Chapter 4
Learning Mathematical Reasoning in a Collaborative Whole-Class Discussion

My interest in pursuing this study was driven primarily by my personal experience as a teacher as well as being faced with a new curriculum that I was not sure how to implement in my classroom. "Where am I going to use this mathematics?" is a popular question among my learners. This question emanates from learners seeing mathematics as unrelated pockets of knowledge rather than a set of related and useful topics. I have also observed that when learners understand and relate a particular topic to their existing knowledge, this question seldom crops up. I believe that learners' inability to see mathematics as a worthwhile human activity is in part due to the low level of mathematical reasoning and collaboration in classrooms. Learners who learn mathematics through mathematical reasoning may find the mathematics more meaningful. Mathematical reasoning allows learners to form connections between new and existing knowledge (Ball and Bass 2003), and this integration of knowledge may support sense-making on the part of learners and the ability to see mathematical activity as worthwhile. Mathematical reasoning enables the development of conceptual understanding and productive disposition (Kilpatrick et al. 2001), which allows learners to draw on their concepts in other situations and experience mathematics as something they can understand and relate to. Learners who engage in mathematical reasoning may be in a better position to connect school mathematical activity to other activity.

I view collaborative learning as a communicative process whereby two or more parties gain new knowledge as a result of their interaction. Collaborative learning not only refers to an exchange of knowledge between the parties, but the interaction itself serves as a catalyst for the formation of new knowledge by the parties concerned (Mercer 1995). In my class, I think of collaborative learning as a joint venture between learner/s and teacher and among learners themselves. This collaboration is governed by the pursuit of knowledge for the development of learner and teacher. How we reason mathematically or allow our learners to reason mathematically is in part dependent on the nature of collaboration between the parties. The nature of the learning that occurs is a complex interplay between individual and social construction (Hatano 1996; Wood et al. 1992).

This chapter represents a response to the new curriculum developments in South Africa. Motivated by a need to teach in a way that will make mathematics more meaningful to my learners and guided by curriculum change, I decided to explore

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the extent to which this could be achieved in my own teaching. In thinking about how to conduct the study, I posed the following questions to myself:

- What do I understand by mathematical reasoning?
- Why pursue the teaching of mathematical reasoning?
- What is collaborative learning and how does it impact on the teaching and learning of mathematical reasoning?

What Is Mathematical Reasoning?

An important part of all learning, including learning how to reason mathematically, is that new knowledge is always connected to current knowledge, and in fact restructures current knowledge if true learning is to occur (Hatano 1996). So, as we try to develop mathematical reasoning among learners, it is important to see whether and how they make these connections and transform their existing ways of reasoning. As discussed in Chap. 1, mathematical reasoning is intertwined with the other strands of mathematical proficiency (Kilpatrick et al. 2001): conceptual understanding, procedural fluency, strategic competence, and productive disposition. These strands suggest that teaching mathematical reasoning requires far more than merely following a “recipe”. If we take seriously the notion of mathematical proficiency, we are faced with an even bigger challenge, the simultaneous development of a range of skills and abilities that is required for learners to be regarded as mathematically proficient.

Kilpatrick et al. (2001) argue that “the strands complement each other but at the same time the reasoning strand, called adaptive reasoning, is the glue that holds everything together” (p. 129). In analysing one learner’s developing reasoning in this chapter, I show how mathematical reasoning provides a link with the other strands, particularly conceptual understanding and procedural fluency. I also draw on the notion of mathematical practices (Ball 2003). These include *representational practices*, *justification*, *generalization,* and *communication*. These practices are seen as vehicles to achieve the mathematical proficiency discussed above.

The Open University (Open University 1997) suggests that mathematical reasoning unfolds as the learner asks and strives towards answering three important questions while engaged in mathematical activity:

- *What is it that is true?* This question arises as the learner looks to find patterns and regularities that can be rendered as evidence to justify an idea. If enough evidence is found to convince the learner, s/he can formulate a conjecture. This is where we see so many of our learners falter and regard the “evidence as proof” (Chazan 1993). Learners may prematurely draw generalized conclusions based on the measurement of a few examples. For example, learners may conclude that the interior angles of a triangle always add up to 180° after having measured only a few or just one set of interior angles of a triangle.

- *How can I be sure?* This question arises as the learner is confronted with the possibility that the evidence collected may not account for all cases. There now exists the need for some reasoning that would include the evidence in the form of a generalized argument or proof. Without the process of gathering evidence and formulating conjectures, the learner at times regards this proof as merely evidence of another case (Chazan 1993). My learners have often viewed my explanation of a proof of a theorem as an example of how to approach the problems in the exercise and not as an explanation of why the theorem is true.
- *Why is it true?* At times, even a logical explanation that explains the truth of a statement is not enough to convince someone as to why something is true. As De Villiers (1990) points out, the explanatory function of proof or arguments is very different from the verification function. It is likely that learners will need to understand *why* something is true in order to accept it, rather than just verification *that* it is true.

All of the above conceptions of mathematical reasoning, as making convincing and explanatory arguments; as intertwined with the other aspects of mathematical proficiency; as involving a number of important practices; and as restructuring current knowledge and practice, informed this study. However, I still had to answer some other important questions, the next one being why should we teach mathematical reasoning?

Why Teach Mathematical Reasoning?

I argued earlier that I view mathematical reasoning as the vehicle to sense-making of and in mathematical activity. I refer to making sense of the mathematics itself, not necessarily to making links with everyday life. My assumption is that only through making sense of the mathematics can we truly move to sense-making as a worthwhile everyday life activity. The National Curriculum Statement (Department of Education 2003) expresses the vision of a learner who is able to “transfer skills from one context to another” and to “think logically and analytically as well as holistically and laterally” (p. 5). This vision suggests a thorough conceptual understanding of mathematics among learners and the capacity to readily identify situations where their knowledge is of relevance.

Boaler (1997) talks about flexible conceptual knowledge. She worked in two schools that were homogeneous in terms of the socio-economic status and educational background of their learners. The only noticeable difference was the way in which the two schools approached the teaching of mathematics. On the one hand, Amber Hill had a typical textbook approach with lessons consisting of rule-based, procedural activities with much drill and practice. “A typical day of maths in the old apartheid days”, was my immediate response. On the other hand, Phoenix Park adopted an open-ended, problem-solving, real-life approach to teaching mathematics, which is what our new curriculum aims at. Boaler’s research concluded that learners gained vastly different experiences of mathematics and developed different forms of mathematical knowledge. The majority of learners from Amber Hill were unable to apply their knowledge to new problems and situations. This suggested that they developed knowledge consisting primarily of memorization and applying rules that could only be applied within a school setting. Learners at Phoenix Park, however, developed more flexible knowledge, the kind of knowledge that enabled them to solve new problems they encountered. Boaler's study inspired me to develop a teaching approach closer to that of Phoenix Park. Collaborative learning was the key to developing mathematical reasoning in this approach.

Collaborative Learning and Mathematical Reasoning

The National Curriculum Statement puts forward the following vision for a post-apartheid South Africa: "To heal the divisions of the past and establish a society based on democratic values, social justice and fundamental human rights" (Department of Education 2003, p. 1). This statement acknowledges diversity and the need for equity, promotes the integrity of each individual with the power to affect decisions and suggests that a way to achieve equity is through the promotion of social justice and fundamental human rights. To achieve this, learners need to "work effectively with others as members of a team, group, organization and community" (p. 2). This important notion is picked up later in a focus on mathematics: "mathematics enables learners to work collaboratively in teams and groups to enhance mathematical understanding" (p. 10). Taking these two assertions together, we see that collaborative learning is both an end and a means (Brodie and Pournara 2005). We need to develop skills and dispositions towards collaboration in learners as democratic citizens and also to use collaborative learning to aid mathematics learning.

Developing a social conscience based on *democratic rule, social justice, and human rights* can be obtained within the context of collaborative learning. It would be difficult if not impossible to teach learners to value other people and their opinions, without learners actually learning together from each other. It is the relevance to the learning of mathematics that tends to be more challenging. There is, from South African classrooms, an evidence of teachers using group work without much mathematics learning happening (Brodie and Pournara 2005). I think of collaborative learning as a joint venture between learner/s and teacher as well as among learners themselves. How we reason mathematically or support our learners to reason mathematically is in part dependent on the interdependence between the parties in collaboration.

Mercer (1995) strengthens my ideas about collaboration with the following quotations: "I suggest that we need to recognize that knowledge exists as a social entity and not just as an individual possession" and "the essence of human knowledge is that it is shared" (p. 66). Mercer's ideas resonate with those of Lave and Wenger (1991) who argue that learning occurs in communities of practice, with shared goals and practices. The idea is to create such a community in the classroom, where the teacher takes a leading role in helping learners to develop interactions and practices as a community. Teaching mathematical reasoning demands that learners be able to voice their mathematical thinking, so that mathematical discussion around their assertions can generate an "intellectual ferment" (Chazan and Ball 1999). Learners need to move away from a dependence on the teacher as the only mathematical authority in the class towards a position that Davis (1997) refers to as a "community-established standard: a collective authority" (p. 369). As argued in Chap. 1, this authority comes from the discipline of mathematics. Developing a broader sense of authority requires changes in the way learners and teachers view their roles in the classroom. Teachers need to become what Davis (1997) terms "hermeneutic listeners" (p. 369), which is genuine listening as a participant in the conversation in order to understand what learners are saying. We refer to this kind of listening "with" learners. This is very different from evaluative listening, i.e. listening *for* the right answer, which many teachers do most of the time.

Listening to learners in better ways does not necessarily help teachers to know how to respond to learners' ideas (Heaton 2000). In their article aptly named "Beyond being told not to tell", Chazan and Ball (1999) suggest practical ways in which teachers can act in classroom discussions, without giving the answers, that may focus and give direction to a particular discussion. These include

- Rephrasing learners' comments and helping the class to hear them
- Asking for clarity when they think learners' assertions are not clear and
- Focussing learners' attention on a particular aspect of a discussion

As teachers do this, focussing learners on the norms of participation is important, particularly sociomathematical norms (Yackel and Cobb 1996, see also Chap. 1), where an explanation consists of a mathematical argument, not simply a procedural description or summary; mathematical thinking involves understanding relationships among multiple strategies; errors provide opportunities to reconceptualize a problem, explore contradictions in solutions, or pursue alternative strategies; and collaborative work involves individual accountability and reaching consensus through mathematical argumentation (Kazemi and Stipek 2001).

Summarizing My Perspective

In the above pages, I have made a number of arguments, which informed how I conducted this study and analysed the data. First, I argued that mathematical reasoning is made up of a number of processes. The learner makes observations, tries to provide evidence and explanations, and through connecting these with existing knowledge, restructures this knowledge (Hatano 1996). Proficiency in "procedural fluency" and "conceptual understanding" (Kilpatrick et al. 2001) is needed for such restructuring. Key to enabling restructuring is explaining, communicating, and justifying conjectures and claims, which are features of "adaptive reasoning" as argued by Kilpatrick et al. During this communicative process, we see the learner evaluating and refining new knowledge.

I further argued that learning mathematical reasoning as part of mathematical proficiency (Kilpatrick et al. 2001) is best achieved through collaboration in communities of practice (Lave and Wenger 1991). Such communities are governed by norms of practice, (Yackel and Cobb 1996) and as teachers, we can and should take the lead in developing classroom norms that deeply engage learners (Kazemi and Stipek 2001). Teachers can listen carefully and make a range of moves (Brodie 2004b) which do engage learners' thinking. Taking account of the above, I embarked on a study to see whether what these researchers are claiming is possible in my classroom in South Africa.

My Classroom

My pseudonym in this study is Mr. Daniels; there is a detailed description of my school context in Chap. 2. I refer to some of it briefly here. This study was conducted with one of my Grade 11 classes, consisting of 35 learners with a range of mathematics abilities. The class was situated within a school of 1,600 learners, which is well integrated in terms of historically racial divisions. A teaching staff of 63 puts the teacher–learner ratio at about 1:23. Actual class sizes average 33 learners per class. Although the school is situated in a middle-class suburb, a large number of the learners travel to school from lower income areas. English is the language of instruction at the school and is not the first language of the majority of the learners. My classroom is relatively well resourced with desks and chairs for every learner. The building structure in general is well maintained. Aside from the writing board, I also have an overhead projector and screen at my disposal.

As part of the collaboration in this project, I worked with two colleagues to develop a series of tasks that we hoped would elicit mathematical reasoning in our Grade 11 classes. We drew on a number of resources, including texts that were in the process of being written for the new curriculum. The tasks that we developed have been analysed in Chap. 2. I planned to use the tasks over a week in my Grade 11 class. I structured the work as follows: learners had some time to work on the tasks themselves, then they came together in small groups of three or four learners to discuss their findings, and finally, the groups reported back to the class and we had a whole-class discussion. The lessons were videotaped and I wrote reflections after each lesson, which helped with my analysis.

The Analysis

There were three important issues in my analysis of the data collected:

- The first was how to *select* certain parts of the data to analyse.
- The second was how to see the analysis in *context*.

- The third involved *structuring the writing* to make it easy for the reader to understand my argument even though I can present only some of the data.

As I struggled to select data, I decided to focus on one learner's development in one lesson. It was not possible to do more in the scope of this study and I believed I could achieve more depth of analysis through focussing on one learner. My decision to focus on Winile in particular was because of her visible participation throughout the lessons, which allowed me to plot a developmental sequence of her learning. The analysis therefore focusses on the development of Winile's reasoning through the lesson and how collaboration with me and other learners made this possible. Focussing on Winile's learning enabled me to understand how her learning as an individual both influenced and was influenced by the social interaction in the class.

To isolate a learner from a whole-class discussion in order to analyse and follow her mathematical reasoning is not entirely possible. This is due to the collaborative learning that takes place. In such an analysis, the question arises as to how to know which statements influence each other. One learner's statement may or may not motivate another learner to say something. To link contributions in discussions to each other is a difficult task, and it is important to always remember that there is a variety of influences on learners' development. This means that the context of any utterance needs to be considered very carefully and from a number of perspectives.

The analysis presented in the next section was obtained from thorough analysis of the video and transcript focussing on the claims that Winile made in one lesson, over a time period of about 36 min. It is clearly not feasible to present all of these data here. So, I need to make another selection, which is how to show the reader what I have seen, in much less time and space than it took me to see it.

Winile's Learning

The analysis focusses on Activity 2. The content of the activity was how to think about the changes affected by the horizontal translation of the graph of $y=x^2$ to the graphs of $y=(x-p)^2$ where p was 3 and −4 respectively. Winile's group had just reported back on their findings and Michelle posed a question, asking why the graph for $y=(x+4)^2$ has a turning point of −4. She suggested that the +4 inside the bracket contradicted a turning point of −4 and asked Winile to explain this. This served as a catalyst for a fervent discussion, which resulted ultimately in Winile formulating her new conceptual frame for understanding graphs and equations. The analysis follows Winile's learning in five steps: (1) making observations; (2) explaining and justifying claims; (3) connecting her claims to the mathematical representations; (4) restructuring conceptual understanding; and (5) using her new conceptual frame to test other claims. I describe each of these and show how the classroom collaboration supported Winile's shifts.

Making Observations

Winile's journey started as I called her group to share their findings with the rest of the class. Winile became the reluctant spokesperson for the group. In the extract below, Winile hesitantly indicated that the turning points of the graphs $y=x^2$, $y=(x-3)^2$ and $y=(x+4)^2$ differ; the y-values of corresponding points stay the same; the x-values change; and the sizes of the graphs are the same and the equations of the graphs differ.

Winile:	We said they are different on the turning point, and the equation, but the y-axis stays the same, and the size of the graph also stays the same, and *(inaudible)*
Mr Daniels:	Okay so the, what stay the same
Winile:	The, the y-axis.
Mr Daniels:	The y-axis. The y-axis stay the same.
Learners:	*talk over each other, inaudible*
Winile:	The y values stay the same but x-axis changes.
Mr Daniels:	Okay, can we speak one at a time. Let's speak respectfully to one another here. If you've got a question, just raise your hand.
Learner:	(*inaudible*)
Winile:	What
Learner:	(*inaudible*)
Winile:	The y-value stays the same, the x-value (*inaudible*) the turning point (*inaudible*)
Mr Daniels:	Okay, so you say the equation changes, the y-value stays the same,
Winile:	And the turning point,
Mr Daniels:	And the turning point stays the same.
Winile:	No, it changes *(shakes her head and looks at her notes)*
Mr Daniels:	The turning point also changes

We see here that Winile's initial claims were merely observational and she did not see a need for justification. In fact, even to enable her to make a proper report back required a lot of support from me. This support was in the form of keeping other learners quiet, establishing social norms so that Winile could be heard, and also helping her to voice her ideas, and in some cases rephrasing (Chazan and Ball 1999) or revoicing them (O'Connor and Michaels 1996).

Explaining and Justifying Assertions Made

After this, Grant, a member of the same group, came up to comment on the next part of the task. Grant tried to explain that since the x-values changed and the y-values stayed the same as the graphs were shifted left or right, the equations must change. I pressed him to say more specifically how the graph had changed and Grant struggled, looking at his book and searching for an explanation. His attempt follows in the next extract:

Grant:	Sir, uh, the graph's position has moved, so when you, however many positions its moved, you either add it or minus it, onto your equation.
Winile:	Can I just make it simple sir, you substitute the x-value with the variable, we change the equation and then the y, uh, variables never changes (*inaudible*).
Mr Daniels:	Yes, okay, now how d'you mean, just explain what you said, he said that it changes, what did you say? I didn't follow nicely.
Grant:	If the graph, the graph's position has changed, on the x-axis
Mr Daniels:	Right.
Grant:	Therefore, so then you either add onto your equation, its moved how many spaces, or you minus it. Now do you understand?

After Grant's initial contribution, Winile stepped in a little more confidently. Her assertion is still vague; however, it does show a different interpretation from Grant's, which is also vague. This response from Winile suggests that she began to acknowledge a need to explain and justify claims, realizing that Grant's claim needed explanation for her and probably the rest of the class.

Winile's move shows how learning collaboratively feeds into the process of mathematical reasoning. Winile did not see a need to clarify or explain her own claims, yet hearing another learner's claims, which she had been party to, prompted a need to explain. In making her explanation, Winile started to make connections between her observations and the equation. In a sense, she was justifying why the equation must change. She was reasoning at a higher level, brought on by realizing the need to explain Grant's claim to the class. My role in this interaction was to press Grant to explain his claim, which also supported Winile to do so.

Connecting Observations with Mathematical Representations

As learners in the class sought more clarity from Winile with regard to Grant's assertions, Winile realized that she needed to switch representations. She came up to the overhead projector and tried to explain her concept as follows:

> It means that this x uh, here, because when you move three times to your right *(writes* $y = x^2 + 3$*)*, or you *(writes* $y = x^2 - 3$*)*, it means that you move to the left, this means when you move to the right three times, that's what we trying to do, that when you move the graph three times, you supposed to add it three times, and when you move it three times to your left then you subtracting three times

These connections between equation and graph are mathematically incorrect. However they do show that she was starting to make conjectures about certain patterns she had observed.

The use of alternate representations by Winile suggests that again she was reasoning at a higher level. Not only was she explaining observations, but she was making connections between her graphical observations and the representations in equations. The need to use a written representation to illustrate the translation of the graph to $y = x^2$ to $y = x^2 - 3$ (translation of 3 units to the left) and to $y = x^2 + 3$ (translation of 3 units to the right), served as a catalyst for making these connections. The need to explain to others more effectively once again served as a catalyst for mathematical reasoning. Winile's reasoning was extended to expressing the changes she observed in an alternate representation. She had progressed to not only connecting various aspects of the mathematics but also producing mathematical representations with which to express these connections. Although these representations were mathematically incorrect, they demonstrate her reasoning in relation to the task.

At this point, Winile was interrupted by Michelle who wanted to ask her a question. The interaction involved a few learners and is captured in the following transcript:

Michelle:	Okay, can I ask a question
Mr Daniels:	Okay.
Michelle:	Okay, look on task one right. You said that if it is a positive, you move to the right and if it is a negative, you move to the left. So now, can you please tell me why on your second drawing, where it says y equals x minus three squared (*looks at Winile*) can you see that? Say yes Winile if you understand.
Winile:	Yes, I can see it.
Michelle:	Alright, so now how come in the bracket there's a negative but where the turning point is, is a positive. That's what I would like to know.
Mr Daniels:	Okay, Lorrayne (*interruption by learners*) Carry on Lorrayne
Lorrayne:	Sir, you have a negative three in the bracket and it's a square, when you square something, remember, Sir said when you square it, it becomes positive.
Learner:	If it's a negative
Lorrayne:	Ja
Michelle:	And then if you look at y equals x plus four, why is it that the turning point is a negative.
Learner:	But the equation is positive
Michelle:	And the drawing is positive.
Learner:	I asked that too. *(Some learners laugh)*.
Learner:	I'm also asking the same question.

In the above extract, Michelle and Lorrayne co-produced an important question relating the equations to the graph. It was a question that had occurred in a number of groups and so was shared by learners. What is notable in this extract is how the learners worked together and spoke to each other, with almost no intervention from me, except to give Michelle permission to talk and to keep the class quiet so Lorrayne could speak. Brodie (2007b) argues in relation to this episode that when learners share important provocative questions, they are more likely to engage in real conversation.

Winile was silent during the above interaction and continued to be silent as a number of other learners discussed Michelle and Lorrayne's question. Winile did not return to her seat however, but took a seat in front of the class where she listened intently to the ensuing discussion. The discussion continued for some time, until I felt the need to intervene and make an important point as follows:

Mr Daniels:	Okay now, what is a point? A point is made up of what?
Learner:	x- and y-co-ordinates.
Mr Daniels:	x- and y-co-ordinates. Good! So what is the x-co-ordinate there?
Learner:	It's minus four is your x co-ordinate
Mr Daniels:	Good. So what is negative there? The turning point is negative or is it one of the co-ordinates that's negative? Okay, let's hear.

Brodie (2007c, see also Chap. 9) argues that although the above interaction might seemed somewhat constrained, in fact it served an important function in moving the learners' discussion and thinking forward, in that it reminded them to consider both co-ordinates of the turning point, rather than only the *x*-value. Up until this point, they had been talking about the turning point as −4, which did not help them to see that a point is a relationship between *x* and *y*, given by the equation. This interpretation is borne out by Winile's following contributions. The intervention helped to support her to move to the next level of her reasoning trajectory.

Reconstructing Conceptual Understanding

Immediately after my intervention above, Winile emerged from being a silent participant with new ideas to contribute:

Winile:	The positive four is not like the x, um, the x, like, the number, you know the x *(showing x-axis with hand)*, it's not the x, it's another number. For that when you do the equation you get some sense from the answer you get, cause without that p, that minus p, your equation will never make sense.
Learners:	*(murmuring)*
Mr Daniels:	Can I just get back to, That's good, Winile
Learners:	Sshh
Mr Daniels:	Does people want to make clear of what Winile is saying?
Learners:	Yes, *mutter, talk over each other as Winile comes up to OHP*
Winile:	You see, Michelle when you've got this *[writes $y=x^2$]*, you substitute this with a number, isn't it. Like you go, whatever, then it gives you an answer. *[substitutes 3 for x and gets 9]*
Learner:	Yes
Winile:	You see when you got this, plus three *[writes $y=x^2+3$]*, you have to substitute this with the, that with like the one, zero, one two, three *[Draws numberline, x axis]*. Your turning point is here. You have to substitute this with this negative one here, plus three. Do you understand? This three *[circles the 3 in $y=x^2+3$]* is not, is not part of the, this x, uh, variables. Its the given (*inaudible*) Get it?

In this extract, Winile justified her claims similarly to how she explained Grant's assertions earlier. First, she made a verbal contribution, which was difficult for others to understand. She then came up to the overhead projector and wrote equations to explain her new understanding. As she explained the second time, her explanation is not only clearer to the listener but her explanation has progressed to become more focussed and connected, even though she is still using the incorrect equation.

The clarity of Winile's mathematical reasoning was evident as she explained that the +4 and the −3 in the equations were not the x-values of the turning point but as she put it "*some other*" values. She affirmed that the equations represented the relationship between the x and y variables and that the x-values must be substituted into the equations to give the y-values. During these assertions, it was evident that Winile was more confident and self-assured that she was on the right track. Winile's reasoning had evolved to a point where she was in a position to evaluate previous claims and adapt them to her understanding. She was now in a position to make the appropriate connections between the value of the turning point and the representational equation. This learning came after a relatively long period of silence from Winile where I can only assume that she was quietly reasoning and adjusting her own understanding as the class discussion involved other learners.

This highlights again the quality of collaborative learning which was present in Winile's reasoning. She was able to modify her assertions by listening to the discussion that prompted her own reasoning. Her explanation to the class facilitated their understanding but also assisted in refining her own understanding of the issue at hand. With this understanding, she confidently answered Michelle and Lorrayne's question.

Testing Other Claims

After this, David indicated disagreement with Winile, arguing that the turning points could be determined by taking out the +4 or the −3 from the bracket, moving them to the other side of the equal sign and changing the signs. Again, she had to justify her ideas, which she did as follows:

> We supposed to get the y, aren't we supposed to get the y, what the y equals. We're not supposed to get what x is equal to, we getting what y is equal to. So we supposed to, supposed to substitute x to get y.

This justification supported Winile to move to yet another level of mathematical reasoning. She emphasized the fact that we use the equation to get the y-value by substituting the x-value into the equation. In doing this, she tested her own conceptual frame against that of David's and used her understanding to extract the weaknesses in David's argument. Winile did not wait to be invited to give a response to David, but confidently and openly engaged David's assertions. She argued (laughing):

> Okay sir, he's just telling us where to put like, the turning point of the graph, and we want to know why, the y-value is, we want to know what the y-value is and you're telling us the x-value.

Winile was using her conceptual understanding to test and spot the failures in David's argument. This places her in a position to challenge David's assertions. She continued to do this for the rest of the lesson.

The Teacher's Role

The above analysis indicates how important the collaborative learning in the class was to Winile's learning. In particular, the role of the teacher was central to this collaboration. In the above analysis, I indicated a number of roles that I played in supporting learners to talk, in steering the collaboration, in pushing for justification, in remaining silent when I needed to, and finally, in making substantial mathematical contributions when necessary. To further analyse my own role, I came up with three main categories, each of which contains some important teacher moves.

Establishing Discourse

By "establishing discourse," I refer to my actions that attempted to create a climate of interaction, which could support the learners to participate in the discussion and to reason mathematically. One way in which I did this was to create social and socio-mathematical norms in the classroom (Yackel and Cobb 1996). Social norms included speaking one at a time; raising one's hand as an indication that one wants a speaking turn; listening to each other; and building on each other's ideas. Socio-mathematical norms refer to the nature of the mathematical interaction. For example, after Michelle and Lorrayne had asked their question, Candy tried the following response:

Candy:	Sir, couldn't it just be like a basic thing, that if it's on the positive side then your equation is negative and if it's on the negative side then your equation is positive? Can't it just be like that *(laughs)*
Michelle:	I can't accept that
Learners:	Mutter, talk over each other
Mr Daniels:	Okay. Let's … Say that again.
Michelle:	I can't just accept that.
Mr Daniels:	So, I'm not expecting you to accept it.
Michelle:	No, I'm just saying that I can't …
Mr Daniels:	That's good. That's what I'm saying. I'm saying it's good that you don't just accept it

Candy was asking whether we should just accept the fact that the signs were different. Michelle indicated that she could not just accept that, implying that she needed a better justification. I praised her position as valid, indicating that I did not expect nor want her to just accept it and the discussion continued to try to find the justification. This helped to establish the socio-mathematical norm of requiring a justification and may have helped Winile to restructure her understanding to include the need for justification.

The second way in which I established a particular kind of discourse is by modelling how to participate. I listened attentively to try to understand what the speaker was saying and I asked questions if I disagreed with learners' assertions or needed some clarity on their ideas. The following extract occured when I asked Michelle to clarify the question she asked Winile.

Michelle:	And then if you look at y equal x plus four, why is it that the turning point is a negative.
Learner:	But the equation is positive
Michelle:	And the drawing is positive.
Learner:	I asked that too. *(Some learners laugh)*.
Learner:	I'm also asking the same question.
Mr Daniels:	What question are you asking?
Michelle:	The question …
Mr Daniels:	Yes.
Michelle:	Look at our drawing where …
Mr Daniels:	Okay. Where's my drawings? (*finds drawings*)
Michelle:	Where it says y equals x plus four on the left hand side.
Mr Daniels:	Right.
Michelle:	Our turning point is a negative four.
Mr Daniels:	Okay
Michelle:	Then Lorrayne that said with the one on the right, where it says y equals x negative three squared, and the turning point is a positive. Because you squaring it, it will become a positive. But what happens with um, the one on the left?
Lorrayne:	The negative one. The equation is positive but the graph is on the negative side.

In the above extract, I model how to listen by asking the learners "what question are you asking", by explicitly showing them that I was looking for my drawings in order to understand their question and by indicating agreement as they spoke and I understood. This is important because many learners have not participated in discussions previously and may not know how to listen and contribute appropriately.

Framing Discussion

By framing discussion, I refer to the actual mathematical content that I used to help the learners make progress. The best example for this is the one quoted above, where I used a sequence of closed and directive questions to remind the learners that a point consists of two co-ordinates. I did not do this because I wanted them to remember that as a fact in and of itself. Rather, it was an important mathematical fact that could help their thinking (see also Brodie 2007c and Chap. 9). By reminding learners that a point consists of a co-ordinate, I focussed their thinking onto the relationship between the *x* and *y*, helping to move the discussion forward.

Lesson Flow or Momentum

With lesson flow, I refer to the movement and progression of discussion. Does the discussion show any progression or is it stagnant on one point, which does not seem to be resolved? Is there any discussion taking place at all? The ability of the teacher to negotiate between speaking turns and free dialogue plays a key role in the lesson flow. The ability to assess when to intervene and when to allow discussion to take its course is an important consideration for a progressive and meaningful lesson flow. For this, the teacher needs to be on the pulse of the discussion, constantly aware of the meanings learners are constructing within the discussion as well as the "social and emotional tone of the discussion" (Chazan and Ball 1999).

Conclusions and Implications

This study set out to analyse the ways in which learners collaboratively engaged in mathematical reasoning and how they learned to reason mathematically through collaboration. The analysis points towards the possibility of such learning. The analysis also provides an argument that this learning was made possible through mathematical processes characterized as follows:

- Making observations
- Connecting observations with various mathematical representations
- Explaining and justifying assertions made
- Reconstructing conceptual understanding
- Using a new conceptual frame to evaluate assertions

My analysis shows how a learner constructed and readjusted her own conceptual understanding of the content, motivated by the collaborative nature of the learning environment. Her learning was not simply learning from her peers but learning with her peers. It could be argued that she might not have moved to a new conceptual frame without the catalyst provided by collaboration characterized by an intellectual ferment (Chazan and Ball 1999) in the classroom discussion. Reflecting on Winile's learning, I can see the important role that collaboration played in her learning.

I have also shown that the teacher is central in collaborative learning. I have shown how I created the conditions of possibility for the collaboration and provided a mathematical voice at certain key moments. From me, there is a strong message to teachers here, which is that this kind of teaching is much harder than traditional teaching. If we are to continue to use the word "facilitate" to describe the teaching we would like to do, we should understand that facilitation requires much more work than we are used to. From this experience, I have seen that lessons in which teachers support mathematical reasoning in their learners through collaborative learning are very time consuming. These lessons are important in that they allow for greater conceptual understanding and reasoning as was shown in the analysis. However, it may be the case that less content is covered, as happened in my class. I suggest that researchers look into developing learning materials and teaching methods that will enable teachers to cover content in a more integrated way, so that more content can be covered, while reasoning is simultaneously developed.

In concluding this chapter, so much is still left unsaid. What is clear to me, however, is that the teaching of mathematical reasoning is achievable through a collaborative learning environment with effective whole-class discussions. A lot of research still needs to be done in establishing sound pedagogy to facilitate this type of teaching. It is my hope that this project will spark a flame in many teachers and researchers to initiate more rigorous research and reflection.