrilaterals by their diagonals (strongly guided version)

Use the five properties: diagonals perpendicular; diagonals intersect; one diagonal bisects the other; both diagonals bisect each other, and diagonals equal; to characterise types of quadrilaterals such as squares, rectangles, parallelograms, kites, darts, and rhombi. What about trapezia?

Fig. 3 Classifying and characterising quadrilaterals by their diagonals (partly guided variant)

Characterise named quadrilaterals solely in terms of properties of their diagonals.

Fig. 4 Classifying and characterising quadrilaterals by their diagonals (unguided variant)

An alternative presentation of the task would be more likely to stimulate exploration (Fig. 3).

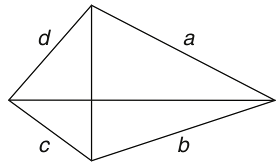
A completely open version might take the form (Fig. 4).

In the first version, each case describes a property of the named quadrilateral and invites proof that the property characterises it, because all quadrilaterals with that property belong to the class with that name and vice versa. Any version of the task could turn into a major project, or could be used simply to highlight the actions of classifying and characterising. Version B would be likely to lead to people asking for examples or direction; version A would be likely to lead to confusion about the meanings of the identified properties; the original version would be likely to lead to people reasoning about the stated cases but not proceeding further without prompts. No version is perfect; each situation suggests a tailor-made version as judged by the proposer.

Curiously we do not have names for the class of all quadrilaterals with equal diagonals nor for the class of all quadrilaterals with perpendicular diagonals. The usual reason for not naming a class is that there are no interesting theorems about all and only those members of the class, however the next task shows that the class of quadrilaterals with perpendicular diagonals does have another interesting property which enables a wider class of polygons to be characterised as well (Fig. 5).

This classifies all quadrilaterals having perpendicular diagonals as also having the alternating sum of squares property (that the alternating sum of squares of edge lengths in cyclic order is zero). Immediately the mathematical thinker asks whether there might be other quadrilaterals with the alternating sum of squares property, or

Show that a quadrilateral with perpendicular diagonals has the property that the alternating sum of the squares of the edges (proceeding around the quadrilateral) is zero.



$$a^2 - b^2 + c^2 - d^2 = 0$$

Fig. 5 Alternating square sums

does that property provide an equivalent characterisation of the property of perpendicular diagonals? The converse is in fact true: if a quadrilateral has the alternating sum of squares property, then its diagonals are perpendicular. Thus the two properties are equivalent, and each characterises the other. Alert readers will notice a parallel with the classification and characterisation of escribed quadrilaterals (those whose edges are all tangent to a single circle). Classification involves associating or relating apparently disparate objects and trying to articulate what properties they share.

However, the characterisation has payoff, for even though it is the case that to have a pair of diagonals perpendicular really requires two diagonals and so a quadrilateral, the property of having alternating sum of squares of edges equalling zero can be extended to any even-sided polygon. This raises the question of what implications there are for perpendicularity of diagonals, and it turns out that polygons with alternating cyclic sums of squared edge lengths being zero characterises polygons which can be dissected into quadrilaterals each with perpendicular diagonals.

With the circle dissections, there may be a lingering doubt that perhaps there is another way to dissect the circle, for example without having the centre of the circle on the boundary of all the pieces. It seems intuitively to be unlikely, but a convincing argument remains elusive. With 12 congruent pieces, there is a dissection in which only half of the pieces have the centre on their boundary, and it remains an open question whether there is any dissection of a circle into congruent pieces for which the centre is not on the boundary of any piece.

To classify mathematically is to isolate or stress a property and to consider the set of all objects which satisfy that property. To characterise is to establish through mathematical reasoning that some other property classifies exactly the same objects. For example, the notion of an *odd number* classifies certain numbers; the description of an odd number as one more than an even number provides a characterisation, as does, ‘leaves a remainder of 1 on dividing by 2’ or ‘does not end in an even digit’.

Lakatos (1976) exposed the role of characterisation in relation to definition in the context of Euler’s relationship between vertices, edges and faces of polyhedra: definitions are chosen so as to make proofs work, and proofs are often or mainly to characterise one property in terms of others. This is particularly true when a definition is given as a global property, but can be characterised in terms of a local property. Thus continuity of functions is at first a global feature of ‘having neither gaps nor inordinate wiggles’, but is also captured locally in terms of continuity at a point; quadrilaterals inscribed in a circle (a global property) are also characterised by having one (and hence both pairs) of opposite angles adding to half a revolution (a local property); quadrilaterals escribed about a circle (a global property) are also characterised by having the sums of their opposite sides being equal (a local property). Considerable mathematical power arises from having both global and local characterisations of properties. As another example, numbers can be classified as rational or irrational, so irrationality is global in the sense of the number not being rational; irrationals can be characterised by the fact that they have a decimal representation with no repeating tail, which is a property local to the decimal presentation. Not all characterisations have the local-global relationship: whole numbers which are one more than the product of four consecutive numbers can be characterised as the squares of numbers which are themselves one less than double a triangular number, and vice versa.

More elementary examples abound in the primary and secondary curriculum: squares can be characterised as rectangles with a pair of adjacent edges equal; rectangles can be characterised as quadrilaterals with two pairs of opposite sides equal and one right angle, or as quadrilaterals with three right angles; circles are the locus of points equidistant from a fixed point, or the locus of points subtending a fixed angle with respect to two fixed points (care is needed to define 'subtended' appropriately!). Note that the property—all chords through a fixed point have a fixed length—does not in fact characterise circles, as other figures have the same property. An obtuse angled triangle is characterised by having the square of one edge greater than the sum of the squares of the other two edges.

None of this is 'advanced' or beyond the reach of children. In order to read, children have to recognise letters despite variations in the way letters are written due to handwriting and fonts. This is a form of unarticulated intuitive classification and characterisation which has been carried out quite spontaneously. If learners of mathematics do not have their attention drawn to classifying and characterising in mathematical contexts then they are being severely short-changed and impoverished in their endeavours.

Methods as Classification of Tasks

The whole essence of a 'method' for solving a class of problems is to recognise when the method is suitable for resolving or contributing to the resolution of a problem. So as soon as a learner can solve a problem, they are ready to be asked what features other problems might have which would make the method suitable for them as well. The observation that giving a child a hammer converts everything into nails has a more positive manifestation in mathematical exercises. As soon as learners have an action which they can 'do', an action they can perform, a method or technique which they can carry out, they are ready to consider associated 'undoing' questions: what other problems of a similar type would have the same answer and what sorts of answers are possible when using the technique, as well as what features of problems make them amenable to the method. Prompting learners to explore classes of similar tasks and to work on 'undoing' problems is a good way, if not the only effective way, to exploit Vygotsky's *zone of proximal development* (van der Veer and Valsiner 1991, p. 334). Carrying out actions triggered by some outside agency ('acting in itself') is transformed into learners being able to initiate actions 'for themselves' when learners become aware of a wider class of tasks all susceptible to the same 'action'. Put another way, such tasks serve to prompt learners to educate their awareness, and their 'awareness of their awareness' (Gattegno 1987) thus contributing to their mathematical thinking.

A very simple version occurs when children are presented with tasks like $3+4=5+?$. Once they have found a way to resolve one such task, they can be asked how they can adapt their method to solve ones with different numbers. Dave Hewitt (Open University 1992) developed this into a lesson in which, starting with simple tasks such as “I’m thinking of a number, I add 2 and the answer is 7; what is my number?”, more and more complexity was added, such as “I’m thinking of a number, I add 2 then divide by 3 then subtract 4 then multiply by 5 and the answer is 13; what is my number?”. Once fluency developed, he focused attention on the relationship between the sequence of actions he described and the sequence of actions performed to reach a solution (doing and undoing). He then moved to a symbolic version in which the arithmetic operations involve numbers represented by letters of the Greek alphabet! The core of the lesson involved characterising the relationship between doing addition and undoing it using subtraction (and vice versa), and doing multiplication and undoing it by division (and vice versa).

A more sophisticated version of generalising a single task so as to become aware of a whole class of similar tasks, which also yields a rich exploration when reversed (an undoing task) is the following (Fig. 6).

It is not difficult to find an expression in the number of rows and the number of columns of the block, and to generalise this to take account of the number of perforations for the horizontal and vertical sides of a stamp as parameters. (Note that for 0 stamps, the usual formulae anticipate a corner perforation!) It is even possible to have the number of perforations in a corner as a third parameter. The act of generalising is a form of classifying a space of tasks each of which has a particular number of perforations horizontally, vertically and in the corners. These tasks can of course all be done in the ‘same way’. It is the ‘same way’ which is important, not the particularities of counting the perforations themselves that contributes to learners’ mathematical thinking. Indeed, the multiplicity of similar tasks suggests finding structural approaches rather than resorting to actual counting.

The calculations alluded to so far are all ‘doing’ calculations, however general. ‘Undoing’ questions take the form of characterising those numbers of perforations which can arise from a particular size of stamp (number of perforations on each side and in the corners of each stamp), and in how many ways. This can be done first for a specified size of stamp and its perforations, and then in general.

The same process applies to routine exercises. A learner, who has not considered and tried to articulate what it is about a problem that makes a particular technique likely to succeed, has not understood or appreciated the technique sufficiently to make use of it in the future. A learner, who has not considered what it is about an ob-

The picture shows a block of stamps with 2 rows and 3 columns, delineated by perforations. How many perforations will be used in a block of *r* rows and *c* columns of stamps?

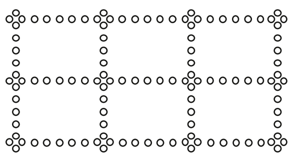


Fig. 6 Perforation count. (See Mason et al. 2005, p. 98)

ject which makes it an instance of a mathematical concept or definition, has not understood or appreciated the concept. An excellent way of promoting this type of classification is to get learners to construct their own examples, whether of problems or indeed of other mathematical objects as well (Watson and Mason 2005, 2006). Rather than having learners beat their way through a collection of prepared and pre-digested exercises, having them construct easy-hard-general problems promotes classification and characterisation, thereby preparing them more surely for tests and examinations. Similarly, tasks asking them to construct examples of objects with specified properties are powerful stimuli for making sense of concepts and properties.

Learners who develop an inter-connected and structured space of examples (and ways to construct examples) associated with each mathematical concept, heuristic, theme, and technique have a rich collection to access when they are trying to make sense of some situation which brings the concept, heuristic, theme or technique to mind (Watson and Mason 2005). Taking every opportunity, whether with routine exercises or with explorations and problems, to try to classify and characterise the associated class of objects stimulates the growth and interconnectedness of learners' accessible example spaces (Goldenberg and Mason 2008). Prompts which promote classification and characterisation include variants of:

- What similar problems give the same answer?
- What answers can be obtained from similar problems?
- What features can be changed in an example of a concept and still it is an example?
- What features can be changed in a task and still it makes use of the same techniques and concepts, the same mathematical themes and heuristics?

By asking themselves what aspects or features are permitted to change, learners explore the *dimensions of possible variation* within which objects remain examples, instances of a property, or which satisfy the conditions of a theorem. By asking themselves what the *range of permissible change* is in each aspect which can change, learners extend and enrich the class of examples to which they may have access in the future (Watson and Mason 2005).

By looking for similar or related tasks, including simplifications, extensions and variations to explore, teachers and learners enrich their sense of the utility (Ainley 1997; Ainley and Pratt 2002) of the techniques and concepts used in the task.

Exploiting Classifying and Characterising in the Classroom

As with any of the many natural powers that learners bring to class, it is useful to bring the power to classify and characterise to teachers' attention by first engaging them in a task for themselves, in which they are highly likely to experience classification and even characterisation, along the lines of tasks in the previous section. The same thing applies to teachers working with learners. The first 'circles' task

(Fig. 1) was of this type, and it has worked well in many situations with teachers. Having generated an experience which can be 'taken-as-shared', teachers and pre-service teachers can be asked to think back to other tasks they have worked on, either for themselves or with learners, which also have potential for highlighting classifying and characterising. The purpose of this reflection is to enrich their awareness of the potential and the ubiquity of classifying and characterising. Then they can be invited to engage in tasks which are more curriculum-based, with the suggestion that they look for opportunities to engage in classifying and characterising. The whole enterprise is, in this case, to sensitise them to possibilities of calling upon their learners' natural powers in the midst of some mathematical topic or other activity. In other words, the power to classify and characterise can be added to the awarenesses which inform planning, and which can then be called upon in the midst of mathematical work. It helps every so often to engage learners in tasks not directly on the curriculum but which challenge and extend the use of specific powers. Thus, tasks can be devised (such as Perforations) which call upon other powers in concert, so providing teachers and pre-service teachers with a rich web of interconnected experiences.

Learners can be engaged in tasks that ask them to classify and characterise mathematical objects (e.g. quadrilaterals, numbers leaving a given remainder on dividing by a given number, number sentences, algebraic expressions and equations), definitions in terms of local properties, unfamiliar properties of familiar objects (such as numbers one more than the product of four consecutive number mentioned earlier or triangles containing an obtuse angle in terms of squares of edge lengths), tasks amenable to a particular approach or method, and methods of approaching tasks. Discovering that there may be many different ways to organise, classify or characterise can only add to their flexibility and appreciation of interconnectedness of mathematics.

The role of the teacher educator working with teachers and pre-service teachers parallels the role of the teacher working with learners. The aim is to initiate tasks which will generate activity in the form of people making use of familiar actions, modified so as to meet fresh challenges. The purpose of the activity is to gain experience; but to learn from experience requires more than the experience itself. 'One thing we do not seem to learn from experience, is that we do not often learn from experience alone' (Mason 1992).

Something more is required. Having engaged people in activity, it is then necessary to draw them out of their immersion so that they can become aware of actions they were using, so as to better able to initiate those actions for themselves. One of my principles (Mason 2002) is that in order to sensitise myself to notice something in others, I need to become aware of and sensitised to my own experience. A very helpful device is to find some resonant label for situations which typically have potential for exploiting classifying and characterising. Examples include 'local-global', 'equivalent', 'sorting' (see below), 'The effect of a label' is to classify appropriate situations, so that recent experience informs future practice. The single most effective aspect of reflection is to imagine yourself as fully and as specifically as possible in a teaching situation making use of some strategy or tactic in a typical situation while having the chosen label in mind. When something in a situation resonates with the label, you may then find the potential action coming to mind.

Sorting and Matching Tasks

To classify an object is to identify properties it satisfies, which is a form of sorting, on the way to characterising. Sorting tasks provide an excellent stimulus for classification. Inviting learners to take a collection of objects and to sort them, implicitly calls upon discerning the way other people have sorted and to try to articulate the basis of their criteria awakens the possibility of different sorting criteria, different distinctions and hence different classifications. Trying to find a succinct but descriptive name for the different classes in a classification is also informative, and can help inform choices in the future. Having some extra objects to insert into the sorting scheme to see if the original sorter agrees is a good way of testing criteria. If the sorting is done by groups of two or more learners, then one person can stay behind and check the conjectures of other visitors to the classification when they try to insert further objects.

As an introduction to sorting or if you have learners from very varied backgrounds, you can use locally available objects such as leaves, pebbles, shells, seeds, etc. Sorting ordinary objects provides direct experience of how human beings naturally classify, and it is salutary to discover that different people see things very differently, and so classify differently.

Where the leader has a preferred sorting or is intending people to encounter a particular way to sort, it is necessary to provide a reason for the sorting which can act as criteria. Sometimes it is worthwhile adding the desired sort to the collection produced by the learners. Thus teachers can be invited to sort a collection of cards with fraction calculations of the form

$$\frac{1}{5}+\frac{3}{5} \qquad \frac{1}{5}+\frac{1}{7} \qquad \frac{2}{5}+\frac{3}{7} \qquad \frac{3}{7}-\frac{2}{5} \qquad \frac{8}{15}+\frac{7}{10}$$

where there are additions and subtractions with same and different denominators, according to different criteria, such as method used, ease of calculation, or likely learner errors. This then provides the basis for discussion which is likely to reveal different pedagogic assumptions and approaches. It is unwise to expect that teachers will suddenly change their approaches or their thinking, but encountering other perspectives opens up the possibility of choosing to think or sort differently next time. Learners could be invited to sort according to similar criteria, or according to whether the calculations give the same answer (calculations need to be carefully constructed), or, accompanied by graphical displays of partly shaded rectangles, according to whether they represent the same calculation. Discussion can then be focused on efficiency of calculation, or on the relationship between the ‘easier’ ones, and the method used for the ‘hardest’ ones, or on complexity of the diagrams. Discussing what it is that makes a question ‘hard’ or ‘easy’ can also be very fruitful.

Swan (2006, 2007) has devised a variety of clever sorting and matching tasks which invite learners to associate expressions or other mathematical objects, so that in the process of sorting, they make connections between previously disparate (re) presentations. For example, cards with graphs, with equations and with tables of values can be sorted for equivalence, as can packs of cards with decimals, fractions and positions on a number line, or cards with rectangular area diagrams, algebraic expressions and instantiations of those expressions.

You can also use the set of exercises at the end of a chapter or of several chapters, or other mathematical objects. Here what is revealed by sorting is what learners stress and consequently may overlook when they tackle an examination question. By exposing them to alternative ways of sorting (the teacher can include their preferred sorting as well) learners can sensitise themselves to the classes of problems they are likely to meet, rather than simply rehearsing particular ones over and over.

If you invite sorting without specifying an aim, then you get insight into the criteria which dominate different teachers' attention; if you invite sorting with a specific aim such as method or assumed difficulty, then you get insight into peoples' pedagogical assumptions and dominant awarenesses. If there are criteria, participants can discuss which sorting is most effective. Sorting and matching tasks can also be used as research probes to reveal the richness of learners' interconnections (Collis 1971; Silver 1979; Zaslavsky and Leiken 2004).

Ordering Tasks

Putting objects in order according to some criteria is often a useful way to provoke learners into finding a simple way to do comparisons. For example, the fraction cards mentioned above could be ordered according to the magnitude of the answer or according to perceived difficulty. A simpler set of fraction cards with single fractions, including ones with common factors between numerator and denominator can be used to direct learner attention to the structure and meaning of the denominator and the numerator by being put in order. Ordering a mixed set of fractions and decimals can be similarly instructive. A set of linear expressions in one variable can be ordered according to the value achieved for a particular value of the variable, and then participants can search for values of the variable that produce each of the possible orders.

Prestage and Perks (2001) provide a variety of strategies for adapting and modifying tasks to make them more focussed and more effective, including ways of working with ordering tasks such as those suggested. Teachers and pre-service teachers can be asked to order tasks from a textbook or work-card according to presumed difficulty (as evidenced by learner scores), or according to the order in which learners would be exposed to questions of that type. Both of these can lead to lively discussions which bring pedagogic assumptions and theories to the surface. Teachers can also be asked to predict how learners (or pre-service teachers) will order a set of tasks, and compare these with the actual scores when they try to do them.

Classifying and Characterising Pedagogically

Some of the power of mathematical characterisation and classification carries over into pedagogical and didactic settings, but not all. For example, analysing research data consists of a mixture of applying a previously determined 'framework' of distinctions while at the same time allowing distinctions to emerge from close reading of the data, as in *grounded theory* (Strauss and Corbin 1990). In other words, sorting is carried out by becoming aware of possible distinctions, and using those distinctions to classify data elements, whether they be transcripted discourse, observed behaviour or responses to probes. Unfortunately, it is often very difficult to be precise about criteria for discerning detail in the way that it is possible within mathematics. Instead of trying to tie down distinctions exactly, researchers resort to triangulation (getting at least two different perspectives on the same situation) or seeking a measure of agreement between different classifiers.

A considerable amount of research in mathematics education has focused on classifying and characterising different forms of learner and teacher responses to probes administered by interview, observation of spontaneous or prompted behaviour, and questionnaire. These probes have variously emphasised cognitive, affective and enactive features. It is assumed that where distinctions can be made which correlate with some observation or desired outcome, those distinctions can be used to inform future practice. For example, tasks can be classified according to the degree of engagement displayed by learners; by the possibilities they are perceived to afford for access to mathematical concepts, themes, heuristics; by the potential use of their powers by learners; by the potential for development of facility and fluency in use of techniques and concepts; by how they promote participation, engagement, discussion and collaboration; and so on. It remains unclear whether having made distinctions, these are easily communicated to and used by teachers to alter their practice, or whether indeed the mere actualising of actions associated with the distinctions is expected to make significant difference to learning.

In searching for ways to improve learner learning, especially in schools, researchers, teachers and policy makers have explored the classroom ethos (socio-mathemtical norms of Yackel and Cobb 1996), classroom rubric (Floyd et al. 1981); teachers' beliefs (Thompson 1984, 1992; Leder et al. 2002; Forgasz and Leder 2009); teachers' mathematical background; textbooks; pedagogical practices (Keller 1968; Hake 2007; and so on); tasks; obstacles, errors and misconceptions; questions and prompts and other interactions between teacher and learners; mathematical sophistication and challenge; and so on. They have made use of more or less elaborate theories of how learning and development take place, and they have imposed a variety of practices on teachers through national curricula and national campaigns for reform.

In each case, it seems as though there is a search for some magic potion which, if instituted, would transform the learning of mathematics for everyone. Tasks, questions, social interactions, texts, mathematical techniques, and obstacles encountered by learners have all been and continue to be classified through increasingly complex taxonomies. The result is a recurrent cycle of amplification of distinctions as items for practitioners to 'tick off', and for inspectors to confirm as having been 'ticked off'. Reliance on cause-and-effect, a mechanism derived from manipulating the material world through the use of machines and derived from Descartes' legacy of fascination with the cuckoo clock as a metaphor for the material world, completely misses the essence of human beings as complex, will-full, intentional, and sometimes aware, organisms. It is true that discoveries arising from behaviourism about stimulus-response are certainly pertinent and useful, but they apply only when the individual or the group is acting mechanically, automatically, and out of habit, rather than responding freshly and mindfully to the situation.

If real progress is to be made against the tide of increasingly fine distinctions redolent of the medieval tendency to dissect and further dissect in lists, then it is necessary to maintain complexity. Proponents of 'complexity theory' (see for example Davis et al. 2006) and of approaches influenced by eastern thought (see for example Brookes 1966; Varela et al. 1991) prefer to work with the mathematical, pedagogic and didactic sensitivities and awarenesses of the teacher (and hence of the learner) rather than on observable and repeatable behaviour *per se*. If teachers are deeply and resonantly aware of pervasive mathematical themes, mathematical powers and heuristics, and especially of their own awarenesses, then they are in the best position to have suitable actions come to mind when they are teaching.

Focusing on the domain where mathematics, pedagogy and didactics intersect, Wheeler et al. (1984) asked for the fundamental awarenesses (the basis for action) which underpin the school curriculum. The idea is that if learners are exposed to these awarenesses, then much of the curriculum becomes instantiation in specific contexts. Gattegno (1987) went so far as to say that the fundamental problem in mathematics education is what to do with learners once you have taught them the whole of the school mathematics curriculum by the age of about 12, since he was convinced that by working with and on awareness you could short-cut the obstacles thrown up by repeatedly but inefficiently and ineffectually teaching concepts that learners have already been taught previously but have not internalised. Simon and Tzur (2004) are similarly interested in the basic awareness, the fundamental shifts needed in order to re-construct for oneself the fundamental ideas of mathematics. Ma (1999) claimed that Chinese primary teachers differ from American teachers in having a *profound understanding of fundamental mathematics*, although other studies (Li et al. 2008) suggest that this profound understanding may not be universal, and may not always penetrate to the fundamental awarenesses, being content to remain at the level of efficient transmission of techniques and procedures.

It seems that although classification can be powerful in mathematics, if over done in domains such as mathematics education, it may become counter productive.

Dangers of Classifying and Characterising

Once classification takes place, a distinction comes into existence, and there is often a considerable degree of ontological commitment to its maintenance. Once a class or property is defined it can be difficult to appreciate overlaps or alternative perspectives which cut across the distinction or even which intentionally blur it. For example, someone committed to class or gender issues tends to 'see' these, tends to use them as the basis for distinctions they make. Their value system is integrated into their distinctions. It can be quite difficult to conduct a conversation much less to reach an agreed analysis where value systems and hence distinctions do not coincide. Someone committed to the importance and centrality of socio-cultural interaction, or to a particular theoretical frame may find it impossible to communicate with someone with a different frame or with a commitment to the idiosyncrasy of individual psychology.

In mathematics, the same object can quite happily be seen as exemplifying quite different constructs as illustrated by the symbols 2/3 mentioned earlier. Flexibility to move between interpretations is vital; being confined to only one or a few can severely limit effective use of the construct. When learners intuitively classify fraction notation as having a specific meaning, they make it difficult for themselves to extend that meaning to encompass other interpretations. The result is that they miss out on the mathematical power of such notation. In mathematics education, although an interaction between a teacher and a learner can similarly be interpreted in many ways, there is a tendency to settle on a single interpretation rather than on an amalgam. The single interpretation promotes reductive simplification; multiple interpretation celebrates and exploits complexity.

As soon as labels appear, ontological commitment sets in, and with it, metonymic association. When someone makes a good conjecture or is slow in picking up an idea, it is tempting to classify their behaviour ('good thinking'; 'slow thinking'; 'low attainment') which is then all too easily transferred to the person ("she is a good thinker"; "he is a slow thinker"; "she is a low attainer"). Labelling negative behaviour can induce adolescents to take on those attributes, so that the label reinforces and amplifies the undesirable behaviour. Once the description moves to the person, it is very difficult to overcome. Indeed is well known that there is often either a self-fulfilling prophecy in that individuals adjust to meet the labels that they are given, or there is a reaction which carries over into other situations and leads to disruption and further labelling. Used with positive behaviour (such as aspects of using their powers to think mathematically) learners may begin to adopt the behaviours of mathematical thinkers.

Whenever teachers make assumptions about what their learners 'can' and 'cannot', or 'will not be able to' do, they limit the opportunity for learners to reveal as yet undeveloped and undisclosed powers. For example, it is well known that a visitor to a class can often spark individuals into uncharacteristic behaviour, suggesting that the 'ability' has been present all along, but the conditions have not been appropriate. Dweck (2000) reports a career-long study into ways to promote learners to switch from a language of "can't" to "didn't but could try harder or differently" in an effort to combat the effect of inappropriate labels being adopted as accurate by individuals.

Conclusion

Classifying and characterising are important natural powers displayed by all human beings through their participation in social interaction and their use of language. Where teachers are able to provoke learners into using those powers, and so into developing and refining their use in making mathematical sense and in making sense of mathematics, learners are likely to appreciate mathematics as a mode of enquiry, as well as succeed in their studies. They will learn how to learn mathematics. Not all classification is helpful, however. Overly refined classification may not be helpful in informing teaching, and where learners are labelled by behaviours they sometimes exhibit, possibilities for learning are likely to be limited if not truncated.

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Designing Tasks that Challenge Values, Beliefs and Practices: A Model for the Professional Development of Practicing Teachers

Malcolm Swan

Introduction

Over the years, we have come to realise that teacher development does not come about through repeated attempts to persuade but through opportunities for individual teachers 'to doubt, reflect and reconstruct' in unhurried, 'safe' environments (Wilson and Cooney 2002, p. 132). We do not seek to change teachers' beliefs so that they behave differently, but rather offer opportunities to behave differently so that their experiences may give them cause to reflect on and modify their beliefs (Fullan 1991, p. 91). In our work with teachers, this experiential approach is conducted in four broad stages:

1. **Recognise existing values, beliefs and practices.** We invite teachers to describe the situations in which they work, and elicit their existing values and beliefs about mathematics, teaching and learning and their classroom practices. We articulate and clarify classroom dilemmas and their underlying causes. During this process, a shared experience begins to emerge.
2. **Analyse discussion-based practices.** Through working on classroom tasks, then watching their use on video, teachers are confronted with practices that contrast with their own. They discuss the research-based principles that underpin these. These provide 'challenge' or 'conflict'. We articulate and address some common objections to these ways of working.
3. **Suspend disbelief and adopt new practices.** Teachers are encouraged to try out the new classroom activities using prepared classroom resources. They are offered a mentor and a network of support as they do this.
4. **Reflect on the experience.** After trying out the activities, teachers are invited to meet together to share their classroom experiences and discuss the pedagogical implications. They are explicitly encouraged to reflect on the growth of new beliefs. Further challenges are provided.

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We have conducted several professional development courses using this model (DfES 2005; NRDC 2006; Swain and Swan 2007; Swan 2005, 2007; Swan and Green 2002). While these courses used a similar structure, their context and content varied considerably. Some courses were aimed at adult numeracy teachers, while others were focused on teachers of 16–19 year-old students of varying levels of attainment. Each course began with a two-day residential workshop in which the first two stages were explored, and were followed up with a series of one-day workshops in which teachers reported back and reflected on their experiences. The total commitment for teachers was from 4 to 6 full days face-to-face contact time, spread over one year. Each time we have done this, teachers, students and independent classroom observers have reported to us that they noticed substantial changes to classroom practices, and to students' attitudes and attainments (Swan 2006a, b). The discursive classroom approaches we advocate are much more complex and challenging for teachers and students than the "explanation, example, exercise" methods used in the past. They also conflict with the prevailing cultures within the institutions within which the teachers work. Teachers have, however, found that they can use many of the tasks with other teachers within their own institutions and in this way, the professional development has spread and continued long after the initial input.

This chapter follows the four-part structure of the above model, and displays some of the tasks we use with teachers. We also report on obstacles and difficulties encountered and on the transformative effects that this process has had.

Four Stages in Professional Development

Recognising Existing Values, Beliefs and Practices

Teachers usually attend professional development sessions in order to gain information, ideas and materials. They expect that an 'expert' will try to persuade them that a new method of teaching is better than their current practice and are surprised when we begin, not by informing, but by asking them to reflect on their existing values, beliefs and practices. The purpose of this is to make teachers more aware of the reasons underlying their classroom actions and to develop a language for values, beliefs and practices. This, we hope, will help them to more able to consciously monitor and control their own behaviours.

We attempt to do this in a non-judgmental atmosphere in a way that will encourage sharing. To begin with, we ask them to work together in pairs or groups of three to discuss a number of beliefs about mathematics, teaching and learning, such as those shown in Table 1.

We ask teachers to sort statements into three piles according to whether they broadly agree with the statements, disagree with them, or whether they cannot decide. They are also encouraged to modify statements, adding amplifications, conditions and caveats. This task usually occupies about 30 minutes.

Table 1 A set of cards to stimulate a discussion of beliefs and practices

Mathematics is a network of ideas. You follow up connections as they arise so lessons are always unpredictable	Mathematics is a hierarchical subject. You need to plan a logical sequence of activities and stick to it
Mathematics is best learned when individuals practice on their own	Mathematics is best learned through discussion in pairs or small groups
It is important to complete the whole syllabus, even if students do not understand it all	It is important that students understand all that they do, even if this means we cannot cover the syllabus
It is best to begin teaching mathematics with easy problems, working gradually up to harder ones; otherwise students make mistakes and lose confidence	It is best to begin teaching mathematics with complex problems, or students won't appreciate the need for it
Mathematics is a creative subject. Students learn best by creating their own questions and methods	Students learn mathematics best by working through carefully constructed exercises. They cannot create these for themselves
It is best to spend time on few questions and solve them in more than one way, even if this slows the lesson down	It is best to cover a wide range of questions, so that students are able to practice the methods intensively
Students are at such different levels of competence that I have to allow them to work at their own pace	I try to teach the whole class at once and keep them at the same pace
I find out which parts of mathematics students already understand and don't teach those parts	I start teaching mathematics from the beginning, assuming they know nothing
I try to avoid students making mistakes when learning mathematics	I encourage my students to make and discuss mistakes when learning mathematics
I prefer to share my objectives the beginning of the lesson so that the class know what it is all about	I prefer to keep quiet about my lesson objectives so that the lesson retains some elements of surprise

Although these statements clearly present false dichotomies (for example, one may argue that Mathematics is both a network and at least *partially* hierarchical) the activity raises a number of common sources of tension in mathematics teaching. For example:

"If I allow time for discussion, how will we cover all the content?"
"If I allow the freedom to explore and work at their own pace, how can I be sure that they will discover anything of significance?"
"If I encourage students to make and discuss mistakes, then how can I be sure that they will not simply become confused?"
"If I tell students how to tackle a problem, they are likely to follow my instructions in a mechanical way, without understanding. If I don't tell them how to tackle a problem, how are they going to make any progress?"

Generally speaking, we have found that teachers' belief orientations may be classified as predominantly *transmission, discovery* or *connectionist* (Askew et al. 1997; Swain and Swan 2007; Swan 2007). Briefly, a *transmission* orientation views mathematics as a series of 'rules and truths' that must be conveyed to students and teaching as explanation, example, exercise until fluency is attained. The *discovery* orientation views mathematics as a human creation and encourages students to learn through individual exploration and reflection, while the teacher adopts a rather passive, reactive, facilitating role. The *connectionist* orientation views mathematics as a network of ideas that the teacher and student must construct together through collaborative discussion. Here, the teacher has a proactive role in challenging students. These are not, of course, exclusive categories but they do offer a useful framework for discussion with teachers (and for the analysis of the effects of the professional development on teachers' beliefs)[1].

As teachers discuss these ideas, they usually realise that different belief orientations are called into play, according to the values and purposes that apply in a particular lesson. A lesson designed to encourage *fluency* in a technique, they argue, will tend to involve students in individual practice, while a lesson intended to develop *interpretations* and *meanings* will tend to involve discussion and debate. A lesson designed to develop *problem solving strategies* may start by offering a non-routine problem and comparing alternative solutions, while a lesson designed to foster an *appreciation of the cultural roots of mathematics* may be introduced with an expository film or a story, followed by a discussion. The issue thus becomes one of *values:* What relative emphasis should/do we place on each of these priorities?

In order to clarify such distinctions, we offer teachers a list of five purposes and invite them to indicate the emphasis that they would *ideally* like to give to each purpose (Table 2), and then the proportion of their time that they *actually* give to each purpose in their daily practice. The difference offers them an indication of the discrepancy between their values and their practices. Examples of classroom activities fulfilling each purpose are given, so that teachers understand the implications of implementing these values.

So far, we have repeated this task with several hundred teachers and the most frequent outcome is that they perceive their current practices to be predominantly concerned with developing fluency through practice, while they would wish to spend much more time working towards other goals, particularly those concerned with interpretations and strategies. They recognise that conflicting goals (even within a single lesson) are at the root of many pedagogical difficulties they face. When they try, for example, to teach Pythagoras' theorem through an open-ended investigation their convergent purpose conflicts with the divergent nature of the task. This creates the dilemma: "Should I let them follow their own line of enquiry, or should I direct them?" It is thus made clear to teachers that the aim of the professional development is to equip them to develop a better match between their values and practices. We emphasise that we are not condemning existing teaching methods, but are rather offering to increase teachers' repertoire of teaching strategies to encompass a wider range of purposes.

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[1] In passing, we note that research evidence suggests that the connectionist orientation is the most effective for conceptual learning, while the discovery orientation is the least effective (Askew et al., 1997)

Table 2 Purposes and related classroom activities. Indicate the approximate percentage of classroom time you would *ideally* give to each purpose. Now repeat this showing the time you *actually do* give

Purpose is to develop	If this purpose is valued, then students will be engaged in ….	% Ideal	% Actual
Fluency in recalling facts and performing skills	• Learning and memorising names and notations • Practicing algorithms and procedures for fluency and 'mastery'		
Interpretations for concepts and representations	• Discriminating between examples and non-examples of concepts • Generating and interpreting representations of concepts • Constructing relationships between concepts • Translating between representations of concepts		
Strategies for investigation and problem solving	• Formulating situations and problems for investigation • Constructing, refining, comparing strategies and solutions • Monitoring their own progress during problem solving and investigation • Interpreting, evaluating solutions and communicating results		
Awareness of maths, learning maths and the values of the educational system	• Learning how maths 'fits together' • Recognising different purposes of learning mathematics • Developing appropriate strategies for learning/ reviewing mathematics • Appreciating aspects of performance valued by the examination system		
Appreciation of the power of mathematics in society	• Appreciating mathematics as human creativity of historical/cultural value • Creating and critiquing 'mathematical models' of situations • Appreciating uses/abuses of mathematics in social contexts • Using mathematics to gain power over problems in one's own life		

Analyse Discussion-Based Practices

In the first stage, as noted above, most teachers express a desire to develop teaching strategies that will foster a greater emphasis on *interpretations for concepts and representations* and improved *strategies for investigation and problem solving* in their students. These therefore form foci for the second stage of the professional development. Now, we attempt to directly challenge the beliefs and practices elicited in the first stage with specific examples of contrasting beliefs and practices. Evidence suggests that as teachers 'suspend disbelief', take risks, implement novel

Table 3 Research-based principles for teaching concepts and strategies. (For research that support these principles see, for example: Askew et al. (1997), Askew and Wiliam (1995), Black and Wiliam (1998), Mercer (2000))

Teaching is more effective when it …	
• Builds on the knowledge students already have	This means developing formative assessment techniques and adapting our teaching to accommodate individual learning needs
• Exposes and discusses common misconceptions	Learning activities should exposing current thinking, create 'tensions' by confronting students with inconsistencies, and allow opportunities for resolution through discussion
• Uses higher-order questions	Questioning is more effective when it promotes explanation, application and synthesis rather than mere recall
• Uses cooperative small group work	Activities are more effective when they encourage critical, constructive discussion, rather than argumentation or uncritical acceptance. Shared goals and group accountability are important
• Encourages reasoning rather than 'answer getting'	Often, students are more concerned with what they have 'done' than with what they have learned. It is better to aim for depth than for superficial 'coverage'
• Uses rich, collaborative tasks	The tasks we use should be accessible, extendable, encourage decision-making, promote discussion, encourage creativity, encourage 'what if' and 'what if not?' questions
• Creates connections between topics	Students often find it difficult to generalise and transfer their learning to other topics and contexts. Related concepts (such as division, fraction and ratio) remain unconnected. Effective teachers build bridges between ideas

classroom practices and reflect on the outcomes, they grow professionally. We begin by addressing two questions: "What general research-based principles should underpin the teaching and learning of mathematical concepts and strategies?" and "What specific types of classroom task are most appropriate and how should these be used?"

Table 3 displays the research-based principles that we introduce to teachers. They are much easer to state than to implement[2]! We therefore illustrate each principle by modelling it in action while teachers engage in a series of practical tasks; we play the role of teacher while they are the students. Typically, we select from the following five generic task-types, spending about one hour on each: *Classifying mathematical objects; Evaluating mathematical statements; Interpreting multiple*

[2] After one extensive professional development program, observations of teachers showed that many succeeded in effectively using 'higher order questions' and 'cooperative small group work' but still had great difficulty in 'building on what students already know' and in 'exposing and discussing common misconceptions' (Swain and Swan 2007).

representations; Creating problems for others to solve; Comparing solution strategies on unstructured problems. These are chosen and adapted to closely fit the curricula needs of the teachers.

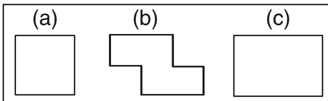
We work with teachers on each task-type using the following four-step procedure[3]. First, teachers work collaboratively on an exemplar task at an appropriately challenging level. As they do this, we model introductions, interventions and discussions that promote learning. Teachers then watch a video of an experienced teacher working with students on a task of the same type (though sometimes with a simpler example). They often express surprise at the high quality of student-student discussions and level of engagement shown on these videos. Thirdly, we discuss how the tasks were constructed to expose existing knowledge and misconceptions and, fourthly, we offer teachers the opportunity to generalise by modifying and constructing new tasks of the same type for themselves and also by considering alternative examples that others have constructed. Finally we challenge teachers to use these tasks in their own classrooms. Through this sequence of experiences, teachers feel what it is like to be a student challenged to think, reason and explain mathematically, appreciate their own role in facilitating this and also begin to understand the importance of careful task design.

Below, we describe the five task-types, offering an example of how each type may be presented first to teachers and then to students. We should point out, however that teachers are also offered detailed sample lesson plans[4].

Classifying Mathematical Objects

Purpose: For teachers to recognise that concepts develop as students discriminate between and recognise properties of mathematical objects.

Sample task for professional development: In each of the triplets below, how can you justify each of (a), (b), (c) as the odd one out? What properties of the objects does this reveal? Create triplets of your own that you could use to stimulate discussion among your students.



(a) $y = x^2 - 6x + 8$
(b) $y = x^2 - 6x + 9$
(c) $y = x^2 - 6x + 10$

[3] It may be noted that there are close similarities between the local four-step procedure being adopted for each task-type and the global four-stage structure outlined for the series of professional development workshops outlined in the introduction. Both are seeking to generate surprise and 'conflict' by confronting current practices and expectations with novel practices and research evidence.

[4] Further examples, including videos and lesson plans may be found in DfES (2005) (see: http://www.nationalstemcentre.org.uk/elibrary/collection/282/improving-learning-in-mathematics) and in Swan (2006a).

Now try this activity with students. Also, ask students to devise their own classifications for mathematical objects (e.g. shapes, formulae). For example, ask students to place cards showing such objects into two-way attribute grids and devise their own alternatives. As they do this, encourage them to articulate meanings of words and develop definitions.

	Large area	Small area
Large perimeter		
Small perimeter		

	Line symmetry	No line symmetry
Rotational symmetry		
No rotational symmetry		

Evaluating Mathematical Statements

Purpose: For teachers to recognise the importance of confronting and discussing common misconceptions with students.

Sample task for professional development: Classify each statement below as always, sometimes or never true. If you think it is always or never true, then try to explain how you can be sure. If you think it is sometimes true, then try to define exactly when it is true and when it is not. Write statements based on common misconceptions that your own students could discuss.

Numbers with more digits are greater in value	When you multiply 12 by a number, the answer is greater than 12
When you cut a piece off a shape, you reduce its area and perimeter	Max gets a 15% pay rise and Jane gets a 10% pay rise So Max gets the greater pay rise

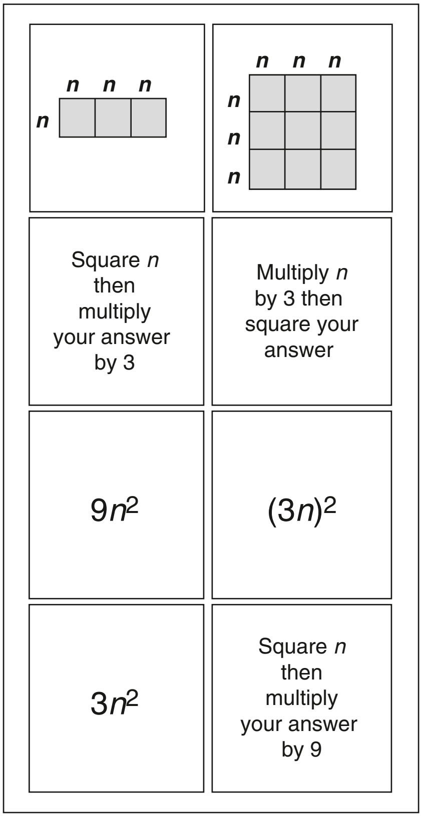
Now try this activity with students. Ask students to decide on the validity of mathematical statements that incorporate misconceptions that they may have. Encourage students to defend their reasoning by devising examples and counterexamples.

Interpreting Multiple Representations

Purpose: For teachers to appreciate the importance of developing mental images for concepts by exploring alternative representations and the multiple connections between them.

Sample task for professional development: Match the cards together if they have equivalent meaning. Now add cards to this set that will force students to distinguish between representations that are often confused (such as $(3n)^2$ and $3n^2$ in the example below). Create a different set of cards that will encourage students to interpret other representations in mathematics. These may include: Words, algebraic symbols, pictures, graphs, tables, geometric shapes….

Now try using your own card sets with students.



Creating Problems for Others to Solve

Purpose: For teachers to appreciate the value of: students seeing mathematical problems as instances of more general structures; offering opportunities for students to creative and communicate mathematically.

Sample task for professional development: Take a typical problem from a textbook or examination paper. Modify the problem to make a new version according to a given constraint (e.g., make it more realistic, change the numbers in the problem and/or the questions asked about the context). Try to make your problem interesting, difficult and solvable. Answer your own question and then give it to someone else to solve. Reflect on the different mathematics the creator and solver employed.

Now try this activity with students. Ask students to create problems for other students to solve. When the 'solver' becomes stuck, ask the problem 'creators' to take on the role of teacher and explainer. For example, one student may create an equation (by starting with the 'answer' $x = 7$, and then building it up step-by-step to give 2(*x* + 10)/3 = 12). Their partner must now tries to solve it by working out the steps involved and 'undoing' each one.

Comparing Solution Strategies on Unstructured Problems

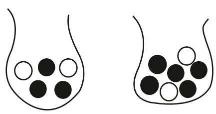
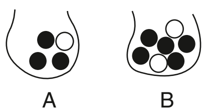
Purpose: For teachers to recognise the importance of removing step-by-step scaffolding in order that students will develop greater autonomy when problem solving.

Professional development task: Take a typical 'closed' textbook task. List all the decisions that are made for students. (For example, tasks that start 'copy and complete the following table' prevent students from being able to choose how they will organise data for themselves). Rewrite the task in a more open, ustructured form so that some of these decisions are handed back to students.

Now try the unstructured task with students. Encourage students to put together their own chains of reasoning. Invite students to share and discuss the different approaches that are taken. Supply further, complete (imperfect) solutions for students to discuss and improve.

Listening to Students

Classrooms are busy places and we find that many teachers spend little time listening probing the understanding of individual students. One activity that we use with teachers to encourage listening and questioning is a role play in which one teacher takes the role of a 'student' with a particular misconception, while the other, the 'teacher', tries to identify it (Swan and Crust 1992). This activity is set up by giving one participant a 'teacher' card describing a general topic area, while the other is given a 'student' card describing a specific misconception within that topic. The 'teacher' must then ask the 'student' a series of questions designed to uncover the problem while the 'student', without articulating the misconception explicitly, answers the questions consistently in the manner suggested by the misconception.

Teacher's card	***Student's card***
Your student answers questions like this sometimes correctly, sometimes incorrectly: "I take a ball out at random from each bag. Which bag gives me the best chance of drawing out a black ball?"	You consistently compare differences rather than proportions when deciding which event is most likely. So bag B would give the better chance of choosing a black ball because there are 3 more blacks than whites in B; while there are 2 more blacks than whites in A.
	

A New Classroom Culture

As we illustrate such tasks in use, with the help of video, teachers gradually realise that we are presenting a different paradigm of teaching to that found in most classrooms. Students are seen articulating misconceptions, discussing these in collaborative groups and teachers are seen intervening in non-directive ways that stimulate further discussion and debate. This presents a challenge to teachers that is welcomed by some and seen as a threatening by others. Objections are often felt, if not always articulated. We therefore provide a task that attempts to expose such objections for scrutiny by offering teachers a series of genuine quotes and asking them to work together to write a reasoned response to each objection:

Control:
" What will other teachers think of the noise?"
" How can I possibly monitor what is going on?"

Views of students:
" My students cannot discuss."
" My students are too afraid of being seen to be wrong."

Views of mathematics:
" In mathematics, answers are either right or wrong – there is nothing to discuss."
"If they understand it there is nothing to discuss. If they don't, they are in no position to discuss anything."

Views of learning:
" Mathematics is a subject where you listen and practise."
" Mathematics is a private activity."

We then continue by showing teachers how they might recognise the qualities of classroom talk that are conducive to learning using, for example, the constructs of *dialogic* and *exploratory* talk (Alexander 2006; Mercer 1995, 2000), and then share strategies on how these forms of talk may be encouraged in mathematics classrooms. Included in this is a discussion of the ways in which 'ground rules' may be established with students and the teacher's role during small group and whole class discussion. We introduce the two tasks as follows:

- Most students (and adults!) do not discuss in helpful ways most of the time. Some are reluctant to talk at all, while others just take over and dominate. Students may therefore need to be *taught* how to discuss. Prepare your own list of 'ground rules' that, in appropriate language, give guidance to pupils on how to talk together profitably. How could you introduce such rules to pupils? How could they be involved in drawing up such a list?
- While pupils are discussing, teachers often find it difficult to define their role. The character and content of pupil–pupil can change dramatically when the teacher listens in! How do you decide when to intervene? What are helpful and unhelpful things to say or do? Describe your own role while pupils talk together.

After each discussion, we ask teachers to compare their suggestions with those offered in Table 4, below.

Table 4 Ground rules for students and the role of the teacher *during* a discussion

Ground rules for students	The role of the teacher
• Give everyone a chance to speak • Listen without interrupting • Check that everyone else listens • Try to understand what is said • Build on what others have said • Challenge what is said • Demand good explanations • Treat opinions with respect • Share responsibility • Try to reach agreement	• Make the purpose of the discussion clear • Keep reinforcing the 'ground rules' • Listen before intervening • Join in, don't interrupt • Don't judge or praise—this discourages contributions • Ask students to describe, explain and interpret • Do not do the thinking for the students • Don't feel you need to resolve everything before leaving a group or before the end of the lesson

Suspending Disbelief and Adopting New Practices

After exposing teachers to the tasks described above, we invite them to temporarily 'put to one side' their doubts and fears and simply implement the tasks we have provided in an open-minded way. This is a risk for them, and we emphasise that it is the pedagogical principles under trial—not the teachers. We offer ongoing advice and support through classroom visits, or where this is impractical through a system of 'telephone mentors'—teachers who have been through the process before. We monitor the implementation through a triangulation of student questionnaires, researcher observation and teachers' own self-reports (diaries, interviews and questionnaires). For a complete set of instruments, see Swan (2006a).

As may be expected, teachers implement the principles in many ways, with differing degrees of success. This appears to be profoundly influenced by teachers' prior beliefs about mathematics, teaching and learning.

Many secondary and adult education teachers on our professional development courses begin with a transmission belief system that they have held for a long time. These beliefs (see Sect. 2 above) are resistant to change and act as filters through which the teachers both anticipate what will happen when they implement the principles (e.g., 'I will lose control'), and interpret lesson outcomes (e.g., 'Discussing misconceptions confused them'). In contrast, connectionist teachers appear to have higher expectations of students (e.g., 'Students can cope with more demanding work') and are enthusiastic when reporting outcomes (e.g., 'I'm very glad they did get confused, because then they started to think'). Other transmission teachers, however, appear more open and able to act in new ways *as if* they believed differently. Some abandon their traditional 'explainer' role completely and move to the opposite extreme playing a passive, reactive role. They thus change their behaviour to act in ways consistent with a *discovery* orientation. It is only later that we find they begin to renegotiate their role and learn how to intervene, provoke and collaborate without 'taking over'. This may go towards explaining why some teachers temporarily (we hope!) become less effective during professional development as they ‘unlearn’ old habits.

As may be expected, interpretations of the principles evolve considerably through the course of professional development. Initially, for example, some appear to interpret our purpose as one of curriculum development rather than professional development. They see the different tasks as providing ‘enrichment’ to existing resources rather than as a generic means to foster conceptual development and different forms of reasoning. They become productive instead of reflective. Some use language imprecisely and do not discriminate between, for example, *mistakes* and *misconceptions* or between *talk* and *discussion*. This causes a number of implementation problems. For example, after exposing an error, the teacher might focus on putting students right, rather than on developing their reasoning. The teacher may ask students to work together, without talking to them about *how* they should work together. Students and teachers clearly need time to explicitly discuss and accommodate new ways of working.

Reflecting on Experience

In our model, teachers meet periodically for follow-up whole day ‘workshops’ to reflect on their classroom experiences. Initially, we ask them to use their classroom ‘diaries’ to recollect what happened when they tried to implement a principle and to report on it descriptively, *without* passing judgments, such as “It went well”. They use student work to help in this process. We organise this in pairs or small groups in the form of informal interviews, with one partner prompting with questions and taking notes, while the other responds in a detailed, vivid way. A typical set of prompting questions is reproduced in Table 5. Towards the end of

Table 5 Suggested prompts for teacher feedback

- What was your purpose for the lesson?
- What were your fears and expectations?
- How did you organise the lesson? How was the classroom arranged? Why did you organise things in this way?
- How did you introduce the lesson? What did you tell the students about: (a) the project; (b) the particular task? (c) the way students should work on the activity; (d) the reasons why you wanted them to work in this way?
- What happened during small group work? What did your students find difficult? What did you find difficult?
  How and when did you intervene?
- What happened during whole class discussions? How did you organise it? Just at the end, or during the lesson?
  What did students find difficult? What did you find difficult?
  Did students report back on their discussions? How did this happen?
  What generalisations/big ideas emerged?
- What issues have arisen for you? What changes should be made to the tasks? What would you do differently next time?

the report, we invite teachers to identify areas with which they are struggling and then plan discussions on these issues for the following workshop. Common requests for help are on managing small group and whole class discussion.

Towards the end of each workshop, we encourage teachers to describe how their perceptions of teaching and learning have changed and the reasons for this. Those that profess the greatest changes are usually those who recognise the inadequacies of their current practices and are willing to persist with the tasks over several lessons. These teachers frequently report both surprise and delight at the improvement in the engagement and attitude of their students:

> There was a significant increase in student involvement…. It took one or two lessons for students to adjust to the new learning style. I have learned to wait and listen to student responses. I like to start lessons with 'what do you know about?' which encourages discussion. I am learning when to stand back at the appropriate time and allow students to reason on their own or with fellows. My teaching style is now far less prescriptive. Exploring what students know encourages a far wider participation. I also believe that allowing students to make mistakes and learn from them is a very powerful technique. (Teacher's written report)

Concluding Remarks

Research evidence to suggest that teachers' values, beliefs and practices are extremely resistant to change (Kagan 1992; Nespor 1987). In a literature review focused on the design of effective professional development, Wilson and Cooney distill three important themes. Firstly, they found focused, specific reflection is necessary in order to avoid teachers merely recalling past events and experiences.

> To accommodate change, teachers need first hand experiences working on specific innovative investigations and activities that they are attempting to use in their classrooms. These experiences, as both students and teachers, influence what teachers ultimately think and do…. *It is through the act of reflecting on specific events that those centrally held beliefs can be affected in fundamental ways* (Wilson and Cooney 2002, p. 142).

Secondly, they comment on the power of encouraging teachers to attend to students' understanding. Encouraging student debate in the classroom not only helps teachers to become sensitized to student understanding, it also emphasises the value of this way of working. Thirdly, they emphasise the importance of teachers sharing the 'authority' of both intellectual and pedagogical issues with students. Teachers thus begin to learn from their students and the environment becomes truly collaborative.

In the sequence I have described, all three themes are evident. Our research suggests that when teachers adopt new practices and reflect upon the often-surprising consequences, their beliefs are changed in profound ways. We also find that teachers welcome an opportunity to clarify and discuss their values and the framework we have provided for this offers one way of understanding the tacit dilemmas that they face every day. Rather than trying to find the 'best way' to teach mathematics, they begin to look for *an appropriate* way to teach each particular lesson according to whether the primary goal is to improve fluency, understanding, strategies, awareness or appreciation. Research may then be used to indicate appropriate teaching principles that may be applied. The tasks and 'lesson plans' we design then offer one way of realizing these principles in practice.

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Pedagogical, Mathematical, and Epistemological Goals in Designing Cognitive Conflict Tasks for Teacher Education

Irit Peled and Anat Suzan

Introduction

The Emergence of Cognitive Conflict as an Instructional Strategy

The analysis of different dimensions of teacher knowledge as described by Shulman (1986) in general and further detailed for teaching mathematics, has called attention to the complexity of the teacher educator's role. Teacher education is expected to promote prospective teacher knowledge in mathematics, in pedagogical content knowledge, psychological aspects of children's learning, existing curricula, and additional important dimensions and issues such as goals and beliefs.

Commonly, the teacher educator (TE) does not have a textbook, and has to exhibit a lot of creativity in designing and implementing tasks to achieve these goals. As a result, teacher educators develop their own task construction principles and strategies that fit with their beliefs about learning, their assumptions about prospective teacher knowledge, and their knowledge about the learning of prospective teachers. Thus, for example, Zaslavsky (2005) designs tasks to evoke teacher uncertainty, and Peled (2007a) uses analogical reasoning in constructing tasks that make prospective teachers aware of children's difficulties.

It is not surprising that in searching for an instructional strategy that has a good potential for promoting change and knowledge growth, the use of cognitive conflict as a strategy comes to mind. Cognitive conflict was originally considered by Piaget (1985, original work 1975) as a pivotal step in his equilibration theory. According to Piaget (1985, original work 1975), development is a constant process of growth motivated by the desire to stay in a state of equilibrium. Piaget suggests that an opportunity for growth arrives when some new experience conflicts and cannot be explained using existing knowledge schemes. When the conflict is strong, it can-

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not be resolved by simply assimilating the new piece of knowledge. The drive for regaining equilibrium results in a stronger effort to find a resolution, leading to the accommodation of existing schemes and the construction of new ones.

One of the main educational implications of Piaget's theory is that in order to encourage and support development, it might be a good idea to *deliberately* create cognitive conflict opportunities. As a result, cognitive conflict has become an instructional strategy.

In its use as an instructional strategy, cognitive conflict has become an immediate association with conceptual change. The conceptual change model suggested by Posner et al. (1982) builds on Piaget's theory (1985, originally1975) elaborating and highlighting the crucial phase of dissatisfaction that generates cognitive conflict creating the need to restructure existing concepts.

Accumulated experience with the cognitive conflict strategy has shown that its implementation and effect are not straightforward and that different considerations should be made. Limón (2001) reviews and analyzes research results, identifying several factors and conditions that determine whether instruction would lead to a meaningful conflict. Some of these factors are motivation, prior knowledge and beliefs. Zohar and Aharon-Kraversky (2005) focus on an important factor that relates to prior knowledge, showing that students' academic levels play a significant role in determining the effect of the cognitive conflict strategy.

Cognitive Conflict in Mathematics and Science Teacher Education

There is more literature on using cognitive conflict with children than on its implementation with teachers. But many considerations that apply in working with children apply to our work with teachers and prospective teachers.

In designing a task that aims to create a cognitive conflict for teachers the teacher educator deals with problems that are similar to what a teacher faces in designing a task for children. Specifically, just as the teacher should know that what she assumes to be evidence that would create conflict for children might not be perceived as such by them, the teacher educator would have to be aware of the possibility that what she views as anomalous data might not be accepted as such by the teacher or prospective teacher.

This obstacle is demonstrated by Peled (2007a) following prospective teacher use of an inappropriate linear model in a task that asks about the number of greeting cards that can be cut from a rectangular cardboard. Some teachers calculated the ratio between the length of the board and the length of a card, and the ratio between their widths. Then instead of multiplying the two values, they gave one of them as the answer. It was speculated that this linear model resulted from the fact that the ratios in both dimensions happened to be the same and because the prospective teachers did not pay much attention to the situation. It was also assumed that different ratios might trigger confusion when one tries to use only one ratio. Following these assumptions and in an effort to refrain from telling the teachers what was wrong, Peled (2007a) designed a similar task with numbers that the yield different ratios. It was expected that these numbers would create conflict and encourage problem re-organization, reflection and self correction. Unfortunately, in the first new task more than half of the prospective teachers who used a linear model did not note or experience any conflict.

The failure to facilitate a conflict often results in cycles of task design efforts in the spirit of design experiments. Thus, realizing the need to make the cognitive conflict strong enough for his prospective primary teachers, Parker (2006) explores the effect of a range of scenarios of increasing depth of conflict in developing teachers' knowledge about a scientific concept (shadow formation).

Some of the work on cognitive conflict in mathematics teacher education is conducted under a more general perspective of a search for powerful tasks to promote learning. In this capacity Zaslavsky's work (2005) that was mentioned earlier offers a reflective account on task design and task implementation in teacher education and arrives at the conclusion that conflict has played a central role in the tasks she has found to be effective.

An example of the effect of a task that evokes uncertainty is demonstrated by Zaslavsky et al. (2002), who draw a graph in a non-homogeneous coordinate system creating conflicting answers about the slope of a graph. This non standard situation, presented to students, teachers, and prospective teachers, creates, on the one hand, a (desired) need to settle definition inconsistencies. On the other hand, it demonstrates that a conflict might lead to frustration instead of facilitating knowledge growth.

Three Examples of Cognitive Conflict Goals

As mentioned earlier, there are many knowledge dimensions on which mathematics teachers and prospective teachers are expected to grow. Our purpose, in this article, is to demonstrate the potential power of the cognitive conflict strategy in a wide range of goals pertaining to teacher knowledge.

A specific task might facilitate the "whole" cognitive conflict process in the sense that it serves both as a source for conflict and instability, and at the same time as an opportunity to reflect, reorganize one's knowledge and construct new knowledge. Yet, the use of a cognitive conflict strategy quite often involves a sequence of activities and not just one task. Tasks can serve different roles in such a process. A task that is used in the initial stage of the process might be designed to make teachers aware of the limitations of their knowledge creating dissatisfaction and motivation to acquire new knowledge. A task that is used at a later point in the sequence might be designed for the purpose of promoting the construction of new knowledge (that, to use Piaget's terms, is expected to resolve the conflict, bringing one back to a stable state).

As will be shown (and compared in the discussion), the following three examples demonstrate the use of the cognitive conflict pedagogical strategy for making different types of changes. At the same time they demonstrate tasks that are used at different points in the process. The first example is an initial task in an activity sequence. It intends to cause dissatisfaction with current knowledge about children's thinking and create motivation to understand the development of children's decimal conception. The other two examples have been designed to serve a double purpose. In addition to creating a conflict, they are expected to help prospective teachers in constructing new knowledge. The second example deals with affecting mathematical and pedagogical content knowledge while the third example deals with prospective teacher epistemological knowledge related to beliefs on the role of mathematics.

All the following examples describe activities and student responses in a teacher education course that deals with psychological and didactical aspects in teaching mathematics (this is also the course's name). The prospective teachers in this course are studying towards their teacher certificate in secondary school mathematics. Most of them are in their third year of undergraduate mathematics and some have already completed their undergraduate degree in mathematics. The course was designed and taught by the first author. The lessons were recorded by the second author as a part of her doctorate research.

A Pedagogical Goal: Trading Places

Background and Design

This task was constructed as a part of a sequence of activities that were designed to promote prospective teacher knowledge on children's thinking about decimal fractions. It was designed to be the initial task the purpose of which was to substantiate the importance of learning about children's development of this mathematical concept.

The task consisted of an episode that showed the results of two tests taken by two (imaginary) children that represent two common types of decimal conceptions. The TE designed the tests using items that can detect the two conceptions. The proportion of the different items was purposefully chosen to create a "trading places" phenomenon. That is, although none of the children undergoes any knowledge change, their grades change dramatically from the first to the second test. Specifically, one of them succeeds in the first test and fails in the second test, while the opposite is true for the second child.

Based on earlier experience with prospective teachers and teachers, the TE's assumption was that, being told that the children did not study between tests, the prospective teachers would expect each child to get similar grades in the two tests, and would be perplexed upon viewing a situation that conflicts with this expectation. This realization was expected to create dissatisfaction with their own conception of children's decimal knowledge and with their knowledge about the nature of tests. As a result it was supposed to generate motivation and curiosity for further learning.

Table 1 The trading places task: a pedagogical goal

Trading Places: Puzzling test results.

Instructions: Below are the results of two tests, given to students on close occasions. The table depicts the test results of two students, William and Frank. Suggest an explanation for the observed phenomenon of "trading places" between these students' test performances.

Test 1				**Test 2**			
		William	Frank			William	Frank
0.2	0.35	+	–	0.23	0.4	–	+
0.35	0.350	+	+	0.2	0.20	+	+
0.72	0.3	+	–	0.35	0.7	–	+
0.842	0.8	+	–	0.837	0.9	–	+
0.32	0.248	–	+	0.12	0.234	+	–
Percent of correct answers		80	40	Percent of correct answers		40	80

+ correct answer – incorrect answer

First explanation:__________________________________

Second explanation:__________________________________

Task Implementation

At the start of the activity sequence on children's decimal conceptions, the prospective teachers were asked to analyze the episode presented in Table 1.

The instructions included an explanation about the task and about the test circumstances. The TE added the following information: The two children, William and Frank, whose tests are presented in Table 1, were in sixth grade. Their teacher gave her class two decimal comparison tests on two consecutive days without any additional instruction between the first and the second tests. In each test the children were asked to compare pairs of decimals and circle the bigger number or mark them as equal. The teacher checked the tests using pluses for correct answers and minuses for incorrect answers (as seen in Table 1). She then calculated the proportion of correct answers for each child in both tests. While doing so, the teacher was especially surprised at the change in performance of two of her students, William and Frank.

The prospective teachers were told that their task is to help the teacher interpret and explain these results. They were told to record their speculations in the space allocated for "first explanation", and informed that they would be asked to suggest a "second explanation" at a later point. During the task implementation, when the prospective teachers were showing signs of puzzlement, they were told that it is "legitimate" to record these feelings in the space assigned for the first explanation.

Based on their first explanations, all the prospective teachers in the course expressed surprise at the fact that there was such a change in grades given there was no change in knowledge. A variety of different speculations was suggested in these explanations, such as: *"Maybe one of them was sick"*, or *"Maybe William remembered answers from the first test"*. While a part of the prospective teachers could not think of any interpretation, some others suggested that William might have used a certain rule in both tests, judging longer numbers to be bigger. A few of the prospective teachers suggested that, contrary to William, Frank judged shorter numbers to be bigger. These explanations remained at a superficial level and did not go any deeper than looking at number lengths.

The emerging conclusion from class discussion was that it was important to find an explanation for the observed episode, and that it could be done by learning about the way children perceive decimals. Thus the discussion set the ground for teacher learning about the development of children's decimal conceptions. In terms of the cognitive conflict process, following the realization that one's knowledge does not suffice in explaining the given case, the first stage was reached. That is, a state of dissatisfaction with one's knowledge was established, as depicted in Table 2.

The more advanced stages of the desired (cognitive conflict based) change process occurred in following tasks that involved reading excerpts of children's interviews, discussing children's conceptions informed by decimal number research (Nesher and Peled 1986), and role playing some of the main conception types. Equipped with new knowledge the prospective teachers were asked to give a second explanation for the trading places episode. At this point they were able to diagnose William as a child who has a whole number conception of decimals. This conception fails him in some cases where he would say "0.35 > 0.7 because 35 > 7" but sometimes results in a correct response (for the wrong reasons) such as "0.35 > 0.2 because 35 > 2". Frank was diagnosed as using a "shorter is bigger" rule that could be derived from several different conceptual sources. The common

Table 2 Components of cognitive conflict in the trading places example

Initial beliefs or knowledge	Cause for equilibrium disruption	Essence of conflict	Expected effect
1. Feeling confident about understanding children's thinking about decimals 2. Believing that a test grade reflects the child's knowledge (and given two test, a child should get similar grades)	A case where children get very different grades in two consecutive decimal tests Specifically, a child with a high grade gets a very low grade, and a child with a low grade gets a very high grade	Conflict between the belief about tests and the observed case Existing knowledge does not provide an explanation	1. Motivation for learning about children's decimal knowledge development. New knowledge is expected to explain the case 2. New conceptions about tests

speculation, based on the most prevailing source, was that Frank holds a decimal fraction unit conception leading him to view tenths as bigger than hundredths disregarding the number of units of each type and thus concluding that 0.2 > 0.35. This strategy "wins" him some points in items such as "0.32 > 0.248" because the number "in hundredths" (0.32) happens to be bigger than the number "in thousandths" (0.248).

With this diagnosis what was still left to discuss was the "trading places" phenomenon from a testing and evaluation perspective. The grounds for this discussion were prepared by one of the activity sequence tasks involving test construction and role playing different conceptions. This task served to demonstrate the tight relation between choice of test items and children's success rate. Specifically, it demonstrated that a test that consists of a large proportion of items that "detect" one conception would result in a low grade for a child with this conception and, possibly, a high grade for a child holding a different conception.

The discussion strengthened teacher awareness of the crucial role of the choice of item frequencies (the proportion of items that were constructed to detect each conceptual model) in determining children's grades. With regard to the episode, it was also realized that the teacher educator's success in designing this task had resulted from the same explanation. That is, that one can design a test that will yield the grades 40, 80 for two children holding some known conceptions, and also design a test where these same children would get the grades 80, 40 correspondingly. This realization promoted a shift towards viewing the role of such tests as a diagnostic rather than a mean for assigning grades.

A Mathematical (and Pedagogical) Goal: Father and Son

Background and Task Design

While the previous example was designed by the teacher educator as a pre-planned activity mainly intended to create doubt and need to learn more about a specific topic, the current activity was created ad-hoc and designed to achieve several goals.

The task was preceded by a sequence of lessons that dealt with complex aspects of percent problems involving a focus on the crucial role of the operator's reference and the use of qualitative argumentation. The prospective teachers were given a homework assignment that included the following problem:

> The price of an adult ticket to an amusement park is 40% lower than the price of a child's ticket. A father and his son paid $128 for their tickets. What was the price of the father's ticket?

As expected, most of the prospective teachers solved the problem correctly, following the choice of the price of the son's ticket as the missing value, x. Yet, two of the prospective teachers chose the father's ticket price as x, and then went on to (incorrectly) claim that both tickets would amount to 240% x, getting the expression 240% x = 128. As can be seen, although their mistake was not in the choice of x but in the wrong inversion of the direction of the percentage relations (i.e. in saying that if A is 40% less than B, then B is 40% more than A), it was this choice that made the inversion necessary to begin with (the common choice of x did not require an inversion).

This was an unexpected and therefore interesting response, since these prospective teachers had coped with problems involving increase and decrease of prices, and had discussed the concept of reference in calculating percentages. The TE was wondering whether these two students had not acquired the mathematical knowledge needed for handling situations with percentages, or perhaps they do hold this knowledge, but the unfamiliar situation enabled their (wrong) intuitions to take control and suppress existing formal knowledge. The TE was also wondering if the prospective teachers that did solve the problem *correctly* would make the correct conversion if they were put in a situation that required inversion of relations.

Following these deliberations, the TE decided to design a "follow up" similar problem in the spirit described by Peled (2007a) with an effort to create a conflict that would promote self reflection. The idea behind the design of the task was to request all prospective teachers to solve the task in two ways, using the convenient choice of x, and the less convenient choice of x, that involves the inversion of the given relation. The TE expected the two teachers who made an earlier incorrect conversion to get a correct solution when they use the "convenient x", and an incorrect solution for the "inconvenient x". Moreover, the TE expected some of the other prospective teachers to (also) make a wrong conversion when they would be forced to use the "inconvenient x". All the teachers who would make the wrong conversion would get two different results, and were expected to become perplexed by these conflicting answers, and engage in an effort to resolve the conflict.

By designing this task the TE had several simultaneous goals in mind. She wanted to investigate the source of teacher incorrect answers and at the same time use the new situation as an additional opportunity to extend teacher knowledge. The task was expected to promote their knowledge about the concept of reference in percentages, and demonstrate the use of a pedagogical approach that promotes self reflection, increasing teacher content knowledge and pedagogical content knowledge (Shulman 1986). It was also intended to serve as an opportunity to discuss the struggle for control between intuitive knowledge and formal knowledge (Fischbein 1987; Fischbein et al. 1985). That is, demonstrate the strength of intuitive knowledge through their own experience and thus facilitate better understanding of the forces that are involved in children's problem solving.

Task Implementation

In the next class period (that happened to be about a month later because of a semester break), the whole class was given the following problem:

The Country Fair Problem: A group of 6 adults and 10 children went on a trip. One of the group members, Dan, paid for the group's entrance to a Country Fair a total amount of $560. When he wanted to collect the money from his mates he realized that he does not remember the different ticket prices. All he remembered was that the price of a child's ticket was 20% lower than an adults' ticket. On the basis of this data Dan tried to reconstruct the ticket prices. **Suggest at least two ways by means of which Dan could have figured out these prices**.

When the task was presented to the prospective teachers, they deliberated on what could be considered as different ways. The TE explained that the difference did not have to be radical, and that the diverse choices of what to represent as x could be considered as different ways. The prospective teachers continued their work. After a while the remark "*but I am getting different solutions*" started coming up from different directions.

As it turned out, all of the prospective teachers, and not only the two teachers who erred in the original assignment, failed in calculating the inverted price relations. They all concluded that if one amount is 20% less then another amount, then the latter is 20% more than the first. Specifically, while with the more convenient choice of x, the price of an adult ticket, there was no problem figuring out that the child's ticket was 0.8x, but, when x stood for the price of a child's ticket, the adult's ticket was incorrectly calculated as being 20% more than x, getting 1.2x.

For some time the prospective teachers worked individually, trying to figure out what went wrong, but could not detect the source of the problem. Thus, at this point they reached the stage of disequilibrium (depicted in Table 3) but did

Table 3 Components of cognitive conflict in the father and son (tickets) example

Type of knowledge	Initial beliefs or knowledge	Cause for equilibrium disruption	Essence of conflict	Expected effect
Content knowledge	Given that a certain amount A is 20% less than B, it can be deduced that B is 20% more than A	Given a problem that should have one specific result, the two different ways to solve the problem yield (unexpected) different results	A feeling that there must be something wrong in the knowledge related to solving the problem	Strengthening formal knowledge related to the reference in calculating percentages
Pedagogical content knowledge	Getting the right answer indicates good formal knowledge, while a wrong answer indicates lack of formal knowledge	Encountering difficulty in spite of possessing relevant formal knowledge	A feeling that different forces take part in their thinking	Understanding that intuitive knowledge might interfere even when one has relevant formal knowledge

not resolve it yet. Only a collective effort through group discussion brought up the problematic inversion. Once it was realized, one of the prospective teachers identified the connection to problems that had been dealt with in previous classes: "*It's like what we talked about! When a product's price increases by 20% and then there's a 20% reduction, it does not return to its original price because the 'whole' changes!*"

The discussion continued, and when it seemed that all prospective teachers agreed about the conversion and arrived at the ratio of $1/0.8 = 1.25$ between the adult's ticket and the child's ticket, they were asked to generalize and interpret this relation. That is, asked whether it is always the case that when one value is 20% less than another, then the latter is 25% more than the first.

At the end of this class session the TE discussed the goals of the problem solving sequence. She highlighted the mathematical difficulty involved with the issue of reference in calculating percentages together with the strong misleading intuitions that keep "creeping up". She pointed out to the teachers that they were influenced and "overpowered" by these intuitions in spite of their formal knowledge and their previous experience with relevant problems; It was, therefore, expected that they would acquire more understanding and compassion towards similar difficulties exhibited in the future by their own students.

In addition to that, the TE detailed the didactical rationale for designing the second problem (the Country Fair problem), explaining her effort to make students aware of this mathematical difficulty by creating a situation that would get them into a cognitive conflict.

Since this task involved several goal levels, and it was important to make sure that the prospective teachers became aware of the different perspectives, a reflection process complemented the task. The prospective teachers were asked in their homework assignment to reflect on the whole process and on the different goals of their teacher, the TE. They were asked to describe what the TE expected would happen (being advised to use the term *cognitive conflict* the TE had used in class), the task characteristics that were expected to cause it, and their own personal experience.

An Epistemological Goal: The Lemonade Stand

Background and Task Design

This example entails goals that have not been discussed in the literature on teacher knowledge, but we believe them to be important. These goals are epistemological, relating to teacher understanding of the role of mathematics in general and in problem solving in particular. Similarly to the "trading places" example, this was a pre-planned example aimed to elicit cognitive conflict leading to change in the prospective teacher beliefs.

As discussed by Peled and Basan-Cincenatus (2005) and by Peled (2007b), our assumption was that the prospective teachers would regard alternative solutions for problems involving normative contexts as mathematically inferior to conventional solutions that use proportion. Our purpose was to make them realize that the different solutions should have the same mathematical status.

To establish prospective teacher prior beliefs we presented them with the ***lottery problem***:

> Two friends, Anne and John, bought a $5 lottery ticket together. Anne paid $3 and John paid $2. Their ticket won $40. How should they share the money?

All the prospective teachers solved the problem by sharing the money according to the ratio of the friends' contributions to the ticket's price. Class discussion showed that while choosing to use proportion, the prospective teachers were not aware of using any assumption. It is only during the discussion that the implicit assumption, that each invested dollar should yield the same profit, became explicitly formulated.

Following this discussion, prospective teachers were presented with children's answers and asked whether these answers were correct and whether they would have accepted them as valid answers:

Aviv: They should split it evenly. 40:2 = 20 so each gets $20.
Ron: Since Anne paid one dollar more than John, she should get $20½ and John should get $19½.
Dona: They should get according to what they paid. 40:5 = 8, 8×3 = 24 8×2 = 16, so Anne should get $24 and John should get $16.

Most of the prospective teachers stated that the only correct answer is the third one. Even though the other two answers were viewed as practically reasonable, they regarded them as mathematically inferior, or, to say it more bluntly, as mathematically incorrect. This activity was conducted in several slight variations. In another class, instead of the lottery ticket, prospective teachers were presented with a shopping problem that involved several options of sharing a purchase price between friends. Similarly to the reaction to the lottery ticket example, prospective teachers believed that there was only one mathematically sound solution, while the rest of the solutions were acceptable morally but not mathematically.

This attitude towards alternative solutions reflected and established teacher epistemological beliefs about the role of mathematics and the meaning of fitting a mathematical model in a given situation. To achieve a change in these beliefs, we designed *the lemonade stand problem*. This task was similar in structure to *the lottery problem*, conventionally considered as a proportion problem. But its expected solution did not involve proportion and we anticipated that it would be more willingly accepted as a sound mathematical solution by the prospective teachers.

It was also expected that a conflict would be created as a result of the prospective teachers' realization that they were inconsistent in their attitudes towards the solutions of similar structure problems.

Task Implementation

The prospective teachers were asked to solve the following problem:

The Lemonade Stand Problem: During the Country Fair Tammy and Abby put up a lemonade stand. Tammy bought disposable cups for $10 and Abby bought concentrated lemon-juice for $14 and mineral water bottles for $36. They sold lemonade for a total of $480. How should they split the money?

Two different solutions were suggested by the prospective teachers. One solution involved regarding the expenditures as investments and splitting the total amount according to the investment ratio. A second solution involved reimbursing each of the two friends for the money that was spent on the products, i.e. $10 for Tammy and $50 for Abby, and splitting the remaining $420 evenly between them.

At this point some of the prospective teachers began noticing that there was some inconsistency in their attitude towards problem solutions. As depicted in Table 4, they were puzzled by the realization that although the problem's structure is similar to the structure of the lottery problem, they regarded its solutions differently. While alternative solutions of the lottery problem were not accepted, here they were ready to accept an alternative solution and thought that it did not have a more inferior mathematical status. They were even beginning to wonder whether this same solution could apply to the lottery problem. That is, that each partner would get what she paid for the ticket and the rest of the money would be split evenly. This, in fact, resulted in the same answer as offered by the second child ($20½ and 19½).

Table 4 Components of cognitive conflict in the lottery vs. lemonade example

Initial beliefs or knowledge	Cause for equilibrium disruption	Essence of conflict	Expected effect
The lottery problem and other problems of a similar structure (where 2 partners make different investments) should only be solved by proportional profit sharing. Alternative solutions might be accepted on realistic basis, but should not be considered as mathematical solutions	A problem that is analogical in structure to the lottery problem (i.e. 2 partners make different investments), and yet what seems like a sound mathematical solution is not based on a mathematical model of proportion	Different solutions and different attitudes towards realistic considerations in supposedly analogical problems	1. Readiness for change in attitude towards solutions that are based on realistic considerations (in terms of their mathematical structure) 2. Motivation for better understanding of mathematization processes

The Nature of Examples: A Comparison

We have chosen to present three examples that demonstrate the wide range of goals for which the cognitive conflict strategy can be used. Obviously, through the use of the same strategy, they all have much in common. All three tasks were designed to facilitate change. They all offer an opportunity for experiencing discomfort and confusion creating motivation to search for a resolution and setting the ground for welcoming new ideas that might offer better answers.

Still, each example has its own special features that we have highlighted and presented in Table 5. Some of the differences are less "dramatic" and simply involve a different focus or goal preference. For example, while all tasks are expected to create motivation for learning new knowledge or new perspectives, the "Trading places" example focuses on this goal in particular. As can be seen in Table 5, although all three tasks were carefully designed, the second example (father and son tickets) was constructed ad hoc triggered by some incorrect prospective teacher solutions to an earlier problem. This difference is also an indication of the type of knowledge that was handled in this example. On the one hand it is less "impressive" because it deals with some very specific knowledge, while the third example is aimed at a more general and abstract goal. On the other hand this example turns up as an opportunity to achieve several goals at the same time:

1. Growth in mathematical knowledge relating to the issue of reference in calculating percentages;
2. Growth in understanding children' difficulties on this issue; and,
3. Learning about cognitive conflict as an instructional strategy.

Table 5 A comparison of the three examples

	Trading places (test grades)	Father and son (tickets)	Lemonade and lottery (profit sharing)
Goal in terms of knowledge dimension/type	Pedagogical	Mathematical and pedagogical	Epistemological
Design circumstances	Planned	Designed ad hoc	Planned
Mathematical context	Decimal fractions	Algebra and percentages	Proportion
Main expected effect	Create motivation for acquiring more pedagogical knowledge. Further change is expected to occur following an instructional trajectory	Acquire new content knowledge and pedagogical content knowledge by collective discussion of task solution	Create disposition for change in attitude and for change in conceptions. Possibly actual change for some of the prospective teachers, while others need additional instruction

The first example stands out in a different way. While the other two examples were administered with the expectation for a "complete" conceptual change process, the trading-places example was only meant to arouse curiosity and prepare the ground for future change. An added value to the motivation for learning is its contribution to better understanding the role of diagnostic tests, when the process of designing the task becomes transparent.

It should be noted that each of the examples was originally designed by the teacher educator to attain a different goal. But because they are all examples of using cognitive conflict as an instructional strategy, each example also models this strategy for teachers. However, to ensure awareness and learning from this experience, each example has to be discussed with the prospective teachers on a meta-level. That is, the teachers have to reflect on the process and to explain the role of the task characteristics in promoting conflict and change.

Concluding Remarks

The use of cognitive conflict as an instructional strategy is common in science education, where data is presented for the purpose of making the learner realize that for her current scientific schema this data is anomalous. Since the schema is not powerful enough to explain the given data, the learner realizes it should undergo a change. The new schema that emerges following further instruction, deliberations and investigations is a schema that can view the originally anomalous data as "normal" data. This instructional process imitates the development of new schemes and new scientific theories in the history of science, where new ideas were triggered by identifying and noticing anomalous data rather than being introduced to it by an instructor.

In the learning of mathematics "anomalous observations" are substituted by data that is considered to be unexpected or in conflict with one's mathematical knowledge. For example, encountering different answers to a problem might be in conflict with one's knowledge in cases where the problem has a structure that is associated with one answer. Similarly, getting an equation that involves multiplication might be in conflict with the solver's qualitative evaluation that "the number should get smaller" if this solver believes that "multiplication makes [the number] bigger" (Fischbein et al. 1985).

In mathematics teacher education anomalous data is substituted by situations that contradict prospective teacher beliefs, knowledge, or conceptions about children's thinking. These situations together with group discussions can trigger cognitive conflict and conceptual change. In this article we have demonstrated the potential use of cognitive conflict tasks in achieving a variety of teacher education goals. However, as is often the case with children, it cannot be guaranteed that the desired processes will indeed occur. Obviously, there is much need for identifying the conditions and factors that will increase their onset chances making them meaningful and effective.

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Working Mathematically on Teaching Mathematics: Preparing Graduates to Teach Secondary Mathematics

Anne Watson and Liz Bills

Introduction

In working with prospective teachers our practice is to start with mathematical tasks, so in this chapter we describe three tasks which we presented to them, the way in which they responded to the tasks, and our interpretation of their learning through these tasks.

We saw the chance to write this chapter together as an opportunity to examine for the first time the way we are working together as teacher educators. We knew already that both of us had a strong commitment to the view that the shared experience of working on mathematical tasks is at the heart of learning mathematics, learning about mathematics, and learning about learning mathematics. Although we have been colleagues in a number of different contexts for many years, this year is the first time that we have worked together with a group of prospective teachers over the period of a year. Our common approach has developed over the period of this year through shared planning and teaching and through observation of each other's teaching. We have spent time discussing the responses of our students to our teaching, especially the way in which we see them working in schools, but had not explicitly compared our teaching approaches.

In this chapter we have written about the mathematical tasks we present to prospective teachers and the work which they have done with the tasks. Most of our taught sessions with the group start from mathematical tasks and move on to pedagogical questions. Occasionally their experience as teachers is used as the starting point. The work we present here is typical of our teaching sessions rather than illustrative of an occasional approach.

The prospective teachers we teach are taking a one year post graduate course which will give them Qualified Teacher Status (necessary for teaching in state funded schools in the UK) as well as academic credits at masters level. Teaching is still not a popular career choice for mathematics graduates in the UK, which means that

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admission to our course is competitive but not highly so. We usually attract around 60 applications for 30 places each year. Nevertheless, our students tend to be well qualified mathematically. They have a good first degree which includes at least 50% mathematics (this might be an engineering degree, for example, where experience of rigorous pure mathematics is limited) and a significant number of them have higher degrees as well. They have mostly been very successful in mathematics at school, but may more recently have felt that they reached some kind of wall in their own learning. Because most of our applicants are academically qualified for the course, we are able to select on the basis of other requirements, which include strong interest in mathematics and evidence of the ability to think critically about teaching and learning.

Some Theoretical Background on Using Tasks to Learn to Teach Mathematics

The approach we take to mathematics teacher education is to offer a sequence of complex mathematical experiences which are designed to expose and bring to articulation ambiguities, distinctions, alternative conceptions, of teaching and of school mathematics. In each session we work on what Thompson and colleagues call 'coherent mathematical meaning' (Thompson et al. 2007) through bringing what is coherent for our students alongside what might be seen as coherent for their learners. In this way, we ask them to appreciate learners' experiences, and to see 'coherence' from the learners' perspective.

This is a delicate task, because as we have worked for many years as teacher educators, some distinctions and constructions are very obvious to us—but this does not mean that they will be helpful for our students. It is a classic temptation in education to teach unifying theories, which make sense to those who already have a lot of relevant knowledge, to novices who do not know what is being unified. Instead, we use their existing mathematical knowledge and experience as learners as a starting point for developing language and realisations about their experience, and then applying those realisations in their teaching. Even with high level qualifications, there is always enough variety in ways of understanding the tasks we give to use diversity, comparison, analysis of implications, and relationships to school mathematics as structuring devices for interactive sessions.

We rarely offer easy closure by giving ways to teach topics, or ways to use ideas. We do not give generalisations about teaching and learning. Instead we work together on tasks, we use their responses to expose pedagogic and didactic details and choices, and we reflect on what is afforded for learners in imagined situations. It is a characteristic of our work that we do this through mathematics, so that the thinking required at every stage is mathematical, that is, concerned with presentation, exploration and perception of variation in questions, examples, diagrams, and other mathematical artifacts. Yet the atmosphere is about pedagogy. For example, in an early session on fractions, several different representations were used, each for a different task for which they were well-suited. The final task was intended to evoke criticism of reliance on limited images. All the representations which had been used so far were offered as a list:

Fraction walls	Folded rectangles
Squares in rectangular arrays	Folded strips
Congruent parts of shapes	Area representations
Shaded parts, not congruent	Shaded elements of set
Slices of pizza	Division sums
Points on a number line	Decimal number
Conventional symbolic form	

The task was:

> Decide the uses and limitations of each representation, bearing in mind that secondary school students have to work with objects which have a 'fraction' structure such as "sine = opposite/hypotenuse".

This end-of-session task provided more complexity than closure, prompting one prospective teacher to say that he thought this was why some teachers only taught procedures—working with images and understanding took a lot longer. Another announced that he was confused, but this is not a problem for us—a sense of confusion reduces as they realise there are no 'right' answers. What we aim to achieve is a shift from an approach characterised by the question, 'How shall I teach so-and-so?' to one of 'What does it take to learn so-and-so?'

We report on some tasks we have used, and how we use them, seen within the holistic nature of our course. School-based experience, mentoring, and university-based teaching are integrated to support the development of complex understanding of teaching mathematics. Key ideas about mathematical pedagogy are raised in practice, in formal sessions, or in small-group tasks or assignments. Within a student's individual trajectory there are opportunities to recognise structures and distinctions, through talking about experiences, which will inform future thinking about teaching. In the task sequences described below, some of these themes can be seen as threads that run through several sessions. Distributivity emerges in work on mental arithmetic and in algebra. Representation is explicit in the session using a line segment, explicit in a session on fractions, and implicit in other sessions. Ratio arises as an example of a shift to be made from additive to multiplicative thinking, but is given a full session of its own later. In a session on 'student errors seen in school' our students find that they learn even more about arithmetic, and we find that they apply a view based on alternative conceptions rather than 'mistakes'. All of this is enacted in schools through observing experienced teachers and by prospective teachers being supported through mentoring. In this way, we manifest

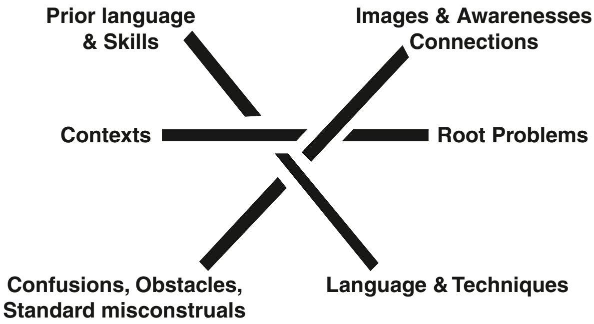


Fig. 1 Preparing to teach framework. (Taken from Mason 2002, p. 191)

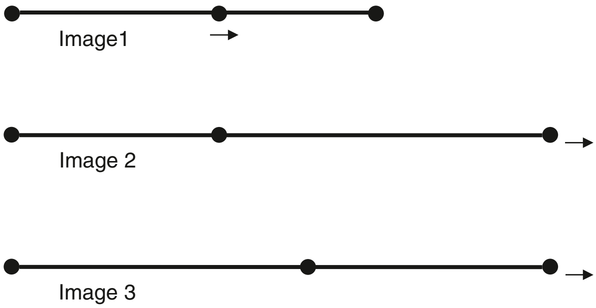
many of the practices which are taken-as-shared internationally (Watson and Mason 2007). Where we might differ from others is in the established, integrated, relationship between all aspects of our course (McIntyre et al. 1994; Furlong et al. 2000). It would be wrong to give an impression that there is a finely-detailed advance plan underneath what we do. Each prospective teacher teaches different years, groups and topics in school, so mentoring is responsive to individuals. Our teaching focuses on coherent mathematical meaning, and is influenced by the 'preparing to teach' frameworks developed at the Open University in the 1980s (e.g. Griffin and Gates 1989). This framework (which is still evolving) offers three dimensions, cognitive awareness, behaviour and emotional engagement, to think about teaching a topic (see Fig. 1).

In our teaching, therefore, we offer opportunities to do some mathematics and talk about it, to articulate their responses to it, and to think about how these would be contextualised for their students in school. For this chapter we observed each other teaching and identified common principles of how we do this. Since we are teaching teachers, we often state openly to them and each other how we have planned our sessions, but what we had not realised until this shared observation and analysis was that we also adhere to similar methods of putting these into practice, using prospective teachers' comments to develop a critical atmosphere.

Typically, we offer a mathematical task or set of tasks which relates directly to the school curriculum, and which can be tackled by all prospective teachers. Often, this task will trigger experiences they have had in school, either as teachers, supporters, or as learners. Soon after this, we give a new task which develops from the earlier one, but which is unexpectedly harder for some reason. It might demand comparisons between tasks or methods. We may have asked an unexpected question in a familiar context, or pushed a mathematical commonplace into an unfamiliar arena, or gone beyond the usual range of numbers or shapes, or questioned something which is often assumed. An example of this might be to ask prospective teachers if it is valid to join the points of a curve which has been generated from integer data. The introduction of such shifts and comparisons generates uncertainty, debate, intrigue, disturbance, which is not publicly resolved but becomes more comfortable through shared perceptions and thought about pedagogical implications.

In the next three sections we present accounts of three teaching episodes and relate them to this theoretical perspective.

Working with a Line Segment to Think About Shifts of Understanding



Static image 1 was projected on the board as prospective teachers entered the room. They were asked to say what they saw. Initial comments were: '65%' and 'golden ratio' and 'a black line with blobs on'. The diagram was then animated by moving the middle dot while maintaining the overall length. I[1] then asked them to say more about what they were seeing.

Sandy: A line of set length which is divided into two sections—two variable lengths—well one is variable and the other is fixed to the variable
Pat: There are two or three lengths, which is the starting length?

I commented that they had shifted from trying to guess what the diagram meant to reporting what they had seen, and that this shift appeared to have come when I animated the diagram and asked them to say what they saw.

Don: Part of the line has a variable length

I asked them to write down something which represented this variable. Eventually someone offered:

$$t = kp + (1 - k)p$$

Someone else observed that this simplifies to $t=p$. The next offering was:

$$x + y = L$$

And I queried the status of each term. L was said to be a constant, or given; x represented one of the lengths, and y was therefore a dependent variable. The letters therefore had three different uses in this statement of a relationship.

Someone then offered two further versions of the same relationship:

$$L - x = y$$

$$L - y = x$$

[1] In these descriptions of episodes the word 'I' refers to the one of us who was teaching at the time.

I said it was important for learners to have a sense of these three representations as a package, as three ways to represent the same relationship. I then described shifts of understanding that had been demonstrated so far:

- from guessing to being analytical about what they were shown
- from descriptive comments to analysing in terms of variables
- to interpreting what is free to move and what is constrained
- from variables to conjectures about variables
- from variables to relationships

I announced that we were about to shift from additive to multiplicative ways of conceiving relationships. I pointed out that $x+y=L$ seemed to be an attempt to record an additive relationship, where the earlier attempt using k seemed to be trying to express a multiplicative relationship. Someone said, 'It is like probability.'

I then animated image 2, the length being extended but the blob which was positioned on the line staying in the same place. This animation creates a different invariant, but is still additive. Finally I animated the line again in the way shown in image 3. Could they all try to express this as a multiplicative relationship? Eventually this was offered:

$$x = k \ \times \text{total length}$$

$$y = (1 - k) \ \times \text{total length}$$

where x and y are the two parts of the total segment. I had been hoping for an expression of direct proportionality such as $x=ky$.

I commented that this group had 'gone into algebra' straight away, but I was not sure that everyone was able to 'see' the relationships they were describing. What question could they ask learners to help them shift from seeing the lines additively to seeing multiplicatively?

The prospective teachers suggested:

How much of …?
What fraction of …?
How many times does this bit go into that bit?
What proportion of…?
What is the ratio of x to y?
Tell me the length of this bit in terms of this bit?

At this point it became clear that one of our students had not noticed *how* the point positioned on the line had moved, so I repeated the animation, asking 'what stays fixed and what changes in each diagram?'

I finished by exemplifying with $7=?\times 3$, asking for three different expressions:

$$7 = k \times 3$$

$$\frac{7}{k} = 3$$

$$\frac{7}{3} = k$$

I repeated the earlier comment about having a 'package' of three ways to express one relationship.

Superficially, the session was about how diagrams can be used as images for algebraic relationships and how focusing on invariance and change in dynamic representations can trigger new ways of seeing. The line segment image is particularly powerful because, seen as a statement about lengths, it carries semantic meaning about addition and multiplication, and it also acts syntactically, in that the three ways of transforming the key algebraic relationships can each be constructed from the diagram itself, rather than only from manipulating the formal expression.

However, this session contained far more about mathematical awareness than 'just' this. For example, someone referred to earlier work about how giving diagrams in particular orientations could be misleading for learners. I also hoped to initiate new awareness which would be revisited later on, and these were that:

- learners have difficulty in shifting from additive to multiplicative understandings of change—and in that respect this session was precursor to considerations of ratio later;
- there are alternative ways to express relationships—and this signalled an approach to algebra as expressing generality, and transforming equations as constructing equivalent expressions;
- letters have various roles; and,
- shifts from thinking about variables to thinking about relationships are important.

Also there had been opportunities during the session for those who were not sure about the mathematics themselves to work alone or with others, either on the direct mathematical tasks or the related pedagogical issues.

Working with Mental Calculations to Explore Links Between Algebra and Arithmetic

The following calculations appear on the screen one at a time without comment, with time for our students to consider each before the next appears.

$5+8$	5×19	25×33	$216-175$
$17\frac{1}{2}\%$ *of* £42.50	$6\div1.5$	$6\div2.5$	

They were asked to work on each individually and make notes about what they did. Next they were asked to compare their methods with their neighbours (there were six per table) and to consider whether they could draw out any mechanisms or principles.

After a few minutes they were asked for comments. The first contribution was about calculating 25×33 by 'multiplying by 100 and dividing by 4'. They called this 'compensation'. I asked for further examples of compensation strategies and these were offered:

Do $216-175$ by subtracting 200 and adding 25
Do 5×19 by multiplying 5 by 20 and then subtracting 5

Tutor: When is it helpful to use compensation?
Val: To break a complicated sum into something easier you can do in your head
Tutor: Let's be fairly specific about this. How do you recognise what is going to be easier?

A series of responses to this mentioned: multiples of 10 and 100, multiplication by single digits, single digits used as 'the adjustment', dealing with decimals by multiplying and dividing by powers of 10, familiarity and 'roundness' and 'splitting things into chunks of some bits that work'.

The discussion continued and touched on the usefulness of powers of ten, the use of 'known' facts and converting between percentages, decimals and fractions. After a few more minutes I asked them to take a few moments to consider whether they could link what had been said so far with things they had read or discussed earlier in the course.

Tutor: Anyone got anything to say?
Will: There is implicit use of the distributive law
Tutor: I thought I might have stopped you from seeing that by choosing 25×33

Will wrote on the board:

$$\begin{aligned}&25\times33\\&=(10+10+5)\times33\\&=10\times33+10\times33+5\times33\\&=330+330+165\end{aligned}$$

Andrew: When you *thought* it, did you think 'bracket ten plus ten plus five'?
Will: I didn't think 'bracket'
Tutor: Did anyone see that idea in any of the others?

Madena wrote on the board:

$$\begin{aligned}&5\times19\\&=5(20-1)\\&=5\times20-5\times1\\&=100-5\end{aligned}$$

Tutor: What if we had used Andrew's approach of $10+9$ is 19?

They nodded. Tansy wrote:

$$\begin{aligned}&17\frac{1}{2}\%\ of\ 42.50\\&=\left(10+5+2\frac{1}{2}\right)\times42.50\times\frac{1}{100}\\&OR=\left(10\%+5\%+2\frac{1}{2}\%\right)\end{aligned}$$

commenting "but it's not nice to write percentages inside brackets."

Sally wrote:

$$25 \times 33 = 25 \times (32 + 1)$$
$$= (25 \times 4) \times 8 + 25$$

Caroline added ‘I started with 25 squared’.

Tutor: So you are using known facts.
Andrew: There are ‘known known facts’ and ‘recently known facts’. For example in the $17\frac{1}{2}\%$ example from 10% you get 5%–it’s recent knowledge. This is different from ‘knowing’ 25×25.

At this point I referred the prospective teachers to a government publication about strategies for mental calculation and we moved on to consider written calculation.

One of the main purposes of this session was to offer the possibility to see algebraic structure in arithmetic. I did this by:

- generating mathematical activity (asking them to do the calculations themselves);
- focusing on sameness and difference (by comparing similar methods for different calculations and different methods for the same calculation); and,
- prompting prospective teachers to connect their recent experience with past experience.

As a result many, perhaps all, were able to see a relationship between distributivity as a property of the number system and a variety of informal methods they had used to calculate. The examples presented made it possible to see the wide application of this structure, not just as the distribution of multiplication over addition.

Beyond this a number of ideas arose from individuals, thus becoming available for the group to work with in this session and subsequently, for example:

- Andrew’s question to Will about what he thought when doing the mental calculation (‘Did you think “bracket”?’) enabled a distinction to be made between informal use and formal expression of structure;
- the importance of ten and its powers in the arithmetic of our decimal system;
- the usefulness of being able to shift from one representation to another (here from percentage to fraction).

The prospective teachers also had the opportunity to experience the variety of valid approaches to the same calculation and learning with and from each other by comparing different ways of seeing.

Exploring the Meaning of Algebra

The prospective teachers were seated at tables in threes or fours. Each table was given a collection of slips of paper on which the following items (questions or expressions) were printed:

A	15 +16 = ☐	15 + ☐ = 31
B	$16 \times 7 = 10 \times 7 + 6 \times 7$	$(a+b) \times c = ac + bc$
C	Find the next term in the sequence **1, 4, 7, 10, ...**	Find the 100th term in the sequence **1, 4, 7, 10, ...**
	Find the *n*th term in the sequence **1, 4, 7, 10, ...**	
D	I asked my grandma to tell me how old she was. She replied, 'If you multiply my age by 3 and then subtract 100 you get the same answer as if you took my age and added 34.' Find grandma's age.	Alan thinks of a number, multiplies it by 7, then adds 13 to the result. The final answer is 69. What number did Alan think of?
E	The spreadsheet formula **= A3*0.15 + 12.50** produced by typing on the keyboard	The spreadsheet formula **= A3*0.15 + 12.50** produced by clicking on A3, and typing
	The spreadsheet formula **= A3*0.15 + 12.50** produced by dragging a formula from a higher cell	

They were asked to negotiate with each other in order to separate the slips into two piles 'Algebra' and 'Not Algebra'. As they talked they tended to consider the items in the pairs or threes in which they are presented above. They also often preferred to talk about items being more or less algebraic rather than 'algebra' or 'not algebra'.

After about ten minutes of group discussion I asked for comments. The first offered was about the expressions in set A above and asserted that the second is algebra but the first is not.

Veronica: It depends on how you present it
Tutor: Does putting a letter in make a difference?
Veronica: No, it's because it is an unknown
Tutor: So what makes it algebra or not?
Andrew: Without the box the first one looks just like a problem
Tutor: If you think of it as something to rearrange it is algebra, but otherwise it is just a 'sum'?

Soon after this Veronica summarised the discussion by saying that algebra is not marks on paper, but an approach that is taken to what is written.

The next comment was on the pair of expressions labelled B. Madena said that the first is a realisation of the axiom expressed in the second, so that neither of them is algebra if algebra is seen as something you do. This allowed us to contrast two meanings of algebra, that is the task of manipulating symbols according to certain rules (manipulative algebra) as opposed to the study of the rules themselves (abstract or axiomatic algebra).

The next remarks were about set C. Saidah said that the first is not algebra because you can 'just add three'. The second is algebra because in order to find the 100th term you need to know how to find the nth term.

Tutor: If you do it by keeping on adding three is it algebra? Or if you add 96 times three is that algebra?

Madena: It depends whether you see algebra in the structure or as something you do. $10\times7+6\times7$ is algebra depending on what you focus on.

David: If they are adding 96 times three they might just see it like that—it's not necessarily algebra—it depends on what they do with what they see

Tutor: So you mean that having structure is not enough. Can you say a bit more?

David: Algebra is about letters, unknowns, generalising, so for me the nth term is algebra, but not the 100th term.

Tutor: You are talking about how they **express** the generality. Would the person who says that you add 96 times three be able to give the 102nd term or the 99th? This is a test of whether they **see** the generality?

In discussion of the 'word problems' (row D) Alastair said, 'The one you can "undo" is not algebra.' Madena added that writing down the expression (perhaps as a function machine) is the same thing as doing it in your head. In the continuing discussion we agreed that neither of the problems 'is algebra' but that algebra provides methods to solve either of them.

The discussion of E was curtailed by shortage of time, only allowing for a brief mention of the difference between using a 'label' consciously or unconsciously.

Later in the same session our students were offered experiences through which to consider the differences between uses of letters as unknowns, variables and generalised numbers. They also were introduced to the six uses of letters identified by Küchemann (1981)[2], to research on understanding of the 'equals' sign and to Gray and Tall's (1994) notion of 'procept'. The next day was spent considering some curriculum materials for teaching and learning algebra. They were asked to work in groups to comment on the materials using the mathematical distinctions they had developed the previous day.

The main intention of the card sorting activity described above was to broaden the prospective teachers' understanding of what might be meant by algebra. Madena

[2] The book from which this comes is a set text for the course.

and David presented two points of view, namely that algebra (at least at school) is about manipulation of expressions involving letters and that expression using letters is the distinguishing feature of algebraic activity. During the discussion these ideas were explicitly challenged by several assertions that algebra is not what is written, but the way in which we think about what is written. The idea that the structure of a problem, relationships between quantities, and generalisation, can be the drivers for algebra was made available. In addition common mathematical experiences were offered from which a language of distinctions could be derived. Our students were also offered another opportunity to experience the usefulness of looking for similarity and difference between mathematical entities.

Coda

The three examples above illustrate our general approach to using mathematical tasks to promote complex thought about what it means to do and learn mathematics. Because this approach is sustained by us throughout our teaching, prospective teachers are being enculturated into ways of thinking about teaching mathematics which persist, by and large, when they are in school. At the start of our course, it is usual for them to want to exchange stories about what they have seen teachers and learners doing in school, whatever the task we give them and whatever we hoped the focus would be. By offering tasks organised to challenge their own instant responses, we support the development of habits of probing mathematical meaning as the starting point for thinking about teaching, rather than trawling memory for associated stories.

We are not claiming that all our students sustain this approach all the time—that would be too hard. However, when we observe them teaching in school and ask about their planning and in-the-moment decisions it is clear that the majority start from wondering about how their students are going to learn and structuring what they do to support this, rather than adopting 'tricks of the trade'. We have little knowledge about how many of them sustain this once they are in their first posts, but we do know that in some of our partnership schools the culture of the mathematics department is to think first about learning, and then about tasks, sequences of tasks and 'coverage'.

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Didactical Variability in Teacher Education

Jarmila Novotná and Bernard Sarrazy

Introduction

Learning mathematics is successful only when the learner is able to identify conditions for the use of algorithms, to take one aspect, in new situations. These conditions, however, are not present in the algorithms and cannot be transferred directly from teachers to their learners. This is one the paradoxes of the didactical contract: "The more the teacher gives in to her demands and reveals whatever the student wants, and the more she tells her precisely what she must do, the more she risks losing her chance of obtaining the learning which she is in fact aiming for." (Brousseau 1997, p. 41).

How can we explain why some students[1] show an ability to use taught knowledge in new contexts, while others although familiar with the taught algorithms are not able to do so? Without a suitable model, these differences are attributed to different individual personalities, to their cognitive skills, or simply to the mysterious mental properties for which teachers have no didactical tools for further transformation or development. Sarrazy (2002) presents such a model: he explains these differences as an effect of the teachers' didactical variability in the domain of setting problem assignments.

Sarrazy's model is based on the following idea: The more versions of realisation a particular form includes, the more uncertainty is attached to this form. To satisfy the teacher's expectations, the student must 'examine' the domain of validity of his/her knowledge much deeper than a student who is exposed to strongly ritualised (repetitive) teaching and therefore considerably reduced variability. In other words, a strongly ritualised teaching lets the student know in advance what he/she must do and thus to behave in an appropriate way. However, as soon as the introduced

[1] In the text the word student refers to school students and teacher trainees to pre-service teachers.

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routines are interrupted the students cannot rely only on their cues (e.g. semantic indicators, triggers) and therefore can neither anticipate nor master the behaviours expected by the teacher.

In Sarrazy (2002) and Sarrazy and Novotná (2005), it is shown that the teachers' variability in the domain of setting problem assignments may be the variable explaining the significant differences in the number of models of the assignment structure created spontaneously by students.[2] This is in accordance with the fact that the students' results differ when they are asked to reproduce only the reference language presented by the teacher from when they get acquainted with several reference languages or even use their own reference languages. In the latter two cases their results are better. In addition, these two cases support the development of the students' cognitive skills and psychological characteristics, mainly their ability to analyse critically and their consciousness of responsibility for their own activity.

Moreover, we believe that the analysis of models created by students enables the teacher to help them when their effort to solve the problem correctly is not successful; in particular, it helps the teacher in determining the type of obstacles the student has faced. This point is elaborated in more details in Novotná (2003).

Our research supports the idea that the more the teacher creates (whether consciously or voluntarily) contextual variations in the organisation of teaching, the more students are guided to meditate upon the content of teaching beyond the formal characteristics of lessons. To illustrate the point, let us imagine a teacher who, after a lesson on multiplication, assigns to his/her students three problems on multiplication; if the teacher behaves systematically in the same way, students will behave economically: The lesson is on multiplication; therefore the problems are to be solved by multiplication! It is not surprising that the students determine the type of calculation according to the semantic indicator ("share", divide) or to the signal ("altogether", addition; "remainder", subtraction; "everybody", division; etc.). It corresponds with the following French proverb: "When the boss points to the moon, the ox sees the finger". On the other hand, if the teacher decides to get his/her students acquainted with several types of reference languages, he/she should be aware that there are not only positive consequences, but also negative ones. One of the important considerations is an increased uncertainty in the less able students who, besides their doubt in their ability to solve the problem correctly, also face the uncertainty about which reference language enables them to solve the problem (Novotná and Sarrazy 2005).

In the following paragraph, the index of teachers' variability developed by Sarrazy is introduced. Then, the consequences of the research results for mathematics teacher training are discussed.

[2] The following terminology is used: *Coding of word problem assignment* is the transformation of the word problem text into a suitable system (*reference language*) in which data, conditions and unknowns can be recorded in a more clearly organized and/or more economical form. The result of this process is called a *model* (in both cases—models taught by teachers or models as results of the inner need of the solver). The reference language contains basic symbols and rules for creating a model. There exist different reference languages for any one type of word problem.

Teachers' Variability

Sarrazy (1996) introduces a model sensitive to differences in students' treatment of problem types. This model facilitates describing the modes of teachers' actions through the following three dimensions:

1. Didactical structure of the lesson (what the teacher really does from the perspective of the knowledge to be taught);
2. Forms of social organisation (the teacher's activities regarding class management);
3. Variability of arithmetical problem assignment.

As far as this set of three domains is concerned, six variables are defined in order to measure variability in organisation and management of the teacher's work during and between lessons:

1. *Didactical structure of the lesson*

v_1 Type of didactical dependence: Does the teacher proceed from simple to more complex tasks or the other way round?

v_2 Place of institutionalisation in the sense of Brousseau (1997): At which moment does the teacher present a model of how to solve such problem? Closer to the beginning or to the end of the lesson? Or only at the beginning or at the end?

v_3 Types of validation: Are the students informed about validity of their answers? Does the teacher always use the same type of evaluation and assessment (e.g. through the milieu, by direct evaluation, by the Topaze effect[3], by peers).

2. *Social organisation (How are exchanges in the classroom organised?)*

v_4 Interaction modes: teacher-student(s), student(s)-student(s), etc.

v_5 Management with regard to the students' groupings: the whole class, small groups, individual work, etc.

3. *Variability*

v_6 The variable is related to editing the problem assignment. It is given by an indicator which measures the teacher's "capacity" to consider diverse modalities of the same didactical variable in the assignment.

Let us recollect here: (1) The rate of variability in the case of word problems is a suitable tool for broadening the register of variables that have the potential to influence the difficulty of the solving procedure. (2) This measure makes it possible to distinguish between the levels of variability of individual teachers. (3) The more a teacher shows the ability to see different ways of wording

[3] ***Topaze effect.*** When the teacher wants the pupils to be active (find themselves an answer) and when they can't, then the teacher suggests disguises the expected answer or performance by different behaviours or attitudes without providing it directly. Example: Teacher: 6×7? Pupils: 56. Teacher: Are you sure?

a problem, the better the teacher can cope with an unexpected and incorrect response from a learner, for example via a “didactic artefact” by immediately finding (“off the cuff”) a counter-example to invalidate (or develop) it. For more details about the construction of this variable, its epistemological status and empirical results whose establishment it enables, see Sarrazy (2002).

Consequences for Teacher Training

The crucial question concerning teacher training is “What professional skills, what attitudes are to be acquired for teaching of mathematics?” Restricted to the teachers’ capacity for variability, the question changes into “What professional skills, what attitudes are to be acquired for developing higher variability in teaching?” Learning to teach requires a balance between teachers’ theoretical and practical knowledge and skills. Therefore the answer to the above posed question has two sides. It covers knowledge of psychology and pedagogy on the one hand and the knowledge and skills supporting the preparation of suitable didactical situations on the other. Our experiences as teacher trainers with both prospective and practicing teachers confirm that there are two crucial phases in the process of developing teachers’ variability: the phase of constructing and solving a mathematical model of the assigned problem and the phase of the *a priori* analysis of the didactical situation. The *a priori* analysis is an important instrument enabling the teacher to manage the didactical situation in all its parts—devolution, a-didactical situation and institutionalization (Brousseau 1997).

Example of an Activity Aiming to Developing Prospective Teacher’ Variability

The following activity was used in training secondary mathematics teachers at Charles University in Prague, Faculty of Education. The activity was a part of the programme of training teachers for teaching mathematics through a foreign language (Content and Language Integrated Learning—CLIL (Favilli 2006)). This two-semester pre-service teacher training course is opened to students from the second year of their studies. It has a form of a seminar, two teaching units per week, with many activities run in the form of a workshop.[4]

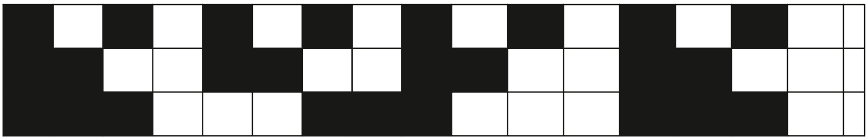
[4] The course was originally designed for teacher training of prospective teachers of mathematics and English language. It is run in English. Regardless of this fact, also students who are prospective teachers of other non-language subjects and foreign languages (and moreover, not only language specialists) participate. This feature enriches the course in the multilingual perspective. It is easily adaptable for the practicing teacher training.

Prospective teachers were asked to choose a mathematical topic to be developed at lower secondary level. At first, they worked with traditional materials, later they decided to adapt one of them and develop it into a lesson plan. The lesson was first simulated in the teacher training course in the form of peer teaching, and later taught in a real classroom.

Original materials come from Mathematical Rally Transalpine (see http://www.rmt-sr.ch/archives.htm):

Bizarre colouring

Maxime is filling in a square grid. In each line, the rule of colouring is different:



He has already filled correctly the first 15 columns. He states that the columns 1, 9 and 13 are fully filled. He continues with column 16.

Will column 83 be fully filled? And column 265?

Explain how you have found the solution.

The mathematical topics within the task included solving word problems, patterns combining arithmetic, algebra, and combinatorics.

The first phase of the activity took place as peer teaching in the training course with the participation of ten prospective teachers, 22–25 years of age. It took place in 45 minute training session during four successive weeks. The programme covered:

- *a priori* analysis of the text of the presented problem which included discussion from the perspective of possible mathematical solutions and the language of the assignment)
- preparation of the lesson.

Reflecting on and analyzing the training lesson: Trainees presented critical remarks both about the wording of the problem and the execution of the lesson plan. The necessity to change the assignment in order to reduce the algorithmic nature of the problem was emphasized. The trainees volunteered to prepare some new teaching material that would better correspond with the learners' age and interests. It resulted in the "Fashion World Magazine" (for the extract, see the Appendix).

The second phase of the activity took place in the classroom in a secondary school in Prague. It took 45 minutes. Both, the original version and the "Fashion World Magazine" were used. The third phase—*a posteriori* analysis of the lesson—took place again in the training course. The discussion was based on observations and the video recording of the whole lesson. The items discussed were: lesson analysis, comments, critical remarks and suggestions for alternatives.

During the process of material adaptation, the trainees modified both the context and the wording of the original problem. The new version offers variability in dealing with the original assignment (see the Appendix). Three different perspectives of dealing with the original situation motivated the trainees to try to increase their variability (see an extract from the “Fashion World Magazine” in the Appendix).

We believe that by means of classroom observations and subsequent analyses, the trainees are encouraged to look for important characteristics of good teaching strategies. Our observations of further work in the course and the prospective teachers’ consequent school practice indicated that their use of variation in problems had increased.

Discussion

The example presented above, and works on didactic diversity[5], make it possible to show the significance of teachers’ variability for the improvement of teaching of mathematics, especially in the perspective of flexibility in the use of taught algorithms.

In the following text, the question discussed is: Is it possible to increase teachers’ variability through training it? This is a legitimate but difficult question. Its difficulty lies in the fact that it cannot be answered directly. In fact, as far as we know, research that would give an answer does not exist. In the text below, we first examine some hypotheses that are related to the effects of the flexibility in students’ use of algorithms; then some propositions for teacher training are presented.

Interpretation of Effects of Variability

How can the effects of didactical variability on students’ achievements be explained from the perspective of knowledge decontextualisation?

This question is investigated from three perspectives:

a1—*Psychological interpretation:* For Richelle (1986) or Drévillon (1980), variability gives priority to the change of students’ operational register by diversifying their relationship to the object of teaching or to their action. In fact, the diversity of modes of relationship to the object of teaching, which is typical for didactical environments with strong variability, brings in an alternation between the phases of knowledge integration and differentiation in their usage. For example, let us imag-

[5] See e.g. Martel (1999). Culturally colored didactics: The sociopolitical at the heart of second/foreign language teaching in Francophone geolinguistic spaces. *Instructional Science,* Vol. 27, Numbers 1–2, pp. 73–96. or http://www.e-learning-baltics.de.

ine two categories of didactical environments which are in contrast in the perspective of variability: in the first category typical for weak variability the classical structure of an additive problem (e.g.: the initial state, the transformation and the final state) is presented with a relative stability—the question posed to the students asks for the final state; in the category developed didactical variability, the question will in different moments ask for the initial state, or the transformation, or the final state. It is clear that the relationship of the students in the latter category to an additive structure will grow stronger and will be diversified; on the contrary, in case of weak variability the students will not feel the need (or will feel it in a very limited extent) to examine relationships between the given numerical data because for these students each linguistic structure has the same model of arithmetical solving.

This alternation facilitates coordination and differentiation of operational schemes—their importance in the development of hypothetic-deductive thinking was shown by Piaget (1975); as a consequence, students would possess a plurality in their access to objects that would be efficient to help "not only to proceed to the operational formal stage but to construct a repertoire of cognitive registers. This repertoire enables a student, if asked or if it is needed, to examine a problem and solve it at the functional level, i.e., practical and objective, or to extract the operational quintessence and thus to construct a more general activity model" (Drévillon 1980, p. 336)[6].

According to Piaget (1975, 1981), it is also possible to consider variability as one of the sources of perturbations resulting from variations of didactical environments; this variability provokes cognitive adaptations (accommodations) and thus increases the student's cognitive register in relation to a conceptual field—e.g., additive and multiplicative structures studied by Vergnaud (1979, 1982, 1994).

A précis of this first aspect as considered from a didactical position can be formulated by changing the frameworks proposed by Douady (1986) in the theory of "dialectic 'tool-object' (outil-objet)": "A student possesses mathematics knowledge if he/she is able to provoke its functioning as explicit tools in problems he/she must solve […] if he/she is able to adapt it when the normal conditions of its use are not exactly satisfactory for interpreting problems or for posing questions with regards to it"[7] (Douady 1986, p. 11).

a2—*Anthropological interpretation:* It is also possible to interpret variability effects in relationship to what could be called the "school culture" of the class. Then vari-

[6] Translation from French by J. Novotná. Original text: « non pas seulement à passer au stade opératoire formel mais à construire un clavier de registres cognitifs. Ce clavier permet à la demande, et en cas de besoin, d'examiner un problème et de le résoudre au niveau fonctionnel, c'est-à-dire pratique et objectif, ou d'en extraire la quintessence opératoire et de construire ainsi un modèle plus général de l'activité. »

[7] Original text: « Un élève a des connaissances en mathématiques s'il est capable d'en provoquer le fonctionnement comme outils explicites dans des problèmes qu'il doit résoudre […] s'il est capable de les adapter lorsque les conditions habituelles d'emploi ne sont pas exactement satisfaites pour interpréter des problèmes ou poser des questions à leurs propos ».

ability creates a characteristic of the environment in which students develop and learn mathematics. On the other hand, knowledge of mathematics that students really learn (e.g. to solve an equation of the first degree, compare two fractions with the same denominator, divide rational numbers), may also be learnt without being it taught. This happens for example in creative, reflexive, ritualised and other activities. In other words, students learn mathematics that they are asked to *do*. On their own they learn to manipulate mathematical contents, to consider them etc. For details about preparation of such activities, see Brousseau's Theory of didactical situations. In order to adapt themselves to the usual teacher's demands, the student develops strategies of *coping* (Woods 1990) trying to answer in accordance with the criteria usually used. It is not unreasonable to think that repetitive teaching, poorly varying in its forms of organisation and in the content, leads the students to a hyper-adaptation[8] of the proposed situations. An example of this is when students can easily detect indicators that allow them to adapt their decisions and their behaviour to their teacher's didactical requests. In that case, students can easily apply suitable behaviour without really understanding the sense of the lesson or of the problem they were assigned. Alternatively, in cases of strong variability, the student cannot rely solely on activity ritualisation because he/she can neither anticipate nor manage the succession of sequences or behaviours expected by the teacher. As stressed by Bru (1991, p. 163), with strong variability, simulation becomes more difficult and the students' engagement in the situation is much more probable.

It is well known that a particular teacher's attitudes create the educational environment, a "climate", a special attitude towards "the life of mind" (Cookson 1988). In individual cases, this climate can support or, on the contrary, block students' future success in developing a productive orientation to learning. Flanders (1966) has shown the influence of teachers' ways of functioning on the "class climate". This climate was defined as "common attitudes that students have, in spite of their individual differences, with respect to the teacher and the class". According to Flanders, these attitudes are firm and influence the way the class functions. Some authors in the domain of didactics of mathematics, e.g., Perrin-Glorian (1993) or Noirfalise (1986), support this observation.

The experience of the authors of this text has shown that, according to their methods of class management, some teachers focus their teaching rather on the content that is to be taught while others prefer to focus on their students. The first mainly look for progress in subject matter and gaining new knowledge, and they appreciate all attitudes with which the students manifest their interest in what they are taught; whereas the latter privilege their relationships with students and the relationships between students. In other words they prefer the production of ideas and communication among students. According (Perrin-Glorian 1993) and (Noirfalise 1986), achievements obtained by students differ significantly according to the con-

[8] Hyperadaptation is a term which refers to features overly well adapted to their present function (Rudnick, D., Burian, R.: Hyperadaptation—Another Missing Term in the Science of Form. http://scholar.lib.vt.edu/theses/available/etd-6797-111852/unrestricted/etd.pdf)

sidered domains: focus on the content appears to favour success in algebra while a focus on students leads to better results in geometry and to making mathematics more attractive for the student.

a3—*Didactical interpretation:* As mentioned in a1, Douady's results (1986) allowed us to clarify the processes which enabled us to report the effects of variability. This research is done in two frameworks: Theory of conceptual fields by Vergnaud (1990) and Theory of didactical situations by Brousseau (1997). For Douady, teaching a mathematical concept requires a transformation, on the completion of which there may even be a rejection by the students of some of their previous knowledge. The assigned problems must be composed in such a way that the students have an opportunity to use at least one basic solving strategy. However, this strategy will be insufficient on its own: the taught knowledge (*object*) must correspond to the *tool* most suitable for solution of the particular problem.

Douady (1986) distinguishes 6 different phases constituting the process of the "dialectic tool-object":

1. *Phase a—Mobilisation of "former knowledge":* Corresponds to the phase of the problem adaptation by the student.
2. *Phase b—"Research":* Corresponds to the phase of action of the Theory of didactical situations (Brousseau 1997). During this phase, students encounter difficulties caused by the insufficiency of their previous knowledge and consequently look for new, better adapted instruments.
3. *Phase c—"Local explication and institutionalisation":* The teacher points out the elements that played an important role in the initial phase and formulates them in terms of the object with the condition of their use at the given moment.
4. *Phase d—"Institutionalisation"* (in the sense of the Theory of didactical situation—Brousseau 1997): The teacher gives a cultural (mathematical) status to the new knowledge and he/she requests memorization of current conventions. He/she structures the definitions, theorems, proofs, pointing out what is fundamental and what is secondary.
5. *Phase e—"Familiarisation-reinvestment":* It concerns the maintenance of what was learned and institutionalised in the various exercises.
6. *Phase f—"Complexification of the task or a new problem":* The aim of this last phase is to allow the students to make use of the new knowledge in order to allow new objects to occupy their position in the students' previous knowledge repertoire.

Douady (1986) states that the aim is to exploit the fact that most mathematical concepts operate in several frameworks—in fact in diverse types of problems. For example, a numerical function can be presented at least in three frameworks: numerical, algebraic, and geometrical. These changes of frameworks ("game of frameworks") allow varying the significances ("supports of significations") for the same concept and avoid the possibility of making them function in partial or over-contextualised ways. The interactions among diverse frameworks allow, according to the Douady, knowledge to progress while maintaining all the conceptual potential of the taught object.

Variability and Teacher Training

The previous section focuses on the advantages associated with variability in teaching mathematics. What remains is to determine the conditions for its development in teacher training.

As mentioned earlier, it is not a simple task; we do not know of any research on the question in the field of mathematics. The only works we know have been published are those of Bru (1991) about teaching written language.

In Bru's model of the variability estimation (not presented in detail in this text) eleven variables are used. The model enables to construct two variability indices (Bru 1991, pp. 122–123): (a) *Index of scheduled variety* corresponding to variations that the teacher is able to foresee when he/she is asked about his/her approach to the lesson, and (b) *index of realised variety* corresponding to variations really implemented when teaching.

Bru shows that when the realised variety is high, then the students' performances are significantly better than when it is low. Moreover, he shows that there is no correlation between the indices of scheduled or planned and realised variations. In other words, if a teacher plans a large number of variations, it is not sure whether he/she will really carry it out; nevertheless, and it is an important result, *the realised variety is effective only if it is associated with a high scheduled variety*.

This last point has direct impact on teacher training; it is very encouraging for orienting it in such a way that it allows teachers:

1. to foresee different modalities of action in the organisation of their teaching (having in mind phases of Brousseau's Theory of didactical situations or on Douady's game of frameworks of the dialectic tool-object);
2. to develop a better knowledge of theories which explain the effects on students' learning.

As already mentioned above, the development of planned variety is necessary but not sufficient for realising it efficiently in lessons. This is an area to be further researched, but the first results encourage us to think that teachers who realise variation in their teaching differ from others (those with a high scheduled - low realised variety) in the educational capacities they possess (e.g., equality, justice, success of all) and by they attempt to promote it in their teaching. Here we recover the ideal that Jan Amos Komenský raised in his fundamental work "Didacta Magna" (1631) but also a non-radical limitation of teacher training. For English translation see (Comenius 1967).

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Appendix: Extract from "Fashion World Magazine"

Show us your CREATIVITY and find your own pattern. BUT DO NOT FORGET. Your design must fit our offered SET!

And now a SUPERB OPPORTUNITY ONLY FOR YOU!

Find the right answers to our quiz, fill in the information sheet, cut it off and send!

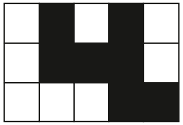
Everybody wins!

EACH CORRECT ANSWER = 10% DISCOUNT

Quiz

Question No.1

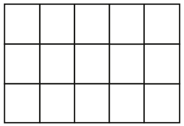
Is this a pattern from our set?



Question No.2

If our catalogue contained all the other patterns, what would the pattern no. 16 be like? (Blacken the appropriate squares!)

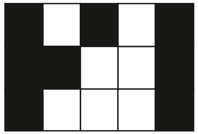
80 81 82 83 84



Question No.3

What is the number of this pattern? (If there are more possibilities, write ALL of them!)

? ? ? ? ?



Question No.4

Choose a pattern from the set and write the appropriate number.

THOSE WHO SOLVE THE QUIZ QUICKLY WILL GET A FREE T-SHIRT!!!

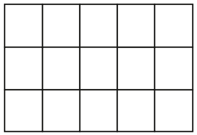
CUT OFF HERE

Name:
Surname:
Date of Birth:
Country:
Address:
Telephone number:
Your size:
Colour: Green Blue Red Grey Black
Pattern No.:

Answer No.1: Yes - No

Answer No.2:

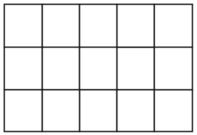
80 81 82 83 84



Answer No.3:

It is the pattern number:

Answer No.4:



I have chosen this pattern.
Its unmber is:

GOOD LUCK!

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Bridging Between Mathematics and Education Courses: Strategy Games as Generators of Problem Solving and Proving Tasks

Nitsa Movshovitz-Hadar

Introduction: The Problem of Bridging Between Mathematics Courses and Education Courses

In many universities the pedagogical, psychological, and didactic courses that constitute a part of the preparation for high-school mathematics teaching are given in the school of education while the mathematics courses are taken in the mathematics department, by students who are future mathematicians and have very limited interest in education. These courses are usually taught by faculty who possess little, if any, awareness of prospective teachers' intellectual needs, and the courses quite often have the structure of "theorem-proof" presentation of a theory.

It has been a challenge for teacher-educators all over the world to create a bridge between the mathematical content, to which prospective teachers are exposed in these university-level mathematics courses, and the pedagogical, psychological, and didactic issues involved in learning and in teaching high-school mathematics, to which prospective teachers are exposed in their education courses. Furthermore, mathematics-education faculty have a commonly agreed-upon goal of providing for a context in which future teachers can grasp the wide-scope nature of mathematics as a problem-posing/conjecturing and problem-solving/proving discipline, as well as the culture, beauty, and intellectual fulfillment of mathematics, so that they develop an enthusiastic attitude towards communicating these values to school children. Solutions to this challenge are eagerly sought. (For a discussion of the different ways in which mathematics needs to be known by teachers of mathematics as opposed to the ways in which mathematicians need to know mathematics see Ball and Bass 2004).

This chapter proposes a four-course series as a possible solution, and elaborates on one of the four.

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Bridging Courses—A Possible Solution

Four courses were specially designed to bridge between the subject-matter courses, pure and applied, taken towards a first degree in mathematics, and the pedagogical, psychological, and didactic courses taken towards the accreditation as a (junior and senior) high-school mathematics teacher. The four courses are non-sequential; all are problem-solving centered; and each is implanted in a different motivating context:

1. Mathematics problems that arise in the context of (strategy) games;
2. Mathematics problems that raise cognitive conflicts (paradoxes)[1];
3. Mathematics problems that had a significant impact on the development of mathematics throughout its history;
4. Mathematics problems related to applications of mathematics and mathematical modeling.

Experimentation

Following the development of preliminary syllabi, challenging activity tasks were drafted so that each one would be suitable for a 90-minute in-class activity and possible follow-up homework assignment. Each class session was designed to be independent of the previous ones, igniting new interest in the relevant activity, irrespective of a student's involvement in the preceding activities. A teaching approach that engages prospective teachers in group-work and reflective discussions was adopted and tried out in all four courses. Analysis of the data accumulated systematically in the naturalistic setting of the courses during the first two semesters of their experimental implementation served as the basis for modifying and improving the tasks, as well as the impact on prospective teachers' preparation for their future professional life.[2]

Assessment

Each student's evaluation and course grade was based upon three factors:

1. Active participation in class sessions;
2. Weekly homework associated with the activity of that week; and,
3. An individual term paper in the spirit of the particular course. Term paper preparation required student's 'minds-on' worksheet design, student's 'heart-on'

[1] Tasks related to the second course were published in Movshovitz-Hadar and Webb (1997).

[2] For publications related to the other courses, please refer to Kleiner and Movshovitz-Hadar (1994); Movshovitz-Hadar et al. (1994); Movshovitz-Hadar (1993a, b); Movshovitz-Hadar and Hadass (1990, 1991); Hadar and Hadass (1981).

experimental work with a group of high-school students, and an in-depth analysis of these experiences.

Theoretical Anchors

The theoretical anchors of such courses are many. The reader is assumed to be familiar with notions such as concept formation; cognitive conflicts; conjecturing and proving; constructivism; contextual learning; history of mathematics and its pedagogical aspects; human rationality; knowledge fragility; motivation and frustration; people's desire to win games; problem solving; spiral learning; setting of role models; the utilitarian value of mathematics; and value system development.

Preparing and Conducting Such Courses

The challenge in preparing such courses is threefold:

- Integrating the mathematical contents of university-level mathematics courses with the pedagogical, psychological, and sociological issues dealt with in the education courses, while revisiting high-school mathematics;
- Finding the right balance between friendliness and mathematical accuracy/rigidity; and,
- Providing a context in which future teachers can grasp the wide-scope nature of mathematics culture, its beauty, and its intellectual fulfillment, so that they develop an enthusiastic attitude towards communicating these values to school children.

It is of particular interest to note that ambiguity, contradictions, surprise, and paradoxes are the common thread of all the activities. These attributes as Byers (2007) suggests, are in use by mathematicians to create mathematics. Conducting such courses effectively requires a great deal of attention to students' fragility of knowledge, and requires coping with occasional frustration students may face during the struggle with problem-solving.[3] Nevertheless, in these courses prospective teachers experience intellectual as well as social mathematical courage (Movshovitz-Hadar and Kleiner 2009), which they will hopefully induce on their school-students.

The next section is devoted to details about, and sample tasks taken from the first course, namely: Mathematics problems that arise in the context of (strategy) games. Some more task-specific details related to task design, and class management strategies aimed at enhancing the underlying pedagogical principles, accompany the sample tasks.

[3] For a discussion of these issues as related to the second course see Movshovitz-Hadar (1993a).

The Case of Strategy Games as Generators of Problem Solving and Proving Tasks

A typical activity in this course is designed as a solo game or a two to four player game. A handout for each game describes the needed materials, the rules of the game, and the definition of winning the game. Following the instructions for the players, the handout includes a series of mathematical questions that come to life in the context of this game. Through solving these problems, prospective teachers are exposed to various problem-solving/proving heuristics, while looking at high-school mathematics from an advanced viewpoint.

Students' activity starts with free play, followed by problem solving, individually and, for more advanced problems, in groups, with possible whole-class negotiations of the solutions guided by the course mentor, who may assign a few parts for follow-up as homework, where appropriate. A reflection on the experience as a whole and on its values for prospective professional mathematics teachers closes the activity. In particular, the joy of problem solving and of learning 'serious' mathematics in the context of strategy games comes to the surface.

The list of games that constitute the syllabus of this course includes:

- Checker board jumps (Quadratics, geometric series, golden ratio)
- Dominoes (Combinatorial reasoning)
- Hex (Game theory)
- Magic tricks (Odd and even numbers)
- Map Coloring (Graph theory)
- Nim games—"Who gets first to 100?" (Number sense, recursion)
- Sir Pinsky's game (Chaos and fractals)
- Sitting and standing (Chinese rings, recursion)
- Sprouts (Strategy game)
- Tax Man (Prime factorization)
- The 15 game (Permutations)
- Tri-square rug game (Pythagorean triplets)

The rest of this chapter is an elaboration on the problem-solving tasks generated by two of these games: "Who gets first to 100?" (Nim games), and "Checker Board Jumps" (also known as "Kangaroo Game").

Sample Task 1: Nim Games

The games in this task require very limited background in mathematics, but they yield a set of problems that demonstrate processes typical of doing mathematics, such as: synthesizing mathematical findings (question 4), inductive inquiry and verification (questions 5, 6), and generalization from a collection of particular cases (question 7). When prospective high-school mathematics teachers are exposed to a sequence of such problems, which they usually find insightful and intellectually fulfilling, they may pick up the general idea and adopt it to their own task design. Making this explicit is the goal of the whole-class discussion that follows the task.

Sample Task 1: Student Handout

Who Gets First to 100?

How to play

Game 1

- Get in pairs, and decide who goes first.
- First player calls his/her choice of a number from 1 to 10.
- Players take turns adding any number from 1 to 10 to the previously-called number, announcing the sum only.
- The winner is the first one to reach 100.

Questions

1. Play several times and see if you can come up with a conjecture about a number x for which the following is true: "To be first to reach 100, it is sufficient to be first to reach x". What value of x did you find sufficient?
2. Can you find a lower value for x?
3. Does it matter if you go first or second?
4. Working with your partner for the game, try to develop a winning strategy, that is, a set of behavior-rules which guarantee that you will win independent of your opponent's moves. Verify your strategy by replaying the game. Write down your strategy clearly.
5. Once you have verified your winning strategy, change the target number to 150 and see if your winning strategy is still working. Play again. Can you make sure you win this new game? How? Verify your strategy by replaying the game.
6. Play again "100 wins" but this time change the rules so that players may add any number from 1 to a different agreed-upon number (e.g., 9 or 12, instead of 10). Can you make sure you win this new game? How? Verify your strategy by replaying the game.
7. Generalize the winning strategy to the case where your target number is T and you can add any number between 1 and N.
8. How do the values of T and N change the level of difficulty of the game?

Game 2

The same rules as in Game 1, except that the first person to reach 100 loses. Play repeatedly. Answer questions 1–7 modified for this game.

Discuss with your peers

- What did YOU learn while playing the games?
- Reflect upon the steps it took you to discover the winning strategy. Do you see why the process is commonly called “backward induction”?
- Consider the appropriate level of the game for age-specified school students.
- Look back at the structure of the task as a whole. Describe the pedagogy underlying its design.
- (Challenge!) Design a learning sequence for high-school students in a mathematics topic of your choice that leads the learners to new findings employing similar pedagogy.

Historical background

This game is a simple example of a group of games known as Nim games. Other versions of Nim games exist in the literature. Although Nim is known to be an old game, its origin is not well established. Charles Leonard Bouton of Harvard University, developed the complete theory of the game in 1901, see: http://www.jstor.org/stable/pdfplus/1967631.pdf.[4]

Sample Task 1: Classroom Management

In presenting games, it is generally not a good practice to assign reading the whole set of rules from a handout. It is much more engaging to students if the instructor starts by playing the game with the class and introduces the rules one by one while demonstrating how to play. It is advisable NOT to employ the winning strategy in the first demonstration game, but rather to let the class (or individual student) win. Here is a sample instructor-class dialogue for introducing the first game in Sample Task 1:

Instructor: We’ll play today a game called “Who gets first to 100?” Sam, will you play with me, please?
Student: OK.
I: I’ll let you go first, Sam. Chose a number between 1 and 10.
S: OK.
I: Tell me your number, please.

[4] One of the reviewers commented: In the form presented here (only one pile, no separation into piles), it was popularized if not invented by Henry Dudeney. Using 31 and numbers 1 to 5 forms the basis for a good deal of Guy Brousseau’s theorizing about didactic situations.

S: 7.
I: It is now my turn, so I add a number between 1 and 10 and tell you my sum—It is 12. Your turn, Sam. Add to 12 any number between 1 and 10 and tell me your sum.
S: 12 + 8 = 20.
I: Good. You don't have to tell me anything except the sum, 20. Now it's my turn, and I say 25.
S: Alright, I say 35.
I: 45.
S: 55.
I: 63.
S: 73.
I: 81.
S: 87.
I: 92.
S: 100.
I: Bravo, Sam—you won. (To class:) Now please get in pairs and repeat the game.

While students are playing, instructor distributes the handout, suggesting to students to follow the questions in the handout.

Sometimes after the first demonstration game, another student asks to play again against the instructor. At this point it would be challenging to the class if the instructor wins, alas. Without employing the whole strategy, just make sure to be first to get to 89. Winning again, against yet another challenged student would then set the stage for students to play in pairs.

Students now play in pairs for a few minutes and soon enough—sometimes immediately after the demo game, if the instructor is not careful enough in choosing the steps—they realize that 89 is a key stepping stone. That is, whoever gets first to 89 can get first to 100 by adding a number between 1 and 10. Hence the game reduces to "Who gets first to 89?". In subsequent rounds of the game, the race to 89 yields 78 as a key value, this then reduces the game to "Who gets first to 78?", and so on.

It is important to note that in any given class, there may be a few students who are familiar with Nim games. This should not inhibit the free play of others. Those few can proceed to the next questions in the handout. It is very unlikely that they know the answers to all of them.

Question 5 which asks for generalizing to "Who gets first to 150?" might prove to be a pitfall. Students who discovered that going first and choosing 1, 12, 23, 34, 45, 56, 67, 78, 89, is a winning strategy for getting first to 100, may extend it to seeing 100, 111, 122, 133, and 144 as the key stops along the way. But in fact, this approach will lead them astray when they attempt playing by this strategy. They will realize that generalization requires some more careful thinking.

Related Task-Design Issues for Sample Task 1

A significant part of this task is the first problem. Many students do not stop to recognize for themselves the recursive meaning of their finding about 89 as a key to winning. They keep playing over and again towards 89, without seeing this as a reduced version of the original game. Once they do realize it, they usually get quite easily down to 78, 67, etc. and finally to 1, which implies that in order to guarantee winning, one needs to be the player who goes first.

Question 6 was added in the revision of this handout, in order to facilitate the transition to the general rule, particularly to seeing that it is $N+1$ that needs to be repeatedly subtracted from T in order to get to the initial step. Game 2 was added in order to introduce a higher level challenge for those who claim they are familiar with game 1 and express a low motivation to play it. (Some may even claim that it is a "trivial game" once you know the winning strategy.) These students may find it satisfying to discover that the variation "the first to 100 loses" is equivalent to "The first to 99 wins". Furthermore, this small change implies a major difference in the winning strategy, because backwards analysis of the winning steps leads to the refreshing conclusion that the winner here is not the first player but the second one.

Students' Response to Sample Task 1

Although the mathematics in the Nim games is limited to simple arithmetic, students related to it favorably, appreciating the "thinking backwards" feature of discovering the winning strategy. "This game is not about drill and practice in addition and subtraction as I thought at the beginning", one of them commented. Quite a few students got really involved in the discussion of Question 8. One, who was impressed by the control over level of difficulty that a change in the target number implies, indicated: "Playing to 150 in adding 7s is a challenge even to high-school kids, and they must be able to do it without a calculator". Interestingly, those students who claimed familiarity with game 1 to start with, and were therefore referred to Game 2, became quite involved in it, and actually found themselves back to Game 1, with no complaint. One of them even admitted: "It was a nice trick to assign Game 2 to us. It made me realize how much I do *not* know about Game 1…".

Sample Task 2: Checker Board Jumps

This activity consists of three handouts. The first introduces the game. The second includes a set of immediate questions about the game in an increasing order of difficulty, and the third is a guided discovery to a surprising result embedded in the game, and to its proof. Solving the problems, or at least reading their solutions, may help the reader see the value of these tasks for prospective teachers, as discussed later on.

Sample Task 2: Student Handout 1 (of 3)

Checker Board Jumps[5] (Alternative name: Kangaroo Game)

Materials for each player
- Two regular 8×8 checker boards placed end to end;
- A set of 24 identical chips;
- Graph paper, a pencil, and an eraser.

Number of players
This is a solo game, with a 2-player optional version. See comments below.

Object of the game
The object of the game is to move chips by legal jumps from the 8×8 Starting Board onto the adjacent 8×8 Target Board, scoring as many points as possible. (It is advisable to make a record of the initial setting and the moves.)

How to play

At the start
Designate the board near you as the Starting Board, and the farther one as the Target Board. (See Fig. 1).

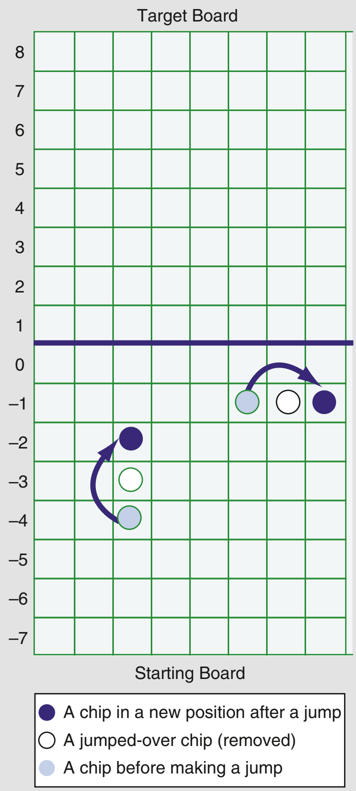
Select any number of chips to start playing with, and place them one by one in different boxes anywhere you like on the Starting Board. Move one chip at a time according to rules.

Rules
- A chip can only move by jumping over another chip to an empty space on the Starting Board or on the Target Board. Jumps may be performed either vertically or horizontally, but not diagonally.
- A jumped-over chip is removed from the board.
- The game ends when no legal move can be performed.
- The final position of the chips is then scored by row position on the Target Board as follows: 1 point for one chip (or more) in the 1st row; 2 points for one chip (or more) in the 2nd row; 4 points for one chip (or more) in the 3rd row; 8 points for one chip (or more) in the 4th row scores; 16 points for one chip (or more) in the 5th row scores; 32 points for one chip (or more) in the 6th row; 64 points for one chip (or more) in the 7th row scores; and 100 points for one chip (or more) in the 8th row.

Your job
Read the rules, then play and record the number of chips you start with, their Starting Board arrangement, and the highest score you were able to reach with them on the Target Board.

[5] Development of this activity was inspired by Honsberger (1976).

Fig. 1 Layout of the two boards and sample jumps



Comments

This game can easily be turned into a two-player game. Each player plays with a different set of colored-chips. The two players play on the same pair of jumping boards, sitting at opposite ends, defining the Starting Board of one player as the Target Board of the other one, and vice versa. The two players play the one-player game with their own colored chips, taking turns in performing one legal move at a time upon their own chips. There are two versions for a two-player game, as described below. You can have students decide with their partner the version they wish to play:

1. The players agree on the outset on the number of chips they both start with, and they each display this number of chips on their Starting Board in any layout they each like. During the game it may happen that two chips, one of each player, are in the same box, but each player may touch only his/her own color chip. The game ends when one of the two players has no way of making another legal move. The winner is the one who reached farther on his or her Target Board. (The game can end in a tie, that is, nobody wins.)

2. The players agree on the target row the winner has to reach. Each is free to decide how many chips to start with and how to display them on the Starting Board. The winner is the first one to reach the target row. If both reached the target row in the same number of moves, the winner is the one who started with a smaller number of chips. (The game can end in a tie.)

Sample Task 2: Student Handout 2 (of 3)

Checker Board Jumps

Part 2

After playing for a while, answer the following questions

(a) (Easy). What is the least number of chips on the Starting Board that will allow a chip to reach the first row on the Target Board? Show how those chips should be arranged at the outset.
(b) What is the least number of chips on the Starting Board that will allow a chip to reach the second row on the Target Board? Show how those chips should be arranged at the outset.
(c) What is the least number of chips on the Starting Board that will allow a chip to reach the third row on the Target Board? Show how those chips should be arranged at the outset.
(d) Based upon your findings so far, what would you expect the least number of chips on the Starting Board to be, so that it would be possible for a chip to reach the fourth row on the Target Board? Check your conjecture. How accurate was it? What is the actual number of chips with which you were able to reach the fourth row? Show how they were initially arranged.
(e) (Challenge!). How many chips do you think it will take to get to the fifth row on the Target Board? See if you can confirm your conjecture by playing this number of chips in some different initial arrangements.

Solutions to the Problems in Student Handout 2

(a) The smallest number of chips needed to get a chip to the Target Board is two, arranged in the same column and in rows 0 and −1.
(b) Four chips are needed to reach row 2 of the Target Board: 3 adjacent chips in row 0 and one chip in row −1 below the extreme left or the extreme right chip in row 0. Then three jumps are needed to get a chip to the Target Board as illustrated in Fig. 2.
(c) Eight chips are needed to get to row 3: five in row 0 and three more in row −1. The first moves are identical to those needed to get a chip to row 2 as shown in Fig. 2. For this, four chips are needed. Four more chips are necessary to get

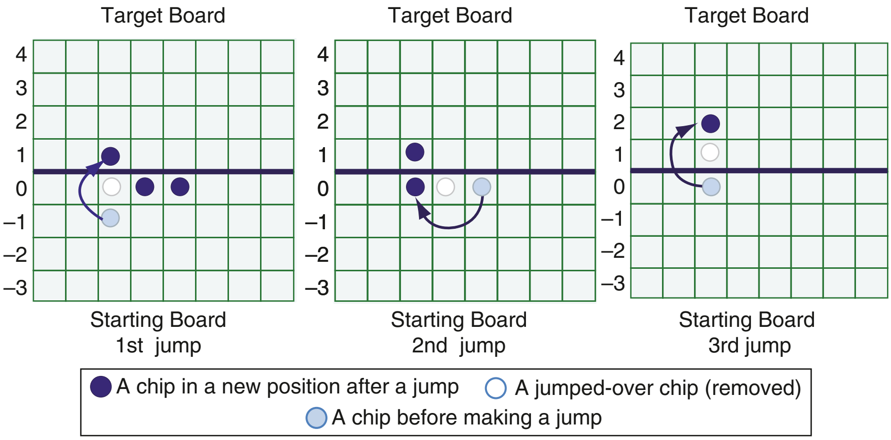


Fig. 2 Initial layout of four chips on the starting board, and the three jumps needed to get a chip to the 2nd row of the target board

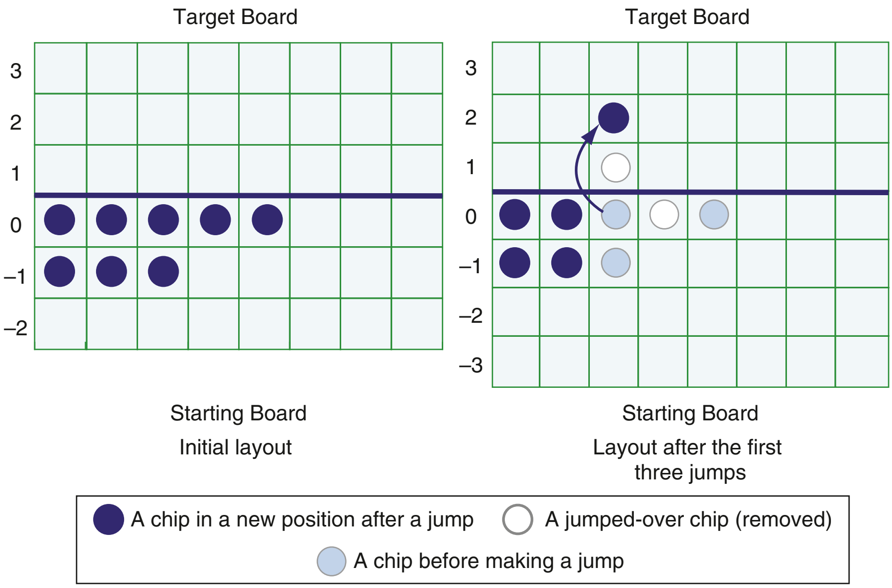


Fig. 3 Initial layout of eight chips on the starting board, and the position after the first three jumps (four more jumps are needed to get a chip to the 3rd row of the target board)

a chip to the third row, afterwards. The illustration in Fig. 3 is an example of a possible arrangement of the eight chips (other arrangements are possible).

(d) On the basis of the fact that to reach row 1, two chips are required at the outset, to reach row 2, four (2^2) chips are required, and to reach row 3, eight (2^3) are needed, one might expect that $2^4=16$ chips would be needed to start with, in order to get to row 4. Actually the least number is 20. Starting with the initial configuration described in Fig. 4, the first 12 jumps are as follows:

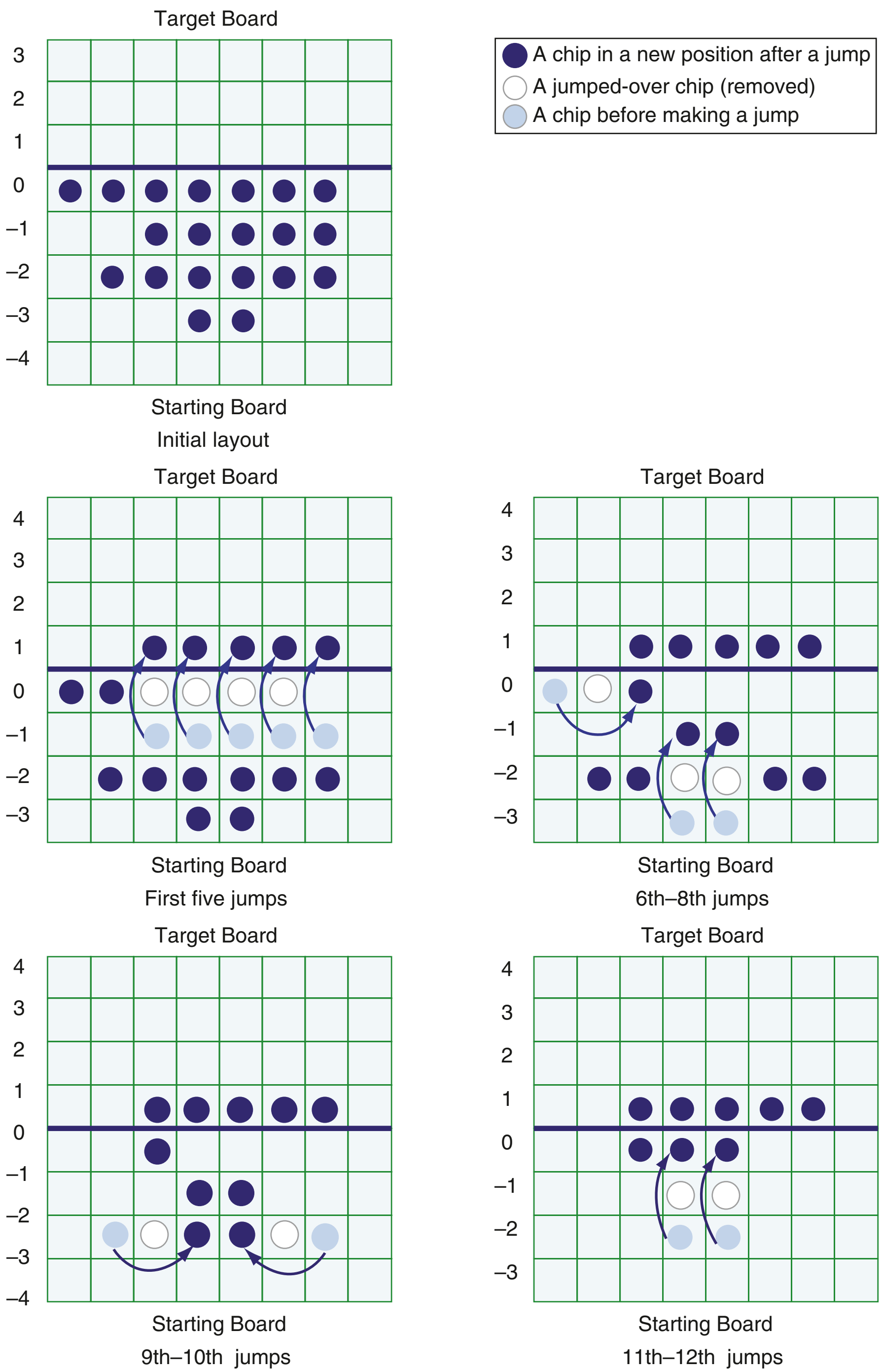


Fig. 4 Initial layout of 20 chips on the starting board, and the first 12 jumps needed to get eight chips arranged in row 0 and 1 to enable a chip to get to the 4th row of the target board

The first twelve moves get us back to the initial configuration of eight chips by which we showed a chip can reach the third row. Since this eight-chips configuration "sits" one row higher, it can get a chip to the fourth row, in a process similar to that shown in answer c. (Note: This does not prove that this is the smallest possible number of chips).

(e) Although the process as described up to here seems to be recursive, it is not. There is no way one could get to the 5th row!!! The proof, due to Conway, is within reach for a high-school student who has had experience with quadratic and higher simple polynomials whose coefficients are all 1 (geometric progression). The proof is introduced to students via a series of advanced problems in Handout 3.

Sample Task 2: Student Handout 3 (of 3)

Checker Board Jumps

Part 3: Advanced problems

(You may wish to verify your findings with your peers or ask for advice at any point.)

Let us get more mathematical, now. The mathematics to which this game gives rise involves quadratic and higher simple polynomials whose coefficients are all 1 (namely, geometric progression). By solving the following problems, you will see how.

(f) Let P designate a particular box on some row you want to reach. We define the distance from P to any other box to be the smallest number of unit-steps in parallel to one of the two axes, needed in order to get from P to that box.
We now assign a position value to every box: It is a power of x with the distance from P as its exponent. (For now, x represents a positive real number, which will be specified later). For instance, P itself has position value x^0. How many boxes have position value x^1? x^2?
Find the position value of all the boxes on the Starting Board and on the Target Board, if P is in the 3rd row and the 5th column of the Target Board.

(g) For a given target point P, let us define the value of a set of chips to be the sum of the position-values of their positions on the board at one time. You may note that this value may change if you perform a legal jump, because in any legal jump, two adjacent boxes are evacuated and a formerly empty box becomes occupied.
There are three different types of changes a legal jump can bring about for the distance of a chip from P:

  (i) It can bring that chip closer to P;
  (ii) It can bring that chip farther from P; and
  (iii) It can keep the distance of that chip from P unchanged

Familiarize yourself with these three kinds by performing examples of each one.
How does each type of move change the value of the whole set of chips?

(h) We now determine the value of x, such that jumps of type (i) will not have any impact on the sum. What is the (positive) value of x that keeps the total value of the set unchanged? In other words, for what value of x does the total change amount to 0?

(i) It is interesting to note that the solution to the previous question is related to the Golden Ratio: $(1+\sqrt{5})/2$.
Actually it is its reciprocal, and x is a positive number smaller than 1.
Now, that you have determined the value of x, such that type (i) jump will have no impact on the total value of the set of chips, consider how each of the other two types of jump will change the value of the set of chips. Will there be an increase, a decrease, or no change in the total value?
From the results you obtained, try to find a lower bound for the total value of the initial set-up of the chips, in order to reach the target point P.

(j) At this point we wonder if there is a row one can never get to, even if the number of chips at start is not limited, nor is the size of the board. To answer this question, consider an arbitrary point P on the 5th row as the target point, starting with an infinite Starting Board fully covered with chips. What would be the total value of this starting layout? (To get it you will have to recall your knowledge of polynomials and geometric series, and choose x such that the value of the sum doesn't change for a jump of the first type. You found this value in part 'g' above. Note that for that particular value: $x^2=1-x$). Now, keeping in mind the lower bound you determined in part 'i', prove that the fifth row cannot be reached, no matter how many chips you start with (and this is true even if the board is infinitely large!).

(k) Phrase the result as a statement you have actually proved through the above series of questions.

Discuss with your peers
Review the proof you constructed in answering items f–j. What properties would you attribute to this proof? Is it elegant? Which part(s) of it in particular deserve a credit of this sort? Which of the steps deserves credit for ingenuity? Would you attribute beauty to the proof? In what way? Is any one of the steps extraordinarily courageous?

Historical Background
This proof was discovered by the prolific mathematician John Horton Conway, born in 1937 in Liverpool England, now at Princeton NJ. Conway is known for many inventions of strategy games, notably Sprouts and The Game of Life. (For more details see: http://www-history.mcs.st-andrews.ac.uk/Biographies/Conway.html.)

Solutions to the Advanced Problems in Student Handout 3

(f) Let P designate a particular box on some row one wants to reach. We defined the distance from P to any other box to be the smallest number of unit-steps in parallel to one of the two axes, needed in order to get from P to that box. We then assigned a position value to every box to be a power of x with the distance from P as its exponent. Thus P has position value x^0 (x itself is yet to be specified).
There are four boxes with position value x^1, and 8 with position value x^2. The position value of all the boxes on the Starting Board and on the Target Board, where P is in the 3rd row and the 5th column of the Target Board, is given in Fig. 5

(g) For a given target point P, we defined the value of a set of chips to be the sum of the position-values of their positions on the board at one time. Since in any legal move two adjacent boxes are evacuated and a formerly empty box becomes occupied, by performing a legal move this value may change.
There are three different types of changes a legal jump can imply on the distance of a chip from P:

(i) Bringing a chip closer to P. This change implies gaining x^n while losing x^{n+1} and x^{n+2}, for some value of n, hence the total change of value of the set in this case is:

$$x^n - (x^{n+1} + x^{n+2}) = x^n(1 - x - x^2)$$

(ii) Bringing a chip farther from P. This change implies gaining x^{n+2} while losing x^{n+1} and x^n, for some value of n, hence the total change of value of the set in this case is:

$$x^{n+2} - (x^{n+1} + x^n) = x^n(x^2 - x - 1)$$

(iii) Keeping the distance from P the same. This happens if the jump is over the same row or over the same column in which P is positioned. The total change of value implied by this type of jump is:

$$x^n - (x^{n-1} + x^n) = -x^{n-1}$$

(h) We now determine the value of x so that jump of type (i) will not have any impact on the sum. To do it, we solve for $x : 1 - x - x^2 = 0$, and get

$$x_{1,2} = \frac{1 \pm \sqrt{5}}{-2}.$$

Considering the positive root we get

$$x = \frac{\sqrt{5} - 1}{2}.$$

Note that this is the reciprocal of the well known Golden Ratio: $(1 + \sqrt{5})/2$.

Fig. 5 : The position value of all the boxes on the Starting Board and on the Target Board, where P is in the 3rd row and the 5th column of the Target Board

Target Board

8	x^9	x^8	x^7	x^6	x^5	x^6	x^7	x^8
7	x^8	x^7	x^6	x^5	x^4	x^5	x^6	x^7
6	x^7	x^6	x^5	x^4	x^3	x^4	x^5	x^6
5	x^6	x^5	x^4	x^3	x^2	x^3	x^4	x^5
4	x^5	x^4	x^3	x^2	x^1	x^2	x^3	x^4
3	x^4	x^3	x^2	x^1	$P = x^0$	x^1	x^2	x^3
2	x^5	x^4	x^3	x^2	x^1	x^2	x^3	x^4
1	x^6	x^5	x^4	x^3	x^2	x^3	x^4	x^5
0	x^7	x^6	x^5	x^4	x^3	x^4	x^5	x^6
–1	x^8	x^7	x^6	x^5	x^4	x^5	x^6	x^7
–2	x^9	x^8	x^7	x^6	x^5	x^6	x^7	x^8
–3	x^{10}	x^9	x^8	x^7	x^6	x^7	x^8	x^9
–4	x^{11}	x^{10}	x^9	x^8	x^7	x^8	x^9	x^{10}
–5	x^{12}	x^{11}	x^{10}	x^9	x^8	x^9	x^{10}	x^{11}
–6	x^{13}	x^{12}	x^{11}	x^{10}	x^9	x^{10}	x^{11}	x^{12}
–7	x^{14}	x^{13}	x^{12}	x^{11}	x^{10}	x^{11}	x^{12}	x^{13}

Starting Board

(i) We have obtained a value of x that is positive and smaller than 1. For this value $x^2 = 1 - x$, hence the change implied by type (ii) jump is:

$$x^n(x^2 - x - 1) = x^n(1 - x - x - 1) = x^n(-2x) < 0$$

Namely, it decreases the total value. Type (iii) jump also decreases the total as $-x^{n-1} < 0$. Hence, to reach P (the value of which$=$1), one must get started from a layout that has total value ≥1. A set with value less than 1 would require an increase in its value in order to reach *P,* and this cannot be realized by any type of jump.

(j) As shown above, the 4th row is within reach by 20 jumps. We now consider reaching the 5th row. If one box on the 5th row can be reached, then any other box on that row can also be reached, as it only takes moving the initial layout to the left or to the right. Hence we consider an arbitrary point *P* on the 5th row as the target point, starting with an infinite Starting Board.
As stated above the total value of the starting layout must be ≥ 1.
Let us calculate the value of the half plane of the Starting Board by columns, starting from the points underneath *P.* As $0<x<1$ the value of this column is:

$$x^5 + x^6 + x^7 + \cdots = \frac{x^5}{1-x}$$

The value of each of the two adjacent columns is:

$$x^6 + x^7 + x^8 \cdots = \frac{x^6}{1-x}$$

Hence their total is $2x^6/(1-x)$. The next pair consists of $x^7 + x^8 + x^9 + \cdots$ each, adding to the total $2x^7/(1-x)$. Similarly, there are pairs of columns on each side of the column of boxes below *P.* Thus we get for the half plane the total value *S,* where

$$\begin{aligned} S &= \frac{x^5}{1-x} + 2\left(\frac{x^6}{1-x} + \frac{x^7}{1-x} + \frac{x^8}{1-x} + \cdots\right) \\ &= \frac{x^5}{1-x} + 2\frac{x^6}{1-x}\left(1 + x + x^2 + \cdots\right) \\ &= \frac{x^5}{1-x} + 2\frac{x^6}{1-x} \cdot \frac{1}{1-x} \end{aligned}$$

Since for the chosen value of x we have $x^2=1-x,$ we get

$$S = x^3 + 2x^2 = x(x^2 + 2x) = x(1 - x + 2x) = x + x^2 = x + 1 - x = 1.$$

Therefore, in order to get to the 5th row we must start with an infinite board in which all the boxes are occupied or else the total value will be less than 1. Now, any finite board, even a fully occupied one, is a part of an infinite board with empty boxes, hence there is no way of getting to the 5th row in a finite game. QED

Sample Task 2: Main Concepts, Skills, and Strategies

This activity involves two levels of un-anticipated results: One in part d, where students are asked to generalize from particular cases, a risk they should not take too lightly. The other one is in part e, where it becomes very difficult to achieve a success empirically, because it is actually impossible, as they will be guided to prove in Student Handout 3. (See the solutions section).

It also provides an opportunity to discuss the following important issues:

- Inductive patterns that do not necessarily generalize;
- Organizing and analyzing data;
- Proving something to be impossible;
- Ingenuity and beauty in mathematics;
- Intellectually courageous moves in mathematics.[6]

Sample Task 2: Classroom Management

In Sample Task 2, as in Sample Task 1 and in general, introducing the game itself is better done before the handout is distributed. In this case, preparing a game board transparency and demonstrating the moves using transparent colored circle cutouts works nicely. (Alternatively, one can use an erasable marker on a transparency where the boards are marked with non-erasable ink). Students' free play can take place with chips on a grid paper, or using pencil marks and an eraser to replace the chips and the jumps. Emphasize that a player may start with as many chips as he or she wishes, and the chips may be placed wherever the player wishes. Players may even extend the boards as much as they wish.

Upon distribution of the second handout, the first two problems can also be solved through discussion with the whole class. The rest should be left for individual or in-pair struggle. It is worth encouraging the less competitive students to get as far as they can on the board, even if the number of chips they start with is not the least possible number. A good goal is to do it. A secondary goal is to do it with less chips. The solution to the advanced problems in the third handout, in particular the last two, may require instructor's intervention, to clarify their solutions.

The most important part of this activity is the discussion that it yields of properties of the proof that follows the game. The sense of beauty and ingenuity this proof brings up are appreciated by the majority of the students and it is believed that it has an influence on their mathematical taste, as well as on their pedagogical ability.

In implementing tasks of this nature one should be careful not to "steal one's thunder", that is, leave students feeling proud of their own work, even if it is not fully polished. It is not necessary for the course instructor to always present the most elegant solution (Resek, D., 2009, Department of Mathematics, San-Francisco State University, personal communication.).

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[6] For an elaborated discussion of Intellectually Courageous Moves in mathematics, see Movshovitz-Hadar and Kleiner (2009).

Sample Task 2: Related Task-Design Issues

Honsberger (1976) presents a problem in checker jumping that inspired the development of Sample Task 2 above. For him, the problem is "to determine the least number of men in the starting zone which permits a man to reach a prescribed height above the x-axis", where the starting zone consists of the half plane of lattice points on and below the x-axis, and the object is "to get a man as far as possible above the x-axis". He then provides, in a few lines augmented by three illustrations, the solutions for the first three levels as 2, 4, and 8, respectively, and then gives the surprising result of the fourth level being 20 rather than 16 as might be expected. Now he introduces the question of getting to the fifth level, stating that: "Incredibly, no arrangement, with however many men, is sufficient to reach level five!" The rest of his exposition is devoted to establishing this incredible result.

It was quite a challenge to turn Honsberger's charming yet concise exposition into a problem-solving task that involves trial-and-error free play to start with, followed by guess-and-test, conjecturing, and proving activities. In doing so, one must be very careful about making things concrete, competitive/collaborative, and playful, while raising curiosity about problems that are inherent in the game and maintaining interest in solving them. It involves a process I call: *Pedagogical cracking of mathematical exposition.*

There are several principles underlying this "cracking" of mathematical exposition, which implies sequencing the steps into a well-structured game task that leads to a surprise, leaves enough freedom for exploration and proof, and emphasizes the ingenuity in the findings. The principles are these: (1) Design a game with clear rules and a well-defined state of winning; (2) dramatize the activity by providing for counterintuitive findings; (3) turn an expository proof into a guided-discovery learning task by breaking the exposition into short paragraphs each addressing one question, thus obtaining a series of questions and answers replacing the exposition; and (4) wherever appropriate, have students work on a "transparent" particular case, i.e., an example that is large enough to mirror the general case, yet is small enough to remain concrete. (For the definition of transparent proof, see: Movshovitz-Hadar 1988, p. 19. For its employment in teaching college mathematics, see Malek and Movshovitz-Hadar (2011))

Consequently, Sample Task 2 took a format of handouts at three different levels: The first one introduces the game (in two versions). The second one includes five questions, intentionally phrased in a very similar way. The first three are straightforward questions about getting to the first, second, and third row of the Target Board, respectively. Each is harder than its predecessor. The fourth question is about getting to the fourth row, yielding a surprising result as compared to the expectations created by the solutions to the first three. The last question is about getting to the fifth row. Having solved the previous four, students do not suspect the fifth row to be inaccessible. Since the fifth row cannot be reached, students clearly will have a hard time here. Needless to say, it is in the hands of the instructor to stop the work on the second handout before too much frustration is accumulated, but without letting out the secret that the last question on that handout is impossible to solve. The third handout starts as if it has nothing to do with the former. It is composed as a guided discovery activity, gradually leading the students to find out that getting to the fifth row is impossible, which brings us back to the last question on the second handout.

Construction of the series of problems started from reading Honsberger's exposition, but contrary to Honsberger's way of telling the results, the series of questions puts the student into an exploratory adventure. For example: Following the definition of a position value to every point in the plane, relative to a distinguished point *P,* Honsberger says:

> Thus *P* itself bears the value x^0 or 1; the four lattice points adjacent to *P* are valued at *x,* the eight lattice points which are two steps from *P* have value x^2, and so on. As a result the rows and columns of lattice points are assigned sequences of consecutive powers of *x.* (See Fig. 19)

Instead, the student's handout calls the student to find "How many boxes have position value x^1? x^2?", and then to get the full picture for a particular point *P:* "Find the position value of all the boxes on the Starting Board and on the Target Board, if *P* is in the 3rd row and the 5th column of the Target Board". Thus, students are active in clarifying the notion of position value, by working on a particular case which is large enough to reflect the general case but small enough to remain concrete.

Note that in the first handout, the game is described in a 'neutral' way, namely without disclosing a hint about the 5th row being inaccessible. Scoring is assigned to getting anywhere from the 1st to the 8th row of the Target Board. This is an example of what I call *poker-face pedagogy.* It appears again in the design of handout 2, where students are asked to conjecture about the number of chips it may take to get to the 5th row, giving no clue to the fact that it might be impossible. This same strategy guides phrasing questions in a "neutral" mode, namely, replacing the phrase "prove that" by "Is it true that…" followed by "If yes, prove it; if not, provide a counter example". This strategy helps reducing the student's dependence upon the instructor's verbal feedback or body language, and supports the development of the student's trust in logical reasoning.

Students' Response to Sample Task 2

Because this is a part of a course that consists of self-contained meetings, each focused on a new game, students come in curious to see what is waiting for them each day. The atmosphere becomes playful and cooperative very quickly. With respect to Sample Task 2, it is of particular interest to note the reactions of students to questions d and e in handout 2, before they realized that part e is impossible, and their discussion of these questions once they did realize it, having completed part 3 of the handout. Some of them may argue that it is somewhat 'unfair' to assign an impossible task, a few may even use expressions like "kind of cheating", and wonder about the message the teacher may send to class in using such a method of problem posing. An in-depth analysis of the resources for such doubts, possibly with opposite views expressed by peers who appreciate the "poker face" strategy of posing such questions, could lead to an acceptance of such an approach. Some participants may suggest school-level situations in which the teacher may assign similar tasks, e.g., to draw a triangle with 1 acute angle, 1 straight angle, and 1 obtuse angle (as a part of a drawing task of various triangles, before learning the angle-sum theorem, of course).

Main Points for Students' Discussion in This Bridging Course

In addition to task-specific points, prospective teachers have the opportunity, throughout this bridging course, to:

- Enjoy refreshing basic mathematical knowledge while playing and solving problems;
- Reflect upon their own experiences in playing strategy games;
- Discuss the breaking of a proof published in a book or a journal into a series of problem solving tasks; Creating good hints; What is it that makes a hint a good one?
- Discuss the merits of learning mathematics in the context of strategy games: Beyond the fun, is it sufficiently rich? Is it an acceptable approach for learning mathematics by the majority, or is it only appropriate for some (elite/disadvantaged) group?
- Consider: Problem posing, hypothesis generation and testing, representations, generalization from particular cases, verifying, defining and modeling, making connections, problem solving and proving, as parts of a mathematical activity.

Wrapping-up

This chapter describes one of four courses for educating prospective high-school mathematics teachers. All four are problem solving courses that consist of a series of stand-alone class meetings. They differ in the contexts that give rise to the problems: games; paradoxes, historically significant problems, and problems related to applications of mathematics and mathematical modeling. All four courses are aimed at bridging the gap between the mathematical courses—pure or applied—and the education courses—psychology, sociology, and philosophy of education—that prospective teachers are required to take in order to become professional high-school teachers. The former are usually taught by research mathematicians whose care for playing a role model of quality teaching is not always present. The latter courses usually are taught in heterogeneous classes of future teacher preparing for teaching various school subjects, by experts whose keenness about mathematics is not their pride, to say the least.

These bridging courses focus on challenge, curiosity, connectivity, and creative thinking, which until recently have been quite rare in the ordinary Algebra, Calculus, Analytic Geometry, and other courses in the mathematics department, but need to be present in high-school if implanting a positive image of mathematics is a preference. They augment the didactic courses (sometimes called Methods or Mathematics Pedagogy courses) in providing the prospective teachers with some exposures to mathematics that make their eyes sparkle, carrying the message that mathematics is a boxful of surprise on the one hand, and on the other hand that in doing mathematics investing effort, being persistent, and not giving-up easily yield a lot of intellectual satisfaction.

Pólya (1962, 1965/1981) envisioned the teacher's job to be to help students "discover by themselves as much as feasible" and develop problem-solving "know-how" (Pólya 1962, 1965/1981, p. 104). It is no secret that the basic university-level preparation towards the school teaching profession is only the initial step. It takes a lot of digestion, classroom experience, sharing of ideas, and further studies to become a knowledgeable teacher. Yet being knowledgeable is not enough. It is becoming a sensitive and creative teacher that will keep the professional teacher from being burned out. The bridging courses, one of which is described in some detail in this chapter, aim at giving the prospective teacher a wide umbrella of resources to hang on to, when in the future she or he might look for ways to enliven and refresh their mathematics classes, thus raising and maintaining their pupils' motivation to succeed in mathematics and making them find it intellectually rewarding. (For publications related to the other courses, please refer to Movshovitz-Hadar and Webb 1997; Kleiner and Movshovitz-Hadar 1994; Movshovitz-Hadar et al. 1994; Movshovitz-Hadar 1993a, b; Movshovitz-Hadar and Hadass 1990, 1991; Hadar and Hadass 1981).

Finally, a personal note: The four problem-solving environments mentioned above were always given the highest student-survey evaluation scores. However, their enduring impact on those who graduated and became practicing mathematics teachers always remained a question I wondered about. While I was composing this chapter, I happened to meet a former student who graduated several years ago. She is now the mathematics department head of a large high-school in Israel. We had a casual conversation about her professional development and personal life. As we were ready to depart from each other, she said, suddenly looking at me very seriously: "May I tell you something personal?" "Surely" I responded, not without quietly wondering what it was that she had been holding back all these years. And she said: "These problem-solving courses I took from you … I keep recalling many of the tasks and have been trying to adopt that spirit to the *daily* planning of my mathematics classes", she said, and I was relieved. "My major effort during all the years I have been teaching mathematics" she added "is to plan *every* lesson to include some mathematical surprise or something dramatic so that the kids will be looking forward to the next lesson, as we were in those courses". This made me very happy, of course, but the best was still coming. "Most of my lessons" she carried on "evolve from a clear statement of a new problem that we deal with and solve during that lesson. I erased from our vocabulary the term 'rehearsal lessons'; there must be something new and interesting in *every* lesson." I thanked her deeply and secretly wished that many other students have similar recollections and attempt to bring to their classes experiences that have similar taste to what they had in these courses.

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Part II

Mediating Mathematics Teaching Development and Pupils' Mathematics Learning: The Life Cycle of a Task

Barbara Jaworski, Simon Goodchild, Stig Eriksen and Espen Daland

Introduction

This chapter addresses the development of mathematics learning and teaching in a four-year developmental research project in which the role of mathematically-related tasks has been significant to developmental practice of the educators taking part in the project. In the chapter we first introduce the project—*Learning Communities in Mathematics* (LCM). We go on to discuss the theoretical development on which the project was based and which proceeded with the project, followed by more detail of project activity. Our next section introduces our use of Activity Theory in analyses of data and to trace development within LCM. We go on then to discuss one example of a mathematical task—the Mirror Task—through which we aim to show how the project used tasks to promote learning of pupils, teachers and didacticians in the project. Finally we return to our activity theory analysis to discuss tensions in mediated activity and learning in the project.

The project, *Learning Communities in Mathematics* (LCM)[1] was founded on principles of *co-learning inquiry* between didacticians[2] in a university and teachers in eight schools in Norway. Its fundamental aim was the enhancement of learning experiences in mathematics for pupils in Norwegian schools from grades 1 to 13. Didacticians wrote the project proposal, gained funding from the research council

[1] The LCM project was funded by the Research Council of Norway (RCN) in their programme Kunnskap, Utdanning og Laering (Knowledge, Education and Learning—KUL): Project number 157949/S20.

[2] Didacticians are university academics who conduct research in mathematics education (matematikk didaktikk, in Norway) and work with teachers to promote development of mathematics learning and teaching in classrooms. The four authors of this paper were didacticians.

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and invited schools to participate. The eight schools who volunteered spanned the full age range of pupils. (See Jaworski et al. 2007, for a comprehensive account of the project as a whole.)

The LCM project involved cooperation between didacticians and teachers to explore development of mathematics teaching to enhance learning opportunities. Thus, its main focus was on mathematics teaching development. Research was seen as a tool both for promoting a developmental process as well as for charting development. Both teachers and didacticians were *insider* researchers in the project, exploring their own practice and its development. Didacticians were also *outsider* researchers studying activity, progress and development in the project (Bassey 1985; Cochran-Smith and Lytle 1999; Goodchild 2007; Jaworski 2003, 2004b).

The main centres of activity in the LCM project were *workshops,* held periodically in the university and attended by all project participants (teachers and didacticians), and *schools* where the school project team (of three or more teachers) were responsible for innovative practice in classrooms. Each school team had an associated didactician team (of three people) some or all of whom visited the school, worked with teachers on planning for the classroom, and collected video data from innovative classroom activity. Four doctoral students, conducting research within the project, offered the main contact with and collected data from schools with which they were associated according to their own specific research questions[3].

An early decision, taken by the didactician team, was to use specially designed mathematical tasks as a basis for workshop activity, an aim of which was to explore the processes through which pupils, teachers and didacticians learn. Tasks were designed to:

(a) build community between teachers and didacticians through engaging in mathematics together;
(b) enable discussion on specific areas of mathematics;
(c) provide a basis for raising didactical and pedagogical issues related to learning mathematics and working with pupils in classrooms;
(d) provide examples from which teachers could design their own tasks for the classroom; and, overall
(e) contribute to the learning of pupils, teachers and didacticians.

The four year project included three phases, each of one school year, of activity between didacticians and teachers. We show in discussion below how these aims (a–e) were achieved during the three phases and indicate the emergence of developmental insights, as well as issues and tensions arising from design and use of tasks.

[3] Stig Eriksen and Espen Daland were two of these doctoral students and are currently completing theses entitled respectively *Mathematical tasks and the building of a learning community of mathematics between teachers and teacher educators* and *Developing learning communities in mathematics: Exploring issues in a mathematics teaching development and research project.*

Theoretical Development

Inquiry Community

Tasks were designed to promote inquiry within the project within three layers:

1. Inquiry in doing mathematics in workshops and classrooms;
2. Inquiry in planning for workshop or classroom and in design of mathematical tasks;
3. Inquiry in the research process of developing teaching and exploring development.

Inquiry, according to Chambers' *English Dictionary* (Schwarz 1988), means to ask a question; to make an investigation; to acquire information; to search for knowledge. Wells (1999, p. 122) speaks of "dialogic inquiry" as "a willingness to wonder, to ask questions, and to seek to understand by collaborating with others in the attempt to make answers to them". He emphasizes the importance of *dialogue* to the inquiry process in which questioning, exploring, investigating, and researching are key activities or roles of teachers and didacticians (and ultimately, we hope, pupils). In LCM we have sought to create *communities of inquiry* in and related to the project in which we use *inquiry as a tool* to develop our thinking and practice, and work towards developing inquiry identities or inquiry *as a way of being in practice* (Holland et al. 1998; Jaworski 2004a).

Inquiry communities can be seen to develop from *communities of practice* as conceptualized by Lave and Wenger (1991) and Wenger (1998). According to Wenger (1998), *belonging* to a community of practice involves *engagement, imagination* and *alignment.* Participants *engage* together in the activity or practice of the specified community with its own purposes and goals. *Engagement* implies meaningful involvement and co-participation. *Imagination* allows individuals to envision their activity and role and to engage meaningfully in the practice. *Alignment* implies literally 'lining up with' the norms and rules of engagement established over time within the practice. According to Wenger (1998, p. 183), these are "modes of belonging which are involved with varying degrees of emphasis in different types of community". From an LCM perspective, we saw our activity in university or in schools to be part of established communities of practice in which we were experienced participants, familiar with and adhering to the norms and rules of practice: didacticians as academics and university teachers, engaging in research and publishing findings as expected by the academic community; teachers engaging in teaching activity in and out of the classroom and according to the systemic functioning of the school as an educative organism.

An inquiry community derives from a community of practice through the introduction of *inquiry* to promote *critical alignment.* Thus, rather than aligning tacitly with the practices of the community of practice, those engaging in critical alignment question their participation, seek to know more about the *hows* and *whys* of participation, create dialogue with peers to recognize and address issues in practice, and open up possibilities for changing or developing aspects of practice

(Jaworski 2006). Rogoff et al. (1996, p. 388) speak of a *learning community* in which "learning involves transformation of participation in collaborative endeavour". The idea of *inquiry* community makes the nature of transformation more explicit: didacticians and teachers (and ultimately pupils) engage together *in inquiry activity* through which new ways of seeing and doing become evident and learning occurs. What such activity should or could consist of, and how it should or could relate to activity in existing communities of practice, the classrooms, schools and university settings was a focus of research in LCM. The use of tasks was seen as central to learning activity.

Project Activity

In order to enable critical alignment with the norms of everyday practice for both teachers and didacticians it seemed important to create situations in which norms of practice could be made evident, alternatives considered and opportunity for innovation become a reality. From a community of practice model it seems clear that world views of those concerned in a developmental project will be strongly related to the norms in established communities of practice—largely, the schools and university where they work. When a project begins, it has yet to develop its own norms. So, planning and design of activity in the project have to encourage growth of new ways of doing and being. However, those planning or designing at any stage are a part of this system, and cannot be seen as standing outside of it. In the beginning, didacticians, according to sound theoretical principles wished to establish practices in which inquiry ways of doing and being could be fostered. They could only think about this from within their own community of practice, albeit within an inquiry frame so modes of practice could be considered, questioned and analysed.

From the perspective of didacticians at the beginning of the project it seemed that carefully designed mathematical tasks could be a basis for enabling teachers and didacticians to work together in an area of common interest and lead to discussion of mathematics learning and teaching in classrooms. Tasks planned for workshops were not intended as recommendations for classroom activity: project design suggested that teachers would design their own tasks for classrooms, related to their own curriculum, with didactician support. However, teachers, from the perspectives of a school community of practice and experience of teacher education events in other contexts expected didacticians to suggest ideas for the classroom. In a focus group interview at the end of Phase 2, one teacher, Agnes, reflecting on her early experience, expressed this as, "I thought very much that you should come and tell us how we should run the mathematics teaching. This was how I thought, you are the great teachers". So, in many cases, tasks experienced in workshops, perhaps seen as recommendations for classroom activity, were used by teachers in classrooms, either directly following their mode of use in the workshop, or modified in some way to suit particular groups of pupils.

In each of the three phases of activity, tasks were designed (mainly by didacticians), used in workshops (didacticians and teachers together) and adapted for the classroom (mainly by teachers). The nature of the tasks used evolved during the three phases. At the beginning of Phase 1, tasks were chosen to be readily accessible to teachers of varying mathematical experience, yet with opportunity for extension to offer serious mathematical challenge for all. They were not necessarily linked directly to particular areas of the curriculum. Teachers characterised such tasks as *general* or *fun* tasks. Some teachers, particularly at upper secondary level, suggested that they needed tasks which were more curriculum related as they did not have time for the so-called *fun* tasks. Responding to comments of this kind, and requests to address certain curriculum topics in workshops, didacticians tried hard to design tasks that had clear relevance to an agreed curriculum area, such as algebra, or probability.

One early, and in retrospect we can see quite rare, example of teachers designing their own tasks came from the team in an upper secondary school. They requested didactician support to design tasks related to the teaching of a topic on linear functions at Grade 11. Two special meetings, one in school and one in the university were organized at which the material of linear functions was discussed with reference to the textbook the teachers used. Discussion went deeply into meanings of linear functions and considered activity that went beyond the textbook. Subsequently, the three teachers designed a set of four tasks to engage pupils in inquiry related to the topic, and each of them used these tasks in a lesson with pupils, video recorded by didacticians (Hundeland et al. 2007; Jaworski 2007). A meeting held after these lessons to discuss outcomes and reflect on activity allowed the teachers and didacticians to consolidate professional relationships that had grown through this collaboration. The teachers agreed to present their activity and issues arising at a workshop—where other teachers were able to try out the designed tasks and share in consideration of issues—and later at a project conference.

The tasks discussed above were mathematical tasks focusing on key concepts in the topic of linear functions. Increasingly as the project progressed, tasks were clearly related to a given mathematical topic such as algebra, geometry or probability, and interpreted at a variety of levels related to the age or grade of pupils with whom teachers worked. Key task-related elements of activity introduced above include:

- initial design of mathematical tasks by didacticians for workshops in which collaborative mathematical activity was desired for reasons given above;
- some teachers' use of such tasks, adapted by teachers for pupils' activity in their classrooms;
- some teachers' design of curriculum-related tasks with didactician support; and, subsequent presentation and discussion of school activity in project workshops and conference, disseminating thinking and activity locally and more widely.

Considering that task design evolved through three years of activity, what we have said here is extremely brief. Three factors emerged as being highly significant for task design related to developmental progress: that is the development of knowledge and practice of teachers and didacticians through their engagement in the project. The first was that teachers were eager for tasks of an inquiry or investigative nature that they could use with pupils, and readily used workshop tasks in a variety of ways in classrooms. Secondly, teachers' own design of tasks in the school setting proved problematic due to difficulties in finding time and opportunity for teachers to meet during the school day. This resulted, in Phase 2, in workshop activity in which teachers from different schools, in same-grade groups, designed activity for the classroom which individuals would then take further in their own contexts. Thirdly, as video material from innovation in classrooms accumulated, it became an important tool for dissemination in the project. Seeing tasks used by teachers in the realities of classrooms led to developing awareness of teachers and didacticians of possibilities for engaging pupils and the associated issues, didactical, pedagogical and systemic (For further detail, see also Bjuland and Jaworski 2009; Daland 2007).

Using Activity Theory to Address Issues from Activity Using Tasks

As the project community matured and inquiry practices developed, teachers and didacticians gained insights into each other's ways of doing and being through project activity and dialogue. A major goal of the project from the start was to achieve *co-learning* (Wagner 1997) between teachers and didacticians: that is our learning through participation and inquiry should have a mutual dimension in which both groups brought knowledge and expertise and both learned from joint activity. We learned to talk to each other in ways that opened up each other's perspectives. Teachers came to use the language of inquiry, and didacticians learned about school systems and structures that influenced teachers' thinking and what was possible in school. At a meeting of all teachers and didacticians to discuss the style and content of Phase 2, Oswald, a teacher at upper secondary level who had been frank in his criticism of what the project was offering him and his colleagues, said,

> …the rumour could easily spread that we are dissatisfied with what is done, or that I am dissatisfied. On the contrary, I will gladly give you praise for what you have done and I think it has been very interesting what we have done so far. (Goodchild 2007, p. 200)[4]

The teacher Agnes, quoted above, went on to say,

> …now I see that my view has gradually changed because I see that you are participants in this as much as we are even though it is you that organise. Nevertheless I experience that you are participating and are just as interested as we are to solve the tasks on our level and find possibilities, find tasks, that may be appropriate for the pupils, and that I think is very nice. So I have changed my view during this time. (Daland 2007, p. 168)

One teacher, working at lower primary level, was inspired by the phrase "inquiry as a way of being" and quoted this phrase at a number of public meetings. He started

[4] This is a translation from Norwegian, as are other teachers' words quoted in the text.

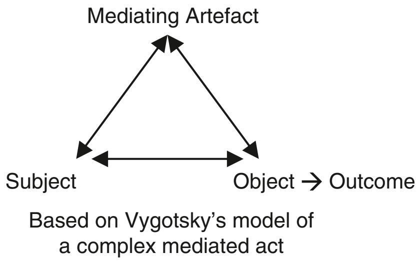

Fig. 1 A simple mediational triangle. (c.f. Vygotsky 1978, p. 40)

to analyse his work with pupils from perspectives of inquiry in collaboration with one didactician (Jørgensen and Goodchild 2007).

Didacticians learned what teachers preferred in terms of group formation in workshops, and planning for the classroom as part of workshop activity (Goodchild 2007). Thus, insights emerged from project activity, and issues and tensions became evident, vested as they usually were in the established practices of teachers and didacticians. The theories of communities of practice and inquiry did not extend to providing an analysis of emergent insights, issues and tensions. For this we turned to activity theory. We provide here a brief explanation of Activity Theory as we saw it relating to the project and being of use in analysis. We later demonstrate our use of it in analysis of the Mirror Task.

The key idea for us here is that human activity is motivated within the sociocultural and historical processes of human life and comprises mediated, goal-directed action. According to Leont'ev (1979, p. 46), "Activity is the non-additive, molar unit of life … it is not a reaction, or aggregate of reactions, but a system with its own structure, its own internal transformations, and its own development". We can see, in these terms, the complexity of the educational system of which the LCM project was a part. To analyse the nature and role of tasks within this complexity, we start from a simple mediational triangle (Fig. 1).

For pupils (as *subject*), engaging in mathematics in classrooms, with *object* and intended *outcome* that of mathematical learning and understanding, the mathematical task designed by the teacher is a mediational *artefact.* Through the task, pupils should have opportunity to engage in mathematics, make sense of mathematics (enjoy mathematics) and become able to use and apply the mathematics they have learned. For teachers (as *subject*), engaging in tasks in workshops, and adapting or designing tasks in consequence, is a mediational process in which tasks are *artefacts.* Through this process, teachers come to understand more about teaching and learning processes and how to engage pupils appropriately, the *object* and intended *outcome* of their activity. For didacticians (as *subject*), designing tasks and working on tasks with teachers in workshops is a mediational process in which tasks and task design are *artefacts.* Through this process, didacticians learn about the contribution of tasks to the learning of both teachers and pupils; they learn to modify their design of tasks according to observation and reflection, and to respond to systemic issues arising when tasks and their design enter into the school community. The *object* of activity for the didacticians was to encourage teachers to design tasks for

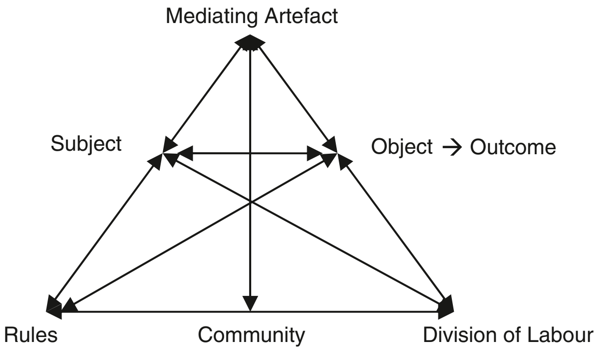

Fig. 2 An expanded mediational triangle. Engeström's (1998) complex model of an activity system

pupils, and the intended *outcome* was the implementation of task design in schools for enhanced learning of pupils. What emerged for didacticians were deeper awarenesses of teachers' thinking, teachers' activity systems in schools, and the systemic complexities in which the developmental process is embedded[5].

We use an expanded version of the mediational triangle (Engeström 1998) to try to capture, theoretically, this complexity (Fig. 2). The simple triangle sits at the top of this new figure, and below it is what Engeström refers to as "the hidden curriculum". *Community,* as expressed above, is central to activity. It affords collaborative development of insights and issues and also constrains what is possible through its established practices which are hard or impossible to shift. These include the *rules* or norms of engaging in the established practice. For example, the project teacher group within the school community affords a collaborative base for working on tasks in school classrooms. However, such collaborative activity has to fit with the educational system (curriculum expectations etc.) and school organisation and ways of being and doing in school which may not fit well with the objectives of the project. The established school system is hard to change, so the project has to adapt to what is seen to be possible in school, albeit with aims for school development through the activity of teachers in the longer term. *Division of labour* includes the differing roles of teachers in the project and their colleagues in schools as well as differences between teachers and didacticians. The rootedness of activity, expectations of practitioners, modes of being and doing are all significant to how individuals or groups can achieve project goals.

Thus, the hidden curriculum captures much of the complexity with which the project engages in design and use of tasks for effective development at a range of levels. The expanded triangle allows identification of norms or modes of activity within the project and the complex mediational factors that influence outcomes. It makes possible the mapping of issues and tensions that both constrain what is possible and lead to possibilities for development. Engeström (1999) emphasises the importance of tensions or contradictions between elements of the hidden curriculum in promoting learning within activity.

[5] See Mason (2008) for a discussion of *awareness.*

Before going further, we offer a more detailed example of one mathematical task, explaining its initiation, design, use and outcomes, in order to exemplify in practice some of these theoretical ideas, and return at the end of this chapter to our Activity Theory analysis of developmental issues and their achievement through mediation in task design.

The Mirror Task

Didacticians' Design of the Task

The task we describe was presented in Workshop 8, the second workshop of Phase 2 of the project (the second year of activity between didacticians and teachers). The preceding workshop had included overt planning of tasks for the classroom, as requested by teachers, and feedback from teachers and didacticians' reflections led us to believe that the format and content had been well received. It had been decided that Workshop 8 would focus on geometry. Prior to the meeting in which the workshop was to be planned, Stig initiated an e-mail discussion in the didactician team that took place over 16 days, involving contributions from seven didacticians.

In the first message, Stig had reminded us of the importance of continued attention to community building and the value of engaging in some mathematics together to achieve this end. He suggested several problems on which we might work: one of these, the basis of what we refer to as 'the Mirror Task', is simply stated: *How tall a mirror must you buy if you want to be able to see your full vertical image?* (Shultz et al. 2003, p. 310). The e-mail discussion considered many of the issues that had emerged in the project over the previous months. The perceived tension between investigative tasks that open up the mathematics and tasks focused on the curriculum and textbooks used by teachers was uppermost. The goal of community building was considered important and the value of mathematics in achieving this accepted; however, if the teachers did not share the didacticians' enthusiasm for a task it might prove counter productive. The teachers had requested that time in workshops could be spent in planning for their classrooms and it was important not to forget this after one, apparently successful, workshop which followed the pattern they had requested. It was believed important to find a rationale that would offer a bridge between what might be conceived as the project's goals and the demands of the curriculum; although from the didacticians' perspective the goals coincided. Furthermore, as the project included teachers of classes from Grade 1 to Grade 13 there was some question about how the task might be interpreted or adapted for pupils at different stages of development, and whether a knowledge of the physics of reflection was required. Stig responded to the challenge by mapping variations of the task to different levels of the school curriculum. This accorded with aims in the project to show the versatility of mathematical tasks and ways in which tasks could be adapted in relation to the differing needs of pupils. It was seen to contribute to community building in helping teachers at different levels to perceive the experience of their pupils more holistically throughout schooling.

Figure 3 shows what Stig presented to didacticians at the planning meeting. The words in italics in this version are written by Stig for didacticians, who will consider the task and possibly help to modify it for use with teachers. The simple statement of the task is the question at the top: *How tall must a mirror must you buy if you want to be able to see your full vertical image?* It can be seen that this question is expanded below into forms of activity that relate to 13 grade levels for pupils from ages 6 to 19. After discussion in the didactician team, the text was modified for presentation in the workshop: the ideas for Grades 12 and 13 became, 'What if the mirror is not vertical?' together with relevant quotations from the national curriculum.

It might therefore be seen as if the task became 13 separate tasks, and indeed this is one way of seeing it since we think of a task as more than just the problem posed. The task includes the problem posed and other factors generating activity in relation to this problem. So, when we talk about *The Mirror Task,* we can be thinking about the whole situation as created by didacticians for use in the workshop, including the 13 problem statements and the modes of activity in the workshop (as described below), or alternatively we can think of the task for any group of teachers in the workshop which possibly involved considering just one of the 13 statements relating to their own pupils.

The Wider Activity Related to the Task

It is sometimes difficult to see where a task begins and ends. The Mirror Task could be seen as the simple question asked at the top of Fig. 3. Or it could be seen as this question together with ideas for addressing the question at different grade levels. However, task-related activity went beyond just these simple words on paper as we explain. When nine didacticians met to plan the workshop several had not contributed to the e-mail discussion but all had been able to read the mails that had been exchanged. Stig opened up the discussion by giving a practical demonstration of the Mirror Task.

Much of the discussion focussed on what would take place in the workshop. It was decided that Stig and Espen would introduce the Mirror Task in a plenary session. Participants would be provided with mirrors and short columns of (multilink) small interlocking cubes to try the task practically during the plenary. Stig would follow this with some open discussion and an exposition of the work he had done on the task and how he had reflected on the way the task might be adapted to various levels of the curriculum. Another didactician would demonstrate how the task might be interpreted and explored using *Cabri-Géomètre,* a dynamic geometry program on the computer. The Mirror Task would be presented orally with reference to the article by Shultz et al. (2003).

The discussion between the didacticians explored deeper issues fundamental to the project. It was decided to recommend teachers to read, in advance of the workshop, two articles with a geometry focus that had appeared in teachers' mathematics

How tall a mirror must you buy if you want to be able to see your full vertical image?

Below you will find one short description of an activity I see as linked both to the 'læreplan' (curriculum) and to the task above for each year. I have tried to keep the surprise part of the original task. I have not discussed HOW to use the presented activities in a "building-community-of-inquiry-way".

Maybe you can take these initial ideas further?

Grade 1
One pupil faces a mirror holding a stick (against his stomach). This pupil directs another, who, using a whiteboard marker, marks the mirror image the first one sees. Compare the original stick with the marks on the mirror. Try different distances from the mirror.

Grade 2
One pupil holds a geometric figure (against the stomach) and explains to another pupil how to draw (on the mirror) the mirror image he sees. Compare.

Grade 3
Measure yourself in centimetres. Measure your mirror image in centimetres. Draw yourself seeing yourself in a mirror.

Grade 4
Have a mirror with a grid. One pupil holds a geometrical figure (against stomach) and explains how another pupil can draw this on the mirror. Count number of squares (area) and compare.

Grade 5
Cut a figure from paper. Measure it and find the perimeter. Find perimeter of mirror image. Compare.

Grade 6
Measure yourself and your mirror image. Draw yourself (simplified) looking in a mirror with the correct ratios (and angles) in your drawing.

Grade 7
Draw model of a figure and an eye and the mirror image the eye sees (keep the eye and the figure at the same distance from the mirror?). Describe lengths and angles. What do you see?

Grade 8
(perspective drawing)
Hold a cube and go close to the mirror. Draw on the lines of the cube on the mirror. What do you see?

Grade 9
How tall a mirror must you buy if you want to be able to see your full vertical image?
- draw model
- describe angles and triangles

Grade 10
How tall a mirror must you buy if you want to be able to see your full vertical image?
- justify your conclusion
- try with objects with different distances from mirror

Grade 11
How tall a mirror must you buy if you want to be able to see your full vertical image?
- justify your conclusion
- try with objects with different distances from mirror
- describe ratios in model

Grade 12
How tall a mirror must you buy if you want to be able to see your full vertical image?
- justify your conclusion
- try with objects with different distances from mirror
- describe ratios in model
- use the cosine rule to derive the height of the actual figure when the height of the mirror image is known

Grade 13
Draw yourself and a mirror in a three dimensional vector space.

Fig. 3 Relating the mirror problem to the curriculum at specific grade levels journals (Karkoutli 1996; Smith 2003). These articles were intended to raise teachers' awareness of pedagogical issues related to the teaching of geometry. It was also decided to send out an analysis of responses to shape and space items that had been part of a longitudinal test that pupils in grades 4, 7, 9 and 11 had taken as part of the project one year earlier. The purpose of sending this analysis was to raise awareness of where pupils' learning opportunities might be improved. Finally it was decided to send out four investigative tasks that teachers might try in advance with the thought that in the workshop they might choose to develop teaching material based on one or more of them. Thus, the wider activity included pre-readings in geometry, problems to engage with in preparation for the workshop, the practical activity in plenary and the example offered in *Cabri-Géomètre.*

The planning meeting also decided that it was important to make clear in the workshop the reasons for sending the materials in advance and for the activities included in the workshop. This would take the form of one didactician making a plenary presentation of about 15 minutes that would once again (the issues had been explained in previous workshops) explain the rationale for the tasks. The reasons included community building, offering an example of how a single task could be interpreted differently to make it appropriate for a variety of grade levels, and to stimulate design activity for the classroom. This presentation would open the workshop, followed by a report from one teacher of his design and implementation of probability tasks inspired by the previous workshop.

We include this detail to indicate the range of activity envisaged by didacticians, with an aim to draw teachers into broader thinking about geometry within which the task could be situated relative to their own level of teaching and their pupils' understanding of geometry. It would have been ideal if teachers could have taken part in such thinking and planning, but time and opportunity for teachers to engage in activity was scarce, and so reserved for workshop participation.

Discussing and Planning in the Workshop

As planned, Stig and Espen first introduced the task in plenary and participants all engaged in practical activity with small mirrors and columns of cubes for 10–15 minutes. Stig then related the task and possible variations to the curriculum, drawing on what is written in Fig. 3 above and encouraging discussion. This was followed by a dynamic demonstration (in *Cabri-Géomètre*) of the relationship between the lengths of objects and their reflected image on the surface of the mirror to show what happened when the mirror or object were moved relative to the observer. After these presentations participants were organised into small groups in which they could develop one or more of the tasks for using with their class. The groups comprised teachers whose classes were at similar grade levels. Groups were given the opportunity to choose which task(s) they wanted to work on. Our account follows one lower secondary school teacher (Trude) who eventually implemented the Mirror Task with her class in Fjellet School.

Trude was in a group that included one other lower secondary school teacher, two upper secondary school teachers and two didacticians. Despite the intentions of the didacticians the group did not engage in doing any mathematics together, rather the teachers led a discussion on their ideas about teaching and largely controlled the discussion. Some of this focused on different opportunities for developing mathematical ideas that existed at the different schools, for example the upper secondary teachers expressed a belief that it was possible to spend more time on some topics in the lower secondary schools. There was some discussion focused on the teaching of Pythagoras' rule, which was current in the experience of one of the teachers although not included in any of the materials sent out in advance of the workshop, or in the preceding plenary presentation. A substantive part of the discussion focused on the Mirror Task and the teaching of similar figures. One teacher wondered whether the Mirror Task might be appropriate for teaching about similarity but Trude and the other lower secondary teacher agreed that it would be better to leave it until later, as an application of the principle. One of the didacticians suggested that an approach to the topic could be to collect similar objects that occur in everyday life.

In the discussion a number of pedagogical issues were considered. For example, the benefit of visualization and motivation at all levels, and Trude reflected on her own teaching and wondered whether she used different examples to support pupils' understanding or to make the mathematics more enjoyable. Trude also observed that pupils want to have a reason for studying a topic, her concern for providing a rationale for her pupils apparently matching the didacticians' concern to provide teachers with reasons for the activities chosen in the workshop. By the end of the group discussion Trude expressed her eagerness to teach the topic of similarity and to use the Mirror Task, but as agreed earlier, she would not use it as an introduction to the topic.

Just over one week later Trude introduced her 8th grade class to similarity. Stig was invited to attend and video record the lesson. Trude had assembled a collection of objects, such as milk cartons and pictures that could be explored for properties of similarity. She had even included an ironing board because the hinged legs formed two similar triangles, one with the surface of the board, the other with the floor. Trude started her lesson by telling the class of an incident where she had requested an enlargement of a photograph. There had been some misunderstanding in the shop: whereas Trude gave the specification in centimetres the enlargement was produced using the same numbers but measured in inches. Following this introduction, despite the intention stated in the workshop, Trude moved on to introduce the Mirror Task. Following the format of the workshop, pupils were supplied with small mirrors and columns of cubes to explore and expose some solution. Pupils worked on the practical task for about 30 minutes.

In Workshop 9, about four weeks later, Trude reported, in plenary, on the activity with her class, she was disappointed that it had not worked out as intended and blamed herself for not making the task sufficiently clear. Trude explained that the pupils had not understood that the mirror needed to be held at face level, and that the column of cubes should be held in the same vertical plane as the observer's eye, and both on the same vertical plane perpendicular to the surface of the mirror. Also the pupils had been able to tilt the small mirrors, which again distorted the results. She reported that at the end of thirty minutes of practical activity some results (length of column, length of image on the surface of the mirror) were collected and entered into a table on the chalk board but it was impossible to discern any relationship because the orientations and positions of the column and the mirrors had not been managed effectively. Despite the activity, she said, pupils remained convinced that the size of the mirror did not matter; it would be possible to see a reflection of their whole body in a small mirror. Trude had provided a larger mirror in the class and invited one pupil to use it to test the conjecture but he refused, embarrassed because of the presence of the video camera. The pupils were left to test their belief at home.

Discussion and Conclusions: Tensions in Mediated Activity

Trude's very frank report from her classroom experience with the Mirror Task seems to indicate a high degree of confidence and trust in the LCM community, especially her willingness to reveal sensitive aspects of her thinking and practice. She spoke forthrightly and with humour, and those listening responded with indications of understanding and sympathy. It seemed that pupils had not gained as she hoped from the mathematical activity and that she saw her own preparation for the activity as (at least partially) responsible. As didacticians, we also reflect on our part in these events and recognize aspects of the activity overall which could have contributed to observed outcomes. Of the five aims expressed in section "Introduction" above, we can see (a) and (c) being clearly addressed in workshops and classroom jointly. However, it is not clear if (b) was addressed, and the extent to which (d) was addressed is limited. This discussion now attends to (e), what has been learned jointly by teachers and didacticians from workshop and classroom events, and draws on Activity Theory constructs to support analysis.

Trude designed her lesson to include a Mirror Task and that was modelled almost exactly on the workshop activity. It seemed clear that it was only when pupils worked with the mirrors and cubes that Trude appreciated the importance of the *rules* of the task, as she expressed them later in her workshop presentation. What do we learn from this? We have some evidence of what Trude learned, since she expressed this in her talk. Through her pupils' engagement, she came to see important conditions that must be satisfied if the Mirror Task is to provide insights to a solution for the mirror problem. Didacticians learn that, despite engagement of Trude and others in Workshop 8 in the Mirror Task, key aspects of the task had not been addressed or understood. Had there been mathematical discussion in the small group, these factors might have become evident. However, the teachers had led, even controlled, this discussion. Should or could the didacticians have made it otherwise? The nature of *community,* with confidence, trust and openness in working together, was established through an effort to create equity between didacticians and teachers in influencing activity. This could have been damaged if didacticians had tried to control the small group discussion.

An *outcome* of workshop activity was that in Fjellet school, one group of pupils worked on a version of the Mirror Task. By Trude's account, and analysis of the video recording of the lesson, we perceive that little was achieved mathematically by the pupils. They seemed to believe something which teachers and didacticians knew to be mathematically false. We can appreciate how such belief came about. A didactician would have hoped to challenge pupils' conceptions in the classroom. However, the means to do this within the classroom ethos are at the teacher's disposal but not necessarily available to an outsider didactician. A clumsy handling of the pupils' activity could be just as damaging for confidence and trust in the classroom as could a clumsy handling of the small group in the workshop. Perhaps the reasons for leaving pupils with their misconceptions would be similar to leaving teachers with avoidance of mathematics in the small group.

We see *the hidden curriculum* deeply exemplified in these situations. In both workshop and classroom, community norms and ethos mediated activity related to the task. Here *community* includes both the established communities and the project (inquiry) community. *Division of labour* figures strongly in decisions made to act or to hold back. The *rules* of engagement derive from established norms or delicately balanced relationships including the interactions of didacticians in workshops and teachers and pupils in schools. The Mirror Task seems central to activity, mediating what takes place, but we see very clearly that the task itself cannot achieve desired outcomes. It is the way the people using the task engage with it and the nature and quality of the engagement that leads to learning. Thus we see tensions arising between elements of the Activity Theory model, which, as Engeström (1999) has expressed, can lead to opportunities for learning.

A central tension relating to knowledge, expertise and experience of both teachers and didacticians can be seen between the elements *community* and *division of labour*. The didacticians are strong in mathematical know-how, with the didactical experience to modify the task and challenge those engaging in it to draw out key mathematical ideas. However, didacticians have no power in schools, little knowledge of established ways of being in the school, and no direct knowledge of the pupils. These are all in the province of teachers. It is the teachers who can work with the pupils and who have to develop both mathematical understandings and the didactical and pedagogic awarenesses that are necessary to offer a task and challenge pupils sensitively in relation to the task. Workshops are the place where such understandings and awarenesses can be fostered. Here didacticians have power to engage teachers in tasks and offer mathematical challenge. Yet, didacticians do not wish to be cast in the role of telling teachers what to do and how to do. We see this tension between community and division of labour to be a manifestation of what John Mason has called the *didactic tension:*

> The more explicit I am about the behaviour I wish my pupils to display, the more likely it is that they will display the behaviour without recourse to the understanding that the behaviour is meant to indicate; that is they will take the form for the substance…The less explicit I am about my aims and expectations about the behaviour I wish my pupils to display, the

> less likely they are to notice what is (or might be) going on, the less likely they are to see the point, to encounter what was intended or to realize what it was all about. (Mason 1988; rooted in Brousseau's Topaze effect (Brousseau 1984), cited in Jaworski 1994, p. 180)

Thus a teacher has to navigate carefully between directing pupils and providing opportunity for their own self-direction. Challenge has to be judged extremely sensitively and it is not always clear what levels of intervention offer appropriate mediation. The same is true for didacticians working with teachers. As we have exemplified above, in the words of Agnes, teachers have a growing awareness of the tension—expressed simply as expecting didacticians to tell teachers what to do and how to do it, versus teachers becoming aware as part of their own growing knowledge and experience of what is needed in classrooms and the challenges they need to face. We see this tension manifested clearly in the activity of the Mirror Task.

From an activity theoretical perspective of learning and development the tension outlined above can be seen to have potential to motivate activity that might result in learning. Didacticians do not want to 'tell' teachers what to do, nor do teachers want to 'tell' pupils, the aim is to provoke engagement with the substance of teaching or mathematics, respectively, not with the surface form that is apparent in 'telling' or giving scripts or recipes for action. Designing a lesson based on a well chosen task will engage teachers in critical reflection about what it is they want to achieve in their classes. A well chosen task will engage pupils in critical reflection on the substance of the mathematical ideas embedded in the task. Thus the task mediates between didacticians and teachers, and between teacher and pupils. The tension that arises from engaging critically with the task—"How do I do this?"—leads potentially to creative thinking, innovation, for the teacher new insights and actions in teaching, for the pupil new understanding and possibilities in mathematics, for the didacticians new awarenesses of school community norms and teachers' ways of thinking.

Didacticians learned a great deal, for example, in terms of project patterns in design and use of tasks. Stig was able to formulate and lead development of the Mirror Task as a result of previous activity involving design and use of tasks, reflection in the didactician team and synthesis of aims and objectives of workshop activity. For didacticians, task design and use has been central to project activity and learning in the project. It is not that any task per se has its own power and developmental properties (although some tasks weather the passage of time and grow into the norms of practice) but power rests in what is learned by practitioners about task design and use, the challenges and tensions that it generates. Didacticians and teachers now know so much more about mediation within the hidden curriculum than they did four years ago when LCM began.

The reality of such events is that a convenient 'closure' is rarely available. There is no closure to the story of the Mirror Task. It was not possible, for example, to analyse Trude's learning, nor provide evidence of outcomes of that learning in future use of the Mirror Task with pupils. The learning of which we speak takes place over time, and this time might be considerable. In the case of the Mirror Task, in its various manifestations, we see an example of growth over time related to the tensions as experienced by didacticians and teacher within our own established communities and the developing inquiry community. The teacher became more aware of the mathematics of the mirror task through her pupils' responses which did not fit with her expectations. Didacticians became more aware of interpreting their aims and intentions in workshop settings through the teachers' activity which did not fit with their expectations. Thus, didacticians can trace their activity from the planning stage to classroom activity and recognize factors which influence directions and outcomes. The tensions, valuably, are a motivating factor in planning for future events and the ongoing *planning* ↔ *activity* process leads to development in knowledge and awareness. This is the main outcome of the developmental process, and its impact for the classroom follows. Research needs now to find ways of tracking this impact. We have written elsewhere about other examples of such learning (Goodchild and Jaworski 2005; Jaworski and Goodchild 2006).

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Learning from the Key Tasks of Lesson Study

Catherine Lewis, Shelley Friedkin, Elizabeth Baker and Rebecca Perry

Introduction

Lesson study is a professional learning approach that originated in Japan and has recently spread among both prospective and practicing teachers in North America. In lesson study, teachers engage in cycles of inquiry in which they collaboratively plan, observe, and discuss classroom "research lessons" in order to improve their shared understanding of teaching, learning, students, and subject matter. These "research lessons" are ordinary lessons in the sense that they are real classroom lessons with students, with the unpredictability and on-the-spot decision-making that all teaching entails. Research lessons are often unusual, however, in that a group of teachers has carefully studied the subject matter and collaboratively considered the lesson design most appropriate to the students, and these teachers (as well as invited colleagues) observe, collect data, and formally discuss how the lesson actually unfolds with students. When practised over time, lesson study is designed to build the skills, habits of mind, tools, and culture for teachers to learn daily from colleagues, students, and curriculum materials. Japanese teachers typically teach one research lesson in their own classroom each year and observe and discuss research lessons in about 10 other classrooms (Fernandez and Yoshida 2004).

This chapter breaks out the five core tasks of the lesson study cycle shown in the left column of Table 1. Typically, lesson study begins with teachers formulating a shared "research theme" that captures their long-term goals for student learning and development. Often this is done by a whole school faculty. Next, teachers break into grade-level or subject matter groups to study the topic they want to teach (often

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Table 1 Five key tasks of lesson study and their impact on teachers

Task	Impact on individual teachers	Impact on teacher community
Task 1: Develop research theme	Consider long-term goals for students Connect daily instruction to long-term goals such as student motivation to learn	Teacher community develops shared long-term vision
Task 2: Solve and discuss mathematics task, anticipate student thinking	Develop own mathematics knowledge Develop knowledge of student thinking	See colleagues as useful resource for understanding mathematics and student thinking
Task 3: Develop shared teaching-learning plan	Refine and build own ideas about mathematics and its teaching- learning by negotiating a shared lesson plan Develop a habit of anticipating student thinking and connecting daily lessons to long-term goals	Negotiation of lesson plan builds shared ideas, reveals differences Written teaching-learning plan enables teachers to see how anticipated and actual student thinking compare Written plan allows teachers to capture their learning and revisit and spread their ideas
Task 4: Collect data during the research lesson	Develop knowledge of student thinking, focus on student thinking, and skill capturing it	Teachers focus on different students, enabling teacher community to construct picture of learning across the class Data on student thinking enables re-design and improvement of teaching-learning plan
Task 5: Conduct a post-lesson discussion	Refine ideas about mathematics teaching and learning by hearing colleagues' perspectives on instruction seen by all Develop habits of lesson analysis and refinement	Develop shared vocabulary for analysis of teaching-learning that is linked to actual instruction Develop sense of shared responsibility for all students' learning

looking at innovative curricula and research related to that topic) and they collaboratively choose or develop a "research lesson" designed to bring to life their long-term goals for student development as well as their goals for student learning about the topic. One team teaches the research lesson in a classroom, with other team members gathering data on student thinking and responses as the lesson unfolds. In the post-lesson discussion, teachers share and discuss the data they collected during the lesson, using these data to consider how the lesson might be improved and more generally to build their knowledge of teaching, learning, students and subject matter (Lewis 2002a; Lewis et al. 2009).

Lesson study originated in Japan but has spread to many other countries in recent years, and is used by both preservice and practicing teachers (Akita 2004, 2007; Cossey and Tucher 2005; Isoda et al. 2007; Lewis et al. 2006; Matoba et al. 2006; Wang-Iverson and Yoshida 2005)

This chapter will explore five tasks that together constitute the major elements of the lesson study cycle. Each task is described briefly, followed by examples from

Alma Middle School (pseudonym), a public lower secondary school serving a racially and socioeconomically diverse student body (ages 11–14). The final section of the chapter makes theoretical conjectures about the contribution of the tasks to teachers' development.

Task 1: Development of a Research Theme

The first task of lesson study is to develop a "research theme" to guide the lesson study work. The research theme allows teachers to voice their long-term aspirations for students and come to a shared set of goals. Figure 1 provides a step-by-step guide to developing a research theme. Each part of the task should be presented separately, before seeing the next part. Typically, the research theme is developed by all the teachers at a school or all the members of a class for prospective teachers, based on careful observation of the strengths and challenges of students they teach. The purpose of development of the research theme is to focus teachers on their long-term goals for student development, and to identify gaps between these goals and students' current characteristics.

While at first blush, the process of developing a research theme may not seem "mathematical," it lays the groundwork for teachers' mathematical lesson study

Part 1:
Think about the students you serve. What qualities would you like these students to have 5-10 years from now? Jot down a list of the qualities you would like your students to have if you were to meet them 5–10 years from now.
Present this prompt separately, verbally or visually, before looking at the prompts below. Have participants discuss their lists.

Part 2:
Once again, call to mind the students you serve. List their *current* qualities. Think about their strengths as well as any qualities you may find worrisome. Make a second list, of your students' current qualities.
Present this prompt separately, verbally or visually, before looking at the prompts below. Have participants discuss their lists.

Part 3:
Compare the ideal and the actual qualities you listed. Identify a gap between the ideal and the actual that you really feel merits your attention as an educator.
Have participants briefly work individually, and then share their ideas with the group.

Part 4:
Collaboratively develop a research theme—that is, a long-term goal—for your lesson study work, by *stating positively* the ideal student qualities you wish to build. For example, teachers at a school serving low-achieving students whose families had suffered discrimination chose the following goal:
"For students to develop fundamental academic skills that will ensure their progress and a rich sense of human rights."

Teachers (for example, a school faculty or a class of prospective teachers) work together develop a shared research theme.

Fig. 1 Development of a research theme work in three important ways. First, teachers focus on qualities crucial to students' *long-term* development as mathematics learners that may be neglected in daily planning, such as curiosity, persistence, or the habit of relating mathematics to daily life. Second, teachers carefully consider their students: Who are they, and what are their strengths and challenges? As they share out ideas, teachers can compare their own views of students with those held by colleagues. For example, science teachers at a California high school were shocked to realize that the teachers of the ninth graders saw incoming students as very curious about the subject matter and eager to learn, but that by twelfth grade students were seen as disaffected. Third, development of the research theme can provide motivational fuel, by connecting teachers' most central goals as teachers—such as building motivation to learn—to the particular topic under study. The long-term focus of the research theme provides a welcome counterbalance to the short-term focus of much educational evaluation, reminding us that it is important not simply whether students have learned to perform a particular procedure, but whether they have learned to do it in a way that fosters mathematical habits of mind more broadly. For prospective teachers, the research theme also provides a way of seeing what they share with colleagues, and a chance to practice negotiating some of their differences of educational goals before entering the challenging realm of planning the research lesson.

As one prospective teacher commented,

> A lot of [American] schools develop mission statements, but we don't do anything with them. The mission statements get put in a drawer and then teachers become cynical because the mission statements don't go anywhere. Lesson study gives guts to a mission statement, makes it real, and brings it to life.[1]

Development of the Research Theme

Mathematics teachers at Alma Middle School have practised lesson study since 2002, and they typically revisit their research theme each year, adjusting it as necessary to fit their current concerns. Of persistent interest to these teachers has been the very large achievement gap among students. Mathematics classes are not tracked, and they include a very wide range of student achievement levels. Hoping to build students' persistence and self-image as mathematics learners, the teachers developed in 2003 the research theme of "helping students learn to have mathematical conversations and reason mathematically." During subsequent years, as teachers noticed continuing achievement gaps among students, they expanded their research theme to include a focus on improved achievement on the state test, and they also focused on ways to increase the "status" of students who might be ignored in classroom conversations because they were not considered mathematically able by their peers. The research theme informed teachers' development of the teaching-learning plan (see Sect. Task 3: Development of a shared teaching-learning plan).

[1] I am indebted to a prospective teacher at Mills College for this remark, January 12, 2001.

For example, in order to build mathematical conversations, the teachers included in one research lesson a large visual representation that would enable students to easily share their thinking with the class. When focused on raising the "status" of low-achieving students, the teachers had students learn certain "expert skills" at the beginning of class that they could share with classmates during the lesson. The research theme helped teachers begin their work from long-term goals, such as having students show persistence and success as mathematics learners, and to consider the intermediate steps, such opportunities to engage in mathematical reasoning and mathematical conversations, that might be designed into research lessons to promote these goals.

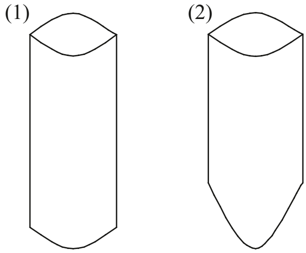
Task 2: Solve the Mathematical Task in Order to Anticipate Student Thinking

A second task of lesson study is for teachers to solve and share their thinking about the task to be given to students during the research lesson, in order to help anticipate a range of student responses. As teachers discuss their approaches, they make their mathematical thinking visible to colleagues, and teachers may expand their knowledge of solution methods in this way. These conversations may also surface difficulties or misunderstandings related to the subject matter, making problematic ideas available for discussion and revision. By solving tasks, sharing solutions, and anticipating student solution methods, teachers may build their own understanding of both mathematics and student thinking.

The following conversation occurred during an hour-long lesson study meeting at a Alma Middle School. These practicing mathematics teachers had just solved a problem from a Japanese textbook (see description of problem in Fig. 2); the problem was provided along with a range of other US and foreign resources in a toolkit designed to support teachers' lesson study on proportional reasoning. Comparing how different curricula (such as those from US and Japan) present a topic can expand teachers' thinking about what is important. When the teachers shared their solutions and anticipated how students might think about the problem, one teacher commented that some students might not distinguish between proportional and non-proportional increases. His comment sparked a conversation about whether students in their school have had an opportunity to learn to make this distinction (Video-recorded teacher meeting on 1.21.08, video timecode: 24:00–35:48):

Teacher 3: So … my belief is that some students when they attempt to answer the question will think, "More water, more depth" for the first, "More water, more depth" for the second, so they'll say both are proportional. More water, more depth: as one goes up, the other goes up, so it's kind of like correlation.

Teacher 5: And the table carries that through …



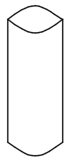
If water is poured into these test tubes, looking at these containers do you think the depth of water will be proportional to the amount of water?

For container (1) ________________

For container (2) ________________

What do you notice about the numbers in the tables below?

(1)	Amount of Water *(dl)*	1	2	3	4	5	6
	Depth of Water *(cm)*	4	8	12	16	20	24



(2)	Amount of Water *(dl)*	1	2	3	4	5	6
	Depth of Water *(cm)*	4	7	10	13	16	19

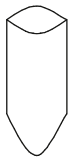


Fig. 2 How things change (Problem reproduced from Book 6A Tokyo Shoseki's Mathematics for Elementary School (p. 72). Copyright 2004 Global Education Resources (myoshida@globaledresources.com). Do not copy, reproduce or distribute without written permission)

Teacher 3: So the constant rate, that's not mentioned anywhere. *You* see the constant but they don't see the constant. Constant in the first one but not in the second one.

Teacher 4: I would really like to be able to have my students in the 7th grade be able to look at the tables and realize that the top one is dealing with equivalent fractions and the bottom one isn't. To know that aspect of proportionality, through ratio tables or, yeah

Teacher 5: Either ways, equivalent fraction or common multiplier … that they should be flexible enough to do that, and know that it doesn't apply to the second table.

Teacher 4 Um hmm. [As if he is a student solving it:] 2 times 3 is 6, 8 times 3 is 24.

Teacher 5: So do you think if you gave this to your seventh graders now, they would have an understanding of it, since you guys have finished your ratio and proportion unit? I don't think my 7th grade intervention class would do very well with this …

Teacher 3: What if it was stripped of the problem context … would they be able to look at the two tables and say which one is proportional?

Teacher 5: That's a good question. I don't know, but I like that thinking. What do you think, Teacher 2?

Teacher 2: I *don't* [think so]. Because I think that's the piece we haven't done. We've done work with ratio tables but we've kind of stated "This is a rate problem, this is a problem where you use proportional reasoning." They haven't done much to determine *when* a situation is proportional or not … when the data follows that. So I think that's sort of where we're heading with the multiple representations: being able to distinguish cases in which it should be proportional and in which it shouldn't.

The preceding conversation illustrates what teachers may learn from solving and discussing a student task and anticipating student responses to the task. Teachers identified a potential difficulty for students (distinguishing between proportional and non-proportional increase) and discussed the implications for their own teaching. Through such discussions, teachers can share and build their knowledge of student thinking. Although the teachers in this lesson study group all seemed to be clear about the difference between proportional and non-proportional increase, in other lesson study groups this task surfaced teachers' own misunderstandings of proportional increase, and enabled discussion of them.

Research suggests that these teachers are quite right in observing that students may have difficulty distinguishing proportional from non-proportional situations (Van de Walle 2007). More generally, research suggests that teachers who ground their instructional decisions in careful analysis of students' current mathematical knowledge may be better able to promote student learning (Peterson et al. 1989) and that orientation to student thinking supports continuing learning by teachers (Franke et al. 2001). The activity of solving and discussing a task in order to anticipate student solutions thus builds a core aspect of teachers' instructional skill.

Task 3: Development of a Shared Teaching-Learning Plan

Development of the Shared Teaching-Learning Plan brings together the research theme (Task 1) and the mathematical topic teachers want to focus on during the lesson (explored in Task 2), as teachers ask, "How can we help students learn about

Teaching-Learning Plan

Date:
Grade:
Subject:
School:
Instructor:
Planning Group:

1. Unit Name

2. Unit Objectives

3. Research Theme (or "Main Aim") of Lesson Study

4. Current Characteristics of Students

5. Learning Plan for Unit:
- Unit Goals or Outcomes (Connections to Standards and Prior and Subsequent Learning, if appropriate)
- Sequence of Lessons in the Unit

Number of Lessons	Content	Points to Notice and Evaluate	Materials, Strategies

- Explanation of Unit "Flow" that will Enable Students to Move From Current Understanding, Motivation, and Skills to Desired Outcomes

Fig. 3 Template for teaching-learning plan

6. Plan for the Research Lesson			
Teacher Activity	**Anticipated Student Thinking and Activities**	**Points to Notice and Evaluate**	**Materials, Strategies**

a. Aims of the Lesson

b. Learning Process for the Lesson (What "drama" of activities and experiences will help students move from their initial understanding to the desired aims?)
(*This chart may continue for several pages.*)

c. Evaluation of This Lesson (Major Points to Be Evaluated)

d. Copies of Lesson Materials (e.g. Blackboard Plan, Student Handouts, Visual Aids)

7. Background Information and Data Collection Forms for Observers (e.g. Seating Chart, Prior Student Work, Note-taking Forms, Information on Particular Students to be Observed)

Fig. 3 (continued)

this topic in a way that supports our research theme?" Figure 3 provides a template for the Teaching-Learning Plan that is developed collaboratively during lesson study. Even when the group starts, as it should, with the best available lesson plan on a topic, it may take two or more meetings to flesh out the Teaching-Learning Plan, which includes elements often omitted from standard US lesson plans—such as anticipated student thinking and data to be collected during the research lesson. Development of a shared teaching-learning plan surfaces teachers' ideas about the important content within a topic *and* about how students best learn mathematics. As teachers' thinking becomes visible, so may differences of opinion among them. The instructional plan represents the thinking of the whole lesson study group about three concentric layers of practice: the lesson itself, the larger unit and academic subject area of which it is part, and the even larger domain of students' long-term development. As a lesson study team moves on to conduct the research lesson, the instructional plan will:

- Support the research lesson teacher, by providing a detailed outline of the lesson and its logistical details (such as time allocation, needed materials and wording of key problems);
- Guide observers' data collection by specifying the "points to notice" and data to be collected;
- Help observers understand the rationale for the research lesson, including the lesson's connection to goals for subject matter and students and the reasons for particular pedagogical choices;
- Record the lesson study group's thinking and planning, so that team members can revisit it after the research lesson and notice where their thinking may have changed.

Because the instructional plan plays several important roles and because it may be quite different from the lesson plans familiar to American teachers (which tend to focus on *teacher* actions), it is useful to examine in some detail instructional plans developed by experienced Japanese or US lesson study practitioners (Global Education Resources 2006; Lesson Study Communities Project in Secondary Mathematics n.d.; Lewis 2002b; Mills College Lesson Study Group n.d., 2005; Teachers' College Lesson Study Research Group n.d.). Team members "become aware of how you think about lessons and about mathematics."[2] as each element of the plan is considered, including anticipated student thinking, the learning flow of the entire unit, how the topic connects to prior and subsequent learning and to long-term goals for students, and the data that will be collected during the lesson.

For example, one teaching-learning plan developed by teachers from Alma Middle School integrated twin goals of providing challenging mathematics tasks and implementing research-based strategies to raise the status of low-achieving students (Cohen 1994). The teachers noted in their pre-lesson brief for observers that the research lesson is designed to "allow more students to contribute *mathematically* … not just I'll be the colourer." One team member commented:

> Some of us have experimented with group roles, and that promotes experimentation, but sometimes the engagement was not at a very high mathematical level; it was "I'll be the record-keeper, and you tell me what to write." There's not much cognitive demand there. So here, it's hopefully are they engaged at a mathematically high level.

[2] Nakamura, T. p. 18, in Zadankai: Shougakkou ni okeru juugyou kenkyuu no arikata wo kangaeru. (Panel Discussion: Considering the nature of lesson study in elementary schools) in Ishikawa et al. 2001.

The team members asked observing teachers to collect data on individual students over the course of the lesson, in order to see whether low-achieving students showed increased mathematical participation after an intervention that taught them a particular "expert skill" (how to represent data in a table).

Development of the teaching-learning plan helps teachers refine their knowledge of mathematics and its teaching and learning by making their own knowledge visible and negotiating with other team members about what constitutes good instruction and important mathematical content. Researchers have documented the cacophony of competing demands on teachers and the very limited opportunity for teachers to integrate and make sense of these demands in the context of actual classroom practice (Elmore 1996). In their lesson study cycles, the Alma teachers have persistently experimented, over multiple lesson study cycles and several years, with strategies to increase the participation of low-achieving students and to build mathematical problem-solving.

Task 4: Enactment of the Research Lesson with Data Collection

As noted earlier, research suggests that teachers who ground their instructional decisions in careful analysis of students' current mathematical knowledge may be better able to promote student learning (Peterson et al. 1989; Franke et al. 2001). The fourth lesson study task, collection and discussion of student data during the research lesson, develops teachers' knowledge of student thinking. Although 4–6 is an optimal number of teachers for lesson planning, additional teachers may be invited to observe and collect data during the research lesson. For example, teachers of algebra may work as a lesson study group to plan a research lesson, and invite teachers of other mathematics classes to observe and discuss the lesson. During the research lesson, team members and invited observers carefully observe selected students throughout the lesson, collecting detailed data on their activities, speech, writing, and use of materials. These data allow the team to construct a detailed record of how the lesson "played" from the point of view of the observed students. How did they initially think about the problem? How did their thinking change or develop over time? What supported or obstructed their progress? What role did the problem design and wording, visual aids, the teacher's interventions, or comments by peers play in the development of their thinking?

Because the data to be collected vary with the specific mathematical topic, there is no single blueprint for data collection, making it one of the most challenging aspects of lesson study. However, some rules usually apply. The thinking and actions of several target students should be documented in as much detail as possible from the beginning to the end of the lesson. The target students should be selected to represent different issues the team wants to understand: for example, how does the lesson look from the point of view of high-, middle-, and low-achieving students, second-language learners, students who show little curiosity about mathematics, or other subgroups of interest. The observers should not teach or help the students or otherwise interfere with the natural flow of the lesson. (It should be explained to students that the teachers are there to investigate the *lesson*—not to evaluate students or to provide help.)

A second general principle is that the lesson should be designed to reveal as much student thinking as possible (Lesh et al. 2000). Gathering students' written work supplements the in-depth observation of selected students and provides a broader picture of learning within the class. For example, the lesson by Alma Middle School teachers, described in the previous section, was designed to help lower-achieving students take a more active role in heterogeneous small groups, by teaching these students certain mathematical "expert skills." Each observer followed a selected student to see whether and how they brought skills from their "expert" groups back to the heterogeneous groups, and how their written work on a proportional reasoning task changed after learning the "expert skill" of making a table to record data. Written work and observational data suggested that the expert skills enabled some students, but not others, to increase their mathematical participation in the heterogeneous groups. The contrast among the students was striking, with some students moving from virtually no written work prior to the "expert skills" experience to extensive written work afterwards, and other students making little apparent advance in their mathematical participation. The contrasts offered a useful reminder of the diverse experiences within a class and the power of data collection.

Task 5: Discussion of the Research Lesson

The fifth task is discussion of the research lesson. The purpose of this task is for teachers to draw conclusions about the strengths and weaknesses of their lesson design, and more generally to refine their ideas about mathematics teaching and learning based on an actual concrete sample of instruction that all members have all just seen. Figure 4 provides a protocol designed to support thoughtful, data-focused discussion of the research lesson. The protocol allows the teacher who taught the lesson to speak first, followed by the team members, who focus on presenting the data they collected on student thinking, rather than on evaluation of the teaching.

The discussion following a research lesson by Alma Middle School teachers illustrates the potential for learning about lesson design and about instruction and student learning more broadly. This proportional reasoning lesson focused on the relationship between the height of a ball's bounce and the height from which it is dropped. Students found it hard to focus on the proportional relationship because they struggled with variations in measurement of the bounce height. The observers of the lesson also noticed that although students efficiently calculated the mean of three bounces, they were not clear about the purpose of calculating the mean as a way to mitigate error. A team of elementary teachers, whose students feed into Alma Middle School, observed the research lesson. Part of the discussion follows.

1. The Instructor's Reflections
The instructor describes the hopes for the lesson, comments on anything that was surprising, and reflects on what was learned in planning and conducting today's lesson.

2. Background Information from the Lesson Study Group Members
Using the instructional plan, the lesson study team members explain their long- and short-term goals, and why they designed the lesson as they did. They may also describe how the lesson changed over time.

3. Presentation of Data from the Research Lesson
Lesson study team members present data on student thinking and behavior from the research lesson (and sometimes the larger unit of which it is part). The data may include observational notes, student work, discussion record, record of the blackboard, etc., that have been agreed upon in advance.

4. Discussion
A brief free discussion period, facilitated by a moderator, may be provided, during which additional participants add their observations. The focus is on student learning and development. Comments of a sensitive nature may be conveyed privately at a later time.

5. Outside Commentator (optional)
An invited outside commentator may discuss the lesson.

6. Thank
If the gathering is large, it is common for an administrator to thank the instructor, planners, and attendees.

Note: The items in bold are the actual agenda items from a faculty discussion following a research lesson. The regular typeface is our commentary. It is common for each speaker to preface his or her comments with an expression of thanks to the teachers who taught, planned, and supported the lesson.

Fig. 4 Agenda for discussion of a research lesson

Alma Teacher 2:	Well, I think that's really interesting [that they didn't grasp the purpose of calculating the mean] because, I think a ton of time at the beginning of the year in seventh grade is spent calculating means, so …
Alma Teacher 5:	And sixth grade.
Elementary Teacher:	And fifth grade and fourth grade.
Alma Teacher 2:	So we didn't say this is why we calculate mean, but the fact that it's not entirely clear to them says that the way we've been teaching it is … you know, I don't. No student said, "How do you calculate mean?" Like they all knew how to do it.
Elementary Teacher:	But … the purpose of doing it was not clear, which is really, sort of diagnostic, you know, do kids make sense of the power of mean not just how to do it.

Discussion of the lesson yielded ideas about how to improve lesson design; in a later version of the lesson, students received data, and were able to focus more clearly on the proportional relationship. In addition, the discussion led both elementary and secondary teachers to consider what kind of instruction would facilitate better understanding of the purpose of calculating means. As this lesson study example illustrates, using data from the lesson may yield implications for the lesson design, and also for the understanding of student learning and instruction more broadly—for example, the idea that students may efficiently calculate means without a good understanding of the purposes for doing so.

How Do the Tasks of Lesson Study Support Teachers' Learning?

Figure 5 reproduces a widely-used framework for understanding mathematics teachers' learning from and in practice (National Research Council 2001). It represents as three points of a triangle the three major types of learning *within practice*—learning from colleagues, learning from students, and learning from mathematics (from curriculum, mathematical tasks, etc.). Lesson study supports learning from each element of practice represented in the triangle. Teachers learn from each other as they consider long-term goals for students, solve and discuss mathematical tasks, collaboratively develop the teaching-learning plan, and share and discuss observations from the research lesson. They learn from students as they observe and collect data during the research lesson, and from mathematics as they study curriculum and solve and discuss the mathematical tasks. Lesson study often brings the points of the triangle into closer relationship so that teachers can draw on colleagues' ideas to help them unpack student thinking and to make sense of the mathematics in the curriculum. For example, one teacher wrote at the end of a lesson study in which she solved several mathematical tasks and then discussed

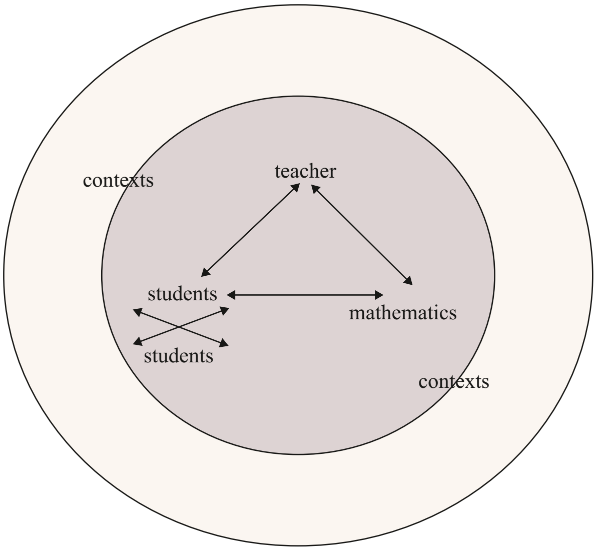


Fig. 5 How teachers learn from and in practice (National Research Council 2001, p. 9.)

them with colleagues, "The discussion with colleagues along with the review of student work opened my eyes to the many possible ways to solve the problem. Many people will have different ways to do things than me and I need to understand that to be a better teacher."[3]

Lesson study makes elements of teachers' thinking visible that might otherwise remain invisible and unexamined. For example, the lesson study cycle "How Many Seats?" (Lewis et al. 2009; Mills College Lesson Study Group 2005) surfaced a disagreement among teachers about whether it was desirable to have students struggle to organize data themselves (rather than be given an empty function table that "spoonfed" them the pattern). Teachers often expand or refine their own thinking as they encounter colleagues' ideas. For example, teachers in the "How Many Seats?" lesson study cycle adopted the idea of examining students' counting methods after watching a colleague use this strategy productively to gain insight into student thinking during a research lesson (Lewis et al. 2009). As noted above, after watching students struggle to describe the relationship between ball bounce and dropped height, the Alma teachers developed a shared realization that students could calculate a mean but did not understand the purpose of doing so.

The five tasks of lesson study described in this chapter are not "one-shot" tasks, but core elements of lesson study cycles that recur throughout one's lesson study work as a prospective and practicing teacher. Table 1 summarizes influences of these tasks on individual teachers and on the teacher community. Over time, these tasks build teachers' knowledge of mathematics, pedagogy, and student thinking, as well as habits of mind that are central to teaching, such as careful observation of students and an inquiry stance toward teaching. Beyond impact on individual teachers, lesson study also impacts the teacher community, as teachers come to share goals for students, ideas about what is good instruction, and a common language for talking about features of teaching and learning.

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Mathematical Problem Solving: Linking Theory and Practice

Berinderjeet Kaur and Tin Lam Toh

Introduction

Mathematical problem solving has been the primary goal of the school mathematics curriculum in Singapore since 1990 (Ministry of Education 1990, 2000, 2006). Such a goal is not unique as there has been a world-wide push for problem solving to be the central focus of school mathematics curriculum since the 1980s. In the UK, the Cockcroft Report emphasized that 'mathematics teaching at all levels should include opportunities for problem solving' (Cockcroft Report 1982, paragraph 249) and that problem-solving ability lies 'at the heart of mathematics' (p. 73), a means by which mathematics can be applied to a variety of unfamiliar situations. In the United States, the principles and standards for school mathematics of the National Council of Teachers of Mathematics (NCTM) stated that "Problem solving should be the central focus of mathematics curriculum" (NCTM 1989, p. 23) as it encompasses skills and functions which are an important part of everyday life. In Australia the 1990 National Statement on Mathematics for Australian Schools stated, as one of the goals, that students should develop their capacity to use mathematics in solving problems individually and collaboratively (Australian Education Council 1990).

In the teaching and learning of mathematics problem solving is critical. It is a vehicle for teaching and reinforcing mathematical knowledge and helping to meet everyday challenges. It is also a skill which can enhance logical reasoning. Individuals can no longer function optimally in society by just knowing the rules to follow to obtain a correct answer. They also need to be able to decide through a process of logical deduction what algorithm, if any, a situation requires, and sometimes need to be able to develop their own rules in a situation where an algorithm cannot be directly applied. For these reasons problem solving can be developed as a valuable skill in itself, a way of thinking (NCTM 1989), rather than just as the means to an end of finding the correct answer.

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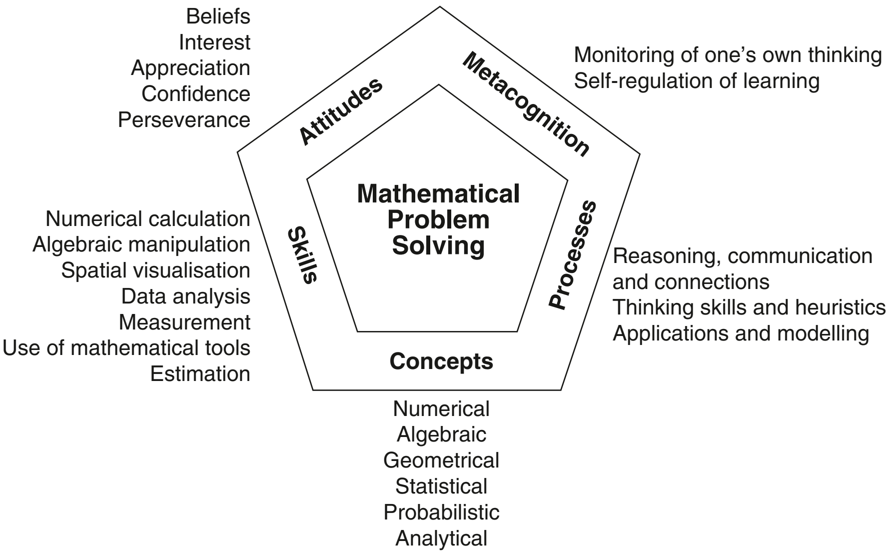


Fig. 1 Framework of school mathematics curriculum

In Singapore, mathematical problem solving is central to mathematics learning at both the primary and secondary school levels. It involves the acquisition and application of mathematics concepts and skills in a wide range of situations, including non-routine, open-ended, and real-world problems (Ministry of Education 2006). Once again, emphasis is placed on mathematical problem solving as the primary outcome of school mathematics. Figure 1 shows the framework of the school mathematics curriculum. The framework guides the teaching and learning of mathematics in Singapore schools and it is imperative that prospective mathematics teachers learn about it as part of their teacher education.

The framework highlights that the development of mathematical problem solving ability is dependent on five inter-related components, namely, Concepts, Skills, Processes, Attitudes and Metacognition, which are detailed in the secondary school mathematics syllabus document (Ministry of Education 2006). From Fig. 1, it is apparent that development in all five components is necessary for students to become successful mathematical problem solvers.

As part of the curriculum studies course, Teaching and Learning of Mathematics, prospective secondary school mathematics teachers are engaged in solving mathematical problems throughout the course. A formal introduction to mathematical problem solving and review of the relevant literature is done at the beginning of the course. As an introduction to mathematical problem solving, we engage our teachers in two tasks, The Circular Flower Bed, shown in Appendix A, and Solve 4 Problems, shown in Appendix B, to jump start discussion on mathematical problem solving and bridge theory into practice. The goals of the tasks are as follows. The Circular Flower Bed task provides prospective teachers an opportunity to engage in problem solving and initiate discussion on the process of finding a solution, specifically the feelings, emotions and regulation of thinking during the process. The Solve 4 Problems task helps prospective teachers to clarify the definition of a problem, distinguish heuristics from strategies and link their "steps taken" during problem solving with Polya's (1973) four phases of problem solving. Elaborations of the terms, heuristics, strategies and Polya's (1973) four phases of problem solving follow in the next section which also outlines the background of the tasks. The tasks are also presented and the nature of responses they illicit from the teachers discussed.

Background to the Tasks

To engage in problem solving one has to confront a task which is a problem. Hence to identify a task as a problem, the definition of a problem as spelt out by Charles and Lester (1982, p. 5) is used, this being a task for which:

- The person confronting it wants or needs to find a solution.
- The person has no readily available procedure for finding the solution.
- The person must make an attempt to find a solution.

This definition emphasizes three crucial components of a problem. Firstly, a desire or need on the part of the problem solver to find a solution to the problem, secondly the solution cannot be obtained directly or immediately by mere recall of knowledge, and thirdly the problem solver must make a conscious attempt to arrive at the solution.

In solving a problem one has to engage in a complex process that requires an individual to coordinate previous experiences, knowledge, understanding and intuition, in order to satisfy the demands of a novel situation. In simple terms it is the mental journey one takes to arrive at a solution starting with the "givens" of a situation. According to Charles and Lester (1982), generally three factors influence the problem-solving process of an individual. They are:

1. experience factors, both environmental and personal, such as age, content knowledge;
2. familiarity with solution strategies, familiarity with problem context and content;
3. affective factors, such as interest, motivation, pressure, anxiety, tolerance for ambiguity, perseverance, and so on;
4. cognitive factors, such as reading ability, spatial ability, analytical ability, logical ability, computational skill, memory, and so on.

In problem solving, the terms *strategies* and *heuristics* are often used to describe certain approaches and techniques used in the solution process. These two terms are often used interchangeably or at times used jointly as "heuristic strategies" to mean the same. In this chapter, we use the word "strategy" to mean an overall plan and heuristic to mean a specific technique or approach. Polya (1973) stated that there were four phases in the process of problem solving. They are: Understand the problem; Devise a plan; Carry out the plan; and Look back. It is important to note that these phases are not linear as a problem solver may proceed from the first to the second and return to the first to check for correctness of his understanding or a problem solver may proceed from the first to the second and on to the third before returning to the first again to clarify some doubts that may have surfaced due to the nature of the resulting answer.

To meet our goals of engaging prospective teachers in problem solving and initiating discussion on the process of finding a solution, specifically the feelings, emotions and regulation of thinking during the process we needed a task that is a problem to most if not all of our prospective teachers. Over the course of our work with many secondary school teachers, we piloted several tasks and found that The Circular Flower Bed Task met our criteria i.e. most of the teachers we asked to solve it were unable to do it like an exercise. Hence we selected it. To be able to find a solution to the problem posed prospective teachers had to:

- accept the challenge and be interested in finding a resolution;
- draw on their mathematical knowledge of concepts and skills;
- use their process skills to analyse, construct logical arguments, and apply mathematical knowledge; and
- engage in metacognition to regulate their thinking.

The prospective teachers were given the Circular Flower Bed task to do at the onset of introduction to mathematical problem solving and as such no prior knowledge of problem solving was reviewed or expected of them. The second task, Solve 4 Problems, was given to the prospective teachers after an extensive class discussion of the solutions to the Circular Flower Bed problem. The formulation of the second task was guided by the definition of a problem (Charles and Lester 1982) and the general strategies (Polya 1973) and heuristics that may be used in the process of problem solving (Ministry of Education 2006).

A Circular Flower Bed

This task, sourced from an internet website, is used to engage the prospective teachers in collaborative problem solving. The mathematical structure of the task draws on some basic mathematical knowledge of geometry and trigonometry which is within the grasp of the teachers. The framing of the task makes it an interesting problem because it is non-routine and the many possible solutions make it a mathematically enjoyable one because it is a closed yet open kind of mathematical task. The instructions posed to the prospective teachers as part of the task are intentional. They are meant to guide the prospective teachers in thinking about aspects of the solution process, in particular their emotions, metacognition, and use of mathematical knowledge and mathematical processes when solving the problem. The prospective teachers are asked to work in pairs as we want them to question each others thinking. This is something we encourage them to do throughout the course. Furthermore we also like them to experience working collaboratively, as many of them will later engage their own students in such activities.

Solve 4 Problems

This task is given to the prospective teachers as an extended piece of work after the solutions to the first task described in this chapter have been extensively discussed in class clarifying the concepts "problem" and "problem solving". The prospective teachers are given this task as an assignment. They spend two weeks working on it before submitting their work for grading and subsequent class discussion on general problem solving strategies and heuristics. The prospective teachers were asked to select four mathematical problems from a collection shown in Appendix C. The mathematical problems are taken from Kaur and Yeap (2006, p. 330).

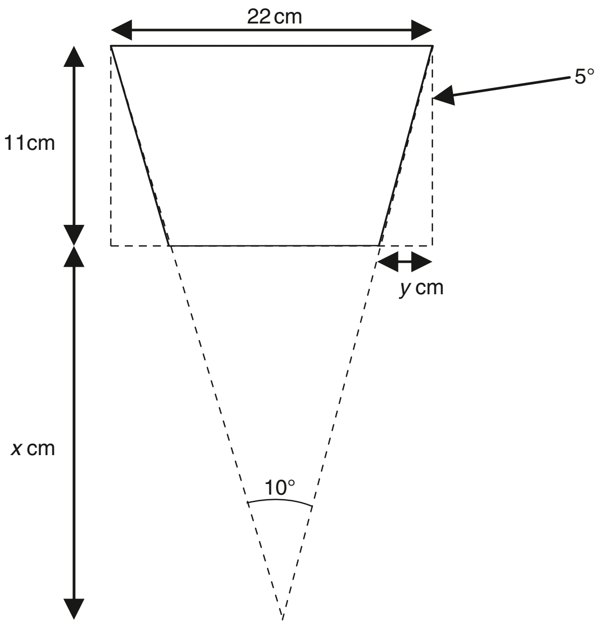
This task requires prospective teachers to select mathematical problems that are problems to them and solve them. They record their attempts, detailing the strategies and heuristics that they used to solve each problem. After solving the four problems they reflect on their problem solving process and generalize how they solve problems. Finally they are asked to compare their generalization of the process with that of Polya's (1973), i.e. the four phases of problem solving: understand the problem, devise a plan, carry out the plan and look back.

Development of the Theory and Practice Linkage

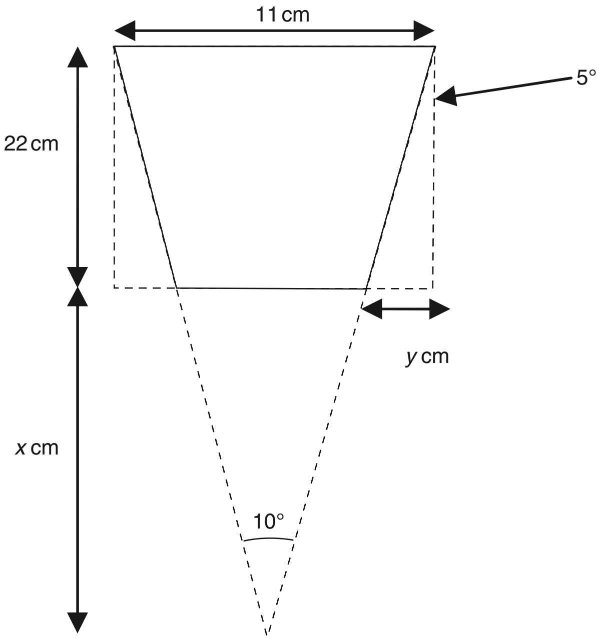
This section discusses the nature of classroom discourse that follows the completion of the above two tasks by prospective teachers. The objective of the classroom discussion is to link their practice to theory.

A Circular Flower Bed

This task takes two lessons to complete. Each lesson is three hours in duration, inclusive of a 15 minutes break. During the first lesson prospective teachers are given 2.5 hours to do the task. After completion of the task prospective teachers display their flower beds on the side boards that line their tutorial rooms. Their models are made to scale. From the flower beds displayed their interpretations of the flower bed are apparent. The main type of flower bed constructed has an annulus of 11 cm, an outer circumference of approximately 36×22 cm and a radius of $x = 11\ (\cot 5° - 1)$.



The not so common flower bed constructed has an annulus of 22 cm, an outer circumference of approximately 36×11 cm and a radius of $x = 11\ (\cot 5° - 2)$.



As mentioned earlier on, in the chapter that this is a mathematically enjoyable task as it has the scope for the teachers to go beyond the above two possibilities. They may decide to cut the 22 cm by 11 cm bricks into more rectangular bricks to make the "circular flower bed". Some prospective teachers have actually shown in their solutions varying sizes of the flower bed as well as the concept that as the width of the brick gets smaller and the number of bricks gets larger the shape of the flower bed tends to a better approximation of a circle.

In the next lesson, the class discussion begins with the question "How did you solve the task?" From their responses, we make notes on the board and draw commonalities amongst their experiences leading to a distinction between having an algorithm to solve the task and exploration (i.e., having no obvious means of resolving the task drawing on prior experience) leading to the solution of the task. From our experience working with prospective teachers, this task is a problem, in the sense of the definition, to many of them.

When sufficient inputs have been drawn from the prospective teachers, we encourage them to define a problem in their own words. After presentations by several teachers of what a problem is, we share with them Charles and Lester's (1982) definition of a problem. Next we focus on the process they undertook to arrive at the resolution of their problem by asking them to share with the class their feelings, emotions and regulation of thinking. The many and varied responses are categorized by us on the chalkboard leading to factors such as *concepts, skills, attitudes, metacognition* and *processes*. When all the inputs from them have been exhausted we share with them the framework of the school mathematics curriculum, shown in Fig. 1, which encapsulates the five factors. To all of them, this pentagonal shape is new knowledge but having drawn on many of the five aspects while solving the task they are often impressed by its succinct representation of mathematical problem solving.

We next ask them to share with the class their list of content knowledge needed to solve the problem. This enables the class to see that the problem can be solved with different levels of content knowledge, such as plane geometry, measures of circles or trigonometry. During the last phase of the discussion we draw on their responses to questions *d* and *e* so that they may see the different perspectives of their classmates or peers and consider the task as an open-closed one which has several solutions depending on the assumptions they make. In particular responses to question *e* uncovers many of their beliefs as students. A common response is "no, the task did not ask for it". The class discussion for the task stops short of introducing the students to the work of Polya (1973) and problem solving heuristics.

Solve 4 Problems

Prospective teachers are given two weeks to complete this task as out of class assignment. Along with the task sheet they are given a sheet of paper containing Polya's (1973) four stages of problem solving. They submit their assignments which are graded by us and returned to them. What follows is a session, three hours in duration inclusive of a 15 minutes break, of class discussion on the four problems, strategies, heuristics and framework for problem solving.

The class discussion is initiated by the question “How did you select your problems?” which lends to the listing of their responses on the board which are in turn summarized and Charles’s and Lester’s (1982) definition of a problem revisited. From the collection of 20 problems (Appendix C), all the teachers do find four problems for themselves to solve. Next, several of them share with the class specific solutions to some of the problems. After every presentation of a solution to a problem by a prospective teacher, others in the class are encouraged to share alternative solutions to the problem. The solutions are studied and strategies as well as heuristics used in the process of problem solving highlighted by us with inputs from the teachers. From the presentations it becomes apparent that heuristics are not problem specific while strategies are general plans that guide the problem solver through the process just like in a cooking lesson—first collect the ingredients, next prepare the ingredients, then cook the dish and finally taste it and see if any thing may be improved or improvised!

The discussion proceeds from solutions of specific problems to generalizations of their problem solving trajectories. We write on the board, “When I am given a problem to solve I …” and ask them to get in groups of four and complete the statement on flip charts. They are given 15 minutes to complete the task before coming forward to the board and displaying their charts along the side boards of the tutorial rooms. Next, the teachers are asked to do a gallery walk before returning to their places. During the last part of the session, they are asked to share with the class how generalizations of their problem solving process compare with that of Polya’s (1973) four stages: Understand the problem; Devise a plan; Carry out the plan; and Look back. Many of the prospective teachers report that their generalizations are lacking of the fourth stage, i.e., looking back. They are often pleased with a solution and this has always been expected of them. The session ends with a discussion of what looking back may entail other than checking for the answer or reasonableness of.

Conclusion

As mathematical problem solving is the primary goal of the school mathematics curriculum, it is essential for our prospective secondary school teachers to clarify the concepts and skills of mathematical problem solving during their teacher education. The first task, A Circular Flower Bed, engages them in solving a problem and initiates an exploration of “what a problem is” and discussion of the feelings, emotions and regulation of thinking during the process of solving it. The second task, Solve 4 Problems, facilitates the clarification of the definition of a problem and distinction of heuristics and strategies. It also engages the teachers in reflecting about their problem solving process and making connections with Polya’s (1973) four phases of problem solving. Being mindful of the fact that our prospective teachers do not develop the conception that problem solving is a topic of the curriculum but rather a part of every topic we engage them in mathematical problem solving throughout the course. At appropriate junctures, we introduce them to the three types of problem solving lessons, that are: teaching for problem solving, teaching about problem solving and teaching via problem solving.

Appendix A

A CIRCULAR FLOWER BED

A landscape gardener uses exactly 36 paving bricks 22 cm by 11 cm to form a "circular" flower bed.

However, the gardener does not want any spaces between each brick, so she cuts the bricks so that the face of each brick is in the shape of an isosceles trapezium.

Determine the shape of the required brick.
Make a model from cardboard.

Give details of the steps taken. State any assumptions that you have made. Clearly show all mathematical calculations.

Source: http//smard.cqu.edu.au/database/junior/space/trigonometry/doc1.rtf (retrieved on 7 April 2008)

Instructions
What you have to do in pairs?

a) Solve the problem.

b) Reflect on your journey of solving the problem and make notes of your feelings? Emotions? How did you regulate your thinking?

c) List all the content knowledge that is needed to solve the problem.

d) Does this problem have a unique solution?

e) Did you attempt to go beyond the first solution you arrived at?

Appendix B

SOLVE 4 PROBLEMS

Select any 4 problems from the given collection.
Make sure that each of these problems is a "**problem**" to you.

Attempt to solve each of the problems.
Note that you may take an extended period of time to solve a problem.
Do not despair.
Record your attempts in detail together with your reflections alongside.
Finally list the strategies and heuristics (rule of the thumb) you used to solve the problem.

Your presentation may be as follows

Problem:	Reflection
1st attempt	
2nd attempt	
nth successful attempt	
Strategies -	
Heuristics –	

After having solved the 4 problems over a period of time you are expected to:

- generalize how you solve problems

compare your generalization with that of Polya (1973)

Appendix C

- How many squares are there on a standard (8×8) chessboard?
- Karen has to number the 396 pages in her biology book. How many digits will she have to write?
- Into how many different plane regions do n lines, no three of which are concurrent and none of which are parallel, separate the plane?
- Find all rectangles with integral sides whose area and perimeter are numerically equal.
- What is the maximum number of regions into which n chords divide a circle?
- In the "equation", $(he)^2 = she$, the letters represent digits and the configurations represent numerals in base 10. Find the replacements for the letters that make the statement true.
- Find the last three digits of 1995^{1995}.
- Two circles are concentric. The tangent to the inner circle forms a chord of 12 cm in the larger circle. Find the area of the "ring" between the two circles.
- A rectangle 4 by 3 has six squares passed through a diagonal. Find the number of squares passed through by a diagonal for a rectangle of size m by n.
- A palindrome is a number that reads the same backwards as forwards. How many 4-digit palindromes are there? Show that all palindromes are divisible by 11.
- Tennis balls are packed in cylindrical cans of three. The balls just touch the sides, top and bottom of the can. How does the height of the can compare with the circumference of the top? What is the ratio of the volume of a ball to that of the can?
- The Tans are having a party. The first time the doorbell rings, a guest enters. On the second ring, three quests enter. On the third ring, five guests enter, and so on. That is, on each successive ring, the entering group is two larger than the preceding group. How many guests will enter on the 15th ring? How many guests will be present after the 15th ring?
- Two towns lie to the south of a straight road, but they are neither connected to it, nor to one another. The citizens of the two towns decide to build two roads, one from each town, to the existing road. They are cost conscious in the choice of the roads. Find the shortest route connecting the two towns via the existing road.
- A curious biological fact is that a male bee has only one parent, a mother, whereas a female bee has both a mother and a father. How many second-generation ancestors (grandparents) does the male bee have? Third-generation ancestors (great-grand-parents)? Fourth? Fifth? Tenth?
- A cake that is in the form of a cube falls into a large vat of frosting and comes out frosted on all faces. The cake is cut into small cubes of the same size. The cake is cut so that the number of pieces having frosting on three faces will be 1/8 the number of pieces having no frosting at all. Find the total number of small cubes.
- Five women are seated around a circular table. Mrs Ong is sitting between Mrs Lim and Miss Mah. Ellen is sitting between Cathy and Mrs Ng. Mrs Lim is between Ellen and Alice. Cathy and Doris are sisters. Betty is seated with Mrs Png on her left and Miss Mah on her right. Match the names with the surnames.
- A new school has exactly 1000 lockers and exactly 1000 students. On the first day of school, the students meet outside the building and agree on the following plan: The first student will enter the school and open all of the lockers. The second student will then enter the school and close every locker with an even number (2,4,6,8, etc.). The third student will then "reverse" every third locker. That is, if the locker is closed, he will open it; if the locker is open, he will close it. The fourth student will reverse every fourth locker, and so on until all 1000 students in turn have entered the building and reversed the proper lockers. Which lockers will finally remain open?

- Twelve golf balls appear identical but 11 weigh exactly the same while 1 is either lighter or heavier than the others. Determine the odd ball and whether it is lighter or heavier than the others in as few weighting of the balls on a balance as possible.
- Sixty-four cubes are assembled to form a large cube. The face of the large cube is then painted. How many of all the small cubes are untouched by paint? How many of the small cubes have (a) one face, (b) two faces, and (c) three faces painted?
- After gathering a pile of coconuts one day three sailors on a deserted island agreed to divide the coconuts evenly after a night's rest. During the night, one sailor got up, divided the coconuts into three equal piles with a remainder of one, which he tossed to a monkey that was conveniently near by, and, secreting his pile, mixed up the others and went back to sleep. The second sailor did the same thing, and so did the third. In the morning, the remaining pile of coconuts (less one) is again divisible by 3. What is the smallest number of coconuts that the original pile could have contained?

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Part III

Guiding Mathematical Inquiry in Mobile Settings

Michal Yerushalmy and Galit Botzer

Introduction

Engaging students in active exploration of real-life scenarios and supporting inquiry processes are major challenges for mathematics teachers. It requires a shift in the teacher's role from lecturing and telling to listening, observing, facilitating, and guiding. It also requires new considerations in choosing curricular materials, with attention to both textual materials and technological support. Important questions focus on the ways in which particular pieces of software, for example, do or do not support particular pedagogical methods and goals or how they work in concert with them. As teachers consider using technology with their students, they make many decisions. What technology should they use? How does it support what they want to do with their students? And, what sorts of tensions might arise from the use of this technology? To answer these questions teachers must use their understanding of the curricular approaches they wish to adopt and must develop their own approaches to teaching. Current teacher education programs, as described by Chazan and Schnepp (2002), "…often devoted substantial attentions to helping pre-service candidates envision the type of teacher that they would like to be" (p. 192). In particular, engaging in tasks within technology enhanced learning environments designed to support inquiry-based learning enable teachers to rethink and revise their pedagogical, curricular, and subject matter knowledge. Use of technology anywhere anytime is one of the notable qualities of mobile learning. Students can use mobile applications to gather, access, and process information outside the classroom, and to bridge school, afterschool, and home activities. In our exploratory work we found that mobile devices can offer a challenging setting for educators to deepen their thinking about sensing mathematics and about socially constructing and mediating mathematical knowledge. The qualities of this unique setting are grounded in the

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mobility that enables learners to share knowledge through mobile applications and tools. We consider mobile learning to be an important aspect of future changes in the curriculum and in the nature of classroom. For these changes to occur, prospective and practicing teachers must learn to interact in this novel setting and develop new communicative skills. Attempting to analyze the challenging aspects of mobile learning and teaching with cellular phones, Shuler (2009) suggested that it is necessary to address the cultural norm and attitude whereby teachers regard mobile phones as distractions and consider that they have no place in school. It is difficult to change this norm because only a few attempts have been made to establish a mobile theory of learning (e.g., Pachler et al. 2010; Sharples 2009), and there are no assessments of sustained effectiveness. In our work we attempt to support prospective teachers, intending to teach in innovative settings, to conduct an effective mathematical mobile discussion with their peers and future students. "Mathematical mobile discussion" refers to learning interactions in which participants use personal mobile devices (cellular phones) to create and manipulate mathematical objects and to communicate about them. Participants can work both in face-to-face and remote settings. In both settings the challenge lies in linking personal and collaborative manipulations and understandings, which requires a moderator to guide the mobile discussion and set norms of collaboration. "Moderator" refers to a teacher educator who directs a discussion among prospective teachers. We attach great importance to developing norms and guidelines for mathematical discussion in a setting that enables practicing teachers to moderate mobile discussions in their future classes.

This chapter focuses on tasks and activities related to the goals of teaching the construction of mathematical models in algebra and calculus. It starts with the theoretical considerations underlying the design of inquiry tasks in mobile settings for prospective and practicing teachers. It then presents mathematical tools and tasks for mobile inquiry and discusses the integration of these tasks in teacher education sessions. In the final section the chapter presents opportunities and challenges for mathematics educators in mobile learning settings, discusses how mobile tools can help social constructivist teachers fulfill their commitment of guiding active inquiry, teaching skills, and covering the curriculum, and suggests further research directions.

Designing Inquiry Tasks in Mobile Settings: Theoretical Considerations

Our goal in designing task-based situations for prospective teachers is twofold. First, we attempt to design innovative examples that create opportunities for educators to learn through mobile guided inquiry and explore the type of teaching in which they would like themselves and their students to be involved. Second, guided inquiry of real-life scenarios and collaborative activities using handheld devices are expected to reveal the socio-cultural aspects of knowledge construction and to be based on these aspects. We suggest that the learning instances created around these specially designed tasks offer opportunities for identifying the teachers' needs and open new vistas for research of mobile learning and teaching.

Modelling and Representing Real-Life Phenomena in Secondary Mathematics

A mathematical model is a mathematical construction that describes a class of phenomena external to mathematics (e.g., temporal phenomena). In this sense, modelling uses mathematical language for reasoning about the phenomenon. There is a wide range of pedagogical approaches concerning the link between the phenomenon and the mathematical model. In the traditional application approach the mathematical symbols and operations are taught first and only later do students interact with the signified physical field to model the situations. Shternberg and Yerushalmy (2003) claim that in this case the link between the physical field and mathematics is weak and even artificial, and the construction of formal mathematical language often remains meaningless and cannot be applied later. Alternative approaches start with applications or daily life situations, assuming that learners can describe the situations in ordinary language and proceed from there to formal mathematical language. Cobb (2002) and Gravemeijer and Stephan (2002) both recommend "guided reinvention," in which formal mathematics should grow out of the students' activity. They describe how the process of modelling grows from construction of *model of* a phenomenon in a specific context which then becomes a *model for* representing a mathematical reality and a point of reference for more formal mathematical reasoning. We regard such process of modelling as an important tool for inquiry based learning mathematics. Familiar situations provide meaning to a mathematical concept, and the mathematical concept facilitates deeper understanding of the situations. Lehrer and Schauble (2000) suggested that models are vehicles for big ideas in mathematics and science, and serve the design of instructional activities. To become more effective and meaningful, mathematics should not be expected to emerge only from specially structured situations and must be accompanied by specially designed mediators. Learning technology can integrate both representational tools for augmenting mathematical cognition and mediation tools for social participation in the practice of mathematics (Roschelle et al. 2007), which requires careful design of inquiry tasks and support for teacher attempts to mediate such activities. Carrejo and Marshall (2007) argued that,

> As teachers immersed in a modelling environment move within the realms of personal experience, mathematics, and science (e.g., physics), emerging tensions in student learning (and their own) could become apparent to them. If teachers are to move effectively between these realms, they must make choices on how to relieve resulting tensions within themselves and their students; such choices have a profound impact on the use of modelling approaches in the classroom (p. 48).

Therefore, by engaging in modelling tasks, mathematics teachers can rethink their pedagogical, curricular, and subject matter knowledge, connect mathematical knowledge with real-life contexts, and interact socially in ways that support the creation of a community of proficient mathematics teachers.

Social Interactions and Mathematics Teaching and Learning

One of the instructional views that requires the teachers' attention and a shift in the habits and goals of mathematics teaching has to do with the belief that knowledge is socially constructed. Construction of knowledge is best supported through collaboration. It should be designed to enable participants to share knowledge and carry out projects that incorporate teamwork, real-world content, and the use of varied information sources (Scardamalia and Bereiter 2002). Leikin (2004) described processes occurring through constructive engagement in tasks that help in the acquisition of knowledge and reinforce individual habits of mind, but also provide opportunities for working together, sharing knowledge, inspiring each other, and applying active social interactions. The social construction of knowledge is of primary significance for mobile learning. Sharples (2000) proposed a theory of personal learning mediated by mobile technology, founded on social constructivist theories. Handheld devices can improve classroom dynamics because their data connectivity supports social interaction and collaboration (Low and O'Connell 2006; Naismith et al. 2004; Hoppe et al. 2003). Naismith et al. (2004) suggested that mobile devices provide a "shared conversation space" (p. 27) that enables people to share their descriptions of the world and construct common understandings and knowledge. The communication capabilities of mobile devices can augment face-to-face interactions (Liu and Kao 2007). In particular, the communication capabilities of mobile phones have promising implication for learning because of their high adoption rate among school-age children and the active part they play in our social practice (Wagner 2005). Wei et al. (2007) reported on the integration of voice conversations through mobile phones into a Web discussion forum enabling learners to extend learning experiences anytime anywhere in order to facilitate the exchange of voice and text knowledge. However, there are only a few reports on using communication capabilities for learning with mobile phones.

Handheld Devices in Mathematics Education

Roschelle et al. (2007) reviewed three successful implementations of handheld devices in mathematics education: graphing calculators, classroom response systems, and probeware, which have produced valuable improvements in school learning. The success of graphic calculators can be attributed to the fact that they provide students with multiple linked representations, especially a combination of linguistic and graphical representations, that can produce significant learning gains. Networked response systems are participatory and feedback tools that with teacher mediation can increase the students' engagement in learning, enhance classroom communication between teacher and students, and provide an opportunity for peer instruction (Mazur 1997). The networked response system enables rapid classroom interaction with a small-size task. For example, students can be asked to draw a graph line, provide their responses as input into a personal computing device such as a graphing calculator or palm computer, after which the teacher's desktop machine collects the students' responses and can reveal common patterns in them (Hegedus and Kaput, 2003). Hegedus and Kaput suggested that the integration of a dynamic software environment with connectivity can dramatically change students' engagement with core mathematics. Probeware uses probes and sensors with associated software. For example, probeware that models motion uses sensors and software to record the motion and provide mathematical representations of it. Nemirovsky and Borba (2003) suggested that "the use of appropriate materials and devices facilitates the inclusion of touch, proprioception (perception of our own bodies) and kinesthesia (self initiated body motion) in mathematics learning" (p. 103). Kaput and Roschelle (1997) emphasized,

> the important role of physical motion in understanding mathematical representations… [whereby] students confront subtle relations among their kinesthetic sense of motion, interpretations of other objects' motions, and graphical, tabular, and even algebraic notations (p. 106).

Probeware incorporates elements of representations (instantly graphing data) and feedback (students quickly obtain feedback on the collected data). *Math4Mobile* applications (Yerushalmy et al. 2006) are similar in their computational capabilities with those of the handheld tools described above, but they offer new learning opportunities in a 1:1 setting. *Math4Mobile* challenges known aspects of social interactivity and connectivity, context sensitivity, and the use of personal technological tools in a new type of learning in which mobility, availability, and flexibility are the key terms.

Mathematical Tools and Tasks: Designing a Setting for Mobile Inquiry

Math4Mobile Applications

In recent years the voice function of mobile phone ceased to be the only dominant one. Textual and visual communications and the use of web resources and applications (online and local) are fast becoming central functions of mobile communication. The ability to use the devices to send graphs and formulas to other students as short text messages (SMS), the communication capabilities of the mobile phone, and the availability of cellular accessories such as cameras can be used to enhance

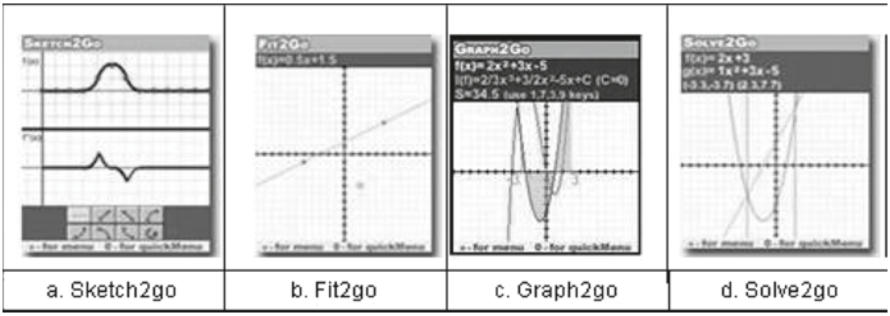
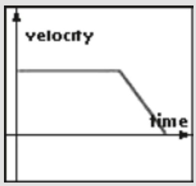


Fig. 1 Graphing applications in the Math4Mobile environment

the learners' engagement with mathematics learning. The graphing applications developed within the *Math4Mobile* project are designed to support the qualitative and quantitative inquiry of temporal phenomena described by single variable functions. According to Johnson et al. (2009), the ability to run third-party applications represents a fundamental change in the way mobile devices will be used in education. We describe here four *Math4Mobile* applications and illustrate each one with a task that can be integrated in teacher education sessions.

Sketch2Go (Fig. 1a) is a qualitative graphing tool based on the results of a long-term research and development project carried out by Schwartz and Yerushalmy (1995) and Yerushalmy and Shternberg (2001), who propose an intermediate bridging representation based on the function and its vocabulary. Graphs are sketched using seven graphic icons that describe the change in both the function and its rate of change. The seven icons represent constant, increasing, and decreasing functions that change at constant, increasing, or decreasing rates. The application provides immediate feedback on the drawn graph by presenting a derivative graph. The following exercise sent as an SMS message demonstrates an activity with the tool.

SMS Sketching Exercise When you receive this velocity vs. time graph by SMS, use the *Sketch2Go* application to sketch a corresponding position vs. time graph. Compare your derivative graph with the given graph to check whether your graph is correct.



Fit2Go (Fig. 1b) is a linear and quadratic function graphing tool and curve fitter that supports data collection and measurement, and highlights the numerical aspects of a phenomenon by proposing a model that can appropriately describe the user's data. The following is an example of a task planned for a face-to-face setting in which learners work on their own with their personal mobile phone:

In-Class Measuring Task: The Motion of a Toy Car

1. Observe the motion of the toy car in the inclined plan. Predict how the position vs. time graph will look and sketch it.
2. Take several measurements of position and time. Use a ruler to mark the line at equal distances on the inclined plan and the stop-watch in your cell phone to measure time.
3. Use the *Fit2Go* application to mark the (x, t) points and find a graph that traverses the points you marked. Compare the resulting graph with the one you predicted.

Graph2Go (Fig. 1c) is a special-purpose graphing calculator that operates on given sets of function expressions. Its contribution to modelling, as that of other graphing tools, can be essential in exploring given or conjectured symbolic models. Its unique feature is enabling the dynamic transformation of functions. Thus, by parameterizing an example, students turn it into a family of functions. Research suggests that this type of dynamic control creates a kinesthetic relation between the user and the object on the screen and can play an important role in developing a deeper understanding of the mathematical concept (Kieran and Yerushalmy 2004; Sever and Yerushalmy 2007). The following task takes advantage of the dynamic control of functions in *Graph2Go*.

Exploration Task: Analyzing a Position vs. Time Function The following expression describes the position vs. time of the motion of a toy car.

$$x(t) = 2t^2 + 3t + 2$$

1. Use *Graph2Go* to explore this function and determine the meaning of each of the coefficients for the motion of the car.
2. Display the graph of the derivative. What features of the motion does the graph of the derivative describe?
3. Change each of the coefficients of the x(t) function, and explore the graph of the derivative, and interpret the graph with reference to the motion of the toy car.

This exploration task, when introduced to prospective teachers motivated a group discussion in which participants use their cellular phones to manipulate the function. It can also be posed in a mobile distance setting in which participants solve the task at their own pace and can receive online support. The moderator can elaborate the mathematical discussion by encouraging the group to collaborate, share information, present and assess each other's solutions, and justify their work.

Solve2Go (Fig. 1d) supports solving equations and inequalities by means of conjectures based on visual comparisons of two processes. Conjectures can be refuted or supported by examples provided by the tool, and ought to be proven using symbolic manipulations on paper.

SMS Solving Exercise The following expressions describe the position vs. time of two race cars.

When will car A pass car B?

$$A: x(t) = t^2 + 3t + 1; \quad B: x(t) = 3t + 5$$

Sequence of Tasks for Secondary Mathematics Teacher Education

The following sequence of tasks was designed to engage teachers in modeling taking advantage of the computational capabilities of *Math4Mobile* applications and of the accessories of mobile phones. The sequence combines activities in a mobile setting with face-to-face activities, and involves both personal and collaborative challenges. Parts of the task presented here were examined in a learning experiment with prospective teachers (Botzer and Yerushalmy 2007; Genossar et al. 2008). The sequence presented here has been extended and refined based on our observations.

Rationale, Goals, and Context

The main consideration that led the design of the following task was to enable teachers to link their everyday experiences with mathematical content and enable them to explore within themselves how they can bring real-life contexts to their classrooms. Although we expect teachers to be familiar with formal mathematics for modelling tasks, we followed the "guided reinvention" approach (Cobb 2002; Gravemeijer and Stephan 2002) in which formal mathematics grow out of the learner's activity. Therefore, the task sequence begins with open-ended tasks that involve documentation and analysis of daily life situations, followed by more structured modelling tasks. This sequence can serve as a model to guide the work of prospective and practicing teachers. Naturally, teachers would have to rewrite these tasks when presenting them to their students (e.g., including more detailed instructions or helping students determine which application to use).

Another important consideration in the design of the tasks was engaging teachers in mathematical discourse. Each task requires learners to present and justify their work, comment on their colleagues' work, and obtain feedback from the moderator and from their colleagues. The unique characteristics of mobile tools and their communication capabilities can contribute to the development of sociocultural norms.

The tasks can be integrated in teaching methods courses in algebra and calculus, with a potential to facilitate prospective teachers' understanding of the potential and complexity of modelling real-life scenarios. The tasks can also be integrated in courses for practicing teachers to encourage them to enrich learning practices in their classrooms.

Task Sequence

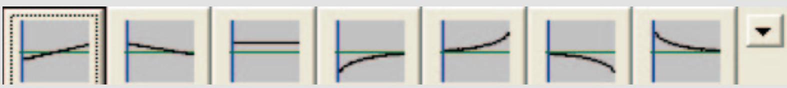
The task sequence includes exploration tasks in a mobile setting followed by face-to–face discussion and by modelling tasks in a mobile setting.

Videotaping Real-Life Phenomena

a. Use the mobile phone camera to document phenomena of change that occur anywhere in your environment (school yard, home, the road, sports field, etc).
b. Send the video clip to the group (your colleagues and the moderator) by MMS. Add a short verbal description of the phenomena with reference to the changing quantity and to the pattern of change. Use one of the *Math4Mobile* applications to sketch a temporal graph and send it to your colleagues and to the moderator.
c. Watch your colleagues' videos and read you're their descriptions. Determine whether the descriptions fit the clips. Justify your position and send your comments to the group.
d. Read the comments of your colleagues and of your moderator, and reconsider your description. Send your comments or the refined description to the group by SMS.
e. Continue to comment on your colleagues' work and refine and consolidate your own work. You may reach a consensus. If not, at the next meeting present the points of disagreement.

Analyzing the Videotaped Phenomena According to the Pattern of Change

1. Present the phenomena that you videotaped and the graph that you sketched to the group, refer to the comments that you received from the moderator and from your colleagues, and explain how they affected the graph you finally sketched.
2. The following icons present 7 different patterns of change:



Interpret each icon and describe the pattern of change that it represents.

3. Describe the group's collection of video clips and additional videos presented by the moderator using the 7 icons.

Modelling Task: Stopping Car A car travels at a speed of 20 meters per second when the driver sees a ball rolling on the road. The driver's reaction time is one second (reaction time is the time that passes between identifying the ball and pressing the brakes.) During that time the car continues at its constant speed. After the driver presses the brakes, the car decelerates for 2 seconds and stops.

1(a) Use *Sketch2Go* to describe in a graph the distance that the car traveled from the time the driver saw the ball until the car stopped. Send the graph to your colleagues and to the moderator by SMS.
1(b) Predict whether the graph will change in each of the following scenarios, and if so how: (i) The driver drove faster; (ii) the driver was drunk; (iii) it was a rainy day. Send your prediction to your colleagues and to the moderator
1(c) Use the dragging capability of *Sketch2Go* to modify the original graph to describe situations i-iii. Send the graphs to your colleagues and to the moderator.

Presenting the Task in Teacher-Education Sessions

The videotaping task takes advantage of the accessories of the mobile phone and of the *Math4Mobile* applications to enrich traditional graphing tasks with real-life context. The main challenge is to videotape a phenomenon that is simple enough to identify the pattern of change. Selection of the phenomena requires an abstraction of reality and focusing on certain aspect of the phenomena. Figure 2 presents phenomena that were videotaped by prospective teachers (Botzer and Yerushalmy 2007; Genossar et al. 2008).

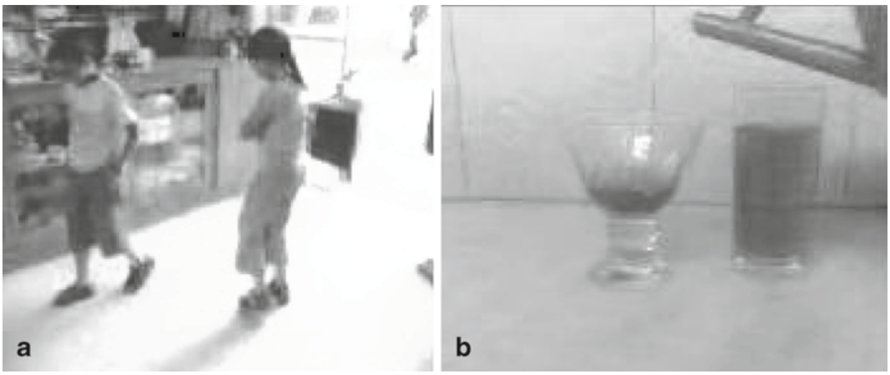


Fig. 2 Videotaped phenomena. **a** Two kids walking; **b** Water being poured into glasses

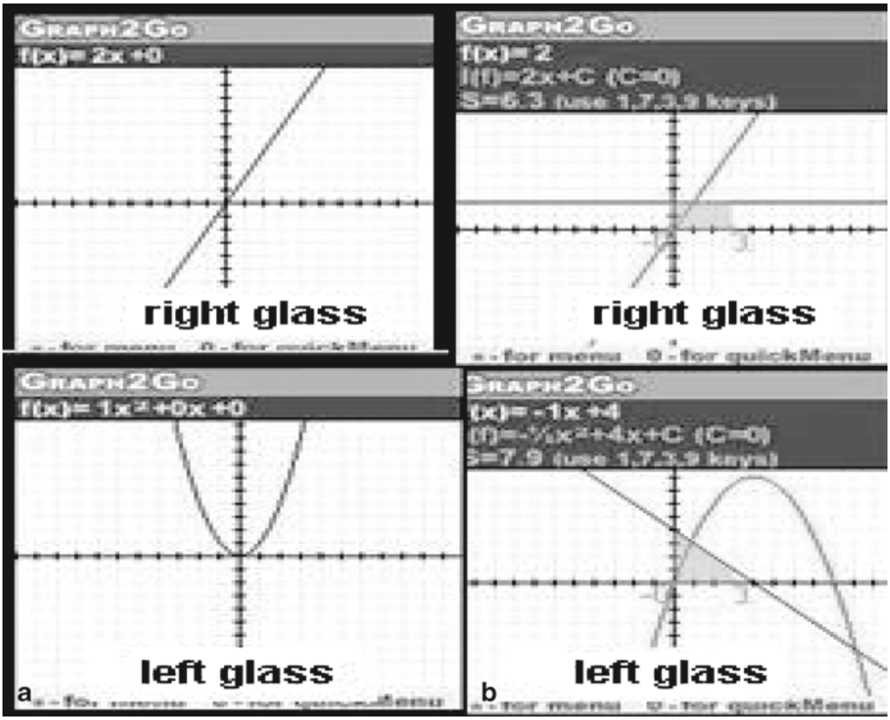


Fig. 3 Two rounds of graphing the water height and its rate of change

Amman (a prospective teacher) videotaped two kids walking: a boy walking forward and a girl walking forward to the middle of the path, then turning and walking backward. Her colleague, Ziva, videotaped water pouring into two glasses of different shapes.

These phenomena are simple enough for a "naked eye" observer to identify a pattern of change. The social interactivity and connectivity of the cellular phone provided an opportunity to exchange the videos together with mathematical objects. Ziva sent the pouring water video clip to her colleagues. Anna and Amman used the *Graph2Go* application to graph water height vs. time for each glass. Their graph for the left glass (Fig. 3a) shows an increasing rate of change rather than a decreasing one. Ziva asked her colleagues to review their graph but did not provide a mathematical justification for her request. Amman then sent another graph representing the possibly correct rate of change vs. time (Fig. 3b). Ziva confirmed that the graphs were correct. She explained in her diary that using *Graph2Go* had the benefit of displaying the water height by graphing the integral function alongside the graph of the rate. The moderator did not intervene during the discussion, although such intervention may have served to elaborate the issues under discussion.

The analyzing task was designed to call the learners' attention to different patterns of change and provide them with a mediating language for modelling. The task was designed for face-to-face settings in which learners use their personal mobile phones. The moderator can take a more active role than in the videotaping task, and may review with the learners the interpretation of each icon, arrange the presentations of the video clips according to the complexity of the videotaped phenomena, and call attention to their features. Personal technological tools at hand may enable learners to move flexibly between face-to-face discussion, personal exploration, and collaborative work. For example, the moderator can present a video clip that has not yet been explored, ask the group to sketch a graph using their personal mobile phones, then share the graphs through SMS. Obtaining a live screen rather than a static graph may encourage collaboration, as several learners must agree about the graph that represents the phenomenon.

The stopping-car task is a prototype for a variety of modelling tasks that require turning a verbal description of phenomena into a qualitative graph. The task was designed to emphasize the benefit of using the graphic icons. The mobile tools enable learners to solve the task wherever they choose, at their own pace, consult with their colleagues and with the moderator, and share mathematical objects. Table 1 presents a hypothetical scenario of mathematical discussion in a mobile setting.

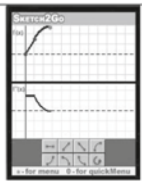
The scenario described here resembles to some extent distant learning models, in particular the asynchronous collaboration model (e.g., Tinker 2001) that involves asynchronous discussions and group problem solving among learners.

Mediating Prospective Teachers' Mathematical Mobile Discussion on Modelling Task: Summary

In outlining the above scenario, we suggest that it is important to go beyond the specific example to understand some general principles we have identified as being important: (a) during the mobile discussion the moderator attempts to set norms for the discussion, for example, sharing personal comments, questions, and solutions with the entire group; (b) the moderator attempts to motivate mathematical discussions to engage learners in conjectures and solutions using terms and representations that appear in the suggested applications; (c) moderator responses of different types implicitly encourage a variety of solutions and avoid direct responses about the truth of any specific suggestion.

Summarizing the communication with the group, the moderator may designate three main issues that require further work. (1) *Challenging the prospective teachers' content knowledge* in this unit of modelling and derivatives (beyond the issues already planned, which appear above). For example, the moderator may devise a similar challenge that would more naturally promote work with single function expressions (as in the toy-car task above). Next, the moderator may introduce several exercises, each including a given pairs of graphs of a function and its derivative, asking the group to explain which graph can describe the phenomena. We expect prospective teachers working in a mobile setting to engage in the modelling of complex real-life scenarios that challenge their mathematical and pedagogical content

Table 1 Hypothetical scenario of mathematical M-discussion

Dan chooses the "stopping car" modelling task from the pool of tasks.	
	Dan attempts to graph with *Graph2Go* by choosing a function's template but notices that the application does not allow working with a split domain.
	Dan sends an SMS to the moderator asking whether he can choose two different templates to describe the two segments of motion.
The moderator sends an SMS massage to the entire group: "Dan has pointed out the difficulty in specifying the different segments of the motion using a single function rule. Sketching (rather than accurate graphing) may be helpful at this stage. You may try *Sketch2Go*."	
Dan sketches the following graph:	Sara is prompted to take up the task and sketches the following graph:
	
Dan and Sara send their graphs to the group by SMS.	
Noa opens her mailbox and finds three massages relating to the stopping car task. She reads the task and examines the graph sketched by Dan.	
Noa sends SMS to the group: "I think that Dan sketched a v-t graph instead of an x-t graph"	
The moderator respond to Noa's comment and send SMS to the group: "There are different sketches that can describe the motion. When you send a graph please let all of us know what it describes, how we are to read the motion, and why."	
Rona uses the graph she received from Sara and modifies it.	
Rona sends the graph to the group with the massage "The function should go upward monotonically since the driver moves forward all the time."	
The moderator sends a massage to the group: "Wow, I wonder how Rona would feel stopping in no time – pretty frightening!"	
Dan has two options: describe the speed and the acceleration in his original graph, confirming Noa's conjecture about his graph, or modify his description to correctly model the distance. Dan uses Rona's graph, modifies it, and sketches the following graph:	
Dan sends the graph to the group with the massage: "Indeed I sketched the v-t graph and I think that it is correct the acceleration graph is not continues but its OK, but the velocity graph must be continues you can't stop at no time. The velocity decreases and not increases as it is in Rona's graph. Hence we should use the concave-down curve rather than the concave up curve.	
The group accepts Dan's graph and explanation.	

knowledge. (2) *Introducing other ICT tools*. We believe that mobile tools should be used as complementary to other ICT applications. For example, the moderator may ask the group to collect data in a mobile setting then analyze the data collaboratively using traditional computerized tools as spreadsheets or simulations. (3) *Explicitly discussing general principles for effective mobile discussion* in order to direct the attention of prospective teachers to the norms of mobile discussion. We believe that implementation of the guided inquiry approach in the classroom using mobile tools requires long-term support to encourage both beginner and professional teachers to take part in a community of practice. The communication capabilities of mobile tools can play an important role in the creation of such a community.

Opportunities and Challenges

The tasks presented above exemplify different teaching methods such as guided exploration of real-life phenomena (the videotaping task), collaborative group discussion (modelling of the motion of a stopping car), or personal exercise (e.g., the racing cars task). These tasks were designed not only to enrich the content knowledge of prospective and practicing teachers but also to illustrate the different commitments of the social constructivist teacher described by Chazan and Schnepp (2002). Chazan and Schnepp referred to a teacher in the calculus class who is

> committed both to helping his students develop substantial understanding of broad issues, like relationships between rates of change and accumulation of totals … and detailed, technical facility with specific techniques of differentiation and integration (ibid, p. 171).

They suggested that while sometimes these commitments reinforce each other, at other times they come into conflict and the teacher may experience tensions. The tasks and teaching scenario we selected for this chapter exemplify three different ways of working in the classroom as described by Marty Schnepp. In the videotaping task the role of the moderator was to support the learners' exploration by listening and assessing their work. Later, during the group discussion about the videotaped phenomena and in the M-discussion we described a more active role of the moderator and showed how the moderator may suggest issues for discussion, introduce mathematical concepts and representations, and call the learners' attention to interesting features of the phenomena. We also presented a sample of exercises in which the mobile tools are used to help the teachers fulfill their commitment to teach skills and cover the curriculum. In regular classroom settings teachers must consider carefully how to shift between these ways of teaching in order to balance the tensions between their different commitments. The mobile setting may reduce these tensions by augmenting face-to-face interactions (Liu and Kao 2007) and expanding the opportunities for learner-learner and learner-teacher social interactions. But this requires reconsideration of socio-mathematical norms (Yackel and Cobb 1996) that may sustain inquiry-based discussion and argumentation in a mobile setting. We suggested some general principles for the moderation of M-discussions with emphasis on setting norms of collaboration, encouraging variety in solutions, and avoiding direct responses regarding the truth of a specific suggestion.

The brief example we presented and the work that has been developed along similar ideas by Daher (2009) illustrate the opportunities that mobile mathematics tools such as *Math4Mobile* can provide for secondary mathematics teaching. Because mobile phones are becoming popular all over the world, the tools and tasks we presented can be easily implemented even in rural area where access to technological learning tools is limited. Similarly to other works that examined the use of handheld devices (Roschelle et al. 2007; Hegedus and Kaput 2003; Vahey et al. 2004), we suggest that the integration of dynamic software tools with social interactions can dramatically change the engagement with core mathematics. Shifting attention from static, inert representations to dynamic personal constructions on the learner's personal device, and sharing these constructions among learners create new challenges for teachers. Substantial teacher knowledge of both content and pedagogy is needed to facilitate this shift and to focus the public mathematical dialogue on the features and meanings of these visually shared objects (Roschelle et al. 2003).

Mobile phones have an advantage over specially designed tools such as the graphic calculator as a platform for the mobile learning environment because they already regularly serve daily out-of-school personal functions of all sorts. Using the cellular accessories such as the camera and the stop watch to enrich traditional inquiry tasks with the documentation and measurements of authentic situations, can upgrade and improve upon previous attempts to use the students' daily exposures and habits of interaction as part of the foundation of math learning.

While mobile applications for cellular phones are promising innovations for mathematics teaching, extensive research of the cognitive processes involved in the use of cell phones in the mathematics classroom is required. As with any pedagogical innovation, use of mobile applications raises a set of questions with regard to the teachers' views about what the technology has to offer. Does it support alternative approaches or simply offers novel solution strategies? Do teachers feel tensions in their instruction as they try to carry out curricular changes supported by out-of-school personal devices? How do they view opportunities to teach in a mode in which every student has a personal mobile phone as a tool for exploring and manipulating in mathematics and is encouraged to use it? And how do they view their new involvement with their students in mathematical inquiries outside the classroom? These questions indicate the complexity that must be addressed when teachers and developers try change and incorporate curricular ideas and technological innovations that support a range of approaches to school mathematics.

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Technology Integration in Secondary Mathematics: Enhancing the Professionalisation of Prospective Teachers

Merrilyn Goos

Introduction

For some time, education researchers have recognised the potential for mathematics learning to be transformed by the availability of digital technologies such as computers, graphics calculators, and web based applications (see Burrill et al. 2002; Hoyles et al. 2006 for recent reviews). These technologies offer new opportunities for students to develop and communicate their mathematical thinking by enabling fast, accurate computation, collection and analysis of data, and exploration of the links between numerical, symbolic, and graphical representations (e.g., Hennessy et al. 2001; see also Yerushalmy and Botzer, this volume). In most parts of Australia, secondary school mathematics curriculum documents now encourage or require teachers to incorporate digital technologies into learning and assessment activities (e.g., Queensland Studies Authority 2008; Victorian Curriculum and Assessment Authority 2005). To meet state mandated curriculum and assessment requirements teachers may need to consider use of:

- general purpose computer software that can be used for mathematics teaching and learning (e.g., spreadsheeting software);
- computer software designed for mathematics teaching and learning (e.g., dynamic graphing software, dynamic geometry software);
- hand held (calculator) technologies designed for mathematics teaching and learning (e.g., graphing calculators with and without algebraic manipulation or dynamic geometry facilities). (Queensland Studies Authority 2008, p. 8)

However, actual use of digital technologies is uneven across Australian schools and many teachers remain unconvinced that technology can help students understand mathematical concepts or explore unfamiliar problems (Goos and Bennison 2004). As a result, technology is more likely to be used merely as a replacement for pen

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and paper calculations than a means of transforming the very nature of mathematics teaching and learning as proposed by researchers in this field. These are issues that need to be dealt with when working with prospective teachers as they learn how to select and use technologies and to design suitable tasks for the students they will teach. Research has also suggested that novice teachers who are nevertheless technologically knowledgeable can act as agents of change in schools by demonstrating effective technology use in practice (Marcovitz 1997; Weinburgh et al. 1997). Thus a central goal of my teacher education course is to prepare graduates who can not only respond to the technology requirements of current curricula, but also anticipate and initiate new developments in technology integration in mathematics education.

This chapter describes an assessment task (the Technology Seminar task) that I have used with prospective teachers since 1998. The task requires them to work in pairs to prepare a technology based learning activity for classroom use with senior secondary mathematics students and to present the activity to an audience of their peers in the form of a professional development seminar. Through this assessment task they learn about the strengths and limitations of different technology tools while experiencing the benefits of sharing ideas with colleagues in a professional development setting. The first part of the chapter describes the conceptual framework underlying the design of the task. This is followed by an outline of the task as it is presented to the prospective teachers, a discussion of how I prepare them to tackle the task and examples of how they have responded. I conclude with reflections on my role as the teacher educator in this process.

Conceptual Framework

My work as a teacher educator is framed by sociocultural theories of learning. Lerman (1996) defined sociocultural approaches to mathematics teaching and learning as involving "frameworks which build on the notion that the individual's cognition originates in social interactions (Harré and Gillett 1994) and therefore the role of culture, motives, values, and social and discursive practices are central, not secondary" (p. 4). The conceptual framework underlying the design of the Technology Seminar task draws on two sociocultural concepts: (a) cultural tools and (b) communities of practice.

Technology as a Cultural Tool

Sociocultural perspectives on learning grew from the work of Vygotsky in the early twentieth century (Forman 2003). Vygotsky's theoretical approach refers to the social origins of higher mental functions, and the mediation of these functions by tools and signs, such as language, writing, systems for counting and calculating, algebraic symbol systems, diagrams, and so on. From a sociocultural perspective, technologies such as computers and graphics calculators are viewed as cultural tools that not only re-organise cognitive processes but also transform classroom social practices (Berger 1998; Resnick et al. 1997).

In research involving experienced mathematics teachers and their senior secondary school classes, my colleagues and I developed metaphors to describe how digital technologies can provide a vehicle for incorporating new teaching roles (Goos et al. 2003). Teachers can see technology as a *master* if their knowledge and competence are limited to a narrow range of operations, especially in situations where external pressures from education systems force implementation. Technology is a *servant* if it is used as a fast, reliable adjunct to pen and paper (e.g., as a tool for drawing graphs or performing numerical calculations), but does not change the nature of classroom activities. However, when teachers develop an affinity for technology as a *partner,* there is potential for students to achieve more power over their own learning by, for example, providing access to new kinds of tasks or new ways of approaching existing tasks. Technology becomes an *extension of self* when seamlessly incorporated into a teacher's pedagogical and mathematical repertoire, such as through the integration of a variety of technology resources into course planning and the everyday practices of the mathematics classroom.

The four modes of working outlined above are not necessarily tied to the level of mathematics taught or to the kinds of technologies available, and teachers do not necessarily remain attached to a single mode of working with technology in the classroom (see Goos et al. 2003, for a classroom case study that illustrates multiple modes of working). Nevertheless, the categories elaborate increasingly sophisticated ways in which teachers may appropriate technology as a cultural tool. In preparing prospective teachers to undertake the Technology Seminar assessment task I emphasise the role of technology as a *partner* in developing secondary school students' mathematical understanding or exploring different perspectives on problems.

Learning to Teach in a Community of Practice

Contemporary sociocultural theory acknowledges that learning involves increasing participation in socially organised practices, and the idea of situated learning in a community of practice composed of experts and novices is now well established (Lave and Wenger 1991; Wenger 1998). A community of practice is a sustained social network of individuals who share common beliefs, values, and practices in the pursuit of a mutual enterprise that is connected to the larger social system in which the community is nested. Such communities have a common cultural and historical heritage, and it is through the sharing and re-construction of this collective knowledge base that individuals come to define their identities in relationship to the community. Because communities of practice evolve over time they also have mechanisms for reproduction through which the community can maintain itself.

The concept of *community of practice* is useful for understanding how teachers gain access to professional knowledge through collaboration with more experienced members of the community (Lerman 2001; Peressini et al. 2004). For prospective teachers, these "more experienced members" are usually limited to the university-based teacher educator and the school-based practicum supervisor; thus an overarching aim of the Technology Seminar task is to connect prospective teachers with the broader professional community of practicing mathematics teachers.

Program Structure

At my university, prospective secondary school teachers enroll in either a one year Graduate Diploma in Education (DipEd) or a four year dual degree program that overlaps an initial non-education degree with a Bachelor of Education (BEd). In the latter case a three year non-education degree provides the disciplinary knowledge for subject specialisation as a secondary school teacher as well as foundation courses in education addressing learning theories, adolescent development, and the sociology of education. Dual degree participants complete the BEd in a fourth year, known as the professional year, which is devoted solely to the study of practical and professional issues in education. Prospective teachers enrolling in the DipEd are typically mature age entrants who are changing careers, having already completed an undergraduate degree in areas such as science, engineering or information technology. The one year DipEd program is identical to the fourth year (professional year) in the BEd dual degree program.

All prospective mathematics teachers complete their curriculum studies as a single class group in a course that lasts for the duration of the professional year (February–October). This period includes fourteen weeks of practicum sessions taken in two blocks of seven weeks each. The mathematics curriculum class meets twice weekly for three hour workshops during the remaining seventeen weeks of the academic year. Assessment tasks for the course typically comprise (a) a review of an article published in a professional journal, (b) a Technology Seminar (described below), and (c) a curriculum planning task.

Design of the Task

I designed the Technology Seminar assessment task with three purposes in mind:

1. to develop prospective teachers' skills in selecting and using digital technologies and in preparing technology based teaching resources;
2. to elicit from prospective teachers demonstrations of how they would use specific digital technologies in teaching mathematics; and,
3. to encourage prospective teachers to share ideas with colleagues in a professional development setting.

These purposes, along with the intended audience for the task, are made explicit in the task instructions provided to the class (see Appendix). Additional guidelines on the task sheet are intended to clarify the scope of the task and highlight examples of past course participants' work as a source of assistance.

The assessment criteria and standards for the task specify the relevant performance dimensions and benchmarks that will be used in making judgments about the quality of the prospective teachers' work (Fig. 2; standards range from A to E, only the A standard descriptors are shown here). My emphasis on using technology as a *partner* is reflected in the standards descriptors for the assessment criteria labelled "Understanding of the use of technology in teaching mathematics" and "Relevance of selected technology based activity".

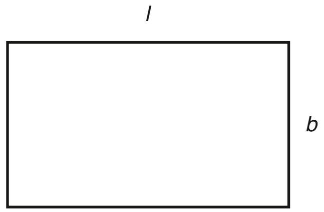
The seminars are presented over two days to simulate the format of a professional development conference. I then help the prospective teachers to share their work with the wider professional community by encouraging them to present their seminars at mathematics teacher conferences and to prepare articles for publication in professional journals, thus offering their work for critical scrutiny by practicing teachers and initiating them into their *community of practice*. An extended example of this process is presented later in the chapter.

Preparing for the Technology Seminar Task

Most prospective mathematics teachers come to my course as quite competent users of general purpose computer software (e.g., word processing, spreadsheets), having gained some experience with these technologies during previous university or school courses. However, very few have been exposed in their own secondary schooling to graphics calculators and data logging peripherals, such as motion detectors and temperature probes. Thus on their first encounters with digital technologies in a mathematics teaching context they are inclined to view technology as a *servant* (a tool for performing calculations quickly and accurately) or as their *master* (when the technology is unfamiliar to them). Rather than giving workshops on how to use specific software applications or hand held devices, I integrate a range of technologies throughout the course in order to serve broader pedagogical purposes. For example, a workshop on teaching geometry typically rotates participants through a menu of tasks that asks them to investigate use of dynamic geometry software (loaded onto a laptop computer) as well as manipulables; a workshop on mathematical modelling may involve use of graphics calculators as part of the modelling process. A low cost hiring scheme provides each prospective teacher with continuous personal access to a Texas Instrument TI-83 or TI-84 graphics calculator for the duration of the course (including practicum sessions). They bring their calculators to all classes so that we can use the technology spontaneously, as well as in workshops specifically planned for this purpose, thus modelling effective pedagogy while also circumventing some of the difficulties in gaining access to computer laboratories that need to be booked for classes some weeks in advance.

Because my research has shown that practicing teachers are more convinced of the benefits of digital technologies for doing numerical calculations or making graphing quicker and easier (technology as *servant*) than for building understanding or exploring unfamiliar problems (Goos and Bennison 2004), I engage prospective teachers in a range of tasks that exemplify the role of technology as *partner*. One way in which I do this demonstrates how to use graphical approaches to build students' understanding before moving into analytical work. For example, graphical treatment of simple optimisation problems—such as finding the maximum area of a rectangle with fixed perimeter—makes this concept accessible to lower secondary students without the need to invoke calculus concepts.

> I have 20m of wire with which to fence a rectangular garden. What are the dimensions of the largest area that can be enclosed?



$$b + l = 10$$

$$b = 10 - l$$

$$A = l \times b$$

$$= l\,(10 - l)$$

The prospective teachers usually begin by randomly choosing a range of lengths and calculating the corresponding breadths and areas, filling in a table of values "by hand" as shown below.

Length (m)	Breadth (m)	Area (m^2)
2	8	16
4	6	24
7	3	21
9	1	9

Allowing them to begin with such an unsystematic approach provides an opportunity for me to demonstrate how to use the graphics calculator to generate these data using lists (Fig. 1, first screen; there is also some discussion about the use of whole number versus decimal values). Through questioning I elicit the independent and dependent variables of interest (length and area respectively), and ask for a prediction of what a scatterplot might look like (observing fingers move through the air to trace out a parabola). The calculator is used to produce the scatterplot (Fig. 1, second screen), and the area function is graphed over these points (Fig. 1, third screen). We then explore various methods for finding the maximum area, such as by tracing along the curve or querying the calculator directly (Fig. 1, fourth screen).

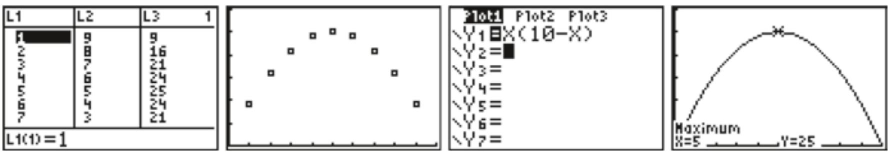


Fig. 1 Finding the maximum area of a rectangle with fixed perimeter

Thus this simple example allows me to introduce graphics calculator lists, statistical plots, function graphing, the trace function, and calculation of specific values, all in the context of a typical textbook problem that taps into significant mathematical concepts. However, the greatest impact comes from seeing prospective teachers—who already "know" the answer to the problem—mesmerised by the visual impact of real time graphing of the quadratic function over the scatterplot.

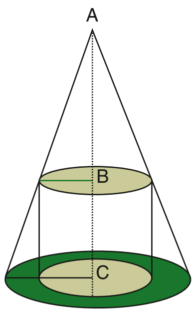
Responses to the Technology Seminar Assessment Task

Three examples of prospective teachers' responses to the Technology Seminar assessment task are provided in this section to illustrate difficulties as well as successes since I started using this task in 1998. This is important because, despite my efforts in preparing the class for this task by emphasising the use of technology as a *partner* in mathematics learning, some prospective teachers design technology based activities that do not give balanced attention to the mathematics, technology, and teaching approach (as required by the assessment criteria shown in the Appendix), and thus they limit the roles of technology to either *master* or *servant*.

Example 1: Unsuccessful Response to the Task (Optimisation Using a Spreadsheet)

A recent example of an unsuccessful response to the task comes from the class of 2007. One pair of prospective teachers designed an activity based on the question shown below and accompanied it with a spreadsheet solution method for calculating cylinder volumes for varying heights and graphing volume versus height as a means of identifying the maximum value. In preparing the activity they had in mind teachers who might not be familiar with spreadsheets. The activity was presented in a computer laboratory so that all participants (i.e., fellow prospective teachers) could attempt a spreadsheet solution for themselves.

Find the maximum volume of a cylinder that can be inserted in a cone of height 24 cm and a base radius 8 cm.



Instead of beginning with a mathematical discussion of possible solution approaches that led naturally to construction of a spreadsheet, the method they demonstrated led participants through a step by step procedure for labeling spreadsheet columns, entering data, and writing Excel formulae (partly illustrated below). Although motivated by the expectation that they would need to explain to teachers how to use a spreadsheet (technology as *master*), they placed undue emphasis on procedural aspects of the technology and thus obscured the mathematical aims of the activity. This approach caused much confusion amongst their peers (the seminar participants) as they tried to implement the demonstrated solution on their own computers, but it nevertheless led to fruitful suggestions for alternative approaches that might provide a better understanding of the mathematical basis for the solution.

Step 1: Open Excel (Start—All programs—Microsoft Office—Microsoft Excel). (Excel: You should have a spreadsheet on your screen.)

Step 2: Label column with information that we need to know. (Excel: To auto size your columns so that your text fits, use your mouse to double click on the right side of the column letter.)

Step 3: Work out the volume of the cone and enter in the Volume Cone column. (Excel: Insert in A2.)

Step 4: Discuss what height the cylinder can take within the cone.

Step 5: Enter height data into Height Cone column. (Excel: click on C2, type 0.1, hold shift and press down until you have highlighted all the rows you wish to fill, then click edit; fill; series; type step value as 0.1 as you are counting down, and then hit OK.)

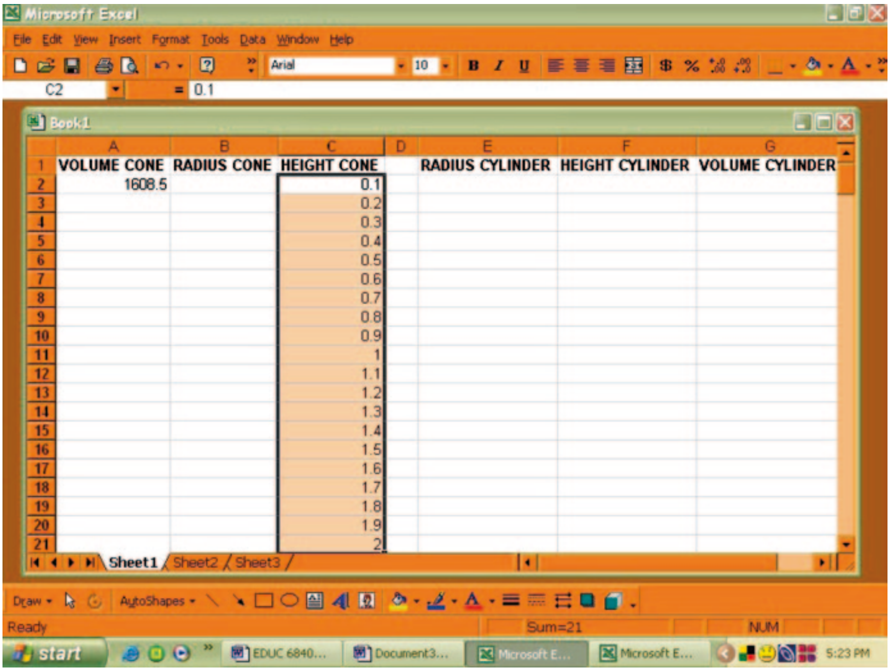


Table 1 Basketball shooting data

Distance (ft)	3	6	9	12	15	18	21	24	27	30	40
Shooting Percentage (%)	62	52	40	32	28	24	21	20	18	17	13

Example 2: Task Response Modified During Seminar Presentation (Fitting a Function to Data)

In 1998, the first year in which I set the task, a pair of prospective teachers designed an activity based on the basketball data shown in Table 1 (Hays 1978) that required participants to find a function that models the relationship between shooting percentage and distance from the basket. Their solution method involved using a graphics calculator to fit regression model equations to the data stored in lists and to calculate the corresponding R-squared values as a measure of goodness of fit. In their seminar presentation they asked each group of participants to investigate one function (e.g., exponential, power, quartic) and they then collated the respective R-squared values on the whiteboard so that the best model could be identified.

One participant challenged this data-driven approach, pointing out that models with high R-squared values did not always satisfy the real life constraints of the problem. For example, a power model (R-squared = 0.9749) predicts that the shooting percentage—which is limited to a maximum value of 100%—becomes infinitely large when the distance from the basket approaches zero (Fig. 2). A quartic model (R-squared = 0.9976) strikes problems for large distances from the basket as it predicts negative values for shooting percentage (Fig. 3). These observations led

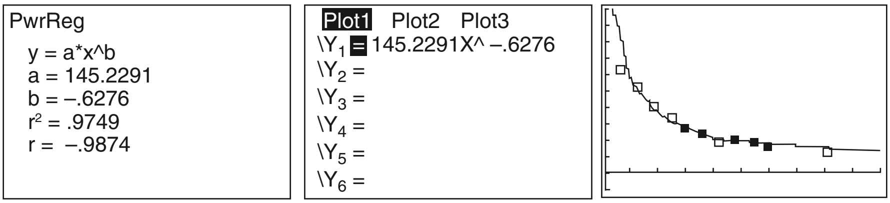


Fig. 2 Power regression model

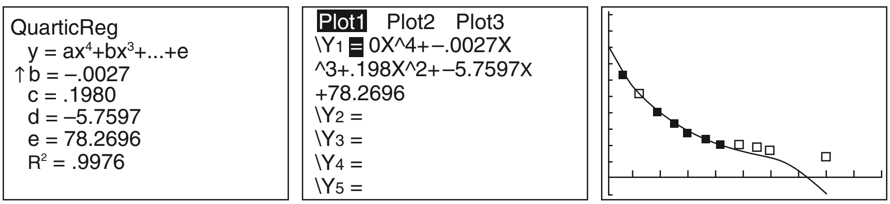


Fig. 3 Quartic regression model

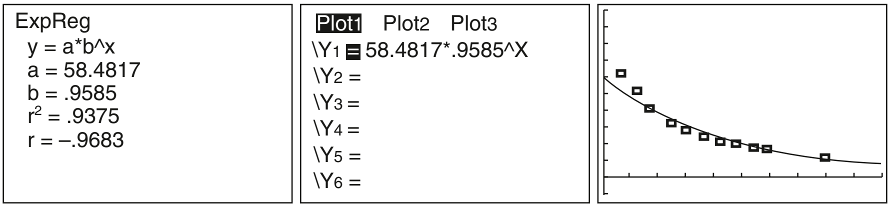


Fig. 4 Exponential regression model

to a lively discussion amongst participants that eventually converged on the view that using only the R-squared value to determine the most appropriate model results in students deferring to the calculator's "black box" algorithms rather than engaging in mathematical reasoning, and possibly choosing a model that does not make sense in the context of the problem. In this case, then, an approach that treated the technology as a *servant* to do the regression calculations was rejected in favour of seeking mathematical understanding. An exponential model (R-squared = 0.9683, Fig. 4) was selected as being the most appropriate because it predicts sensible values for shooting percentages when the distance from the basket is small (approaching 60% for zero distance) and large (approaching 0%).

After listening to the debate amongst the prospective teachers I proposed an alternative approach that relies even more explicitly on exponential reasoning and thus treats technology as a *partner* for investigating mathematical concepts. This method seeks the real world meaning for a and b in the function $y = ab^x$, uses the data to estimate reasonable values for a and b, and then tests and adjusts these values by graphing the function over the scatterplot of data points. Here, a represents the shooting percentage at zero distance and b the constant ratio between successive values for the shooting percentage. Estimating $a = 75$ and $b = 0.94$ gives a reasonably good fit to the data provided, and arriving at these values via experimentation with the graphics calculator requires mathematical understanding of the effects of both parameters on the y-intercept and the gradient of the graph.

Example 3: Successful Response to the Task (Modelling with a Spreadsheet)

This third example illustrates what I regard as a successful response to the Technology Seminar assessment task, where "success" is evaluated via the sociocultural criteria of tool use (mode of working with technology) and initiation of the prospective teachers into a community of professional practice. This activity was devised in 2004 by a pair of prospective teachers to model the transit of Venus, a rare astronomical event that occurred most recently on 8 June that year. Earlier in the year, one of the pair had attended a presentation at a local astronomical society at which he discovered that transits occurred in a regular cycle but with an unusual pattern.

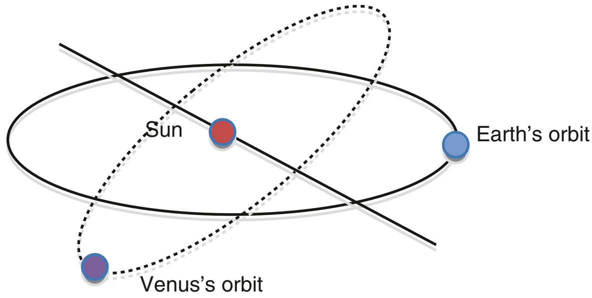


Fig. 5 Orbits of Venus and Earth

Transits are separated by periods of eight years, followed by a gap of 105 years, then eight years, and then 121 years. After an extensive internet search yielded over 5000 entries related to the transit phenomenon but none that provided a satisfying mathematical explanation for the sequence of transits, the prospective teachers decided to investigate whether this intriguing cycle could be modelled using mathematics. A summary of their investigation is given below.

The orbit of Venus lies closer to the Sun than Earth's orbit. A transit of Venus occurs when Venus passes between the Sun and the Earth, or, as seen from Earth, seems to pass directly in front of the Sun. Because Venus and Earth have different orbital periods, it is of interest to know when such planetary alignments occur. Frequency of alignments can be calculated from the planets' respective orbital periods as 583.9 days. However, an alignment does not imply a transit because Venus's orbital plane is inclined with respect to Earth's orbital plane (called the plane of the ecliptic), as shown in Fig. 5.

An initial condition for a transit is that Venus, Earth, and Sun must be collinear, and this occurs when Venus *crosses the ecliptic* at alignment. But since the Sun appears as such a large disk in the sky, Venus need not be *exactly* crossing the ecliptic at this time—there is some margin involved (shown in Fig. 6) that can be calculated using simple trigonometry as 0.1929 million km.

Thus the underlying problem requires identifying how far above or below the ecliptic Venus is at each alignment and whether this distance is within the margin defining a transit. If we assume that Venus travels in a roughly circular orbit at constant speed, for half its orbit it is above the ecliptic and for half below. This oscillatory behaviour can be modelled using the trigonometric function

$$y = A \sin Bx$$

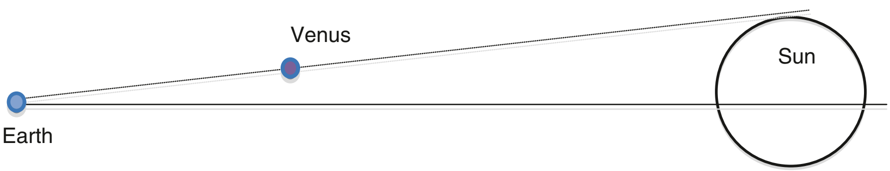


Fig. 6 Margin for height of Venus above (or below) ecliptic

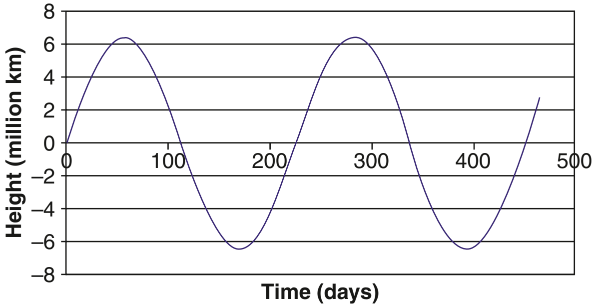


Fig. 7 Graphical model of the transit of Venus

where A is the amplitude (maximum distance of Venus above the ecliptic), B the period (determined by Venus's orbital period), and y the height of Venus above the ecliptic at time x, the number of days into Venus's orbit measured from a nominated starting point. We now inspect the graph of this function to find every alignment (i.e., every 583.9 days along the x-axis) when Venus is within the calculated margin above or below the ecliptic (±0.1929 million km along the y-axis). However, the prospective teachers found that the axis scales make the graph difficult to read in this manner whether it is produced on a graphics calculator or with a spreadsheet, as in Fig. 7. Instead, they created a sophisticated spreadsheet (part of which is displayed in Fig. 8) that calculated past and future transit dates predicted by their

Plus 400 years			Alignment (days past node)	Years	Year	Ht above ecliptic
56756.82	30-Oct	1759	-89340.18	-244.8	1759	3.564432
57340.75	5-Jun	1761	-88756.25	-243.2	1761	0.191539
57924.67	10-Jan	1762	-88172.33	-241.6	1762	-3.876225
59676.44	28-Oct	1767	-86420.56	-236.8	1767	3.784414
60260.36	3-Jun	1769	-85836.64	-235.2	1769	-0.077023
60844.28	8-Jan	1770	-85252.72	-233.6	1770	-3.659033
98215.34	4-May	1873	-47881.66	-131.2	1873	-3.386996
98799.26	9-Dec	1874	-47297.74	-129.6	1874	-0.402549
99383.18	15-Jul	1876	-46713.82	-128.0	1876	4.042279
101134.95	1-May	1881	-44962.05	-123.2	1881	-3.611938
101718.87	6-Dec	1882	-44378.13	-121.6	1882	-0.134212
102302.80	12-Jul	1884	-43794.20	-120.0	1884	3.830413
145513.08	2-Nov	2002	-583.92	-1.6	2002	3.659155
146097.00	8-Jun	2004	0.00	0.0	2004	0.076875
146680.92	12-Jan	2006	583.92	1.6	2006	-3.784294
147264.85	19-Aug	2007	1167.85	3.2	2007	6.083329
147848.77	25-Mar	2009	1751.77	4.8	2009	-6.118357
148432.69	30-Oct	2010	2335.69	6.4	2010	3.876343
149016.61	5-Jun	2012	2919.61	8.0	2012	-0.191687
149600.54	10-Jan	2014	3503.54	9.6	2014	-3.564309
150184.46	17-Aug	2015	4087.46	11.2	2015	5.993791
150768.38	23-Mar	2017	4671.38	12.8	2017	-6.192589
151352.30	28-Oct	2018	5255.30	14.4	2018	4.086719
151936.23	3-Jun	2020	5839.23	16.0	2020	-0.459912
152520.15	8-Jan	2022	6423.15	17.6	2022	-3.338059
153104.07	15-Aug	2023	7007.07	19.2	2023	5.893718

Fig. 8 Spreadsheet model of the transit of Venus

model. These corresponded exactly to dates of previously observed transits (thus validating the model) as well as to future dates calculated by contemporary astronomers.

In terms of the conceptual framework presented at the start of the chapter, the abbreviated account of this investigation makes it clear that technology was used as a *partner* to develop the mathematical model as the use of a spreadsheet made it possible to explore a complex real world phenomenon. The other aspect of the framework—learning to teach in a *community of practice*—is illustrated in the three sequels to the Technology Seminar assessment task that took the prospective teachers' work to a wider audience of teachers.

- They presented a one hour workshop at the annual conference of the state mathematics teachers' professional association.
- They published a refereed journal article outlining the development of the Transit of Venus model (Quinn and Berry 2006).
- They published another short article in a professional journal describing the genesis and dissemination of the modelling task through the phases of *conception* (initial curiosity about the transit phenomenon), *birth* (preparation of the Technology Seminar as a university assessment task), *development* (presentation of a conference workshop to an authentic audience of experienced mathematics teachers), and *maturity* (transformation of the workshop into an academic paper for an unknown but critical professional audience) (Berry and Quinn 2005).

The purpose of the latter article was to encourage other teachers to share their successful classroom activities with the professional community. The words of the two prospective teachers capture their excitement in transforming the original idea:

> … from a paper manufactured to satisfy the needs of an undergraduate assignment to gain those elusive passing grades, to a workshop tailored to the needs of practising teaches working in front of actual students whom we did not know (apart from our cloistered sessions of practicum experience at real schools), to the broadest of all audiences, the great mass of interested practising mathematics teachers in the workplace, who have the need to present real problems to students on an everyday basis and keep it interesting. (Berry and Quinn 2005, p. 18)

Teacher Educator Reflections

My own role as the teacher educator is critical to the success of this task, although, as the examples presented above demonstrate, success is not always guaranteed. This role can be analysed with respect to the conceptual framework that guided the design of the Technology Seminar assessment task. First, I model the use of technology as a cultural tool that has the potential to transform students' mathematics learning and teachers' classroom practice by emphasising a mode of working with technology as a *partner* for building understanding rather than simply as a *servant* for performing calculations or for checking work done first by hand. Often this may involve using standard textbook tasks, such as the gardening problem presented earlier, to show how technology can make mathematical concepts more accessible to students. However, I gain more pleasure from designing my own tasks, testing these in teacher education and professional development settings, and publishing them as resources for a wider audience of practicing teachers (e.g., Goos 2000a, b). Second, I claim membership of the *community of practice* composed of professional mathematics teachers, and I regard my teacher education course as being nested within this larger community. Part of my role, therefore, is to maintain this professional community by bringing in new members who are prospective teachers and connecting them to more experienced members. I do this by selecting prospective teachers to attend and present workshops at professional development conferences and helping them publish their work in professional journals. Because these activities are framed by my sociocultural perspective on learning they give coherence to my work as a researcher and a teacher educator.

Appendix

Assignment 2 – Technology Seminar

Task Specifications

Purposes:

To develop skills in preparing technology resources.

To demonstrate ways of using computer software and graphics calculators in teaching mathematics.

To share ideas with colleagues in a professional development setting.

Audience:

Peers (i.e., teachers). Technology Seminar sessions will be organised during class time.

Task: Work in pairs to complete this assignment

(a) Choose a topic from the Senior Mathematics A, B or C syllabuses, and prepare a technology based activity designed to teach some aspect of the topic.

(b) Present the activity to an audience of your peers. This presentation will take the form of a professional development workshop lasting 30–40 minutes. Allow 20–30 minutes for participants to try the activity, and a further 10 minutes for questions and discussion.

(c) To accompany your oral presentation, provide a handout containing information that will enable another teacher to implement the activity. Your handout should include:

- the year level and syllabus topic
- the source of the activity

- a description of any modifications made to the original form of the activity
- a rationale which explains the purpose of the activity and its relevance to the mathematical topic
- a statement of the problem to be solved or task to be completed
- possible solutions
- teaching notes

Due Date: 24 and 25 July 2007 (during class time)

Weighting: one third

Length: 40 minute workshop plus 1000 word handout

Some Guidelines

1. The technology based activity could be one which you used or observed during the practicum, or one which you have adapted from another source (human, print, or electronic). It should **not** be an activity supplied or demonstrated during curriculum workshops. Neither should it simply reproduce material produced by someone else. Remember that the work of others must be fully acknowledged.
2. For the purposes of this assignment, "technology" means:
   (a) **Computer software.** Your activity must be demonstrated with the software legally available in the Faculty computer laboratories (e.g., Excel, TI-Interactive). Activities developed for other types of software will need to be appropriately modified if you wish to use them. Note that many computer-based activities can be adapted for use with graphics calculators.
   (b) **Graphics calculators.** Your activity must be suitable for the TI-83 PLUS. A viewscreen will be available for you to use during your seminar.
   (c) **Data logging equipment.** CBR and CBL are available for borrowing and use in your seminar.
3. You and your partner will be assessed **as a pair**, on both the seminar presentation and the written handout. You will need to provide copies of the handout for every member of the class.
4. Samples of past course participants' work are available for borrowing from the lecturer. You may also visit the UQEdMaths website (groups.yahoo.com/group/UQEdMaths) and download examples of past course participants' technology activities that have been edited and polished for publication (both online and in print). Go to Files → Course information → Sample assignments → Technology seminars.

Assessment Criteria	Descriptors for 'A' standard work
Understanding of the use of technology in teaching mathematics	Seminar handout deals comprehensively with relevant aspects of implementing the activity, giving balanced attention to mathematics, technology, and teaching approach.
Relevance of selected technology based activity	Activity is imaginatively developed, selected, or adapted, and highly appropriate for developing students' understanding of the mathematical topic. Rationale clearly explains relevance and purpose of the activity.
Structure and organisation of seminar presentation	Logically structured; activity stimulates student interest and actively engages them with the technology and the mathematics; resources are used with imagination and flair; pacing makes best use of the time available.
Quality of oral communication	Communicates with clarity; good use of variation; demonstrates well developed questioning and explanation skills with individual students; uses students' questions and comments to orchestrate whole class discussion of activity.
Quality of written communication	Writing is concise, well-structured and error-free. Format and structure of handout make it easy to follow and practical to use.

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Mathematical Machines: From History to Mathematics Classroom

Michela Maschietto and Maria G. Bartolini Bussi

Introduction

Mathematical machines are cultural artefacts that draw on centuries (and even millennia) of tradition. Briefly, a mathematical machine is a tool that forces a point to follow a trajectory or to be transformed according to a given law. These machines are collected in the Laboratory of Mathematical Machines at the Department of Mathematics of the University of Modena and Reggio Emilia (MMLab: http://www.mmlab.unimore.it). The Laboratory is a well known research centre for the teaching and learning of mathematics by means of artefacts (Maschietto 2005).

Familiar examples of mathematical machines are the standard compass (that forces a point to go on a circular trajectory, Fig. 1) and the Dürer's glass (Fig. 2) used as a perspectograph (that transforms a point into its perspective image on a glass from a given point).

As argued by Bartolini Bussi and Maschietto (2006), they are part of the historical phenomenology of geometry: ruler and compass are at the roots of elementary geometry (e.g., Euclid); curve drawing devices are at the roots of algebraic geometry (e.g., Descartes, van Schooten, Newton); and, perspectographs are at the roots of projective geometry (e.g., Desargues). They are linked to the cultural development of mankind in a sense that does not consist merely of mathematics but encompasses also art and technology. They are concretely manipulable, in order to produce the intended effect. In a nutshell, they are good candidate to equip the mathematics classroom for meaningful mathematical experiences, where practice (manipulation and real experiments) and theory (elaboration of definitions, production of conjectures and construction of proofs) are strictly interlaced within a historic-cultural perspective, up to the present modelling of concrete machines by means of Dynamic Geometry Environments (DGE).

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Fig. 1 The compass

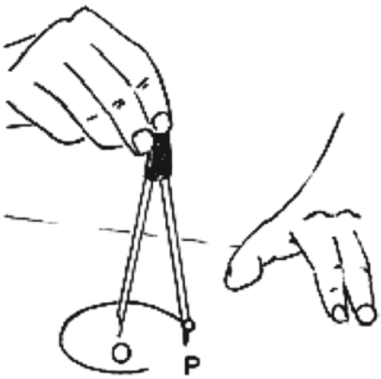
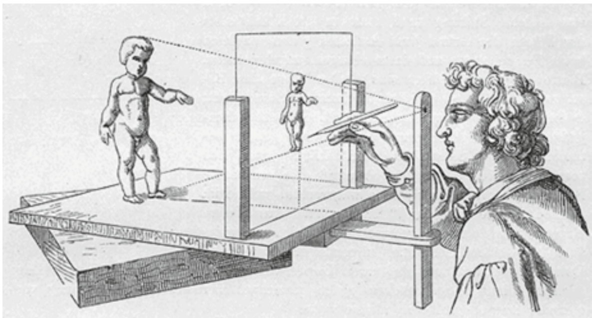


Fig. 2 Dürer's glass



All the above activities are consistent with the idea of mathematical laboratory: this idea has a long tradition not only in the professional mathematical practice—as we have said above—but also in the history of mathematics education (see for instance Maschietto and Martignone 2008; Bartolini Bussi in press). The laboratory activity is a great challenge for teachers. In this chapter, we discuss some kinds of activity concerning a particular mathematical machine as paradigmatic examples of mathematical laboratory activities. They are proposed to prospective teachers:

- to be experienced in a mathematical laboratory session;
- to provide a model that might serve for future class activity; and,
- to make them think over the relationships between manipulative and theoretical aspects in doing mathematics, on the basis that only manipulation is not enough to construct mathematical knowledge.

These activities can be transferred to students' classes because of the availability of materials (working sheets[1] and artefacts that can be reconstructed, using plastic or cardboard bars, by students too).

[1] For the Italian version see http://www.mmlab.unimore.it/on-line/Home/VisitealLaboratorio/Materiale.html

The chapter is composed of four sections. The first section presents some elements concerning the idea of mathematical laboratory connected to teacher education, then the theoretical background developed within a Vygotskian perspective. The other three sections propose three different activities about van Schooten's ellipse drawing device, according to three different dimensions: in particular, the second session focuses on historical sources (historic-epistemological dimension); the third section on the manipulation of the mathematical machine (manipulative dimension) and the fourth section on the construction of a model of the same mathematical machine by a DGE (digital dimension).

Some Theoretical Elements

Mathematical Laboratory and Teachers Education

The Italian Mathematical Union has drawn on the ancient idea of the mathematical laboratory, when the new mathematics standards for 5–18 years old students were prepared (Anichini et al. 2004). The document reads:

> A mathematics laboratory is not considered a place (e.g., a computer classroom) but rather a methodology, based on various and structured activities, aimed to the construction of meanings of mathematical objects. A mathematics laboratory activity involves people, structures, ideas. We can imagine the laboratory environment as a Renaissance workshop, in which the apprentices learned by doing, seeing, imitating, communicating with each other, in a word: practicing. In the laboratory activities, the construction of meanings is strictly bound, on one hand, to the use of tools, and on the other, to the interactions between people working together. It is important to bear in mind that a tool is always the result of a cultural evolution, and that it has been made for specific aims, and insofar, that it embodies ideas. This has a great significance for the teaching practices, because the meaning can not be only in the tool per se, nor can it be uniquely in the interaction of student and tool. It lies in the aims for which a tool is used and in the schemes of use of the tool itself. (p. 60)

In this quotation, the last sentences evoke the distinction between artefact and instrument (Rabardel 1995). The *instrument* (to be distinguished from the artefact) is defined as a hybrid entity made up of both artefact-type components and schematic components that are called *utilization schemes*. The utilization schemes are progressively elaborated when an artefact is used to accomplish a particular task; thus, the instrument is a construction of an individual. It has a psychological character and it is strictly related to the context within which it originates and its development occurs. The elaboration and evolution of the instruments is a long and complex process that Rabardel names *instrumental genesis*. Instrumental genesis can be articulated into two coordinated processes: *instrumentalisation*, concerning the emergence and the evolution of the different components of the artefact, drawing on the progressive recognition of its potentialities and constraints; *instrumentation*, concerning the emergence and development of the utilization schemes.

According to the Italian governmental regulations issued in 1998, teacher education (including mathematics teachers education) is organized around three main kinds of activities: lectures (for large groups of prospective teachers, up to 100 and more), in-school apprenticeship (individual participation in standard classroom activities, under the supervision of expert teachers) and laboratories (with a number of prospective teachers around 25, i.e., the standard size of a classroom). In these laboratories, prospective secondary mathematics teachers come personally into contact with new methodologies, with new tools that offer innovative models for their future teaching practice: the personal experience is accompanied by a reflection of the possible application in secondary school teaching. The laboratory activity is a great challenge for teachers, as it requires specific professional competences, which cannot be taken for granted. Some authors have discussed the domains of professional knowledge for teachers. For instance, Ball et al. (2008), suggest at least the following domains, as a refinement of Shulman's (1986) categories of Subject Matter Knowledge and Pedagogical content knowledge:

- the *common content knowledge*, i.e., the mathematical knowledge at stake in the material to be taught;
- the *knowledge of content and students*, related to the prediction and interpretation of students' processes when a task is given;
- the *knowledge of content and teaching*, related to the teacher's actions aiming at the students' construction of mathematical meaning; and,
- the *specialised content knowledge*, that is the mathematical knowledge and skill uniquely needed by teachers in the conduct of their work.

Elsewhere Bartolini Bussi and Maschietto (2008) have linked the analysis of Ball et al. (2008) to the model developed in the Laboratory of Mathematical Machines (MMLab) for teacher education, as both encompass the needed complex and systemic approach. Our aim is to put the prospective teacher in a situation where the artefacts of the Laboratory (either mathematical machines or computers) are used according to an approach based on the Vygotskian perspective of semiotic mediation (details in the quoted paper and in Bartolini Bussi and Mariotti 2008). In this way, prospective teachers can experiment with both exploration processes (that could be activated in their students) and a model of didactic management of activities with artefacts used as tools of semiotic mediation (by the teacher educator).

Tools of Semiotic Mediation

The theoretical construct of semiotic mediation draws on Vygotsky's papers[2] published in the Thirties (for an English translation, see Vygotsky 1978). It has been elaborated and applied to mathematics education by some authors. In this chapter

[2] See Goos' contribution in this volume for other elements concerning the socio-cultural perspective and cultural tools.

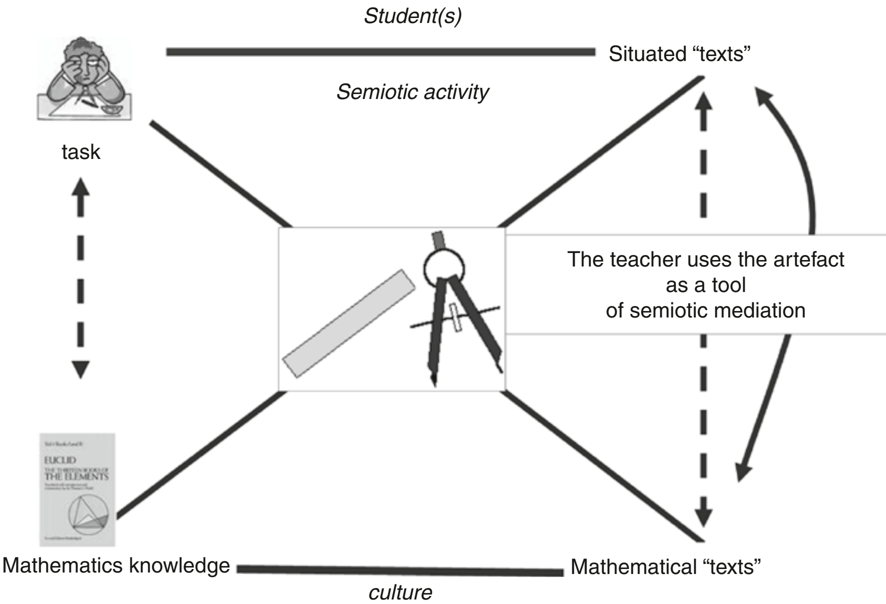


Fig. 3 Semiotic mediation diagram

we follow the elaboration of Bartolini Bussi and Mariotti (2008) which is shortly outlined below.

The process of semiotic mediation may be described schematically by means of the following drawing (Fig. 3).

A learner (either a secondary student or a prospective teacher) is given a task (left-top vertex of the rectangle of the Fig. 3; for an example, see below), to be solved by means of a specific artefact (e.g., the pair straightedge and compass, centre of the rectangle of the Fig. 3). The piece of mathematics knowledge at stake may concern the meaning of circle and of straight line and geometrical properties of some figures.

In the resolution process of the given task, two levels can be distinguished. At the first level, a technical solution of the task may be given using the artefact mechanically, i.e., repeating, in automatized way, a set of instruction, without wondering why the geometrical construction works. At the second level, a solution becomes “meaningful” (in the etymological sense) when it is justified and commented with reference to the properties of circles, triangles and so on, as, in this way, the meaning of geometrical construction is approached at and enriched. This meaning is a piece of mathematics knowledge (left-bottom vertex of the rectangle of the Fig. 3).

If the activity stays on the technical plane (task, artefact and situated texts triangle in the Fig. 3), the justification of the correctness may be not at stake. The control by either perception or measuring might be enough, to agree that the solution is correct. The justification belongs to the theoretical plane. The technical description answers the question, “How?”, whilst the theoretical description is the first step to

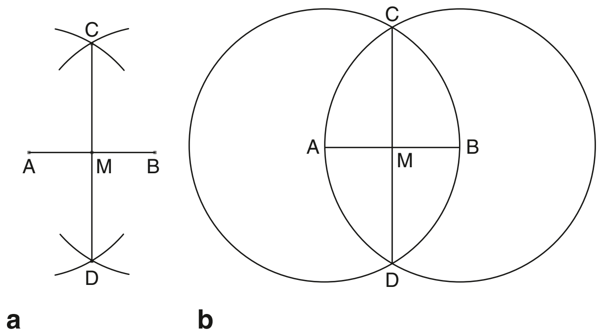


Fig. 4 **a** Set the needlepoint of the compass on *A* and the lead point on *B* and draw a small arc on each side of the line *AB*. Set the needlepoint of the compass on *B* and the lead point on *A* and draw a small arc on each side of the line *AB*. Mark by means of a pencil the points *C* and *D* where the arcs intersect each other. Put the edge of the ruler on *C* and *D* and draw by means of a pencil a line r. Mark by means of a pencil the point *M* where the line r intersects *AB* (Java animation: http://www.mathopenref.com/constbisectline.html. Accessed February 2010). **b** Draw a circle with centre *A* and radius *AB*. Draw a circle with centre *B* and radius *BA*. Find the intersection *C* and *D* of the two circles. Draw a straight line r joining *C* and *D*. Find the intersection *M* of r and *AB*

answer the question, "Why?". The path towards the justification is neither simple nor fast. For example, from the initial situated expressions which refer to the actual use of the ruler and the compass, the reference to the artefact disappears, remaining embodied or evoked in the straight line and in the circle, i.e., the geometrical objects traced by means of them.

Consider, for instance, the task to bisect a given finite straight line[3] by ruler and compass, and compare the following texts, that accompany similar (yet not identical) drawings (Fig. 4a, b). In the two texts, the artefact is the same (the pair ruler and compass). On the left, there is an evident reference to the physical operations to be performed by means of the concrete available tools, whilst on the right the reference is to geometrical objects that evoke their geometrical properties. The two sets of instructions are different: the left one evokes a text of technical drawing or engineering[4], whilst the right one evokes Euclid's construction. A novice might be at ease with the left set of instructions, whilst an expert might be annoyed by it. In the left set of instructions, the characteristic properties of the circle are not explicitly evoked, at both linguistic and graphical level. The text only mentions (and the drawing only contains) a "small arc" instead of a "circle". In the right list of instructions, the references to the circle and its properties are explicit. The text on the left is situated, whilst the text on the right is decontextualised (hence, it is a mathematical text). This may be interpreted, after Rabardel (1995, see Section "Mathematical

[3] This construction problem is taken from the First Book of Euclid's elements (Proposition 10, see Heath 1956, p. 267). The solution we propose is a bit different from Euclid's one.

[4] http://www.tpub.com/engbas/4.htm. Accessed February 2010.

Laboratory and Teachers Education" above), saying the authors are referring to two different instruments.

The reader might be interested to write, for the same task, the instructions for another artefact, e.g., a DGE like *Cabri* or *Geometer's Sketchpad*. The situated text in this case is different, as the reference is to the commands available on the menus. For instance, in DGE there is no needlepoint and arcs can be drawn only after having drawn the whole circle. Yet the Euclid-style text on the right can serve still as a geometrical reference text.

Whichever is the artefact (the concrete pair ruler and compass on the paper, but also the virtual commands on the screen in the case of a DGE), the mathematics teacher's aim is not (only) the technical process, but also the geometrical process that evokes the properties (either definitions or theorems) of geometrical objects. The artefacts allow the implementation of concrete actions (i.e., they are outward oriented) and, on the other hand, they allow the formation of the subject's plane of consciousness (i.e., they are inward oriented). In this second case, culturally based psychological processes are created (Vygotsky 1978), in the sense that by means of the physical activity (either ruler and compass or the menu commands) the user is constructing the meanings of circles and lines. According to Vygotskian approach, within the social use of artefacts in the accomplishment of a task, shared signs are generated. These signs are related to the accomplishment of the task and to the used artefact, on the one hand, and they may be related to the content that is to be mediated, on the other hand. They can be intentionally used by the teacher to exploit semiotic processes, aiming at guiding the evolution of meanings by the evolution of signs centred on the use of an artefact within the class community. In other words, the teacher acts as mediator using the artefact to mediate mathematical content to the students. In this sense, the teacher uses the artefacts as tools of semiotic mediation (Bartolini Bussi and Mariotti 2008, Fig. 3).

The ruler and the compass are the most known drawing devices. In the following sections we study the case of another drawing device, based on the geometrical properties of antiparallelogram, i.e., a quadrilateral in which the pairs of nonadjacent sides are congruent, but in which the pairs of opposite sides intersect (unlike in a parallelogram). The analysis is distinguished into three parts: the historic-epistemological dimension concerning textual descriptions; the manipulative dimension involving material copies and the digital dimension based on simulations by a DGE. For all dimensions, the focus is on tasks for teacher's education.

Historic-Epistemological Dimension

The Background

The ruler and the compass have been used from Euclid's era to solve construction problems in plane geometry. The discussion about acceptable tools to solve construction problems was raised in the classical age (Heath 1956) and later at-

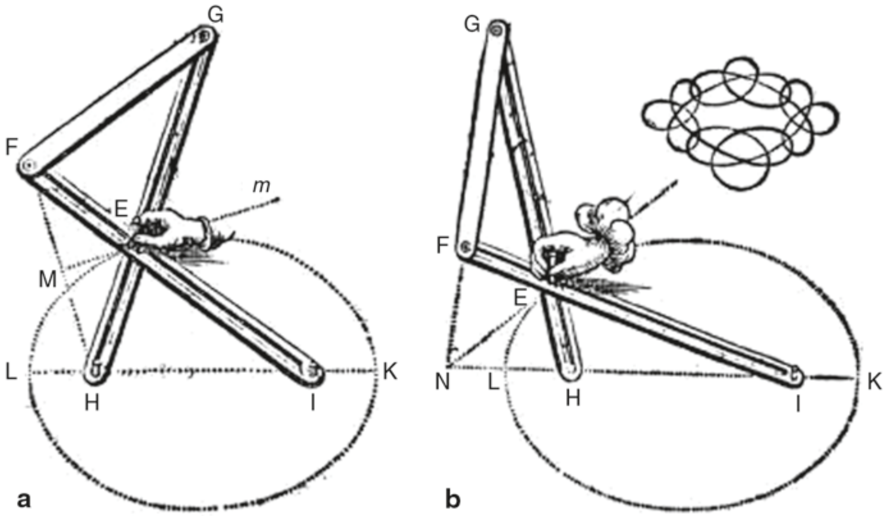


Fig. 5 van Schooten's antiparallelogram. (van Schooten 1657)

tacked directly by Descartes, in the XVII century, when he wrote the *Géométrie* (Descartes 1637), i.e., the appendix to the *Discourse de la Méthode*. His aim was to delineate the frontier between those curves that are acceptable in geometry, which Descartes called "geometric", and the rest, which he called "mechanical" (Bos 2001; Dennis and Confrey 1995; see also Bartolini Bussi 2001). As said above, in the classical age the "identification" of curves and artefacts (drawing devices) had been realized for straight line (ruler) and circles (compass). Conics were rather considered as solid curves (conic sections), i.e., curves obtained by cutting a cone. Yet conics and other curves could be used to solve construction problems (e.g., the trisection of an angle, see Heath 1956) that could not be solved using only straight lines and circles. Descartes looked for artefacts able to draw curves by a continuous motion: in this way the perceptual evidence of intersection between curves could be used to state the existence of a rigorous solution of a construction problem (Lebesgue 1950). Van Schooten followed him in the same direction: he translated Descartes' *Géométrie* into Latin and appended commentaries (*Exercitationes*) about curve drawing devices. The Fig. 5 shows an articulated antiparallelogram used as a curve drawing device from van Schooten (1657). The Fig. 7 shows students using a modern wooden reconstruction of it.

Drawings and Texts as Artefacts

With respects to artefacts, Wartofsky (1979) distinguished primary, secondary and tertiary artefacts:

> What constitutes a distinctively human form of action is the creation and use of artifacts, as tools, in the production of the means of existence and in the reproduction of the species. *Primary artifacts* are those directly used in this production; *secondary artifacts* are those used in the preservation and transmission of the acquired skills or modes of action or praxis by which this production is carried out. Secondary artifacts are therefore representations of such modes of actions. (Wartofsky 1979, p. 200 ff.)

In this chapter, we have examples of primary artefacts (the antiparallelogram of the Fig. 7) and of secondary artefacts (drawings and text from van Schooten's book). There is also another class of artefacts (tertiary artefacts):

> (…) which can come to constitute a relatively autonomous 'world', in which the rules, conventions and outcomes no longer appear directly practical, or which, indeed, seem to constitute an arena of non-practical, or 'free' play or game activity. This is particularly true (…) when the relation to direct productive or communicative praxis is so weakened, that the formal structures of the representation are taken in their own right as primary, and are abstracted from their use in productive praxis. (Wartofsky 1979, p. 208 ff.)

Mathematical theories are examples of tertiary artefacts, organizing the models constructed as secondary artefacts. Mathematical theories have the potential of being expanded to create something anew, that maintains links with practical and representative activities.

The two drawings of the Fig. 5 (van Schooten 1657) show two different positions (like two 'frames' in a modern motion picture) of the articulated antiparallelogram. They seem realistic (bars, pivots, and even the hands), but we discuss this point below. Beside the locus of E also the tangent line in E is drawn.

Van Schooten's text follows (the reference is to the Fig. 5a, b):

> *Chapter VIII. About the way of tracing ellipses in a plane, when the foci and the vertices are given*.
> There are several ways to trace ellipses: the one when foci and vertices are given is not more complex than others […]. Given in a plane the foci H and I, a vertex L and the other vertex K, so that LK is the transverse axis, to trace, in the same plane, the drawing of an ellipse, with those vertices and foci. To prepare, in either brass or wood or other hard material, three bars HG, GF and FI, with HG and FI equal to LK, whilst FG is equal to the distance HI between the two foci. Besides, let the bars HG and FI be fissured (along their length) by two runners with the same width of the diameter of the cylindrical stylus, that will be inserted into them to trace the elliptical drawing. Let each of the bars HG and FI be drilled at the ends H and I, to insert the hinges pegged down in the foci H and I; the ends G and F of the same bars will be hinged on the ones of the bars FG, to create the configuration of the figure. That done, if the stylus inserted in both runners (i.e., in the point E where the bars HG and FI intersect each other) is moved, it will drag the bars Hg and FI, which will rotate on the points H and I: moving it from L to K the stylus will trace half (LEK) of the elliptical drawing. In the same way the other half will be traced (van Schooten 1657, p. 339, translated by the authors).

The Task

Van Schooten's text hints at the process of instrumental genesis for both the coordinated processes of instrumentalisation and instrumentation (see Section Mathemati-

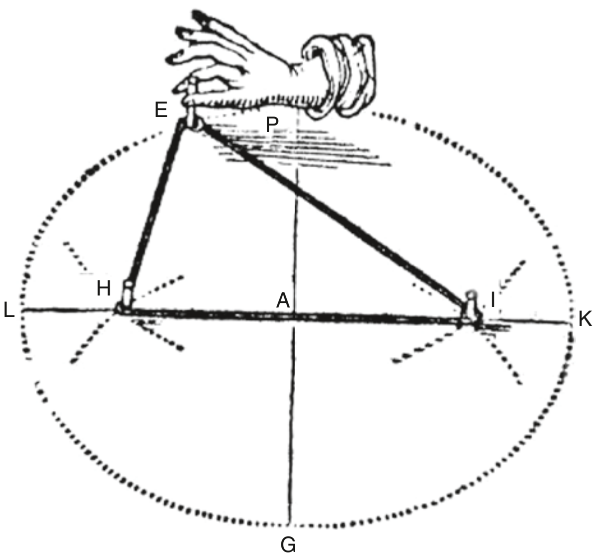


Fig. 6 Drawing of gardener's string. (van Schooten 1657)

cal Laboratory and Teacher Education above). This suggests the following task for prospective teachers, as an example of analysis of a secondary artefact:

> Read van Schooten's texts about the ellipse drawing device by antiparallelogram. Find the parts concerning the components of the artefact and the constraints for its points and the parts concerning the utilization schemes[5] of the artefact.

The antiparallelogram construction is related yet different from the better known string construction of ellipses (or gardener's string construction, see the Fig. 6, taken from van Schooten 1657). An additional task may be designed, for prospective teachers, as a comparison between them:

> Compare van Schooten's text about antiparallelogram and the gardener drawing of the artefact pencil-string, with regard to the components of the artefacts and the utilization schemes.

[5] Béguin and Rabardel (2000) define *instrumentation* as follows:

> Utilization schemes have both a private and a social dimension. The private dimension is specific to each individual. The social dimension, i.e., the fact that it is shared by many members of a social group, results from the fact that schemes develop during a process involving individuals who are not isolated. Other users as well as the artefact's designers contribute to the elaboration of the scheme. (Bèguin and Rabardel 2000, p. 182)

In the antiparallelogram there is a linkage whose motion is perfectly determined by the physical constraints, whilst in the gardener's string construction the string has to be taut by the user by means of a pencil during the process. Hence, in the former the motion is controlled by the artefact, whilst in the latter is controlled by the user. The hand in the former has mainly the function to keep the pencil in the right position, although the motion might be given to the artefact pushing other points of the bars (e.g., G, F and others); the hand in the latter has both functions: it holds the pencil and moves it as well, keeping the string taut.

Following Ball et al.'s (2008) approach (see Section "Mathematical Laboratory and Teachers Education" above), these tasks are related to the specialised content knowledge. In fact, they concern the mathematical knowledge needed for teaching: for instance, social dimension of the instrumental genesis and different instruments (artefacts + utilisation schemes) related to the same mathematical meanings. In this case, they also contribute to enrich the knowledge of content and teaching.

Manipulative Dimension

The Background

In the MMLab there are more than two hundred working reconstructions (based on the original sources) of mathematical artefacts taken from the history of geometry. Some of them (e.g., van Schooten's antiparallelogram, see Fig. 7) are reproduced

Fig. 7 The concrete artefact

in multiple copies to allow small groups (four or five people) use them in the same session (with either secondary school students or prospective teachers). Afterwards, we present the features of a mathematical laboratory session. The structure of a session and the working sheet result from a long process of revision and refinement, based on our analysis of the laboratory sessions realised in the MMLab (for both students and prospective teachers).

The Task

The working sessions are usually split into three parts:

- historical introduction for the whole group;
- small group work on the linkage, by means of a working sheet; and,
- collective work on the solutions for the given tasks.

In the second part, a copy of the van Schooten's antiparallelogram (considered as a primary artefact) with an exploration sheet (Fig. 8), where a schema of the artefact is drawn, is given to each group. Each group is asked to write its answers to the questions. Each working sheet contains several different questions that support the exploration process of the mathematical machine. They take into account on one hand the process of instrumental genesis (Rabardel 1995), on the other hand our intention to foster the processes of both production of conjecture and construction of proof, beyond the pure manipulation. In fact, questions concern not only how the artefact is made and works, but also the properties of the drawn curve and the characteristics of the device permitting to draw that curve. The proposed sequence of questions considers the temporal commitment of two hours (at the maximum) for a session, in order to permit a suitable work.

1. How many rigid rods make up the linkage?
2. Measure the lengths of the individual rods.

Which figures do the rods form?

3. Which are the elements of the instrument which are fixed at the plan?

Move the linkage.

4. Which are the segments that do not change in length during the movement?
5. Which are the segments that change their length during the movement?
6. Which variable length segments are equal?

This instrument has three tracer points: Q, R and T.
Answer the following questions:

7. Which curves do the points Q and R trace?
8. Put your pencil in T and draw a part of a curve. Which is the property of the curve plotted by the point T?
9. Choose a suitable Cartesian axes system. Write the equations of the curves plotted by the points Q, R and T.

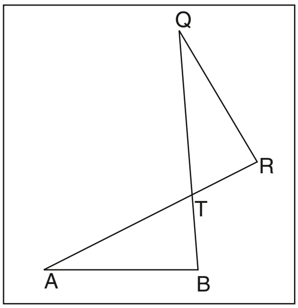


Fig. 8 Working sheet

Questions 1, 2 and 3 aim at highlighting the physical features of the given artefact (the emergence of the components in the instrumentalisation process). In particular, Question 2 offers elements to justify the functioning of the linkage and the property of the drawn curve. Questions 4, 5 and 6 require the movement of the quadrilateral and aim at highlighting some invariants in its structure during this movement. In this request, the first elements of the instrumentation process are in play, because the users have to choice a pilot point, often in an implicit way. This instrument has three tracer points (Q, R and T), but there is only a hole for pencil in T. So, the trajectories of Q and R could not be really traced, but only supposed. Question 8 concerns the instrumentation process. It also requires to explicit the property of the drawn curve, on the basis of the exploration. The definition of ellipse as a locus of points in a plane such that the sum of the distances to two fixed points (foci A and B) is a constant is expected. In particular, Question 8 prompts a process of conjecture production (*what*) and proof construction (*why*). Question 9 imposes the passage to the analytic geometrical register. The best choice for the Cartesian axes system is as follow: straight line containing the line segment AB as x-axis, the perpendicular bisector of the line segment AB as y-axis. Furthermore, the solver can choice the distance AB as a and the distance AR as b in writing the required equations.

The collective part of the session (third part) concerns the shift from the texts (right-top vertex in the Fig. 3) produced by the prospective teachers towards mathematical texts with definition and properties of ellipse (right-bottom vertex in the Fig. 3). In particular, Questions 2 and 7 are interesting to be developed in a collective discussion, because the former is related to the mathematical meaning of tangent line to ellipse and the latter to a definition of ellipse different from the definition evoked by Question 8. As regards to Question 2, if the quadrilateral ABRQ is recognize as an isosceles trapezoid, its symmetry axis is the tangent line to ellipse at its point T (as it appears in van Schooten's drawings, Fig. 5). Question 7 allows attention being paid to the relationship between the circle with centre on focus A and the point T. In fact, T is a point at the same distance from the focus B and the circle traced by R with centre on focus A (in other term, ellipse as a locus of points in a plane such that the distances to a fixed point and to a circle with centre on another fixed point is equal). The circle with centre A is named "directrix".

In a mathematics laboratory session, the teacher educator uses the artefact as a tool of semiotic mediation. At the same time, prospective teachers are involved and test an example of didactic management of this session.

Digital Dimension

The Background

Dynamic Geometry Environments (DGE; e.g., *Cabri*) are used in MMLab as modelling contexts for dynamic artefacts. Prospective secondary mathematics teachers, after having explored the physical drawing device, are asked to produce a digital model of it. This task represents a challenge for prospective and practicing teachers. In fact, the main idea is to use DGE not to explore open problems or as a model for theoretical systems (for a discussion, see Laborde 2000), but as a modelling environment.

The Task

Prospective teachers are given again a working copy of the drawing device (Fig. 7) and the following task:

> Construct on the *Cabri* screen a model of the drawing device, that may be piloted in order to work in the same way of the physical one.

In this case, the artefact is DGE (i.e., *Cabri*) and the prospective teacher has the possibility to use the menus to solve the task. Some different solutions emerge: we illustrate only two solutions[6] (Fig. 9 and Fig. 10) and discuss the difference.

First Solution

Line segments are assembled to produce an antiparallelogram.

Two prototypes of the bar are drawn (AB and CD) (Fig. 9).

1. compass: AB in H
2. intersection: I
3. compass: CD in H: select G on the circle
4. compass: CD in I
5. compass: AB in G
6. intersection: F
7. segment: IF
8. segment: HG
9. intersection: FI and HG: E
10. intersection: L and K
11. drag G to pilot E.

[6] We refer in a short way to the *Cabri* commands. Legend:

- *compass*: to transport the given segment with a vertex in a given point (the software draws a circle);
- *intersection*: to find the intersection point of two objects on the screen;
- *intersection* (after *compass* command): to intersect the circle with another object on the screen;
- *segment*: to draw a segment joining two points;

The others (*axis, locus, symmetrical point*) hint at geometrical meanings, and are realized by means of the available commands.

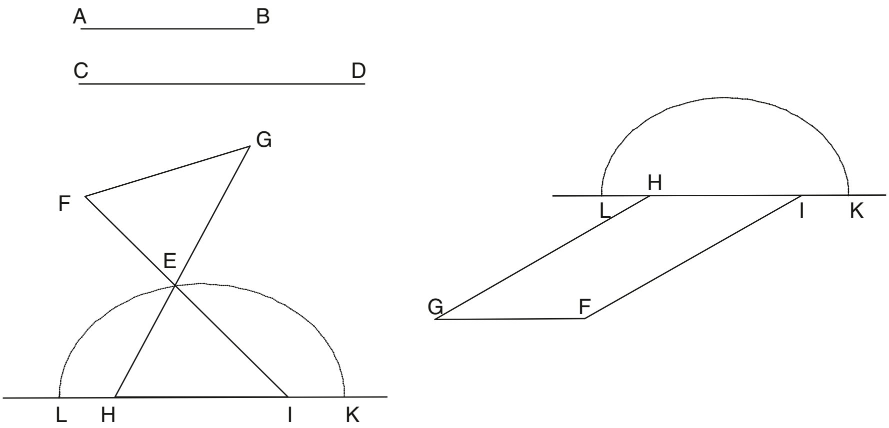


Fig. 9 A technical solution

Locus: the same as the one drawn by the physical device; different from van Schooten drawing (see Fig. 5).

Motion: when the point G is dragged on the circle suddenly the antiparallelogram unknits and becomes a parallelogram.

If one recognizes that the quadrilateral HFGI is an isosceles trapezoid, whose HG and FI are its diagonals, he/she is able to design a digital antiparallelogram, satisfying the two previous conditions. In this case, the symmetry axis of the antiparallelogram is the tangent line to the ellipse in each point, as van Schooten's drawing clearly shows. The second solution is described below.

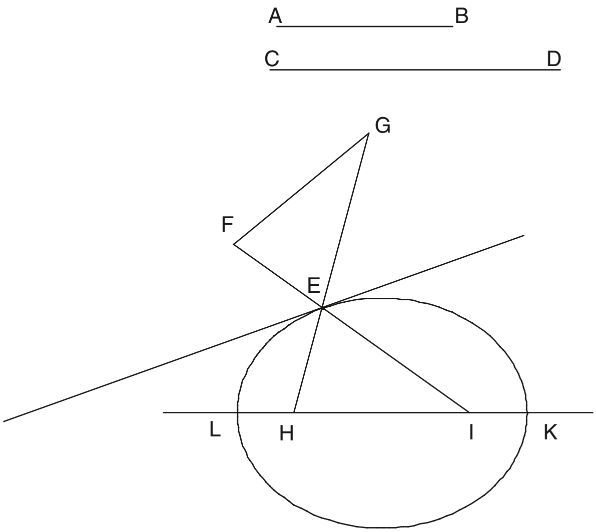
Second Solution

A geometric property of antiparallelogram is used.

Two prototypes of the bar are drawn (AB and CD) (Fig. 10)

1. compass: AB in H
2. intersection: I
3. compass: CD in H: select G on the circle
4. axis of GI
5. symmetrical point of H with respect to the axis: F
6. segment: IF
7. segment: HG
8. intersection: FI and HG: E
9. intersection: L and K
10. drag G to pilot E

Fig. 10 A geometrical solution



Locus: the same as van Schooten drawing (see Fig. 5); different from the drawing produced by the physical device.

Motion: when the point G is dragged on the circle the antiparallelogram is maintained.

In the two solutions, the same commands (artefacts) are instrumented in different ways.

In all the cases a difference emerges. The task is impossible if it is taken literally. Actually, it is not possible to design a model that works exactly like the physical one. As we have observed, the physical artefact can be moved pushing many points of the bars, provided that the pencil is firmly inserted into the moving hole E. This cannot be realised with *Cabri*. Every construction is ordered: the user has to define which is the starting point (G in the above constructions), to be assumed as independent variable, and what follows is strictly dependent on this choice. This is a general property. If one wishes to select the point E as the piloting point, she/he should produce a different set of instruction where E is a piloting point (independent variable) and the others are dependent on E. The choice of the piloting point (a point with one degree of freedom) has to be done explicitly before starting the *Cabri* construction. This means looking at the antiparallelogram according to the constraints of *Cabri* (and the same is true for whichever other DGE). The second construction produces van Schooten's model, but does not work as the linkage. The first construction (with adjustment) is closer to the linkage but produces only a part of the ellipse.

If one goes back to the schema of the Fig. 3, the first solution may be described by means of a situated text (right-up vertex of the rectangle of the Fig. 3): copies of the prototypes of the bars are assembled as in a meccano setting. The names used are bars rather than straight lines. The observation of prospective teachers at work shows that they try to mime the rotation of the bars GH and FI on the screen with fingers, pointing with thumb in H and I and with forefingers in G and F, and look for a position where FG has the given length. The second solution, instead, hints at a non- transparent property of the artefact (the presence of a symmetry axis), that is better acknowledged when a static frame is considered. This is not a spontaneous solution, as the manipulation of the concrete artefact suggests rather the first one. Yet, as soon as the second solution is found, a new exploration of van Schooten's antiparallelogram may be started on the screen, to highlight the tangent line and the relationships between the length of the longest bar and the major axis of the ellipse (as said in van Schooten's 1657 text).

Concluding Remarks

In this paper we have presented and discussed three different ways of introducing a mathematical machine (i.e., a curve drawing device, that produces an elliptical trajectory) into the mathematical laboratory of a secondary teacher education program: the discussion and the interpretation of an artefact given by the pair text and drawings from a XVII century treatise; the manipulative exploration, according to a working sheet, of a material copy of the ancient artefact; the production of a digital simulation of the ancient artefact. Additional tasks may be designed (e.g., building a material copy, drawing on van Schooten's 1657 description) and analysed as well. All these activities can be carry out in two hours (at the maximum) sessions. For this reason, they can be easily proposed in both teacher training and students' mathematics course. Nevertheless, a systematic use of mathematical machines for all conic sections needs a careful planning and it represents a methodological choice of the teacher.

In all cases the instrumental genesis (according to Rabardel 1995) is at work, yet in different ways. In the first case the prospective teacher is invited to recognize in the text hints at the instrumentation and the instrumentalisation process concerning the task of drawing an ellipse: as usual in most ancient treatises, the two processes are intertwined and not easily separable from each other. In the second case the prospective teacher is invited to experience in a personal way the instrumental genesis working with suitable tasks on a material model: the tasks are similar to the ones that he/she might give to his/her students. In the third case the curve drawing device is paired with another artefact (i.e., a DGE), that introduces additional strong constraints which force a new exploration of the material artefact and produce another way for drawing the same curve. The expert geometer might say that what is focused is "the same" artefact, i.e., van Schooten's ellipse drawing device by means of an antiparallelogram. Actually the artefacts are different. According to Wartofsky's classification (1979) in the first case it is a secondary artefact, used in transmission of modes of actions; in the second case it is a primary artefact that is directly used, although the justification required to introduce also secondary and tertiary artefacts; in the third case what is called into play is a tertiary artefact, i.e., the geometrical properties (referred to a mathematical theory) of the figure "antiparallelogram" in the *Cabri* setting. From a didactical perspective, the instruments (Rabardel 1995) are different in the three cases because of different utilization schemes and constraints (material or digital) as well.

This experience shows to be paradigmatic for prospective teachers, in order to make them aware that, in spite of some widespread simplifications (see for instance the National Library of Virtual Manipulatives, http://nlvm.usu.edu/) it is quite different to operate on textual descriptions (even with "realistic" drawings), on material copies, on digital simulations. This is obviously true not only for the van Schooten's parallelogram but also for other teaching aids that may exist in either descriptive or material or digital forms. In every case, for every task, a careful analysis of the instrumental genesis and of its relationships with the construction of mathematical meaning is needed for the use in the mathematical laboratory with secondary school students.

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Part IV

Using Video Episodes to Reflect on the Role of the Teacher in Mathematical Discussions

João Pedro da Ponte

Introduction

This chapter presents a practice-based teacher education task and reflects on its actual use with a group of experienced teachers. First, it draws the rationale for using such kind of tasks. Next, it presents the context in which this task was designed and reports on its actual enactment with a group of middle school teachers. And, finally, it concludes with a discussion about what I learned from using this task, the conditions necessary to make it successful, and its potential for teacher education of prospective and practicing teachers.

Exploratory Tasks in the Mathematics Classroom

Traditionally, the prevailing mode of work in Portuguese classrooms involves two steps: in the first step, the teacher introduces a new topic, concept or procedure, and provides one or more examples. In the second second, the teacher assigns a set of exercises for the students to do and, finally, provides the solutions on the blackboard.

Many teachers refer to this mode of work as presenting "theory" and providing moments of "practice". The presentation of "theory", lecturing or "exposition" can be done in many ways—sometimes the students are asked questions, and short discussions take place during the presentation of the new material. During "practice", in solving the exercises, the students usually work individually, but, in some cases, they are allowed to check their solutions with those of their nearby colleagues. The solutions for the exercises are presented on the board, sometimes by the teacher, other times by a student who volunteers or who the teacher asks to present his/her

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work to the classmates. Working in this way has been has been labelled "expository teaching" or "direct teaching" (Brooks and Suydam 1993) and has been common in mathematics classrooms for many years (Fey 1979).

Recently, an alternative mode of work is emerging in Portugal and elsewhere: the teacher introduces a task for the students to work for some period of time and, in a second step, the students present their solutions to the whole class and discuss the solutions of their classmates. This second mode of working is being increasingly used at grades 1–4 (1st cycle of basic education), 5–6 (2nd cycle) and 7–9 (3rd cycle)[1]. The students usually work in pairs or in small groups (often with four students). Sometimes they are asked to write a report with their strategies and solutions, in other cases they present it orally during the discussion. This mode of work requires suitable tasks to propose to students—exploratory, inquiry, or investigative tasks that lead them to do substantial work and from which they can learn new mathematics. It also requires that teachers support students working during an extended period and, later, conduct a productive discussion during which mathematics ideas are raised, clarified, and, finally, formalized. This vision of the mathematics classroom fit with what many documents refer to as "reform mathematics education" (NCTM 2000), or "inquiry-based mathematics classroom" (Battista 1999; Kazemi and Stipek 2001), or "exploratory mathematics learning" (Ponte 2005).

Worthwhile mathematics tasks are a necessary condition for good mathematics teaching (NCTM 1991). However, appropriate tasks for one class may not be appropriate for another. Thus, the teacher needs to know how to select the right kind of task for his/her students. Tasks vary in a number of dimensions. For example, some tasks are very structured, indicating exactly what is given and what is asked, and even sometimes suggesting what is to be done. Other tasks are more open, requiring some degree of interpretation from the students concerning the question to address the givens, conditions and strategies (Ponte 2005). Some tasks begin with structured questions but then continue with rather open questions. This may help the students to get started in the conceptual field related to the task, thus providing some direction for the subsequent work.

Another important dimension of tasks is the degree of mathematical challenge (Potari and Jaworski 2002). If a task is perceived as too difficult, the students rather quickly will likely give up working on it. If it is perceived as too easy, the students will not invest much energy and creativity on it. This, of course, creates a serious difficulty for the teacher, given the heterogeneity of student ability in a regular mathematics classroom.

Tasks still differ in other dimensions. For example, one needs to consider the context—mathematical or non-mathematical—and the time required to complete a given task; this may range from a few minutes to some days, weeks, months or even

[1] Students at 1st cycle of basic education (grades 1–4), are aged 6–9 years, at 2nd cycle (grades 5–6), 10–11 years, and at 3rd cycle (grades 7–9), 12–14 years. In fact, students are often retained at a particular grade, and, therefore, in a given class it is frequent to see students with the expected age together with students that are 1 or 2 years or more older.

more (Ponte 2005). The critical role of open and challenging tasks in mathematics teaching has been recognized by many mathematics educators (e.g., Skovsmose 2001; Sullivan et al. 1997) and indeed already plays significantly in some countries (Boaler 1998).

However, appropriate mathematical tasks do not work just by themselves. The role of the teacher in the classroom is critical at every step (Stein and Smith 1998). The task needs to be presented in a way that is appealing and interesting to students and these must be supported during their work. However, as students ask for help, the teacher needs to restrain from giving directions immediately that would take away all the challenge from the task. Rather, he/she needs to understand well what the students' difficulty is and find a subtle way of putting them back on track. Of course, also very important is the development of a classroom atmosphere of sharing, discussion, argumentation, and critical analysis, so that the mathematical work done in these tasks is socialized by all the students and institutionalized as valid classroom knowledge.

In Portugal, a new mathematics curriculum for basic education—grades 1–9—was recently approved. Mathematics content is organized in four main themes: Number and operations, Geometry, Algebra, and Data Handling. In contrast with the former curriculum, this puts more emphasis on algebraic thinking (in the former curriculum the emphasis was on algebraic computation), as well as on geometric transformations (isometries, similarities) and on data handling (designing investigations and representing, analysing and drawing conclusions from data). It also stresses three transversal capacities: problem solving, reasoning, and communication. This curriculum has several levels of objectives:

- general objectives for mathematics teaching for all basic education;
- general objectives for each cycle and theme (that is, algebra has a set of general objectives at the 2nd cycle and another set at the 3rd cycle); and,
- specific learning objectives in each cycle.

This new curriculum encourages teachers to propose exploratory tasks in mathematics classrooms. It suggests that rich exploratory tasks and whole class discussions are important elements in the students' learning experiences but it leaves to the teacher to decide about the appropriate balance of classroom working modes.

Such exploratory tasks are quite demanding on teachers in several respects: Their selection involves a high level of understanding of the mathematics ideas involved as well as an in-depth knowledge about students' abilities and interests. In supporting students, teachers have to restrain themselves from saying too much, at the risk of taking away the need for students' thinking and thus trivializing the moment of final discussion. This discussion moment, on the other hand, requires that teachers are able to orchestrate the classroom discourse, providing opportunities for all students to intervene, stimulating moments of controversy and argumentation, and also moments of systematization and formalization of mathematical ideas. Several other countries have curriculum documents with similar orientations regarding classroom work (Ponte et al. 2006).

Teacher Education to Transform Classroom Practice

Teacher education has been strongly critiqued because of its inability to have any impact on classroom practice (Lampert and Ball 1998). For some time, many teacher educators put special attention on teachers' beliefs, conceptions, and knowledge regarding mathematics and mathematics teaching. The implicit underlying assumption was that if those could be changed, then teachers' classroom practice would also change. Now, it is becoming quite clear that if the goal is to have a real impact on teachers' classroom practice, then classroom practice needs to play a key role in teacher education (Ball and Cohen 1999; Smith 2001).

This leads to the consideration of practice-based teacher education. However, this notion may have several meanings. At a first level, teacher educators may seek to recognize the existing problems in the practical situation that the teachers experience and frame some possible strategies to deal with them, perhaps taking into account educational theory. Such strategies are then exemplified by some materials constructed on purpose that are then used in teacher education settings. At another level, teacher education may be situated in practice. That means that the materials that represent the teaching activity and their results (for example, mathematical tasks, records of students' work, classroom episodes) are used as opportunities for critique and investigation. Teachers then develop knowledge analyzing real situations that they may use later in their actual teaching practice. On a third level, teacher education may be based on teachers' own practice. In this case, teachers collect data from their practice and reflect about them with support of the teacher education setting, that includes the teacher educator, other teachers, and possibly other resources.

The first level is already oriented towards practice, but one works with "artificial" materials, constructed on purpose for teacher education. On the second level, one works with material drawn from actual classrooms that may be more or less familiar to the teachers that will analyze it. On the third level, teachers use material collected from their practice as the basis for their reflections and analysis. This is very powerful, but requires a lot of effort in planning for data collection, collecting data, and making it suitable to use in teacher education. All three levels have their specific strengths and weaknesses. The second level—that informs this chapter—is a good choice for a small teacher education activity, when there is not much time for teachers to collect data from their classes, but that seeks to pay attention to issues related to classroom events.

The Teacher Education Task and Context

The General Setting

The new Portuguese mathematics curriculum will be used by schools from September 2009 onwards, but the Ministry of Education decided that the preparation of teachers should begin immediately. A first group of about 50 middle/lower secondary school teachers (from 2nd and 3rd cycles) was invited to act as teacher educators in workshops that were carried from March to June 2008. Each workshop lasts for 25 hours, including 6 sessions over a period of 5 months. The workshops include the analysis of the curriculum, doing mathematical tasks, preparing and conducting a small classroom experiment, and discussing and sharing experiences.

These teachers will now act part of their time as workshop leaders.[2] In order to assist them in preparing these workshops, a small activity (15 hours) was conducted over two days February 2008. The task reported in this chapter was presented in the first day. This is a task for teachers (Fig. 2) that is based on the analysis of a classroom episode based on a mathematics task for students (Fig. 1). This student task was designed and proposed to a grade 8 class by a mathematics teacher, Idália Pesquita[3]. I found that the work that went on in the classroom is very interesting and could be used as a basis for a teacher education activity. So, next I describe the task and the situation that Idália experienced in her class and then I come back to the teacher education setting.

The Classroom Situation

The mathematics task proposed to students concerns working with a pattern (Fig. 1). The pattern may be seen as representing a (growing) linear model, with 4 as the first element and increments of 3. However, there are other ways to regard such pattern, for example, assuming that it is a repeating pattern in which the three given terms repeat themselves, or assuming that it alternates increasing and decreasing sections. Therefore, there are many possible answers to the questions posed and all of them might be accepted if justified on the basis of a proposed pattern.

At first sight, it appears as a very simple problem, but the disposition of the elements in the pictures makes it a little tricky, especially for those students (and teachers) who have little experience in working in this kind of problems. First, the grade 8 students are just asked to continue the pattern, assuming that it represents the growing linear model, but then other questions encourage them to find a generalization. Such generalization may be formulated in a number of ways, notably in words (using natural language) or with symbols (using algebraic language).

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[2] The participants in this teacher education activity continue to have their regular teacher duties, teaching their own classes. They will act as workshop leaders just for about 5 hours a month, during 5 months. Therefore, in this chapter, they will be referred to as "participants" or "teachers".

[3] The initiative for the design of the task, the recording of classroom work, and its analysis was part of the activities of the Portuguese Group of International Comenius Project PDTR—Professional Development of Teachers Researchers. I am the national coordinator of this group and the project, besides Portugal, involves groups of teachers from Poland, Hungary, Italy, and Spain.

1. Observe the following sequence of pictures:

a) How would you construct the 4th picture of this sequence?
b) Using 25 stars, what is the order of the picture that you could construct? All the stars would be necessary?
c) How many stars are necessary to construct the 10th picture?
d) Describe a method to indicate the number of stars in the 40th picture. What about the 100th picture?
e) How many stars are necessary to construct the *n*th picture?
f) The sequence will include a picture with exactly 150 stars?

Fig. 1 Students' task: generalizing a pattern

The students worked on this task in groups for about 45 minutes. Then Idália, the classroom teacher, conducted a classroom discussion, in which all students had a chance to participate. The whole class was video recorded and later transcribed.

The Workshop Setting

The participants in the activity that I describe in this chapter were 20 teachers (10 of 2nd cycle and 10 of 3rd cycle) who were preparing workshops on Numbers and Operations and Algebra. The group included a mixture of experienced teachers and younger teachers, many of whom, however, had completed post graduate studies in mathematics education (master degrees). These teachers come from all parts of the Portuguese territory and constitute a group committed to contribute to improve mathematics teaching and learning in this country. I was the teacher educator in this activity, assisted by Hélia Sousa, a 1st cycle teacher acting in this case also as teacher educator. We had as main aim for this activity making participants aware of the power of exploratory tasks to foster students' learning and to become aware of the role of the teacher in conducting classroom discussions. In addition, we wanted to provide a model of how classroom situations can be used in teacher education, and also to show the relevance of tasks involving the study of patterns for the development of students' algebraic thinking. Our expectation was that the participants would use similar activities in the workshops they would be leading later and also that they would have an opportunity o reflect on the importance of algebraic thinking and classroom discussions for their own teaching practice.

The Teacher Education Task

This task has three parts (Fig. 2) and was planned to be carried out in 3 hours and a half.

Part 1. The first part involves an *a priori* analysis of the task. The teachers are asked first to solve the task, then to analyse what objectives of the curriculum it may help to achieve—general objectives, theme objectives, and specific objectives. Next, the teachers are encouraged to think how this task could be used in the classroom, especially how to organize students and how to manage time. This part took 2 hours. The aim of this part is to have the teachers analysing the characteristics of this task (structure, degree of challenge) and its fit of this task to the new curriculum, and, at the same time, recognizing the structure and content of this curriculum.

This part ended with a collective discussion about these issues and the possible reactions from the students. The teachers had no trouble in identifying the links with the curriculum objectives and to suggest group work as suitable setting to develop the activity. However, they were rather pessimistic about the way the students would handle the task:

1. Solve the task presented in figure 1, intended for grade 8 students. Answer the following questions:
   a) This task is related to some general objectives for mathematics teaching, general objectives of the cycle/theme and specific objectives of the mathematics curriculum and the 2nd or 3rd cycle?
   b) How this task be used in the classroom? How to organize students? What time should they be given to solve it? And for a final discussion?
   c) What difficulties may the students feel in doing it?
2. Observe the video with students discussing this task as well as the transcript
   a) Identify and analyse then roles assumed by the teacher.
   b) Identify and analyse the interventions of the teacher.
   c) What important decisions the teacher assumes during this segment?
   d) Identify and analyse the roles assumed by students.
3. Final reflection.
   a) Discuss if what you saw in this episode is in line with your initial expectations.
   b) Indicate the aspects that you find important that the teacher may have into account in order that this kind of task is successful in class?
   c) What other comments and suggestions can you draw from this episode?

Fig. 2 Teachers' task: analysing a classroom discussion

Transcript of the classroom discussion (excerpts)

Students' explanation 1 (question a)

Teacher: Look (with emphasis), I need to hear … Say.
Joana: Is the star on the middle and then we add groups of 4 stars horizontally at the right and left horizontal and downwards vertically.
Teacher: How is that? Say in very slowly…
Catarina: We have a star at the middle.
Teacher: A star at the middle…
Catarina: Then...
Ana: Groups of 4 stars...
Teacher: Groups of 4 stars....
Joana: At the right...
Ana: Horizontally and at the left horizontally...
Teacher: And then?
Ana and Catarina: Downwards vertically.
Sofia: Ours is better explained.
Inês: Ours too.
Teacher: So, Inês, how did you explain?... How did you explain?
Inês: Pedro explains.

Fig. 3 First excerpt of the classroom discussion

- They will have trouble in seeing any regularity …
- I can them with a lot of trouble in formulating a generalization.
- Maybe they will draw a table… And represent the number of starts at each stage …
- For me the most difficult to the students is to explain how each picture is constructed from the former …

Part 2. Then, on a second part, the teachers were shown an extract of the video record of the lesson (7 minutes), covering different moments of the final discussion of the task. Along with the video record they could read the corresponding full transcript (parts of which are shown in Figs. 3 and 4).

The video shows first a group of students responding to question (a) and describing verbally the rule of this pattern. Idália, the classroom teacher, finds their presentation as not clear and encourages them several times to go on and explain better their idea (Fig. 3).

Then, another group indicates that they "have a better explanation" takes the lead presenting their ideas (Fig. 4). These students indicate verbally their solution "is 3 times the number of the picture plus 1" and finally provide the algebraic expression "$3x+1$". Uncomfortable with the way some students were participating in the discussion, the teacher gives some indications about how they must sit and suspends the discussion of this solution.

After, we see another group of students presenting the way they responded to question (b). Several students ask their colleagues about clarifications, indicating that they had done it differently.

Transcript of the classroom discussion (excerpts)

Students' explanation 2

Teacher: Pedro, how did your group explain?
Pedro: It is 3 times the number of the figure plus 1
Teacher: Ah… you decided to say something else right way... How did you say it? Picture 4… [Yes] This is picture 4… How have you done it?
Pedro: It is picture 4 times 3 plus 1.
Teacher: … Picture 4…
Ana: That is the number of stars, teacher!
Teacher: … Times 3 plus 1. Is it this? Then explain it to me.
Inês: No Pedro, that is what you concluded yourself!
Sofia: I wrote 3x+1.
Hugo: No, teacher.
Teacher: Wait. Just a moment. Hey children … Look. I think that it may be better that you turn yourselves to the front. Yes…Turn to the front. You are sitting back to the blackboard and that is not a very good idea.

Fig. 4 Second excerpt of the classroom discussion

Next, the teacher asks about the 12th picture (this was not mentioned in the written task) and one student draws it on the board. There is some discussion and the students concluded that what was shown was the 11th picture and added a few stars to make it the 12th. The teacher then inverted the question and asked the students how many stars were necessary to draw the 12th picture and this originated another exchange participated by several students.

Finally, the teacher asked what would be the general method to know the number of stars in the 40th picture and the same students that had already indicated the generalization indicated it again.

The role of the teacher was mostly that of giving voice the students and encouraging them to clarify and justify their explanations. However, there was a critical moment, when one group of students presented a generalization formulated in rather abstract terms, and Idália decided to postpone for a latter discussion. This decision enabled the pursuing of a detailed discussion with all the students. Seeing the video, reviewing collectively the transcript and interpreting the episodes took about 30 minutes.

Part 3. The final part of the task is a discussion about this class. The first part aimed at leading the participants to recognize key elements of the roles and interventions of Idália and the students during this lesson, as well as important decisions taken by the classroom teacher. And, finally, teachers are encouraged to contrast their initial expectations regarding what would happen with what they actually saw on the video. The aim here is that the participants recognize some of the key features of the work in such an environment, and make them aware of the challenges it poses to teachers. The teachers first discussed this in small groups (3 elements), for about 15 minutes, and them there was a final collective discussion, for about 45 minutes.

In the collective discussion, the teachers recognized that the students were able to do much more than they initially expected. But the style of teacher in leading the discussion and some critical moments, in which the teacher posed key questions, were also noted:

- This is not just the teacher asking questions and the students providing answers…
- In fact we see the students arguing with each other…
- Sometimes the teacher poses questions such as, "How have you done?" or "Why did you do 25 minus 1?" Other times the teacher seems to keep encouraging the students to go on with their reasoning, just repeating what they said.

Those students had a complete response to the question. But the teacher did very well in holding their intervention, to allow the students who had just partial responses to speak first.

Discussion

This task has a high potential for teacher education of prospective and practicing teachers. In fact, the way the teachers participated in the final discussion showed that they felt they learned a lot from it. This task proved to be quite successful in a number of respects, as it was apparent from the high involvement of the teachers during its realization by the frequent number of cases that were referred to in later moments, and the interest that it promoted in the participants to look at classroom situations as teacher education activities. It is more difficult to know in what measure it led these teachers to become more aware of issues on algebraic thinking or in leading classroom discussions, but my perception is that at least it was helpful in increasing their interest for these issues.

Some conditions that seemed important for the success of this teacher education task include: (1) its clear relation to a curriculum topic (algebra) and to specific learning objectives (solving problems involving patterns); (2) the fact that it included detailed elements about the classroom activity on the mathematical task; (3) the teachers' perception of ecological validity in terms of the usual teaching conditions (the time available for the students to carry out the task, the number of students in the class, students' characteristics, etc.); and, (4) the fact that the issues raised in this tasks resonate with broader curriculum orientations and existing literature on the topic.

What did I learn from this teacher education activity?

In Relation to Algebraic Thinking

I noticed that the participating teachers have very little experience in carrying out this kind of mathematical tasks (searching patterns, generalizing). In fact, working with sequences at grade 8 is part of the former curriculum but it is a topic that tends to be given very little attention. In consequence, the teachers had some trouble in understanding the questions posed to the students and took more time to do the different parts of the task than I initially foresaw. Also, as the curriculum is a new document, relating the questions to the topics and objectives required some time and effort.

Teachers' Perceptions of Students' Difficulties and About Classroom Discussions

The teachers tend to stress the difficulty that the mathematical task would have for the students. They indicated that students would have difficulty in understanding the questions, in finding the pattern, and, most especially, in providing the aimed generalization. Of course, all of these were difficulties for the students, but the video of the classroom shows that they were overcome with much more ease than the teachers indicated. This happened, in great measure, because of the questioning style of Idália, the classroom teacher, showing the importance of the way the teacher conducts classroom discussions.

This activity shows that the teachers are not used to analysing classroom discussions. In Portugal, the normal routine of school activity does not involve teachers observing each other's classes. Even when they do it (for example, in teacher education for prospective teachers), they tend to look issues such as the general atmosphere, use of teaching materials, use of the blackboard, management of the time, or mathematical mistakes, and not at classroom discourse, paying attention to the nature of the interventions of teacher and students.

The Value of This Teacher Education Activity

The participating teachers were strongly impressed by seeing and analysing an actual mathematics classroom episode. This is very unusual in mathematics teacher education in our country. The technological apparatus (showing video excerpts of the classroom on a data projector connected to a computer, using sound columns, etc.) was intriguing. But the most important was the fact that the teachers could relate to the actual situation, had the time to discuss it in small groups with a few colleagues and finally had the opportunity to discuss it in the whole group.

This activity was carried out during 3.5 hours as planed. Some participating teachers found this time too short to do everything that was asked. Some of them did not finish working on the mathematics task and thinking about how to use it in the classroom (Part 1); some others said they needed more time to reflect on the episodes and the review the transcripts (beginning of Part 3). However, the way the task was structured, and the review that was made at the beginning of Part 2 and Part 3 helped to maintain all the teachers "on board" and enabled them to spend the necessary time on Part 3, the crucial part of the activity.

Conclusion

Looking on the work done with this teacher education task, I become more convinced that this format proved very useful to provide teachers with a reflection about exploratory algebraic tasks and classroom discussions and the problems that such activities pose to the teacher. The importance given in this case to the Portuguese curriculum document derives from the fact that we just introducing it in schools. In other situations, perhaps less stress should be put in similar documents, even if some relation is desirable to make sure that the task is related to significant curriculum objectives and to significant mathematics concepts, processes and ideas.

Similar activities may be also of much value with beginning teachers and prospective mathematics teachers. Looking at actual mathematics teaching situations, especially at situations that may provide useful models for successful mathematics teaching, may help them to realize that these are not just abstract models or utopian theories impossible to put into to practice in the classroom. However, with prospective mathematics teachers perhaps some more structure or some reading assignments could be useful to help them to deal with the complexity of the classroom situations.

The preparation of such teacher education tasks is quite demanding in terms of planning, recording data and transcribing and analysing it. In this case, it was an output of another teacher education project. Using and evaluating such material in teacher education is an important activity for teacher educators involved in research.

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Sensitivity to Student Learning: A Possible Way to Enhance Teachers' and Students' Learning?

Ulla Runesson, Angelika Kullberg and Tuula Maunula

Introduction

It has been suggested that learning to teach implies to shift from a simple to a more complex understanding of the phenomenon (Wood 2000). To plan for and evaluate student learning is complex in terms of aspects that needs to be taken into consideration by the teacher. This chapter will highlight some, which we consider to be more significant than others. However, first we invite the reader to stop reading here and carefully considering the task in following:

> Imagine you are planning for teaching addition and subtraction with negative numbers like:
>
> $$-5 + (-3) =$$
> $$3 + (-5) =$$
> $$-5 - (-3) =$$
> $$5 - (-3) =$$
> $$3 - (-5) =$$
>
> Suppose the topic is new to the students. They have not got any formal teaching before.
>
> What would you take into consideration when planning and evaluating the lesson?
>
> What do you think is necessary for learning this?
>
> What could be critical?
>
> What will you do if they do not learn what you had intended to?

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Perhaps you would consider the activities, for instance how to make the lesson interactive, if the students should be working in pairs/groups or individually, what your role as a teacher should be, what tasks, devices or hands on material to use and so on. Perhaps you would also consider the time allocated for learning and practicing. Maybe you also would consider whether you should 'teach for understanding' or if it sufficient for the students to learn this by applying 'rules' and practice these on various examples.

All this we think is necessary for a teacher to consider and carefully plan for. It is true that the character of the activity in the lesson is important just as the task and learning devices the students encounter and work with and the amount of time allocated. Although we consider these as necessary, we do not think they are sufficient. To all these, we want to add some more things that we believe are significant also. We would suggest that *how the students understand and experience that which is taught and learned* must, in our view, be one of the most important aspects that the teacher must have in mind when planning and evaluating learning (cf., Ausubel 1968). However, when we talk about 'sensitivity to students' learning here we do not refer to sensitivity in general, but sensitivity to students' learning a particular topic, concept or skill. Trying to understand the learners' ways of understanding could be a demanding task for the teacher. It is not easily understood why students come up with answers like '$-18>-3$', for instance. How do they make sense of this? What rationales lie behind that way of reasoning? However, just being sensitive and understand how the learner understands is not sufficient. We think this must be related to the idea of what is critical for learning. I order to learn something, for instance how to calculate '$-5-(-3)=$', there are certain critical features that must be discerned by the learner. The awareness of what those aspects may be, we would advocate, is necessary for a teacher to take into consideration when planning for teaching and learning.

To be able to handle this complexity—the relation between what the learner learns and what is critical for learning—must be one of the core competencies of a teacher, we believe. One way to develop this could be made by a systematic inquiry into the teaching—learning process, preferably in a collaborative process among a group of teachers and teacher educators. In this chapter we will illustrate how such a joint collaboration could be done by reporting on a group of experienced mathematics teachers in a Swedish comprehension school working with a particular approach of 'plan-teach-review' process—Learning study—aiming at enhancing student learning. The particular study took place in a combination of a research and developmental project among teachers and educators from the university. The aim of a Learning study is forefront and most to enhance students' learning. However, Learning study includes the teachers' learning as well (Gustavsson 2008). Although this particular study involved experienced teachers, Learning study is appropriate for a prospective context also. Davies and Dunhill (2008), for instance, report about the implementation of this approach in a two year project in the initial teacher education programme in the UK. They conclude that this promoted a more complex understanding of teaching among the prospective teachers.

Learning Study: A Systematic Enquiry Approach into Students' Learning and Understanding

In a Learning study a group of teachers investigate the nature of learning a particular concept or skill and how to promote learning this. The capability the students are supposed to develop *and* how they conceptualise this is in focus; hence, Learning study is both content and learner oriented. The ultimate aim of Learning study is to enhance students' learning. However, in order to provide the best learning opportunities and to make learning possible, the teachers themselves must learn about the nature of the capability they want the students to develop. For instance, if the aim is that students should learn how to calculate '$-5-(-3)=$', and similar tasks with negative numbers, the teachers deeply investigate what it implies to be able to do this operation, what the learning difficulties for the students may be, and what features of this that the students must be aware of. So, Learning study is about learning on two levels: the teachers' and the students' learning.

In Learning study an iterative process of planning, observing and revising is used. This process is similar to Lesson study (Lewis 2002; Yoshida 1999). However, whereas the aim of Lesson study could be to implement curriculum or a particular teaching arrangement, Learning study is always focused on a particular 'object of learning', a capability that the students should develop, and therefore the organisation of the lesson, methods or other teaching arrangements is not the main issue of concern. Furthermore, to meet the learners' difficulties and to provide better learning opportunities, in Learning study a theoretical framework serves as a guiding principle in the process. This framework called variation theory (Marton and Booth 1997) is used both for structuring and designing the lessons and for understanding students' learning outcomes. Variation theory is mainly a framework for learning, but has been used as an analytical tool when studying teaching and designing for learning and has been demonstrated to be powerful and appropriate to better understand how teachers' actions affect what is made possible to learn (Lo et al. 2006; Marton and Tsui 2004). In variation theory *what* the learner learns and how this is perceived, understood or experienced is fundamental. Thus, being sensitive to the learners' ways of understanding becomes very important in Learning study. 'The learner's perspective' is central and paid attention to on two levels. The teachers analyse *what is made possible to learn* in the lesson (what the learners should encounter and what they actually encounter in the lesson) by carefully observing recorded lessons and *how the learners make sense of that which is learned* (in the lessons and on pre- and post-tests/interviews).

From a variation theory perspective, learning is seen as a process of differentiation, thus to be able to discern similarities and differences (Gibson and Gibson 1955). However, what is critical for learning, for instance to calculate '$-5-(-3)=$', can probably not be prescribed on a general level, or be derived from mathematical theory alone but must include the learner and what she/he brings into the learning situation in terms of previous experiences and how she/he understands what is be learned. The way the learner perceives, understands or experiences that which is learned is due to what extent the critical aspects are discerned by the learner. A student's failure or lack of understanding can be understood in the light of un-discerned aspects; for instance if the learner does not differentiate the double meaning of the 'minus' sign in the operation above. So, the *discernment* of critical aspects is essential for learning. From this theoretical point of departure, in Learning study, the teachers try to find out what the critical aspects are and how they should be brought out in the learning situation in a way that makes discernment possible.

A Learning Study About Negative Numbers

The Learning study reported here was included in a project combining school development and research. The authors of this chapter were all involved in the project, two as researchers and one as a teacher. One of the aims of the research project was to gain insights into the relation teaching and learning. This was done by studying the way the object of learning was handled in the classroom, thus what was made possible to learn and what students actually learned in the lesson. Although the researchers and the teachers had a common goal—to promote students' learning—they had different roles in the project. The teachers decided about the object of learning, thus what capability they wanted the students to develop, and they planned and revised the lessons mainly on their own decisions. The researcher's role was to support the teacher group with video recording the lessons, testing and, if needed, literature. She/he was also a discussant partner in the meetings, but since it was important that the teachers themselves had the ownership of the study (another aim of the research project was to investigate in what ways a group of teachers by the help of the iterative process and guided by variation theory can investigate and improve their teaching to maximize the learning of all students in the class) the researcher did not try to impose her/his ideas nor did she/he reject to ideas brought up by the teachers.

The Learning study reported here lasted about one semester. There were six meetings with the teachers (lasting about one and a half hours each) before and after the lessons and four lessons (about 60 min) were conducted. The four participating teachers were all experienced and well-educated mathematics teachers and took part on a voluntary basis in the project that involved two other Learning study groups at their school. Three of the teachers in the study reported here taught grade 7, the other one grade 8 (13–14 years old). All lessons were video recorded and the pre- and post-meetings with the teachers were audio recorded. The analysis presented in this chapter is based on those and on results from the pre- and post-test from all four participating classes. Each class had only one lesson in the Learning study cycle. It should be noted that the students had little or no experience of negative numbers before the intervention lesson. In Sweden usually this topic is taught in grade 8 as a teaching unit of about two weeks.

Although team-work among teachers is common in Swedish schools, co-operation about a particular topic is rare among mathematics teachers. Most teachers in

Sweden are dependant on the textbook. Furthermore, in order to cater for individual differences the students mostly work individually on their own pace with tasks in the textbooks. Whole class discussions about a common mathematical topic are not so frequent. So, this way of collectively going deeply into teaching and learning about addition and subtraction of negative numbers and to teach a common theme for a whole class was in to some extent a new experience for the teachers.

Anticipating Learning Difficulties and Planning of the First Lesson in the Cycle

A Learning study starts with deciding about the object of learning—a capability to be developed—mainly something the teachers find problematic to teach and students to learn. In this case the teachers had previously experienced that addition and subtraction with negative numbers was very hard for the students to learn. In the first two meetings previous teaching experiences and information from research findings was the topic discussed. The teachers demonstrated their sensitivity to students' learning by anticipating learning difficulties that students may encounter when calculating with negative numbers. Why is, for instance, the task '$-5-(-3)=$' so difficult? Is '$5-(-3)=$' an easier task? And what about '$-3-(-5)=$'? How do the students make sense of this? The teachers wanted to teach for understanding, not just to get the students to come up with the correct answer. With the background of considering such issues, the teachers designed a pre-test and gave it to the students to find out about the particular learning difficulties their students may have.

Testing students before a teaching unit, thus investigating the learners' pre-knowledge, very seldomly occurs in the Swedish mathematics classrooms. However, the teachers realised that a careful analysis of the pre-test results would give them valuable information if they tried to make sense of why the students had failed on certain tasks and how they had explained their ways of reasoning. This gave the teachers deeper insights into what could be problematic for the students and subsequently about possible ways to teach the topic. They found, for example, that many students could solve the tasks with problem solving skills or by using the rule 'two minus signs make plus' and without an understanding of addition and subtraction with negative numbers. This 'rule' was often of no meaning to the students and was used as a method of a procedure. They also realised that the different meanings of the operational sign for subtraction and the 'minus' sign for a negative number probably was confusing for the students (cf., Ball 1993; Vlassis 2004). In Sweden the signs look and sounds the same (minus) and could easily be interpreted as such[1]. The teachers considered possible solutions to this problem: different words for the number (e.g., 'negative three') and the operation (subtraction or minus) could be used, the two signs could be separated by putting the sign for the negative number

[1] $-5-(-3)=$ is read 'minus five minus minus three' equals. In Swedish: 'minus fem minus minus tre är lika med'.

'higher up' than the operational sign. They discussed whether concrete representations and metaphors, for example temperature (–3 as three degrees below zero) and debt, would be a limitation for a deeper understanding. These representations seemed to be of use for solving some of the tasks but not all. How could, for instance subtraction task '5−(–3)=', be represented in a good way? And how could the different tasks '5−3=' and '5−(–3)=' be told apart. The fact that this topic in mathematics is hard to represent in the different combinations (e.g., a+−b, –a+–b, –b+a, –b+–a, a−–b, –a−b, –a−–b, –b−a, –b−–a) made the teachers to consider an approach without concrete representation at all.

With the experience of the students' pre-knowledge the team jointly planned the first lesson (L1) that aimed for that the students would understand addition and subtraction with negative numbers. The teachers had an idea that the lesson should focus on opposite numbers, an idea they found in a mathematics text book. They wanted the students to see that an addition of the opposite numbers, e.g. +5 and −5 equals zero. "We have to show that the rules of operation have to be the same on the negative side as on the positive one", one of the teachers said. In order to draw the learner's attention to this, the team decided to teach negative numbers as 'patterns' (cf., Freudental 1983). Paradoxically, although the teachers were very much in to avoiding teaching about rules, the main idea for the first lesson was to use 'patterns' to make the students "invent" a rule or to "see the consequence of a rule". "At least with this, we will have shown how the rules work. At least shown the rule in one way", T1 said. "They [the students] will ask about this rule about minus and minus [becoming plus]", T2 added. "No, my soul, I will avoid that", T1 replied. To overcome learning difficulties they were very much into 'avoiding things'. They wanted to make the lesson as simple as possible without any examples that would mix up things for the students. For instance, although they were aware of that subtraction could be seen as a difference between two numbers, they thought that this would make it more complicated. "Subtraction means 'taking away'. It is stupid to call it 'difference'. Let us just say subtraction. They know what that is", T2 suggested.

The team was very aware that the two different meanings of the minus-sign might be confusing to the learners. They discussed how they systematically should vary the meanings by using examples like '8−3=5' and '3−8=–5' to draw the learners awareness to the different meanings of the sign for a positive or a negative number, but still keep the sign for the operation the same.

Implementing the First Lesson in the Cycle (L1)

The first lesson brought up the feature of opposite numbers as was planned. This was an introduction to find and discuss negative numbers in relation to the positive numbers and let the learners themselves discover the rule "adding (subtracting) a negative number is the same as subtracting (adding) its opposite" (Freudental 1983, p. 437). The teacher in this lesson wanted the students to find the answers to the following questions: "What happens when we add the opposite number to five with

Pattern A	Pattern B	Pattern C
5 + 5 = 10	5 + 5 = 10	5 + 5 = 10
5 + 4 = 9	5 + 4 = 9	5 + 4 = 9
5 + 3 = 8	5 + 3 = 8	5 + 3 = 8
5 + 2 = 7	5 + 2 = 7	5 + 2 = 7
5 + 1 = 6	5 + 1 = 6	5 + 1 = 6
5 + 0 = 5	5 + 0 = 5	5 + 0 = 5
5 + (–1) = 4	5 + (–1) = –4	5 + (–1) = 6
5 + (–2) = 3	5 + (–2) = –3	5 + (–2) = 7
5 + (–3) = 2	5 + (–3) = –2	5 + (–3) = 8
5 + (–4) = 1	5 + (–4) = –1	5 + (–4) = 9
5 + (–5) = 0	5 + (–5) = 0	5 + (–5) = 10

Fig. 1 Three contrasting patterns used in lesson 1 for the students to explore "What happens in addition when we go below zero?" (Note, pattern B and C are incorrect)

five, '5+(–5)'?" and "What happens if we go below zero, '5+(–1)'?" This was for the students to explore in pairs during the first lesson. The students came up with different possible solutions to the questions (see Fig. 1). The point with bringing up different solutions of the same task (pattern A, B, C) was done with the intention of contrasting different ways of understanding what happened with the operation when counting with negative numbers. In this way the learners' interpretations of the operations was elicited and discussed, thus they became visible. In this case pattern A (correct) and B (incorrect) came from the students, but pattern C (incorrect) was introduced *deliberately by the teacher*. To demonstrate an in-correct pattern to the learners could be considered as un-appropriate to do. Is it not a risk that the students will learn the 'wrong thing'? However, the teacher held the idea (in line with variation theory) that, in order to know what something *is*, you must know what it *is not*. To make the student aware of aspects concerning the nature of negative numbers he contrasted it with an example that demonstrated feature that did not belong to negative numbers.

In a second phase of the lesson the patterns of addition was connected to patterns in subtraction. The teachers wanted the student to find out that subtraction with a negative number could be substituted with addition of the opposite number. Furthermore, the different meaning of the 'minus-sign', for instance in '5–(–5)=' was elicited in the lesson. The teacher said:

> T: The minus sign does not stand for exactly the same thing. Uh…this [points] tells us, this minus sign tells us that it is an operation, for example this [pointing to the first minus sign] is the operation. For example this [points to + in another expression] is different …[they are] symbols for the operation.

The teacher pointed out one of the different meanings of the minus sign by comparing it to the '+' in addition, the operational meaning. The minus sign as indicating positive/negative numbers was elicited by contrasting the same digit (5, (–5)) and varying the sign.

Analysing the First and Planning the Second Lesson: Subtraction Has a Double Meaning

Analysing the video recorded lesson together with the results from the post-test made the teachers realise that this way of teaching was not successful in all respects. The progress concerning addition was better than subtraction. On subtraction tasks with two negative numbers, the students in class 1 (i.e., lesson 1) showed a *lower* result on the post-test (29% correct answers) than on the pre-test (35% correct answers). Why? The teachers together with the researcher carefully watched the recorded lesson from the point of view of in what way it was made possible for the students to learn about subtraction of negative numbers. So, by analysing the lesson in the light of the learning outcomes (on the post-test and in the lesson) they tried to make sense of how the way the topic was handled affected students' learning. This is of course not an easy task. Learning outcomes can not be explained by a simple cause–effect explanation. Therefore it is important to take the nature of the capability into consideration. What is necessary for the students to be aware of in order to learn? Are there aspects of subtracting negative numbers that has been taken for granted (when planning and in the lesson) that should not have been taken for granted? These were questions posed by the researcher in order to make them discern if aspects of adding/subtracting negative numbers that students ought to be aware of were made possible to experience in the lesson. They drew the conclusion that using 'patterns' (Fig. 1) did not give the students the opportunity to discern certain features of negative numbers and the operation that probably were necessary to discern for learning about subtraction. They realised that the way the topic was handled did not bring out the critical features of subtracting negative numbers. There must be some necessary conditions missing in the lesson.

From these insights, they decided to take quite another approach for the second lesson. Instead of using patterns to solve mathematical expressions, they planned to contrast addition to subtraction. However from observing the recorded lesson they also noticed another thing. One student brought out something which they had decided to avoid when planning the lesson. The student remarked that "minus could be seen as a difference, not just as take away". When planning the first lesson they decided not to mix up things to much and not to vary the 'meaning' of the operation sign. This idea was really challenged now, not by the researcher or by anyone of the teachers, but by one of the students! The statements from the student made the teachers aware of the advantage of seeing subtraction as a 'difference' *and* as 'take away' and that this 'double meaning' of the minus sign probably is a critical aspect of subtraction. Thus, when being sensitive to the students' interpretation the meaning of the minus-sign, they realised how their aspiration to help the learners by just using 'take away', might have had the opposite effect. Therefore they decided to teach '5−(−5)=' is the 'distance' between them [the numbers]. But one problem still remained; when using difference as a metaphor, the students would get both positive and negative differences due to which number was placed first in the expression. If the smallest number was first, the difference would be negative. "What is a negative difference? Is it wrong to discuss that during the lesson?" one of the teachers wondered. In order to make things less difficult for the students the team decided to just try to avoid examples with a negative difference in lesson 2.

Implementing the Second Lesson in the Cycle (L2)

From the recording of lesson 2 it is possible to see that the decision in the post-session meeting after lesson 1—to focus on subtraction as 'difference'—had a great influence on how the topic was handled in the lesson. For instance the teacher made the students aware that the meaning of 'minus' may be taken for granted, when the teacher said:

> T: Today I want you to think about the minus sign as something, something new, something different…for example a difference, a difference between two different things, between two numbers [writes 'difference' on the board]

The rest of the lesson was focused on 'the distance' between two numbers. A number line was used as a teaching aid. A pair of numbers was compared, for instance:

> T: If you think like this…the number five [writes 5] and the number two [writes 2 and a minus between them]. Then it must be…what is the answer then, Erica?
> E: Three
> T: Then you can think about the minus sign as the difference between five and two…it is three.

During the second lesson, the teacher in different ways drew the students' attention to experiencing subtraction as a 'difference' instead of as a 'take away' only. This was done by a comparison between numbers on the number line. However, this was also in accordance with the planning—in all the examples taken, the difference between the numbers were always a positive number (e.g., '5 − (−7) =' or '8 − (−3) =').

However, one aspect which was present in lesson 1 was missing in lesson 2; the minus sign as both a sign for operation and the number value (cf. lesson 1 above). This absence was not planned, so he may just have forgotten about it.

Analysing the Second and Planning the Third Lesson: 'A View Turn'

After having analysed the second lesson and the results from the post tests it became clear to the team that lesson 2 was not successful either. The post-test showed that the students performed better on tasks concerning subtraction, especially to subtract a negative number from a positive number. Certainly the results increased from 41% of correct answers on the pre-test to 76% on the post-test, but results from subtraction tasks with two negative numbers '(−5) − (−2) =', did not increase so much (from 41% to 65%). Since the lesson paid little attention to addition with negative numbers the increase in results concerning addition was small. So, having planned and conducted two lessons in the cycle without getting the outcome they had expected, they realised that there were still some critical aspects that remained to be discovered.

Neither using a pattern (as in lesson 1) nor comparing pairs of numbers with the help of a number line (lesson 2) seemed to give the students possibilities to learn what was intended. All aspects critical for learning this had not been brought out in these lessons. But still there were a few things in lesson 2 that they were satisfied with. The post-tests had shown better results for subtraction compared to lesson 1. The teachers considered what effect that 'negative difference' never appeared in the lesson may have had: "You never showed a negative difference between numbers. $5-1$ should have been followed by $1-5$," T3 said. "We had decided to try to avoid that. We felt that it would mess up things for them", T2, the teacher who had implemented the lesson, said. A long discussion about whether they should try to avoid or focus on expressions with a negative difference followed. Would it be confusing for the students or was it necessary to bring this up? Could the difference between '$5-4=$' 'and $4-5=$' be a critical aspect (i.e., the law of commutativity is not valid for subtraction)? "You could start with $4-5=$ and discuss what happens in that case. With '$5-4=$' it will be one left, but with '$4-5=$' it will be one missing. That difference is understood by every little child," the researcher said. "Yes, and still we ignored that since we felt that we could not explain that," T1 replied.

At this point there was a dramatic view turn. Instead of discussing in terms of *avoiding* negative differences, which they anticipated would be problematic for the learners, they now considered teaching this. Thus, they reconsidered their view of how to help the learners; from avoiding to confronting. To facilitate learning, however, it was suggested to connect the examples to an every day context which would be familiar to the learners. For example, they suggested to compare the age of two persons differing nine years in age, one will be the younger and the other the older. How you represent that (positive or negative difference) depends on whose perspective you take. For instance: John is 12 and David is 9 years old. Starting with the oldest ($12-9=3$) you say: "John is 3 years older". If you do it the opposite way ($9-12=-3$) you say "David is 3 years younger than John". The possibility of using different metaphors like 'longer/shorter', 'smaller/bigger' and so on were discussed. Subtraction as a difference was still meant to be in focus. They considered using another metaphor for negative numbers: debt, as a possible way to show subtraction as a difference between two numbers; for instance to give a scenario of two persons sharing their economies (addition) and comparing them (subtraction).

Implementing the Third Lesson in the Cycle (L3)

Due to an unexpected incident (lightening) happened the third lesson was interrupted. From the video recording of the lesson it could be noted that this incident draw the students' attention away from mathematics, and we have therefore chosen not to present details and tests from this lesson. However the video recorded lesson and the results on the post-test was analysed by the teachers and served as a ground for considerations about the last lesson in the cycle.

The lesson started with discussing the difference between '8 − 3 =' and '3 − 8 =' (just as planned). This was followed by an example comparing persons of different lengths and having different amount of money. The students were asked to consider who is the tallest/shortest and richest/poorest. By this, the teacher wanted to show that there is an 'in-built perspective' for instance in 87 − 85 and 85 − 87. In the first case you could say that 87 is bigger/taller/richer than 85. In the second 85 is smaller/shorter/poorer than 87. Due to the incident of the lightening, this was not followed up. Neither was subtraction as 'difference' brought out in the lesson. However, the different meaning of the minus sign (operation and value) was an aspect present in lesson 3.

Analysing the Third and Planning the Last Lesson: The Critical Aspects Emerge

In the post-session after lesson 3 the task of comparing tallness was much paid attention to. The teachers thought that they had found something critical for learning to subtract with negative numbers when they discussed that the difference in tallness is dependent of from which 'perspective' you are looking at it from. For instance they said: "If you think about the difference between 9 and 15 is 6, but whether it is −6 or 6 depends upon the perspective you take". They decided to make 'the perspective' in subtraction very clearly to the students in the next lesson. This should be done by contrasting the difference between addition and subtraction at the beginning of lesson 4 and to use the metaphor of debt in the context of shared and compared economies. In addition to this they explicitly pointed out that it was important to make sure that the critical aspects identified earlier in the process really would be present in the fourth lesson.

Implementing the Fourth Lesson in the Cycle (L4)

The last lesson in the cycle became a synthesis of all the three previous lessons and the conclusions drawn from the analysis from them. In this lesson the teacher tried to direct the students' attention to all the critical aspects that she knew about. During this lesson it was possible for the students to experience:

1. *the difference between the two signs for subtraction and for a negative number;*
2. that *subtraction could be both seen as 'a take away' and as 'a difference' between numbers*; and,

3. that it matters from which point of view you regard subtraction; *it is always from the perspective from the 'first position'*. For example, if, Liza has 5 crowns and Bill −10 crowns (5−(−10)=15), the difference between their economies is that Liza has 15 more than Bill (i.e., positive or negative difference).

These were the critical aspects the teacher group had identified so far. However, during lesson 4 something happened which illustrates the importance of being sensitive to students' learning and understanding. During lesson 4 the teacher asked the students to come up with two negative numbers that in a subtraction equals one (___−___=1). This task caused a lot a discussion among the students. Initially, the teacher did not understand what the problem was, but perhaps due to the fact that she was 'tuned to be sensitive' to the learners' ways of experiencing that which is learned, she successively understood what was problematic for the students—they were not sure about which was the biggest number −2 or −1. The teacher said:

> T: I know what your problem is, and it was stupid of me not having considered this before. We have to find out which of the two numbers (−1) and (−2) is the biggest number.

On the question "which is the biggest number −1 or −2?" the majority of the students incorrectly answered "–2". Why? Probably, those students believed that, starting from the point zero on the number line, the positive numbers 'get bigger the more to the right you get' and the negative numbers get bigger 'the more to the left from zero you get'. Experiencing the number system like this, it is reasonable to think that '−18 is a bigger number than −3'. One could assume that understanding how the number system is structured, thus realising that −18 is a smaller number compared to −3 and 18 for instance, is critical for understanding and calculating negative numbers. However, that the students could have the opposite and in-correct understanding was never anticipated by the teachers, neither when they planned or observed the three previous lessons. It was not until the students in the last lesson in the cycle gave expressions of being confused and the teacher really tried to understand what this confusion was about, she became aware of this important feature of learning to calculate negative numbers. It became clear to her that 'understanding opposite numbers' was a critical feature for learning to add and subtract negative numbers. This was probably a critical feature that had been taken-for-granted by the teachers when planning and revision the lessons and thus, had not been brought out in the previous lessons. So, besides the three aspects mentioned above, yet another one was possible to discern in the last lesson in the cycle, namely:

4. *The structure of the number system*

The post-test from lesson 4 showed the best results compared to the other classes for the learning outcomes. After this lesson there was an increase from 29% correct answers on the pre-test to 81% on the post-test on tasks concerning subtraction of negative numbers. The results on addition tasks with negative numbers are also showing the same increase. Our interpretation is that there were other and better possibilities to experience aspects critical for learning provided in this lesson compared to the previous ones.

Through the cycle the awareness of aspects critical for learning to add and subtract negative numbers grow among the teachers. The teacher who implemented the lesson plan in the last class (lesson 4 in the cycle) had, as a consequence of the in-depth analysis of teaching and student learning, a more developed understanding of what must be brought out in the lesson to provide better learning. She was more prepared to bring out the identified critical aspects in her class and managed to do so also.

Sensitivity and a Systematic Approach to Enhance Student Learning

To summarise, in a Learning study the aim is try to connect students' learning (how they conceptualise and understand what is learned) with what it takes to know something. There is a collaborative inquiry into what kind of architecture the specific capability has, what aspects of the object of learning are critical and are necessary to discern in order to learn. Once we think we know about these, we try to draw the students' awareness to them in the lesson and plan activities that will make them possible to discern for the students. Hence, the focus is therefore on the critical aspects and the activities and organisation of teaching are seen as means to make it possible to discuss and discern the aspects.

Our interpretation is that the process of analysing lessons collaboratively with the aim to improve students' learning in regard to a specific object of learning, what was made possible to learn and what was actually learned was, in this case, a fruitful and rewarding experience both for the teachers and the learners. The teachers deeply investigated how the learners understand and solve tasks like '$5-(-2)=$' or '$-3-(-4)=$' for instance, and what it takes to learn this. They explored and identified what was critical for the students' learning. You could say that the teachers learned about the students' learning and this learning made them able to refine and develop the lesson plan in terms of how to handle the content. Through out the process the students' learning, *what* they learned, what particular combinations of addition and subtraction with negative numbers they failed or succeeded with, was the main concern. In that sense, the assessment of students' learning outcomes was qualitative and formative, thus used for refining the lesson. Students' failure was never discussed in terms of attributes or shortcomings among the learners which, in our experience, is common, but rather to deficiencies in teaching. However, being sensitive to students' learning in mathematics, for these teachers, did not just refer to how they interacted with their students and whether they cared for them or not. Sensitivity implied to learn more about *why* students may have learning difficulties, how they perceive and conceptualise that which is learned and how that was related to what was made possible to learn in the classroom, thus their teaching.

The teachers had a true ambition to help the learners to understand, not just to rely on 'tricks' and rules that were meaningless to them; unfortunately a common way to teach negative numbers in Sweden. It is easy to be wise afterwards and classify the first lessons as being poor, and since they were not so successful, considering the teachers as unskilled. Remember however, that going so deeply into how to teach and learn a topic was not just a new experience for them, it was also a challenge. Usually they rely on the text book. Here they were confronted with their own ways of teaching and what effect that had on students' learning. They had to consider their knowledge of the subject matter, their students' understandings and learning as well as of their own teaching skills. The sensitivity to the students' understanding before, in and after the lesson gave information about the way of handling the content and how that might provide possibilities for learning.

However, it is worth noticing that the teachers' ambition to facilitate learning, at some occasions, had the opposite effect. Avoiding negative difference and not bringing out the double meaning of subtraction, for instance, seemed to have been counter-productive for providing learning possibilities. The teachers successively came to realising that there must be certain conditions met in the lesson in order for their students to learn to add and subtract negative numbers. Certain aspects were critical for these students' possibility to learn. Such critical aspects identified by the teachers were the *'minus-sign' as operational sign for subtraction compared to the 'minus' sign for a negative number, different semantic meaning of 'minus'* (e.g., subtraction as 'take away' or 'difference'), that *$a-b$ does not equal $b-a$,* thus the law of commutability is not valid in subtraction, and understanding *the order of negative and positive numbers* (e.g. $-3>-18$) (Maunula, 2006). We have doubts about whether the teachers would have come to these conclusions on their own and without this systematic and cyclic approach of investigating teaching and learning.

Learning Study: A Possibility for Learning to Teach and for Professional Development?

Learning study is a labour-intensive and time consuming activity, is it really possible to do this on regular basis concerning all the other obligations teachers have? We think this is a powerful tool for teachers' professional development and an appropriate form of a learning community in teacher education as well.

Considering the demands mathematics teachers face from politicians and the public to improve students' mathematical learning (e.g., as reactions to international comparisons of mathematical performance in TIMSS and PISA) they must be given time and other opportunities to deepen and develop knowledge about how their activities shape students' learning and if that which is intended to be learned is made possible to learn from their teaching. This is valid for prospective teacher education as well. Learning study as an approach in teacher education, Davies and Dunhill (2008) points to, requires "substantial organization and effort, particularly in setting up lessons, working with mentors" (p. 15), However, they argue, the prospective teachers' experience of Learning study changed their understanding of teaching and this shaped their teaching.

"Teachers often feel that learning outcomes are un-predictable, mysterious and uncontrollable," Nuthall argues (2004, p. 276). We can probably not predict learning, but at least (among other things) be able to analyse and design learning situations from the point of view of what is necessary for learning and how to make this learning possible. To improve knowledge about that is a life long professional learning process. In Learning study teachers with different teaching experiences can learn with and from each other by observing themselves and other teachers teaching *the same topic*.

In Learning study we value teachers' experience and contributions. It is believed that, given opportunities, teachers could develop expertise and valuable knowledge. Such knowledge is not just a private experience but could be disseminated and communicated to other teachers in networks as examples of good practice. A Learning study is a way for teachers to conceptualise and talk about teaching and learning. We assume it can help teachers to develop a research stance and an investigative approach to their teaching (Lo et al. 2006), something Davies and Dunhill's (2008) study indicated. They found that a number of teacher students began to "recognise explicitly and relish the research element of the demand that was being placed upon them" (p. 15). Learning study gives a possibility to continue to learn and improve, we believe. In this process, students' learning, teacher learning and what is learned are interconnected.

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Overcoming Pedagogical Barriers Associated with Exploratory Tasks in a College Geometry Course

Norma Presmeg

Focus and Significance

Because the objects of mathematics cannot be apprehended directly by the senses, the role of mediating signs (Peirce 1998) is crucial in all mathematical activity, including its thinking and learning. However, it has been documented (Presmeg 1997, 2006) that compartmentalized thinking may prevent learners of mathematics from making the connections amongst mathematical signs that will facilitate their learning. In particular, when students have satisfied themselves by empirical means—for example, using measurement, or by dragging points in a dynamic geometry environment—that a geometric drawing satisfies some geometrical principle, then their sense of certainty and closure may constitute a barrier to understanding the deep structure of the mathematical principle or principles that underlie the construction.

The environments of geometric constructions and geometric proof may be viewed as different registers, to use Duval's (1999) terminology. In overcoming the barriers of compartmentalized thinking, educators of prospective high school teachers need to model and facilitate ways that students may connect these registers and build the integrated knowledge of deep structure that enables learners to understand *why* constructions work, and how it may be proved that they are rigorous. This chapter illustrates one such task, taken from a geometry content course for prospective high school teachers.

Theoretical Perspectives

Theoretical lenses that are useful in this context are Peirce's (1998) triadic semiotic system, and Duval's (1999) theory of conversions within and amongst mathematical registers. In Peirce's system, a sign consists of three components, namely, an

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object, a representamen (which I prefer to call a sign vehicle—Presmeg 2006) that stands for the object in some way, and an interpretant, which is the result of interpreting the relationship between the sign vehicle and its object. Peirce (1998) also introduced the notion of *abduction*, as one of three modes of logical reasoning. He described abduction as follows.

> Abduction is the process of forming an explanatory hypothesis. It is the only logical operation which introduces any new idea; for induction does nothing but determine a value and deduction merely evolves the necessary consequences of a pure hypothesis. Deduction proves that something *must* be, Induction shows that something *actually is* operative, Abduction merely suggests that something *may* be. Its only justification is that from its suggestion deduction can draw a prediction which can be tested by induction and that, if we are ever to learn anything or to understand phenomena at all, it must be by abduction that this is to be brought about. (p. 216)
>
> An *Abduction* is a method of forming a general prediction without any positive assurance that it will succeed either in the special case or usually, its justification being that it is the only possible hope of regulating our future conduct rationally, and that Induction from past experience gives us strong encouragement to hope that it will be successful in the future. (p. 299)

The purpose of using these theoretical perspectives is to attain a finer grain of analysis of the processes involved, by taking into account ways that students interpret mathematical relationships. Duval's theory is significant because it provides a construct—register—that is broader in its connotations than the terms representation or sign vehicle. As a mode of representation, examples of registers used in mathematics are diagrams, algebraic symbols, and graphs. Dynamic geometry modes and classical Euclidean constructions may be viewed as different registers. However, both of these registers may be contrasted with the register of rigorous deductive reasoning involved in proof of theorems in classical Euclidean geometry. It is the connections amongst these registers that are the focus of this chapter. Specifically, the goal of this chapter is to illustrate some of the abductions and logical thought processes required to unpack why a particular geometric construction is successful, and to address implications of the fact that none of the prospective high school mathematics teachers in the college geometry course concerned, felt the need to undertake this unpacking.

Tasks from a College Geometry Textbook

A college geometry course, *Euclidean and non-Euclidean Geometry*, taught by the author in fall of 2007, had the following description:

> This course is designed to provide geometry content background for students preparing to teach mathematics at the middle school or high school levels. The primary purpose of the course is to involve the participants in thinking about, working on, and communicating about Euclidean and non-Euclidean geometries. Although this is not a course about the teaching of geometry, we may occasionally discuss related topics. More specifically, the course will involve activities and discussions in each of the following general categories:

- Inductive, deductive, and abductive reasoning (in all areas listed below)
- Purposes and methods of proving in geometry
- Euclidean geometry
- Classical geometry
- Transformations
- Use of dynamic geometry software
- Non-Euclidean geometries
- Three-dimensional geometry and spatial reasoning
- Miscellaneous topics in geometry.

The four-semester-hour (i.e., four hours per week for fifteen weeks) course is typically taken by undergraduates in the second or third year of a four-year mathematics teacher preparation program, after they have taken an introductory two-semester-hour Methods course in their first year. The final year is fully taken up with two further Methods courses, clinical experiences, and student teaching.

The specific aspect of the course that is the focus of this chapter is the barrier created to student learning when students have a ready-made protocol or formula for carrying out a task, which eliminates their need for understanding the deep structure that would explain *why* the protocol gives a solution to the task. This phenomenon, which may occur in either a dynamic geometry environment using computers or when students are doing geometric constructions using Euclidean tools, may inhibit the need for proof in geometry. The following task provides an example of this effect.

Several weeks into the course, the undergraduates had revised various basic Euclidean constructions using compasses and straightedge. The following assignment was given from the course textbook (Reynolds and Fenton 2006, p. 23).

30 a. Find a construction to inscribe an equilateral triangle in a circle. Do the same for a square and for a regular hexagon.

b. Here is a construction to inscribe a regular pentagon in a circle: Construct a diameter AB of the circle. At the centre, C, construct a perpendicular line and let D be one of the line's intersections with the circle. Let E be the midpoint of CD. Bisect angle AEC, and let F be the intersection of this bisector with the diameter AB. Construct a line *l* through F that is perpendicular to AB. The points where *l* intersects the circle, together with A, begin the pentagon. Carry out this construction and finish the pentagon.

Because the use of Euclidean tools consisting of compasses and unmarked straightedge is equivalent to using the dynamic geometry software of *Geometer's Sketchpad* (Jackiw 1991), using only "Circle by Centre+Point" and "Line" from the Construct menu, I illustrate the constructions involving the equilateral triangle and the regular pentagon in this GSP environment (Figs. 1a, b and 3, respectively). The constructions to inscribe the square and the regular hexagon in a circle are straightforward and are not addressed further, because few students had difficulty connecting these constructions with the register of proof (Duval 1999), linked with the registers of constructions using computers and Euclidean tools in which they were working.

In contrast to the construction for the regular pentagon—which was presented in the textbook in cookbook fashion—the construction for the equilateral triangle was presented in the textbook as an investigation for undergraduates. The circle is the given starting point of these constructions, and the measure of the radius of this circle defines the side of the inscribed equilateral triangle or regular pentagon.

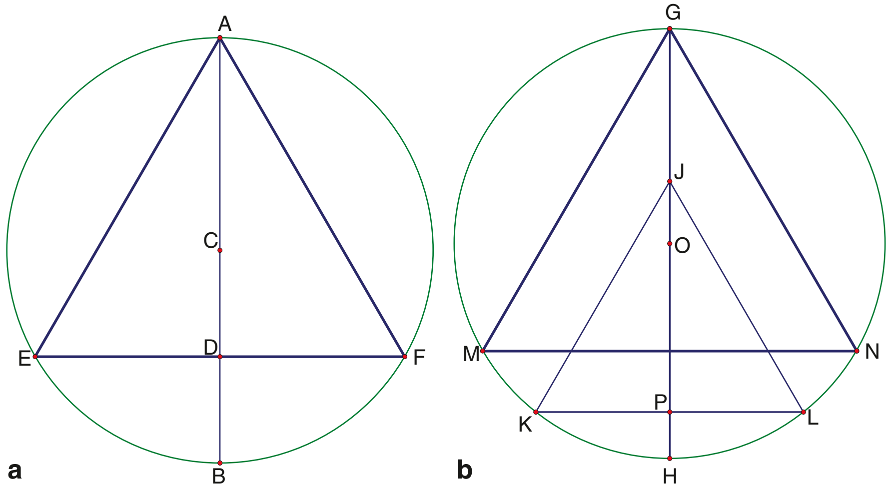

Fig. 1 **a**, **b** An equilateral triangle inscribed in a circle

Before proceeding, the reader might want to investigate how the construction for the equilateral triangle may be done, and why it works.

A key to one solution to this problem (Fig. 1a) is the theorem that states that the point of concurrency of the medians (the centroid) of any triangle divides each median in the ratio 2:1. Students had proved this theorem previously, and those who remembered this theorem could deduce that starting with a diameter AB of circle centre C, and bisecting the radius CB in the diagram, would provide the required ratio. Then drawing chord EF perpendicular to AB at D would provide one side of the required triangle. Joining E and F to A would then complete the equilateral triangle, using the property of symmetry. Because of the properties of the centroid in an equilateral triangle (that it is also the circumcentre, the orthocentre, and the incentre), this construction provides no barrier to the proof that triangle AEF is in fact equilateral.

In constructing an equilateral triangle in a circle, a solution that is accessible to students who do not recall the centroid properties is illustrated in Fig. 1b. Any chord (such as KL in the diagram) has a diameter of the circle as its perpendicular bisector (GH in the diagram). An equilateral triangle may be constructed on side KL. Triangle JKL is not the required inscribed triangle. However, this construction provides the correct angle, and by drawing parallels to the sides of triangle JKL through G, the positions of M and N are determined. Then GMN is the required inscribed equilateral triangle. Again, it is the properties of circles and equilateral triangles that provide entry to the construction through reasoning. The same principles apply if a radius is used to construct a first equilateral triangle with one vertex at the centre of the circle, and then parallel lines are constructed to obtain the required triangle using the method of Fig. 1b.

A memory image of the diagram for the proof that the medians of a triangle are concurrent (Fig. 2) could be the prompt for the construction in Fig. 1a. In Fig. 2, BE

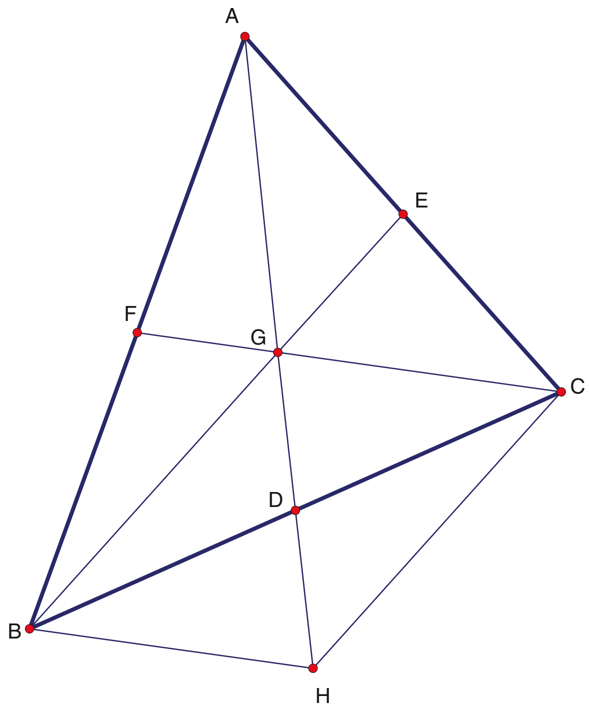

Fig. 2 Diagram for a proof that the medians of a triangle are concurrent

and CF are given as medians of triangle ABC, and G is their point of intersection. By joining AG and extending it its own length to H, the midpoint theorem may be invoked to prove that FG and BH are parallel. Similarly GE and HC are parallel. Thus GBHC is a parallelogram, and its diagonals bisect each other. Thus AD is a median (proving that the medians are concurrent at G), and it also follows that the ratio AG:GD = 2:1.

Because the inscribed equilateral triangle construction (as presented in the textbook) is left to the undergraduates to complete as an investigation, their solutions and the underlying principles and properties of the figures become explicit topics in the ensuing whole-class discussions of the task. Several ways of proceeding are possible[1], and abductions of students (e.g., that starting with any chord and drawing an equilateral triangle will facilitate a solution after further reasoning) are encouraged and pursued in small groups and in whole-class deliberations.

The construction of the inscribed regular pentagon, by way of contrast, is provided in the text as a protocol to be followed, rather than an investigation. This protocol enables undergraduates to construct the required pentagon in the circle, and to verify by measurement that it is in fact a regular pentagon. But *why* is this

[1] John Mason reported that he completed the construction by inscribing a regular hexagon in the circle, using the radius as a side of the hexagon, and then joining alternate vertices of the hexagon to form the inscribed equilateral triangle.

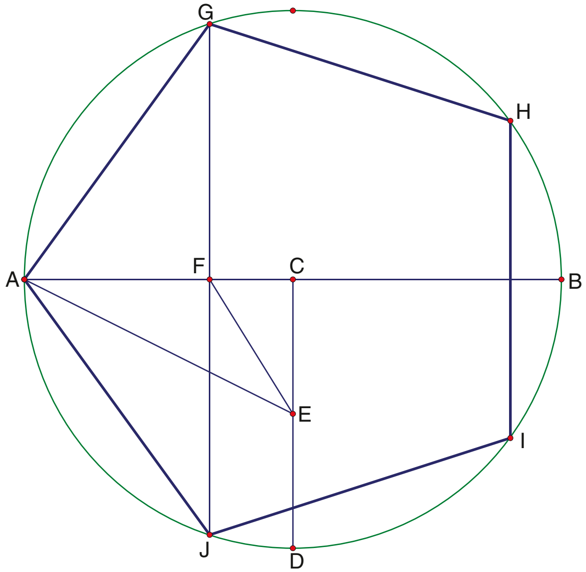

Fig. 3 A regular pentagon inscribed in a circle

so? The sense of closure in this method and the empirical checking of its accuracy give no hints about why the construction works or which properties of the figures involved would be useful in proving that the pentagon obtained is regular. More than that, many undergraduates are convinced by their empirical measurements that the construction generates a regular pentagon in a circle, and for them the need for a formal deductive proof may disappear in this process. Not only is the "cookbook recipe" then a barrier to further learning, but an opportunity to investigate some very interesting properties of the Golden Ratio is thereby lost.

The reader might want to try to figure out why this construction works (see Fig. 3) before proceeding further.

Preliminary Pedagogical Considerations

Although all of the 38 undergraduates in the two sections of the *Euclidean and Non-Euclidean Geometry* course completed the inscribed pentagon task using DGS in the computer laboratory, not one of them raised the question of *why* the construction works. There was no concern at all regarding the need to prove, or at least understand the structure and reasons, why the textbook "recipe" produced the required inscribed regular pentagon. As the instructor of the course, walking around and helping the undergraduates as they worked on individual computers, I felt disturbed on two counts. There was no doubt (and measurement confirmed) that the resulting figure was the required pentagon. But, firstly, I myself could not immediately decipher the underlying structures that revealed the properties behind the construction.

Secondly, it was disconcerting that none of the undergraduates expressed a need to find out why. An overarching theme in the course (and also in the two Methods courses that followed in the final year of the program) was that we should accept no rules without reasons. This pedagogical principle was intended to model teaching that we hoped the prospective teachers would take into their own professional careers in high schools, reflecting a view of the nature of mathematics as logically integrated and structured, rather than a set of techniques to be memorized. As the teacher of the course, I could have asked the undergraduates to investigate why the construction worked. But I did not know the answer to this question myself, and the process of finding out turned out to be complex and time-consuming, although finally the revealed structure was immensely rewarding and pleasurable. The effort required in this process raises the question of the degree to which it is possible for mathematics teachers to adhere consistently to the principle of always requiring reasons for rules used. With the pressures and constraints of high school classrooms, a measure of balance is probably the best compromise, and the culture of the classroom is a relevant factor, although this aspect is beyond the scope of the present chapter.

The following is an account of the logical and abductive processes that I went through in the effort to find out why the construction for the inscribed regular pentagon is successful. The principles involved could form the basis for a formal proof.

Abductive Processes

In trying to decipher the inner structure of the regular pentagon construction, one may recall that the angles subtended by the sides of a regular pentagon at the centre of the circle are one fifth of a revolution, i.e., 72°, which is reminiscent of the base angles of the Golden Triangle (Fig. 4). Abduction suggests that this triangle may

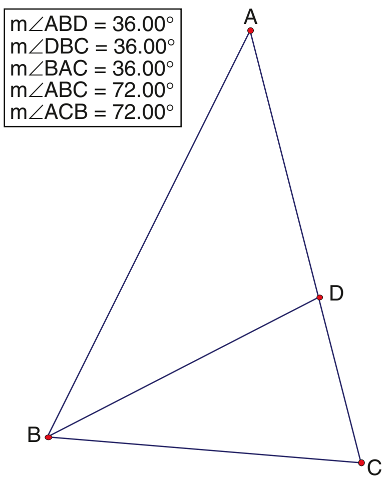

Fig. 4 The Golden Triangle

be relevant. In Fig. 4, because BD is the bisector of angle ABC, triangles ABC and BCD are similar. Then if side BC has length 1, it follows from the angle measures that BD and AD must also have length 1. Now if the length of AB is x, then the length of AC is also x, and thus the length of DC is (x − 1). Then in the two similar triangles,

$$x/1 = 1/(x - 1)\,.$$

Solving the resulting quadratic equation gives x = (1 + square root of 5)/2, the Golden Ratio.

Returning to the pentagon construction (Fig. 3), some further abductions are needed. Right triangle ACE in the pentagon construction looks promising, because if the radius of the circle with centre C is taken to be 2, then the two legs of this triangle, AC and CE, are 2 and 1 respectively, and the hypotenuse AE must then be the square root of 5 (by the Pythagorean theorem). Tantalizing as these numerical ratios are, they are still not enough to unlock the inner structure of why the construction produces a regular pentagon. Something more is needed.

There is one more aspect of the construction that has not been considered yet, and that is the angle bisector, EF. We might recall that the angle bisector theorem[2] states that the bisector of an angle of a triangle divides the opposite side of the triangle in the ratio of the two sides of the bisected angle (a theorem that the students had already proved). So in triangle ACE, the ratio AF:FC = AE:EC = sq.rt.5:1.

One could consider right angled triangle GFC in its own right (Fig. 5).

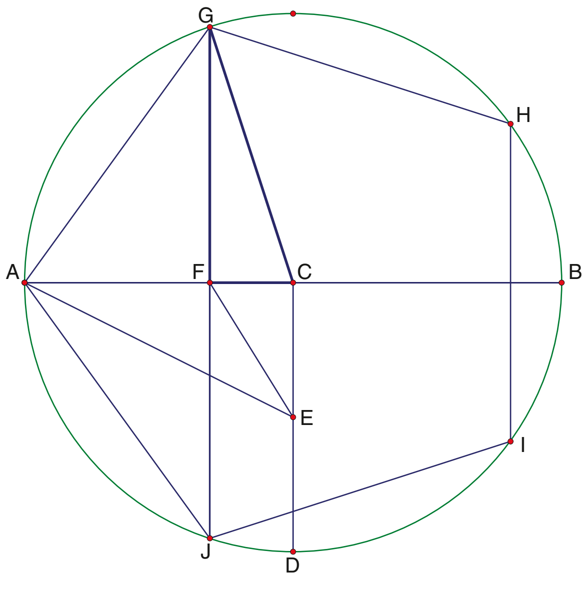

Fig. 5 A key triangle in justifying the construction

[2] I am indebted to John Mason for the insight that the angle bisector theorem provides a missing link in the logical processes involved.

GC is a radius of the circle. If FC is taken to be 1, then by the foregoing argument, the radius GC = AC = 1 + sq.rt.5, which is double the Golden Ratio. Can we link these relationships with the ratios in the Golden Triangle? Yes! The ratio GC:FC is then 1 + sq.rt.5, which is exactly what would be obtained if we dropped a perpendicular to the base of the Golden Triangle and calculated the corresponding ratio. We know that the base angles of the Golden Triangle measure 72°, thus the measure of angle GCF must be 72°. Alternatively, the secant of this angle is 1 + sq.rt.5, which confirms that the angle measure of angle GCF is 72°. Thus each of the congruent sides of pentagon AGHIJ subtends an angle of measure 72° at the centre of the circle, and this fact tells us why the construction produces the required inscribed regular pentagon. The abduction that the relationships in the Golden Triangle might be useful in the pentagon construction, turned out to be a powerful one, but deductive reasoning based on the angle bisector theorem gave the key to the inner structure, as explored further in the next section.

Hindsight and Pedagogical Implications

The earlier abduction that the justification of the construction involved the ratio of the sides of right triangle ACE, in which the legs AC and CE were in the ratio 2:1 by construction, seemed an obvious one. Constructing the bisector of angle AEC had to be involved in some way, but the implications of this part of the construction were not immediately apparent. Sometimes talking with a knowledgeable friend, who views the problem through new lenses, may provide a breakthrough. In sharing my puzzlement, it was the insight of John Mason that the angle bisector theorem could be brought to bear on the situation, which resolved the impasse. This point speaks in favor of collaborative pedagogy in which students use small-group and whole-class discussions to provide such insights.

Once the ratio AF:FC was determined using the angle bisector theorem, the way was opened for the further abduction that right triangle GFC could be thought of as "half" of the Golden Triangle. Knowing the properties of the Golden Ratio served as a prompt for this abduction. All abductions had to be confirmed by deductive reasoning, and all fell into place.

The insights that enabled me to see the inner structure of this construction came after the course was over, too late to share them with the two sections of my class, or rather, to suggest activities that would facilitate undergraduates being able to prove for themselves that the construction produced a regular inscribed pentagon. These prospective teachers were content without a proof: they had verified empirically, in the register of the dynamic geometry environment, that the pentagon was regular, and that it was inscribed. In hindsight, I would not follow the textbook in providing a ready-made construction, which created barriers to deep learning. The following are tasks that could be used as forerunners to this construction.

1. Investigate the ratio of the sides of a right triangle in which one leg is double the length of the other leg. (The Pythagorean theorem is involved.) If the angle formed by the smaller leg and the hypotenuse is bisected, in what ratio does this bisector cut the larger leg of the triangle? (The angle bisector theorem is useful here.)
2. Investigate the properties of the Golden Triangle. What is the measure of the base angles of the Golden Triangle? How is the Golden Ratio implicated? If a regular pentagon were inscribed in a circle, what is the measure of the angle that the sides of the pentagon would subtend at the centre of the circle?
3. Suppose that you have inscribed a regular pentagon in a circle. Investigate how the Golden Ratio could be applied to a radius of this circle.

There would still be abductions to be made before such tasks could be connected to the formal construction. However, such tasks could provide the basis for such abductions, just as the theorem concerning the centroid dividing medians in the ratio 2:1 provided the basis for a possible construction in the case of the inscribed equilateral triangle.

Connecting the registers of geometric constructions with that of formal proof involving deductive reasoning can be exciting and challenging (Mason 1989; Mariotti 2002). Researchers such as Mason and Mariotti are aware of the difficulties regarding generality that may accrue when students are given the tools of dynamic geometry software. What is gained through the use of computer constructions in which shapes may be varied by dragging points, is the ability to vary the sign vehicles—the geometric inscriptions—quickly and efficiently, and to verify measures of angles and lengths of segments using the tools provided. Some students may in this process see "the general in or through the particular" (Mason 1989, p. 45). All geometric sign vehicles are by their nature particular, but they refer to a general object, which cannot be seen, but must be inferred through an interpretation process, or interpretant in Peirce's (1998) terminology. Dragging points in a dynamic geometry register may facilitate the construction of such a mental interpretant. But these advantages also point to a potential loss: the very sense of certainty given by these measures may obviate students' need to know *why* the procedures are successful.

The issues involved concerning geometric construction tasks described in this chapter suggest that an investigative approach is potentially fruitful. Investigative tasks provide one way of overcoming the barrier to understanding of deep structure that may be occasioned by giving students procedures or protocols for geometric constructions.

Although my class was over by the time I had finally worked out a satisfying justification for the pentagon construction, there are a number of further issues that could be investigated in future research. Some of these were suggested by Barbara Jaworski, whose co-authored chapter in this book (Jaworski, Goodchild, Eriksen, and Daland) made use of a structure of levels of learning by researchers (or "didacticians" as they are called in their chapter), by teachers or prospective teachers (such as my undergraduate students), and potentially by students in mathematics classes in schools (such as the learners in the classes that they will teach in the future). Their threefold learning structure is reminiscent of the "multi-tiered teaching experiments" described as a research methodology by Lesh and Kelly (2000), in which researchers, teachers, and their students, are all learning, thus constituting three tiers of investigation going on simultaneously. It would be useful to conduct future research to investigate the following questions, amongst others.

- What kinds of investigations are effective in encouraging students to seek justification for constructions in registers of dynamic geometry software and classical Euclidean tools?
- What are the relationships amongst abductive, inductive, and deductive reasoning in investigating geometric constructions?
- How do the learning processes of prospective mathematics teachers in investigating geometric constructions relate to those they may encounter in classroom situations when they themselves are teachers?

One final thought is captured in a citation from Mason (1989), whose words are eminently quotable:

> Geometrical activity is one excellent way of gaining access to that world [the world of mathematical objects], through the power to form mental images, through seeing through diagrams to the world of generality which can be read in them. It is one way to encounter the discipline of mathematics, where convincing people *why something must be a fact* is as important as finding out what the *fact* is. (p. 44, emphasis in the original)

Investigations that encourage students to generate abductions and explore the consequences of pursuing them, *may* be productive pedagogical means (as future research may tell) of helping them find out why something must be a fact, and subsequently convincing others—thus providing the basis for mathematical proof.

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Using a Model for Planning and Teaching Lessons as Part of Mathematics Teacher Education

Peter Sullivan, Robyn Jorgensen and Judith Mousley

Introduction

The task for teachers that forms the basis of this chapter asks them to use a specific model for planning and teaching mathematics lessons. The model is based on a particular approach to choosing classroom tasks for students, and implementing pedagogies that are appropriate for the type of classroom task.

One of our assumptions is that the use of appropriate mathematics classroom tasks with associated interactions and activities is a key to successful teaching and learning of mathematics. It is through and around classroom tasks that teachers and students work towards the development of understandings, skills and knowledge. The teacher education task that is the focus of this chapter is educating prospective and practicing teachers to use a specific model for planning and teaching mathematics lessons to create productive experiences that can be successful with the diversity of learners and learning that exists in most classrooms.

There are some other assumptions that underpin our teacher education task. These are:

- the process of creating effective lessons from interesting tasks is far from trivial even though many commentators do not acknowledge this;
- the creation of lessons, both hypothetically and actually, offers an ideal milieu in which theoretical perspectives can be juxtaposed with practical considerations;
- given that the prospective teachers have had plenty of experience with the concept of a lesson throughout their school careers, the unit of the lesson offers a comfortable base from which some pre-existing conceptions of learning and teaching can be challenged; and,
- offering teachers a model for planning and teaching helps them to cope with the complexity of lessons and classrooms and offers them a language that can facilitate both collaborative planning and reflective review.

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Table 1 Percentage of students from particular socio-economic backgrounds in highest and lowest levels of PISA mathematical literacy achievement

	Percent at highest level	Percent not achieving level 2
Low SES quartile	6	22
High SES quartile	29	5

We note that while the focus of the following discussion in on prospective teachers, the same process can provide a suitable stimulus for the learning of practicing teachers, and we have used this focus in this way.

Another important focus of our work with prospective teachers is to allow consideration of ways that mathematics can create major barriers to learning for disadvantaged students. One of the current challenges confronting Australian educators is differences in achievement among particular groups of students. For example, the following data are extracted from the report on the PISA 2006 results (Thompson and De Bortoli 2007) relating to mathematical literacy of 15 year old students. They compared the responses to the PISA questions commonly discussed equity groups. Table 1 compares the achievement of students based on the socio-economic background of their parents.

A first implication is that it is difficult to teach such a diverse range of students within the one class. Those not achieving level 2 are responding at a very low level, yet those achieving at the highest level are progressing at the best international levels. Any model for planning and teaching mathematics needs to support teachers in strategies for teaching such diverse classes.

A second implication is that SES background seems very much related to achievement at school, which is contrary to a fundamental ethos of Australian education, that of creating opportunities for all students. The planning and teaching model also specifically seeks to address the needs of these particular groups.

We consider that it is imperative that graduating teachers feel able to address underlying causes of inequalities, partly so that they can actually deal with those difficulties when they are teachers, and partly so that they will feel able to apply for and succeed in schools with high proportions of such students.

Our fundamental approach is to support prospective teachers in designing effective classroom experiences. It is difficult, though, to identify unequivocal research results that can assist teachers in creating effective lessons in the everyday complexity and multidimensionality of mathematics classrooms. We acknowledge, for example, the importance of factors such as classroom resources, classroom organisation, climate, interpersonal interactions and relationships, social and cultural contexts, student motivation and sense of their futures, family expectations, and organisation of schools. Nevertheless, we argue that the key components of understanding teaching and improving mathematics learning are identification of the types of tasks that prompt engagement, thinking, and the making of cognitive connections, as well as the associated teacher actions that support the use of such tasks, especially addressing the needs of individual learners.

The following first summarises the key aspects of the research that informed the development of the planning and teaching model, then illustrates the key elements of the model. This is followed by some specific examples of lessons observed as part of our research. The final section outlines the task that we set prospective teachers: that of implementing the model to plan and teach mathematics lessons that use particular types of classroom tasks and which incorporate the elements that are implied by the model.

The Research that Informed the Planning and Teaching Model

The planning and teaching model described below was a product of research that sought to address tensions in complex learning environments. The research was conducted in classrooms that typically had significant numbers of students who were from disadvantaged backgrounds. These were usually students who were from Indigenous backgrounds, working-class families, rural families or some combinations of these groups.

Initially, our research involved teachers and others in helping to identify and describe aspects of classroom teaching that may act as barriers to mathematics learning for some students, particularly when open-ended tasks were used by mathematics teachers. We also used a range of focus groups to suggest strategies for overcoming such barriers (see Sullivan et al. 2002). This provided a strong framing for our research, particularly as we were exploring the ways in which aspects of the environment and pedagogy mediated learning (Zevenbergen et al. 2004). Next, we created some partially scripted experiences to be taught by participating teachers and analysed by us (see Sullivan et al. 2004). This analysis allowed reconsideration of the emphasis and priority of respective teaching elements. We found that it was possible to create sets of experiences that could be taught as intended by teachers, and that many of these experiences had the effect of including most students in rich, challenging mathematical learning. Arising from this work, we developed a model comprising five key elements of planning and teaching mathematics. Our later research focused on teachers' use of this model in the planning, teaching and reflective evaluation of their own lessons, examples of which are described below.

The Elements of the Planning and Teaching Model

There are five key elements in the model for planning and teaching mathematics that are described in the following section.

The Classroom Tasks and Their Sequence

Of course the first step in designing a lesson is to choose a classroom task(s) that should result in the desired learning opportunities for the students. Within a socio-cultural perspective, the role of mediating tools is central in the learning process—and classroom tasks are a key tool in any mathematics classroom. For us, it is through and around classroom tasks that teachers and students communicate and learn mathematical ideas. The nature of tasks shapes forms of communication, so the classroom tasks used by the teachers become key mediating tools. There is good support in the literature for this point of view. Christiansen and Walther (1986), drawing on the work of Leont'ev (1978), argued that the classroom tasks set and the associated activity form the basis of the interaction between teaching and learning. Similarly, Brousseau (1997) proposed that, "the teacher must imagine and present to the students situations within which they can live and within which the knowledge will appear as the optimal and discoverable solution to the problems posed" (p. 22). Hiebert and Wearne (1997) also suggested that, "instructional tasks and classroom discourse moderate the relationship between teaching and learning" (p. 420). In other words, the classroom tasks used by the teachers to engage students in classroom activity and interactions become the key means for facilitating learning about specific mathematical concepts and skills.

We focus in particular on open-ended tasks because of their particular potential to contribute to mathematics learning. Stein and Lane (1996), for example, noted that student performance gains were greater with relatively open-ended tasks, when "tasks were both set up and implemented to encourage use of multiple solution strategies, multiple representation and explanations" (p. 50). Boaler (2002) provided further evidence of open-ended tasks being a key to progress when she compared the activity, operations, and achievement outcomes in two schools. The schools were chosen to represent similar socio-economic mixes of students; but in one school, the teachers based their teaching on open-ended tasks and in the other traditional text-based approaches were used. After working on an "open, project based mathematics curriculum" (p. 246) in mixed ability groups in the former school, the relationship between social class and achievement was much weaker after three years, whereas the correlation between social class and achievement was still high in the latter school where teachers used traditional approaches. Further, the students in the school adopting open-ended approaches "attained significantly higher grades on a range of assessments, including the national examination" (p. 246). Boaler argued that her project demonstrated the "particular teaching practices that need to be considered in mathematics classrooms and the effectiveness of teachers who are committed to equity and the goals of open-ended work" (p. 254). In other words, the use of open-ended tasks proved effective in improving mathematics learning and overcoming disadvantage, but it took commitment from the teachers as well as the adoption of particular teaching strategies.

The type of classroom tasks used by teachers mediates the learning between the subject (student) and object (mathematics). We propose that open-ended classroom tasks offer greater opportunities to scaffold learning opportunities for students than do closed tasks. Essentially, we assume that working on open-ended tasks can support mathematics learning by fostering operations such as investigating, creating, problematising, communicating, generalising, and coming to understand procedures—as distinct from merely recalling them. There is substantial support for this assumption. Examples of researchers who have found that classroom tasks or problems that have many possible solutions contribute to such activity and learning include those working on investigations (e.g., Wiliam 1998), and those using problem fields (e.g., Pehkonen 1997). It has been shown that opening up classroom tasks can engage students in productive exploration (Christiansen and Walther 1986), enhance motivation through increasing the students' sense of control (Middleton 1995), and encourage pupils to investigate, make decisions, generalise, seek patterns and connections, communicate, discuss, and identify alternatives (Sullivan 1999). Open-ended classroom tasks have been shown to be generally more accessible than closed examples, in that students who experience difficulty with traditional closed and abstracted questions can approach such tasks in their own ways (see Sullivan 1999). Well-designed open-ended classroom tasks also create opportunities for extension of mathematical operations and dimensions of thinking, since students can explore a range of options as well as considering forms of generalised response. We have found that many such classroom tasks lead students to make important abstractions and generalizations.

We encourage prospective teachers to use a particular form of such open-ended tasks that can be readily incorporated in conventional mathematics programs. We describe our classroom tasks as *content specific*. The nature of content specific open-ended tasks can be illustrated by some examples:

> Give the co-ordinates of two points on a line with a gradient of 4. List some other pairs of points for this line.
> What are some events with outcomes that are equally likely?
> A ladder reaches 10 metres up a wall. How long might be the ladder, and what angle might it make with the wall?
> What are some functions that have a turning point at (1,2)?
> On the train to Melbourne, the probability that a passenger is reading a newspaper is 2/3, and the probability that a passenger is female is 1/2. How many passengers might be on the train? How many males might be not reading the newspaper?

Such tasks are *content specific* in that they address the type of mathematical operations that form the basis of textbooks and the conventional mathematics curriculum. The learning that results from such tasks about specific mathematical content is at least what would be expected from completion of a typical text-book-based exercise, so teachers can include these as part of their teaching without jeopardising students' performance on subsequent internal or external mathematics assessments.

These classroom tasks are open-ended in that there is a variety of possible operations and ways of communicating responses. Emphasis is taken off particular solution strategies and specific examples, and put on to general properties. There is a sense of open entry with relatively simple responses as well as extension possibilities.

As discussed above, open-ended classroom tasks create opportunities for personal constructive activity by students directed at mathematical objects. We also consider that careful sequencing of classroom tasks can maximise learning. This relates closely to what Simon (1995) described as a hypothetical learning trajectory that

> … provides the teacher with a rationale for choosing a particular instructional design; thus, I (as a teacher) make my design decisions based on my best guess of how learning might proceed. This can be seen in the thinking and planning that preceded my instructional interventions … as well as the spontaneous decisions that I make in response to students' thinking. (pp. 135–136)

Simon (1995) noted that such a trajectory is made up of three components: the learning goal that determines the desired direction of teaching and learning, the activities to be undertaken by the teacher and students, and a hypothetical cognitive process, "a prediction of how the students' thinking and understanding will evolve in the context of the learning activities" (p. 136).

During our research, we found that the use of well sequenced open-ended classroom tasks promoted understanding, as evidenced by students' participation in discussions, improved students' engagement by their time on task, and successful completion of the teaching and learning activities. The use of such classroom tasks also had an impact on participating teachers' notion of mathematical activity.

Enabling Prompts

Perhaps the key component of our teaching model is our proposition that teachers offer enabling prompts to allow students experiencing difficulty to engage in active experiences that are closely related to the overall classroom task(s). These prompts can involve slightly lowering an aspect of the task demand, such as the form of representation, the size of the number, or the number of steps, so that a student experiencing difficult can proceed at that new level. After success at this level, the student can proceed with the original task. This approach can be contrasted with the more common requirement that such students (a) listen to additional explanations; or (b) pursue goals substantially different from the rest of the class.

The approach has substantial support in the literature. Christiansen and Walther (1986), for example, argued that, "One of the many aims of the teacher is … to differentiate according to the different needs for support but to ensure that all learners recognise that these …actions are created deliberately and with specific purposes" (p. 261). Similarly, Griffin and Case (1997) described teaching as involving knowing what individual learners understand, being aware what knowledge is within their developmental zone, providing carefully constructed tasks to engage students in learning, helping learners as they construct their knowledge, and "constantly shifting or changing the 'bridge' to accommodate the learners' growing knowledge" (p. 4).

The notion of adapting classroom tasks has also been a consistent theme in advice to teachers. For example, the Association of Teacher of Mathematics (ATM 1988) detailed 14 specific suggestions to support students experiencing difficulty, 7 of which relate to task adaptation.

We have found that in every classroom the use of enabling prompts has resulted in students experiencing difficulties being able to start (or restart) work at their own level of understanding, has enabled them to overcome barriers met at specific stages of the lessons, has led to increased student engagement in discussions, and has generally resulted in satisfactory task completion.

Extending Prompts

Very much related to this is the proposition that teachers pose prompts that extend the thinking of students who complete tasks readily in ways that do not make them feel that they are merely getting more of the same. Students who complete the planned classroom tasks quickly are posed supplementary tasks or questions on the same topic and concepts that extend their reflections and understanding, that continue to engage and challenge them, and that serve to enrich later classroom discussions.

In practice, extending prompts have proved effective in keeping higher-achieving students profitably engaged as well as supporting their development of generalisable understandings that we associate with higher order learning.

Explicit Pedagogies

A further step in the planning and teaching model is for teachers to make explicit for all students the usual practices, organisational routines, and modes of communication that impact on approaches to learning. These include ways of working and reasons for these, types of responses valued, views about legitimacy of knowledge produced, and the responsibilities of individual learners. As Bernstein (1996) noted, through different methods of teaching and different backgrounds of experience, groups of students receive different messages about the overt and the hidden curriculum of schools.

We have listed, and used in research, a range of specific strategies that teachers can use to make implicit pedagogies more explicit and so address aspects of possible disadvantage of particular groups (Sullivan et al. 2002). We have found that making expectations explicit enables a wide range of students to work purposefully, with teachers involved in the research commenting positively about the resulting relatively low levels of teacher-student friction.

Learning Community

A deliberate intention in our model for planning and teaching is that all students progress through learning experiences in ways that allow them to feel part of the class community and contribute to it, including being able to participate in reviews and summative class discussions about the work. To this end, the model is based on an assumption that all students benefit from participation in at least some core classroom tasks that can form the basis of common discussions and shared social and mathematical experience, as well as a common basis for any following lessons and assessment items on the same topic.

We have found the use of classroom tasks and prompts that support the participation of all students has resulted in classroom interactions that have a sense of learning community (Brown and Renshaw 2006), with wide-ranging participation in learning activities as well as group and whole-class discussions.

An Example of an Implementation of the Planning and Teaching Model

To illustrate the ways that the planning and teaching model works in practice, the following is a description of two sequential lessons that were designed to address aspects of the relationships between dimensions and 3D shapes, and also the relationship between nets of shapes and the shapes they create. Note that we would use records of lessons that had been taught using the planning and teaching model as part of the initial phases of our teacher education program.

Both lessons were taught to a small group of eight Year 9 and 10 students (7 boys and 1 girl, all aged about 15) in a school serving only Indigenous Australian students. In discussion with the teachers at the school, it seemed that the students had had interrupted school attendance, did not see schooling as creating opportunities, did not usually come to class with the appropriate resources, and seemed to have substantial gaps in the prior knowledge if described in terms of the conventional mathematics curriculum. The teachers at the school were experienced and committed. The overwhelming impression, from discussion and from observation, is that the mathematics teachers in the school struggle to maintain the students' interest. The curriculum of the school has been adapted to the backgrounds of the students, while still preserving the option for some students to progress to a mainstream public school on leaving this school after four years of secondary education. Most of the students do not progress beyond Year 10 (age 15), but some do.

The teacher of the two lessons, Mr Smith (pseudonym), was not a teacher at the school, but was experienced with teaching similar students and also familiar with the project and the teaching model described above. His role was to demonstrate the model to the mathematics teachers in the school and to discuss its perceived viability in this relatively challenging context. The two lessons were taught one after the other, after lunch on a Friday afternoon—a demanding time for any teacher. There was a trained observer present, as was the students' usual mathematics teacher. The lessons had been planned by the research team beforehand, and Mr Smith and the observer met with the class teacher before and after each lesson.

The following report is an amalgamation of the lesson plans, Mr Smith's recollections captured after the lesson, and the observer's notes written at the time. It is presented in this way to illustrate the way the model operates in a difficult classroom context, and to exemplify the ways that enabling prompts can be used to support learning.

Lesson One

The aim of the sequence of two lessons was to further develop the students' understanding of measurement. For the first lesson, the teacher intended that the students work on the following classroom task (see also Watson and Sullivan 2008, for a different discussion of this particular problem):

You have a box that needs 1 m of string to tie it up like this. What might be the dimensions of the box?

Assume that 30 cm is needed to make the bow.

The class teacher had earlier suggested both the lesson focus and this context of wrapping presents, and so it was assumed that it was familiar to the students, and it proved to be so.

The classroom task is illustrative of the type that formed the basis of the project, in that:

1. it addresses challenging and useful mathematics, specifically visualising objects, using dimensions to describe and quantify rectangular prisms, and exploring the relationships between those dimensions, and focuses on mathematics concepts that may have applicability in other contexts;
2. there are many ways of solving the task, and different possible interpretations of the task demand, and so it could be assumed that most students would be willing to make an attempt, given that they have some choice over their approach, and most would be able to do so because of different potential entry levels;
3. the students would have the opportunity to find a solution and to describe their solution to the class, allowing opportunities for problem solving and for considering challenges in communicating the solution as well as experiencing the advantage of being part of a community working on the same learning task;
4. the range of possible solutions, when viewed together, could allow students to see the potential variability in the box within the constraints of the tasks, and that the key dimensions (L, W, H) can vary;
5. there would be limited need to listen to explanations by the teacher at the start, avoiding the potentially disengaging effect of teacher explanations; and,
6. because of the openness of the task, students who finish quickly could be posed extension exercises readily.

As a first step in the lesson, Mr Smith gave each pair of students a box wrapped and tied with string as in the photograph above, and asked them to calculate the length of the string without untying it. Mr Smith later explained that he considered this to be a way of introducing the students to the key concepts without requiring extensive explanations, and the demands of the task would be readily communicated to the students.

When Mr Smith stated that the bow used 30 cm of string, the observer recorded the students' response as follows:

> The students refused to believe that the bow was actually 30 cm. (As in the photo, it did not look like 30 cm.) The teacher asked the students to assume that it was. The students still did not start on the task, so the teacher untied the bow and asked one of the students to measure the loose ends. Each end of the string was 15 cm, and so the students then agreed that the bow used 30 cm of string.

We stress that it is important to clarify aspects of the problem, including specific language. Note that this is different from telling the students how to do the problem. In the rest of this lesson, the consideration of the bow was no longer an issue. This is an example of how the sequencing of tasks can contribute to the success of the lesson, and how in this case an appropriate choice of a preliminary problem allowed the students to engage with the context for themselves, and also prepared them for the task that was to intended to be the focus of the lesson.

As enabling prompts for this preliminary classroom task, Mr Smith had ready some other boxes and loose string for students who might need to tie up a box, a box covered in a streamer that could be cut into sections, and a box covered in plain paper but with no string. Only this latter prompt was used in the class. Note that the initial task posed could also serve as an enabling prompt.

After the students had worked on this preliminary task, Mr Smith invited individual students, on behalf of their pair, to explain their methods of solution to the group. The observer recorded this as follows:

> One student explained that they had measured separately each section of "string", written down the lengths and added them up. There were 4 sections on the top (from each edge to the centre where the string crosses), 4 on the bottom and 4 on the sides.
>
> Another student explained that they had measured each of the lengths and then coloured in the string to show they had done it, adding the lengths as they went.
>
> Another student said all the pieces of string on the sides (meaning not the top or bottom) were the same. There was some debate about this (there were slight differences depending on whether the wrapping on the box was neat and regular), and (Mr Smith) explained that for this purpose those sections were the same. The student then said that the top and the bottom was the same, and that this is how they worked it out.

This initial task was a key part of the lesson, because the subsequent task required the assumption of regularity of the box. This is an example of the teacher considering the trajectory of the class learning and sequencing tasks accordingly.

The classroom task that formed the focus of the lesson, as presented above, was then posed. Terms and assumptions were clarified, and the students invited to ask questions, then they set to work. In normal circumstances it is assumed that the teacher will have already established the appropriate classroom norms for behaviour. Note that all of the boxes in the previous task required more than 2 m of string, and so did not detract from the visualisation needed to address the problem. All pairs retained their box and could have chosen to use it, if they wished.

The observer noted:

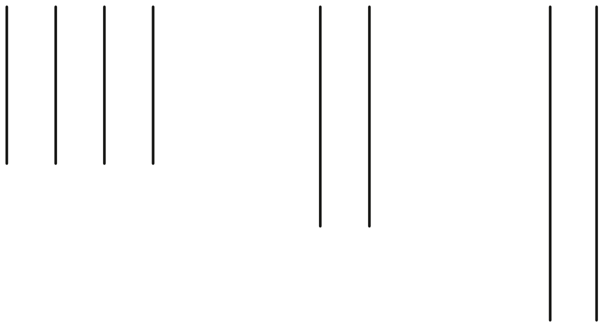
> Again, the students worked in pairs. The students required constant monitoring in that their interactions with each other were confrontational but they engaged with the task, and knew what to do.

Mr Smith had prepared some enabling prompts for students who may have experienced difficulty in engaging with this task, including a box covered in white paper on which they could draw, and a box that used approximately 1 m of string (as mentioned above). He gave the first of these boxes to one group of students, and they used it effectively in coming to understand the nature of the problem they had been set.

Mr Smith later commented that he had also asked students experiencing difficulty to try out a particular dimension, such as assuming that the height was 5 cm. This has the effect of reducing an aspect of the task demand, and so is an example of an enabling prompt.

The students, in pairs, used a variety of approaches to solving the problem. The observer recorded the summary class discussion of the task as follows:

> One student reported that the group had assumed that it was a cube, and that the lengths would add to 70 cm, and that there would be 8 equal lengths but did not progress beyond that.
>
> Another student reported that his pair had said the 4 sides were the same, the 2 lengths the same, and the 2 widths. He represented this as follows on the whiteboard:



> He said that the total had to be 70, and that they were working out what numbers to use to make the total 70. They had first tried unrealistic numbers, but using trial and error the lengths turned out to be 5, 10 and 15 cm respectively, which made 70 together. (Mr Smith) asked the other students to ask any questions.
>
> Another student drew a table of values, with columns headed length, width and height, and made guesses of the dimensions.

In summary, this lesson is illustrative of the planning and teaching model. There was a potentially engaging classroom task, whose openness indicated the possibility of student control and choice, a hypothetical trajectory to lead students to key aspects of the focus classroom task, both pre-prepared and impromptu enabling prompts for students experiencing difficulty, and the potential for ready extension, if needed. In terms of the particular phase of the research, it seems that the model did describe key aspects of the planning, and it was possible to implement each of these elements in this class.

Lesson Two

The second lesson was to the same group of students, immediately following the first, after a short break. The task around with the lesson was planned was posed as follows:

It is possible to make an open top box by cutting squares from the corners of a rectangular card, and then folding up the sides.

If the card is 20 cm × 16 cm, describe the dimensions of some boxes you can make.

Mr Smith said the words, but drew a diagram similar to the one above on the white-board.

The characteristics of this classroom task are similar to those described in the first lesson. As an introductory problem, Mr Smith gave each of the pairs an open topped box made from a net on dotted squared paper. The students were asked to cut out the net, using a different sheet of dotted squared paper.

Mr Smith explained later that he considered this to be enough of a problem to engage the students, but that he had expected they would be able to do it. The observer recorded this aspect of the lesson as follows:

> (Mr Smith) moved from group to group, encouraging the students. He suggested to some students that they open out the box and trace around it. To another student, he said "Have you counted the dots?" They did this quite quickly and showed each other the nets they had cut. (Mr Smith) talked about the shape of the net, and pointed out the missing squares in the corners.

In other words, Mr Smith offered prompts to some students as a way of supporting their engagement with the preliminary task.

The classroom task as presented above was then posed. Mr Smith emphasised that he wanted the students to imagine what might happen if squares are cut out of the corners, without actually cutting them out. He also emphasised that there is more than one possible answer, and that he hoped the students would come up with a number of answers. This is an example of making explicit aspects of the pedagogy.

The observer recorded the next aspect of the lesson as follows:

> The students did not settle down to this. Some seemed to want to continue with the dot paper task. There were only two of the students who got started. (Mr Smith) asked the others to use dotted square paper to cut out some rectangles that were 20 cm long and 16 cm wide, which they did readily. (Mr Smith) asked them to cut squares of the same size from the corners, and pointed to the diagram. They worked by themselves on this. He asked some students to fold the box once they had cut the net.

This alternate suggestion is an example of an impromptu enabling prompt. The classroom task had not engaged a number of the students, so Mr Smith posed the similar alternate task that had potential to lead the students toward the original task.

It had the effect of engaging the students in the concept, albeit at a slightly lower level than originally intended.

Two of the students worked on the original task. One student identified two possible solutions, and the other identified multiple possibilities, including some using decimal lengths of the side of the square (e.g., 15 cm long, 11 cm wide, and 2.5 cm high). Mr Smith commented to this student, “I wonder how many possibilities there are. How big could the cut out rectangles be?” This is an example of an extending prompt. Later the class teacher was enthusiastic about the response of the two students, and especially one who did not usually engage in any way and who in this case had produced creditable responses, even though they did not actually reach the ultimate task. It would have been possible, though, for the teacher to continue with the task in the next lesson, while allowing the students who had identified some solutions to develop further responses, discuss the common properties, and come to some general whole-class conclusions.

Even though the end-point of the trajectory was not reached in the given time, the teacher was generally happy with the lesson and it does illustrate elements of the model described: a sequence of open-ended tasks, some enabling prompts, and some extending prompts. The openness of the classroom tasks allowed all students to engage effectively—especially given the difficult time of the week. The full planned trajectory of the tasks worked for some students and, with enabling prompts, the others were also engaged in meaningful mathematical activity that focused on the same concepts and, in fact, all students were all working on the same tasks. This establishes the important foundation for the specific attention to building a classroom community, although this would require additional actions by the teacher.

These two lessons exemplify the respective elements of the teaching and planning model. They illustrate how the model contributed to the structuring of workable mathematics lessons, and how each element of the model contributed to the whole. This forms part of the rationale given to prospective teachers for requiring them to use the planning and teaching model as part of their learning to plan and teach inclusive mathematics lessons.

The Task for Prospective Teachers

The planning and teaching model described above can be used as a focus for particular consideration of theoretical perspectives associated with mathematics teaching. Classroom tasks are the basis of the social interactions that lead to learning, so they offer ways of considering alternate approaches to learning mathematics and even of the nature of mathematics itself. The sequencing of tasks allows consideration of psychological perspectives on how learning develops, the forms of communication and classroom activity that are most likely to promote the learning process, and philosophical considerations of what it means to come to know. Consideration of enabling prompts allows the study of how to overcome aspects of learning that contribute to student difficulties, including common misconceptions. Extending prompts offer ways of catering for diverse student needs and rich opportunities to raise the level of class discussions. The building of community and being explicit about specific pedagogies can form a useful framework for study of approaches to teaching and learning generally, but especially those than can represent barriers to particular groups of students.

The task that we set for our prospective teachers is to use the planning and teaching model as the basis of, first, hypothetical, and, second, actual classroom lessons, followed by structured reflection on the planning and teaching experience. Essentially the task for teacher education is a process. The use of the planning and teaching model in this way incorporates some elements of approaches to teacher learning from the study of practice such as learning study (Runesson 2008), study groups (Arbaugh 2003), and Japanese lesson study (e.g., Stigler and Stephenson 1994). Each of these involve collaborative reflective study of aspects of practice, with a fundamental assumption that planning, trialling, and reflecting on aspects of practice is a powerful learning opportunity for teachers.

Some of the key phases in this process include prospective teachers:

- Studying the nature of classroom tasks, and especially ways in which non-routine classroom tasks are different mathematically and pedagogically from conventional tasks. This can involve working through tasks can considering both strategies and solutions, sorting tasks into categories, and identifying commonalities and differences in types of tasks.
- Considering the affordances and constraints in using non-routine classroom tasks. This can include consideration of what makes a particular task non-routine, what might contribute to the complexity of a particular task for students, and what might make teaching with a particular task difficult.
- Experiencing the planning and teaching model through a mathematics "lesson" taught to the prospective teachers. This involves the lecturer teaching a lesson that involves mathematics that is new for most students, incorporating all aspects of the model such as open-ended tasks appropriately sequenced, enabling and extending prompts, and specific actions to build the group as a community of learners. After the lesson, the prospective teachers are invited to comment on their mathematics learning, their affective response, and any aspects that were noteworthy for them.
- The prospective teachers are formed into small groups (maximum 3) for collaborative planning of hypothetical lessons, with no intention that the lessons be taught, with critical review of those plans, with particular attention to whether each of the aspects of the planning model are incorporated, and what particular pedagogies may be necessary to implement the lesson as intended.
- Using the same small groups, the prospective teachers plan and then teach a lesson based on the planning and teaching model in a real classroom, incorporating iterative processes for review. This can take the form of one prospective teacher teaching the lesson, then having a group review, with the plan revised, and re-taught by a different group member, and so on.

- Ultimately, it is necessary to create opportunities for review and reflection not only on the teaching and planning model but also on the teacher learning process itself. The intention is the task of using the planning and teaching model that becomes part of the prospective teachers' future practice and forms the basis of future teacher learning opportunities.

Of course, the same process can be adapted for use with practicing teachers. The fundamental proposition is that the task of planning and teaching lessons is complex and can be the object of specific study. The planning and teaching model provides a suitable framework for this.

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Building Optimism in Prospective Mathematics Teachers

Psychological Characteristics Enabling Flexible Pedagogy

Gaye Williams

Introduction

> [I learnt] not to narrow a student's way of thinking based on the ways in which I think…I'm not sure how she did it, but [my lecturer]…taught me not to block student thinking; something that has made an incredible difference to the way I approach teaching [mathematics]. (Hayley, 'P-M-T', Undertook 'Mathematics Curriculum Studies' the Previous Year)

The 'prospective mathematics teacher' (P-M-T) quoted above perceived herself to have developed a greater awareness that mathematical problems can be solved in multiple ways, and that her own students should be encouraged to think for themselves rather than only follow her lead. Mathematics Curriculum Studies (MCS) had raised her awareness of the need to respond flexibly to students' mathematical thinking. This chapter helps to illuminate ways this was achieved and the psychological characteristics P-M-Ts need to enable this.

The context in which MCS is undertaken is described, and its purposes identified. The focus of the chapter is not on flexible pedagogies these P-M-Ts become more likely to use (see quote: Hayley) after participation in this subject, but rather on how a three-task-sequence that connects geometric representations can build psychological characteristics that enable flexible pedagogical moves: 'optimism' ('resilience'). An optimistic orientation increases the ability to overcome adversity (Seligman et al. 1995), and teaching mathematics by flexibly responding to student responses can be considered a situation of adversity as many attempts may be made whilst searching for successful pedagogical 'moves' to elicit further student thinking. The chapter also elaborates theory that guides MCS pedagogy through a framework that focuses on building mathematical and pedagogical understandings, and optimism. P-M-Ts identified this task sequence as particularly significant to their developing pedagogical realizations. Thus it has been used to elaborate the theoretical perspective that guides my pedagogy. Tasks are described and analyzed

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to illuminate the fit between task activity and optimism building. Creative thinking and spontaneous student responses crucial to optimism building are illustrated.

The Context

MCS is a two-semester subject sequence studied by P-M-Ts, undertaking different courses (approximately 30 students). P-M-Ts vary in their mathematical backgrounds and personal histories. There are local Australian, and international, participants who have undertaken an undergraduate degree that included at least four units of mathematics. These P-M-Ts undertake MCS as part of a two-year primary/secondary, or one year secondary, teacher qualification. There are also P-M-Ts in their second or third year of a four-year undergraduate teaching qualification undertaken simultaneously with a content related degree (e.g., Arts, or Science). They have completed two, or four, mathematics subjects from that degree when they participate in MCS. There are extremes in the mathematical backgrounds of these P-M-Ts in geometrical topics (geometric construction, congruence, and deductive proof) because most of the local P-M-Ts undertook secondary mathematics in the interval in Victoria when geometric proof was almost completely eliminated from the secondary mathematics curriculum. Other P-M-Ts were generally exposed to geometry through rules and procedures.

More geometry has recently been reintroduced to the secondary curriculum so these P-M-Ts will be expected to teach these topics. Thus, the cohort in MCS provides an amplified version of common differences existing between P-M-Ts in tertiary settings. Their varying mathematical backgrounds, and personal histories influence their mathematical confidence.

MCS aims to increase P-M-Ts' understandings of:

- The mathematics they will teach;
- The difficulties their students might encounter with this mathematics;
- Pedagogies that could increase student understanding; and,
- Psychological characteristics of students that enable exploratory activity.

In addition, MCS as implemented is intended to build psychological characteristics in P-M-Ts that will increase their flexibility in responding to their students. Theory (Seligman et al. 1995) suggests that embedding P-M-Ts in classroom activities (explorations) in which groups can spontaneously focus their own challenges, and experience intensity, excitement, and pleasure as they achieve successes, should build optimism.

Theoretically Framing the MCS Pedagogy

This section describes the Engaged to Learn Model (Williams 2000, 2005) used to represent conditions to elicit intensity and pleasure during deep learning for secondary students overcoming self-selected challenges. It also describes how and why

I recently extended this to the Engaged to Learn ('Mathematical and Pedagogical', MAP) Model (to represent the greater magnitude of overall mathematical and pedagogical challenge faced by MCS participants when these secondary tasks were adapted for their use).

Engaged to Learn Model

Flow (Csikszentmihalyi 1992) is a state of high positive affect during creative activity that occurs when a person or group perceive the need to develop new skills to overcome self-set challenges. During flow, all sense of time, self, and the world around is lost as all energies focus on the task at hand. This is the overarching theory that frames the Engaged to Learn Model. With mathematical problem solving, flow conditions are created when a student group, or an individual student, idiosyncratically identifies an unfamiliar mathematical complexity that was not apparent to them at the commencement of the task, and decide to explore it (Williams 2005, 2007). High positive affect that can accompany the development of insights has been demonstrated to occur during primary, and secondary mathematical problem solving (Williams 2000, 2005, 2007), and problem solving in engineering (Wood et al. 2008).

Figure 1 represents flow conditions during mathematical problem solving. M represents a 'comfortable' level of challenge and A represents the level of skills and concepts perceived known. For flow to occur, participants *spontaneously* focus an intellectual mathematical challenge of magnitude MN (vertical axis) that can be overcome by developing new mathematical ideas and concepts *just beyond* their present understanding (magnitude AB, horizontal axis). The shaded region between the parallel lines in Fig. 1 represents the state of flow (pursued challenge MN building knowledge AB). When the challenge is too low/too high, and/or skills and concepts required are already known/too far out of reach, different affective states can occur: boredom, apathy, anxiety and panic (see Fig. 1). Once students have experienced flow in attaining the positions N and B (Fig. 1), they 'fall out of flow' unless they focus a new challenge NP building concepts BC (flow parallelogram now within parallel lines beyond shaded region).

Inclination to undertake (or not undertake) exploratory activity associated with flow is linked to psychological factors that affect how people respond to successes and failures (Seligman et al. 1995; Williams 2003). During 'not yet successful' problem solving attempts, successful problem solvers use their 'failures' to get closer to success (but do not call them 'failures'). This is a characteristic of 'optimism' along the dimension Pervasive-Specific (Seligman et al. 1995; Williams 2005). An optimistic person examines their lack of present success to find what to change to increase their likelihood of success. In contrast, a person who does not possess optimistic indicators along this dimension perceives their failure as characteristics of self: "I failed, I am dumb". The enacting of optimism is represented in Fig. 1 by black arrows extending beyond M and A: 'inclining to explore the unknown'.

Where optimistic people feel comfortable working outside their present understanding as they 'set up' new challenges, those who are not yet optimistic do not.

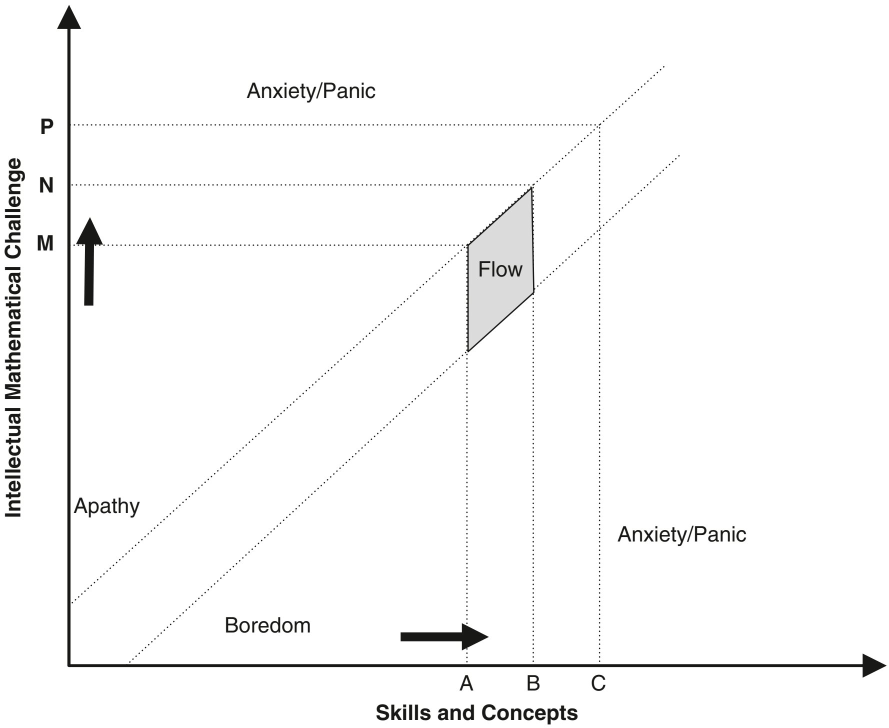

Fig. 1 Diagrammatic representation of conditions for flow and other affective states during the learning of mathematics

An optimistic person perceives successes as permanent (they will be able to do this again), personal (it was achieved through their own effort), and pervasive (it was achieved because of characteristics they possess). They perceive 'failures' as temporary (able to be overcome), external (including some external factors out of their control), and specific (including factors they can vary). Seligman has shown that optimism can be built through experiencing successes during flow situations. The pedagogical approach in MCS provides opportunities for P-M-Ts to set up flow situations to enable successes (develop insights) that can build optimism.

Flow conditions experienced by P-M-Ts and the optimistic dimensions they start to build are illustrated through the following two email reflections during, and after these two students completed this subject:

> The classes were engaging and you really did feel proud of your accomplishment (Bianca, P-M-T during MCS).

Bianca captures the pervasive ('feel proud') and personal ('you') nature of success. She attributed these successes to characteristics of herself rather than to something external. She felt proud of *herself* for what *she* had accomplished: perceived accomplishment as confirmation of her capability.

> You were right when you said to me in first semester that the maths would come back, it did!!! I have had a number of 'magic moments' when I have realized why things work (Debra, 'Beginning Mathematics Teacher', B-M-T, who undertook MCS the previous year).

Barnes' (2000) term 'magic moments' captures intense affect associated with developing insight. Debra (B-M-T), who had just completed both MCS and her teaching qualification, used the term 'magic moments' to capture her excitement and pleasure as she overcame self-set mathematical challenges associated with finding 'why' not just 'how'. Similar pedagogy to that in MCS was used in my senior secondary mathematics class studied by Barnes. Both included small group and whole class interactions as students explored mathematical complexities in tasks. MCS tasks are adaptations of the complex tasks designed for learning secondary mathematics. They differ in that attention is also drawn to pedagogy associated with task implementation.

I had not consciously considered that a double focus on mathematics and pedagogy increased the overall challenge until Liz (B-M-T) raised my awareness. I had consulted her about an assignment that initially caused anxiety for some, but was eventually found to be extremely useful by most (see Fig. 1):

> I felt like there were several components that were really difficult to work out (Liz, B-M-T).

Liz elaborated 'several components' as both mathematics, and how to teach it. I suddenly realized that *simultaneous* mathematical and pedagogical challenges were harder to overcome than these challenge occurring one after the other. I extended the Engaged to Learn Model to the Engaged to Learn Model: Mathematical and Pedagogical or Engaged to Learn (MAP) (Fig. 2) to capture this new realization and remind myself of how important it was to reduce this multiple challenge.

In Fig. 2, two vertical and two horizontal axes are used to represent the simultaneous mathematical and pedagogical challenges and simultaneous building of mathematical and pedagogical insights respectively. In designing and implementing tasks, I now knew I needed to reduce the magnitude of the overall simultaneous challenges or eliminate their simultaneity.

Different aspects of the implementation of the three-task sequence herein illustrate ways this was achieved. In particular, although the tasks still request dual focus, the implementation is responsive to what P-M-Ts focus upon. The following excerpts illustrate email feedback from P-M-Ts who identified this task sequence as significant for them. Jayde saw the task sequence as a turning point during which she became more resilient (her words), Bianca identified features of the classroom culture that contributed to this, and Michael developed confidence through participation in such tasks across the year. Quotes are used to elaborate the nature of optimism.

Key: '…' wording omitted that does not alter meaning.

'[text]': text added by researcher to elaborate.

> At the beginning [of the year] I was quite frustrated and anxious about the way we were being taught [in MCS] … I'm pleasantly surprised how much I have grown this semester. … looking at the big picture, I can see how much it has benefited me. I had to be patient, ponder over things, and let it eventually fit into place in my own head rather than wanting it to happen straight away (Jayde, P-M-T during MCS).

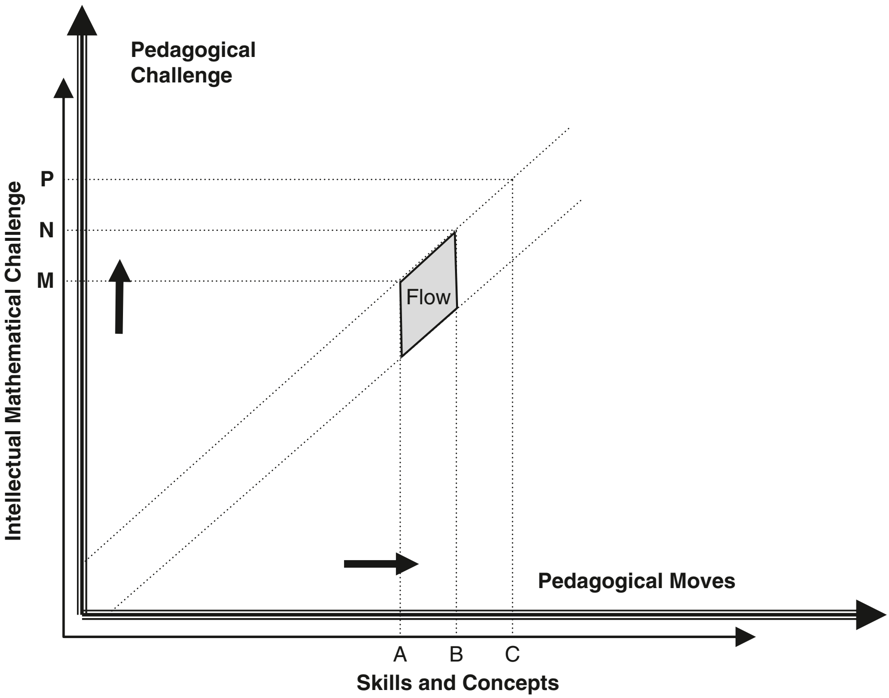

Fig. 2 Engaged to Learn (MAP)

Jayde possessed optimistic indicators on two of the three dimensions (permanent-temporary, personal-external) when she commenced MCS. She saw failure to understand straight away as temporary and able to be overcome through her personal effort of 'pondering over things'. Her successes achieved during this task sequence in which she was not 'told' but struggled in 'working it out' increased her confidence in her capacity to cope with uncertainty (Success as Pervasive; seeing success as due to characteristics of self "how much I have grown"). Bianca identified crucial aspects of the developed classroom culture:

> I loved the Determining Triangles, Construction, and Proof Sequence. I really enjoyed working the construction and triangles out for myself/in the group … [and] how the class dynamic became co-operative and reflective … [even] though we were working in separate groups … it was also amazing that the classroom atmosphere became so accepting of different opinions (Bianca, P-M-T undertaking MCS).

Bianca captures the development of a 'community of inquiry', which did not judge what was contributed but rather reflected on how it could help. The types of activities she identified are illustrated later in this chapter. Bianca's reflections pinpoint the accepting of all contributions (not judging some as 'dumb') as a feature of the learning environment.

Finally achieving successes in this culture helped Michael build confidence:

> [I] noticed that my knowledge of mathematics was always being tested [challenge opportunities] and that near the end of the year I was feeling confident enough to lead in discussions

> and believed that my personal knowledge of math's increased due to our class discussions (Michael, P-M-T, previous MCS participant).

The role of successfully overcoming mathematical challenges contributed to Michael perceiving Success as Pervasive. This increased his perceived mathematical ability, and thus confidence to lead mathematical discussions. The tasks are now described.

Three-Task Sequence

This sequence of 'complex tasks' (Table 1) was designed to connect several mathematical topics in unfamiliar ways, build P-M-Ts' awareness of the affective value of such activities, and build their own optimism through this experience. Mathematically, the sequence:

- Builds understandings of congruent and similar triangle properties;
- Elicits thinking about how to create geometric constructions (e.g., bisecting a line); and,
- Links these ideas by asking for proof that the constructions always work.

These are links that are not traditionally made between topics in Victorian mathematics classrooms. The tasks were presented one after another in the ninth and tenth of eighteen three-hour MCS classes. Task wording is purposefully tentative, to give P-M-Ts the autonomy to focus their own explorations and indicate they are not expected to progress rapidly in a linear fashion to a previously determined endpoint. Features crucial to each task are now discussed and linked to Tables 2, 3 and 4, which include excerpts of activity that led to insights that should build optimism.

Task 1 Features and Enactment

The sequence leads naturally to the need for geometric notation, logical argument, and communication of these arguments in succinct ways. By focusing on pathways secondary students might take, P-M-Ts' own lack of mathematical knowledge need not be revealed. Thus, P-M-Ts are more inclined to contribute in groups and report to the class because any mistakes are attributed to what secondary students might do. P-M-Ts commence the task thinking as students but become engrossed in new ideas themselves.

At intervals, groups gave 1.5-minute reports to the class on one or some of:

1. Ways secondary students might work with the task.
2. Something interesting students might find, and/or that you found.
3. Some generalization students might develop/you developed.
4. Insights that might develop for students or for you.
5. What you as teacher could do to progress learning of your students.
6. Difficulties the teacher might encounter in implementing the task.
7. Ways you have identified to help to overcome such difficulties.
8. Anything else you decide could be interesting to class as a whole.

Table 1 Sequence of complex tasks: secondary three task sequence adapted as oral MCS task

Task 1: Determining triangles (see Williams 1996)

Given sets of information about triangles: (a) two side lengths; (b) three side lengths; (c) three angles; or (d) a mixture of angles and side lengths in various combinations and relative positions, work in groups of three/four to decide which triangles are 'determined'. Determined is taken to mean: only one triangle can be made with this information. Groups will share their ideas with class at intervals. (Specific example given, see Table 2, example in (d))

Task 2: Constructing

Lecturer: Constructing involves making accurate drawings without measuring. It involves using any or all of the following: (a) Straight edge of a ruler (not the measures); (b) pencil; (c) compass; and (d) set square (without using right angle, other angles or edge as measures)

Work in groups of 3 or 4 on the task: Can you find how to construct:

1. A line that bisects another line
2. A line perpendicular to another line at a point other than midpoint
3. A line that bisects a given angle
4. A line parallel to another line
5. Line segmented into n equal parts where n is given?

As soon as you think you have found how to achieve one of these constructions, come out to board and show the class your finding

Task 3: Prove These Constructions Work

Lecturer introduces:

1. 'Axiom' and illustrates with corresponding angle equality
2. Proofs as arguments supported by evidence (illustration not given)
3. Possibility of arguments as oral, diagrammatic, and/or symbolic or any other type of argument the group decides is appropriate
4. Restriction: can use anything developed through this sequence and any theorems as long as they are proven before used the first time
5. Suggestion: bisector of angle construction could be a useful start
6. Orchestrates reporting: order reports to progressively build ideas

P-M-Ts:

1. Work in groups of three/four for 5 minutes to develop proofs that each of the constructions in Task 2 always work
2. Each group is to explain to the class some of their thinking so far (1.5 mins each). Prime your reporter first by hearing/adjusting report
3. Return to groups and work on with your ideas. Decide what is and is not relevant to your group from reports of others
4. Report progress to the class through reporters selected by the group (each group member must report before one reports a second time)
5. Continue this cycle of group work and reporting for four cycles

By focusing the class on these possibilities for reporting, optimistic aspects of problem solving (e.g., 2, 3, 4, 5, 7) are highlighted. If not knowing (mathematically or pedagogically) is a situation of failure, then 2, 3, 4, 5, and 7 emphasize failure as temporary and 2, 3, 5, and 7 emphasize success as personal. Rather than using the term 'failure', we use terms like 'not knowing yet' or 'not quite there' emphasizing its temporary nature.

Reporting new insights (successes) to others can increase perceptions that success is pervasive (due to characteristics of self) where class members show appre-

Table 2 Excerpts of activity during Task 1 and how they help build mathematical knowledge and optimism

Excerpt Task 1	Descriptions of types of activity occurring Key: 'L' lecturer; 'P-M-T' prospective mathematics teacher	Intended purpose	Optimism building
Given: Specific example of two sides and non-included angle. Is triangle determined?	Given: AB 6 units, BC 4 units long, Angle BAC 40°. Is triangle determined? Triangle has fixed lengths AB, and BC relative to AB and fixed angle at A, but position of BC and length of AC are not given. (a) One or two groups sometimes identify two triangles during initial group work and report this to class (b) If not, L progressively asks questions of groups over time but walks away without waiting for answers: "Are you sure?" "What is given and what could you change?" "Is that the only triangle you can make if you are allowed to change any of those things you have identified?" (c) Groups report when at least one group has found the second triangle. Intense interest and spontaneous discussion follows (d) L asks questions in rapid succession even though many groups will already have spontaneously focused their own exploration and will continue to pursue that direction (without being aware of L's questions) as in state of flow: "How did that happen? Are there any more such examples? Is it always possible to make two triangles? Why or why not? Can you find a way to clearly explain what you find? Can you show what you are saying is always true? Can you express a generalization for what you have found?" (e) L selects the order of reports so P-M-Ts who have not yet developed insight can think ahead during reports. Reports generally include: (i) Right angled case (ii) The other specific example of the two possible triangles (iii) Partial argument: how two cases are linked (accompanied by hand gestures drawing attention to the relative sizes of sides given) (iv) Partial argument: why the second case sometimes does not exist (v) Compass construction (centre B and span length BC) and accompanying oral argument: why sometimes two and sometimes one case	Engineering surprise: two triangles not one or many found L asked questions to focus P-M-Ts attention on what could vary to increase likelihood of discovering second triangle Groups may decide to pursue other parts of Task 1, rather than focus around questions L asks Questions may help stimulate interest for some who have not yet decided their focus	Surprise as impetus to explore; thus opportunity for flow. Success from flow builds optimism Flow conditions require spontaneously focused question rather than one dictated externally. Richness of task, sharing of findings, and L's questions with no requirement to respond to them, add to likelihood groups idiosyncratically focus own directions including: "Are there more?" "Why are there only two?" "Did we miss something last time?" "Can we show there can only be two triangles?" Reports of partial findings lead to exclamations and intense discussions amplifying usefulness of successes achieved L's questions focus on 'failure' of not knowing about the complexity as temporary/and able to be overcome through personal effort, and on what can be varied

Table 3 Excerpts of activity in Task 2 related to building mathematical knowledge and optimism

Excerpt from T2	Illustration of aspects of activity of one MCS cohort Key: 'L' lecturer; 'P-M-T' prospective mathematics teacher	Intended purpose	Optimism building elements
Cut a given line into *n* equal segments	(a) P-M-T 1: Used compass length, cuts *n* equal segments on *part* of line (b) L: Requests responses. P-M-T 2 recognized requirements not met (c) L suggested idea might be useful later, acknowledged P-M-T 1's part in clarify what was required. Asked for other ideas (d) P-M-T 2: "I can get evens but not odds". Iterates bisecting a line (e) Ingenuity explicitly appreciated by class and L (f) L asked class about odd numbers of segments and pauses (g) No ideas, L added second line at an angle and paused (h) L drew P-M-Ts attention to idea of P-M-T-1 and asked: "Could it be useful?" (i) P-M-Ts became more alert, began volunteering ideas, and adding them to the diagram on the board. Atmosphere in room was intense. Focus on making useful connections between the two lines. P-M-Ts' questions often structured their further explorations. Contributions to diagram, and questions asked included: (i) P-M-T 1: Segment the new line instead (ii) P-M-T 3: Join final compass mark on new line to end of original line "But how do we link other compass cuts to original line?" (iii) P-M-T 4: Parallel lines could be involved "But, how do we construct the parallel lines we want?" (iv) Several P-M-Ts try without success to orient setsquare and ruler. (v) P-M-T 5 completes process	Accessible by drawing on cognitive artifacts from earlier parts in sequence P-M-Ts focus exploration. L facilitates sharing of ideas L adds information but not procedures L does not interrupt group flow. P-M-Ts come out and contribute when they have ideas, without waiting to be asked. L does not comment on correctness but adds questions to sustain intensity: "You think parallel lines should help, and have tried to construct them. Can anyone help?"	Draws attention to identifying specific elements required (Modeling Failure as Specific) Spontaneous linking of several parts of what previously explored produced insights associated with flow As spontaneity is a condition for flow, L needed to sustain spontaneous nature of interactions when not providing additional information L asks questions to sustain intensity, or where P-M-Ts do not pose their own questions to structure future exploration. L did not contribute information *during final* flow interval ((g), (i–v))

Table 4 Excerpts of activity during Tasks 3 and how they help build mathematical knowledge and optimism

Excerpt from T3	Illustration of aspects of activity of one cohort Key: 'L' lecturer; 'P-M-T' prospective mathematics teacher	Intended purpose	Optimism building elements
Prove procedure for constructing bisector of a line:	(a) Procedure found to Prove 'Construct CD bisecting AB': Use compass length longer than half AB Drawing arcs above and below AB using centers A and B Arcs intersect at C and D above and below AB respectively Join C and D with line intersecting with AB at P (b) Generally P-M-Ts focus on right-angled triangles ACP and BCP, find two pieces of information, but not third L: "Are those triangles the only things you can work with?" (c) P-M-Ts are then usually able to 'see' the larger triangles and use them to find more information about smaller ones (d) If not, L gives strategy: "Look for other triangles that may help give more information in those triangles"	Recognize congruent triangles can be used. L does not focus in specifically on using other triangles at this stage. If necessary, L gives strategy without identifying what triangles to use and how	Flow from identifying complexities around congruence and overlapping triangles 'Big ideas' developed in Task 1 and 2 can contribute to overcoming later challenges and contribute to magnitude of pleasure associated with connecting several (big ideas)

ciation of original findings. Such appreciation can arise when reporting findings in 2, and 3, or a generalization arising from 3 or 7. P-M-Ts tend to exclaim when something unexpected and/or elegant is found. Thus, optimistic activity is valued (not praised) because P-M-Ts and/or lecturer's responses make explicit how findings contribute to class development of ideas. Although P-M-Ts may not recognize the subtly of the pedagogy in MCS *during* task implementation, reflections of beginning teachers illustrate that awareness develops over time. Liz illustrates this:

> You took the back seat and modeled the reporting classroom and encouraged us to interact with each other. It's such a subtle difference, and so obviously important, but something that you can easily forget to do when you're so used to classrooms with students facing the teacher in a two-way interaction. (Liz, B-M-T)

The lecturer's role in Task 1 is to ask questions to elicit further thinking, help P-M-Ts clarify ideas, and orchestrate reporting sessions at regular intervals. Liz shows an awareness of sources of control important to the role.

Although P-M-Ts can focus on pedagogical aspects (e.g., 5–7 above), they tend to focus intently on the mathematics as shown in Table 2. For example, P-M-Ts expressed surprise at finding two possible triangles when they had expected to find one, or many. I now realize this focus on mathematics only, not pedagogy as well, occurred because simultaneous focus was too great a challenge in this task sequence where students had little or no understanding of the geometry involved at the start of the sequence.

I no longer focus P-M-Ts on pedagogy when they are intensely engaged in developing mathematical ideas. Instead, I draw attention to pedagogy after they gain insight (achieve learning success). As P-M-Ts are embedded in the pedagogy during the task sequence, and many experience high positive affect as they learn geometry they did not know, they want to know more about how such situations are created (so they can create them for their own students). A discussion of what was happening pedagogically ensues with students initially focusing the attention. These post-task discussions show pedagogical realizations develop then too.

Task 2 Features and Enactment

Some P-M-Ts may have seen some of these constructions before, but it is unlikely that they have seen them all. The competitive aspect of letting groups report constructions as soon as they find them saves time and leads to appreciation of constructions found by those still struggling to find some. P-M-Ts tend to exclaim about new interesting ideas, and sometimes comment on how they might use them. This can add to the pervasiveness of success: by emphasizing what aspects of successes were valued by others. Table 3, Task 2 (cutting lines into n equal segments) provides an example of multiple small successes contributing to new insights. The optimistic characteristic Success as Pervasive can be built through such multiple opportunities for successes. The intensity of this collaborative whole class activity is demonstrated by the spontaneity with which students kept coming to the board to build on ideas or add new ideas.

Task 3 Features and Enactment

Because proofs of constructions (see Table 4, Task 3) were not part of the secondary mathematics curriculum in Victoria, many opportunities existed to discover complexities. Implementing Task 3 through group work, and sharing ideas with class at intervals, can gradually shift P-M-Ts from intuitive verbal arguments based on visual images, to analysis and rigorous justification (Dreyfus 1994). New mathematical ideas developed in Tasks 1 and 2 (congruence, similarity, constructing) are used in unfamiliar sequences/combinations to develop unfamiliar geometric proofs.

Task 3 provides many opportunities for positive affect and optimism building as progressive small successes bring the class closer and closer to proving why each construction works. Activity associated with the bisecting a line proof (Table 4, Task 3) created surprises that: (a) congruent triangles could be used to develop proofs; and/or (b) overlapping triangles can be used together to prove equalities and demonstrate congruence.

As using congruence for proving is unfamiliar to most or all P-M-Ts (either not understood, or never learnt), the lecturer sometimes needs to contribute an idea without giving specific information about how it might be used (e.g., drawing attention to the possibility of other triangles, see Table 4). Otherwise, the size of the challenge can be too great and panic or anxiety can result (see Fig. 1). The nature and timing of possible lecturer interventions is crucial to retaining spontaneity (for flow).

Once one group recognizes congruence as a tool to show equality of features, the first reporting session starts (to share this idea while it is still in un-crystallized form). This gives other groups an idea to pursue; find whether/how to use it. Initial reports tend to include ideas like: (a) a verbal clarification of what needs to be shown; (b) construction lines added to diagram; (c) identifying what is already known; or (d) labeling to make communication easier. In this way, students begin to develop strategies for undertaking and communicating geometric proofs. During group activity, if some have not spontaneously focused their questions the lecturer asks groups clarifying questions like: "What are you trying to prove?" and "What do you know that could help you to show what you have stated you want to show (e.g., those two angles are equal)". The lecturer also asks questions to stimulate P-M-Ts' evaluations of the reasonableness of their findings; rather than affirming or disputing them. The lecturer decides on the order of reporting so every group has something new to add.

Meta-cognitive Overlay

As P-M-Ts do not tend to focus on pedagogical aspects of the tasks (during task completion) for reasons discussed earlier (see Engaged to Learn (MAP) in Fig. 2), focus on pedagogy occurs after new mathematical ideas have developed. At that time, the lecturer focuses a 'meta-cognitive overlay', which involves drawing attention to the pedagogical strategies that contributed the development of insight. Initially P-M-T's are asked to reflect on the experience and identify what assisted their learning. Then the lecturer identifies aspects that were not 'seen'. P-M-Ts appreciate that something significant has occurred for them during this task sequence and want to know more about how to recreate this for their students.

Discussion

This section discusses how mathematical knowledge, pedagogical knowledge, and optimism were built through this three-task sequence. The Engaged to Learn (MAP) Model (Fig. 2) frames the discussion.

Mathematical Knowledge

Learning through the Engaged to Learn approach requires tasks that encourage P-M-Ts to step outside their present understanding. The tasks enable access to creative thinking through experimentation, and the use of ideas developed earlier in the task sequence. Students can gain understanding of mathematics that was previously not known, or only known as fragmented rules and procedures. By using mathematical ideas they possessed, in unfamiliar combinations and sequences, P-M-Ts built knowledge of congruence, similarity, and geometric constructions, and linked them through proofs as they developed mathematical insights.

P-M-Ts did not focus on pedagogical aspects of the task at that time because they became so engaged in the mathematics that this was sufficient challenge on its own. Table 2, Task 1, shows examples of where flow occurred when the second triangle was found, and students began to wonder: "Why?" "Are there more?" "Are there always two triangles?" "Can we generalize?" The spontaneity of their actions was demonstrated in the questions they asked themselves when a second triangle was discovered. Success in this case included: recognizing only two triangles were possible, why, and hopefully generalizing the conditions.

P-M-Ts' understanding of the meaning of congruence and purposes for learning about it developed through these tasks. For example, they found that overlapping triangles can sometimes be useful for identifying equalities, and that this can be a strategy for identifying sufficient equalities to demonstrate congruence. In addition, they found that the construct of congruence could be used in formulating proofs. Crucial to the mathematical learning that occurred was tasks that stimulated interest beyond P-M-Ts present understanding and elicited spontaneously focused questions that structured their future exploration (e.g., Table 2, Task 1, Column 4).

Pedagogical Knowledge

Task 1 was intended to allow P-M-Ts to explore both mathematical and pedagogical challenges. Instead, they disregarded pedagogical challenges because they became so focused on mathematical challenges. Considering both simultaneously was 'too big a leap'. Guided by the Engaged to Learn (MAP) Model in Fig. 2, I learnt to watch for indications of flow and wait for successful outcomes before focusing a metacognitive overlay around pedagogy that enabled the development of their insights.

Many P-M-Ts spontaneously state, after this task sequence, that they can now see how such an approach is possible. Their intent post-task attention to, and reflections on, the pedagogy in which their activities were embedded helped to build their own pedagogical understandings. As they want to provide similar pleasurable experiences (for their own students) in overcoming mathematical struggles, they are interested in analyzing pedagogy that appeared to them at the time to have minimal lecturer intervention.

The spontaneous whole class brainstorming in Task 2 (see Table 3, Column 2, (i)) gave P-M-Ts opportunity to experience intensity during the collaborative development of new knowledge. Task 3 provided opportunities for 'big leaps' in mathematical knowledge: through intense focus accompanied by high positive affect as mathematical connections were progressively made (see Table 4, Task 3, Columns 2, 3, 4).

Formulating the Engaged to Learn (MAP) Model has raised my sensitivity to simultaneous mathematical and pedagogical challenges that can be too great for some P-M-Ts to overcome. Intense P-M-T focus on mathematics now alerts me to the need to delay drawing attention to pedagogy. The pedagogical understandings developed through this subject have confirmed that it is often appropriate to delay focus on pedagogy until P-M-Ts have built deeper mathematical understandings.

Optimism Building

High positive affect of different types can result from various activities occurring during this task sequence. For example:

- Intensity during search for ways forward (e.g., Table 3, Task 2, how to complete the construction);
- Excitement at recognizing some previously known mathematics may be useful (e.g., Table 4, Task 3, congruent triangles help); and,
- Pleasure when everything suddenly becomes clear (e.g., Table 2, Task 1, why there are sometimes two triangles)

Identifying complexities that were not apparent earlier in a task occurred during each activity, and flow situations were a feature of each. Thus, the situations are expected to be optimism building. The idiosyncratic questions P-M-Ts asked themselves sustained the exploratory processes. Where P-M-Ts were unable to sustain their explorations alone, the lecturer asked questions that structured *future* spontaneous exploration, or *sometimes* needed to add partial information to elicit further thinking. These interventions were followed by further spontaneous activity confirming opportunities for spontaneity were not eliminated by the type of intervention provided. For example, when the lecturer added the additional construction line to the diagram to assist P-M-Ts to find ways to cut lines in to n equal parts (see Table 3, Task 2, Column 2, (g)), many later spontaneous explorations drew upon this added information. This led to many small successes, and resulted in collaborative solving of problems (see Table 3, Column 2, (i) (i–v)). Lecturer questions sometimes focused on the optimistic activity of examining the situation to see what can be varied to increase chances of success (Table 3, Task 2, Columns 2 and 3) and sometimes on eliciting generalizations leading to insights (e.g., Table 2, Task 1, Column 2, (d). Modeling of optimistic activity by the lecturer (and other P-M-Ts) should help to build optimism in more P-M-Ts over time as they learn these new strategies that can reduce the magnitude of challenges they face.

Excitement that accompanies the development of insight is amplified when reporting of these insights to others, and receiving acknowledgement of their usefulness. This consolidates the personal and pervasive dimensions of optimism. Thus, ongoing building of optimism occurs through flow situations and the amplification of these optimism-building aspects during reporting sessions. Due to the varied nature of activity in this task sequence, many different students contributed new ideas that add to the class achieving their goals. Thus many different students had opportunities for optimism building experiences.

Through attention to psychological factors, many P-M-Ts change their perceptions about the nature of mathematics teaching (see opening quote from Hayley), and their implementation capacity because optimism has built (see quote from Jayde). It is hoped that sharing these findings will lead to further conversations with other secondary mathematics educators that will assist me to develop my pedagogy further, and also stimulate ideas of others.

Optimism building through overcoming mathematical challenges should increase P-M-T's inclination to overcome both mathematical and pedagogical challenges because optimism is not content specific (Seligman et al. 1995). Overcoming challenges during pedagogical situations includes responding flexibly to student responses and to other difficulties that may arise during the learning of mathematics. Building optimistic P-M-Ts, and examining the effects of this increased optimism upon the pedagogy they employ is an important area for further study. I look forward to hearing from others who try some of these ideas in their classes.

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